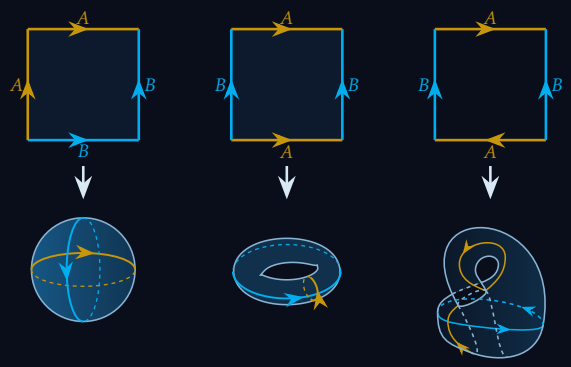
Global Analysis

An introduction to nonlinear analysis and its variational methods on Riemannian manifolds



$$\int_M \left\langle \nabla^s F, G \right\rangle_{g,h_E} d\lambda_g = (-1)^s \int_M \left\langle F, \left(\operatorname{tr}_g \circ \nabla\right)^s G \right\rangle_{g,h_E} d\lambda_g + \sum_{j=0}^{s-1} (-1)^{s-1-j} \int_{\partial M} \left\langle \nabla^j F, \iota_\nu \left(\left(\operatorname{tr}_g \circ \nabla\right)^{s-1-j} G \right) \right\rangle_{g,h_E} d\lambda_{\widetilde{g}}$$

Integration-by-parts formula for vector bundles

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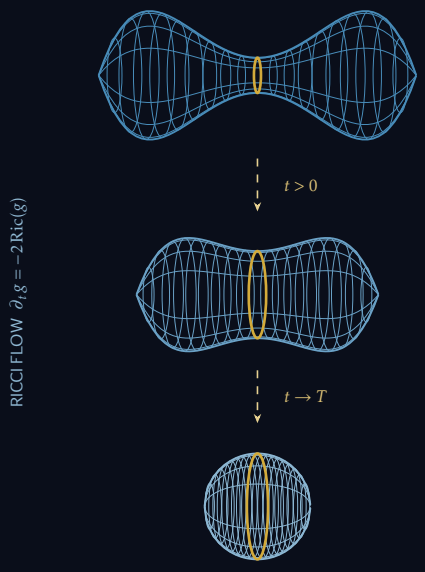

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> *"It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It only gives nominal definitions for them, while the essential means of determining them appear in the form of axioms. The relationship of these presumptions is left in the dark; one sees neither whether and in how far their connection is necessary, nor a priori whether it is possible. From Euclid to Legendre, to name the most renowned of modern writers on geometry, this darkness has been lifted neither by the mathematicians nor the philosophers who have laboured upon it."*
>
> — **Bernhard Riemann**

List of symbols

Sets and general conventions

$\mathbb{K}$ Scalar field, equal to $\mathbb{R}$ or $\mathbb{C}$

$\mathbb{N}$ Set of natural numbers

$\mathbb{N}_0$ The set $\mathbb{N} \cup \{0\}$

$\mathbb{Z}$ Set of integers

$\mathbb{R}^n$ Real Euclidean space of dimension n

$\mathbb{C}^n$ Complex vector space of dimension n

$\mathbb{H}^n$ Upper Euclidean half-space of dimension n

$\mathbb{S}^n$ Unit sphere of dimension n

e_j The jth vector of the standard basis of $\mathbb{R}^n$

$B_d(x, r)$ Open ball with center x and radius r for the metric d

$\overline{B}_d(x, r)$ Closed ball with center x and radius r for the metric d

$A \Subset U$ The set A is compactly contained in U

$\operatorname{supp}(f)$, $\operatorname{supp}(\mathbf{f})$ Support of the function f or the section $\mathbf{f}$

$\mathbf{1}_A$ Indicator function of the set A

$\operatorname{span}(A)$ Vector subspace spanned by A

$\sum_{i=1}^{k} a_i$ Sum of the terms a_i

$\prod_{i=1}^{k} a_i$ Product of the terms a_i

$\coprod_{j \in I} A_j$ Disjoint union of the family $(A_j)_{j \in I}$

$\binom{n}{k}$ Binomial coefficient

α Multi-index, in the context of coordinate calculations

$D^\alpha u$ Partial derivative of u associated with the multi-index α

Topology and functional analysis

(X, τ) Topological space X with topology τ

$\overline{A}$ Closure of the set A

$\sim$ Equivalence relation

$\overline{X}^{\|\cdot\|}$ Completion of the normed space $(X, \|\cdot\|)$

X' Continuous dual of the topological vector space X

V^* Algebraic dual of the vector space V

$\overline{V}$ Complex conjugate vector space of V

$\mathrm{Hom}(V, W)$ Space of linear transformations from V to W

$\mathcal{L}(X, Y)$ Space of continuous linear operators from X to Y

$\langle \cdot, \cdot \rangle$ Inner product or duality pairing, according to context

$\|\cdot\|$ Norm of the space under consideration

$\|\cdot\|_p$ Norm of a Lebesgue space

$x_j \rightharpoonup x$ Weak convergence of x_j to x

$x_j \overset{*}{\rightharpoonup} x$ Weak-$*$ convergence of x_j to x

$\sigma(X, Y)$ Weak topology on X induced by duality with Y

$(X, \mathcal{A}, \mu)$ Measure space with σ-algebra $\mathcal{A}$ and measure μ

λ_n Lebesgue measure on $\mathbb{R}^n$

$L^p(X, \mathcal{A}, \mu)$ Lebesgue space on the measure space $(X, \mathcal{A}, \mu)$

$M_{m\times n}(\mathbb{K})$ Matrices of size $m \times n$ with entries in $\mathbb{K}$

A^T Transpose of the matrix A

A^* Hermitian adjoint of a matrix; for operators, only when the context specifies the Hilbert inner products

$T^\dagger$ Bounded Hilbert space adjoint of T when it must be distinguished from the differential formal adjoint

$\mathcal{D}(A)$ Domain of the linear operator A

$\mathcal{G}(A)$ Graph of the linear operator A

$M^\perp$ Orthogonal complement of the subset M in a Hilbert space

$\mathcal{K}(X, Y)$ Space of compact linear operators from X to Y

$\mathcal{L}^m(X_1,\dots,X_m;Y)$ Space of continuous m-linear operators from $X_1\times\cdots\times X_m$ to Y

$\mathcal{L}^m_s(X;Y)$ Subspace of continuous symmetric m-linear operators

$D^m f(x)$ Fréchet derivative of order m of f at x

$\delta^m f(x)$ Gâteaux derivative of order m of f at x

$\mathrm{Car}(\Omega\times\mathbb{R}^m,\mathbb{R}^\ell)$ Carathéodory functions with the indicated domains and codomains

$\delta_j f(x)$ Partial Gâteaux derivative of f with respect to the jth variable

$D_j f(x)$ Partial Fréchet derivative of f with respect to the jth variable

$\mathcal{N}_f$ Nemytski or superposition operator associated with f

$\mathrm{Iso}(X,Y)$ Set of continuous linear isomorphisms from X onto Y

$(X_0,X_1)_{\theta,q}$ Real interpolation space of the compatible couple (X_0,X_1)

$[X_0,X_1]_\theta$ Complex interpolation space of the compatible couple (X_0,X_1)

$K(t,u;X_0,X_1)$ Lions–Peetre K-functional

$J(t,u;X_0,X_1)$ Lions–Peetre J-functional

$\mathcal{S}(\mathbb{R}^n)$ Schwartz class on $\mathbb{R}^n$

$\mathcal{S}'(\mathbb{R}^n)$ Space of tempered distributions

$\mathcal{F}u=\widehat{u}$ Fourier transform of u with unitary normalization

X/Y Quotient of the normed space X by the closed subspace Y

$BC^j(U,F)$, $BC^\infty(U,F)$ Functions of class C^j with bounded derivatives up to order j; the infinite-order class is the intersection over all orders

$BUC^j(U,F)$, $BUC^\infty(U,F)$ Subspaces of BC^j whose derivatives up to order j are uniformly continuous

Differential geometry

$C^k(M,N)$ Maps of class C^k between the smooth manifolds M and N

$C^k_c(M)$ Functions of class C^k with compact support in M

$C^\infty(M)$ Smooth real-valued functions on M

TM Tangent bundle of the smooth manifold M

T^*M Cotangent bundle of the smooth manifold M

T_pM Tangent space of M at the point p

$\mathfrak{X}(M)$ Space of smooth vector fields on M

∂_i Coordinate vector field $\frac{\partial}{\partial}x^i$

$\mathbf{d}x^i$ Coordinate covector field

df Differential of the function f

$T^{(k,l)}(TM)$ Bundle of k-contravariant and l-covariant tensors

$F^{i_1\dots i_k}_{j_1\dots j_l}$ Coordinate components of a tensor of type (k,l)

$\otimes$ Tensor product

$\wedge$ Wedge product

$\bigwedge^k(T^*M)$ Bundle of alternating k-tensors

$\Omega^k(M)$ Space of differential k-forms on M

$d\omega$ Exterior derivative of the differential form ω

$F\restriction_A$ Restriction of F to the subset A

∂M Boundary of a manifold with boundary M

Riemannian geometry

$(M,\mathbf{g})$ Riemannian manifold M with metric $\mathbf{g}$

g_{ij} Local coefficients of the Riemannian metric $\mathbf{g}$

g^{ij} Coefficients of the inverse matrix of (g_{ij})

$\langle\cdot,\cdot\rangle_{\mathbf{g}}$ Inner product induced by the metric $\mathbf{g}$

$|\cdot|_{\mathbf{g}}$ Pointwise norm induced by the metric $\mathbf{g}$

$\lambda_{\mathbf{g}}$ Riemann–Lebesgue measure associated with $\mathbf{g}$

$d\lambda_{\mathbf{g}}$ Integration element with respect to the Riemann–Lebesgue measure

$dV_{\mathbf{g}}$ Volume form of $\mathbf{g}$ when an orientation has been fixed

$d_{\mathbf{g}}$ Riemannian distance

$L_{\mathbf{g}}(\gamma)$ Riemannian length of the curve γ

$\mathrm{inj}(M,\mathbf{g})$ Injectivity radius of the Riemannian manifold

$\mathrm{grad}_{\mathbf{g}} f$ Riemannian gradient of f

$\mathrm{div}_{\mathbf{g}} X$ Riemannian divergence of the vector field X

$\Delta_{\mathbf{g}}$ Laplace–Beltrami operator, $\mathrm{div}_{\mathbf{g}}\,\mathrm{grad}_{\mathbf{g}}$

$\mathrm{Rm}_{\mathbf{g}}$ Riemann curvature tensor of the metric $\mathbf{g}$

$\mathrm{Ric}_{\mathbf{g}}$ Ricci tensor of the metric $\mathbf{g}$

$R = R_{\mathbf{g}}$ Scalar curvature of the metric $\mathbf{g}$

Vector bundles and sections

$\mathbf{E} \longrightarrow M$ Vector bundle $\mathbf{E}$ over the manifold M

$\mathbf{E}^*$ Finite-rank dual vector bundle of $\mathbf{E}$, in accordance with the finite-dimensional convention

$\overline{\mathbf{E}}$ Complex conjugate vector bundle of $\mathbf{E}$

$\Gamma(\mathbf{E})$ Space of smooth sections of $\mathbf{E}$

$\Gamma^k(\mathbf{E})$ Space of sections of class C^k of $\mathbf{E}$

$\Gamma_c^k(\mathbf{E})$ Compactly supported sections of class C^k of $\mathbf{E}$

$\Gamma_c(\mathbf{E})$ Smooth compactly supported sections of $\mathbf{E}$

$\mathbf{h}_{\mathbf{E}}$ Bundle metric on the vector bundle $\mathbf{E}$

$\boldsymbol{\mathcal{R}}_{\mathbf{E}} \colon \overline{\mathbf{E}} \to \mathbf{E}^*$ Complex-linear Riesz isomorphism, $\boldsymbol{\mathcal{R}}_{\mathbf{E}}(\overline{v})(w) = \mathbf{h}_{\mathbf{E}}(w, v)$

$\langle \cdot, \cdot \rangle_{\mathbf{h}_{\mathbf{E}}}$ Pointwise inner product induced by $\mathbf{h}_{\mathbf{E}}$

$|\cdot|_{\mathbf{h}_{\mathbf{E}}}$ Pointwise norm induced by $\mathbf{h}_{\mathbf{E}}$

$\nabla^{\mathbf{E}}$ Connection on the vector bundle $\mathbf{E}$

∇_i Covariant derivative in the direction of the coordinate vector field ∂_i

$\nabla^k \mathbf{F}$ The kth covariant derivative of the tensor field or section $\mathbf{F}$

$\nabla_w^k \mathbf{F}$ The kth weak covariant derivative of $\mathbf{F}$

$\boldsymbol{\mathcal{D}}(M, \mathbf{E})$ Space of test sections of the bundle $\mathbf{E}$

$\boldsymbol{\mathcal{D}}'(M, \mathbf{E}, W)$ W-valued distributions in the vector bundle $\mathbf{E}$

$\boldsymbol{\mathcal{E}}'$ Continuous dual bundle of a Banach vector bundle $\boldsymbol{\mathcal{E}}$

$V_e\mathbf{E}$, $V\mathbf{E}$ Vertical space at $e \in \mathbf{E}$ and vertical subbundle of $T\mathbf{E}$

w_e^{V} Vertical lift of $w \in \mathbf{E}_{\pi_{\mathbf{E}}(e)}$ at e

$K^{\nabla^{\mathbf{E}}} \colon T\mathbf{E} \longrightarrow \mathbf{E}$ Connection map induced by $\nabla^{\mathbf{E}}$

$H_e^{\nabla^{\mathbf{E}}}\mathbf{E}$, $H^{\nabla^{\mathbf{E}}}\mathbf{E}$ Horizontal space at e and horizontal subbundle induced by $\nabla^{\mathbf{E}}$

X_e^{H} Horizontal lift of $X \in T_{\pi_{\mathbf{E}}(e)}M$ at e

$\pi^*\nabla^{\mathbf{E}}$ Pullback connection on $\pi^*\mathbf{E}$

∇^{H}, ∇^{V} Horizontal and vertical covariant derivatives on the cotangent bundle

Sobolev spaces

$L^p(\Omega)$ Lebesgue space on the open subset $\Omega \subseteq \mathbb{R}^n$

$L^p(M)$ Lebesgue space with respect to the measure $\lambda_{\mathbf{g}}$

$L^p(M,\mathbf{E})$ Space of p-integrable sections of the bundle $\mathbf{E} \longrightarrow M$

$L^p(M,\mathbb{R}^n)$ Space of measurable functions from M to $\mathbb{R}^n$ whose Euclidean bundle norm is p-integrable

$W^{m,p}(\Omega)$ Sobolev space of order m on Ω

$W^{m,p}(M)$ Sobolev space of order m on the manifold M

$W^{m,p}(M,\mathbf{E})$ Sobolev space of sections of the bundle $\mathbf{E}$

$W^{m,p}_{\mathrm{loc}}(M,\mathbf{E})$ Local Sobolev space of sections of the bundle $\mathbf{E}$

$W^{m,p}_0(M)$ Closure of $C^\infty_c(\operatorname{Int} M)$ in $W^{m,p}(M)$

$W^{m,p}_0(M,\mathbf{E})$ Closure of $\Gamma_{c,\operatorname{Int}(M)}(\mathbf{E})$ in $W^{m,p}(M,\mathbf{E})$

$H^{s,p}(M)$ Bessel potential space of order s on the manifold M

$H^{s,p}(M,\mathbf{E})$ Bessel Sobolev space of sections, defined by uniform localization

$H^s_{\mathrm{loc}}(M,\mathbf{E})$ Local Sobolev space of sections of the bundle $\mathbf{E}$

$H^{s,p}_L(M)$ Intrinsic Bessel scale associated with $L = \Delta_B = -\Delta$

$H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})$ Intrinsic Bessel scale associated with the connection Laplacian $L_{\mathbf{E}}$

p^* Critical Sobolev exponent corresponding to the context

Tr, $\mathbf{Tr}^{\mathbf{E}}$ Trace operators for functions and for sections of the bundle $\mathbf{E}$, respectively

$H^{s,p}(\mathbb{R}^n)$ Bessel potential space of order s

$W^{s,p}(\Omega)$ Sobolev–Slobodeckij space of noninteger order s

$W^{s,p}(M,\mathbf{E})$ Intrinsic Sobolev–Slobodeckij space of sections of the bundle $\mathbf{E}$, defined using covariant derivatives and parallel transport

$[u]_{W^{s,p}(\Omega)}$ Gagliardo–Slobodeckij seminorm

$B^s_{p,q}(M)$, $B^s_{p,q}(M,\mathbf{E})$ Besov spaces of functions and sections on M

$F^s_{p,q}(M)$, $F^s_{p,q}(M,\mathbf{E})$ Triebel–Lizorkin spaces of functions and sections on M

$J^s = (I-\Delta)^{\frac{s}{2}}$ Bessel operator of order s

Differential operators

$\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ Partial differential operators of order at most m from $\mathbf{E}$ to $\mathbf{F}$

P^*_h Hermitian formal adjoint of the differential operator P, with respect to the bundle metrics and the Riemannian measure

P' Complex-bilinear formal transpose, $P'\colon \Gamma(\mathbf{F}^*) \to \Gamma(\mathbf{E}^*)$

$\sigma_m(P)$ Principal symbol of order m of the operator P

$d^{\mathbf{E}}$ Exterior derivative with values in the bundle $\mathbf{E}$

$\delta^{\mathbf{E}}$ Hermitian formal adjoint of $d^{\mathbf{E}}$

Δ_B Bochner Laplacian with the positive sign convention, $(\nabla)_h^*\nabla$

Δ_H Hodge Laplacian

$\Delta_{p,g}$ p-Laplacian associated with the metric g

$S^m_{1,0}$ Hörmander symbol class of order m and type $(1,0)$

S^m_{cl} Class of classical symbols of order m

$\Psi^m(M;\mathbf{E},\mathbf{F})$ Pseudodifferential operators of order m from sections of $\mathbf{E}$ to sections of $\mathbf{F}$

$\mathrm{WF}(u)$ Wavefront set of the distribution u

$S^{-\infty}(U\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$ Euclidean smoothing symbols

$\Psi^{-\infty}(M;\mathbf{E},\mathbf{F})$ Properly supported smoothing operators on M

$\Psi^m_{\mathrm{u},\rho}(M;\mathbf{E},\mathbf{F})$ Radius-ρ stage of the uniform pseudodifferential calculus

$\Psi^{-\infty}_{\mathrm{u}}(M;\mathbf{E},\mathbf{F})$ Uniformly smoothing operators of finite propagation

Infinite-dimensional geometry

X, Y Banach spaces serving as model spaces

$DI(u)$ Fréchet derivative of the functional I at u

$T'M$ Continuous cotangent bundle of a Banach manifold

dF_p Differential of a map between Banach manifolds

$T_p^{\mathrm{kin}}M$ Kinematic tangent space of the Banach manifold M

$T_p^{\mathrm{op}}M$ Operational tangent space of the Banach manifold M

T_pN Tangent space of the Banach submanifold N at p

$\mathrm{GL}(E)$ Group of continuous invertible linear operators on the Banach space E

$\flat_{\mathbf{g}}$, $\sharp_{\mathbf{g}}$ Musical morphisms induced by a strong Riemannian metric

F_x, F_x^* Finsler norm on T_xM and its dual norm on the continuous dual $T_x'M$

d_F Length distance induced by a Banach–Finsler structure

$W^{r,p}(M,N)$ Manifold of Sobolev maps from M to N

$\mathcal{A}^r(\mathbf{P})$ Affine Sobolev space of connections on a principal bundle $\mathbf{P}$

$\mathcal{G}^{r+1}(\mathbf{P})$ Gauge group completed in Sobolev regularity

$\mathrm{Diff}^s(M)$ Sobolev completion of the diffeomorphism group

$H^\infty = \bigcap_{s \geq s_0} H^s$ Smooth core of a Hilbert scale

$\varprojlim H^s$ Projective limit of a Hilbert scale

$\mathrm{Met}^r(M)$ Open space of Riemannian metrics of Sobolev regularity r

$C^j_b(M, \mathbf{E})$ Sections of class C^j whose covariant derivatives up to order j are bounded

$C^{\infty,r}_b(M, N)$ Smooth maps whose covariant derivatives of df up to order $r-1$ are bounded

$\mathrm{Met}(I, B_k)(M)$ Complete metrics with positive injectivity radius and curvature bounded up to order k

Variational methods

I Variational functional or energy functional

$DI(u) = 0$ Critical point condition for the functional I

$J^{-1}(c)$ Level set at level c of the functional J

$\mathcal{N}$ Nehari manifold associated with a functional

$I \restriction_{\mathcal{N}}$ Restriction of the functional I to the Nehari manifold

$T_u\mathcal{N}$ Tangent space of the Nehari manifold at u

λ Lagrange multiplier, when it occurs in a variational constraint

K_c Set of critical points at level c

$I^{\leq a}$ Closed sublevel set $\{u \mid I(u) \leq a\}$

$I^{<a}$ Open sublevel set $\{u \mid I(u) < a\}$

$(PS)_c$ Palais–Smale condition at level c

Φ_t Time-t map of a deformation flow

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Introduction

When we begin studying a differential equation on a manifold, something familiar happens: we choose a chart and, for a moment, the problem becomes one defined on an open subset of $\mathbb{R}^n$. There we can differentiate, integrate, and use many of the tools of Euclidean analysis. The difficulty arises when we want to leave that chart. We must then know which parts of the calculation depended on coordinates, which quantities have intrinsic meaning, and how to combine the estimates obtained locally to draw a conclusion about the entire manifold.

Global analysis studies precisely this passage from the local to the global. Coordinates do not disappear: they remain an indispensable computational tool. What matters is understanding which information survives a change of chart and how geometry modifies arguments we already know in $\mathbb{R}^n$. A connection allows us to differentiate sections of a bundle; a metric allows us to measure and integrate them; Sobolev spaces make it possible to work with solutions that are not smooth; spectral theory and heat relate local operators to global information; and variational methods turn many equations into energy problems on infinite-dimensional spaces.

This is the viewpoint that guides the book. Rather than presenting differential geometry, functional analysis, and partial differential equations separately, we want to see how these theories meet in the study of a concrete problem on a manifold. Throughout the text, we will therefore return to the same pattern: first formulate the object intrinsically, then express it in coordinates when a calculation is needed, and finally check which hypotheses allow us to pass from local estimates back to a global statement.

Much of this book grew out of teaching in the graduate program in Mathematical Sciences, in the courses *Nonlinear analysis on Riemannian manifolds I and II* and *Variational methods on Riemannian manifolds I and II*. These courses were led by Dr. María de los Ángeles Sandoval Romero, who taught the nonlinear analysis courses jointly with Carlos Daniel Velázquez Mendoza, M.Sc., and the variational courses jointly with Dr. Romulo Diaz Carlos.

In the classroom, many arguments can be completed with an explanation at the blackboard, an example, or a reference to something covered a few weeks earlier. Turning that material into a book required these connections to be put into writing. This is the source of our concern to state the hypotheses of each result carefully, recover the intermediate steps of proofs, and show explicitly where geometry enters an argument that might at first appear purely analytic.

Note to the reader. This manuscript is a work in progress and may still undergo minor corrections and editorial refinements. Comments, suggestions, and corrections are very welcome.

From local calculus to intrinsic geometry

The starting point is differential calculus. Newton and Leibniz developed, in different languages, tools for studying velocities, tangents, areas, and accumulated quantities. Over time, these ideas acquired increasingly precise formulations: first in one variable, then in several variables, and finally in more general spaces. The derivative ceased to be viewed merely as a computational rule and came to be understood as the linear approximation describing the first-order behavior of a nonlinear object.

Differential geometry then required an answer to a new question: what does differentiation mean when the space on which we work is curved? Gauss showed that the curvature of a surface can be recognized from measurements made within the surface itself, without reference to the ambient space. Riemann extended this idea to arbitrary dimensions and proposed describing geometry through a way of measuring lengths and angles infinitesimally. Later, Poincaré brought another difficulty to the forefront: knowing each small region of a space well is not necessarily enough to understand its global shape. The work of Gauss, Riemann, and Poincaré thus marks the passage from calculus on embedded figures to intrinsic geometry, and then to the study of truly global properties [65, 121, 118].

Isaac Newton
(1642–1727)

G. W. Leibniz
(1646–1716)

C. F. Gauss
(1777–1855)

B. Riemann
(1826–1866)

H. Poincaré
(1854–1912)

Figure 1. From infinitesimal calculus to global geometry. Newton and Leibniz established the fundamental languages of calculus; Gauss discovered the intrinsic nature of surface curvature; Riemann extended metric geometry to arbitrary dimensions; Poincaré made the global organization of spaces a central mathematical problem.

Gauss, Riemann, and the intrinsic viewpoint

In his *Disquisitiones generales circa superficies curvas* of 1828, Gauss studied the geometry of a surface through its first fundamental form and established that Gaussian curvature is intrinsic. Riemann's habilitation lecture of 1854, published posthumously in 1868, changed the scale of the problem: a metric could be considered on spaces of any dimension, and curvature became a property of the space itself. The tensor language and connections subsequently developed by Christoffel, Ricci, Levi–Civita, and Cartan gave this vision an operational form. In this book, that history reappears concretely: tangent spaces and tensors are constructed first; the metric, connection, geodesics, and curvature are then introduced; and finally these structures become coefficients, operators, and energies.

This historical development also explains the order we will follow. The covariant derivative, Laplacian, Sobolev norms, and principal symbol will be introduced, whenever possible, as intrinsic objects. Coordinates will appear when they are needed for calculations, comparison with a Euclidean model, or the proof of an estimate. We do not seek to avoid local calculations, but to use them without confusing the coordinate expression with the geometric object it represents.

The emergence of function spaces

Geometry is not the only difficulty. When the unknown in an equation is a function or a section, the set of possible solutions no longer lies in a finite-dimensional space. To study a minimizing sequence, we need to know in what sense it can converge; to discuss weak derivatives, we need a space that records both size and regularity; to apply variational methods, we need a structure in which continuity, compactness, and differentiation make sense. This was one of the fundamental changes in twentieth-century analysis. In 1906, Fréchet introduced a systematic theory of metric spaces and functional calculus [62], and in 1932 Banach organized a decisive portion of linear analysis around complete normed spaces [18]. Functions could then be studied as points of a space that, although infinite-dimensional, had an appropriate geometry and topology.

Sobolev showed that a function can have derivatives in a sense compatible with integration by parts even when it is not classically differentiable, and his embedding theorems related the number of integrable derivatives to the regularity of the function [136]. Shortly afterward, Laurent Schwartz developed distribution theory, which allows singular objects such as the Dirac delta to be treated rigorously and extends differentiation to a much broader class [131]. These ideas made it natural to begin with weak solutions, obtain estimates, and then ask whether the equation itself forces regularity to be recovered.

Sobolev and Schwartz: regularity without classical smoothness

The conceptual change lay not in abandoning derivatives but in recognizing them through their integral effects. In a Sobolev space, the derivative is identified by an integration-by-parts formula; in distribution theory, that same formula becomes the definition of differentiation. This perspective allows us to begin with data of low regularity, obtain a priori estimates, and then prove that the solution has more regularity than initially seemed available. The route we will follow retraces this development: weak derivatives, Sobolev spaces, distributions, Fourier analysis, interpolation, fractional regularity, and, later, their geometric versions.

What makes global analysis global

In this context, the adjective *global* does not simply mean that the manifold is large or cannot be covered by a single chart. It means that the answer may depend on how the entire space is organized. An elliptic equation is written locally, yet the dimension of its kernel may contain topological information; a Sobolev inequality is proved in charts, yet its constants may depend on volume, curvature, or behavior at infinity; a geodesic satisfies an ordinary differential equation, yet its extendibility for all time is a global property; Ricci flow is a parabolic equation for a metric, yet the study of its singularities ultimately reveals the topology of the manifold.

This relationship between analysis and topology became particularly clear through Hodge theory, Morse theory, and the study of elliptic operators. The Atiyah–Singer theorem subsequently gave that relationship a much more general form: the analytic index of an elliptic operator agrees with an index constructed from its symbol and the topology of the manifold [11]. Microlocal analysis, the pseudodifferential calculus, and K-theory arise naturally on the way to this statement. Instead of presenting them as separate topics, we will develop them as parts of a single chain of ideas.

Another way to obtain global information is to let the geometry evolve. In 1982, Hamilton introduced Ricci flow to deform a metric according to its Ricci curvature [75]. Shortly afterward, DeTurck identified the gauge that reveals a strictly parabolic system behind the diffeomorphism invariance [43]. Two decades later, Perelman intro-

duced monotone quantities, the noncollapsing principle, and new tools for studying singularities and completing the geometrization program in dimension three [114, 116, 115]. This example captures the spirit of geometric analysis: geometry suggests an equation, analysis makes it possible to control its solutions, and the behavior of those solutions ultimately provides information about the original space.

Atiyah–Singer: a bridge between two ways of counting

For a Fredholm operator, the index is the difference between the dimension of the solution space and the dimension of the obstruction space. At first sight, it is an analytic number: it is obtained by solving an equation. The Atiyah–Singer theorem asserts that this same integer can be calculated from topological data of the principal symbol. The result does not erase the distance between analysis and topology; it explains why they describe the same phenomenon from different ends. For this reason, the part devoted to index theory does not begin with the final formula: it first constructs the pseudodifferential calculus, Fredholm theory, the symbol class, Bott periodicity, and the Thom isomorphism.

A contemporary perspective

The preceding questions remain open in many directions and today constitute a broad part of global and geometric analysis. Elliptic and spectral theory study how geometry influences the kernel of an operator, its spectrum, and heat. On noncompact manifolds, bounded geometry allows local estimates to be recovered with constants independent of the point. In the presence of a boundary, traces, boundary terms, and the need for precise control of the normal direction arise. Spaces of maps, connections, and metrics lead naturally to infinite-dimensional geometry, while geometric flows use evolution, rescaling, and compactness to study curvature and singularities.

Recent problems directly related to the themes of the book include new forms of approximation by smooth objects in Sobolev spaces on manifolds [33]; the development of Sobolev, Besov, and Bessel potential scales on manifolds with boundary, bounded geometry, or singularities [6]; characterizations of traces that incorporate the connection and curvature [127]; and intrinsic formulations of integration by parts and Sobolev spaces on vector bundles [41]. These examples are different, but they share a common concern: determining precisely which geometric hypotheses allow a familiar analytic tool to be transferred to the global setting.

For this reason, throughout the book we will not treat settings requiring different arguments as interchangeable. Compactness can replace uniform control at infinity; a boundary introduces traces and boundary terms; the curvature of a connection reappears when derivatives are commuted; and a weak metric in infinite dimensions does not automatically possess the properties of a strong metric. Distinguishing these hypotheses is more than a matter of precision in statements: it is often exactly what makes it possible to understand why a proof works.

The approach to exposition

The book is written so that, whenever possible, the reader can follow the full development of a tool from the Euclidean case to its geometric version. First, the local result is formulated with the hypotheses it actually needs; we then study what happens under multiplication by cutoff functions, changes of coordinates, or passage from one trivialization to another; finally, the result is globalized. Vector bundles also bring bundle metrics and connections into play; noncompact geometry requires uniform control of constants; and in the presence of a boundary, adapted charts and Fermi coordinates allow the normal direction to be separated from the tangential directions.

For the same reason, many proofs are developed in greater detail than would be necessary in a text intended solely as a reference. When an argument is localized, the cutoffs, supports, and norms being compared are specified; when a partition of unity is used, the contribution of its terms is shown; and when a proof depends on an external result, the point at which it is used is identified. The aim is for the logical chain to be reconstructible from the text itself, so that phrases such as “the argument is local” or “this follows by a partition of unity” do not conceal the steps where the real substance lies.

The longer chapters can be read at different levels of depth. A first reading allows the reader to follow the motivation, definitions, and main results; a closer reading allows the proofs to be reconstructed and the roles of the metric, connection, compactness, and boundary to be identified. The examples and exercises then help verify that the constructions are more than formal. Rather than memorizing a list of theorems, the goal is to recognize a few ideas that recur throughout the book in different forms.

Organization of the book

The first part assembles the geometric language used throughout the book. It begins with smooth manifolds, maps, tangent spaces, submanifolds, vector bundles, flows, differential forms, and Lie groups and algebras. Riemannian metrics, principal and associated bundles, connections, parallel transport, geodesics, distance, curvature, and integration then follow. These tools also allow the construction of Clifford algebras, Spin and Spin^c structures, spinor bundles, and their connections. Dirac operators are introduced later, once a general theory of differential operators on bundles is available. The general references for this part are [94, 95, 109, 21].

The second part serves as a Euclidean laboratory. Integer-order Sobolev spaces are constructed from weak derivatives, distributions and Fourier analysis are developed, and interpolation is introduced. Bessel potentials and Slobodeckij, Besov, and Triebel–Lizorkin spaces then organize regularity of real order. This is the setting in which we prove the multiplication, localization, change-of-variables, embedding, compactness, and trace results needed for the passage to manifolds [1, 61, 97, 98, 20, 145, 146, 130].

With these tools available, the third part turns to analysis on manifolds and vector bundles. Sobolev spaces are defined through weak covariant derivatives, and differential operators and bundle-valued distributions are developed. The integration-by-parts formula, Kato and Poincaré inequalities, embeddings, compactness, and trace theorems appear here. The connection Laplacian and the heat semigroup then provide an intrinsic construction of spaces of fractional order. Finally, bounded geometry allows these tools to be retained on noncompact manifolds with uniform estimates, and traces on submanifolds and boundaries to be studied [14, 79, 78, 53, 109, 69, 72, 6].

The fourth part is devoted to index theory. The pseudodifferential calculus allows parametrices for elliptic operators to be constructed and their regularity to be studied; Fredholm theory turns this information into an analytic index; K-theory organizes symbol classes; and Bott periodicity together with the Thom isomorphism leads to the topological index. These constructions finally come together in the Atiyah–Singer theorem for closed manifolds and several of its fundamental geometric examples.

The fifth part concerns geometric evolution equations. We begin with a parabolic theory for quasilinear systems on closed manifolds and apply it to Ricci flow. Diffeomorphism invariance leads to DeTurck’s procedure; curvature evolution equations, maximum principles, a priori estimates, and rescalings prepare the study of singularities. The part concludes with Hamilton compactness and Perelman’s tools: monotone quantities, noncollapsing, κ-solutions, and surgery.

The sixth part changes scale and takes geometry into infinite dimensions. Fréchet calculus on Banach spaces leads to Banach manifolds, bundles, connections, flows, and Hilbert or Banach–Finsler structures. Spaces of maps,

metrics, and connections, gauge groups, and diffeomorphism groups then enter the picture, with fixed regularity carefully distinguished from ILH scales. The application chapters show how harmonic maps, Yang–Mills theory, and the Euler equations can be viewed as geometric problems on configuration spaces [28, 31, 91, 85, 46].

The seventh part brings together the variational methods we will use to prove the existence of solutions. Weak formulations and Nemytski operators allow energy functionals to be constructed; the direct method combines coercivity, reflexivity, and semicontinuity; the Palais–Smale condition and pseudogradients allow sublevel sets to be deformed; and regular constraints and the Nehari method allow the search for nontrivial critical points. The emphasis is on how these techniques adapt to the function space and the geometry of the problem, rather than on treating them as isolated procedures [138, 112, 45, 13, 87].

The appendices collect the tools of topology, mathematical analysis, linear algebra, and tensor notation consulted throughout the book. Their purpose is to let the main development remain continuous without concealing the auxiliary results that support the proofs.

Reading paths

Although the chapters are arranged for continuous reading, it is not necessary to work through the entire book before reaching a particular topic. For Sobolev spaces and variational problems on compact manifolds, a natural route passes through Chapters 1–9 and continues with Chapters 29–32. For trace theory and bounded geometry, it is useful to add Chapters 12–14. Index theory uses the Clifford geometry of the first part, the differential construction of Dirac operators in Chapter 8, and Chapters 11 and 15–19. Ricci flow requires Riemannian geometry and elliptic and Sobolev theory, and is developed in Chapters 20–22. Infinite-dimensional geometry can be studied starting in Chapter 23, although its applications frequently use the Sobolev spaces constructed earlier.

These paths continually intersect. Integration by parts reappears in the definition of weak derivatives, formal adjoints, and Euler–Lagrange equations; compactness underlies Rellich–Kondrashov, Fredholm theory, and Palais–Smale; a connection serves both to differentiate sections and to construct Laplacians, study heat, or describe configuration spaces; and the principal symbol begins as a local object and ends by relating regularity, parametrices, and the topological index. An important part of the book consists precisely in making these connections visible.

The book is intended for advanced undergraduate and graduate students, as well as readers approaching global analysis from geometry or differential equations. Basic knowledge of real analysis, topology, linear algebra, and multivariable calculus is assumed; when an additional tool is needed, it will be recalled in the text or a reference will be given to the appropriate appendix. The aim is that, as readers progress through these pages, they will be able to move naturally between three levels that recur throughout: the equation to be studied, the function space in which it is formulated, and the geometry of the space on which it is defined.

Part I

Riemannian manifolds

Introduction to Part I

Before studying equations, Sobolev spaces, or differential operators on a manifold, we must address a more elementary question: how can we do calculus on a space that generally has no single coordinate system? Locally, a manifold resembles an open subset of $\mathbb{R}^n$, so many constructions can be expressed in coordinates. Yet the geometric object is not the chart we choose to describe it. The first aim of this part is to learn how to pass from a local description to a formulation independent of that choice.

Chapter 1 constructs the smooth structure and the objects that arise from it: tangent and cotangent spaces, submanifolds, vector bundles, flows, differential forms, Lie groups, and smooth actions. Chapter 2 adds the Riemannian structure. Connections, parallel transport, geodesics, distance, curvature, Riemannian measure, and the basic differential operators then enter the picture. The part concludes by introducing Clifford algebras, the groups Spin and Spin^c, and spinor bundles. Dirac operators will be studied later, once the general theory of differential operators on bundles is available.

These constructions are linked by very concrete needs. Charts allow us to calculate, but the tangent space identifies the directions that exist independently of coordinates. A vector bundle brings together data that vary from point to point; a connection makes it possible to compare and differentiate them; the metric measures lengths, angles, and energies; curvature records the discrepancy that arises when transporting and differentiating in different directions. The intention in this part is for these ideas to emerge as natural responses to geometric problems, rather than as an isolated sequence of definitions.

A shared geometric tradition

Gauss showed that a surface has geometric properties that can be recognized without reference to the space in which it is drawn. Riemann extended this idea to arbitrary dimensions and made the metric the starting point of intrinsic geometry. In the following decades, Christoffel, Ricci, and Levi–Civita developed tensor calculus and the notion of a connection, while Cartan reorganized much of the theory using differential forms and moving frames. The modern language retains all these perspectives. We will therefore use coordinates, tensors, connections, and differential forms as the problem requires, without treating them as competing languages.

1

Fundamental notions of smooth manifolds

"One geometry cannot be more true than another; it can only be more convenient."
— Henri Poincaré, *Science and Hypothesis* (1902)

Differential geometry begins with a simple observation: a space may look almost Euclidean when viewed up close, yet have a global shape that no single coordinate system can describe. A sphere, a torus, or a surface with boundary admits local calculus, but its most important features emerge only when different regions are compared. The notion of a smooth manifold gives precise form to this interplay between the local and the global.

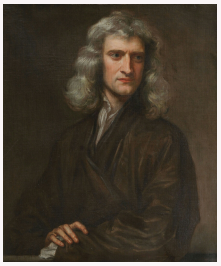
This first chapter develops the language that will accompany the reader throughout the book. We begin with derivatives in Euclidean spaces, move on to charts and changes of coordinates, and arrive at tangent spaces, vector bundles, submanifolds, flows, and differential forms. Each definition addresses a concrete geometric need and prepares an analytic tool for later use. The exposition follows primarily Lee [94, 95] and draws on Nicolaescu's approach [109], particularly where smooth geometry begins to provide the foundations for global analysis.

1.1 From calculus to differential geometry

Differential geometry extends a central idea of calculus: approximating a nonlinear object locally by linear data. Newton and Leibniz independently developed the foundations of differential and integral calculus; Leibniz's notation and Newton's dynamical interpretation continue to be reflected in the modern language of derivatives, curves, and fields.

The subsequent formalization of calculus led to the modern notions of limit, continuity, and differentiability. On a manifold, coordinates allow us to transfer a problem temporarily to an open subset of $\mathbb{R}^n$; compatibility between charts ensures that the resulting objects do not depend on this local choice. The derivative is then understood as more than a slope: it is the linear map that approximates a nonlinear map to first order.

Before introducing smooth manifolds, we recall this formulation of the derivative in Euclidean spaces. Fréchet's definition expresses linear approximation precisely and anticipates the version we will later use in Banach spaces.



Isaac Newton
(1642–1727)

Gottfried Wilhelm Leibniz
(1646–1716)

Figure 1.1. Newton and Leibniz developed the foundations of differential and integral calculus in different mathematical languages.

Definition 1.1.1. Let $U \subseteq \mathbb{R}^n$ be an open set and let $f\colon U \longrightarrow \mathbb{R}^m$ be a function. We say that f is Fréchet differentiable, or simply differentiable, at $x_0 \in U$ if there exists a linear map $L\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $\lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)-L(h)}{\|h\|} = 0$. We denote this linear map by df_{x_0} or by $Df(x_0)$. The map df_{x_0} is unique and is called the derivative of f at x_0. If f is differentiable at every point of U, we say that f is differentiable on U, or simply differentiable when the domain is understood.

In the preceding definition, it is essential that the domain of f be open. Since U is open, there exists $r > 0$ such that $B_{\text{euc}}(x_0, r) \subseteq U$. Therefore, if $h \in \mathbb{R}^n$ and $\|h\| < r$, then $x_0 + h \in U$, and the linear approximation to f can be tested in every direction in $\mathbb{R}^n$. This condition allows us to recover a unique linear map associated with f at the point x_0.

The situation changes when the domain is not open. In that case, the expression $f(x_0+h)$ makes sense only for those h for which x_0+h belongs to the domain of f. Thus the linear approximation detects the behavior of f only along directions contained in the domain. As a result, two distinct linear maps may give the same approximation on the set where the function is defined.

Example 1.1.2. Let $A = \{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and let $f\colon A \longrightarrow \mathbb{R}$ be given by $f(x,0) = x$. Consider the point $0 = (0,0) \in A$. Let $L, \widetilde{L}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the linear maps given by $L(x,y) = x$ and $\widetilde{L}(x,y) = x + y$.
If $h = (h_1, 0) \in A$, then $f(0+h) - f(0) - L(h) = h_1 - h_1 = 0$. Therefore,

$$\lim_{\substack{h\to 0 \\ h\in A}} \frac{f(0+h)-f(0)-L(h)}{\|h\|} = 0.$$

Similarly, if $h = (h_1, 0) \in A$, we have $f(0+h) - f(0) - \widetilde{L}(h) = h_1 - \widetilde{L}(h_1, 0) = h_1 - h_1 = 0$. Hence,

$$\lim_{\substack{h\to 0 \\ h\in A}} \frac{f(0+h)-f(0)-\widetilde{L}(h)}{\|h\|} = 0.$$

Thus the linear maps L and $\widetilde{L}$ satisfy the same approximation condition on A, although they are distinct as linear maps on $\mathbb{R}^2$. The reason is that the set A allows us to approach the origin only through vectors of the form $(h_1, 0)$. Consequently, the function does not detect the component transverse to the x-axis: L and $\widetilde{L}$ agree on A but differ in the transverse directions.

The preceding example explains why the usual definition of the derivative is formulated for open domains. In differential geometry, however, we need to consider functions defined on subsets that are not open in $\mathbb{R}^n$, as happens in the study of manifolds with boundary. In these cases, a natural way to define regularity is to require the existence of smooth local extensions.

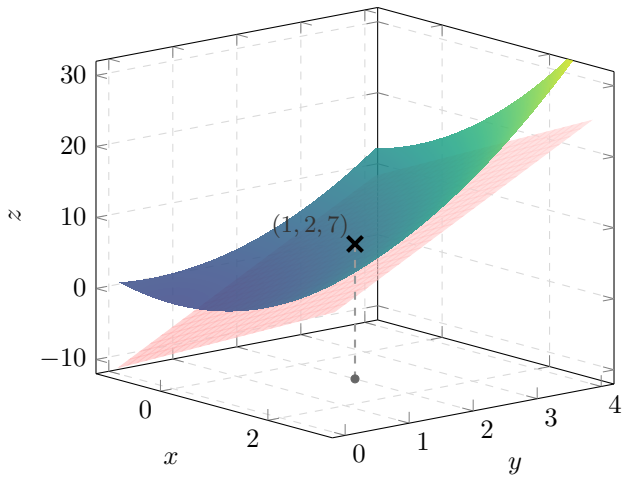
Definition 1.1.3. (Differentiability by local extension) Let $A \subseteq \mathbb{R}^n$, let $x_0 \in A$, and let $f : A \longrightarrow \mathbb{R}^m$. We say that f is differentiable at x_0 by local extension if there exist an open set $U_{x_0} \subseteq \mathbb{R}^n$, with $x_0 \in U_{x_0}$, and a function $\widetilde{f} : U_{x_0} \longrightarrow \mathbb{R}^m$ differentiable at x_0, such that $\widetilde{f} \restriction_{U_{x_0} \cap A} = f \restriction_{U_{x_0} \cap A}$.
If the extension $\widetilde{f}$ can be chosen to be of class C^k in a neighborhood of x_0, we say that f is of class C^k at x_0 by local extension.

Remark 1.1.4. The preceding definition provides a notion of regularity for functions defined on subsets of $\mathbb{R}^n$, but it does not, by itself, define an intrinsic derivative of f at x_0 as a linear map $\mathbb{R}^n \to \mathbb{R}^m$. The reason is that the derivative may depend on the chosen local extension.
The preceding example illustrates this dependence. The function $f : A \longrightarrow \mathbb{R}$ admits at least the extensions $\widetilde{f_1}(x, y) = x$ and $\widetilde{f_2}(x, y) = x + y$. Both agree with f on A, but $d(\widetilde{f_1})_{(0,0)}(v_1, v_2) = v_1$, whereas $d(\widetilde{f_2})_{(0,0)}(v_1, v_2) = v_1 + v_2$. Thus the derivatives of the extensions do not agree. This shows that, for arbitrary subsets, the existence of smooth local extensions allows us to define regularity, but does not necessarily yield a well-defined derivative.

The derivative is interpreted as the *best linear approximation to the function f at x_0*: if a function $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $x_0 \in U$, then the graph of the function $H : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $H(x) = df_{x_0}(x - x_0) + f(x_0)$ is the tangent hyperplane to the graph of f at the point $(x_0, f(x_0))$.

Example 1.1.5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by $f(x, y) = x^2 + xy + y^2$. We know that f is differentiable and that $df_{(x,y)}(u, v) = (2x + y)u + (x + 2y)v$. The tangent hyperplane to the graph of f at $(x, y, f(x, y))$ is given by $H(u, v) = df_{(x,y)}((u, v)-(x, y))+f(x, y) = (2x+y)(u-x)+(x+2y)(v-y)+x^2+xy+y^2 = 2xu+yu+xv+2yv-x^2-y^2-xy$, or, in Cartesian form, is the plane $w = 2xu + yu + xv + 2yv - x^2 - y^2 - xy$.

Surface $f(x, y) = x^2 + xy + y^2$ and its tangent plane at $(1, 2, 7)$



The notion of a tangent hyperplane illustrates especially well the power of the *derivative* to provide linear approximations to nonlinear objects, the idea underlying what is known as *nonlinear analysis*.

We can define the tangent hyperplane to a surface if we know a function describing the surface, either as the graph of the function or through a parametrization. However, the notion of a *surface* should not depend on a function describing it: the surface should be a well-defined mathematical object with fixed topological properties. As we will see later, the concept of a *differentiable manifold* resolves these difficulties.

Other approximations to a differentiable function are obtained using polynomial functions. These approximations need not be linear, but they approximate the function increasingly well, thanks to what is known as *Taylor's theorem*. The coefficients of the Taylor expansion are given by higher-order partial derivatives. These derivatives will also help us study the regularity of functions that need not be differentiable, using Sobolev spaces and the concept of a *weak derivative*. Their properties will be essential there, since several properties of weak derivatives are inherited from ordinary partial derivatives. We therefore review these derivatives and the notion of a ***multi-index*** for readers unfamiliar with this notation.

1.1.1 Higher-order partial derivatives

Before iterating partial derivatives, it is useful to recall the directional derivative. This notion connects Euclidean calculus with a construction that will later appear on manifolds: differentiating a field or a section in the direction of a vector by means of a connection.

Definition 1.1.6. (directional derivative) Let $U \subseteq \mathbb{R}^n$ be open, let $f : U \longrightarrow \mathbb{R}^m$ be a function, and let $x_0 \in U$ and $v \in \mathbb{R}^n$. We say that f is differentiable at x_0 in the direction of v if

$$\lim_{t\to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exists. This limit is called the ***directional derivative of*** f ***in the direction of*** v and is denoted by $D_v f(x_0)$. The directional derivative for $v = 0$ always exists and equals 0.
If f is differentiable at every point $x \in U$ in the direction of v, we simply say that f has a directional derivative in the direction of v or with respect to v.

These directional derivatives can be iterated to define *higher-order directional derivatives*.

Definition 1.1.7. Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $x_0 \in U$. We define higher-order directional derivatives recursively.

(a) If $v_1, v_2 \in \mathbb{R}^n$ and there exists $r_1 > 0$ such that $B_{\text{euc}}(x_0, r_1) \subseteq U$ and $D_{v_1} f(x)$ exists for every $x \in B_{\text{euc}}(x_0, r_1)$, we say that f has a second directional derivative with respect to v_1, v_2 at x_0 if

$$\lim_{t\to 0} \frac{D_{v_1} f(x_0 + tv_2) - D_{v_1} f(x_0)}{t}$$

exists, and we denote this limit by $D_{v_2,v_1} f(x_0)$.

(b) If $v_1, \ldots, v_{k+1} \in \mathbb{R}^n$ and there exists $r_k > 0$ such that $B_{\text{euc}}(x_0, r_k) \subseteq U$ and $D_{v_k,\ldots,v_1} f(x)$ is defined and exists for every $x \in B_{\text{euc}}(x_0, r_k)$, we say that f has a $(k+1)$st directional derivative with respect to

$v_1,\dots,v_{k+1}$ at x_0 if

$$\lim_{t\to 0}\frac{D_{v_k,\dots,v_1}f(x_0+tv_{k+1})-D_{v_k,\dots,v_1}f(x_0)}{t}$$

exists, and we denote this limit by $D_{v_{k+1},\dots,v_1}f(x_0)$.

If, for $v_1,\dots,v_k$, $D_{v_k,\dots,v_1}f(x)$ exists for every $x\in U$, we say that f has a directional derivative with respect to $v_1,\dots,v_k$.

Although the notion of a directional derivative is pointwise, if a function has a directional derivative at every point of its domain in the same direction (or directions), this defines a function with the same domain and codomain as the original function.

Definition 1.1.8. Let $f\colon U\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^m$ and $v_1,\dots,v_k\in\mathbb{R}^n$. If $D_{v_k,\dots,v_1}f(x)$ exists for every $x\in U$, we denote by $D_{v_k,\dots,v_1}f$ the function assigning to each $x\in U$ the value $D_{v_k,\dots,v_1}f(x)$.

This allows us to define higher-order partial derivatives, which have various uses, including polynomial approximations to sufficiently regular functions.

Definition 1.1.9. Let $f\colon U\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^m$ and $x_0\in U$. Let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{R}^n$, and let $i_1,\dots,i_k\in\{1,\dots,n\}$. If $D_{e_{i_k}\dots e_{i_1}}f(x_0)$ exists, we define the kth partial derivative of f with respect to $x^{i_1},\dots,x^{i_k}$ at x_0, denoted by $\dfrac{\partial^k f}{\partial x_{i_k}\cdots\partial x_{i_1}}(x_0)$, by $\dfrac{\partial^k f}{\partial x_{i_k}\cdots\partial x_{i_1}}(x_0):=D_{e_{i_k}\dots e_{i_1}}f(x_0)$.

If $i_1=\cdots=i_k=i$, we simply denote it by $\dfrac{\partial^k f}{\partial x_i^k}(x_0)$ and call it the kth partial derivative of f with respect to x^i at x_0. The number k is called the ***order of the partial derivative***.

If $k=0$, we simply write $\dfrac{\partial^k f}{\partial x_i^k}(x_0)=f(x_0)$.

Multi-index notation provides a compact way to express the existence of all partial derivatives up to a given order.

Definition 1.1.10. An element $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$ is called a **multi-index** of length n. Its order is $|\alpha|:=\alpha_1+\cdots+\alpha_n$. If $f\colon U\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^m$ and the indicated derivatives exist, we write

$$D^\alpha f:=\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},$$

with differentiation understood componentwise when $m>1$, and set $D^0f:=f$. We denote by e_j the multi-index whose entries are all zero except the jth, which equals one. We also set

$$\alpha!:=\alpha_1!\cdots\alpha_n!,\qquad x^\alpha:=(x^1)^{\alpha_1}\cdots(x^n)^{\alpha_n},$$

and write $\beta\le\alpha$ when $\beta_j\le\alpha_j$ for every j.

The class of functions having continuous partial derivatives up to order k will be important in defining differentiable structures.

Definition 1.1.11. (functions of class C^k) Let $U \subseteq \mathbb{R}^n$ be an open set and let $f\colon U \longrightarrow \mathbb{R}^m$. We say that $f \in C^0(U,\mathbb{R}^m)$ if f is continuous. For $k \in \mathbb{N}$, we say that $f \in C^k(U,\mathbb{R}^m)$ if, for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$, the partial derivative $D^\alpha f$ exists on U and is continuous on U. Finally, we say that $f \in C^\infty(U,\mathbb{R}^m)$ if $f \in C^k(U,\mathbb{R}^m)$ for every $k \in \mathbb{N}_0$.
If $x_0 \in U$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, we say that f is of class C^k at x_0 if there is an open neighborhood $V \subseteq U$ of x_0 such that $f\restriction_V \in C^k(V,\mathbb{R}^m)$. Functions of class C^∞ are also called *smooth*.
We write

$$C^k(U,\mathbb{R}^m) := \{f\colon U \longrightarrow \mathbb{R}^m \mid f \text{ is of class } C^k\}.$$

If $m = 1$, we write $C^k(U) := C^k(U,\mathbb{R})$.

The following result, known as the *Clairaut–Schwarz theorem*, gives local conditions under which mixed second partial derivatives commute.

Theorem 1.1.12. (Clairaut–Schwarz) Let $f\colon U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $x_0 \in U$. Suppose that there exists $r > 0$ such that $B_{\text{euc}}(x_0,r) \subseteq U$. If $i,j \in \{1,\dots,n\}$ and the second partial derivatives $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x)$ and $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(x)$ exist for every $x \in B_{\text{euc}}(x_0,r)$ and are continuous at x_0, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_0).$$

Proof. It suffices to work componentwise, so we assume that $m = 1$. Let $h,k \neq 0$ be small enough that the rectangle with vertices

$$x_0, \qquad x_0 + he_i, \qquad x_0 + ke_j, \qquad x_0 + he_i + ke_j$$

is contained in $B_{\text{euc}}(x_0,r)$, and define

$$R(h,k) := f(x_0 + he_i + ke_j) - f(x_0 + he_i) - f(x_0 + ke_j) + f(x_0).$$

We apply the mean value theorem, first in the e_i direction to

$$t \longmapsto f(x_0 + te_i + ke_j) - f(x_0 + te_i),$$

and then in the e_j direction. There exist $\theta_{h,k}$ between 0 and h and $\eta_{h,k}$ between 0 and k such that

$$\frac{R(h,k)}{hk} = \partial_j \partial_i f(x_0 + \theta_{h,k} e_i + \eta_{h,k} e_j).$$

Reversing the order of the two applications gives $\widehat{\theta}_{h,k}$ between 0 and h and $\widehat{\eta}_{h,k}$ between 0 and k for which

$$\frac{R(h,k)}{hk} = \partial_i \partial_j f(x_0 + \widehat{\theta}_{h,k} e_i + \widehat{\eta}_{h,k} e_j).$$

As $(h,k) \to (0,0)$, all the intermediate points converge to x_0. Continuity of the two mixed derivatives at x_0 proves the desired equality. □

Corollary 1.1.13. Let $f: U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be of class C^2 at $x_0 \in U$. Then, for any $i, j \in \{1, \ldots, n\}$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_0).$$

Proof. Since f is of class C^2 at x_0, there is a ball centered at x_0 on which both mixed derivatives exist and are continuous. Apply Theorem 1.1.12. □

The preceding notation simplifies higher-order partial derivatives. We give a few examples of its use.

Example 1.1.14. (a) Let $f(x_1, x_2, x_3) = x_1^2 x_2^3 \sin(x_3)$. Consider the multi-index $\alpha = (2, 1, 1)$. Then

$$D^\alpha f = \frac{\partial^4 f}{\partial x_1^2 \partial x_2 \partial x_3}.$$

Differentiating successively, we obtain:

$$\frac{\partial f}{\partial x_3} = x_1^2 x_2^3 \cos(x_3), \quad \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_3}\right) = 3x_1^2 x_2^2 \cos(x_3),$$

$$\frac{\partial}{\partial x_1}\left(3x_1^2 x_2^2 \cos(x_3)\right) = 6x_1 x_2^2 \cos(x_3), \quad \frac{\partial^2}{\partial x_1^2}\left(3x_1^2 x_2^2 \cos(x_3)\right) = 6x_2^2 \cos(x_3).$$

(b) If $f(x, y) = e^{xy}$ and we use the multi-index $\alpha = (1, 2)$, then

$$D^\alpha f = \frac{\partial^3 f}{\partial x_1 \partial x_2^2} = \frac{\partial^3 f}{\partial x \partial y^2}.$$

We compute:

$$\frac{\partial f}{\partial y} = x e^{xy}, \quad \frac{\partial^2 f}{\partial y^2} = x^2 e^{xy}, \quad \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}(x^2 e^{xy}) = (2x + x^2 y) e^{xy}.$$

(c) Let $f(x_1, x_2) = \ln(1 + x_1^2 + x_2^2)$ and let $\alpha = (1, 1)$. Then

$$D^\alpha f = \frac{\partial^2 f}{\partial x_1 \partial x_2}.$$

We compute:

$$\frac{\partial f}{\partial x_2} = \frac{2x_2}{1 + x_1^2 + x_2^2}, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{2x_2}{1 + x_1^2 + x_2^2}\right) = \frac{-4x_1 x_2}{(1 + x_1^2 + x_2^2)^2}.$$

(d) If we take the multi-index $\alpha = (0, \ldots, 0, k, 0, \ldots, 0)$ with a k in the ith position, then

$$D^\alpha f(x) = \frac{\partial^k f}{\partial x_i^k}(x),$$

which is simply the derivative of order k with respect to the variable x^i.

We use multi-index notation to present the most important properties of higher-order partial derivatives for sufficiently regular functions. Among the properties we will study is the familiar Leibniz rule for products. It

is useful first to recall the result for functions from $\mathbb{R}$ to $\mathbb{R}$. Its proof uses properties of binomial coefficients, in particular the following result:

Theorem 1.1.15. (Pascal's triangle) For all integers $n \geq 1$ and $1 \leq k \leq n$, the following identity between binomial coefficients holds:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This identity reflects the construction of Pascal's triangle: each number is the sum of the two numbers directly above it.

Proof. If $1 \leq k \leq n-1$, then

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

For $k = n$, using the convention $\binom{n-1}{n} = 0$, the right-hand side is $\binom{n-1}{n-1} = 1 = \binom{n}{n}$. □

The Leibniz rule states the following:

Theorem 1.1.16. (generalized Leibniz formula) Let $n \in \mathbb{N}$ and $f, g \colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be n-times differentiable functions. Then fg is n-times differentiable and

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Proof. We proceed by induction on n. The case $n = 1$ states that

$$(fg)' = fg' + f'g,$$

which is the usual product rule. Suppose that the assertion holds for a fixed $n \geq 1$, and now let f and g be $(n+1)$-times differentiable functions. The induction hypothesis gives

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Then

$$\begin{aligned}
(fg)^{(n+1)} &\overset{\text{induction hypothesis}}{=} \left[\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \right]' \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n+1-k)}
\end{aligned}$$

$$= \binom{n}{0} f^{(0)} g^{(n+1)} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] f^{(k)} g^{(n+1-k)} + \binom{n}{n} f^{(n+1)} g^{(0)}$$

$$\overset{\text{Theorem 1.1.15}}{=} \binom{n+1}{0} f^{(0)} g^{(n+1)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} + \binom{n+1}{n+1} f^{(n+1)} g^{(0)}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)}.$$

Thus the assertion holds for $n+1$, and the principle of induction shows that it holds for every $n \in \mathbb{N}$. □

For functions of class $C^{|\alpha|}$, multi-index notation gives a formula similar to the preceding one. We conclude this section by presenting this formula together with other properties of higher-order derivatives.

Proposition 1.1.17. (Properties of partial derivatives in multi-index notation) Let $\alpha, \beta \in \mathbb{N}_0^n$, $f, g\colon U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be functions of class $C^{|\alpha|}$, and let $h\colon U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class $C^{|\alpha+\beta|}$, where $|\alpha+\beta| = |\alpha|+|\beta|$. Then the following properties hold:

(a) Linearity:
$$D^\alpha(\lambda f + \mu g) = \lambda D^\alpha f + \mu D^\alpha g, \quad \text{for any } \lambda, \mu \in \mathbb{R}.$$

(b) $D^\beta(D^\alpha h) = D^{\alpha+\beta} h = D^\alpha(D^\beta h)$.

(c) Product rule (Leibniz rule):
$$D^\alpha(fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta f \cdot D^{\alpha-\beta} g,$$
where $\binom{\alpha}{\beta} = \prod_{i=1}^{n} \binom{\alpha_i}{\beta_i}$ is the binomial coefficient and $\beta \le \alpha$ if and only if $\beta_j \le \alpha_j$ for every $j \in \{1, \dots, n\}$.

Proof. (a) This follows immediately from the linearity of partial derivatives.

(b) Suppose that h is of class $C^{|\alpha|+|\beta|}$. Repeated application of Theorem 1.1.12 to lower-order derivatives allows us to interchange adjacent partial derivatives. Therefore,

$$D^\beta(D^\alpha h) = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \left(\frac{\partial^{|\alpha|} h}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right) = \frac{\partial^{|\alpha|+|\beta|} h}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n} \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

$$= \frac{\partial^{|\alpha+\beta|} h}{\partial x_1^{\alpha_1+\beta_1} \cdots \partial x_n^{\alpha_n+\beta_n}} = D^{\alpha+\beta} h.$$

which is what we wanted to prove. Interchanging α and β in the same calculation and applying the Clairaut–Schwarz theorem again gives $D^\alpha(D^\beta h) = D^{\alpha+\beta} h$.

(c) We proceed by induction on $|\alpha|$. For $|\alpha| = 0$, the sum contains only $\beta = 0$ and the identity is $fg = fg$. Suppose the formula has been proved for all multi-indices of order r, and let $|\alpha| = r+1$. Choose j with $\alpha_j > 0$ and set $\alpha' := \alpha - e_j$. By the induction hypothesis, the usual product rule, and the preceding

item,

$$D^\alpha(fg) = D^{e_j}\left(\sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}D^\beta f\,D^{\alpha'-\beta}g\right)$$
$$= \sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\left(D^{\beta+e_j}f\,D^{\alpha'-\beta}g + D^\beta f\,D^{\alpha'-\beta+e_j}g\right).$$

In the first sum we set $\delta = \beta + e_j$ and in the second $\delta = \beta$. With the convention that a multi-index coefficient is zero when any component of the lower index is negative or exceeds the corresponding component of the upper index, the coefficient of $D^\delta f\,D^{\alpha-\delta}g$ is

$$\binom{\alpha'}{\delta - e_j} + \binom{\alpha'}{\delta}.$$

The factors corresponding to coordinates other than j agree, and in coordinate j Pascal's identity gives

$$\binom{\alpha_j - 1}{\delta_j - 1} + \binom{\alpha_j - 1}{\delta_j} = \binom{\alpha_j}{\delta_j}.$$

Therefore,

$$D^\alpha(fg) = \sum_{\delta\le\alpha}\binom{\alpha}{\delta}D^\delta f\,D^{\alpha-\delta}g,$$

which completes the induction.

□

Motivated by the multinomial theorem, we can state a version of the Leibniz rule for several factors:

Proposition 1.1.18. (Leibniz rule for several factors) Let $\alpha \in \mathbb{N}_0^n$ and let $m \in \mathbb{N}$ and $f_1, \dots, f_m \colon U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be functions of class $C^{|\alpha|}$. Then

$$D^\alpha\Big(\prod_{r=1}^m f_r\Big) = \sum_{\substack{\beta_1,\dots,\beta_m\in\mathbb{N}_0^n\\ \beta_1+\cdots+\beta_m=\alpha}} \frac{\alpha!}{\beta_1!\cdots\beta_m!}\prod_{r=1}^m D^{\beta_r}f_r.$$

Proof. We proceed by induction on m. For $m = 2$, the formula is precisely the Leibniz rule in Proposition 1.1.17.

Suppose the assertion holds for m factors and consider $f_1, \dots, f_{m+1}$. Write $F := \prod_{r=1}^m f_r$, so that $\prod_{r=1}^{m+1} f_r = F f_{m+1}$.

Applying the Leibniz rule for two factors,

$$D^\alpha(F f_{m+1}) = \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}D^\gamma F\,D^{\alpha-\gamma}f_{m+1}.$$

By the induction hypothesis,

$$D^\gamma F = \sum_{\substack{\beta_1,\dots,\beta_m\in\mathbb{N}_0^n\\ \beta_1+\cdots+\beta_m=\gamma}} \frac{\gamma!}{\beta_1!\cdots\beta_m!}\prod_{r=1}^m D^{\beta_r}f_r.$$

Substituting and using $\binom{\alpha}{\gamma}\gamma! = \dfrac{\alpha!}{(\alpha-\gamma)!}$, we obtain

$$D^\alpha\left(\prod_{r=1}^{m+1} f_r\right) = \sum_{\gamma\le\alpha} \sum_{\substack{\beta_1,\dots,\beta_m\in\mathbb{N}_0^n \\ \beta_1+\cdots+\beta_m=\gamma}} \frac{\alpha!}{\beta_1!\cdots\beta_m!(\alpha-\gamma)!}\left(\prod_{r=1}^{m} D^{\beta_r} f_r\right) D^{\alpha-\gamma} f_{m+1}.$$

Now define

$$\beta_{m+1} := \alpha - \gamma.$$

Since $\gamma \le \alpha$, we have $\beta_{m+1} \in \mathbb{N}_0^n$ and, conversely, given any family $\beta_1,\dots,\beta_{m+1} \in \mathbb{N}_0^n$ such that

$$\beta_1 + \cdots + \beta_{m+1} = \alpha,$$

if we set

$$\gamma := \beta_1 + \cdots + \beta_m,$$

then $\gamma \le \alpha$ and $\beta_{m+1} = \alpha - \gamma$. This correspondence is a bijection between the pairs

$$(\gamma, \beta_1, \dots, \beta_m) \quad \text{with} \quad \beta_1 + \cdots + \beta_m = \gamma \le \alpha$$

and the families

$$(\beta_1, \dots, \beta_{m+1}) \quad \text{with} \quad \beta_1 + \cdots + \beta_{m+1} = \alpha.$$

Reindexing the sum by this bijection gives

$$D^\alpha\left(\prod_{r=1}^{m+1} f_r\right) = \sum_{\substack{\beta_1,\dots,\beta_{m+1}\in\mathbb{N}_0^n \\ \beta_1+\cdots+\beta_{m+1}=\alpha}} \frac{\alpha!}{\beta_1!\cdots\beta_{m+1}!}\prod_{r=1}^{m+1} D^{\beta_r} f_r,$$

which completes the induction.

□

The following multi-index form of the chain rule gives explicit control over the partial derivatives of a composition.

Theorem 1.1.19. (Faà di Bruno formula for multi-indices) Let $f\colon U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g\colon V \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$, with $f(U) \subseteq V$, both be of class C^k. Write $f = (f_1,\dots,f_m)$. Then $g\circ f$ is of class C^k and, for $x_0 \in U$ and $\alpha \in \mathbb{N}_0^n$ with $1 \le |\alpha| \le k$,

$$D^\alpha(g\circ f)(x_0) = \sum_{\substack{\beta\in\mathbb{N}_0^m,\ 1\le|\beta|\le|\alpha| \\ \gamma=(\gamma_1,\dots,\gamma_{|\beta|}),\ \gamma_i\in\mathbb{N}_0^n,\ |\gamma_i|>0,\ \sum_{i=1}^{|\beta|}\gamma_i=\alpha \\ l=(l_1,\dots,l_{|\beta|}),\ l_i\in\{1,\dots,m\} \\ \#\{i\mid l_i=j\}=\beta_j,\ 1\le j\le m}} c_{\alpha,\beta,\gamma,l} D^\beta g(f(x_0)) \prod_{i=1}^{|\beta|} D^{\gamma_i} f_{l_i}(x_0).$$

The coefficients $c_{\alpha,\beta,\gamma,l}$ are nonnegative combinatorial constants depending only on the indicated indices.

Proof. We proceed by induction on $q := |\alpha|$. If $q = 1$, there exists $r \in \{1, \dots, n\}$ with $\alpha = e_r$, and the chain rule gives

$$D^{e_r}(g \circ f)(x_0) = \sum_{j=1}^{m} (D^{e_j} g)(f(x_0)) D^{e_r} f_j(x_0).$$

This is the formula in the statement with $\beta = e_j$, $\gamma = (e_r)$, $l = (j)$, and coefficient 1.
Suppose the assertion holds for multi-indices of length q, and let α' have length $q+1$. Choose r with $\alpha'_r > 0$ and set $\alpha = \alpha' - e_r$. Each term in the inductive formula has the form

$$c\,(D^{\beta} g)(f(x)) \prod_{a=1}^{|\beta|} D^{\gamma_a} f_{l_a}(x), \qquad \sum_{a=1}^{|\beta|} \gamma_a = \alpha.$$

Differentiating with respect to x^r, the derivative of the first factor yields

$$c \sum_{j=1}^{m} (D^{\beta + e_j} g)(f(x)) D^{e_r} f_j(x) \prod_{a=1}^{|\beta|} D^{\gamma_a} f_{l_a}(x).$$

This adds e_r to the family (γ_a), adds j to (l_a), and replaces β by $\beta + e_j$. If the derivative acts on the factor with index a, it yields

$$c\,(D^{\beta} g)(f(x)) D^{\gamma_a + e_r} f_{l_a}(x) \prod_{\substack{b=1 \\ b \neq a}}^{|\beta|} D^{\gamma_b} f_{l_b}(x).$$

Here γ_a is replaced by $\gamma_a + e_r$. In both cases, the new multi-indices sum to α', and all the restrictions in the statement are preserved. Like terms are collected; their coefficients are added and remain nonnegative combinatorial constants. This completes the induction. The same construction shows, order by order, that the resulting derivatives are continuous, so $g \circ f$ is of class C^k. □

1.2 Smooth manifolds

A smooth manifold formalizes the idea of a space that may be globally curved or topologically complicated, yet admits Euclidean coordinates around each point. The sphere and the torus are familiar examples: no single chart describes either one completely, but every sufficiently small region can be compared with an open subset of the plane.

The dimension of these spaces is intrinsic. Although the usual sphere is drawn inside $\mathbb{R}^3$, only two local coordinates are needed to move along it. Requiring a space to be locally homeomorphic to an open subset of $\mathbb{R}^n$ captures this topological observation; requiring smooth compatibility between charts then makes it possible to transport calculus. This concept is fundamental in nonlinear analysis, since many partial differential equations and variational problems are naturally formulated on spaces that are not open subsets of $\mathbb{R}^n$. For example:

- Models in mathematical physics whose domain is a curved surface or a space with variable curvature (waves on the sphere, diffusion on the torus).
- Problems in differential geometry involving the boundary of a manifold, such as trace theorems for Sobolev functions defined on the boundary.

Élie Cartan (1869–1951)

Figure 1.2. Élie Cartan was a central figure in the development of modern differential geometry. His work on differential forms, moving frames, connections, and curvature made decisive contributions to the intrinsic study of manifolds and established part of the language that now underlies Riemannian geometry and global analysis.

- Elliptic and parabolic equations on compact spaces with or without boundary, where the Riemannian metric directly influences the differential operators.

The notion of a smooth manifold allows us to extend differential calculus to these settings. Once manifolds are equipped with a Riemannian metric, we will have the tools to define gradients, divergences, Laplacians, and elliptic operators intrinsically. This will allow us to develop the study of Sobolev spaces on manifolds, state and prove trace theorems, and analyze partial differential equations in a general geometric setting.

We will present the modern definition of a *differentiable manifold*, first introducing several preliminary concepts.

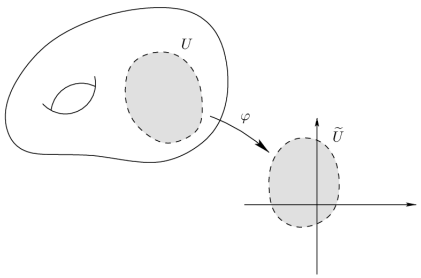
Definition 1.2.1. Let M be a topological space. We say that M is a **topological manifold without boundary of dimension** n if:

(a) M is Hausdorff.

(b) M is second countable.

(c) For every $p \in M$, there exist an open set $U \subseteq M$ and a homeomorphism $\varphi\colon U \longrightarrow \varphi(U)$, where $\varphi(U) \subseteq \mathbb{R}^n$ is open (M is locally homeomorphic to $\mathbb{R}^n$), such that $p \in U$.

To become familiar with the property of *being locally homeomorphic to* $\mathbb{R}^n$, it is useful to consider the following picture:



Remark 1.2.2. To simplify notation, the open set U and the homeomorphism

$$\varphi\colon U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$$

are usually considered together as an ordered pair (U,φ), called a *coordinate chart*, or simply a *chart*.

The topological structure captures the locally Euclidean character of the space, but does not yet allow differentiation. To transport calculus between charts, we must require the changes of coordinates to be smooth. A compatible atlas thus assembles the local information into a coherent global structure.

Definition 1.2.3. Let M be a topological manifold without boundary of dimension n.

(a) We say that two charts (U,φ) and (V,ψ) are smoothly compatible if $U \cap V = \varnothing$ or if the transition map

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a diffeomorphism of class C^∞ (henceforth called smooth).

(b) An atlas for M is a collection of charts $\mathcal{A}$ such that $\bigcup_{(U,\varphi)\in\mathcal{A}} U = M$.

(c) A smooth atlas is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible.

(d) A smooth structure on M is a smooth atlas $\mathcal{A}$ maximal with respect to $\subseteq$ among the smooth atlases of M.

We can now define a *differentiable manifold (or smooth manifold)*:

Definition 1.2.4. A smooth manifold without boundary of dimension n is a topological manifold M of dimension n equipped with a smooth structure $\mathcal{A}$. We write $(M,\mathcal{A})$ when we wish to refer explicitly to the smooth structure.

Remark 1.2.5. We can also define manifolds without boundary of class C^k by requiring the transition maps

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

to be diffeomorphisms of class C^k, that is, functions of class C^k whose inverses are of class C^k.

Topological manifolds (and therefore smooth manifolds) have several interesting topological properties:

Proposition 1.2.6. Let M be a topological manifold without boundary.

(a) M is locally path connected.

(b) M is connected if and only if it is path connected.

(c) The connected components of M coincide with its path components.

(d) M has countably many connected components, each of which is an open subset of M and a connected topological manifold.

Proof. Let $p \in M$ and let U be an open neighborhood of p. Choose a chart (V, φ) with $p \in V$. The open set $\varphi(U \cap V)$ contains an open Euclidean ball B centered at $\varphi(p)$. Then $\varphi^{-1}(B)$ is a neighborhood of p, is contained in U, and is path connected because B is convex. This proves (1).
Now fix $p \in M$ and let C_p be the set of points that can be joined to p by a path. By (1), C_p is open. The same argument at each point of $M \setminus C_p$ shows that its complement is also open. If M is connected, it follows that $C_p = M$. Since every path connected space is connected, this proves (2). Applying (2) to each connected component gives (3), and local path connectedness also shows that every component is open.
Finally, let $\mathcal{B}$ be a countable basis for M. Each connected component C is open and nonempty, and therefore contains some nonempty basis element $B_C \in \mathcal{B}$. Distinct components contain distinct basis elements, since they are disjoint; hence the family of components is countable. Restricting the charts of M makes each component a connected topological manifold. This proves (4). □

Manifolds are also locally compact:

Proposition 1.2.7. Every topological manifold with or without boundary is locally compact.

Proof. Let $p \in M$ and choose a chart $\varphi\colon U \to \varphi(U) \subseteq \mathbb{H}^n$ around p, where $\mathbb{H}^n = \mathbb{R}^n$ if M has no boundary. Since $\varphi(U)$ is open in $\mathbb{H}^n$, there exists $r > 0$ such that

$$\overline{B}_{\text{euc}}(\varphi(p), r) \cap \mathbb{H}^n \subseteq \varphi(U).$$

The preimage

$$K := \varphi^{-1}\left(\overline{B}_{\text{euc}}(\varphi(p), r) \cap \mathbb{H}^n\right)$$

is compact, since it is homeomorphic to a closed and bounded subset of $\mathbb{R}^n$, and it contains the relatively open neighborhood

$$\varphi^{-1}\left(B_{\text{euc}}(\varphi(p), r) \cap \mathbb{H}^n\right)$$

of p. □

Another covering property of manifolds is paracompactness, which will be very useful in geometric analysis:

Theorem 1.2.8. (Manifolds are paracompact) Let (M, τ_M) be a topological manifold with or without boundary. Given an open cover $\mathcal{U} \subseteq \tau_M$ of M and any basis $\mathcal{B}$ for τ_M, there exists a countable, locally finite open refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.

Proof. We first construct a compact exhaustion. Since M is locally compact and second countable, it has a countable basis $(V_j)_{j\geq 1}$ of precompact open sets. Set $K_0 := \varnothing$ and $K_1 := \overline{V}_1$. Once K_j has been constructed, each point of the compact set $K_j \cup \overline{V}_{j+1}$ has a precompact open neighborhood. A finite subcover of these neighborhoods has union W, and the union of their closures defines a compact set K_{j+1} satisfying

$$K_j \cup \overline{V}_{j+1} \subseteq W \subseteq \operatorname{Int} K_{j+1}.$$

Recursively, we obtain compact sets $(K_j)_{j\geq 1}$ such that

$$K_j \subseteq \operatorname{Int} K_{j+1}, \qquad V_1 \cup \cdots \cup V_j \subseteq K_j.$$

Since the sets V_j cover M, we have $M = \bigcup_{j\geq 1} K_j$.

Define

$$C_0 := K_1, \qquad W_0 := \operatorname{Int} K_2,$$

and, for $j \geq 1$,

$$C_j := K_{j+1} \setminus \operatorname{Int} K_j, \qquad W_j := \operatorname{Int} K_{j+2} \setminus K_{j-1}.$$

Each C_j is compact and contained in W_j. For $j \geq 0$ and $x \in C_j$, choose $U_x \in \mathcal{U}$ with $x \in U_x$ and, since $\mathcal{B}$ is a basis, an element $B_x \in \mathcal{B}$ such that

$$x \in B_x \subseteq U_x \cap W_j.$$

Compactness of C_j allows us to retain finitely many of these open sets. The union of the finite families obtained for $j = 0, 1, 2, \ldots$ is countable and refines $\mathcal{U}$.

We show that this family covers M. Given $x \in M$, there exists $j \geq 0$ such that $x \in \operatorname{Int} K_{j+1}$; choose the smallest such index. If $j = 0$, then $x \in C_0$. If $j \geq 1$, minimality implies $x \notin \operatorname{Int} K_j$, while $x \in K_{j+1}$, and therefore $x \in C_j$.

It remains to verify local finiteness. Let $x \in M$ and choose $N \geq 1$ with $x \in \operatorname{Int} K_N$. If $j \geq N+1$, then $K_N \subseteq K_{j-1}$ and

$$\operatorname{Int} K_N \cap W_j = \varnothing.$$

Thus the neighborhood $\operatorname{Int} K_N$ of x can intersect only elements arising from the indices $0, \ldots, N$, and for each of these we chose a finite family. The refinement is locally finite.

The proof uses only the fact that charts provide a basis of precompact open sets; it therefore applies unchanged to charts taking values in relatively open subsets of $\mathbb{H}^n$. □

Various coordinate systems can be defined on a smooth manifold. Among them is the notion of a *regular coordinate ball*, which generalizes the notion of a *ball* in Euclidean spaces to manifolds.

Definition 1.2.9. If M is a smooth manifold without boundary, we use the following terminology:

(a) $B \subseteq M$ is a coordinate ball centered at x (or at $\varphi^{-1}(x)$) if there exists a homeomorphism $\varphi\colon B \longrightarrow B_{\mathrm{euc}}(x, r)$, where

$$B_{\mathrm{euc}}(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| < r\},$$

for some $r > 0$ and $x \in \mathbb{R}^n$, such that (B, φ) is a smooth chart of M.

(b) $B \subseteq M$ is a regular coordinate ball (centered at x) if there exists a coordinate ball B' such that $\overline{B} \subseteq B'$ and there is a smooth coordinate map $\varphi\colon B' \longrightarrow \mathbb{R}^n$ such that, for some positive numbers $r' > r > 0$, we have $\varphi(B) = B_{\mathrm{euc}}(x, r)$, $\varphi(\overline{B}) = \overline{B}_{\mathrm{euc}}(x, r)$, and $\varphi(B') = B_{\mathrm{euc}}(x, r')$.

If M has boundary, we use the same terminology, replacing Euclidean balls by the relative balls $B_{\mathrm{euc}}(x, r) \cap \mathbb{H}^n$ in a boundary chart and taking closures in $\mathbb{H}^n$.

Remark 1.2.10. Every regular coordinate ball (B, φ) has the property that $\overline{B}$ is compact and φ extends to $\overline{B}$. We will call domains of this type ***compact regular coordinate balls***.

Regular coordinate balls are important because of their topological properties. Moreover, every smooth manifold admits a countable atlas of regular coordinate balls, as the following proposition shows:

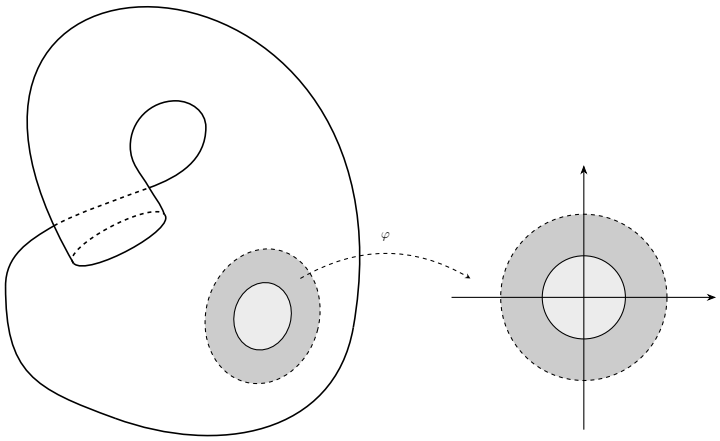


Figure 1.3. A precompact regular coordinate ball whose coordinate map extends to the boundary.

Proposition 1.2.11. Every smooth manifold with or without boundary has a countable basis of regular coordinate balls, and therefore admits a countable, locally finite open cover by regular coordinate balls.

Proof. By second countability, we extract a countable subcover from the cover by chart domains. Within each coordinate image, we take Euclidean balls, or half-balls in a boundary chart, with rational centers and radii, whose closures lie in slightly larger balls in the same coordinate domain. Their preimages form a countable family of regular coordinate balls and constitute a basis for M. Applying Theorem 1.2.8 to this basis and to the cover $\{M\}$ gives a countable, locally finite subfamily covering M. □

Finally, to complete the definitions in this section, we specify the appropriate model for a manifold with boundary:

$$\mathbb{H}^n := \mathbb{R}^{n-1} \times [0,\infty), \qquad \operatorname{Int} \mathbb{H}^n = \mathbb{R}^{n-1} \times (0,\infty), \qquad \partial \mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\}.$$

All open subsets of $\mathbb{H}^n$ are understood in the relative topology. A function defined on one of them is smooth if, in a neighborhood of each point, it is the restriction of a smooth function defined on an open subset of $\mathbb{R}^n$.

Definition 1.2.12. Let M be a topological space.

(a) We say that M is a **topological manifold with boundary of dimension** n if it is Hausdorff and second countable, and each point has an open neighborhood homeomorphic to a relatively open subset of $\mathbb{H}^n$.

(b) A chart for M is a pair (U,φ), where $U \subseteq M$ is open and $\varphi\colon U \longrightarrow \varphi(U) \subseteq \mathbb{H}^n$ is a homeomorphism onto a relatively open subset.

(c) A chart (U,φ) is an **interior chart** if $\varphi(U) \subseteq \operatorname{Int} \mathbb{H}^n$; in that case, its image is an open subset of $\mathbb{R}^n$. It is a **boundary chart** if $\varphi(U) \cap \partial \mathbb{H}^n \neq \varnothing$. The domain of a boundary chart may contain both interior and boundary points.

(d) An atlas for M is a collection of charts $\mathcal{A}$ such that $\bigcup_{(U,\varphi)\in\mathcal{A}} U = M$.

Compatibility and maximality for charts of a manifold with boundary are defined as follows:

Definition 1.2.13. Let M be a topological manifold with boundary of dimension n.

(a) We say that two charts (U,φ) and (V,ψ) are smoothly compatible if $U\cap V=\varnothing$ or if the transition map

$$\psi\circ\varphi^{-1}\colon \varphi(U\cap V)\longrightarrow \psi(U\cap V)$$

is a diffeomorphism of class C^∞.

(b) A smooth atlas is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible.

(c) A smooth structure on M is a smooth atlas $\mathcal{A}$ maximal with respect to $\subseteq$ among the smooth atlases of M.

Definition 1.2.14. A smooth manifold with boundary of dimension n is a topological manifold with boundary of dimension n equipped with a smooth structure.

Theorem 1.2.15. (Smooth invariance of the boundary) Let (U,φ) and (V,ψ) be smoothly compatible charts of a manifold with boundary, and let $p\in U\cap V$. Then

$$\varphi(p)\in\partial\mathbb{H}^n \quad\Longleftrightarrow\quad \psi(p)\in\partial\mathbb{H}^n.$$

Consequently, the notions of interior point and boundary point obtained by examining the image of a point do not depend on the chart.

The proof follows the strategy of Lee [94].

Proof. Suppose, by contradiction, that $x:=\varphi(p)$ is interior and that $y:=\psi(p)$ belongs to $\partial\mathbb{H}^n$. Restricting the charts, we may take a Euclidean ball B around x, contained in $\varphi(U\cap V)\cap\operatorname{Int}\mathbb{H}^n$. The change of coordinates

$$F:=\psi\circ\varphi^{-1}\colon B\longrightarrow\mathbb{H}^n$$

is smooth, and its inverse admits a smooth extension near y to an open subset of $\mathbb{R}^n$. The identity $F^{-1}\circ F=\mathrm{Id}$ on B and the chain rule show that dF_x has a left inverse; since its domain and codomain have dimension n, dF_x is invertible. The inverse function theorem implies that $F(B)$ contains a Euclidean neighborhood of y. This is impossible because $F(B)\subseteq\mathbb{H}^n$ and no Euclidean neighborhood of a point of $\partial\mathbb{H}^n$ is contained in $\mathbb{H}^n$. Interchanging the charts rules out the opposite possibility. □

Definition 1.2.16. Let M be a smooth manifold with boundary. A point $p\in M$ is an **interior point** if $\varphi(p)\in\operatorname{Int}\mathbb{H}^n$ in one —and, by the preceding theorem, in every— chart containing it. It is a **boundary point** if $\varphi(p)\in\partial\mathbb{H}^n$. We denote the corresponding sets by $\operatorname{Int}M$ and ∂M.
A boundary chart is **centered at** $p\in\partial M$ if, after a translation in the first $n-1$ coordinates, $\varphi(p)=0$. This condition concerns the image of the distinguished point; it does not mean that the entire chart domain

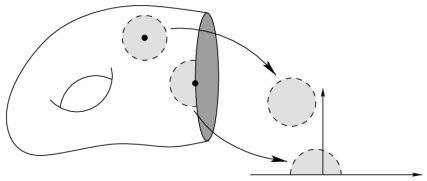
consists of boundary points.

Remark 1.2.17. (Convention for finite-dimensional manifolds) Throughout the finite-dimensional treatment, the term manifold includes the Hausdorff and second countability assumptions in the preceding definitions; we will not repeat them in every statement.

Remark 1.2.18. The following properties hold:

(a) The boundary ∂M and the interior $\operatorname{Int} M$ are disjoint and their union is M. Moreover, $\operatorname{Int} M$ is open and ∂M is closed.

(b) A smooth manifold may be regarded as the special case of a smooth manifold with empty boundary, or one in which every point of M is an interior point.



The abstract definition is especially useful when a space is constructed by gluing local models. The following lemma collects the conditions under which a family of coordinates uniquely determines the topology and smooth structure of the resulting set.

Lemma 1.2.19. (Smooth chart lemma) Let M be a set, and suppose we have a collection $\{U_\alpha\}_{\alpha\in A}$ of subsets of M and maps $\varphi_\alpha\colon U_\alpha \longrightarrow \mathbb{R}^n$ (or $\varphi_\alpha\colon U_\alpha \longrightarrow \mathbb{H}^n$) such that:

(a) For each $\alpha \in A$, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ (or a relatively open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{H}^n$).

(b) For each $\alpha, \beta \in A$, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$ (or relatively open in $\mathbb{H}^n$).

(c) If $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.

(d) There exists a countable $N \subseteq A$ such that $M = \bigcup_{\alpha\in N} U_\alpha$.

(e) If $p, q \in M$ and $p \neq q$, then either there exists $\alpha \in A$ such that $p, q \in U_\alpha$, or there exist $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta = \varnothing$ with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique topology and a unique smooth structure such that M is a smooth manifold (with boundary) and $(U_\alpha, \varphi_\alpha)$ is a smooth chart for every $\alpha \in A$.

Proof. We write $\mathbb{E}^n$ for the appropriate model space, $\mathbb{R}^n$ or $\mathbb{H}^n$, always using the relative topology in the latter case. Consider the family

$$\mathcal{C} := \left\{ \varphi_\alpha^{-1}(V) \,\middle|\, \alpha \in A,\ V \subseteq \varphi_\alpha(U_\alpha) \text{ open} \right\}.$$

This family covers M. If p belongs to $\varphi_\alpha^{-1}(V)\cap\varphi_\beta^{-1}(W)$, then

$$\varphi_\alpha^{-1}(V)\cap\varphi_\beta^{-1}(W)=\varphi_\alpha^{-1}\left(V\cap(\varphi_\beta\circ\varphi_\alpha^{-1})^{-1}(W)\right).$$

The set in parentheses is open in $\varphi_\alpha(U_\alpha)$: we first restrict to the open set $\varphi_\alpha(U_\alpha\cap U_\beta)$ and then use continuity of the change of coordinates. Therefore, $\mathcal{C}$ is a basis for a topology τ on M.
By construction, each U_α is open and φ_α is a homeomorphism from U_α onto its image. We show that (M,τ) is Hausdorff. If two distinct points p,q belong to the same U_α, we separate their images in the Hausdorff space $\mathbb{E}^n$ and take preimages. If we do not use a common chart, the last assumption provides disjoint domains U_α and U_β containing p and q, respectively. In both cases we obtain disjoint open neighborhoods.
Let $N\subseteq A$ be countable, as in the fourth assumption. Each $\varphi_\alpha(U_\alpha)$, with $\alpha\in N$, has a countable basis; the preimages of all these bases form a countable family. This is a basis for M, since the sets U_α, $\alpha\in N$, cover the space. Thus (M,τ) is second countable and locally Euclidean, or locally modeled on $\mathbb{H}^n$.
For $U_\alpha\cap U_\beta\neq\varnothing$, the third assumption, together with the same assumption with α and β interchanged, shows that $\varphi_\beta\circ\varphi_\alpha^{-1}$ is a diffeomorphism between the corresponding open sets. The given charts therefore form a smooth atlas. The maximal atlas containing it defines a smooth structure on M. In the case of $\mathbb{H}^n$, the same construction yields boundary charts, and all local extensions involved in the notion of smoothness are taken on open subsets of $\mathbb{R}^n$; thus the argument also covers manifolds with boundary.
Finally, any other topology making all the maps φ_α homeomorphisms onto their images would necessarily have $\mathcal{C}$ as a basis, and would therefore coincide with τ. A smooth structure for which the given charts were smooth would have the same maximal atlas, since a chart belongs to this atlas if and only if it is smoothly compatible with every chart φ_α. This proves both uniqueness assertions. □

1.3 Smooth maps

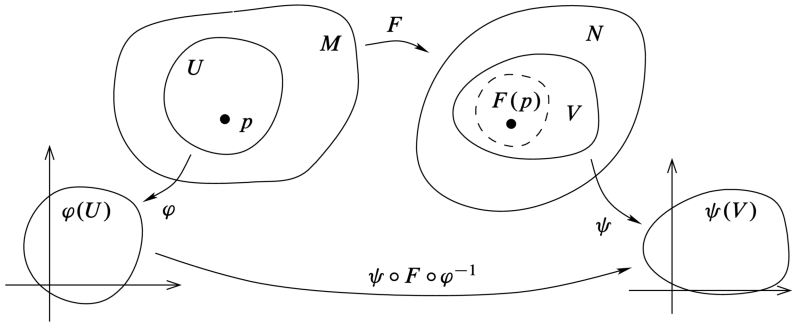
A smooth structure allows us not only to discuss functions on a manifold, but also to compare manifolds through maps that respect calculus. The condition is checked in coordinates, but its validity must be independent of the chosen charts.

Given charts around a point and its image, the map temporarily becomes a function between Euclidean open sets. We require this coordinate representation to be smooth. Compatibility of the atlases ensures that the definition does not depend on the pair of charts used.

Definition 1.3.1. Let M and N be smooth manifolds with or without boundary, and let $F\colon M\longrightarrow N$ be a function. We say that F is a smooth map if, for every $p\in M$, there exist coordinate charts (U,φ) and (V,ψ) containing p and $F(p)$, respectively, such that $F(U)\subseteq V$ and the map $\psi\circ F\circ\varphi^{-1}$ is a smooth function from $\varphi(U)$ to $\psi(V)$.
We say that the map is of class C^k if $\psi\circ F\circ\varphi^{-1}$ is of class C^k.

Definition 1.3.2. If M and N are smooth manifolds with or without boundary and $k\in\mathbb{N}\cup\{\infty\}$, we write $C^k(M,N):=\{f:M\longrightarrow N\mid f \text{ is of class } C^k\}$. If $N=\mathbb{R}$, we simply write $C^k(M):=C^k(M,\mathbb{R})$.

Remark 1.3.3. The function $\psi\circ F\circ\varphi^{-1}$ is called the *coordinate representation of* F and is sometimes denoted by $\hat{F}:=\psi\circ F\circ\varphi^{-1}$.



A partition of unity turns local constructions into global objects while retaining control over supports. It will be our basic tool for gluing functions, metrics, sections, and estimates defined in different charts.

Definition 1.3.4. Let M be a topological space and let $\mathscr{U} = \{U_\alpha \mid \alpha \in J\}$ be an open cover of M. A partition of unity subordinate to $\mathscr{U}$ is a family of continuous functions $(\psi_\alpha)_{\alpha\in J}$, $\psi_\alpha : M \longrightarrow \mathbb{R}$, such that:

(a) $0 \le \psi_\alpha(x) \le 1$ for every $\alpha \in J$ and every $x \in M$.

(b) $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for each $\alpha \in J$.

(c) The family $(\operatorname{supp}(\psi_\alpha))_{\alpha\in J}$ is locally finite.

(d) $\sum_{\alpha\in J} \psi_\alpha(x) = 1$ for each $x \in M$.

We have the following well-known result.

Theorem 1.3.5. (Existence of partitions of unity) Suppose that M is a smooth manifold with or without boundary and that $\mathscr{U} = (U_\alpha)_{\alpha\in J}$ is an open cover of M. Then there exists a partition of unity consisting of smooth functions and subordinate to $\mathscr{U}$.

Proof. We apply the construction in Theorem 1.2.8, choosing, in each of the compact sets occurring in its proof, two concentric regular coordinate balls

$$B_i \Subset B_i' \Subset U_{a(i)}, \qquad a(i) \in J,$$

so that the inner balls $(B_i)_{i\ge1}$ cover M and the family of outer balls $(B_i')_{i\ge1}$ is locally finite. To do this, for each point of the corresponding compact set we choose a ball whose closure is contained in another ball in the same coordinate domain, and then take a finite subcover. Since the outer balls remain within the open sets W_j used in that proof, local finiteness is preserved.
Recall the elementary construction of a cutoff function. Let

$$\vartheta(t) := \begin{cases} e^{-\frac{1}{t}}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

All its derivatives at the origin vanish, so ϑ is smooth. If $0 < r < R_0 < R$, the function

$$\chi_{r,R_0}(x) := \frac{\vartheta(R_0^2 - \|x\|^2)}{\vartheta(R_0^2 - \|x\|^2) + \vartheta(\|x\|^2 - r^2)}$$

is smooth, equals one on $\overline{B}_{\text{euc}}(0,r)$, and has support contained in $\overline{B}_{\text{euc}}(0,R_0) \subset B_{\text{euc}}(0,R)$. The denominator never vanishes. After translating the center in coordinates, restricting this function to $\mathbb{H}^n$ gives the same construction for a relative ball in a boundary chart.
Transporting these functions through the charts and extending them by zero, we obtain $f_i \in C^\infty(M)$ such that

$$0 \le f_i \le 1, \qquad f_i = 1 \text{ on } \overline{B}_i, \qquad \operatorname{supp} f_i \subseteq B_i'.$$

The extension is smooth because the support of the coordinate expression is contained in the interior of B_i'. The family of supports is locally finite, so

$$f := \sum_{i=1}^{\infty} f_i$$

is a smooth function. Since the sets B_i cover M, we have $f > 0$. Hence $g_i := \frac{f_i}{f}$ is smooth, the family $(\operatorname{supp} g_i)$ is locally finite, and $\sum_{i=1}^{\infty} g_i = 1$.
For $\alpha \in J$, define

$$\psi_\alpha := \sum_{\substack{i \in \mathbb{N} \\ a(i) = \alpha}} g_i,$$

interpreting the sum as zero if there are no corresponding indices. All these sums are locally finite. Moreover, the union of the supports appearing in each ψ_α is closed by local finiteness and is contained in U_α; consequently, $\operatorname{supp} \psi_\alpha \subseteq U_\alpha$. It is immediate that $0 \le \psi_\alpha \le 1$ and that $\sum_{\alpha \in J} \psi_\alpha = 1$. We have constructed the subordinate partition. Restricting the Euclidean cutoff functions directly proves the case with boundary. □

The existence of partitions of unity has the following important consequence:

Proposition 1.3.6. (Existence of bump functions) Let M be a smooth manifold with or without boundary. For any closed set $A \subseteq M$ and any open set $U \subseteq M$ such that $A \subseteq U$, there exists a smooth function $\psi \colon M \longrightarrow \mathbb{R}$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ on A, and $\operatorname{supp}(\psi) \subseteq U$.

Proof. Apply Theorem 1.3.5 to the cover $\{U, M \setminus A\}$. Let (ψ_0, ψ_1) be a subordinate partition, with $\operatorname{supp} \psi_0 \subseteq U$ and $\operatorname{supp} \psi_1 \subseteq M \setminus A$. On A we have $\psi_1 = 0$ and, since $\psi_0 + \psi_1 = 1$, we have $\psi_0 = 1$. Therefore, $\psi := \psi_0$ has all the required properties. □

This result allows us to extend smooth functions defined in a neighborhood of a given point to all of M as follows:

Theorem 1.3.7. (Extension of smooth functions) Let M be a smooth manifold with or without boundary and let $U \subseteq M$ be open. If $p \in U$ and $f \in C^\infty(U)$, then there exist an open set $W \subseteq M$ such that $p \in W \subseteq \overline{W} \subseteq U$ and a smooth function $\tilde{f} \colon M \longrightarrow \mathbb{R}$ such that $\tilde{f} \restriction_W = f$.

Proof. Let $U \subseteq M$ be open, and let $p \in U$ and $f \in C^\infty(U)$. Since M is a regular topological space, there exists an open set $W \subseteq M$ such that $p \in W \subseteq \overline{W} \subseteq U$. By Proposition 1.3.6, there exists a smooth function $\psi\colon M \longrightarrow \mathbb{R}$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $\overline{W}$, and $\operatorname{supp}(\psi) \subseteq U$. Define

$$\tilde{f}\colon M \longrightarrow \mathbb{R}, \qquad \tilde{f}(q) = \begin{cases} \psi f(q) & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}.$$

Then $\tilde{f}$ is smooth because $\operatorname{supp}(\psi) \subseteq U$ and $\tilde{f}\restriction_W = f$. □

1.4 The tangent bundle

Approximating a nonlinear object by linear data is one of the central ideas of analysis. In Example 1.1.5, we saw that the graph of a differentiable function is approximated near each point by its tangent plane. On an abstract manifold, we need to preserve this intuition without relying on an ambient space.

In multivariable calculus, many manifolds can be described locally as the image of a *parametrization*

$$\psi\colon \widetilde{U} \subset \mathbb{R}^k \longrightarrow \mathbb{R}^n,$$

where $\widetilde{U} \subseteq \mathbb{R}^k$ is open, $k \leq n$, and $d\psi_x$ has rank k for each $x \in \widetilde{U}$. For example, the sphere, cylinder, and torus admit descriptions of this kind in small neighborhoods of each point.

Fix $x_0 \in \widetilde{U}$ and let $p = \psi(x_0)$. The Fréchet derivative of ψ at x_0,

$$\mathrm{d}\psi_{x_0}\colon \mathbb{R}^k \longrightarrow \mathbb{R}^n,$$

is a linear map that describes how the coordinates of $\mathbb{R}^n$ change when we vary the coordinates of $\widetilde{U}$ slightly around x_0. The image of $\mathrm{d}\psi_{x_0}$ is a vector subspace of $\mathbb{R}^n$ of dimension k that "best approximates" $\psi(\widetilde{U})$ near p. The following example illustrates this:

Example 1.4.1. Consider the parametrization $\psi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$\psi(x, y) = (\cos x,\ \sin x,\ y).$$

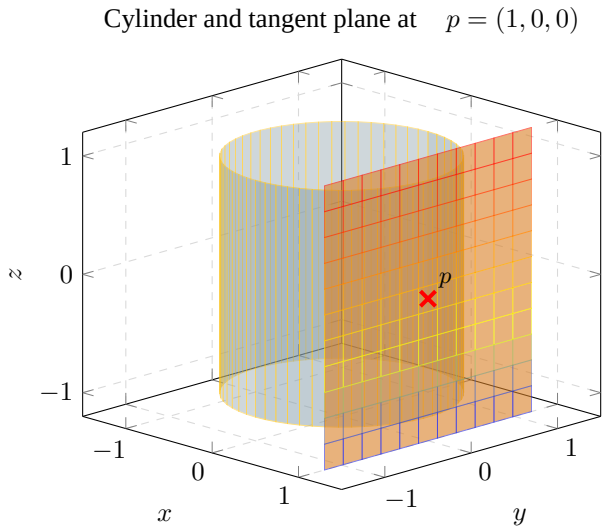
This function describes a cylinder of radius 1 in $\mathbb{R}^3$. Fix $x_0 = (0, 0)$, so that $p = (1, 0, 0)$. The Fréchet derivative at $(0, 0)$ is

$$\mathrm{d}\psi_{(0,0)}(h_1, h_2) = (-\sin 0 \cdot h_1,\ \cos 0 \cdot h_1,\ h_2) = (0,\ h_1,\ h_2).$$

The space

$$\operatorname{Im}(\mathrm{d}\psi_{(0,0)}) := \{(0, y, z) \mid y, z \in \mathbb{R}\}$$

is a vector subspace of $\mathbb{R}^3$ of dimension 2. Translating it by the vector $(1, 0, 0)$ gives the tangent plane to the cylinder at the point $(1, 0, 0)$, given by $\Pi := \{(1, y, z) \in \mathbb{R}^3 \mid y, z \in \mathbb{R}\}$, as illustrated below:



This subspace has basis

$$\beta = \left\{ \frac{\partial \psi}{\partial x}(x_0), \frac{\partial \psi}{\partial y}(x_0) \right\} = \{(0,1,0),(0,0,1)\}.$$

Under these assumptions, we refer to $\mathrm{Im}(d\psi_{x_0})$ as the tangent space to $\psi(\widetilde{U})$ at $p = \psi(x_0)$.
The local tool that makes this description precise is the following theorem from differential calculus.

Theorem 1.4.2. (Rank theorem) Let $f\colon \widetilde{U} \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^n$ be a function of class C^1, and let $x_0 \in \widetilde{U}$. Suppose there exists $r > 0$, with $B_{\mathrm{euc}}(x_0, r) \subseteq \widetilde{U}$, such that df_x has constant rank m for every $x \in B_{\mathrm{euc}}(x_0, r)$. Then there exist open sets $\widetilde{V}_1, \widetilde{V}_2 \subseteq \mathbb{R}^k$ and $V_1, V_2 \subseteq \mathbb{R}^n$, with $x_0 \in \widetilde{V}_1$, $0 \in \widetilde{V}_2$, $f(x_0) \in V_1$, and $0 \in V_2$, and diffeomorphisms of class C^1

$$\Phi\colon \widetilde{V}_2 \longrightarrow \widetilde{V}_1, \qquad \Psi\colon V_2 \longrightarrow V_1,$$

such that

$$\Psi^{-1} \circ f \circ \Phi(x_1, \ldots, x_k) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

In particular, near x_0, the map f can be expressed in suitable coordinates on the domain and codomain as projection onto the first m coordinates followed by the natural inclusion into $\mathbb{R}^n$.

Proof. We apply Lemma 1.5.5, whose proof follows directly from contractions and does not use the rank theorem. After permuting coordinates in the domain and codomain, we may assume that the minor formed by the first m rows and columns of df_{x_0} is invertible. Write $f = (f^1, \ldots, f^n)$ and define

$$H(x) := (f^1(x), \ldots, f^m(x), x^{m+1}, \ldots, x^k).$$

The differential dH_{x_0} is invertible. By Lemma 1.5.5, after restricting the domain, $\widehat{H}(x) := H(x) - H(x_0)$ is a diffeomorphism onto a neighborhood of 0. Let $\Phi = \widehat{H}^{-1}$. In the new coordinates,

$$f \circ \Phi(y) = \big(y^1 + f^1(x_0), \ldots, y^m + f^m(x_0), h^{m+1}(y), \ldots, h^n(y)\big).$$

The first m rows of its Jacobian matrix contain the block $(I_m\ 0)$. Since the total rank is m, necessarily

$$\frac{\partial h^a}{\partial y^b} = 0, \qquad a = m+1,\dots,n, \quad b = m+1,\dots,k.$$

Restrict the neighborhood to a box. Integrating these derivatives along segments parallel to the last $k-m$ axes shows that each h^a depends only on $y^1,\dots,y^m$.
Let $\widehat{h}^a$ be the resulting functions and define, near $f(x_0)$,

$$K(z) := \big(z^1 - f^1(x_0),\dots,z^m - f^m(x_0), z^{m+1} - \widehat{h}^{m+1}(z^1 - f^1(x_0),\dots,z^m - f^m(x_0)),\dots,z^n - \widehat{h}^n(z^1 - f^1(x_0),\dots,z^m - f^m(x_0))\big).$$

This map is a diffeomorphism: its inverse is obtained by adding $f^a(x_0)$ in the first m coordinates and, in the remaining coordinates, the functions $\widehat{h}^a$ evaluated at those first normalized coordinates. More precisely,

$$K^{-1}(w) = \big(w^1 + f^1(x_0),\dots,w^m + f^m(x_0), w^{m+1} + \widehat{h}^{m+1}(w^1,\dots,w^m),\dots,w^n + \widehat{h}^n(w^1,\dots,w^m)\big).$$

By construction,

$$K \circ f \circ \Phi(y) = (y^1,\dots,y^m,0,\dots,0).$$

Taking $\Psi = K^{-1}$ and restricting the open sets so that all the compositions are defined gives the statement. □

Proposition 1.4.3. (Local description by curves) Let $\psi\colon \widetilde{U} \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be of class C^1, let $x_0 \in \widetilde{U}$, and assume only that $d\psi_{x_0}$ has rank k. Then there exists an open neighborhood $V \subseteq \widetilde{U}$ of x_0 such that $\psi \restriction_V\colon V \to \mathbb{R}^n$ is an embedding of class C^1. If $p = \psi(x_0)$, every such neighborhood satisfies

$$\operatorname{Im}(d\psi_{x_0}) = \left\{ \gamma'(0) \;\middle|\; \begin{array}{l} \gamma\colon (-\varepsilon,\varepsilon) \to \mathbb{R}^n \text{ is of class } C^1, \quad \gamma(0) = p, \\ \gamma((-\varepsilon,\varepsilon)) \subseteq \psi(V) \text{ for some } \varepsilon > 0 \end{array} \right\}.$$

Proof. Since $d\psi_{x_0}$ has rank k, some minor of order k of its Jacobian matrix is nonzero. By continuity, this minor remains nonzero in a neighborhood of x_0, where the rank is therefore constantly k. Theorem 1.4.2, followed by a restriction of the neighborhood, gives open sets $V \ni x_0$ and $W \ni p$ and coordinates in which

$$\psi \restriction_V (x^1,\dots,x^k) = (x^1,\dots,x^k,0,\dots,0).$$

Thus $\psi \restriction_V$ is an injective immersion and a homeomorphism onto $\psi(V)$ with the subspace topology; that is, it is an embedding of class C^1.
Let $v = d\psi_{x_0}(w)$. If $w = 0$, the constant curve represents v. If $w \neq 0$, choose $\varepsilon > 0$ so that $x_0 + tw \in V$ whenever $|t| < \varepsilon$, and set $\gamma(t) = \psi(x_0 + tw)$. Then $\gamma'(0) = v$.
Conversely, if γ is a curve as in the statement, the inverse C^1 of the local embedding allows us to define

$$\delta = (\psi \restriction_V)^{-1} \circ \gamma.$$

The chain rule gives $\gamma'(0) = d\psi_{x_0}(\delta'(0))$, and hence $\gamma'(0) \in \operatorname{Im}(d\psi_{x_0})$. □

Remark 1.4.4. (Why the assertion is local) We cannot replace $\psi(V)$ by the entire image $\psi(\widetilde{U})$. An immersion may have self-intersections: a curve passing through $p = \psi(x_0)$ along another branch may have a velocity that does not belong to $\operatorname{Im}(d\psi_{x_0})$. The choice of V separates the branch determined by x_0.

In what follows, we return to considering manifolds in $\mathbb{R}^n$; taking $k \le n$ and considering it to have dimension k served to illustrate what happens when the manifold lies in a possibly larger ambient space.

The preceding motivation leads to an intrinsic definition that works unchanged at interior and boundary points alike.

Definition 1.4.5. Let M be a smooth manifold with or without boundary and let $p \in M$. A **derivation at** p is a linear map $v\colon C^\infty(M) \longrightarrow \mathbb{R}$ such that

$$v(fg) = f(p)v(g) + g(p)v(f)$$

for any $f, g \in C^\infty(M)$. The set of all derivations at p is denoted by T_pM and is called the **tangent space to** M **at** p. Its elements are the tangent vectors at p.

Remark 1.4.6. With addition and scalar multiplication defined pointwise, T_pM is a real vector space. If $v \in T_pM$, then $v(1) = 0$ and therefore $v(c) = 0$ for every constant function c.

At interior points, we recover the geometric description using two-sided curves.

Definition 1.4.7. (Tangent space defined by curves) Let M be a smooth manifold with or without boundary and let $p \in \operatorname{Int} M$. Consider smooth curves $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$ with $\gamma(0) = p$. Two such curves are equivalent if, for some interior chart φ around p, we have $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$. The chain rule shows that this condition is independent of the chart. We denote the class of γ by $[\gamma]$.

Lemma 1.4.8. (Locality of derivations) Let M be a smooth manifold with or without boundary, let $p \in M$, and let $v \in T_pM$. If $f, g \in C^\infty(M)$ agree in a neighborhood of p, then $v(f) = v(g)$. Consequently, v can be evaluated unambiguously on the germ of a smooth function defined only near p.

Proof. Set $h = f - g$ and choose an open set W containing p on which $h = 0$. By Proposition 1.3.6, there exists $\chi \in C^\infty(M)$ with $\chi(p) = 1$ and $\operatorname{supp}\chi \subseteq W$. Since $\chi h = 0$ and $h(p) = 0$, the Leibniz rule gives

$$0 = v(\chi h) = \chi(p)v(h) + h(p)v(\chi) = v(h).$$

Therefore, $v(f) = v(g)$. If a function is defined only near p, Theorem 1.3.7 provides a global extension agreeing with it on a smaller neighborhood; the equality just proved shows that the value is independent of the extension. □

The identification between curves and derivations uses the following two formulas from Euclidean calculus.

Theorem 1.4.9. (Taylor's theorem with multi-indices) Let $f \in C^{k+1}(U)$ and $a \in U$. We define the Taylor

polynomial of f of order k centered at $a \in U$ by

$$P_{k,a,f}(x) := \sum_{|\alpha| \le k} \frac{1}{\alpha!} D^\alpha f(a)(x-a)^\alpha,$$

where $\frac{1}{\alpha!} := \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_n!}$. Define $R_{k,a,f}(x) = f(x) - P_{k,a,f}(x)$ and call it the remainder of order k of f at a. Then:

(a) If $x \in U$ is such that $[a,x] \subseteq U$, there exists $\xi \in [a,x]$ such that

$$R_{k,a,f}(x) = \sum_{|\alpha|=k+1} \frac{1}{\alpha!} D^\alpha f(\xi)(x-a)^\alpha.$$

(b) If $x \in U$ is such that $[a,x] \subseteq U$, then

$$R_{k,a,f}(x) = \sum_{|\alpha|=k+1} \frac{k+1}{\alpha!} (x-a)^\alpha \int_0^1 (1-t)^k D^\alpha f(a+t(x-a))\,dt.$$

(c) We have

$$\lim_{x \to a} \frac{f(x) - P_{k,a,f}(x)}{\|x-a\|^k} = \lim_{x \to a} \frac{R_{k,a,f}(x)}{\|x-a\|^k} = 0$$

and $P_{k,a,f}(x)$ is the unique polynomial of degree at most k with this property.

Proof. Fix $x \in U$ with $[a,x] \subseteq U$, set $h = x - a$, and define $\gamma(t) := f(a+th)$. Induction on j, together with the chain rule, the Clairaut–Schwarz theorem, and the multinomial formula, gives

$$\gamma^{(j)}(t) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} D^\alpha f(a+th) h^\alpha, \qquad 0 \le j \le k+1. \tag{1.1}$$

Indeed, if the equality holds for j, differentiation gives

$$\sum_{|\alpha|=j} \sum_{r=1}^{n} \frac{j!}{\alpha!} D^{\alpha+e_r} f(a+th) h^{\alpha+e_r}.$$

For a multi-index β of length $j+1$, the coefficient of the corresponding term is

$$\sum_{r:\beta_r>0} \frac{j!}{(\beta-e_r)!} = \frac{j!}{\beta!} \sum_{r=1}^{n} \beta_r = \frac{(j+1)!}{\beta!},$$

which completes the induction.

The one-dimensional Taylor formula with integral remainder, obtained by successive applications of the fundamental theorem of calculus, states that

$$\gamma(1) = \sum_{j=0}^{k} \frac{\gamma^{(j)}(0)}{j!} + \frac{1}{k!} \int_0^1 (1-t)^k \gamma^{(k+1)}(t)\,dt.$$

Substituting (1.1), the finite sum is $P_{k,a,f}(x)$, and the last term is

$$\sum_{|\alpha|=k+1} \frac{k+1}{\alpha!} h^\alpha \int_0^1 (1-t)^k D^\alpha f(a+th)\,dt.$$

This proves (2). The mean value theorem for integrals gives $\theta \in [0,1]$ such that the one-dimensional remainder is $\gamma^{(k+1)}(\theta)/(k+1)!$. Using (1.1) again and taking $\xi = a+\theta h$ gives (1).
The derivatives of order $k+1$ are bounded on a closed ball centered at a and contained in U. For x near a, the integral formula implies

$$|R_{k,a,f}(x)| \le C\|x-a\|^{k+1},$$

and proves the limit in (3). If Q is another polynomial of degree at most k with this property, then

$$\frac{(Q-P_{k,a,f})(x)}{\|x-a\|^k} \longrightarrow 0.$$

If $S := Q - P_{k,a,f} \neq 0$, let S_j be its first nonzero homogeneous part, of degree $j \le k$, and choose v with $S_j(v) \neq 0$. Then

$$\frac{S(a+tv)}{\|tv\|^k} = \frac{t^j S_j(v) + O(t^{j+1})}{t^k \|v\|^k}$$

does not tend to zero as $t \to 0^+$: it is unbounded if $j < k$, and tends to $S_k(v)/\|v\|^k \neq 0$ if $j = k$. This contradiction proves uniqueness. □

Lemma 1.4.10. (Finite-dimensional Hadamard lemma with integral remainder) Let $U \subseteq \mathbb{R}^n$ be open, let $x_0, x \in U$, and suppose that $x_0 + t(x-x_0) \in U$ for every $t \in [0,1]$. If $F\colon U \longrightarrow \mathbb{R}^N$ is of class C^1, then

$$F(x) - F(x_0) = \int_0^1 DF\big(x_0 + t(x-x_0)\big)[x-x_0],dt. \tag{1.2}$$

In particular,

$$F(x) - F(x_0) = \sum_{j=1}^n (x^j - x_0^j) F_j(x,x_0), \qquad F_j(x,x_0) := \int_0^1 \frac{\partial F}{\partial x^j}\big(x_0 + t(x-x_0)\big),dt. \tag{1.3}$$

If F is of class C^2, $F(x_0) = 0$, and $DF(x_0) = 0$, then

$$F(x) = \sum_{j,k=1}^n (x^j - x_0^j)(x^k - x_0^k) \int_0^1 (1-t) \frac{\partial^2 F}{\partial x^j \partial x^k}\big(x_0 + t(x-x_0)\big),dt. \tag{1.4}$$

The same formulas hold for a family $F(z,x)$ depending smoothly on a finite-dimensional parameter z: the functions defined by the integrals above are jointly smooth in (z,x,x_0) as long as the segments under consideration remain in the domain.

Proof. Define $g(t) := F(x_0 + t(x-x_0))$. The chain rule gives

$$g'(t) = DF\big(x_0 + t(x-x_0)\big)[x-x_0].$$

The fundamental theorem of calculus, applied componentwise in $\mathbb{R}^N$, proves $F(x)-F(x_0)=g(1)-g(0)=\int_0^1 g'(t)\,dt$ and, upon expressing the differential in the standard basis, yields (1.3).
Now suppose that $F(x_0)=0$ and $DF(x_0)=0$. For each $t\in[0,1]$, applying the preceding formula to DF on the subsegment joining x_0 to $x_0+t(x-x_0)$ gives

$$DF\big(x_0+t(x-x_0)\big)[x-x_0]=t\int_0^1 D^2F\big(x_0+st(x-x_0)\big)[x-x_0,x-x_0]\,ds.$$

Substitute this identity into (1.2). The change of variable $\tau=st$ and Fubini's theorem on the triangle $0\le\tau\le t\le 1$ yield

$$F(x)=\int_0^1 (1-\tau)D^2F\big(x_0+\tau(x-x_0)\big)[x-x_0,x-x_0]\,d\tau,$$

which is precisely (1.4) after expanding the bilinear form in coordinates. The assertion with parameters follows by differentiating under the integral over the compact interval $[0,1]$. □

Proposition 1.4.11. (Equivalence between curves and derivations) Let M be a smooth manifold with or without boundary and let $p\in \operatorname{Int} M$. The map

$$\{\text{classes of two-sided curves at } p\}\longrightarrow T_pM,\qquad [\gamma]\longmapsto v_{[\gamma]},$$

where

$$v_{[\gamma]}(f):=\frac{d}{dt}(f\circ\gamma)(t)\Big|_{t=0},\qquad f\in C^\infty(M),$$

is a canonical bijection.

Proof. If $[\gamma_1]=[\gamma_2]$, choose an interior chart $\varphi\colon U\to\mathbb{R}^n$ around p. For $F=f\circ\varphi^{-1}$, the chain rule gives

$$\frac{d}{dt}(f\circ\gamma_i)(t)\Big|_{t=0}=DF(\varphi(p))[(\varphi\circ\gamma_i)'(0)],\qquad i=1,2.$$

The coordinate velocities agree, so $v_{[\gamma]}$ is well defined. Linearity and the product rule show that

$$v_{[\gamma]}(\lambda f+\mu g)=\lambda v_{[\gamma]}(f)+\mu v_{[\gamma]}(g),\qquad v_{[\gamma]}(fg)=f(p)v_{[\gamma]}(g)+g(p)v_{[\gamma]}(f).$$

Therefore, $v_{[\gamma]}\in T_pM$.
Write $\varphi=(x^1,\dots,x^n)$. By Lemma 1.4.8, the coordinate functions can be evaluated as germs, and

$$(\varphi\circ\gamma)'(0)=\big(v_{[\gamma]}(x^1),\dots,v_{[\gamma]}(x^n)\big).$$

Thus $v_{[\gamma_1]}=v_{[\gamma_2]}$ implies $[\gamma_1]=[\gamma_2]$, and the map is injective.
For surjectivity, let $v\in T_pM$ and set $x_0=\varphi(p)$ and

$$w^i:=v(x^i),\qquad w=(w^1,\dots,w^n).$$

Since p is interior, for sufficiently small $|t|$ the curve

$$\gamma(t) := \varphi^{-1}(x_0 + tw),$$

is defined, and $(\varphi \circ \gamma)'(0) = w$. Let $f \in C^\infty(M)$ and write $F = f \circ \varphi^{-1}$. Apply Taylor's theorem with integral remainder at the point x_0:

$$R_{1,x_0,F}(x) = \sum_{|\alpha|=2} \frac{2}{\alpha!}(x - x_0)^\alpha \int_0^1 (1-t) D^\alpha F\big(x_0 + t(x - x_0)\big)\, dt,$$

and therefore

$$F(x) = F(x_0) + \sum_{i=1}^n \frac{\partial F}{\partial x_i}(x_0)(x_i - x_i^{(0)}) + R_{1,x_0,F}(x).$$

Taking $x = \varphi(q)$ for q near p, we obtain:

$$f(q) = f(p) + \sum_{i=1}^n \frac{\partial F}{\partial x_i}(x_0)\big(x^i(q) - x^i(p)\big) + \underbrace{R_{1,x_0,F}\big(\varphi(q)\big)}_{=:\, r(q)}.$$

We can rewrite this as

$$r(q) = \sum_{|\alpha|=2} \big(\varphi(q) - \varphi(p)\big)^\alpha h_\alpha(q), \qquad h_\alpha(q) := \frac{2}{\alpha!} \int_0^1 (1-t) D^\alpha F\big(\varphi(p) + t(\varphi(q) - \varphi(p))\big)\, dt,$$

where each h_α is smooth and well defined near p.
The vector v annihilates the remainder term r. For each multi-index α with $|\alpha| = 2$, let

$$m_\alpha(q) := \big(\varphi(q) - \varphi(p)\big)^\alpha = \prod_{i=1}^n \big(x^i(q) - x^i(p)\big)^{\alpha_i}.$$

Then $m_\alpha(p) = 0$, and m_α can be viewed as a product of at least two factors vanishing at p, since $|\alpha| = 2$ forces either two of the α_j to be 1 or one to be 2. By repeated application of the Leibniz rule,

$$v\big(m_\alpha\big) = 0 \quad \text{and} \quad v\big(m_\alpha h_\alpha\big) = m_\alpha(p) v(h_\alpha) + h_\alpha(p) v(m_\alpha) = 0.$$

Summing over $|\alpha| = 2$, we conclude that $v(r) = 0$. Applying v to the decomposition of f gives

$$v(f) = \sum_{i=1}^n \frac{\partial F}{\partial x^i}(x_0) v\big(x^i\big).$$

On the other hand,

$$v_{[\gamma]}(f) = \frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=0} = \frac{d}{dt} F(x_0 + tw)\Big|_{t=0} = \sum_{i=1}^n w^i \frac{\partial F}{\partial x^i}(x_0),$$

and since $w^i = v(x^i)$, we have $v(f) = v_{[\gamma]}(f)$ for every $f \in C^\infty(M)$. Thus $v = v_{[\gamma]}$, and every derivation arises

from an equivalence class of curves. We conclude that the map

$$\{\text{classes of two-sided curves at } p\} \longrightarrow T_pM, \qquad [\gamma] \longmapsto v_{[\gamma]},$$

is bijective when $p \in \operatorname{Int} M$. □

At a boundary point this equivalence no longer holds for two-sided curves contained in M. We will return to this distinction after describing the coordinate basis of T_pM.

There are other equivalent descriptions of the tangent space. One uses *germs of smooth functions*, which reflect the local nature of differentiation.

Definition 1.4.12. Let M be a smooth manifold with or without boundary and let $p \in M$. Define an equivalence relation on $C^\infty(M)$ by

$$f \sim g \iff \text{there exists a neighborhood } U \text{ of } p \text{ such that } f\restriction_U = g\restriction_U.$$

We denote the equivalence class of f by $[f]$ and call it the *germ of f at p*. The set of all such classes is denoted by $C^\infty_p(M)$ and is called the *ring of germs of smooth functions at p*.

Proposition 1.4.13. (Derivations of the ring of germs) Let M be a smooth manifold with or without boundary and let $p \in M$. There is a canonical linear isomorphism

$$T_pM \longrightarrow \operatorname{Der}\big(C^\infty_p(M), \mathbb{R}\big), \qquad v \longmapsto \big([f] \longmapsto v(f)\big).$$

Its inverse sends a derivation D of germs to the derivation $v_D(f) := D([f])$ of $C^\infty(M)$.

Proof. Addition and multiplication of germs are defined by $[f]+[g] = [f+g]$ and $[f][g] = [fg]$; these operations are independent of representatives because equality of germs is preserved by addition and multiplication of functions. Moreover, $f(p)$ depends only on $[f]$. Therefore, the rule

$$D([f][g]) = f(p)D([g]) + g(p)D([f])$$

is well defined.
If $v \in T_pM$, Lemma 1.4.8 shows that $\widetilde{v}([f]) := v(f)$ is independent of the representative. Linearity and the Leibniz rule for $\widetilde{v}$ follow from those for v. Conversely, if $D \in \operatorname{Der}(C^\infty_p(M), \mathbb{R})$, then $v_D(f) := D([f])$ is linear and

$$v_D(fg) = D([f][g]) = f(p)v_D(g) + g(p)v_D(f),$$

so $v_D \in T_pM$. Finally, $v_{\widetilde{v}}(f) = \widetilde{v}([f]) = v(f)$ and $\widetilde{v_D}([f]) = v_D(f) = D([f])$. The two constructions are inverse to each other and require no choices. □

With the tangent space defined, we can now introduce the tangent map, or differential, of a smooth function.

Definition 1.4.14. Let M and N be smooth manifolds with or without boundary, and let $F\colon M \longrightarrow N$ be a

smooth map. For each $p \in M$, define the map

$$dF_p \colon T_pM \longrightarrow T_{F(p)}N$$

called the **tangent map or differential of** F **at** p, by

$$dF_p(v)(f) := v(f \circ F), \qquad v \in T_pM, \quad f \in C^\infty(N).$$

Indeed, $dF_p(v)$ is linear and, for $f, g \in C^\infty(N)$,

$$dF_p(v)(fg) = f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f),$$

so it belongs to $T_{F(p)}N$.

The tangent map, or differential, has the following properties:

Proposition 1.4.15. Let M, N, and P be smooth manifolds with or without boundary, and let $F \colon M \longrightarrow N$ and $G \colon N \longrightarrow P$ be smooth maps. Let $p \in M$.

(a) $dF_p \colon T_pM \longrightarrow T_{F(p)}N$ is linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p \colon T_pM \longrightarrow T_{G(F(p))}P$.

(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.

(d) If F is a diffeomorphism, then $dF_p \colon T_pM \longrightarrow T_{F(p)}N$ is an isomorphism and

$$(dF_p)^{-1} = d(F^{-1})_{F(p)}.$$

Proof. Linearity follows directly from the linearity of each derivation. If $v \in T_pM$ and $h \in C^\infty(P)$, then

$$d(G \circ F)_p(v)(h) = v(h \circ G \circ F) = dF_p(v)(h \circ G) = dG_{F(p)}(dF_p(v))(h),$$

which proves the chain rule. The formula for the identity is the immediate case $h \circ \mathrm{Id}_M = h$. If F is a diffeomorphism, apply the chain rule to $F^{-1} \circ F = \mathrm{Id}_M$ and to $F \circ F^{-1} = \mathrm{Id}_N$; thus $d(F^{-1})_{F(p)}$ is the inverse of dF_p. □

Since T_pM is a vector space, a basis is useful for describing it. In the Euclidean case, $T_p\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ for every $p \in \mathbb{R}^n$, and tangent vectors may be viewed as directional derivatives at p in the direction of $v \in T_p\mathbb{R}^n$. This leads to the following proposition:

Proposition 1.4.16. Let M be a smooth manifold with or without boundary of dimension n, and let $p \in M$. Then T_pM is a vector space of dimension n, and for each smooth chart (U, φ) such that $p \in U$, with $\varphi = (x^1, \ldots, x^n)$, the vectors

$$\partial_i|_p(f) := \frac{\partial \widetilde{F}}{\partial x^i}(\varphi(p)), \qquad F = f \circ \varphi^{-1},$$

form a basis for T_pM. Here $\widetilde{F} = F$ in an interior chart; in a boundary chart, $\widetilde{F}$ is any smooth extension of F

to a Euclidean neighborhood of $\varphi(p)$.

Proof. In a boundary chart, the definition of a smooth function provides the extension $\widetilde{F}$. If two extensions agree on the relatively open set $\varphi(U) \subseteq \mathbb{H}^n$, their partial derivatives agree at interior points and, by continuity, also at boundary points. Thus the formula is independent of the extension. Euclidean linearity and the Leibniz rule show that each $\partial_i|_p$ is a derivation at p.
Let $v \in T_pM$ and write $x_0 = \varphi(p)$. Restrict the chart, if necessary, so that segments starting at x_0 remain in its image. Hadamard's lemma 1.4.10, applied to a Euclidean extension when $p \in \partial M$, gives smooth functions F_i such that, near x_0,

$$F(x) - F(x_0) = \sum_{i=1}^{n} (x^i - x_0^i) F_i(x), \qquad F_i(x_0) = \frac{\partial \widetilde{F}}{\partial x^i}(x_0).$$

Lemma 1.4.8 allows us to apply v to this identity of germs. Since each $x^i - x_0^i$ vanishes at p, we obtain

$$v(f) = \sum_{i=1}^{n} v(x^i) \frac{\partial \widetilde{F}}{\partial x^i}(x_0) = \left(\sum_{i=1}^{n} v(x^i)\, \partial_i|_p \right)(f).$$

Thus the coordinate vectors span T_pM. If $\sum_{i=1}^{n} a^i\, \partial_i|_p = 0$, evaluation on the germ of x^j gives $a^j = 0$. They are therefore linearly independent and form a basis; in particular, $\dim T_pM = n$.
If $(y^1, \dots, y^n)$ is another coordinate system around p, the expansion just proved, applied to $v = \partial_i|_p$, gives the change-of-basis formula

$$\partial_i|_p = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i}(p)\, \partial_j\Big|_p. \tag{1.5}$$

□

Remark 1.4.17. (Two-sided curves at the boundary) Suppose that $p \in \partial M$ and take a boundary chart with $\mathbb{H}^n = \{x^n \geq 0\}$ and $x^n(p) = 0$. If $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is smooth and $\gamma(0) = p$, the function $t \mapsto x^n(\gamma(t))$ is nonnegative and attains a minimum at 0; therefore, $(x^n \circ \gamma)'(0) = 0$. The velocities of two-sided curves contained in M range over precisely the coordinate subspace

$$\operatorname{span}\left\{ \partial_1|_p, \dots, \partial_{n-1}|_p \right\},$$

which will later be identified with $T_p\partial M$, rather than all of T_pM. The reverse inclusion follows by using the coordinate curves $t \mapsto x_0 + t(w^1, \dots, w^{n-1}, 0)$.
With this convention, a vector $v = v^i\, \partial_i|_p$ points inward if $v^n > 0$, outward if $v^n < 0$, and is tangent to the boundary if $v^n = 0$. The sign is independent of the chart: for two normal boundary coordinates x^n and y^n, Hadamard's lemma gives $y^n = a x^n$ in a relative neighborhood of p, with $a(p) > 0$. For this reason, the primary definition of T_pM at a boundary point is the definition by derivations, rather than the description by two-sided curves.

This notation is suggestive because it essentially transfers the concept of a partial derivative from the Euclidean setting to manifolds. The following definition makes this precise:

Definition 1.4.18. Let M be a smooth manifold with or without boundary, let (U,φ) be a smooth chart of M, with $\varphi = (x^1,\dots,x^n)$, and let $f : U \longrightarrow \mathbb{R}$ be a smooth function. Given a multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$, we define:

(a) For each $p \in U$, the partial derivative of f associated with α at p by $\partial^\alpha f(p) := \partial^\alpha (f \circ \varphi^{-1})(\varphi(p))$.

(b) The partial derivative function of f associated with α on U by $\partial^\alpha f : U \longrightarrow \mathbb{R}$, given by $p \mapsto \partial^\alpha f(p)$.

The tangent bundle, which gives this section its name, is defined as follows:

Definition 1.4.19. Let M be a smooth manifold with or without boundary. Define the **tangent bundle of** M by

$$TM := \coprod_{p\in M} T_pM$$

Elements of TM have the form (p,v) with $p \in M$, $v \in T_pM$. The following notations are commonly used for an element of TM: (p,v), v_p, and v, the last when we do not wish to emphasize the base point. By Proposition 1.4.16, every element of T_pM can be written in the form (using Einstein notation)

$$v_p = v^i \partial_i \Big|_p ,$$

which gives another way to express elements of TM.

The tangent bundle can be equipped with a smooth manifold structure, with or without boundary according to the type of manifold we start with. The following proposition describes the construction:

Proposition 1.4.20. If M is a smooth n-manifold with or without boundary, the tangent bundle TM has a natural topology and smooth structure making it a smooth $2n$-manifold with or without boundary. With respect to this smooth structure, the projection $\pi : TM \longrightarrow M$ is smooth. The coordinate charts for this smooth structure are given as follows:

(a) If (U,φ) is an interior chart of M, then $\widetilde{\varphi} : \pi^{-1}(U) \longrightarrow \mathbb{R}^{2n}$, given by

$$\widetilde{\varphi}\left(p, v^i \partial_i \Big|_p\right) = (x^1(p),\dots,x^n(p),v^1,\dots,v^n)$$

makes $(\pi^{-1}(U),\widetilde{\varphi})$ an interior chart of TM.

(b) If (U,φ) is a boundary chart of M, then $\widetilde{\varphi} : \pi^{-1}(U) \longrightarrow \mathbb{H}^{2n}$, given by

$$\widetilde{\varphi}\left(p, v^i \partial_i \Big|_p\right) = (v^1,\dots,v^n,x^1(p),\dots,x^n(p))$$

makes $(\pi^{-1}(U),\widetilde{\varphi})$ a boundary chart of TM.

Proof. For each chart (U,φ), the map $\widetilde{\varphi}$ is a bijection onto $\varphi(U)\times\mathbb{R}^n$ in the interior case and onto $\mathbb{R}^n \times \varphi(U)$ in the boundary case; the latter set is open in $\mathbb{H}^{2n}$ under the convention in the statement. If (V,ψ) is another chart, write $y = \psi \circ \varphi^{-1}(x)$. The change-of-basis formula (1.5) shows that, on the overlap, the components

of a vector satisfy

$$w^j = \sum_{i=1}^{n} \frac{\partial y^j}{\partial x^i}(x) v^i.$$

Thus, apart from explicitly interchanging the order of the two blocks in a boundary chart, the change of coordinates on TM is

$$(x, v) \longmapsto \big(y(x), Dy(x)v\big).$$

It is smooth, and its inverse is obtained by replacing y by $x(y)$. An overlap between an interior chart and a boundary chart contains no points of ∂M, so the same formula is interpreted on Euclidean open sets.
The smooth chart lemma proved above, applied to this family, provides a unique Hausdorff, second countable topology and the stated smooth structure on TM. In these coordinates, the projection is $(x, v) \mapsto x$ (with the corresponding interchange of blocks at the boundary), so it is smooth. If M has no boundary, all charts are interior charts; if it has boundary, the boundary charts described above determine the boundary of TM. This also completes the uniqueness of the structure for which the charts in the statement are smooth. □

1.5 Immersions, submersions, and embeddings

As in the Euclidean case, the differential of a function between manifolds, which plays the role of the usual derivative, describes the best linear approximation to a smooth map near each point. Thus many local properties of a map can be studied through the linear properties of its differential. This motivates the following definition.

Definition 1.5.1. Let M and N be smooth manifolds with or without boundary, and let $F\colon M \longrightarrow N$ be a smooth map. We define the rank of F at a point $p \in M$ as the rank of the linear map

$$dF_p\colon T_pM \longrightarrow T_{F(p)}N.$$

Equivalently, the rank of F at p is the dimension of the image of dF_p. In coordinates, this number equals the rank of the Jacobian matrix of a coordinate representation of F. If F has the same rank at every point of its domain, we say that F has constant rank.

The notions of immersion and submersion generalize, respectively, injectivity and surjectivity of linear maps to the smooth setting.

Definition 1.5.2. Let M and N be smooth manifolds without boundary, and let $F\colon M \longrightarrow N$ be a smooth map. We say that:

(a) F is an immersion if dF_p is injective for every $p \in M$;

(b) F is a submersion if dF_p is surjective for every $p \in M$.

Being an immersion or a submersion is a local property. If the differential of a map is injective or surjective at a point, the same property holds throughout some open neighborhood of that point.

Proposition 1.5.3. Let M and N be smooth manifolds without boundary, let $F\colon M \longrightarrow N$ be a smooth map, and let $p \in M$.

(a) If dF_p is injective, there exists an open neighborhood U of p such that $F\restriction_U$ is an immersion.

(b) If dF_p is surjective, there exists an open neighborhood U of p such that $F\restriction_U$ is a submersion.

Proof. In charts around p and $F(p)$, the differential is represented by the Jacobian matrix. If dF_p is injective and $m = \dim M$, some minor of order m has nonzero determinant. By continuity, this minor remains nonsingular in a neighborhood of p; there the rank is m, so the differential is injective. If dF_p is surjective and $n = \dim N$, the same argument applies to a minor of order n, and the differential is surjective in a neighborhood of p. □

We now give some examples of immersions and submersions.

Example 1.5.4. (a) If $M_1,\dots,M_k$ are smooth manifolds without boundary, each projection $\pi_i\colon M_1 \times \cdots \times M_k \longrightarrow M_i$ is a submersion. For each $(p_1,\dots,p_k) \in M_1 \times \cdots \times M_k$, its differential satisfies $d(\pi_i)_{(p_1,\dots,p_k)}(v_1,\dots,v_k) = v_i$. Indeed, given $w \in T_{p_i}M_i$, the vector $(0,\dots,0,w,0,\dots,0)$ in the tangent space to the product is sent to w; therefore, the differential is surjective.

(b) The map $F\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$, given by $F(x^1,\dots,x^m) = (x^1,\dots,x^m,0,\dots,0)$, with $m \le n$, is an immersion. Its differential is the natural linear inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$, which is injective.

(c) The map $G\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, given by $G(x,y,z) = (x,y)$, is a submersion, since $dG_{(x,y,z)}(a,b,c) = (a,b)$ for every $(x,y,z) \in \mathbb{R}^3$, and this linear map is surjective.

(d) The map $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, given by $\gamma(t) = (\cos t, \sin t)$, is an immersion, since $\gamma'(t) = (-\sin t, \cos t) \neq 0$ for every $t \in \mathbb{R}$.

(e) The map $H\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$, given by $H(x,y) = x^2 + y^2$, is not a submersion on all of $\mathbb{R}^2$, since $dH_{(0,0)} = 0$. It is, however, a submersion on $\mathbb{R}^2 \setminus \{(0,0)\}$.

(f) Consider the map $K\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, given by $K(x,y) = (x^2 - y^2, 2xy)$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, this map corresponds to $z \mapsto z^2$. A direct calculation shows that $DK(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$, whose determinant is $4(x^2+y^2)$. Thus K is both an immersion and a submersion at every point other than the origin, but has neither property at $(0,0)$.

The most important theorems of multivariable differential calculus also hold for smooth manifolds and will be useful throughout the book. We present them next.

Lemma 1.5.5. (Euclidean inverse function theorem) Let $\Omega \subseteq \mathbb{R}^n$ be open, let $f \in C^1(\Omega, \mathbb{R}^n)$, and let $a \in \Omega$. If $Df(a)$ is invertible, there exist connected open sets $U \subseteq \Omega$ and $V \subseteq \mathbb{R}^n$, with $a \in U$ and $f(a) \in V$, such that $f\restriction_U\colon U \longrightarrow V$ is a diffeomorphism of class C^1. If f is of class C^r, with $2 \le r \le \infty$, its inverse is also of class C^r.

Proof. Set $A := Df(a)$ and $c := \|A^{-1}\|_{\mathrm{op}}$. By continuity of Df, there exists $r > 0$ such that $\overline{B}(a,r) \subseteq \Omega$ and

$$\|I - A^{-1}Df(x)\|_{\mathrm{op}} \le \frac{1}{2}, \qquad x \in \overline{B}(a,r). \tag{1.6}$$

Let $\delta := \frac{r}{2c}$ and fix $y \in B(f(a),\delta)$. Define

$$T_y(x) := x - A^{-1}(f(x) - y), \qquad x \in \overline{B}(a,r).$$

The fundamental theorem of calculus applied to the segment joining x and z, followed by (1.6), gives

$$\|T_y(x) - T_y(z)\| \le \frac{1}{2}\|x - z\|, \qquad x, z \in \overline{B}(a,r).$$

Moreover,

$$\|T_y(x) - a\| \le \|T_y(x) - T_y(a)\| + \|A^{-1}(y - f(a))\| < \frac{r}{2} + \frac{r}{2} = r.$$

Thus T_y maps the closed ball into itself and is a contraction.
If $x_0 \in \overline{B}(a,r)$ and $x_{k+1} := T_y(x_k)$, then

$$\|x_{k+1} - x_k\| \le 2^{-k}\|x_1 - x_0\|.$$

Summing these bounds shows that (x_k) is Cauchy. The closed ball is complete, so $x_k \to x$, and continuity of T_y gives $T_y(x) = x$. If x' is another fixed point, then $\|x - x'\| \le \frac{1}{2}\|x - x'\|$, whence $x = x'$. Denote this unique fixed point by $g(y)$. The equation $T_y(g(y)) = g(y)$ is equivalent to $f(g(y)) = y$.
If $y, z \in B(f(a),\delta)$, the contraction inequality gives

$$\begin{aligned}\|g(y) - g(z)\| &\le \|T_y(g(y)) - T_y(g(z))\| + \|T_y(g(z)) - T_z(g(z))\| \\ &\le \frac{1}{2}\|g(y) - g(z)\| + c\|y - z\|.\end{aligned}$$

Thus

$$\|g(y) - g(z)\| \le 2c\|y - z\|, \tag{1.7}$$

and g is continuous.
Let $V := B(f(a),\delta)$ and set $U := f^{-1}(V) \cap B(a,r)$. Each $y \in V$ has exactly one preimage in $\overline{B}(a,r)$, and the preceding strict estimate shows that this preimage belongs to $B(a,r)$. Hence $f\restriction_U$ is a bijection onto V, with inverse g. The set U is open and connected, since $U = g(V)$ and V is connected.
It remains to prove regularity of the inverse. From (1.6), we obtain $A^{-1}Df(x) = I - R_x$ with $\|R_x\| \le \frac{1}{2}$. The geometric series $\displaystyle\sum_{k=0}^{\infty} R_x^k$ converges in operator norm and is the inverse of $I - R_x$; thus $Df(x)$ is invertible for $x \in U$. Fix $y \in V$, $x := g(y)$, and write $h := g(y+k) - g(y)$ when $y + k \in V$. Differentiability of f at x gives

$$k = Df(x)h + r_x(h), \qquad \frac{\|r_x(h)\|}{\|h\|} \longrightarrow 0 \quad (h \to 0).$$

By (1.7), $\|h\| \le 2c\|k\|$. Consequently,

$$\frac{\|g(y+k)-g(y)-Df(x)^{-1}k\|}{\|k\|} \le 2c\|Df(x)^{-1}\|_{\mathrm{op}}\frac{\|r_x(h)\|}{\|h\|} \longrightarrow 0.$$

Thus $Dg(y) = Df(g(y))^{-1}$. Continuity of Df, of g, and of inversion on the set of invertible matrices shows that Dg is continuous. If f is of class C^r for some $r \ge 2$, the preceding formula and smoothness of the inversion map on $\mathrm{GL}(n,\mathbb{R})$ allow induction on j: if $g \in C^j$ and $1 \le j < r$, then $Df \in C^{r-1}$ implies $Df \circ g \in C^j$; hence $Dg \in C^j$ and $g \in C^{j+1}$. Starting from $g \in C^1$, the formula successively yields $g \in C^2, \dots, C^r$. For $r = \infty$, apply the argument at every finite order. This completes the proof. □

Theorem 1.5.6. (inverse function theorem) Let M and N be smooth manifolds without boundary, and let $F\colon M \longrightarrow N$ be a smooth map. If $p \in M$ is such that dF_p is invertible, there exist connected open neighborhoods U_0 of p and V_0 of $F(p)$ such that $F\restriction_{U_0}\colon U_0 \longrightarrow V_0$ is a diffeomorphism.

Proof. Since dF_p is a bijection, we have $\dim(M) = \dim(N)$. Consider smooth charts (U,φ) and (V,ψ) centered at p and $F(p)$, respectively, with $F(U) \subseteq V$. Then $\widetilde{F} := \psi \circ F \circ \varphi^{-1}$ maps the open set $\widetilde{U} := \varphi(U) \subseteq \mathbb{R}^n$ into the open set $\widetilde{V} := \psi(V) \subseteq \mathbb{R}^n$.
Since φ and ψ are diffeomorphisms, the differential

$$d\widetilde{F}_{\varphi(p)} = d\psi_{F(p)} \circ dF_p \circ d(\varphi^{-1})_{\varphi(p)}$$

is an isomorphism. Lemma 1.5.5, applied to $\widetilde{F}$, gives connected open sets $\widetilde{U}_0 \subseteq \widetilde{U}$ and $\widetilde{V}_0 \subseteq \widetilde{V}$, with $\varphi(p) \in \widetilde{U}_0$ and $\widetilde{F}(\varphi(p)) \in \widetilde{V}_0$, such that $\widetilde{F}\restriction_{\widetilde{U}_0}\colon \widetilde{U}_0 \longrightarrow \widetilde{V}_0$ is a diffeomorphism.
Defining $U_0 := \varphi^{-1}(\widetilde{U}_0)$ and $V_0 := \psi^{-1}(\widetilde{V}_0)$, we obtain connected open neighborhoods of p and $F(p)$, respectively, such that $F\restriction_{U_0}\colon U_0 \longrightarrow V_0$ is a diffeomorphism. □

Remark 1.5.7. The preceding statement requires the manifolds to have no boundary. If boundary is allowed, the result may fail. For example, the inclusion $\iota\colon \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ has invertible differential at every point but is not a local diffeomorphism at points of $\partial\mathbb{H}^n$.

This motivates the concept of a local diffeomorphism:

Definition 1.5.8. Let M and N be manifolds with or without boundary. A function $F\colon M \longrightarrow N$ is called a local diffeomorphism if every $p \in M$ has an open neighborhood U such that $F(U)$ is open in N and $F\restriction_U\colon U \longrightarrow F(U)$ is a diffeomorphism.

Local diffeomorphisms have the following useful property.

Proposition 1.5.9. Let M and N be smooth manifolds without boundary, and let $F\colon M \longrightarrow N$ be a function.

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim(M) = \dim(N)$ and F is a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Proof. If F is a local diffeomorphism, it has a smooth inverse around each $p \in M$. The chain rule shows that dF_p and $d(F^{-1})_{F(p)}$ are inverse to each other; thus F is an immersion and a submersion.
Conversely, if F is an immersion and a submersion, each dF_p is an isomorphism. Theorem 1.5.6 provides neighborhoods on which F is a diffeomorphism, so F is a local diffeomorphism. This proves (1). When the dimensions are equal, every injective linear map between the tangent spaces is surjective, and conversely; (2) follows from (1). □

The rank theorem (Theorem 1.4.2) has an analogue for smooth manifolds, bearing the same name.

Theorem 1.5.10. (Rank theorem) Let M and N be smooth manifolds without boundary of dimensions m and n, respectively, and let $F\colon M \longrightarrow N$ be a smooth map of constant rank r. Then, for every $p \in M$, there exist smooth charts (U,φ) of M centered at p and (V,ψ) of N centered at $F(p)$ such that $F(U) \subseteq V$ and the coordinate representation $\widetilde{F} := \psi \circ F \circ \varphi^{-1}$ has the form $\widetilde{F}(x^1,\dots,x^m) = (x^1,\dots,x^r,0,\dots,0)$.
In particular, if F is an immersion, then $r = m$ and $\widetilde{F}(x^1,\dots,x^m) = (x^1,\dots,x^m,0,\dots,0)$. Likewise, if F is a submersion, then $r = n$ and $\widetilde{F}(x^1,\dots,x^m) = (x^1,\dots,x^n)$.

Proof. Fix $p \in M$. Take smooth charts (U_0,φ_0) of M around p and (V_0,ψ_0) of N around $F(p)$, with $F(U_0) \subseteq V_0$. Consider the coordinate representation $f := \psi_0 \circ F \circ \varphi_0^{-1}\colon \varphi_0(U_0) \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Since F has constant rank r, the derivative of f also has constant rank r. Moreover, since F is smooth, f is of class C^∞. Applying the smooth version of Theorem 1.4.2 gives open sets $\widetilde{W}_1, \widetilde{W}_2 \subseteq \mathbb{R}^m$ and $W_1, W_2 \subseteq \mathbb{R}^n$, together with smooth diffeomorphisms $\Phi\colon \widetilde{W}_2 \longrightarrow \widetilde{W}_1$ and $\Psi\colon W_2 \longrightarrow W_1$, such that

$$\Psi^{-1} \circ f \circ \Phi(x^1,\dots,x^m) = (x^1,\dots,x^r,0,\dots,0).$$

Define $U := \varphi_0^{-1}(\widetilde{W}_1)$ and $V := \psi_0^{-1}(W_1)$. Restricting U if necessary, we may assume that $F(U) \subseteq V$. Now take the charts $\varphi := \Phi^{-1} \circ \varphi_0 \restriction_U$ and $\psi := \Psi^{-1} \circ \psi_0 \restriction_V$. Then (U,φ) is a smooth chart of M centered at p, and (V,ψ) is a smooth chart of N centered at $F(p)$. In these charts,

$$\psi \circ F \circ \varphi^{-1} = \Psi^{-1} \circ \psi_0 \circ F \circ \varphi_0^{-1} \circ \Phi = \Psi^{-1} \circ f \circ \Phi.$$

Therefore, $\widetilde{F}(x^1,\dots,x^m) = (x^1,\dots,x^r,0,\dots,0)$.
If F is an immersion, then $r = m$, and the preceding expression becomes $\widetilde{F}(x^1,\dots,x^m) = (x^1,\dots,x^m,0,\dots,0)$.
If F is a submersion, then $r = n$, and the expression reduces to $\widetilde{F}(x^1,\dots,x^m) = (x^1,\dots,x^n)$. □

1.6 Submanifolds

A submanifold allows us to treat a subset as a smooth space in its own right. The definition must reconcile two features: the induced topology and injectivity of the differential. The following figures motivate this condition before we formulate it precisely.

Definition 1.6.1. (Embeddings and submanifolds) Let M be a smooth manifold without boundary.

(a) A smooth map $F\colon N \to M$ is a **smooth embedding** if it is an injective immersion and a homeomorphism from N onto $F(N)$ with the topology induced by M.

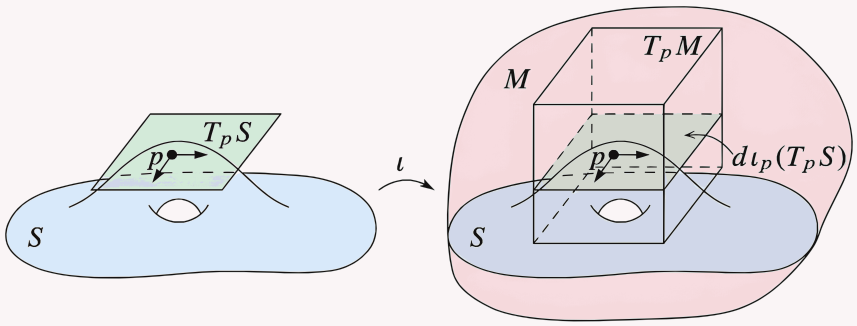


Figure 1.4. Tangent space to a submanifold.

(b) A subset $S \subseteq M$, equipped with a smooth manifold structure, is an **immersed submanifold** if the inclusion $\iota\colon S \hookrightarrow M$ is an immersion.

(c) It is an **embedded submanifold** if ι is a smooth embedding. It is **properly embedded** if, in addition, ι is proper.

For an immersed submanifold, the topology on S may be finer than the subspace topology; for an embedded submanifold, the two coincide.

Definition 1.6.2. (Slice chart) Let M be a smooth manifold without boundary, and let $S \subseteq M$ and $p \in S$. A chart (U, φ) of M is **adapted to** S **at** p, or is a **slice chart**, if $p \in U$, $\varphi(p) = 0$, and, for some k,

$$\varphi(U \cap S) = \varphi(U) \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right).$$

The last equality includes the local topology of the subset, not merely an equality of tangent spaces.

Theorem 1.6.3. (Local slice criterion) A subset S of a smooth n-manifold without boundary admits an embedded submanifold structure of dimension k if and only if each $p \in S$ has a slice chart of dimension k. When it exists, this smooth structure is unique.

Proof. If slice charts exist, the maps

$$\varphi_S := \mathrm{pr}_{\mathbb{R}^k} \circ \varphi \restriction_{U \cap S}$$

are homeomorphisms onto open subsets of $\mathbb{R}^k$. Their changes of coordinates are restrictions of the smooth changes of charts on M, and hence are smooth. Since S carries the subspace topology, it is Hausdorff and second countable. The smooth chart lemma gives it the structure of a k-manifold, and in these coordinates the inclusion is $x \mapsto (x, 0)$; it is therefore an embedding.
Conversely, if $\iota\colon S \hookrightarrow M$ is an embedding, Theorem 1.5.10 applied to ι gives this normal form in a neighborhood of each point. Being a homeomorphism onto its image allows us to shrink the neighborhood in M until no other parts of S occur there. This gives the slice equality. Finally, any structure making S an

embedded submanifold has these same restricted charts, which proves uniqueness. □

Proposition 1.6.4. (Tangent space of an embedded submanifold) Let M be a smooth manifold without boundary, let $S \subseteq M$ be an embedded submanifold without boundary of dimension k, and let $\iota\colon S \hookrightarrow M$ be the inclusion. Then $d\iota_p$ is injective and gives the identification

$$T_pS \cong d\iota_p(T_pS) \subseteq T_pM.$$

In a slice chart $(x^1,\dots,x^n)$,

$$d\iota_p(T_pS) = \operatorname{span}\left\{\partial_1|_p,\dots,\partial_k|_p\right\}.$$

Proof. In slice coordinates, ι is the linear inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$. Its differential has the same form, proving both injectivity and the formula. □

Proposition 1.6.5. (Smooth graphs) Let M and N be smooth manifolds without boundary. If $U \subseteq M$ is open and $f : U \to N$ is smooth, then

$$\Gamma_f := \{(p, f(p)) \mid p \in U\} \subseteq M \times N$$

is an embedded submanifold diffeomorphic to U. At the point $(p, f(p))$,

$$T_{(p,f(p))}\Gamma_f = \{(v, df_p v) \mid v \in T_pM\}.$$

Proof. The map $F\colon U \to M\times N$, $F(p) = (p, f(p))$, is injective, and $dF_p(v) = (v, df_p v)$ is injective. The restriction of the first projection to Γ_f is the continuous inverse of F, so F is an embedding. The tangent-space formula follows from the preceding proposition. □

Definition 1.6.6. Let $F\colon M \to N$ be a smooth map between manifolds without boundary, and let $q \in N$. We say that q is a **regular value** of F if dF_p is surjective for each $p \in F^{-1}(q)$. If the preimage is empty, the condition is considered satisfied.

Theorem 1.6.7. (Regular preimage theorem) Let M^m and N^n be smooth manifolds without boundary, and let $F\colon M \to N$ be smooth. If q is a regular value, then $F^{-1}(q)$ is either empty or a properly embedded submanifold of dimension $m-n$, and

$$T_pF^{-1}(q) = \ker dF_p, \qquad p \in F^{-1}(q).$$

Proof. At each point of the level set, Theorem 1.5.10 gives coordinates in which $F(x^1,\dots,x^m) = (x^1,\dots,x^n)$. The level set is then the slice $x^1 = \cdots = x^n = 0$, and the preceding local criterion gives it its embedded structure. The same coordinate expression identifies its tangent space with the kernel of dF_p. Since $\{q\}$ is closed, the level set is a closed subset of M; the inclusion of a closed subset is proper over every compact set. □

Definition 1.6.8. (Submanifold with boundary) A smooth manifold S with boundary contained in a smooth manifold M is an **embedded submanifold with boundary** if the inclusion $S \hookrightarrow M$ is a smooth embedding. If M has no boundary, a **half-slice chart** of dimension k for S is a chart of M in which

$$S \cap U \longleftrightarrow \varphi(U) \cap \left(\mathbb{H}^k \times \{0\}^{n-k}\right),$$

where $\mathbb{H}^k = \mathbb{R}^{k-1} \times [0, \infty)$.

Theorem 1.6.9. (Local criterion with boundary) Let M be a smooth manifold without boundary and let $S \subseteq M$. Then S is an embedded submanifold with boundary of dimension k if and only if each interior point of S has a slice chart and each point of ∂S has a half-slice chart. This characterization uniquely determines the smooth structure of S.

Proof. First suppose the indicated charts exist. In a slice chart, define

$$\varphi_S := \mathrm{pr}_{\mathbb{R}^k} \circ \varphi \restriction_{U \cap S},$$

and in a half-slice chart use the same formula, with image in $\mathbb{H}^k$. These maps are homeomorphisms onto open subsets of $\mathbb{R}^k$ or relatively open subsets of $\mathbb{H}^k$. If two of them overlap, the change of coordinates is obtained by restricting a change of charts on M to $\mathbb{R}^k \times \{0\}^{n-k}$. More precisely, if F is the ambient change of coordinates, the change on S is represented by

$$u \longmapsto \mathrm{pr}_{\mathbb{R}^k} F(u, 0).$$

This expression extends smoothly to an open subset of $\mathbb{R}^k$, as does its inverse. Thus the restricted charts form a smooth atlas with boundary on S. In these charts, the inclusion has the expression $u \mapsto (u, 0)$, so it is an immersion. Since S carries the subspace topology, the inclusion is also a homeomorphism onto its image and is therefore an embedding.

Conversely, suppose the inclusion $\iota\colon S \hookrightarrow M$ is an embedding. At an interior point, Theorem 1.5.10 applies, as in Theorem 1.6.3. Now let $p \in \partial S$. Choose a boundary chart

$$\theta\colon W \longrightarrow O \cap \mathbb{H}^k$$

of S, centered at p, where $O \subseteq \mathbb{R}^k$ is open, and a chart ψ of M centered at p. After shrinking W and the domain of ψ, the coordinate map

$$f := \psi \circ \iota \circ \theta^{-1}$$

admits a smooth extension $\widehat{f}\colon \widehat{O} \to \mathbb{R}^n$ to an open set $\widehat{O} \subseteq \mathbb{R}^k$ containing the origin. Since $d\iota_p$ is injective, $D\widehat{f}(0)$ has rank k. Some minor of order k remains nonsingular in a neighborhood of 0; hence $\widehat{f}$ has constant rank k there. The Euclidean rank theorem gives local diffeomorphisms α and β, with $\alpha(0) = 0$, such that

$$(\beta \circ \widehat{f} \circ \alpha^{-1})(u) = (u, 0)$$

for u in a neighborhood of 0 in $\mathbb{R}^k$.

It remains to straighten the image under α of the half-space constituting the original domain. In a neigh-

borhood of 0, define

$$r(u) := \big(\alpha^{-1}(u)\big)^k.$$

Since α is a diffeomorphism, $dr_0 \neq 0$. Choose linear forms $\ell^1, \dots, \ell^{k-1}$ such that $\ell^1, \dots, \ell^{k-1}, dr_0$ form a basis for $(\mathbb{R}^k)^*$, and define

$$\eta(u) := (\ell^1(u), \dots, \ell^{k-1}(u), r(u)).$$

The differential of η at 0 is invertible. By the inverse function theorem, there are open neighborhoods A_η and B_η of the origin such that

$$\eta \colon A_\eta \longrightarrow B_\eta$$

is a diffeomorphism. Shrink the domains of α and β so that the image of α is contained in A_η. Then

$$\eta\big(\alpha(O \cap \mathbb{H}^k)\big) = \eta(\alpha(O)) \cap \mathbb{H}^k$$

locally, since the last component of $\eta \circ \alpha$ is the last coordinate of the original domain. After also restricting the normal variables, choose a neighborhood C of the origin in $\mathbb{R}^{n-k}$ and define the diffeomorphism

$$\widetilde{\eta} \colon A_\eta \times C \longrightarrow B_\eta \times C, \qquad \widetilde{\eta}(u, z) := (\eta(u), z).$$

Set $\alpha_1 := \eta \circ \alpha$ and $\beta_1 := \widetilde{\eta} \circ \beta$, after the corresponding restriction of the domain of β. Then

$$\begin{aligned}(\beta_1 \circ \widehat{f} \circ \alpha_1^{-1})(u) &= \widetilde{\eta}\big(\beta \circ \widehat{f} \circ \alpha^{-1}(\eta^{-1}(u))\big) \\ &= \widetilde{\eta}(\eta^{-1}(u), 0) = (u, 0).\end{aligned}$$

Thus the ambient chart $\beta_1 \circ \psi$ sends the local image of S to $\mathbb{H}^k \times \{0\}^{n-k}$. Since W is open in S and S has the topology induced by M, there exists an ambient open set whose intersection with S is contained in W. Restricting the chart once more excludes the points of S outside the local image and gives a half-slice chart centered at p.

The construction also proves uniqueness. In any smooth structure with boundary for which the inclusion is an embedding, charts obtained by restricting slice charts and half-slice charts are smooth. Conversely, these charts generate the smooth structure because they cover S. Thus every structure with the stated property has the same maximal atlas. □

Theorem 1.6.10. (The boundary as a submanifold) If M is a smooth manifold with boundary of dimension n, then ∂M has a unique smooth manifold structure without boundary of dimension $n-1$ for which the inclusion $\iota_\partial \colon \partial M \hookrightarrow M$ is a proper embedding. In a boundary chart $(x^1, \dots, x^n)$,

$$d(\iota_\partial)_p(T_p \partial M) = \operatorname{span}\left\{ \partial_1|_p, \dots, \partial_{n-1}|_p \right\}.$$

Proof. For each boundary chart (U, φ), the map

$$\varphi_\partial := \operatorname{pr}_{\mathbb{R}^{n-1}} \circ \varphi \restriction_{U \cap \partial M}$$

is a homeomorphism onto an open subset of $\mathbb{R}^{n-1}$. Smooth invariance of the boundary ensures that the changes of coordinates preserve $x^n = 0$; their restrictions, and those of their inverses, are smooth. These

charts give ∂M the asserted structure. In these charts, the inclusion is $(x',0) \mapsto (x',0)$, proving the tangent-space assertion and uniqueness. The inclusion is an embedding because we use the subspace topology. Finally, ∂M is closed in M, so its intersection with any compact set is compact and the inclusion is proper. □

Theorem 1.6.11. (Regular preimage in a manifold with boundary) Let M^m be a smooth manifold with boundary, let N^n be a manifold without boundary, and let $F\colon M \to N$ be smooth. If $q \in N$ is a regular value of both F and $F\restriction_{\partial M}$, then $F^{-1}(q)$ is a properly embedded submanifold with boundary of dimension $m-n$, and

$$\partial\big(F^{-1}(q)\big) = F^{-1}(q) \cap \partial M, \qquad T_p F^{-1}(q) = \ker dF_p.$$

Proof. At interior points, Theorem 1.6.7 applies. Let $p \in F^{-1}(q) \cap \partial M$. Take boundary coordinates $(y^1,\dots,y^{m-1},t)$ centered at p, with $t \geq 0$, and coordinates on N centered at q. Denote the coordinate representation of F by f. Surjectivity of

$$d(F\restriction_{\partial M})_p\colon T_p\partial M \longrightarrow T_qN$$

implies that, after reordering the tangential variables, the matrix $\partial(f^1,\dots,f^n)/\partial(y^1,\dots,y^n)$ is invertible at the origin. Consider

$$\Psi(y,t) := \big(f(y,t), y^{n+1},\dots,y^{m-1},t\big).$$

Its Jacobian at the origin is invertible. Extend f smoothly to negative values of t and apply the inverse function theorem to the extension of Ψ. Since the last component of Ψ is t, the resulting restriction is a diffeomorphism between relative neighborhoods in the half-space and preserves its boundary. In the new coordinates $z = \Psi(y,t)$, we have

$$f(z) = (z^1,\dots,z^n).$$

Consequently, the level set $f^{-1}(0)$ is described by $z^1 = \cdots = z^n = 0$ and $z^m \geq 0$; it is a half-slice of dimension $m-n$, whose boundary corresponds exactly to $t = 0$.
Together with the slices obtained at interior points, Theorem 1.6.9 gives $F^{-1}(q)$ the structure of an embedded submanifold with boundary. In the preceding coordinates, the tangent space to the level set is the kernel of the linear projection representing dF_p, whence $T_pF^{-1}(q) = \ker dF_p$. Finally, $F^{-1}(q)$ is closed because $\{q\}$ is closed; thus the preimage of a compact set under the inclusion is a closed subset of a compact set, and is therefore compact. The inclusion is proper. □

The double replaces a manifold with boundary by a manifold without boundary containing two copies of the original manifold. This construction is useful, for example, for extending smooth objects across the boundary and locally applying results formulated for manifolds without boundary.

Theorem 1.6.12. (Collar and smooth double) Let M be a smooth manifold with boundary.

(a) There exist $\varepsilon > 0$ and a smooth embedding

$$c\colon \partial M \times [0,\varepsilon) \longrightarrow M, \qquad c(x,0) = x,$$

whose image is a neighborhood of ∂M in M.

(b) If M_+ and M_- are two copies of M, the quotient

$$DM := (M_+ \sqcup M_-)/\big(x_+ \sim x_- \text{ for } x \in \partial M\big)$$

admits a smooth manifold structure without boundary. The inclusions of both copies are smooth embeddings with boundary, and their boundaries have the same image, an embedded hypersurface of DM.

The diffeomorphism type of DM does not depend on the chosen collar.

Proof. Part (a) is the collar theorem [94, Theorem 9.25]. For part (b), apply the smooth manifold gluing theorem to the two copies of M and the identity diffeomorphism of ∂M; see [94, Theorem 9.29 and Example 9.32]. These results construct the smooth structure on the quotient, prove the assertions about the inclusions, and show that two choices of collar produce diffeomorphic doubles. □

Corollary 1.6.13. (Smooth extension to the double) Let M be a smooth manifold with boundary and let DM be its double. Every smooth function and every smooth tensor field on one copy of M admit a smooth extension to DM. If $\mathbf{g}$ is a Riemannian metric on M, a symmetric tensor extension $\widetilde{\mathbf{g}}$ can be chosen that is Riemannian on an open neighborhood of that copy.

Proof. Identify M with the copy $M_+ \subset DM$ and let $\mathbf{T}$ be a smooth tensor field on M_+. By the definition of smoothness up to the boundary, for every $p \in M_+$ there exist an open set $V_p \subset DM$ containing p and a smooth tensor field $\widetilde{\mathbf{T}}_p$ on V_p whose restriction to $V_p \cap M_+$ agrees with $\mathbf{T}$. At interior points, we simply take $V_p \subset \mathrm{Int}(M_+)$; at boundary points, we extend the local components of $\mathbf{T}$ to negative values of the normal coordinate in a chart of the double.
The family

$$\{V_p : p \in M_+\} \cup \{DM \setminus M_+\}$$

is an open cover of DM. Let $(\chi_p)_{p\in M_+} \cup \{\chi_0\}$ be a smooth, locally finite partition of unity subordinate to this cover. Each product $\chi_p \widetilde{\mathbf{T}}_p$, extended by zero outside V_p, is a smooth tensor field on DM. By local finiteness, the sum

$$\widetilde{\mathbf{T}} := \sum_{p\in M_+} \chi_p \widetilde{\mathbf{T}}_p$$

is well defined and smooth. If $x \in M_+$, then $\chi_0(x) = 0$, and each $\widetilde{\mathbf{T}}_p(x)$ occurring in the sum agrees with $\mathbf{T}(x)$; consequently,

$$\widetilde{\mathbf{T}}(x) = \left(\sum_{p\in M_+} \chi_p(x)\right) \mathbf{T}(x) = \mathbf{T}(x).$$

The same argument, with tensors of type $(0,0)$, treats functions.
If $\mathbf{T} = \mathbf{g}$, replace the resulting extension by its symmetric part. The extension still agrees with $\mathbf{g}$ on M_+. Since the set of positive definite symmetric bilinear forms is open in each fiber and the extension is continuous, the set of points of DM where $\widetilde{\mathbf{g}}$ is positive definite is an open set containing M_+. Therefore, $\widetilde{\mathbf{g}}$ is a Riemannian metric on an open neighborhood of that copy. □

1.7 Vector bundles and the cotangent bundle

The tangent bundle suggests a more general construction. Over each point of a manifold we may place a vector space, provided these fibers can be identified locally in a smooth manner and the identifications are linear on each fiber. This leads to vector bundles, the natural setting for fields, differential forms, tensors, spinors, and the geometric unknowns that will appear later.

Definition 1.7.1. Let M be a topological space. A **real vector bundle of rank k over M** is a topological space $\mathbf{E}$ together with a continuous surjective function $\pi\colon \mathbf{E} \longrightarrow M$ such that:

(a) For each $p \in M$, the fiber $\mathbf{E}_p = \pi^{-1}(p)$ over p is equipped with the structure of a real vector space of dimension k.

(b) For each $p \in M$, there exist a neighborhood U of p in M and a homeomorphism $\Phi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ such that $\pi_U \circ \Phi = \pi$, where $\pi_U\colon U \times \mathbb{R}^k \longrightarrow U$ is the projection, and for each $q \in U$, the restriction of Φ to $\mathbf{E}_q$ is an isomorphism from $\mathbf{E}_q$ to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$. ($\Phi$ is called a **local trivialization of $\mathbf{E}$ over U.**)

Definition 1.7.2. Let M and $\mathbf{E}$ be smooth manifolds with or without boundary such that $\pi\colon \mathbf{E} \longrightarrow M$ is a continuous surjective map making $\mathbf{E}$ a real vector bundle of rank k over M. We say that $\mathbf{E}$ is a **smooth vector bundle of rank k over M** if

(a) π is a smooth map.

(b) For each $p \in M$, there exist a neighborhood U of p and a local trivialization $\Phi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ of $\mathbf{E}$ over U such that Φ is a diffeomorphism. This map is called a **smooth local trivialization of $\mathbf{E}$ over U**.

Definition 1.7.3. (Smooth complex vector bundle) Let M be a smooth manifold with or without boundary. A *smooth complex vector bundle of rank r* over M is a smooth map $\pi\colon \mathbf{E} \to M$ admitting, around each point of M, a smooth trivialization

$$\Phi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r$$

whose restriction to each fiber is complex linear. In particular, the total space is regarded as a smooth real manifold, and the transition functions take values in $GL(r, \mathbb{C})$.

Remark 1.7.4. When no ambiguity arises, we omit the projection when referring to a vector bundle, and may even omit the base space M. Thus, if $\pi\colon \mathbf{E} \longrightarrow M$ is a vector bundle, we write $\mathbf{E} \longrightarrow M$ or $(\mathbf{E}, M)$ or simply $\mathbf{E}$.

The simplest example of a vector bundle is what is known as the *trivial bundle of rank k over M*.

Example 1.7.5. If M is a smooth manifold with or without boundary, the bundle $\underline{\mathbb{R}}^k_M = M \times \mathbb{R}^k$ is a smooth vector bundle of rank k over M. It is called the ***trivial bundle of rank k over M***.

The vector bundle in this example is trivial in the sense that it admits a global trivialization (a local trivialization with $U = M$).

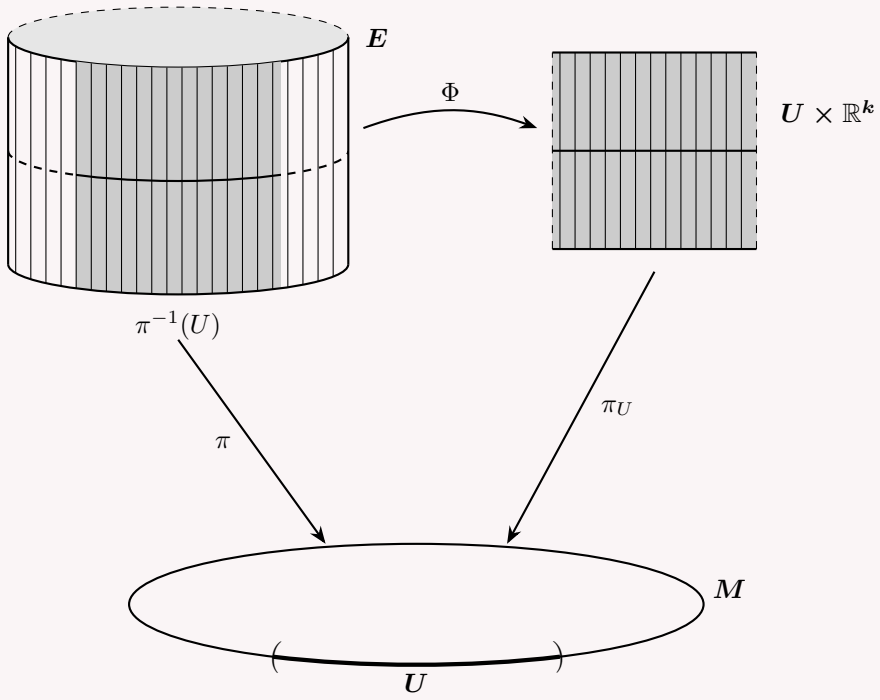


Figure 1.5. Local trivialization of a vector bundle.

The tangent bundle provides another example of a vector bundle.

Proposition 1.7.6. Let M be a smooth manifold with or without boundary of dimension n. Its tangent bundle TM is a smooth vector bundle of rank n over M.

Proof. Given a coordinate chart (U,φ) of M such that $\varphi = (x^1,\dots,x^n)$ and $p \in U$, define the map $\Phi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ by

$$\Phi\left(p, v^i \partial_i\Big|_p\right) = (p,(v^1,\dots,v^n)).$$

The map Φ is linear on the fibers of π and satisfies $\pi_U \circ \Phi = \pi$. If (U,φ) is an interior chart, the tangent-bundle chart in Proposition 1.4.20 satisfies

$$\widetilde{\varphi} = (\varphi \times \mathrm{Id}_{\mathbb{R}^n}) \circ \Phi.$$

If it is a boundary chart, the tangent-bundle chart places the vector coordinates first, followed by those of the base point. Therefore,

$$\widetilde{\varphi} = S \circ (\varphi \times \mathrm{Id}_{\mathbb{R}^n}) \circ \Phi, \qquad S(x,v) = (v,x).$$

The interchange S is a linear isomorphism of $\mathbb{R}^{2n}$ and sends $\varphi(U) \times \mathbb{R}^n$ diffeomorphically onto $\mathbb{R}^n \times \varphi(U)$, a relatively open subset of $\mathbb{H}^{2n}$. In both cases, Φ is a composition of diffeomorphisms, and hence is a smooth

local trivialization. □

Every nontrivial vector bundle requires more than one local trivialization. The following lemma shows that the composition of two smooth local trivializations has a simple form where they overlap.

Lemma 1.7.7. Let M be a smooth manifold with or without boundary and let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank k. Suppose $\Phi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ and $\Psi\colon \pi^{-1}(V) \longrightarrow V \times \mathbb{R}^k$ are smooth local trivializations of $\mathbf{E}$ with $U \cap V \neq \varnothing$. Then there exists a smooth map $\tau\colon U \cap V \longrightarrow GL(k,\mathbb{R})$ such that

$$\Phi \circ \Psi^{-1}\colon (U \cap V) \times \mathbb{R}^k \longrightarrow (U \cap V) \times \mathbb{R}^k$$

has the form

$$(\Phi \circ \Psi^{-1})(p,v) = (p, \tau(p)v),$$

where $\tau(p)v$ denotes the usual action of the matrix $\tau(p)$ in $k \times k$ on $v \in \mathbb{R}^k$.

Proof. The following diagram commutes:

$$\begin{array}{ccccc} (U \cap V) \times \mathbb{R}^k & \xleftarrow{\Psi} & \pi^{-1}(U \cap V) & \xrightarrow{\Phi} & (U \cap V) \times \mathbb{R}^k \\ & {\scriptstyle \pi_1} \searrow & \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \pi_1} & \\ & & U \cap V & & \end{array}$$

where the maps at the top are interpreted as the restrictions of Ψ and Φ to $\pi^{-1}(U \cap V)$. Commutativity of the diagram tells us that

$$\pi_1 \circ (\Phi \circ \Psi^{-1}) = \pi_1,$$

where π_1 is projection onto the first coordinate.
This implies that there exists a map $\sigma\colon (U \cap V) \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ such that

$$(\Phi \circ \Psi^{-1})(p,v) = (p, \sigma(p,v)).$$

By construction, σ is the second coordinate function of $\Phi \circ \Psi^{-1}$; that is,

$$\sigma = \pi_2 \circ (\Phi \circ \Psi^{-1}),$$

where π_2 is projection onto the second coordinate.
For each $p \in U \cap V$, the map $v \mapsto \sigma(p,v)$ from $\mathbb{R}^k$ to itself is linear and invertible, since Φ and Ψ are local trivializations; hence $\Phi \circ \Psi^{-1}$ is a fiberwise diffeomorphism preserving the first coordinate. In particular, there is a unique matrix $\tau(p) \in GL(k,\mathbb{R})$ such that

$$\sigma(p,v) = \tau(p)v \qquad \text{for every } v \in \mathbb{R}^k.$$

We now show that $p \mapsto \tau(p)$ is smooth.
Fix the standard basis $(e_1,\dots,e_k)$ of $\mathbb{R}^k$. Since $v \mapsto \sigma(p,v)$ is linear, it is determined by its values on the basis

vectors, and we define $\tau(p)$ as the matrix with these values as columns:

$$\tau(p)e_j = \sigma(p, e_j) \qquad \text{for } j \in \{1, \dots, k\}.$$

The map

$$p \longmapsto \sigma(p, e_j) = \pi_2\big((\Phi \circ \Psi^{-1})(p, e_j)\big)$$

is a composition of smooth functions (since (p, e_j) is constant in the second coordinate and $\Phi \circ \Psi^{-1}$ is smooth), and is therefore a smooth map from $U \cap V$ to $\mathbb{R}^k$ for each $j \in \{1, \dots, k\}$.
Denoting projection onto the ith coordinate by $\mathrm{pr}_i \colon \mathbb{R}^k \longrightarrow \mathbb{R}$, the entries of the matrix $\tau(p) = (\tau_{ij}(p))$ are given by

$$\tau_{ij}(p) = \mathrm{pr}_i\big(\sigma(p, e_j)\big), \qquad i, j \in \{1, \dots, k\}.$$

Each function τ_{ij} is smooth on p, so $p \mapsto \tau(p)$ is a smooth map.
Consequently,

$$(\Phi \circ \Psi^{-1})(p, v) = (p, \tau(p)v)$$

with $\tau \colon U \cap V \longrightarrow GL(k, \mathbb{R})$ a smooth function, as desired. □

The following result allows us to construct smooth vector bundles.

Lemma 1.7.8. (Smooth vector bundle lemma) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let M be a smooth manifold with or without boundary, and for each $p \in M$ let $\mathbf{E}_p$ be a vector space of fixed dimension r over $\mathbb{K}$. Set $\mathbf{E} = \coprod_{p \in M} \mathbf{E}_p$ and let $\pi \colon \mathbf{E} \longrightarrow M$ be the map sending each element of $\mathbf{E}_p$ to p. Suppose we are given:

(a) An open cover $\{U_\alpha\}_{\alpha \in A}$ of M.

(b) For each $\alpha \in A$, a fiber-preserving bijection

$$\Phi_\alpha \colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{K}^r$$

whose restriction to $\mathbf{E}_p$ is a $\mathbb{K}$-linear isomorphism from $\mathbf{E}_p$ onto $\{p\} \times \mathbb{K}^r$.

(c) For $U_\alpha \cap U_\beta \neq \varnothing$, a smooth map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow GL(r, \mathbb{K})$ such that

$$(\Phi_\alpha \circ \Phi_\beta^{-1})(p, z) = (p, g_{\alpha\beta}(p)z).$$

Then $\mathbf{E}$ has a unique topology and a unique real smooth structure making it a smooth manifold with or without boundary, for which π is smooth and the maps Φ_α are smooth, $\mathbb{K}$-linear local trivializations. Addition $\mathbf{E} \times_M \mathbf{E} \to \mathbf{E}$ and scalar multiplication $\mathbb{K} \times \mathbf{E} \to \mathbf{E}$ are smooth. In particular, $\mathbf{E} \to M$ is a smooth real vector bundle if $\mathbb{K} = \mathbb{R}$ and a smooth complex vector bundle if $\mathbb{K} = \mathbb{C}$.

Proof. Let $d = \dim_{\mathbb{R}} \mathbb{K}$. For each α and each chart (V, φ) of M with $V \subseteq U_\alpha$, define

$$\widetilde{\varphi}_{\alpha, V} := (\varphi \times \mathrm{Id}_{\mathbb{K}^r}) \circ \Phi_\alpha \colon \pi^{-1}(V) \longrightarrow \varphi(V) \times \mathbb{K}^r.$$

Identify $\mathbb{K}^r$ with $\mathbb{R}^{dr}$. If (V, φ) is an interior chart, its image is open in $\mathbb{R}^{n+dr}$; if it is a boundary chart, reordering the coordinates identifies $\varphi(V) \times \mathbb{R}^{dr}$ with a relatively open subset of $\mathbb{H}^{n+dr}$.

Consider two charts (V,φ) and (W,ψ) subordinate, respectively, to U_α and U_β. On the intersection, the change of coordinates from the second chart to the first is

$$(y,z) \longmapsto \left(\varphi\circ\psi^{-1}(y), g_{\alpha\beta}(\psi^{-1}(y))z\right).$$

It is smooth as a real map, and its inverse is obtained by interchanging α,β; indeed, the given bijections imply $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$. The images of the intersections are open because they have the form $\varphi(V\cap W)\times\mathbb{K}^r$.
This family of charts on the total space has a countable subfamily covering $\mathbf{E}$: simply choose a countable subcover of the domains V on the base, since M is second countable. Two points of $\mathbf{E}$ lying in the same fiber belong to a common chart $\pi^{-1}(V)$. If their base points are distinct, first choose disjoint coordinate domains in M, restricted to their respective sets U_α, and then take their preimages. This verifies the second countability and separation assumptions of Lemma 1.2.19. The lemma provides a Hausdorff, second countable topology and a smooth structure on $\mathbf{E}$ for which the maps $\widetilde{\varphi}_{\alpha,V}$ are charts.
Each Φ_α is a diffeomorphism: after restriction to $\pi^{-1}(V)$, with $V\subseteq U_\alpha$, its expression in the newly constructed charts and the product chart of $V\times\mathbb{K}^r$ is the identity. In the same coordinates, π is $(x,z)\mapsto x$, and hence is smooth. Fiberwise addition and scalar multiplication have the expressions

$$(x,z,w)\longmapsto(x,z+w), \qquad (\lambda,x,z)\longmapsto(x,\lambda z),$$

which are smooth as real maps and $\mathbb{K}$-linear in the fiber variables.
If another topology and smooth structure had the properties in the statement, all the maps Φ_α would be diffeomorphisms. Consequently, the sets $\Phi_\alpha^{-1}(O)$, with O open in $U_\alpha\times\mathbb{K}^r$, would necessarily form a basis for the same topology, and the charts $\widetilde{\varphi}_{\alpha,V}$ would belong to the same smooth structure. This proves uniqueness.
In the case with boundary, the construction also shows that $\partial\mathbf{E} = \pi^{-1}(\partial M)$.
In the complex case, the same construction is carried out on the underlying real manifold and preserves the complex linearity of the transition maps. □

This lemma produces nontrivial vector bundles, as the following example shows:

Example 1.7.9. (Möbius bundle) Define on $\mathbb{R}^2$ the relation

$$(x,y)\sim(x',y') \Longleftrightarrow (x',y') = (x+n,(-1)^n y), \quad n\in\mathbb{Z}.$$

Let $\mathbf{E} = \mathbb{R}^2/\sim$ and let $q\colon\mathbb{R}^2\longrightarrow\mathbf{E}$ be the quotient projection. Geometrically, $\mathbf{E}$ is obtained from $[0,1]\times\mathbb{R}$ by identifying $(0,y)$ with $(1,-y)$; the subset $q([0,1]\times[-r,r])$ is the compact Möbius strip of width $2r$.
Let $\pi_1(x,y) = x$ and $\varepsilon(x) = e^{2\pi i x}$. If U is a proper open arc of $\mathbb{S}^1$ admitting a continuous branch of the argument, each component $\widetilde{U}$ of $\varepsilon^{-1}(U)$ is an open interval and $\varepsilon\restriction_{\widetilde{U}}\colon\widetilde{U}\to U$ is a diffeomorphism; its inverse is obtained by dividing that branch of the argument by 2π. Since $\varepsilon(x+n) = \varepsilon(x)$, the map $\varepsilon\circ\pi_1$ is constant on the equivalence classes of $\sim$. Therefore,

$$\pi\colon\mathbf{E}\longrightarrow\mathbb{S}^1, \qquad \pi([(x,y)]) = \varepsilon(x),$$

is well defined and satisfies $\pi\circ q = \varepsilon\circ\pi_1$. This is summarized by the diagram:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{q} & \mathbf{E} = \mathbb{R}^2/\sim \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow{\varepsilon} & \mathbb{S}^1 \end{array}$$

Let $p \in \mathbb{S}^1$ and choose $x_0 \in \mathbb{R}$ such that $\varepsilon(x_0) = p$. First characterize the real numbers x with $\varepsilon(x) = p$. Since $\varepsilon(x) = e^{2\pi i x}$, the equality $\varepsilon(x) = \varepsilon(x_0)$ is equivalent to $e^{2\pi i(x-x_0)} = 1$. The real solutions of $e^{2\pi i t} = 1$ are precisely the integers $t \in \mathbb{Z}$, so $\varepsilon(x) = \varepsilon(x_0)$ if and only if $x - x_0 \in \mathbb{Z}$, that is, if and only if $x = x_0 + n$ for some $n \in \mathbb{Z}$. Consequently,

$$\pi^{-1}(p) = \{[(x,y)] \in \mathbf{E} \mid \varepsilon(x) = p\} = \{[(x_0+n, y)] \mid n \in \mathbb{Z},\ y \in \mathbb{R}\}.$$

We can rewrite this fiber by recalling the definition of $\sim$. We know that $(x_0+n, y) \sim (x_0, (-1)^n y)$ for any $n \in \mathbb{Z}$ and $y \in \mathbb{R}$. This allows us to conclude that

$$\pi^{-1}(p) = \{[(x_0, (-1)^n y)] \mid n \in \mathbb{Z}, y \in \mathbb{R}\},$$

which can be rewritten as $\pi^{-1}(p) = \{[(x_0, t)] \mid t \in \mathbb{R}\}$.
Define $j_p \colon \pi^{-1}(p) \longrightarrow \mathbb{R}$ by $j_p([(x_0,t)]) = t$.
We show that this does not depend on the chosen t: if $(x_0,t) \sim (x_0,t')$, there exists $n \in \mathbb{Z}$ such that $(x_0,t') = (x_0+n, (-1)^n t)$, which gives $n = 0$ and therefore $t = t'$.
The function j_p is injective because $j_p([(x_0,t)]) = j_p([(x_0,s)])$ implies $t = s$, and surjective because $a = j_p([(x_0,a)])$ for every $a \in \mathbb{R}$. Equip $\pi^{-1}(p)$ with the vector space structure given by

$$[(x_0,t)] + [(x_0,s)] := [(x_0, t+s)]$$

and

$$\lambda[(x_0,t)] = [(x_0, \lambda t)].$$

These operations are well defined by the uniqueness of the representative of the form (x_0,t) proved above. With these operations, j_p preserves addition and scalar multiplication, and is therefore a linear isomorphism. If another lift $x_0' = x_0 + m$ is chosen, with $m \in \mathbb{Z}$, then

$$[(x_0,t)] = [(x_0', (-1)^m t)].$$

The corresponding isomorphism j_p' satisfies $j_p' = (-1)^m j_p$. Since multiplication by $(-1)^m$ is a linear isomorphism of $\mathbb{R}$, the operations of addition and scalar multiplication transported to the fiber are unchanged. Its vector space structure is therefore independent of the chosen lift.
Take the cover

$$U_\pm = \mathbb{S}^1 \setminus \{(\pm 1, 0)\}.$$

Fix components

$$\widetilde{U}_- = (-\frac{1}{2}, \frac{1}{2}) \subset \varepsilon^{-1}(U_-), \qquad \widetilde{U}_+ = (0,1) \subset \varepsilon^{-1}(U_+),$$

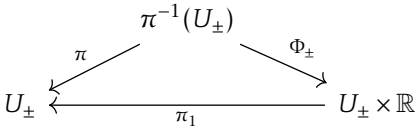
so that $\varepsilon|_{\widetilde{U}_\pm} \colon \widetilde{U}_\pm \longrightarrow U_\pm$ are homeomorphisms. Define the trivializations

$$\Phi_\pm \colon \pi^{-1}(U_\pm) \longrightarrow U_\pm \times \mathbb{R}, \quad [(x,y)] \mapsto (\varepsilon(x), y), \ \ x \in \widetilde{U}_\pm,$$

Both satisfy $\pi = \pi_1 \circ \Phi_\pm$. For $p \in U_\pm$, let $x_\pm \in \widetilde{U}_\pm$ be the unique point with $\varepsilon(x_\pm) = p$. If $x_0 \in \mathbb{R}$ is any other lift of p, there exists $m \in \mathbb{Z}$ such that $x_0 = x_\pm + m$, and then

$$\Phi_\pm([(x_0, t)]) = \Phi_\pm([(x_\pm, (-1)^m t)]) = (p, (-1)^m t).$$

In particular, using the representative with $x_0 = x_\pm$, the second component is t. With any other lift, it changes only by the linear isomorphism $t \mapsto (-1)^m t$. Thus the restriction of $\Phi_\pm$ to each fiber is a vector space isomorphism.



We next compute the transition maps. Let $(p, v) \in (U_+ \cap U_-) \times \mathbb{R}$. To invert Φ_-, choose the representative

$$\Phi_-^{-1}(p, v) = [(x_-, v)] \quad \text{where} \quad x_- = (\varepsilon|_{\widetilde{U}_-})^{-1}(p) \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}.$$

To then apply Φ_+, express the same class using a representative whose first component lies in $\widetilde{U}_+ = (0, 1)$:

$$x_+ = \begin{cases} x_- + 1, & \text{if } x_- \in (-\frac{1}{2}, 0), \\ x_-, & \text{if } x_- \in (0, \frac{1}{2}), \end{cases} \qquad [(x_-, v)] = \begin{cases} [(x_+, -v)], & \text{if } x_- \in (-\frac{1}{2}, 0), \\ [(x_+, v)], & \text{if } x_- \in (0, \frac{1}{2}), \end{cases}$$

since $(x_-, v) \sim (x_- + 1, -v)$ by the equivalence relation. Consequently,

$$(\Phi_+ \circ \Phi_-^{-1})(p, v) = \begin{cases} (p, -v), & \text{if } x_- \in (-\frac{1}{2}, 0), \\ (p, v), & \text{if } x_- \in (0, \frac{1}{2}). \end{cases}$$

This can be written as

$$(\Phi_+ \circ \Phi_-^{-1})(p, v) = (p, \tau_{+-}(p)v), \qquad \tau_{+-} \colon U_+ \cap U_- \longrightarrow GL(1, \mathbb{R}),$$

where $\tau_{+-}(p) = -1$ on the component corresponding to $x_- \in (-\frac{1}{2}, 0)$ and $\tau_{+-}(p) = 1$ on the component corresponding to $x_- \in (0, \frac{1}{2})$. The image of τ_{+-} is the subset $\{+1, -1\} \subset GL(1, \mathbb{R})$, and τ_{+-} is smooth because it is constant on each connected component of $U_+ \cap U_-$.

These trivializations and transition maps satisfy the assumptions of Lemma 1.7.8. Therefore, $(\mathbf{E}, \pi, \mathbb{S}^1)$ is a smooth vector bundle of rank 1. To verify that it is nontrivial, suppose it admits a nowhere vanishing continuous section. Lifting it through q represents the section by a continuous function $a \colon \mathbb{R} \to \mathbb{R}$ satisfying

$$a(x + 1) = -a(x),$$

because of the relation $(x, y) \sim (x + 1, -y)$. If $a(0) > 0$, then $a(1) < 0$, and if $a(0) < 0$, then $a(1) > 0$; in both cases, the intermediate value theorem gives $x \in [0, 1]$ with $a(x) = 0$. The case $a(0) = 0$ is immediate. Thus every continuous section vanishes somewhere. A trivial rank-one bundle has the nowhere vanishing global section $p \mapsto (p, 1)$; therefore, $\mathbf{E}$ is nontrivial.

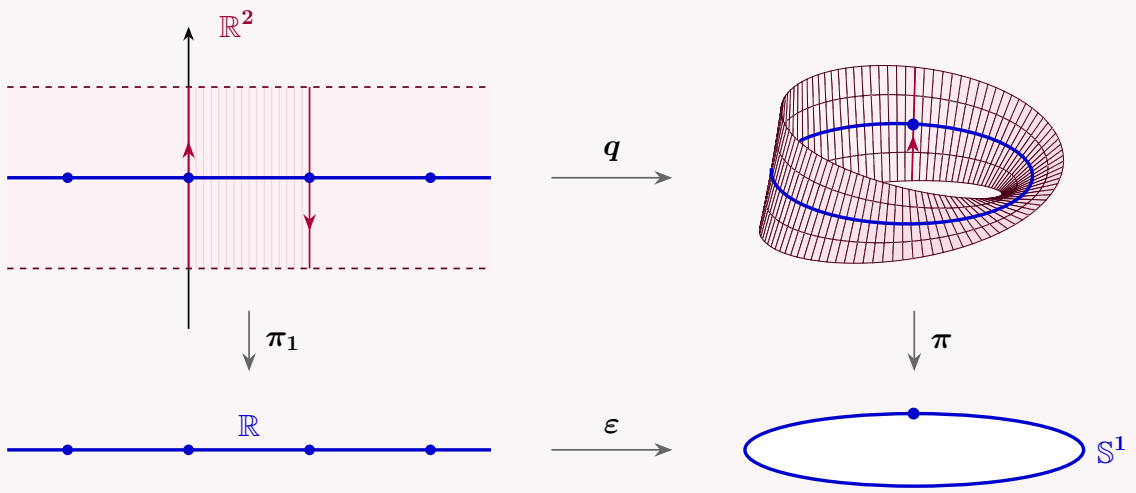


Figure 1.6. The Möbius bundle and $[0,1]\times\mathbb{R}$ as a fundamental domain representing it.

Operations on vector spaces also produce new bundles, as follows.

Definition 1.7.10. Given a smooth manifold M with or without boundary and smooth vector bundles $\pi'\colon \mathbf{E}' \longrightarrow M$ and $\pi''\colon \mathbf{E}'' \longrightarrow M$ of ranks k' and k'', respectively, define the **Whitney sum of $\mathbf{E}'$ and $\mathbf{E}''$** by $\mathbf{E}' \oplus_W \mathbf{E}'' := \coprod_{p\in M} (\mathbf{E}'_p \oplus \mathbf{E}''_p)$, with projection $\pi\colon \mathbf{E}'\oplus\mathbf{E}'' \longrightarrow M$ given by $\pi(p,(v',v'')) = p$ for each $p \in M$.

Proposition 1.7.11. Let M be a smooth manifold with or without boundary. The Whitney sum of two smooth vector bundles over M of ranks k' and k'' is a smooth vector bundle over M of rank $k'+k''$.

Proof. For each $p \in M$, consider an open neighborhood U of p over which there exist local trivializations (U,Φ') of $\mathbf{E}'$ and (U,Φ'') of $\mathbf{E}''$, and define $\Phi\colon \pi^{-1}(U) \longrightarrow U\times\mathbb{R}^{k'+k''}$ by $\Phi(v',v'') = (\pi'(v'),(\pi_{\mathbb{R}^{k'}}\circ\Phi'(v'),\pi_{\mathbb{R}^{k''}}\circ\Phi''(v'')))$.
Suppose we are given another pair of local trivializations $(\widetilde{U},\widetilde{\Phi}')$ and $(\widetilde{U},\widetilde{\Phi}'')$. Let $\tau'\colon (U\cap\widetilde{U}) \longrightarrow GL(k',\mathbb{R})$ and $\tau''\colon (U\cap\widetilde{U}) \longrightarrow GL(k'',\mathbb{R})$ be their respective transition functions. Then the transition function for $\mathbf{E}'\oplus_W\mathbf{E}''$ is given by:

$$(\widetilde{\Phi}\circ\Phi^{-1})(p,(v',v'')) = (p,\tau(p)(v',v'')),$$

where $\tau(p) = \begin{bmatrix} \tau'(p) & 0 \\ 0 & \tau''(p) \end{bmatrix}$, is smooth. Thus Lemma 1.7.8 ensures that $\mathbf{E}'\oplus\mathbf{E}''$ is a vector bundle of rank $k'+k''$ over M. □

We now introduce *local and global sections* and *local and global frames*. These will allow us to describe fully the vector bundles commonly used on a smooth manifold M. They will also make it easier to define vector fields and tensor fields.

Definition 1.7.12. Let $\pi\colon \mathbf{E} \longrightarrow M$ be a vector bundle. A **local section of $\mathbf{E}$** is a continuous map $\sigma\colon U \longrightarrow \mathbf{E}$ defined on an open set $U \subseteq M$ such that $\pi\circ\sigma = \mathrm{Id}_U$. If $U = M$, we also call σ a **global section of $\mathbf{E}$**.

Definition 1.7.13. Let M be a smooth manifold with or without boundary and let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle. A **smooth (local or global) section of $\mathbf{E}$** is a (local or global) section that is a smooth map from its domain to $\mathbf{E}$.

Remark 1.7.14. For some purposes, we need to consider functions that satisfy the conditions for a section except for continuity. A function satisfying all the conditions for a section other than continuity will be called a ***rough (local or global) section of*** $\mathbf{E}$.

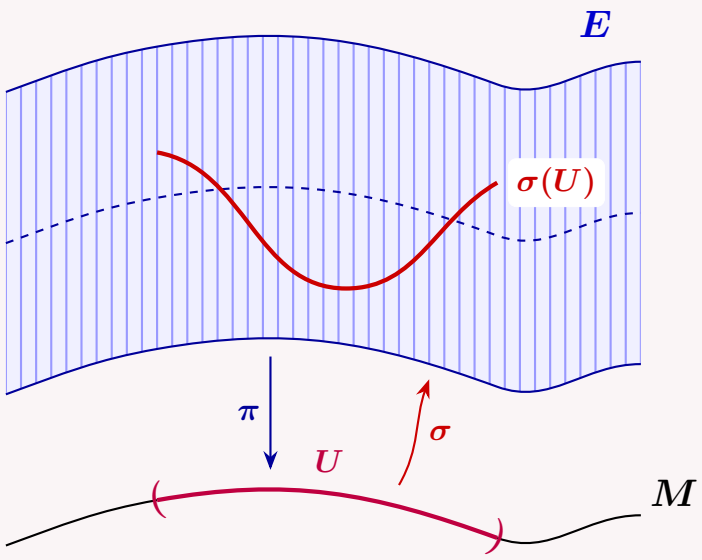


Figure 1.7. Local section of a vector bundle.

Since we will consider sections of various degrees of regularity, it is useful to have notation that simplifies specifying the differentiability class of a section.

Definition 1.7.15. Let M be a smooth manifold with or without boundary and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Define

$$\Gamma^k(\mathbf{E}) := \{\sigma\colon M \longrightarrow \mathbf{E} \mid \sigma \text{ is a section of class } C^k\}.$$

We will sometimes need to work with compactly supported sections, so we also define the support of a section. If $\sigma\colon M \longrightarrow \mathbf{E}$ is a section (possibly rough), write

$$\operatorname{supp}(\sigma) := \overline{\{p \in M \mid \sigma(p) \neq 0_p\}}$$

and also define

$$\Gamma_c^k(\mathbf{E}) := \{\sigma \in \Gamma^k(\mathbf{E}) \mid \sigma \text{ has compact support}\}.$$

If $k = \infty$, we simply write

$$\Gamma(\mathbf{E}) := \Gamma^\infty(\mathbf{E}) \qquad \text{and} \qquad \Gamma_c(\mathbf{E}) := \Gamma_c^\infty(\mathbf{E}).$$

When we need to specify the manifold on which sections are being considered, for example when working with several bundles over different smooth manifolds, we write $\Gamma^k(M,\mathbf{E})$, $\Gamma(M,\mathbf{E})$, $\Gamma_c^k(M,\mathbf{E})$, and $\Gamma_c(M,\mathbf{E})$ with the same meanings as above.

We now define local and global frames. Among other uses, these will give an important criterion for determining when a section is smooth.

Definition 1.7.16. Let $\pi\colon \mathbf{E} \longrightarrow M$ be a vector bundle, let $U \subseteq M$ be an open subset, and let $(\sigma_1,\dots,\sigma_k)$ be a k-tuple of local sections of $\mathbf{E}$ over U. We say that $(\sigma_1,\dots,\sigma_k)$ is a **local frame for E over U** if $\{\sigma_1(p),\dots,\sigma_k(p)\}$ is a basis for $\mathbf{E}_p$ for each $p \in U$. If $U = M$, the local frame $(\sigma_1,\dots,\sigma_k)$ is called a **global frame for E over** U. To simplify notation, we write $(\sigma_i)_{i=1}^k := (\sigma_1,\dots,\sigma_k)$ or simply $(\sigma_i) := (\sigma_1,\dots,\sigma_k)$ when the index set is clear.

Definition 1.7.17. Let M be a smooth manifold with or without boundary. If $\pi\colon \mathbf{E} \longrightarrow M$ is a smooth vector bundle, we say that a local frame $(\sigma_i)_{i=1}^k$ defined on an open subset U of M is a **smooth local frame** if σ_i is a smooth section for every $i \in \{1,\dots,k\}$.

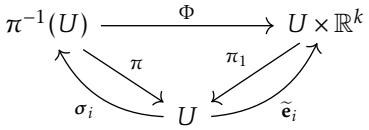
Smooth local frames and smooth local trivializations of a smooth vector bundle are related as follows:

Proposition 1.7.18. Let M be a smooth manifold with or without boundary. If $\pi\colon \mathbf{E} \longrightarrow M$ is a smooth vector bundle of rank k, then:

(a) Every smooth local trivialization $\Phi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ induces a smooth local frame given by $\sigma_i = \Phi^{-1}(p,\mathbf{e}_i)$ for each $p \in U$.

(b) Every smooth local frame (σ_i) defined on an open set $U \subseteq M$ induces a smooth local trivialization.

Proof. (a) If $\Phi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ is a smooth local trivialization of $\mathbf{E}$, we can construct a smooth local frame for $\mathbf{E}$ over U. For each $i \in \{1,\dots,k\}$, define maps $\sigma_1,\dots,\sigma_k\colon U \longrightarrow \mathbf{E}$ by $\sigma_i(p) = \Phi^{-1}(p,\mathbf{e}_i) = \Phi^{-1} \circ \widetilde{\mathbf{e}_i}(p)$, where $\widetilde{\mathbf{e}_i}\colon M \longrightarrow M \times \mathbb{R}^k$, $\widetilde{\mathbf{e}_i}(p) = (p,\mathbf{e}_i)$, and $\{\mathbf{e}_i\}_{i=1}^k$ is the standard basis of $\mathbb{R}^k$. The family $(\widetilde{\mathbf{e}_i})_{i=1}^k$ forms a smooth global frame for the bundle $\pi_1\colon M \times \mathbb{R}^k \longrightarrow M$, where π_1 is projection onto the first coordinate.



Then σ_i is smooth because Φ is a diffeomorphism; moreover, $\pi_1 \circ \Phi = \pi$, so

$$\pi \circ \sigma_i(p) = \pi \circ \Phi^{-1}(p,\mathbf{e}_i) = \pi_1(p,\mathbf{e}_i) = p,$$

which says that σ_i is a section. To see that $(\sigma_i(p))$ forms a basis for $\mathbf{E}_p$, note that Φ restricts to an isomorphism from $\mathbf{E}_p$ to $\{p\}\times\mathbb{R}^k$ and that $\Phi(\sigma_i(p)) = (p,\mathbf{e}_i)$; hence Φ sends $(\sigma_i(p))$ to the standard basis for $\{p\} \times \mathbb{R}^k$. Therefore, (σ_i) is a smooth local frame for $\mathbf{E}$, called **the smooth local frame associated with the smooth local trivialization** Φ.

(b) Suppose $(\sigma_i)_{i=1}^k$ is a smooth local frame for $\mathbf{E}$ defined on an open set $U \subseteq M$. Define $\Psi\colon U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$ by

$$\Psi(p,(v^1,\dots,v^k)) = v^i \sigma_i(p).$$

Since $\{\sigma_i(p) \mid i \in \{1,\dots,k\}\}$ forms a basis for $\mathbf{E}_p$ for each $p \in U$, Ψ is bijective. Moreover, $\sigma_i = \Psi \circ \widetilde{\mathbf{e}}_i$. If we prove that Ψ is a diffeomorphism, then Ψ^{-1} will be a smooth local trivialization whose associated smooth local frame is $(\sigma_i)_{i=1}^k$. Since Ψ is bijective, it suffices to show that it is a local diffeomorphism. Let $q \in U$. By the definition of a smooth vector bundle, there exist an open set $V_0 \subseteq M$ with $q \in V_0$ and a smooth local trivialization $\Phi\colon \pi^{-1}(V_0) \longrightarrow V_0 \times \mathbb{R}^k$. Replacing V_0 by $V := V_0 \cap U$ and restricting Φ gives a smooth local trivialization over an open set $V \subseteq U$ containing q. Since Φ is a diffeomorphism, it suffices to prove that

$$(\Phi \circ \Psi)\restriction_{V\times\mathbb{R}^k}\colon V \times \mathbb{R}^k \longrightarrow V \times \mathbb{R}^k$$

is a diffeomorphism.

For each $i \in \{1,\dots,k\}$, the map $(\Phi \circ \sigma_i)\restriction_V\colon V \longrightarrow V \times \mathbb{R}^k$ is smooth. Thus there exist smooth functions $\sigma_i^1,\dots,\sigma_i^k\colon V \longrightarrow \mathbb{R}$ such that $(\Phi \circ \sigma_i)(p) = (p,(\sigma_i^1(p),\dots,\sigma_i^k(p)))$ for every $p \in V$. Consequently,

$$(\Phi \circ \Psi)(p,(v^1,\dots,v^k)) = (p,(v^i\sigma_i^1(p),\dots,v^i\sigma_i^k(p))),$$

so $(\Phi \circ \Psi)\restriction_{V\times\mathbb{R}^k}$ is smooth.

For each $p \in V$, the matrix $(\sigma_i^j(p))$ is invertible because $\{\sigma_i(p) \mid i \in \{1,\dots,k\}\}$ is a basis for $\mathbf{E}_p$. Denote its inverse matrix by $(\tau_i^j(p))$. Since inversion on $GL(k,\mathbb{R})$ is smooth, the functions τ_i^j are smooth and

$$(\Phi \circ \Psi)^{-1}(p,(w^1,\dots,w^k)) = (p,(w^i\tau_i^1(p),\dots,w^i\tau_i^k(p))).$$

Therefore, $(\Phi \circ \Psi)^{-1}$ is smooth. This proves that Ψ is a local diffeomorphism around q, and since $q \in U$ was arbitrary, the proof is complete.

□

Remark 1.7.19. In particular, Proposition 1.7.18 tells us that there always exists an open cover $(U_\alpha)_{\alpha\in J}$ by domains of smooth local frames.

We obtain the following corollary:

Corollary 1.7.20. Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank k, let (V,φ) be a smooth chart on M with coordinate functions $\varphi = (x^1,\dots,x^n)$, and suppose we have a smooth local frame $(\sigma_i)_{i=1}^k$ for $\mathbf{E}$ over V. Define $\widetilde{\varphi}\colon \pi^{-1}(V) \longrightarrow \varphi(V) \times \mathbb{R}^k$ by

$$\widetilde{\varphi}(v^i\sigma_i) = (x^1(p),\dots,x^n(p),v^1,\dots,v^k).$$

Then $(\pi^{-1}(V),\widetilde{\varphi})$ is a smooth coordinate chart of $\mathbf{E}$.

Proof. The frame determines the smooth trivialization

$$\Psi\colon V\times\mathbb{R}^k \longrightarrow \pi^{-1}(V), \qquad \Psi(p,v)=\sum_{i=1}^{k} v^i \boldsymbol{\sigma}_i(p),$$

by Proposition 1.7.18. The function in the statement is $\widetilde{\varphi}=(\varphi\times \mathrm{Id}_{\mathbb{R}^k})\circ\Psi^{-1}$; it is therefore a diffeomorphism onto $\varphi(V)\times\mathbb{R}^k$. □

We now have a smoothness criterion for rough local sections.

Proposition 1.7.21. Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E}\longrightarrow M$ be a smooth vector bundle, and let $\boldsymbol{\tau}\colon M\longrightarrow \mathbf{E}$ be a rough section. If $(\boldsymbol{\sigma}_i)$ is a smooth local frame for $\mathbf{E}$ defined on an open set $U\subseteq M$, then $\boldsymbol{\tau}$ is smooth on U if and only if its component functions with respect to $(\boldsymbol{\sigma}_i)$ are smooth.

Proof. Let Ψ be the trivialization associated with the frame. If $\boldsymbol{\tau}=\sum_{i=1}^{k}\tau^i\boldsymbol{\sigma}_i$, then

$$(\Psi^{-1}\circ\boldsymbol{\tau})(p)=\big(p,(\tau^1(p),\dots,\tau^k(p))\big).$$

Since Ψ and Ψ^{-1} are smooth, $\boldsymbol{\tau}$ is smooth if and only if all its component functions τ^i are smooth. □

Multiplication by smooth functions makes the space of sections of a vector bundle a $C^\infty(M)$-module. This algebraic structure expresses the fact that the coefficients of a section can vary smoothly over the base.

Proposition 1.7.22. Let $\pi\colon \mathbf{E}\longrightarrow M$ be a smooth vector bundle over a smooth manifold M with or without boundary. Define an action of $C^\infty(M)$ on $\Gamma(\mathbf{E})$ by

$$(f\boldsymbol{\sigma})(p) \;:=\; f(p)\boldsymbol{\sigma}(p), \qquad f\in C^\infty(M),\ \boldsymbol{\sigma}\in\Gamma(\mathbf{E}),\ p\in M.$$

Then $\Gamma(\mathbf{E})$, with the usual addition of functions and this action, is a $C^\infty(M)$-module.

Proof. Let $(U_\alpha)_{\alpha\in J}$ be an open cover of M such that, for each $\alpha\in J$, there exists a smooth local frame $(\boldsymbol{\sigma}_{\alpha,i})_{i=1}^{k}$ of $\mathbf{E}$ defined on U_α. If $\boldsymbol{\sigma}\in\Gamma(\mathbf{E})$, on each U_α we can write

$$\boldsymbol{\sigma}\restriction_{U_\alpha}=\sum_{i=1}^{k} a_\alpha^i\boldsymbol{\sigma}_{\alpha,i}, \qquad a_\alpha^i\in C^\infty(U_\alpha),$$

where k is the rank of $\mathbf{E}$. For $f\in C^\infty(M)$, we have

$$(f\boldsymbol{\sigma})\restriction_{U_\alpha}=\sum_{i=1}^{k}\big(f\restriction_{U_\alpha}\cdot a_\alpha^i\big)\boldsymbol{\sigma}_{\alpha,i}.$$

By Proposition 1.7.21, $f\boldsymbol{\sigma}$ is smooth on U_α because $f\restriction_{U_\alpha}\cdot a_\alpha^i$ is smooth for every i. Since this holds for every $\alpha\in J$, we conclude that $f\boldsymbol{\sigma}\in\Gamma(\mathbf{E})$.

The module identities are

$$(f+g)\sigma = f\sigma + g\sigma, \qquad f(\sigma+\tau) = f\sigma + f\tau, \qquad (fg)\sigma = f(g\sigma), \qquad 1\cdot\sigma = \sigma,$$

To verify them, evaluate each side at a point $p \in M$ and use the vector space identities in the fiber $\mathbf{E}_p$. For example,

$$((fg)\sigma)(p) = f(p)g(p)\sigma(p) = \big(f(g\sigma)\big)(p).$$

The other three equalities follow in the same way. Therefore, $\Gamma(\mathbf{E})$ is a $C^\infty(M)$-module. □

The natural notion of a map between two vector bundles should cover a map between their base spaces and be linear on each fiber. This leads to the following definition.

Definition 1.7.23. If $\pi\colon \mathbf{E} \longrightarrow M$ and $\pi'\colon \mathbf{E}' \longrightarrow M'$ are vector bundles over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a continuous function $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ is called a ***bundle homomorphism*** if there exists a function $f\colon M \longrightarrow M'$ such that $\pi' \circ F = f \circ \pi$, that is, such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{F} & \mathbf{E}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

with the property that, for every $p \in M$, $F\restriction_{\mathbf{E}_p}\colon \mathbf{E}_p \longrightarrow \mathbf{E}'_{f(p)}$ is $\mathbb{K}$-linear. We express the relation between F and f by saying that F ***covers*** f.
If $\mathbf{E}$, $\mathbf{E}'$, and F are smooth, we say that F is a ***smooth bundle homomorphism***. When both bundles have the same base and the covered map is the identity, we write

$$\underline{\mathrm{Hom}}(\mathbf{E}, \mathbf{E}') := \{F\colon \mathbf{E} \longrightarrow \mathbf{E}' \mid F \text{ is a smooth bundle homomorphism over } M\}.$$

We have not required f to be continuous; this follows from continuity of F, which also determines it uniquely.

Proposition 1.7.24. Let $\pi\colon \mathbf{E} \longrightarrow M$ and $\pi'\colon \mathbf{E}' \longrightarrow M'$ be vector bundles, and let $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ be a bundle homomorphism covering $f\colon M \longrightarrow M'$. Then f is continuous and uniquely determined by F. If, in addition, M and M' are smooth manifolds with or without boundary, the bundles are smooth, and F is smooth, then f is also smooth.

Proof. We have $f = \pi' \circ F \circ \zeta$, where $\zeta\colon M \longrightarrow \mathbf{E}$ is such that $\zeta(p) = 0_{\mathbf{E}_p} \in \mathbf{E}_p$ for each $p \in M$. To verify this, note that $(\pi' \circ F \circ \zeta)(p) = \pi'(F(0_{\mathbf{E}_p})) = \pi'(0_{\mathbf{E}_{f(p)}}) = f(p)$. Thus f is continuous, and if the bundles and F are smooth, then π' and ζ are smooth, so f is smooth.
Moreover, f is uniquely determined by F, since every f making the diagram commute must equal that composition. □

Definition 1.7.25. If $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ is a bijective bundle homomorphism whose inverse is a bundle homomorphism, we say that F is a *bundle isomorphism*. If F is a diffeomorphism, we say that F is a *smooth bundle isomorphism*.

The next exercise shows that, over the same base, bijectivity of a smooth homomorphism suffices for its inverse

to be smooth.

Exercise 1.7.26. Let M be a smooth manifold with or without boundary. Prove that if $\mathbf{E}$ and $\mathbf{E}'$ are smooth vector bundles over M and $F \in \underline{\mathrm{Hom}}(\mathbf{E}, \mathbf{E}')$ is bijective, then it is a smooth bundle isomorphism.

Remark 1.7.27. If $\mathbf{E}, \mathbf{E}' \longrightarrow M$ are smooth vector bundles over a smooth manifold M with or without boundary and $F \in \underline{\mathrm{Hom}}(\mathbf{E}, \mathbf{E}')$, then composition with F induces a map

$$\widetilde{F} \colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E}')$$

as follows:

$$\widetilde{F}(\sigma)(p) = (F \circ \sigma)(p) = F(\sigma(p)).$$

We have $\widetilde{F}(\sigma) \in \Gamma(\mathbf{E}')$, since it is smooth as a composition of smooth functions. If the bundles are defined over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, linearity of F on each fiber implies that $\widetilde{F}$ is $\mathbb{K}$-linear on sections. Moreover, for $f \in C^\infty(M)$,

$$\widetilde{F}(f\sigma)(p) = F_p\big(f(p)\sigma(p)\big) = f(p)F_p\big(\sigma(p)\big) = \big(f\widetilde{F}(\sigma)\big)(p),$$

so $\widetilde{F}$ is $C^\infty(M)$-linear; that is, $\widetilde{F} \in \mathrm{Hom}_{C^\infty(M)}(\Gamma(\mathbf{E}), \Gamma(\mathbf{E}'))$.

The preceding remark tells us that every bundle homomorphism induces a homomorphism between the modules of smooth sections. The converse will also hold: every module homomorphism between the sections of the bundles induces a bundle homomorphism. To prove this, we first need a lemma.

Lemma 1.7.28. (Section with prescribed value in a fiber) Let M be a smooth manifold with or without boundary, let $\pi \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle, let $p \in M$, and let $v \in \mathbf{E}_p$. Then there exists a smooth global section $\sigma \in \Gamma(\mathbf{E})$ with $\sigma(p) = v$. Moreover, if U is a neighborhood of p over which $\mathbf{E}$ admits a local trivialization, we can choose σ with $\operatorname{supp}\sigma \subset U$.

Proof. Let (U, Φ) be a local trivialization of $\mathbf{E}$ on an open neighborhood of p, with

$$\Phi \colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k.$$

Write $\Phi(v) = (p, u_0)$ with $u_0 \in \mathbb{R}^k$. By Proposition 1.3.6, we can choose a bump function $\chi \in C^\infty(M)$ with $\chi(p) = 1$ and $\operatorname{supp}\chi \subset U$. Define

$$\sigma(q) = \begin{cases} \Phi^{-1}\big(q, \chi(q)u_0\big), & q \in U, \\ 0_q, & q \notin U, \end{cases}$$

where 0_q denotes the zero vector in the fiber $\mathbf{E}_q$. On U, the composition $q \mapsto \chi(q)u_0$ is smooth and, since Φ^{-1} is smooth, $\sigma \restriction_U$ is smooth. If $q \notin \overline{U}$, then σ agrees with the zero section, which is smooth.
To justify gluing at the boundary, we use the condition $\operatorname{supp}\chi \subset U$. By definition, $M \setminus \operatorname{supp}\chi$ is open and contains ∂U, so for each $q_0 \in \partial U$ there exists an open neighborhood $V_{q_0} \subset M$ such that $V_{q_0} \cap \operatorname{supp}\chi = \varnothing$, and in particular $\chi \restriction_{V_{q_0}} = 0$. Consequently, for $q \in V_{q_0} \cap U$ we have

$$\sigma(q) = \Phi^{-1}\big(q, \chi(q)u_0\big) = \Phi^{-1}(q, 0_{\mathbb{R}^k}) = 0_q,$$

and for $q \in V_{q_0} \setminus U$ we also have $\sigma(q) = 0_q$ by definition. In other words, on an open neighborhood of each point of ∂U, both expressions for σ agree exactly with the zero section. Therefore, σ is smooth on M.
By construction, $\sigma(p) = \Phi^{-1}(p, u_0) = v$. Moreover, if $q \in U$ and $\chi(q) = 0$, then $\sigma(q) = 0_q$, and if $q \notin U$, we also have $\sigma(q) = 0_q$, so $\operatorname{supp}\sigma \subset \operatorname{supp}\chi \subset U$. □

Lemma 1.7.29. (Characterization of bundle homomorphisms) Let M be a smooth manifold with or without boundary, and let $\pi\colon \mathbf{E} \longrightarrow M$ and $\pi'\colon \mathbf{E}' \longrightarrow M$ be smooth vector bundles. A map

$$\mathcal{F}\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E}')$$

is $C^\infty(M)$-linear if and only if there exists a *unique* smooth bundle homomorphism $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ over M such that

$$\mathcal{F}(\sigma) = F \circ \sigma, \qquad \forall \sigma \in \Gamma(\mathbf{E}).$$

Proof. If $F\colon \mathbf{E} \to \mathbf{E}'$ is a smooth homomorphism over M, then $\sigma \mapsto F \circ \sigma$ is $C^\infty(M)$-linear by linearity of each F_p.
Conversely, let $\mathcal{F}\colon \Gamma(\mathbf{E}) \to \Gamma(\mathbf{E}')$ be a $C^\infty(M)$-linear map. For $p \in M$ and $v \in \mathbf{E}_p$, choose $\sigma \in \Gamma(\mathbf{E})$ with $\sigma(p) = v$, whose existence is guaranteed by Lemma 1.7.28, and define

$$F_p(v) := \mathcal{F}(\sigma)(p).$$

We show that the value is independent of σ. It suffices to consider a section τ with $\tau(p) = 0$ and verify that $\mathcal{F}(\tau)(p) = 0$. Take a local frame $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ of $\mathbf{E}$ on an open set U containing p, and a bump function χ supported in U and equal to one in a neighborhood of p. On U, write $\tau = \sum_{i=1}^{r} a^i \mathbf{e}_i$, where $a^i(p) = 0$. Also choose a bump function ρ supported in U and equal to one on $\operatorname{supp}\chi$. Extend $\widetilde{a}^i := \chi a^i$ and $\widetilde{\mathbf{e}}_i := \rho \mathbf{e}_i$ by zero. These are, respectively, smooth global functions and sections, $\widetilde{a}^i(p) = 0$, and $\chi\tau = \sum_{i=1}^{r} \widetilde{a}^i \widetilde{\mathbf{e}}_i$. $C^\infty(M)$-linearity gives

$$\mathcal{F}(\chi\tau)(p) = \sum_{i=1}^{r} \widetilde{a}^i(p)\mathcal{F}(\widetilde{\mathbf{e}}_i)(p) = 0.$$

The section $(1-\chi)\tau$ vanishes in a neighborhood of p. If η is a bump function with $\eta(p) = 1$ and support contained in that neighborhood, then

$$\mathcal{F}((1-\chi)\tau)(p) = \eta(p)\mathcal{F}((1-\chi)\tau)(p) = \mathcal{F}(\eta(1-\chi)\tau)(p) = 0.$$

Thus $\mathcal{F}(\tau)(p) = 0$, and F_p is well defined. Linearity of F_p follows by choosing sections with prescribed values and using linearity of $\mathcal{F}$.
Define $F\colon \mathbf{E} \to \mathbf{E}'$ by $F(v) = F_{\pi(v)}(v)$. To prove smoothness, let $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ and $(\mathbf{e}'_1, \dots, \mathbf{e}'_s)$ be local frames defined on an open set U around p. Choose an open set V such that $p \in V$ and $\overline{V} \subseteq U$, and a bump function $\rho \in C^\infty(M)$ with $\rho = 1$ in a neighborhood of $\overline{V}$ and $\operatorname{supp}\rho \subseteq U$. For each i, define

$$\widetilde{\mathbf{e}}_i(q) = \begin{cases} \rho(q)\mathbf{e}_i(q), & q \in U, \\ 0_q, & q \notin U. \end{cases}$$

The support condition on ρ shows, as in Lemma 1.7.28, that $\widetilde{\mathbf{e}}_i$ is a smooth global section; moreover, it agrees with $\mathbf{e}_i$ on V. There exist smooth functions A_i^a such that

$$\mathcal{F}(\widetilde{\mathbf{e}}_i) = \sum_{a=1}^{s} A_i^a \mathbf{e}'_a$$

on V. If $q \in V$ and $v = \sum_{i=1}^{r} z^i \mathbf{e}_i(q)$, the independence just proved gives

$$F_q(v) = \sum_{a=1}^{s} \sum_{i=1}^{r} A_i^a(q) z^i \mathbf{e}'_a(q).$$

In trivializations, the total map thus has the expression $(q, z) \mapsto (q, A(q)z)$, which is smooth. By construction, it covers the identity, is linear on each fiber, and satisfies $F \circ \sigma = \mathcal{F}(\sigma)$ for every $\sigma \in \Gamma(\mathbf{E})$.
If G is another homomorphism with the same action on sections, given $p \in M$ and $v \in \mathbf{E}_p$, choose $\sigma(p) = v$ and obtain $F_p(v) = (F \circ \sigma)(p) = (G \circ \sigma)(p) = G_p(v)$. Therefore, $F = G$. □

The spaces of linear maps between the fibers themselves form a vector bundle. The following proposition makes this construction precise.

Proposition 1.7.30. Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \to M$ and $\mathbf{E}' \to M$ be smooth vector bundles of ranks k and l. Define

$$\operatorname{Hom}(\mathbf{E}, \mathbf{E}') := \coprod_{p \in M} \operatorname{Hom}(\mathbf{E}_p, \mathbf{E}'_p), \qquad \pi_{\operatorname{Hom}} \colon \operatorname{Hom}(\mathbf{E}, \mathbf{E}') \longrightarrow M, \quad (p, T_p) \mapsto p,$$

with $T_p \in \operatorname{Hom}(\mathbf{E}_p, \mathbf{E}'_p)$. Then $\operatorname{Hom}(\mathbf{E}, \mathbf{E}')$ admits a unique topology and a unique smooth structure making it a smooth vector bundle of rank kl over M for which the matrix trivializations induced by smooth local frames of $\mathbf{E}$ and $\mathbf{E}'$ are smooth.

Proof. Let $(U_\alpha)_{\alpha \in J}$ be an open cover of M such that, for every $\alpha \in J$, there exist smooth local frames

$$(\sigma_{\alpha,i})_{i=1}^{k} \text{ of } \mathbf{E} \text{ on } U_\alpha, \qquad (\sigma'_{\alpha,j})_{j=1}^{l} \text{ of } \mathbf{E}' \text{ on } U_\alpha.$$

The associated trivializations

$$\Phi_\alpha^{\mathbf{E}} \colon \pi_{\mathbf{E}}^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k, \qquad \Phi_\alpha^{\mathbf{E}'} \colon \pi_{\mathbf{E}'}^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^l$$

are characterized by

$$\Phi_\alpha^{\mathbf{E}}\big(\sigma_{\alpha,i}(p)\big) = (p, e_i), \qquad \Phi_\alpha^{\mathbf{E}'}\big(\sigma'_{\alpha,j}(p)\big) = (p, e'_j),$$

where $(e_i)_{i=1}^{k}$ and $(e'_j)_{j=1}^{l}$ are the standard bases of $\mathbb{R}^k$ and $\mathbb{R}^l$.
Given $p \in U_\alpha$ and $T_p \in \operatorname{Hom}(\mathbf{E}_p, \mathbf{E}'_p)$, write its matrix in the local frames as

$$T_\alpha(p) = \big(T_\alpha(p)_i^j\big)_{j,i} \in M_{l \times k}(\mathbb{R}),$$

defined by

$$T_p(\sigma_{\alpha,i}(p)) = \sum_{j=1}^{l} T_\alpha(p)_i^j \sigma'_{\alpha,j}(p).$$

In particular, the linear map $T_\alpha(p)\colon \mathbb{R}^k \longrightarrow \mathbb{R}^l$ satisfies

$$T_\alpha(p)(e_i) = \sum_{j=1}^{l} T_\alpha(p)_i^j e'_j.$$

Then define the local trivialization of $\mathrm{Hom}(\mathbf{E}, \mathbf{E}')$ by

$$\Phi_\alpha^{\mathrm{Hom}}(T_p) := (p, T_\alpha(p)).$$

This function is bijective, with inverse given by

$$(p,S) \longmapsto \left(u = \sum_{i=1}^{k} u^i \sigma_{\alpha,i}(p) \mapsto \sum_{j=1}^{l}\sum_{i=1}^{k} S_i^j u^i \sigma'_{\alpha,j}(p) \right), \qquad S \in M_{l\times k}(\mathbb{R}).$$

Here S is an arbitrary matrix in $M_{l\times k}(\mathbb{R})$. If $S = T_\alpha(p)$, the reconstructed map agrees exactly with T_p; and if we start from an arbitrary S and then apply $\Phi_\alpha^{\mathrm{Hom}}$, we recover (p,S). This shows that the two functions are indeed inverse to each other.
In particular, for each $p \in U_\alpha$, the restriction

$$\Phi_\alpha^{\mathrm{Hom}} \restriction_{\mathrm{Hom}(\mathbf{E}_p, \mathbf{E}'_p)}\colon \mathrm{Hom}(\mathbf{E}_p, \mathbf{E}'_p) \longrightarrow \{p\} \times M_{l\times k}(\mathbb{R})$$

is a linear isomorphism of real vector spaces.
If $U_\alpha \cap U_\beta \neq \varnothing$, Lemma 1.7.7 says that the changes of trivialization of $\mathbf{E}$ and $\mathbf{E}'$ are described by smooth functions

$$\tau_{\alpha\beta}^{\mathbf{E}}\colon U_\alpha \cap U_\beta \longrightarrow GL(k,\mathbb{R}), \qquad \tau_{\alpha\beta}^{\mathbf{E}'}\colon U_\alpha \cap U_\beta \longrightarrow GL(l,\mathbb{R}),$$

which, by Proposition 1.7.18, are characterized by

$$\sigma_{\beta,i} = \sum_{m=1}^{k} (\tau_{\alpha\beta}^{\mathbf{E}})_i^m \sigma_{\alpha,m}, \qquad \sigma'_{\beta,j} = \sum_{n=1}^{l} (\tau_{\alpha\beta}^{\mathbf{E}'})_j^n \sigma'_{\alpha,n}.$$

With this notation, if the matrix of T_p in U_β is $T_\beta(p)$, its matrix in U_α is

$$T_\alpha(p) = \tau_{\alpha\beta}^{\mathbf{E}'}(p) T_\beta(p) \big(\tau_{\alpha\beta}^{\mathbf{E}}(p)\big)^{-1}.$$

Consequently, the change of trivialization of $\mathrm{Hom}(\mathbf{E}, \mathbf{E}')$ is described by the function

$$\tau_{\alpha\beta}^{\mathrm{Hom}}\colon U_\alpha \cap U_\beta \longrightarrow GL(kl,\mathbb{R}),$$

defined at each $p \in U_\alpha \cap U_\beta$ by

$$\tau_{\alpha\beta}^{\mathrm{Hom}}(p)(S) = \tau_{\alpha\beta}^{\mathbf{E}'}(p) S (\tau_{\alpha\beta}^{\mathbf{E}}(p))^{-1}, \qquad S \in M_{l\times k}(\mathbb{R}).$$

Indeed,

$$(\Phi_\alpha^{\mathrm{Hom}} \circ (\Phi_\beta^{\mathrm{Hom}})^{-1})(p,S) = \Phi_\alpha^{\mathrm{Hom}}\left(u = \sum_{i=1}^{k} u^i \sigma_{\beta,i}(p) \;\mapsto\; \sum_{j=1}^{l}\sum_{i=1}^{k} S_i^j u^i \sigma'_{\beta,j}(p)\right) = (p, T_\alpha(p))$$

$$= (p, \tau_{\alpha\beta}^{\mathbf{E}'}(p) S \left(\tau_{\alpha\beta}^{\mathbf{E}}(p)\right)^{-1}).$$

For each p, the map $S \mapsto \tau_{\alpha\beta}^{\mathrm{Hom}}(p)(S)$ is linear. Moreover, it depends smoothly on p, since matrix multiplication and inversion are smooth and the functions $\tau_{\alpha\beta}^{\mathbf{E}}$ and $\tau_{\alpha\beta}^{\mathbf{E}'}$ are smooth.
Consequently, the transition functions of $\mathrm{Hom}(\mathbf{E},\mathbf{E}')$ are smooth and linear on each fiber. Lemma 1.7.8 provides the unique topology and smooth structure making this space a smooth vector bundle of rank kl over M for which the maps $\Phi_\alpha^{\mathrm{Hom}}$ are smooth local trivializations. The same transition calculation applies to any other pair of smooth local frames of the original bundles. Their matrix trivializations are therefore smooth for this structure, and the asserted uniqueness follows from uniqueness for the family already constructed. □

We use the notation $\mathrm{End}(\mathbf{E}) := \mathrm{Hom}(\mathbf{E},\mathbf{E})$.
The concepts introduced above are related as follows:

Proposition 1.7.31. Let M be a smooth manifold with or without boundary, and let $\pi\colon \mathbf{E} \longrightarrow M$ and $\pi'\colon \mathbf{E}' \longrightarrow M$ be smooth vector bundles. Then we have the following natural identifications:

(a) Each smooth bundle homomorphism $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ over M determines a smooth section $p \longmapsto \big(F_p\colon \mathbf{E}_p \longrightarrow \mathbf{E}'_p\big)$ of the bundle $\mathrm{Hom}(\mathbf{E},\mathbf{E}')$. Conversely, every section $\mathbf{A} \in \Gamma(\mathrm{Hom}(\mathbf{E},\mathbf{E}'))$ defines a bundle homomorphism $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ by $F(v) = \mathbf{A}(\pi(v))(v)$ for every $v \in \mathbf{E}$. This gives a natural bijection

$$\underline{\mathrm{Hom}}(\mathbf{E},\mathbf{E}') \cong \Gamma\big(\mathrm{Hom}(\mathbf{E},\mathbf{E}')\big),$$

where $\underline{\mathrm{Hom}}(\mathbf{E},\mathbf{E}')$ denotes the set of bundle homomorphisms over M.

(b) Every section $\mathbf{A} \in \Gamma(\mathrm{Hom}(\mathbf{E},\mathbf{E}'))$ induces a map $\mathcal{F}_{\mathbf{A}}\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E}')$ given by $\big(\mathcal{F}_{\mathbf{A}}(\sigma)\big)(p) = \mathbf{A}(p)(\sigma(p))$ for $\sigma \in \Gamma(\mathbf{E})$. This map is $C^\infty(M)$-linear. Conversely, if $\mathcal{F}\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E}')$ is $C^\infty(M)$-linear, there exists a unique section $\mathbf{A} \in \Gamma(\mathrm{Hom}(\mathbf{E},\mathbf{E}'))$ such that $\mathcal{F} = \mathcal{F}_{\mathbf{A}}$. This gives a natural isomorphism of $C^\infty(M)$-modules

$$\Gamma\big(\mathrm{Hom}(\mathbf{E},\mathbf{E}')\big) \cong \mathrm{Hom}_{C^\infty(M)}\big(\Gamma(\mathbf{E}),\Gamma(\mathbf{E}')\big).$$

(c) Composing the two correspondences above gives a natural identification

$$\underline{\mathrm{Hom}}(\mathbf{E},\mathbf{E}') \cong \mathrm{Hom}_{C^\infty(M)}\big(\Gamma(\mathbf{E}),\Gamma(\mathbf{E}')\big),$$

compatible with composition of bundle homomorphisms and composition of $C^\infty(M)$-linear maps.

Proof. (a) Let $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ be a bundle homomorphism over M, that is, a smooth map such that $\pi' \circ F = \pi$. For each $p \in M$, the restriction $F_p\colon \mathbf{E}_p \longrightarrow \mathbf{E}'_p$ is linear. Define a section $\mathbf{A}_F \in \Gamma(\mathrm{Hom}(\mathbf{E},\mathbf{E}'))$ by $\mathbf{A}_F(p) := F_p$. To see that $\mathbf{A}_F$ is smooth, it suffices to work in local trivializations.
On an open set U_α with smooth local frames $(\sigma_{\alpha,i})_{i=1}^k$ of $\mathbf{E}$ and $(\sigma'_{\alpha,j})_{j=1}^l$ of $\mathbf{E}'$, every vector $v \in \mathbf{E}_p$ can be

written as $v = \sum_{i=1}^{k} u^i \sigma_{\alpha,i}(p)$. Denote by $T_\alpha(p) = \big(T_\alpha(p)_i^j\big) \in M_{l\times k}(\mathbb{R})$ the *matrix* of F_p with respect to these frames; that is,

$$F_p\big(\sigma_{\alpha,i}(p)\big) = \sum_{j=1}^{l} T_\alpha(p)_i^j \sigma'_{\alpha,j}(p) \quad \text{for each } i \in \{1,\dots,k\}.$$

Then, for $v \in \mathbf{E}_p$,

$$F_p(v) = \sum_{i=1}^{k} u^i F_p\big(\sigma_{\alpha,i}(p)\big) = \sum_{i=1}^{k} u^i \sum_{j=1}^{l} T_\alpha(p)_i^j \sigma'_{\alpha,j}(p).$$

Since F is smooth, the associated matrix $T_\alpha(p)$ depends smoothly on p, and by Lemma 1.7.21, $\mathbf{A}_F$ is a smooth section of $\operatorname{Hom}(\mathbf{E},\mathbf{E}')$.

Conversely, let $\mathbf{A} \in \Gamma(\operatorname{Hom}(\mathbf{E},\mathbf{E}'))$. For each $p \in M$ and $v \in \mathbf{E}_p$, define $F_{\mathbf{A}}(v) := \mathbf{A}(p)(v)$. Then $F_{\mathbf{A}}\colon \mathbf{E} \longrightarrow \mathbf{E}'$ is a smooth map, linear on each fiber, and moreover $\pi'(F_{\mathbf{A}}(v)) = p = \pi(v)$, so $F_{\mathbf{A}}$ is a smooth bundle homomorphism over M.

If we start with F and construct $\mathbf{A}_F$, then $F_{\mathbf{A}_F}(v) = \mathbf{A}_F(p)(v) = F_p(v) = F(v)$ for every $v \in \mathbf{E}_p$, so $F_{\mathbf{A}_F} = F$.

In the other direction, if we start with $\mathbf{A}$ and construct $F_{\mathbf{A}}$, for each $p \in M$ we have $\mathbf{A}_{F_{\mathbf{A}}}(p) = (F_{\mathbf{A}})_p\colon \mathbf{E}_p \longrightarrow \mathbf{E}'_p$ and $(F_{\mathbf{A}})_p(v) = F_{\mathbf{A}}(v) = \mathbf{A}(p)(v)$ for every $v \in \mathbf{E}_p$. Since $\mathbf{A}_{F_{\mathbf{A}}}(p)$ and $\mathbf{A}(p)$ agree on every vector in $\mathbf{E}_p$, we conclude that $\mathbf{A}_{F_{\mathbf{A}}}(p) = \mathbf{A}(p)$ for every $p \in M$, and hence $\mathbf{A}_{F_{\mathbf{A}}} = \mathbf{A}$.

We conclude that the correspondences $F \mapsto \mathbf{A}_F$ and $\mathbf{A} \mapsto F_{\mathbf{A}}$ are inverse to each other, proving the bijection

$$\underline{\operatorname{Hom}}(\mathbf{E},\mathbf{E}') \cong \Gamma\big(\operatorname{Hom}(\mathbf{E},\mathbf{E}')\big).$$

(b) Define

$$\Theta\colon \Gamma(\operatorname{Hom}(\mathbf{E},\mathbf{E}')) \longrightarrow \operatorname{Hom}_{C^\infty(M)}\big(\Gamma(\mathbf{E}),\Gamma(\mathbf{E}')\big), \qquad \Theta(\mathbf{A})(\sigma)(p) := \mathbf{A}(p)\big(\sigma(p)\big).$$

By Remark 1.7.27, $\Theta(\mathbf{A})$ is $C^\infty(M)$-linear.

We show that Θ is $C^\infty(M)$-linear in $\mathbf{A}$. Let $f \in C^\infty(M)$ and $\mathbf{A},\mathbf{B} \in \Gamma(\operatorname{Hom}(\mathbf{E},\mathbf{E}'))$. We have

$$\Theta(f\mathbf{A}+\mathbf{B})(\sigma)(p) = (f\mathbf{A}+\mathbf{B})(p)(\sigma(p)) = (f(p)\mathbf{A}(p)+\mathbf{B}(p))(\sigma(p))$$

$$= f(p)\mathbf{A}(p)(\sigma(p)) + \mathbf{B}(p)(\sigma(p)) = (f\Theta(\mathbf{A}))(\sigma)(p) + \Theta(\mathbf{B})(\sigma)(p),$$

and therefore $\Theta(f\mathbf{A}+\mathbf{B}) = f\Theta(\mathbf{A}) + \Theta(\mathbf{B})$.

For surjectivity, let $\mathcal{F}\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E}')$ be a $C^\infty(M)$-linear map. By Lemma 1.7.29, there exists a unique bundle homomorphism $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ such that $\mathcal{F}(\sigma) = F \circ \sigma$ for every $\sigma \in \Gamma(\mathbf{E})$. By (a), F corresponds to a section $\mathbf{A}_F \in \Gamma(\operatorname{Hom}(\mathbf{E},\mathbf{E}'))$ and, by definition, $\Theta(\mathbf{A}_F) = \mathcal{F}$.

For injectivity, suppose that $\Theta(\mathbf{A}_1) = \Theta(\mathbf{A}_2)$. Then, for every section $\sigma \in \Gamma(\mathbf{E})$ and every $p \in M$, we have $\mathbf{A}_1(p)\big(\sigma(p)\big) = \mathbf{A}_2(p)\big(\sigma(p)\big)$. Now let $v \in \mathbf{E}_p$. By Lemma 1.7.28, there exists a section $\sigma \in \Gamma(\mathbf{E})$ with $\sigma(p) = v$. Substituting into the preceding equality gives $\mathbf{A}_1(p)(v) = \mathbf{A}_2(p)(v)$. Since this holds for every $v \in \mathbf{E}_p$ and every $p \in M$, it follows that $\mathbf{A}_1 = \mathbf{A}_2$.

Thus Θ is an isomorphism of $C^\infty(M)$-modules.

(c) The identification in part (c) is obtained by composing those in (a) and (b). Compatibility with composition is justified as follows. Let $F\colon \mathbf{E} \longrightarrow \mathbf{E}'$ and $G\colon \mathbf{E}' \longrightarrow \mathbf{E}''$ be smooth bundle homomorphisms over M. Their composition $G \circ F\colon \mathbf{E} \longrightarrow \mathbf{E}''$ is also a bundle homomorphism, and the associated section

is $\mathbf{A}_{G\circ F}(p) = (G\circ F)_p = G_p \circ F_p$ for $p \in M$.
On the other hand, the operator induced by $\mathbf{A}_{G\circ F}$ satisfies

$$\Theta(\mathbf{A}_{G\circ F})(\sigma)(p) = \mathbf{A}_{G\circ F}(p)(\sigma(p)) = (G_p \circ F_p)(\sigma(p)).$$

Writing $\tau = F\circ\sigma \in \Gamma(\mathbf{E}')$, we have $\Theta(\mathbf{A}_{G\circ F})(\sigma)(p) = G_p(\tau(p)) = \Theta(\mathbf{A}_G)(\tau)(p) = \Theta(\mathbf{A}_G)(F\circ\sigma)(p)$. Since $F\circ\sigma = \Theta(\mathbf{A}_F)(\sigma)$, it follows that

$$\Theta(\mathbf{A}_{G\circ F})(\sigma) = \Theta(\mathbf{A}_G)(\Theta(\mathbf{A}_F)(\sigma)) = (\Theta(\mathbf{A}_G)\circ\Theta(\mathbf{A}_F))(\sigma).$$

Therefore, $\Theta(\mathbf{A}_{G\circ F}) = \Theta(\mathbf{A}_G)\circ\Theta(\mathbf{A}_F)$, proving that the correspondence is compatible with composition of bundle homomorphisms.
Moreover, since (b) showed that Θ is an isomorphism of $C^\infty(M)$-modules, the identification is also compatible with the action of smooth functions; that is, $\Theta(\mathbf{A})(f\sigma) = f\cdot\Theta(\mathbf{A})(\sigma)$ for $f \in C^\infty(M)$ and $\sigma \in \Gamma(\mathbf{E})$.
Consequently,

$$\underline{\mathrm{Hom}}(\mathbf{E},\mathbf{E}') \cong \mathrm{Hom}_{C^\infty(M)}\big(\Gamma(\mathbf{E}),\Gamma(\mathbf{E}')\big)$$

is a natural identification, compatible with both composition and the $C^\infty(M)$-module structure. □

The dual bundle is the special case of the homomorphism bundle in which the target bundle is the trivial rank-one bundle.

Definition 1.7.32. (Dual bundle) Let M be a smooth manifold with or without boundary, and let $\mathbf{E}\to M$ be a smooth real vector bundle. Its *dual bundle* is

$$\mathbf{E}^* := \mathrm{Hom}(\mathbf{E},\underline{\mathbb{R}}_M).$$

In particular, $(\mathbf{E}^*)_p = \mathrm{Hom}_{\mathbb{R}}(\mathbf{E}_p,\mathbb{R})$ for every $p\in M$.

Remark 1.7.33. (Dual frames and transition matrices) Let $\mathbf{E}\to M$ be a smooth vector bundle of rank k, and let $(\sigma_1,\dots,\sigma_k)$ be a smooth local frame of $\mathbf{E}$ defined on an open set $U\subseteq M$. The *dual local frame* $(\sigma^1,\dots,\sigma^k)$ of $\mathbf{E}^*$ on U is characterized by the condition

$$\sigma^i(\sigma_j) = \delta^i_j, \qquad 1\le i,j\le k.$$

If $(\tau_1,\dots,\tau_k)$ is another smooth local frame of $\mathbf{E}$ on U, there exist smooth functions $A^j_i : U\longrightarrow\mathbb{R}$ such that

$$\tau_i = \sum_{j=1}^{k} A^j_i\sigma_j, \qquad i\in\{1,\dots,k\}.$$

Thus the transition matrix between frames of $\mathbf{E}$ is $A = (A^j_i)\in GL(k,\mathbb{R})$.
The dual frame $(\tau^1,\dots,\tau^k)$ on $\mathbf{E}^*$ must satisfy

$$\tau^i(\tau_j) = \delta^i_j,$$

which implies, in index notation,

$$\boldsymbol{\tau}^i = \sum_{j=1}^{k} (A^{-1})^i_j \boldsymbol{\sigma}^j.$$

In matrix form, if $[\boldsymbol{\tau}] = A[\boldsymbol{\sigma}]$ denotes the relation between frames of $\mathbf{E}$, the dual frames satisfy

$$[\boldsymbol{\tau}^T] = [\boldsymbol{\sigma}^T]A^{-1},$$

so the transition matrix of $\mathbf{E}^*$ is

$$B = (A^{-1})^T.$$

Thus the transition matrices of the dual bundle $\mathbf{E}^*$ are the *inverse transposes* of the transition matrices of $\mathbf{E}$.

Conjugate spaces and bundles. In the complex case, we need to distinguish the ordinary dual from conjugation. We first specify these constructions without imposing an identification between a vector space and its dual.

Definition 1.7.34. (Conjugate vector space) Let V be a complex vector space. Its *conjugate space* $\overline{V}$ is the same abelian group as V, with elements denoted by $\overline{v}$ and scalar multiplication

$$\lambda \cdot \overline{v} := \overline{\overline{\lambda}\, v}. \tag{1.8}$$

The ordinary complex dual V^* continues to mean $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

Proposition 1.7.35. (Functorial properties of conjugation) The preceding structure makes $\overline{V}$ a complex vector space, and the map

$$c_V \colon V \longrightarrow \overline{V}, \qquad c_V(v) = \overline{v},$$

is additive and antilinear. The map

$$j_V \colon \overline{\overline{V}} \longrightarrow V, \qquad j_V(\overline{\overline{v}}) = v,$$

is a canonical complex-linear isomorphism.
If $A \colon V \to W$ is complex linear, there exists a complex-linear map

$$\overline{A} \colon \overline{V} \longrightarrow \overline{W}, \qquad \overline{A}(\overline{v}) = \overline{Av}.$$

We have

$$\overline{B \circ A} = \overline{B} \circ \overline{A}, \qquad \overline{\mathrm{id}_V} = \mathrm{id}_{\overline{V}}, \qquad \overline{A^{-1}} = (\overline{A})^{-1}$$

when A is invertible, and $j_W \circ \overline{\overline{A}} = A \circ j_V$.

Proof. Addition is that of the underlying abelian group; that is, $\overline{v} + \overline{w} = \overline{v + w}$. For $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$, the definition gives

$$\begin{aligned}
\lambda \cdot (\overline{v} + \overline{w}) &= \overline{\overline{\lambda}(v + w)} = \lambda \cdot \overline{v} + \lambda \cdot \overline{w}, \\
(\lambda + \mu) \cdot \overline{v} &= \overline{(\overline{\lambda} + \overline{\mu})v} = \lambda \cdot \overline{v} + \mu \cdot \overline{v}, \\
\lambda \cdot (\mu \cdot \overline{v}) &= \overline{\overline{\lambda}\,\overline{\mu}\, v} = (\lambda\mu) \cdot \overline{v},
\end{aligned}$$

$$1 \cdot \overline{v} = \overline{v}.$$

Together with the additive group axioms, these are all the vector space identities. Moreover, $c_V(v+w) = c_V(v) + c_V(w)$ and $c_V(\lambda v) = \overline{\lambda v} = \overline{\lambda} \cdot \overline{v}$, so c_V is additive and antilinear. Under double conjugation,

$$j_V\big(\lambda \cdot \overline{\overline{v}}\big) = j_V\big(\overline{\overline{\lambda} \cdot_{\overline{V}} \overline{v}}\big) = j_V(\overline{\overline{\lambda v}}) = \lambda v,$$

which proves linearity of j_V; its inverse is $v \mapsto \overline{\overline{v}}$.
For $\overline{A}$, if $\lambda \in \mathbb{C}$, then

$$\overline{A}(\lambda \overline{v}) = \overline{A(\overline{\lambda} v)} = \overline{\overline{\lambda} A v} = \lambda \overline{A v}.$$

The remaining identities are checked on elements $\overline{v}$, and hence hold as identities of maps. □

Proposition 1.7.36. (Duality and conjugation) There is a natural complex-linear isomorphism

$$\kappa_V : \overline{V^*} \longrightarrow (\overline{V})^*, \qquad \kappa_V(\overline{\ell})(\overline{v}) := \overline{\ell(v)}. \tag{1.9}$$

In particular, V^*, $\overline{V}$, $\overline{V^*}$, and $(\overline{V})^*$ are distinct objects; only $\overline{V^*} \cong (\overline{V})^*$ are canonically identified by (1.9).

Proof. For $\mu \in \mathbb{C}$,

$$\kappa_V(\overline{\ell})(\mu \overline{v}) = \overline{\ell(\overline{\mu} v)} = \mu \, \overline{\ell(v)},$$

so $\kappa_V(\overline{\ell}) \in (\overline{V})^*$. Likewise,

$$\kappa_V(\lambda \overline{\ell})(\overline{v}) = \kappa_V(\overline{\overline{\lambda} \ell})(\overline{v}) = \lambda \, \overline{\ell(v)},$$

so κ_V is complex linear. The formula

$$\ell(v) := \overline{L(\overline{v})}, \qquad L \in (\overline{V})^*,$$

defines a complex-linear functional, since $\overline{\lambda v} = \overline{\lambda} \cdot \overline{v}$ in $\overline{V}$, and is the two-sided inverse of (1.9). Finally, if $A : V \to W$ is complex linear and $A^* \ell = \ell \circ A$, then

$$\kappa_V \circ \overline{A^*} = (\overline{A})^* \circ \kappa_W,$$

as one checks by evaluating both sides at $\overline{v}$; this proves the asserted naturality. □

For complex bundles, homomorphisms are complex linear on each fiber, in accordance with Definition 1.7.23. The construction of $\mathrm{Hom}(\mathbf{E}, \mathbf{F})$ in Proposition 1.7.30 applies unchanged, replacing $\mathbb{R}$ by $\mathbb{C}$ and requiring complex linearity. Likewise, for a complex bundle we set

$$\mathbf{E}^* := \mathrm{Hom}_{\mathbb{C}}(\mathbf{E}, \underline{\mathbb{C}}_M).$$

Definition 1.7.37. (Conjugate bundle) Let M be a smooth manifold with or without boundary and let $\pi : \mathbf{E} \to M$ be a smooth complex vector bundle. The *conjugate bundle* $\overline{\mathbf{E}} \to M$ has the same total space, underlying smooth real manifold, projection, and fiberwise addition as $\mathbf{E}$. We write $\overline{v}$ when an element

$v \in \mathbf{E}$ is regarded as an element of $\overline{\mathbf{E}}$, and define scalar multiplication by

$$\lambda \cdot_{\overline{\mathbf{E}}} \overline{v} := \overline{\overline{\lambda}\, v}, \qquad \lambda \in \mathbb{C}. \tag{1.10}$$

Proposition 1.7.38. (Construction and naturality of the conjugate bundle) Let M be a smooth manifold with or without boundary and let $\mathbf{E} \longrightarrow M$ be a smooth complex vector bundle. Each fiber $\overline{\mathbf{E}}_p$ is the conjugate vector space of $\mathbf{E}_p$, and the projection $\overline{\mathbf{E}} \to M$ is smooth. If

$$\Phi_\alpha : \mathbf{E}|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}^r, \qquad \Phi_\alpha(v) = (p, z),$$

is a smooth complex trivialization of $\mathbf{E}$, then

$$\overline{\Phi}_\alpha : \overline{\mathbf{E}}|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}^r, \qquad \overline{\Phi}_\alpha(\overline{v}) := (p, \overline{z}),$$

is a smooth complex trivialization of $\overline{\mathbf{E}}$. If the transition functions of $\mathbf{E}$ are given by

$$(\Phi_\beta \circ \Phi_\alpha^{-1})(p, z) = (p, g_{\beta\alpha}(p) z),$$

the transition functions computed for $\overline{\mathbf{E}}$ are $\overline{g_{\beta\alpha}}$. Consequently,

$$\overline{g_{\gamma\beta}}\,\overline{g_{\beta\alpha}} = \overline{g_{\gamma\alpha}}, \qquad \overline{g_{\alpha\beta}} = (\overline{g_{\beta\alpha}})^{-1}.$$

The construction is independent of the chosen trivializations, since it is defined on the same total space by the intrinsic operation (1.10) on each fiber. The assignment on sections

$$\Gamma(\mathbf{E}) \longrightarrow \Gamma(\overline{\mathbf{E}}), \qquad \mathbf{u} \longmapsto \overline{\mathbf{u}}, \qquad \overline{\mathbf{u}}(x) := \overline{\mathbf{u}(x)},$$

is an additive, antilinear bijection compatible with restrictions.
If $A : \mathbf{E} \to \mathbf{F}$ is a homomorphism of complex bundles, then

$$\overline{A} : \overline{\mathbf{E}} \longrightarrow \overline{\mathbf{F}}, \qquad \overline{A}(\overline{v}) = \overline{A(v)},$$

is a smooth homomorphism of complex bundles. Conjugation preserves identities, compositions, and inverses, and there is a natural identification $j_{\mathbf{E}} : \overline{\overline{\mathbf{E}}} \xrightarrow{\cong} \mathbf{E}$. Finally,

$$\kappa_{\mathbf{E}} : \overline{\mathbf{E}^*} \longrightarrow (\overline{\mathbf{E}})^*, \qquad \kappa_{\mathbf{E}}(\overline{\ell})(\overline{v}) = \overline{\ell(v)},$$

is a smooth complex-linear bundle isomorphism.

Proof. Proposition 1.7.35, applied fiberwise, proves the complex vector space axioms on $\overline{\mathbf{E}}_p$. The underlying real manifold and the projection are unchanged; therefore, $\overline{\mathbf{E}} \to M$ remains a smooth map.
Let $\Phi_\alpha(v) = (p, z)$. Since conjugation $z \mapsto \overline{z}$ is a real-linear isomorphism of $\mathbb{C}^r$, the map $\overline{\Phi}_\alpha$ is a real diffeomorphism. If $\lambda \in \mathbb{C}$, then

$$\overline{\Phi}_\alpha(\lambda \cdot_{\overline{\mathbf{E}}} \overline{v}) = \overline{\Phi}_\alpha(\overline{\overline{\lambda} v}) = (p, \lambda \overline{z}),$$

so it is complex linear on each fiber. Thus these functions are smooth complex trivializations of the struc-

ture already defined.
To compute their transition maps, start with a vector whose coordinate is w in the conjugate trivialization $\overline{\Phi}_\alpha$. The same element, regarded as an element of $\mathbf{E}$, has coordinate $\overline{w}$; hence

$$(\overline{\Phi}_\beta \circ \overline{\Phi}_\alpha^{-1})(p,w) = \big(p, \overline{g_{\beta\alpha}(p)\overline{w}}\big) = \big(p, \overline{g_{\beta\alpha}(p)}w\big).$$

The two identities in the statement now follow by conjugating the elementary identities already satisfied by the transition functions of $\mathbf{E}$. These were not used to construct $\overline{\mathbf{E}}$: they are consequences of the intrinsic construction. Since the total space, the projection, and scalar multiplication (1.10) are all independent of any chart, independence of the trivializations is immediate.
If u_α are the local components of $\mathbf{u}$, then $u_\beta = g_{\beta\alpha}u_\alpha$. Therefore, $\overline{u_\beta} = \overline{g_{\beta\alpha}}\,\overline{u_\alpha}$, which are exactly the transformation rules for a section of $\overline{\mathbf{E}}$. The reverse argument uses conjugation once more. The scalar multiplication rule on $\overline{\mathbf{E}}$ proves $\overline{f\mathbf{u}} = \overline{f}\,\overline{\mathbf{u}}$ for $f \in C^\infty(M,\mathbb{C})$; compatibility with restrictions is pointwise.
In local frames, if A has matrix $A_\alpha(x)$, then $\overline{A}$ has matrix $\overline{A_\alpha(x)}$; the change-of-frame rules show that these matrices glue, and smoothness is immediate. Complex linearity of $\overline{A}$ is checked using (1.10). The functorial properties and the formula $j_{\mathbf{E}}(\overline{v}) = v$ are verified fiberwise by Proposition 1.7.35; in conjugate trivializations, $j_{\mathbf{E}}$ has the identity matrix and is smooth. The last assertion is the bundle version of (1.9); its local formula and that of its inverse have constant matrices, so both maps are smooth. □

Corollary 1.7.39. (Transition functions of dual and homomorphism bundles) Let M be a smooth manifold with or without boundary, and let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth complex vector bundles. With the preceding convention, the transition functions of $\mathbf{E}^*$ are $(g_{\beta\alpha}^{-1})^T$, those of $\overline{\mathbf{E}}$ are $\overline{g_{\beta\alpha}}$, and those of $\mathrm{Hom}(\mathbf{E},\mathbf{F})$ act on a matrix A by

$$A \longmapsto h_{\beta\alpha}Ag_{\beta\alpha}^{-1}.$$

Conjugating them gives precisely the transition functions of the corresponding conjugate bundles. These formulas do not identify $\mathbf{E}$ with $\mathbf{E}^*$.

Proof. The first formula follows by requiring the evaluation $\mathbf{E}^*\otimes\mathbf{E} \to \underline{\mathbb{C}}$ to be independent of the frame. The second is the definition, and the third follows from $A_\beta(g_{\beta\alpha}v) = h_{\beta\alpha}A_\alpha(v)$. Conjugating each equality gives the last assertion. □

Certain sections of the tangent bundle TM receive a special name because of their importance in the study of smooth manifolds. These sections are called vector fields:

Definition 1.7.40. Let M be a smooth manifold with or without boundary. A **vector field on** M is a section $\mathbf{X}\colon M \longrightarrow TM$ of the map $\pi\colon TM \longrightarrow M$. We usually write $\mathbf{X}(p)$ as $\mathbf{X}_p$ for each $p \in M$.

Note 1.7.41. The set of smooth vector fields defined on a smooth manifold M with or without boundary is usually denoted by $\mathfrak{X}(M)$.

By Proposition 1.4.16, every vector field $\mathbf{X}\colon M \longrightarrow TM$ can be written in terms of a smooth chart (U,φ) with $\varphi = (x^1,\ldots,x^n)$ as $\mathbf{X} = X^i\partial_i$, where, for each $i \in \{1,\ldots,n\}$, $\partial_i\colon U \longrightarrow TU$ is the vector field defined on U by $\partial_i(p) := \partial_i\Big|_p$ for each $p \in U$. The functions X^i are called the ***component functions of*** $\mathbf{X}$ ***on*** U.
We have the following criterion for smoothness of a vector field:

Proposition 1.7.42. Let M be a smooth manifold with or without boundary and let $\mathbf{X}\colon M \longrightarrow TM$ be a rough vector field. If $(U,(x^i))$ is any smooth coordinate chart on M, then $\mathbf{X}\restriction_U$ is smooth if and only if its component functions with respect to this chart are smooth.

Proof. In the coordinate trivialization of $TM\restriction_U$, the section $\mathbf{X}\restriction_U$ is represented precisely by the map $p \mapsto (X^1(p),\dots,X^n(p))$. Proposition 1.7.21 states that the section is smooth if and only if this map is smooth, which is equivalent to smoothness of each of its component functions. □

The preceding proposition tells us that the coordinate vector fields ∂_i are smooth vector fields on U.

An essential property of vector fields is that they define operators on spaces of smooth real-valued functions. If $\mathbf{X} \in \mathfrak{X}(M)$, $U \subseteq M$ is open, and $f \in C^\infty(U)$, we obtain a new function $\mathbf{X}(f)\colon U \longrightarrow \mathbb{R}$ defined by

$$(\mathbf{X}(f))(p) = \mathbf{X}_p(f).$$

This terminology gives another smoothness criterion for vector fields.

Proposition 1.7.43. Let M be a smooth manifold with or without boundary, and let $\mathbf{X}\colon M \longrightarrow TM$ be a rough vector field. The following are equivalent:

(a) $\mathbf{X}$ is smooth.

(b) For each $f \in C^\infty(M)$, the function $\mathbf{X}(f)$ is smooth on M.

(c) For every open subset $U \subseteq M$ and every $f \in C^\infty(U)$, the function $(\mathbf{X}\restriction_U)(f)$ is smooth on U.

Proof. If $\mathbf{X} = X^i\partial_i$ is smooth in a chart U, then for $f \in C^\infty(U)$ we have

$$(\mathbf{X}\restriction_U)(f) = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i} \in C^\infty(U).$$

This proves $(a) \Rightarrow (c)$, and $(c) \Rightarrow (b)$ follows by taking $U = M$.

Suppose (b) holds. Fix $p \in M$, a chart (U,x) around p, and an open set W with $p \in W \Subset U$. Let $\chi \in C^\infty(M)$ equal 1 on $\overline{W}$ and have support contained in U. The function defined by χx^i on U and by zero outside U is smooth on M. On W,

$$\mathbf{X}(\chi x^i) = \mathbf{X}(x^i) = X^i.$$

Assumption (b) shows that each component X^i is smooth near p. Since p was arbitrary, Proposition 1.7.42 proves $(b) \Rightarrow (a)$. □

Finally, we introduce the notions of a cotangent vector, cotangent space, and cotangent bundle of a smooth manifold M with or without boundary. We have the following definitions:

Definition 1.7.44. Let M be a smooth manifold with or without boundary. For each $p \in M$, define the **cotangent space to** M **at** p, denoted by T_p^*M, as the dual space of T_pM,

$$T_p^*M := (T_pM)^*.$$

Remark 1.7.45. For each $p \in M$, if (U, φ) is a coordinate chart on an open subset $U \subseteq M$ with $p \in U$, the basis $\left(\partial_i\Big|_p\right)$ gives a dual basis for T_p^*M. Denote this basis by $(\lambda^i|_p)$. Now recall the differential of a function and that, if $f \in C^\infty(M)$, then $T_{f(p)}\mathbb{R} \cong \mathbb{R}$. We may therefore regard $df_p(v) \in \mathbb{R}$ for each $v \in T_pM$ and each $p \in M$. Then $df_p \in T_p^*M$ for each $f \in C^\infty(M)$. We know that $df_p = A_i(p)\lambda^i|_p$ for some function $A_i \colon U \longrightarrow \mathbb{R}$. By the definition of the differential and the fact that $(\lambda^i|_p)$ is the dual basis of $\partial_i\Big|_p$,

$$A_i(p) = df_p\left(\partial_i\Big|_p\right) = \partial_i\Big|_p(f) = \frac{\partial f}{\partial x^i}(p)$$

so

$$df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i|_p$$

In particular, taking $x^j \in C^\infty(U)$, for each $p \in U$ we have $\mathbf{d}x^j_p \in T_p^*(M)$, so

$$\mathbf{d}x^j|_p = \frac{\partial x^j}{\partial x^i}(p)\lambda^i|_p = \delta_i^j\lambda^i|_p = \lambda^j|_p$$

Thus the dual basis of $T_p^*(M)$ is in fact $(\mathbf{d}x^i|_p)$. The assignment $p \mapsto \mathbf{d}x^i|_p$ defines a covector field (for now, a rough one) on U called a ***coordinate covector field***.

We define the cotangent bundle:

Definition 1.7.46. If M is a smooth manifold with or without boundary, define the **cotangent bundle of** M by

$$T^*M := \coprod_{p\in M} T_p^*M.$$

Of course, for the cotangent bundle we can define an analogue of vector fields on the tangent bundle.

Definition 1.7.47. Let M be a smooth manifold with or without boundary. A local or global section of T^*M is called a **covector field**. A **rough covector field** is a local or global section that need not be continuous, and a **smooth covector field** is a smooth local or global section.

The smooth structure is obtained by dualizing the changes of basis in the tangent bundle.

Proposition 1.7.48. Let M be a smooth manifold with or without boundary of dimension n. With the standard projection $\pi \colon T^*M \longrightarrow M$ such that $\pi(\omega_p) = p$ if $\omega_p \in T_p^*M$, and with the vector space structure on each fiber of π inherited from that of T_pM, the cotangent bundle T^*M has a unique topology and a unique smooth structure making it a smooth vector bundle of rank n over M for which the coordinate covector fields are smooth local sections. Moreover, if $U \subseteq M$ is the domain of a smooth chart with coordinates $\varphi = (x^1, \ldots, x^n)$, the map

$$\widetilde{\varphi}(\xi_i\mathbf{d}x^i|_p) = (x^1(p), \ldots, x^n(p), \xi_1, \ldots, \xi_n)$$

is a smooth coordinate chart for T^*M.

Proof. For a chart (U,x), with $x=(x^1,\dots,x^n)$, define the trivialization

$$\Phi_x\colon \pi^{-1}(U)\longrightarrow U\times\mathbb{R}^n,\qquad \Phi_x\left(\sum_{i=1}^n \xi_i\,\mathbf{d}x^i|_p\right)=(p,(\xi_1,\dots,\xi_n)).$$

It is a fiber-preserving bijection and is linear on each fiber. Let (U,x) and (V,y) be two charts, and set

$$J(p):=D(y\circ x^{-1})(x(p)).$$

The change-of-basis formula (1.5) says that the components of a vector satisfy $v_y=Jv_x$. The identity $\omega(v)=\xi_x^Tv_x=\xi_y^Tv_y$ for every v then implies

$$\xi_y=(J^{-1})^T\xi_x.$$

Consequently, the change of trivialization of the cotangent bundle is

$$(\Phi_y\circ\Phi_x^{-1})(p,\xi)=\big(p,(J(p)^{-1})^T\xi\big).$$

The map $p\mapsto(J(p)^{-1})^T$ is smooth and takes values in $GL(n,\mathbb{R})$. Lemma 1.7.8 gives a unique topology and a unique smooth structure for which the maps Φ_x are local trivializations. In a chart of the total space, the projection is $(x,\xi)\mapsto x$ and the fields $\mathbf{d}x^i$ have constant components; hence the projection and the coordinate covector fields are smooth. The same calculation works on relatively open subsets of $\mathbb{H}^n$ and covers the boundary case. □

We have the following smoothness criterion for a covector field:

Proposition 1.7.49. Let M be a smooth manifold with or without boundary and let $\omega\colon M\longrightarrow T^*M$ be a rough covector field. The following statements are equivalent:

(a) ω is smooth.

(b) In every coordinate chart, the component functions of ω are smooth.

(c) Every point of M lies in some coordinate chart in which ω has smooth component functions.

(d) For every smooth vector field $\mathbf{X}\in\mathfrak{X}(M)$, the function $\omega(\mathbf{X})$ is smooth on M.

(e) For every open subset $U\subseteq M$ and every smooth vector field $\mathbf{X}$ on U, the function $\omega(\mathbf{X})\colon U\longrightarrow\mathbb{R}$ is smooth on U.

Proof. In a chart (U,x), write $\omega=\omega_i\,\mathbf{d}x^i$. The component criterion in Proposition 1.7.21 proves the equivalence of (a), (b), and (c).
If (a) holds and $\mathbf{X}=X^i\partial_i$ is smooth on U, then $\omega(\mathbf{X})=\omega_iX^i$ is smooth. Therefore, $(a)\Rightarrow(e)\Rightarrow(d)$.
Suppose (d) holds. For a chart (U,x), a point $p\in U$, and an open set W with $p\in W\Subset U$, choose $\chi\in C^\infty(M)$ equal to 1 on $\overline{W}$ and supported in U. The field defined by $\chi\partial_i$ on U and by zero outside U is smooth on M. On W,

$$\omega(\chi\partial_i)=\omega_i.$$

Assumption (d) shows that the components are smooth near every point, proving (c). □

This has the following consequence:

Corollary 1.7.50. Let M be a smooth manifold with or without boundary. The differential of every function $f \in C^\infty(M)$ is a smooth covector field.

Proof. In coordinates,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\, \mathbf{d}x^i.$$

Its components are smooth, so the preceding proposition applies. □

Finally, before concluding this section, we introduce the concept of a smooth vector subbundle:

Definition 1.7.51. If $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ is a vector bundle, a **vector subbundle of E** is a vector bundle $\pi_{\mathbf{D}} \colon \mathbf{D} \longrightarrow M$, where $\mathbf{D} \subseteq \mathbf{E}$ carries the subspace topology and $\pi_{\mathbf{D}}$ is the restriction of $\pi_{\mathbf{E}}$ to $\mathbf{D}$, such that, for each $p \in M$, $\mathbf{D}_p = \mathbf{D} \cap \mathbf{E}_p$ is a vector subspace of $\mathbf{E}_p$ and all the fibers $\mathbf{D}_p$ have the same dimension. If M is a smooth manifold with or without boundary and $\mathbf{E} \longrightarrow M$ is a smooth vector bundle, a **smooth vector subbundle of E** is a smooth vector bundle that is a vector subbundle of $\mathbf{E}$ and an embedded submanifold of $\mathbf{E}$.

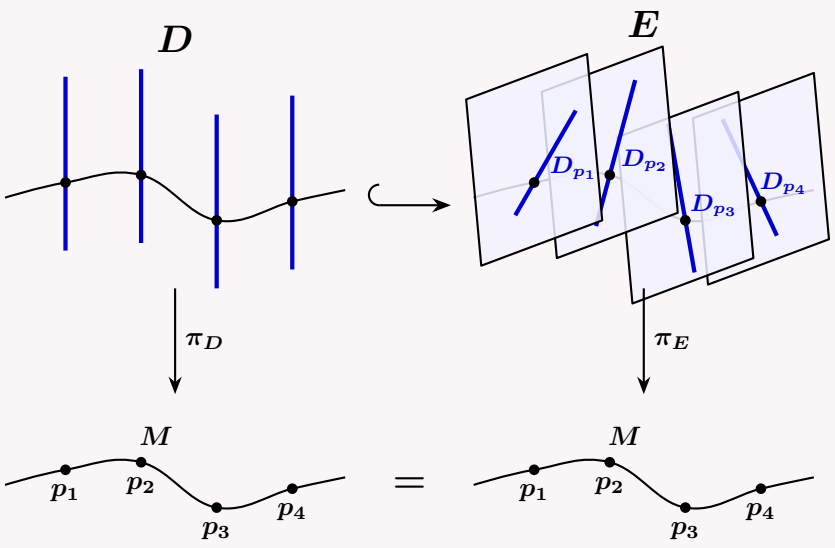


Figure 1.8. Smooth vector subbundle.

Lemma 1.7.52. (Smooth local frame criterion for subbundles) Let M be a smooth manifold with or without boundary, and let $\pi \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle. Suppose that, for each $p \in M$, we are given a vector subspace of fixed finite dimension m, $\mathbf{D}_p \subseteq \mathbf{E}_p$. Then $\mathbf{D} = \bigcup_{p \in M} \mathbf{D}_p \subseteq \mathbf{E}$ is a smooth vector subbundle of $\mathbf{E}$ if and only if every point of M has a neighborhood U on which there exist smooth local sections $\sigma_1, \ldots, \sigma_m \colon U \longrightarrow \mathbf{E}$ such that $\sigma_1(q), \ldots, \sigma_m(q)$ form a basis for $\mathbf{D}_q$ for each $q \in U$.

Proof. If $\mathbf{D}$ is a smooth vector subbundle, a local trivialization of $\mathbf{D}$ gives a smooth local frame $\sigma_1,\dots,\sigma_m$. The smooth inclusion $\mathbf{D} \hookrightarrow \mathbf{E}$ allows us to regard its elements as smooth sections of $\mathbf{E}$, and their values form a basis for $\mathbf{D}_q$.

Conversely, suppose such sections exist around $p \in M$. Choose a local frame $\boldsymbol{e}_1,\dots,\boldsymbol{e}_r$ of $\mathbf{E}$. In the fiber over p, extend $\sigma_1(p),\dots,\sigma_m(p)$ to a basis for $\mathbf{E}_p$ using some of the vectors $\boldsymbol{e}_j(p)$. The determinant of the matrix of these vectors in the frame $(\boldsymbol{e}_j)$ is nonzero at p; by continuity, it remains nonzero after shrinking the neighborhood. We thus obtain a local frame of $\mathbf{E}$

$$\sigma_1,\dots,\sigma_m,\tau_{m+1},\dots,\tau_r.$$

In the trivialization determined by this frame,

$$\mathbf{D}\restriction_U \longleftrightarrow U \times (\mathbb{R}^m \times \{0\}) \subseteq U \times \mathbb{R}^r.$$

Therefore, $\mathbf{D}\restriction_U \to U$ is a smooth vector bundle of rank m, and its total space is an embedded submanifold of $\mathbf{E}\restriction_U$. On intersections, changes of frame between the first m sections take values in $GL(m,\mathbb{R})$ and are smooth, so these local structures glue. Thus $\mathbf{D}$ is a smooth vector subbundle of $\mathbf{E}$. □

1.8 Integral curves and flows

A vector field can be read as a law of motion: at each point, it prescribes a velocity. The trajectories obeying this law are its integral curves. Local existence and uniqueness for ordinary differential equations allow us to assemble them into a family of diffeomorphisms between open subsets of the manifold, called a flow.

The existence time may depend on the initial point, so a flow need not be defined for every $t \in \mathbb{R}$ or on the entire manifold. When it is, it determines a smooth action of $\mathbb{R}$ on M.

The next definition specifies this relation between velocity and vector field.

Definition 1.8.1. Let M be a smooth manifold without boundary and let $\mathbf{X} \in \mathfrak{X}(M)$. We say that a smooth curve $\gamma\colon I \longrightarrow M$ is an integral curve of $\mathbf{X}$ if, for every $t \in I$,

$$\gamma'(t) = \mathbf{X}(\gamma(t)).$$

If $0 \in I$, we call $\gamma(0)$ the *initial point of* γ.

Example 1.8.2. (a) Consider $\mathbb{R}^2$ with standard coordinates (x,y), and let $\mathbf{X} = \dfrac{\partial}{\partial x}$. The integral curves of $\mathbf{X}$ are precisely the lines parallel to the x-axis, parametrized as

$$\gamma(t) = (a+t, b).$$

In particular, there is a unique integral curve starting at each point of the plane, and the images of distinct integral curves either coincide or are disjoint.

(b) Let $\mathbf{W} = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ on $\mathbb{R}^2$. If $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ is a smooth curve given by $\gamma(t) = (x(t), y(t))$, the condition

$\gamma'(t) = \mathbf{W}(\gamma(t))$ becomes

$$x'(t)\frac{\partial}{\partial x}\bigg|_{\gamma(t)} + y'(t)\frac{\partial}{\partial y}\bigg|_{\gamma(t)} = x(t)\frac{\partial}{\partial y}\bigg|_{\gamma(t)} - y(t)\frac{\partial}{\partial x}\bigg|_{\gamma(t)}.$$

Equating components gives the system of ordinary differential equations

$$\begin{cases} x'(t) = -y(t), \\ y'(t) = x(t). \end{cases}$$

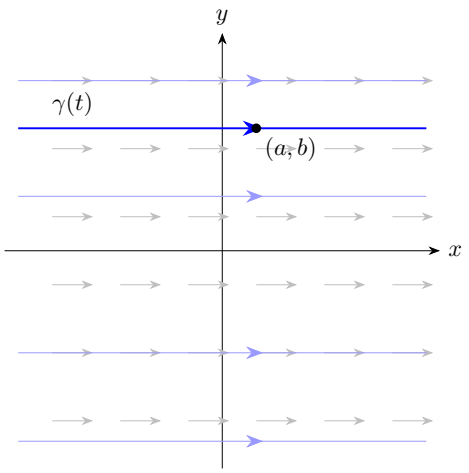
Its solutions are

$$x(t) = a\cos(t) - b\sin(t), \qquad y(t) = a\sin(t) + b\cos(t), \qquad a, b \in \mathbb{R}.$$

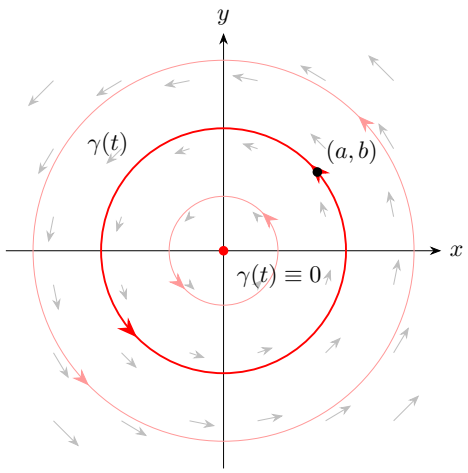
Thus the integral curves are given by

$$\gamma(t) = (a\cos(t) - b\sin(t),\, a\sin(t) + b\cos(t)).$$

When $a = b = 0$, we obtain the constant solution $\gamma(t) \equiv 0$; otherwise, the trajectories are circles centered at the origin, traversed counterclockwise. Since $\gamma(0) = (a, b)$, there is a unique integral curve starting at each point of $\mathbb{R}^2$, and the images of integral curves either coincide or are disjoint.



(a) Integral curves of $X = \frac{\partial}{\partial x}$



(b) Integral curves of $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

As the examples illustrate, finding integral curves "reduces" to solving a system of ordinary differential equations in a chart. Suppose that $\mathbf{X} \in \mathfrak{X}(M)$ and that $\gamma\colon I \longrightarrow M$ is a curve.

Let (U, φ) be a chart such that $\gamma(I) \subset U$, and write $\varphi \circ \gamma = \widetilde{\gamma} = (x^1(t), \ldots, x^n(t))$. Also write the vector field $\mathbf{X}$ in coordinates as $\mathbf{X} = X^i\partial_i$.

The condition that γ be an integral curve of $\mathbf{X}$, namely $\gamma'(t) = \mathbf{X}(\gamma(t))$, becomes in coordinates

$$\dot{x}^i(t)\partial_i\Big|_{\gamma(t)} = X^i(x(t))\partial_i\Big|_{\gamma(t)}.$$

Equating components gives the system of ordinary differential equations

$$\begin{aligned} \dot{x}^1(t) &= X^1(x^1(t),\dots,x^n(t)),\\ &\vdots\\ \dot{x}^n(t) &= X^n(x^1(t),\dots,x^n(t)). \end{aligned}$$

This shows that finding integral curves is equivalent, in coordinates, to solving an autonomous system of ordinary differential equations. The terminology comes precisely from this observation: finding integral curves corresponds to "integrating" the system.

The following result on ordinary differential equations will be fundamental for the subsequent development.

Theorem 1.8.3. (Fundamental theorem for autonomous systems) Let $U \subseteq \mathbb{R}^n$ be open and let $V \in C^\infty(U,\mathbb{R}^n)$. For $(t_0,x_0) \in \mathbb{R}\times U$, consider the initial value problem

$$\begin{cases} \dot{y}^i(t) = V^i(y^1(t),\dots,y^n(t)), \quad i \in \{1,\dots,n\},\\ y(t_0) = x_0. \end{cases} \tag{1.11}$$

Then the following assertions hold:

(a) (Local existence) For each $(t_0,x_0) \in \mathbb{R}\times U$, there exist an open interval $J_0 \subseteq \mathbb{R}$ with $t_0 \in J_0$ and an open set $U_0 \subseteq U$ with $x_0 \in U_0$ such that, for each $x \in U_0$, problem (1.11) with initial condition $y(t_0) = x$ admits a solution $y\colon J_0 \longrightarrow U$ of class C^1.

(b) (Uniqueness) If y_1 and y_2 are two solutions of (1.11) defined on intervals containing t_0 and having the same initial condition, they agree on the intersection of their domains.

(c) (Smooth dependence) If J_0 and U_0 are as in (a), the map

$$\theta\colon J_0\times U_0 \longrightarrow U, \qquad \theta(t,x) = y(t),$$

where y is the unique solution of (1.11) with $y(t_0) = x$, is of class C^∞.

Proof. Apply Theorem B.13.1 with $E = \mathbb{R}^n$ and $F(t,y) = V(y)$. Since V is of class C^1, it is locally Lipschitz; the theorem gives a unique local solution. Proposition B.13.2 extends uniqueness to every common interval. Theorem B.13.5, applied without additional parameters, states that the natural domain

$$\mathcal{E} := \{(t,a) \in \mathbb{R}\times U : t \text{ belongs to the maximal interval of the solution with } y(t_0) = a\}$$

is open and that $(t,a) \mapsto y(t;t_0,a)$ is smooth. Since $(t_0,x_0) \in \mathcal{E}$, there exist an interval $J_0 \ni t_0$ and an open set $U_0 \ni x_0$ with $J_0\times U_0 \subseteq \mathcal{E}$. This gives the uniform choice in (a), uniqueness in (b), and smooth dependence in (c). □

We now establish existence of integral curves.

Proposition 1.8.4. Let M be a smooth manifold without boundary and let $\mathbf{X} \in \mathfrak{X}(M)$. For each $p \in M$, there exist $\varepsilon > 0$ and a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$ such that γ is an integral curve of $\mathbf{X}$ starting at p.

Proof. Let $p \in M$. Take a smooth chart (U, φ) such that $p \in U$, and write $\varphi = (x^1, \dots, x^n)$. Since $\mathbf{X} \in \mathfrak{X}(M)$, on U it can be written as $\mathbf{X} = X^i \partial_i$, where each $X^i\colon \varphi(U) \longrightarrow \mathbb{R}$ is smooth.
Consider the map $V\colon \varphi(U) \longrightarrow \mathbb{R}^n$ given by $V(x) = \big(X^1(x), \dots, X^n(x)\big)$, which is smooth. Let $x_0 := \varphi(p) \in \varphi(U)$. By Theorem 1.8.3, applied to the system $\dot{y}^i(t) = V^i(y^1(t), \dots, y^n(t))$, $i \in \{1, \dots, n\}$, with initial condition $y(0) = x_0$, there exist $\varepsilon > 0$ and a smooth curve $y\colon (-\varepsilon, \varepsilon) \longrightarrow \varphi(U)$ such that

$$\dot{y}^i(t) = X^i(y^1(t), \dots, y^n(t)), \quad i \in \{1, \dots, n\}, \qquad y(0) = x_0.$$

Define $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$ by $\gamma(t) := \varphi^{-1}(y(t))$. Then γ is smooth and $\gamma(0) = p$.
To verify that γ is an integral curve of $\mathbf{X}$, note that $\varphi \circ \gamma = y = (y^1, \dots, y^n)$, so

$$\gamma'(t) = \dot{y}^i(t)\partial_i\Big|_{\gamma(t)} = X^i(y(t))\partial_i\Big|_{\gamma(t)} = \mathbf{X}(\gamma(t)).$$

Consequently, γ is an integral curve of $\mathbf{X}$ starting at p. □

We now examine how affine reparametrizations affect integral curves:

Lemma 1.8.5. (rescaling) Let M be a smooth manifold without boundary, let $\mathbf{X} \in \mathfrak{X}(M)$, let $I \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon I \longrightarrow M$ be an integral curve of $\mathbf{X}$. For any $a \in \mathbb{R}$, the curve $\widetilde{\gamma}\colon \widetilde{I} \longrightarrow M$ defined by $\widetilde{\gamma}(t) = \gamma(at)$ is an integral curve of $a\mathbf{X}$, where $\widetilde{I} = \{t \in \mathbb{R} \mid at \in I\}$.

Proof. Let $t_0 \in \widetilde{I}$ and let f be a smooth function defined on a neighborhood of $\widetilde{\gamma}(t_0)$. Then

$$\widetilde{\gamma}'(t_0)f = \frac{d}{dt}\Big|_{t=t_0} (f \circ \widetilde{\gamma})(t) = \frac{d}{dt}\Big|_{t=t_0} (f \circ \gamma)(at) = a\frac{d}{dt}\Big|_{t=at_0} (f \circ \gamma)(t).$$

Since γ is an integral curve of $\mathbf{X}$, we have

$$\frac{d}{dt}\Big|_{t=at_0} (f \circ \gamma)(t) = \mathbf{X}_{\gamma(at_0)}f = \mathbf{X}_{\widetilde{\gamma}(t_0)}f.$$

Therefore,

$$\widetilde{\gamma}'(t_0)f = a\mathbf{X}_{\widetilde{\gamma}(t_0)}f,$$

proving that $\widetilde{\gamma}$ is an integral curve of $a\mathbf{X}$. □

Lemma 1.8.6. (translation) Let M be a smooth manifold without boundary, let $\mathbf{X} \in \mathfrak{X}(M)$, let $I \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon I \longrightarrow M$ be an integral curve of $\mathbf{X}$. For any $b \in \mathbb{R}$, the curve $\widetilde{\gamma}\colon \widetilde{I} \longrightarrow M$ given by $\widetilde{\gamma}(t) = \gamma(t + b)$ is an integral curve of $\mathbf{X}$, where $\widetilde{I} = \{t \in \mathbb{R} \mid t + b \in I\}$.

Proof. Let $t_0 \in \widetilde{I}$ and let f be a smooth function defined on a neighborhood of $\widetilde{\gamma}(t_0)$. Then

$$\widetilde{\gamma}'(t_0)f = \frac{d}{dt}\bigg|_{t=t_0} (f \circ \widetilde{\gamma})(t) = \frac{d}{dt}\bigg|_{t=t_0} (f \circ \gamma)(t+b) = \frac{d}{dt}\bigg|_{t=t_0+b} (f \circ \gamma)(t).$$

Since γ is an integral curve of $\mathbf{X}$, we have

$$\frac{d}{dt}\bigg|_{t=t_0+b} (f \circ \gamma)(t) = \mathbf{X}_{\gamma(t_0+b)}f = \mathbf{X}_{\widetilde{\gamma}(t_0)}f.$$

Therefore,

$$\widetilde{\gamma}'(t_0)f = \mathbf{X}_{\widetilde{\gamma}(t_0)}f,$$

proving that $\widetilde{\gamma}$ is an integral curve of $\mathbf{X}$. □

Smooth functions allow us to relate vector fields through the notion of F-related fields, which we introduce next.

Definition 1.8.7. Let M and N be smooth manifolds without boundary, let $F\colon M \longrightarrow N$ be a smooth map, and let $\mathbf{X} \in \mathfrak{X}(M)$ and $\mathbf{Y} \in \mathfrak{X}(N)$. We say that $\mathbf{X}$ and $\mathbf{Y}$ are *F-related* if

$$dF_p(\mathbf{X}_p) = \mathbf{Y}_{F(p)} \quad \text{for every } p \in M.$$

Proposition 1.8.8. Let M and N be smooth manifolds without boundary and let $F\colon M \longrightarrow N$ be a smooth map. Then $\mathbf{X} \in \mathfrak{X}(M)$ and $\mathbf{Y} \in \mathfrak{X}(N)$ are F-related if and only if, for every integral curve γ of $\mathbf{X}$, the curve $F \circ \gamma$ is an integral curve of $\mathbf{Y}$.

Proof. First suppose that $\mathbf{X}$ and $\mathbf{Y}$ are F-related, and let $\gamma\colon I \longrightarrow M$ be an integral curve of $\mathbf{X}$. Define $\sigma := F \circ \gamma$. Then

$$\sigma'(t) = (F \circ \gamma)'(t) = dF_{\gamma(t)}(\gamma'(t)) = dF_{\gamma(t)}(\mathbf{X}_{\gamma(t)}) = \mathbf{Y}_{F(\gamma(t))} = \mathbf{Y}_{\sigma(t)},$$

so σ is an integral curve of $\mathbf{Y}$.

Conversely, suppose F sends integral curves of $\mathbf{X}$ to integral curves of $\mathbf{Y}$. Let $p \in M$ and let $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$ be an integral curve of $\mathbf{X}$ with $\gamma(0) = p$. Then $F \circ \gamma$ is an integral curve of $\mathbf{Y}$ with $(F \circ \gamma)(0) = F(p)$, and therefore

$$\mathbf{Y}_{F(p)} = (F \circ \gamma)'(0) = dF_p(\gamma'(0)) = dF_p(\mathbf{X}_p).$$

This shows that $\mathbf{X}$ and $\mathbf{Y}$ are F-related. □

We now study the notion of a *flow*. To do so, we focus on the family of integral curves associated with a vector field.

Let M be a smooth manifold without boundary and let $\mathbf{X} \in \mathfrak{X}(M)$. Suppose that, for each $p \in M$, $\mathbf{X}$ has a unique integral curve defined for every $t \in \mathbb{R}$ and starting at p, which we denote by $\theta^{(p)}\colon \mathbb{R} \longrightarrow M$.

For each $t \in \mathbb{R}$, define a map $\theta_t\colon M \longrightarrow M$ by $\theta_t(p) := \theta^{(p)}(t)$. Intuitively, each θ_t moves points of M along their integral curves for time t.

The translation lemma implies that $t \mapsto \theta^{(p)}(t+s)$ is an integral curve of $\mathbf{X}$ starting at $q = \theta^{(p)}(s)$. Uniqueness

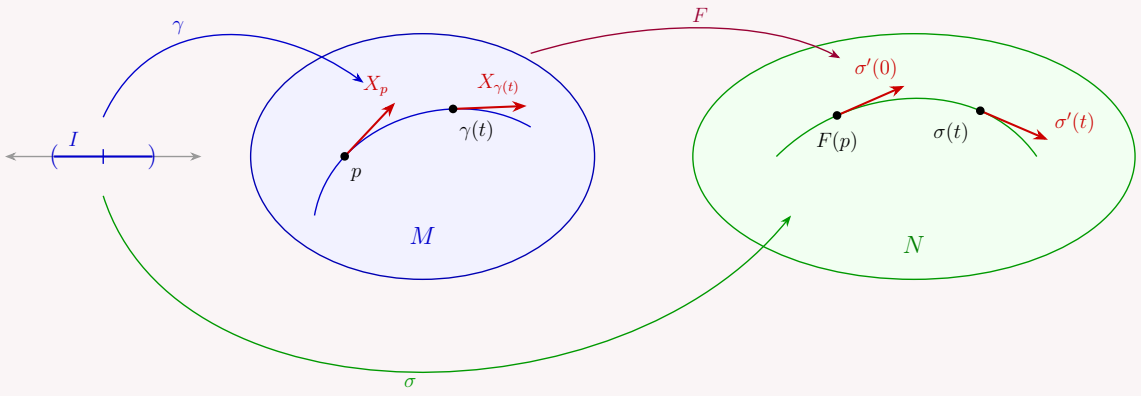


Figure 1.9. F-related vector fields.

of integral curves gives $\theta^{(q)}(t) = \theta^{(p)}(t+s)$. In terms of the maps θ_t, this becomes

$$\theta_t \circ \theta_s = \theta_{t+s}.$$

Moreover, $\theta_0(p) = \theta^{(p)}(0) = p$, so the map $\theta\colon \mathbb{R}\times M \longrightarrow M$, given by $\theta(t,p) = \theta_t(p)$, defines an action of the additive group $\mathbb{R}$ on M.

With this in mind, we give the following definition.

Definition 1.8.9. Let M be a smooth manifold without boundary. A ***global flow on*** M is a continuous action of the additive group $\mathbb{R}$ on M, that is, a map $\theta\colon \mathbb{R}\times M \longrightarrow M$ such that, for any $s,t\in\mathbb{R}$ and $p\in M$,

$$\theta(t,\theta(s,p)) = \theta(t+s,p), \qquad \theta(0,p) = p.$$

If θ is smooth, we say that θ is a ***smooth global flow on*** M.
Given a global flow θ on M, define:

(a) For each $t\in\mathbb{R}$, the map $\theta_t\colon M\longrightarrow M$ given by $\theta_t(p) = \theta(t,p)$. Then $\theta_t\circ\theta_s = \theta_{t+s}$ and $\theta_0 = \mathrm{Id}_M$, so each θ_t is a homeomorphism; if θ is smooth, θ_t is a diffeomorphism.

(b) For each $p\in M$, the curve $\theta^{(p)}\colon \mathbb{R}\longrightarrow M$ given by $\theta^{(p)}(t) = \theta(t,p)$. The image of this curve is called the orbit of p.

The following proposition shows that every smooth global flow arises from the integral curves of a smooth vector field.

Definition 1.8.10. Let M be a smooth manifold without boundary and let $\theta\colon \mathbb{R}\times M\longrightarrow M$ be a smooth global flow. For each $p\in M$, define a tangent vector $\mathbf{X}_p\in T_pM$ by

$$\mathbf{X}_p = (\theta^{(p)})'(0).$$

The assignment $p \mapsto \mathbf{X}_p$ is a rough vector field on M, called the ***infinitesimal generator of*** θ.

Proposition 1.8.11. Let M be a smooth manifold without boundary and let $\theta\colon \mathbb{R}\times M \longrightarrow M$ be a smooth global flow. The infinitesimal generator $\mathbf{X}$ of θ is a smooth vector field on M, and each $\theta^{(p)}$ is an integral curve of $\mathbf{X}$.

Proof. To see that $\mathbf{X}$ is smooth, Proposition 1.7.43 says it suffices to show that $\mathbf{X}f$ is smooth for every $f \in C^\infty(U)$, with $U \subseteq M$ an arbitrary open set. We have

$$\mathbf{X}f(p) = \mathbf{X}_p f = (\theta^{(p)})'(0)f = \frac{d}{dt}\bigg|_{t=0} f(\theta^{(p)}(t)) = \frac{\partial}{\partial t}\bigg|_{(0,p)} f(\theta(t,p)).$$

Since $f(\theta(t,p))$ is a smooth function of (t,p) by composition, its partial derivative with respect to t is also smooth. Thus $\mathbf{X}f(p)$ depends smoothly on p, so $\mathbf{X}$ is smooth. Finally, to see that the curve is an integral curve of $\mathbf{X}$, we must prove that $(\theta^{(p)})'(t) = \mathbf{X}_{\theta^{(p)}(t)}$ for every $p \in M$ and every $t \in \mathbb{R}$. Let $t_0 \in \mathbb{R}$ be arbitrary and let $q = \theta^{(p)}(t_0) = \theta_{t_0}(p)$. The group laws satisfied by θ give, for every $t \in \mathbb{R}$,

$$\theta^{(q)}(t) = \theta_t(q) = \theta_t(\theta_{t_0}(p)) = \theta_{t+t_0}(p) = \theta^{(p)}(t+t_0).$$

Therefore, for any $f \in C^\infty(U)$ where $q \in U$,

$$\mathbf{X}_q f = (\theta^{(q)})'(0)f = \frac{d}{dt}\bigg|_{t=0} f(\theta^{(q)}(t)) = \frac{d}{dt}\bigg|_{t=0} f(\theta^{(p)}(t+t_0)) = (\theta^{(p)})'(t_0)f.$$

□

We have seen that every smooth global flow gives rise to a smooth vector field whose integral curves are exactly the curves defined by the flow. Conversely, we would like to say that every smooth vector field is the infinitesimal generator of a smooth global flow. This is not always true, however, since some smooth vector fields have integral curves that are not defined for every $t \in \mathbb{R}$, as the following example shows:

Example 1.8.12. Let $M = \mathbb{R}^2$ and consider the vector field

$$\mathbf{X}(x,y) = x^2 \frac{\partial}{\partial x}.$$

The integral curve starting at $(1,0)$ is given by

$$\gamma(t) = \left(\frac{1}{1-t}, 0\right),$$

which solves the differential equation $x' = x^2$ with initial condition $x(0) = 1$.
Note that $\gamma(t)$ cannot be extended beyond $t = 1$, since $\dfrac{1}{1-t} \to +\infty$ as $t \to 1^-$. Thus the integral curve is not defined for every $t \in \mathbb{R}$, and consequently $\mathbf{X}$ does not generate a global flow.

For this reason, we need to restrict the domains on which flows are defined while retaining their characteristic properties.

Definition 1.8.13. Let M be a smooth manifold without boundary. A flow domain for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ such that, for each $p \in M$, the set

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} \mid (t,p) \in \mathcal{D}\}$$

is an open interval containing 0. A ***flow*** on M is a continuous function $\theta\colon \mathcal{D} \longrightarrow M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, satisfying the following group laws:

(a) For every $p \in M$, $\theta(0,p) = p$.

(b) For every $s \in \mathcal{D}^{(p)}$ and every $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$, $\theta(t, \theta(s,p)) = \theta(t+s,p)$.

We sometimes call θ a ***local flow*** to distinguish it from global flows. If θ is a flow, define $\theta_t(p) = \theta^{(p)}(t) = \theta(t,p)$ if $(t,p) \in \mathcal{D}$. For $t \in \mathbb{R}$, define

$$M_t := \{p \in M \mid (t,p) \in \mathcal{D}\},$$

so that $p \in M_t \iff t \in \mathcal{D}^{(p)} \iff (t,p) \in \mathcal{D}$. If θ is smooth, the infinitesimal generator of θ is defined by $\mathbf{X}_p := (\theta^{(p)})'(0)$.

Proposition 1.8.14. Let M be a smooth manifold without boundary. If $\theta\colon \mathcal{D} \longrightarrow M$ is a smooth flow, the infinitesimal generator $\mathbf{X}$ of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of $\mathbf{X}$.

Proof. Since $\mathcal{D}$ is open and contains $\{0\} \times M$, the map

$$p \longmapsto \mathbf{X}_p := d\theta_{(0,p)}(1,0)$$

is a smooth section of TM: in a chart of M, its components are the partial derivatives with respect to time of the components of θ, evaluated at $(0,p)$. This proves smoothness of $\mathbf{X}$.
Fix $p \in M$ and $t_0 \in \mathcal{D}^{(p)}$. Since $\mathcal{D}^{(\theta(t_0,p))}$ is an open interval containing zero, for s sufficiently close to zero the flow law gives

$$\theta\big(s, \theta(t_0,p)\big) = \theta(t_0 + s, p).$$

Differentiating with respect to s at $s = 0$, we obtain

$$\mathbf{X}_{\theta(t_0,p)} = \frac{d}{ds}\bigg|_{s=0} \theta(t_0 + s, p) = \frac{d}{dt}\theta^{(p)}(t_0).$$

Since t_0 was arbitrary, $\theta^{(p)}$ satisfies the integral-curve equation throughout its domain $\mathcal{D}^{(p)}$. □

Existence and uniqueness of the flow induced by a vector field is the most important result in flow theory. It will be useful later when we consider certain flows in the study of Sobolev spaces.

Theorem 1.8.15. (fundamental theorem of flows) Let M be a smooth manifold without boundary and let $\mathbf{X} \in \mathfrak{X}(M)$. Then there exists a unique maximal smooth flow $\theta\colon \mathcal{D} \longrightarrow M$ whose infinitesimal generator is $\mathbf{X}$ and which has the following properties:

(a) For each $p \in M$, the curve $\theta^{(p)}\colon \mathcal{D}^{(p)} \longrightarrow M$ is the unique maximal integral curve of $\mathbf{X}$ starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \{t - s \mid t \in \mathcal{D}^{(p)}\}$.

(c) For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t \colon M_t \longrightarrow M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

Proof. For each $p \in M$, local existence and uniqueness in charts allow us to join all integral curves starting at p. This gives a unique maximal integral curve

$$\gamma_p \colon I_p \longrightarrow M, \qquad 0 \in I_p, \qquad \gamma_p(0) = p.$$

Define

$$\mathcal{D} := \{(t,p) \in \mathbb{R} \times M : t \in I_p\}, \qquad \theta(t,p) := \gamma_p(t).$$

Fix $(t_1, p_1) \in \mathcal{D}$. The image of the compact portion of I_{p_1} between 0 and t_1 is covered by finitely many chart domains. In each chart, Theorem 1.8.3 provides a common existence interval and smooth dependence on the initial data. Divide the interval between 0 and t_1 into finitely many subintervals subordinate to those domains. On the first, the final value depends smoothly on the initial point; this value serves as the initial data for the second, and so on. Composing finitely many of these solution maps gives an open set $U_1 \ni p_1$ and an interval $J_1 \ni t_1$ with

$$J_1 \times U_1 \subseteq \mathcal{D}$$

such that θ is smooth on $J_1 \times U_1$. Therefore, $\mathcal{D}$ is open and θ is smooth.
Let $s \in I_p$ and $q = \gamma_p(s)$. The curves

$$t \longmapsto \gamma_p(t+s), \qquad t \longmapsto \gamma_q(t)$$

are integral curves of $\mathbf{X}$ and take the value q at $t = 0$. By uniqueness, they agree on their common domain. Maximality in both directions implies

$$I_q = I_p - s, \qquad \gamma_q(t) = \gamma_p(t+s), \quad t \in I_p - s.$$

This proves (2), the flow laws, and $\theta^{(p)} = \gamma_p$, which is (1). Moreover,

$$\left.\frac{d}{dt}\right|_{t=0} \theta(t,p) = \gamma_p'(0) = \mathbf{X}_p,$$

so the generator is $\mathbf{X}$.
Since $\mathcal{D}$ is open, each $M_t = \{p : (t,p) \in \mathcal{D}\}$ is open. If $p \in M_t$ and $q = \theta_t(p)$, then $I_q = I_p - t$, so $q \in M_{-t}$, and the flow law gives $\theta_{-t}(q) = p$. The reverse argument starts with $q \in M_{-t}$. Thus $\theta_t \colon M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}, proving (3).
Any other smooth flow with generator $\mathbf{X}$ has integral curves of $\mathbf{X}$ as its orbits. If it is maximal, each orbit agrees with γ_p; hence both the domain and the flow agree. This proves uniqueness. □

The flow whose existence is guaranteed by Theorem 1.8.15 is called the *flow generated by* $\mathbf{X}$. The term *infinitesimal generator* is justified by the fact that the vector field $\mathbf{X}$ describes the instantaneous variation of the flow, in the sense that

$$\mathbf{X}(p) = \left.\frac{d}{dt}\theta(t,p)\right|_{t=0}.$$

In a smooth chart, this means that, at each point, the vector $\mathbf{X}(p)$ determines the tangent direction of the integral curve through p, and the flow is obtained by integrating these directions over time. Flows are natural with respect

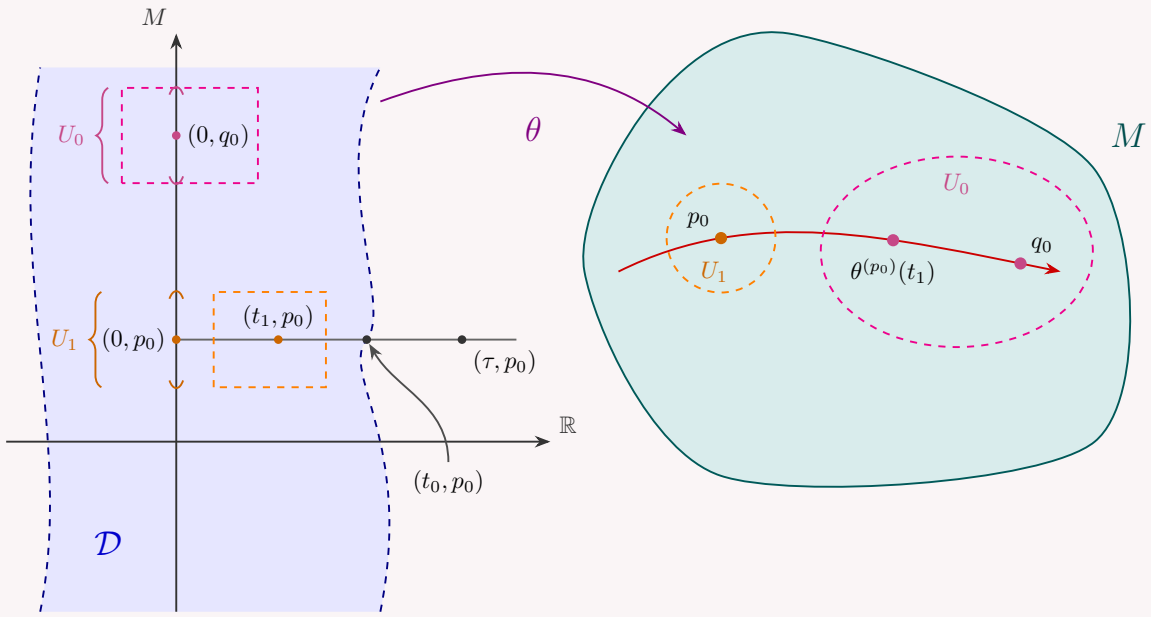


Figure 1.10. Argument showing that $\mathcal{D}$ is open.

to smooth functions in the following sense:

Proposition 1.8.16. Let M and N be smooth manifolds without boundary, let $F\colon M \longrightarrow N$ be a smooth map, and let $\mathbf{X} \in \mathfrak{X}(M)$ and $\mathbf{Y} \in \mathfrak{X}(N)$. Let θ be the flow of $\mathbf{X}$ and η the flow of $\mathbf{Y}$. If $\mathbf{X}$ and $\mathbf{Y}$ are F-related, then, for each $t \in \mathbb{R}$, $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:

$$\begin{array}{ccc} M_t & \xrightarrow{\theta_t} & M_{-t} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ N_t & \xrightarrow{\eta_t} & N_{-t} \end{array}$$

Proof. By Proposition 1.8.8, for any $p \in M$, the curve $F \circ \theta^{(p)}$ is an integral curve of $\mathbf{Y}$ starting at $F(\theta^{(p)}(0)) = F(p)$. By uniqueness of integral curves, the maximal integral curve $\eta^{(F(p))}$ must be defined at least on $\mathcal{D}^{(p)}$, and $F \circ \theta^{(p)} = \eta^{(F(p))}$ on that interval. This means that

$$p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} \Rightarrow t \in \mathcal{D}^{(F(p))} \Rightarrow F(p) \in N_t,$$

equivalently $F(M_t) \subseteq N_t$, and

$$F(\theta^{(p)}(t)) = \eta^{(F(p))}(t), \qquad \forall t \in \mathcal{D}^{(p)},$$

equivalently $\eta_t(F(p)) = F(\theta_t(p))$ for every $p \in M_t$. □

This gives the following corollary:

Corollary 1.8.17. (Invariance of flows under diffeomorphisms) Let M and N be smooth manifolds without boundary, let $F\colon M \longrightarrow N$ be a diffeomorphism, and let $\mathbf{X} \in \mathfrak{X}(M)$ have flow θ. Then the vector field $F_*\mathbf{X} \in \mathfrak{X}(N)$ defined by

$$(F_*\mathbf{X})(q) = dF_{F^{-1}(q)}\big(\mathbf{X}(F^{-1}(q))\big)$$

has flow given by

$$\eta_t = F \circ \theta_t \circ F^{-1},$$

with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Proof. Set $\mathbf{Y} := F_*\mathbf{X}$. By the definition of the pushforward field,

$$dF_p(\mathbf{X}(p)) = \mathbf{Y}(F(p)), \qquad p \in M,$$

so $\mathbf{X}$ and $\mathbf{Y}$ are F-related. Proposition 1.8.16 applied to F gives

$$F(M_t) \subseteq N_t, \qquad \eta_t \circ F = F \circ \theta_t \quad \text{on } M_t.$$

Apply the same proposition to F^{-1}, observing that $(F^{-1})_*\mathbf{Y} = \mathbf{X}$. This gives

$$F^{-1}(N_t) \subseteq M_t.$$

Applying F to this inclusion yields $N_t \subseteq F(M_t)$; therefore,

$$N_t = F(M_t).$$

Finally, composing $\eta_t \circ F = F \circ \theta_t$ on the right with F^{-1} gives

$$\eta_t = F \circ \theta_t \circ F^{-1} \quad \text{on } N_t.$$

□

We record another important property of integral curves:

Lemma 1.8.18. (escape) Let M be a smooth manifold without boundary and let $\mathbf{X} \in \mathfrak{X}(M)$. If $\gamma\colon J \longrightarrow M$ is a maximal integral curve of $\mathbf{X}$ and its right endpoint $b := \sup J$ is finite, then, for every $t_0 \in J$, the tail $\gamma([t_0, b))$ is not contained in any compact subset of M. The assertion for the left endpoint follows by reversing the parameter: if $a := \inf J > -\infty$, no tail $\gamma((a, t_0])$ is contained in a compact set.

Proof. Suppose that $\gamma([t_0, b)) \subseteq K$ with K compact. For a sequence $t_j \to b^-$, compactness gives a subsequence, not relabeled, such that $\gamma(t_j) \to p \in K$. The local existence theorem and dependence on the initial data provide a neighborhood U of p and a common $\varepsilon > 0$ for all initial data in U. For sufficiently large j, $\gamma(t_j) \in U$ and $b - t_j < \frac{\varepsilon}{2}$. The local solution starting at $\gamma(t_j)$ at time t_j agrees by uniqueness with γ on their common domain and extends it to $t_j + \varepsilon > b$, contradicting maximality. For the left endpoint, apply the same argument to the field $-\mathbf{X}$ and the time-reversed curve. □

We give some local estimates for partial derivatives of a flow, which will be useful later when studying Sobolev

spaces on manifolds of bounded geometry.

Lemma 1.8.19. (Grönwall's inequality) Let $u\colon [0,T] \longrightarrow [0,\infty)$ be a continuous function, let $a \geq 0$, and let $b\colon [0,T] \longrightarrow [0,\infty)$ be a continuous function. Suppose that

$$u(t) \leq a + \int_0^t b(\tau)\, u(\tau)\, d\tau, \qquad \forall t \in [0,T].$$

Then

$$u(t) \leq a \exp\left(\int_0^t b(\tau)\, d\tau\right), \qquad \forall t \in [0,T].$$

Proof. Define

$$v(t) := a + \int_0^t b(\tau) u(\tau)\, d\tau.$$

Then $u \leq v$, $v(0) = a$, and

$$v'(t) = b(t)u(t) \leq b(t)v(t).$$

If $a > 0$, we have $v(t) > 0$, so

$$\frac{d}{dt} \log v(t) \leq b(t).$$

Integrating between 0 and t,

$$v(t) \leq a \exp\left(\int_0^t b(\tau)\, d\tau\right),$$

and the same bound holds for u. If $a = 0$, apply the preceding argument, for each $\varepsilon > 0$, to

$$v_\varepsilon(t) := \varepsilon + \int_0^t b(\tau) u(\tau)\, d\tau.$$

This gives

$$u(t) \leq v_\varepsilon(t) \leq \varepsilon \exp\left(\int_0^t b(\tau)\, d\tau\right).$$

Letting $\varepsilon \to 0^+$ yields $u(t) = 0$, which is the assertion in this case. □

Lemma 1.8.20. Let M be a smooth manifold without boundary of dimension n, and let $\mathbf{X} \in \mathfrak{X}(M)$ be a smooth vector field. Let

$$\Phi\colon D \subseteq \mathbb{R} \times M \longrightarrow M$$

be the flow of $\mathbf{X}$.
Let (U, φ) be a chart of M such that $\overline{U}$ is compact. Suppose that $p \in U$, that $t \geq 0$, and that

$$(\tau, p) \in D \quad \text{and} \quad \Phi(\tau, p) \in U, \qquad \tau \in [0,t].$$

Then, for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$, there exists a continuous function

$$\mathrm{Expr}_\alpha\colon [0,\infty) \times [0,\infty) \longrightarrow [0,\infty),$$

nondecreasing in each variable and depending only on α and n, such that

$$\left\|\partial_x^\alpha(\varphi\circ\Phi(t,\cdot)\circ\varphi^{-1})(\varphi(p))\right\| \le \operatorname{Expr}_\alpha\left(\sup_{\substack{\tau\in[0,t],\ \beta\in\mathbb{N}_0^n\\ 1\le|\beta|\le|\alpha|}}\left\|\partial_x^\beta(\varphi_*\mathbf{X})(\varphi(\Phi(\tau,p)))\right\|, t\right).$$

The vector norms are Euclidean. Moreover, if $|\alpha|=1$, we may take

$$\operatorname{Expr}_\alpha(M,t) = e^{\sqrt{n}Mt},$$

while for $|\alpha|\ge 2$ we can choose $\operatorname{Expr}_\alpha$ recursively.

Proof. Define

$$V := \varphi(U)\subseteq\mathbb{R}^n$$

and consider the vector field in coordinates

$$\widetilde{X} := \varphi_*\mathbf{X}\in\mathfrak{X}(V).$$

Since $\mathbf{X}$ is smooth and φ is a diffeomorphism onto its image, the field $\widetilde{X}$ is smooth.
Also define

$$\widetilde{D} := \left\{(\tau,x)\in\mathbb{R}\times V \;\middle|\; (\tau,\varphi^{-1}(x))\in D \text{ and } \Phi(\tau,\varphi^{-1}(x))\in U\right\}$$

and

$$\widetilde{\Phi}\colon \widetilde{D}\longrightarrow V, \qquad \widetilde{\Phi}(\tau,x) := \varphi\big(\Phi(\tau,\varphi^{-1}(x))\big).$$

The function $\widetilde{\Phi}$ is smooth. We will show that, for each $x\in V$, the curve

$$\tau\longmapsto\widetilde{\Phi}(\tau,x)$$

satisfies the differential equation associated with $\widetilde{X}$.
Fix $x\in V$ and write $q := \varphi^{-1}(x)$. Then

$$\frac{\partial}{\partial\tau}\widetilde{\Phi}(\tau,x) = d\varphi_{\Phi(\tau,q)}\left(\frac{\partial}{\partial\tau}\Phi(\tau,q)\right).$$

Since Φ is the flow of $\mathbf{X}$,

$$\frac{\partial}{\partial\tau}\Phi(\tau,q) = \mathbf{X}_{\Phi(\tau,q)}.$$

Therefore,

$$\frac{\partial}{\partial\tau}\widetilde{\Phi}(\tau,x) = d\varphi_{\Phi(\tau,q)}(\mathbf{X}_{\Phi(\tau,q)}) = \widetilde{X}(\widetilde{\Phi}(\tau,x)).$$

Moreover,

$$\widetilde{\Phi}(0,x) = x.$$

Thus $\tau\mapsto\widetilde{\Phi}(\tau,x)$ solves

$$z'(\tau) = \widetilde{X}(z(\tau)), \qquad z(0)=x.$$

Take

$$x_0 := \varphi(p), \qquad Z(\tau) := \widetilde{\Phi}(\tau, x_0).$$

Then

$$Z'(\tau) = \widetilde{X}(Z(\tau)), \qquad Z(0) = x_0,$$

and integration gives

$$Z(t) = x_0 + \int_0^t \widetilde{X}(Z(\tau))\, d\tau.$$

We proceed by induction on $|\alpha|$.

Case $|\alpha| = 1$**.** Define

$$Y_k(t) := \partial_{x_{0,k}} \widetilde{\Phi}(t, x_0).$$

Differentiating the integral equation,

$$Y_k(t) = e_k + \int_0^t d\widetilde{X}_{Z(\tau)}(Y_k(\tau))\, d\tau,$$

and therefore

$$Y_k'(t) = d\widetilde{X}_{Z(t)}(Y_k(t)), \qquad Y_k(0) = e_k.$$

Let

$$M_1(t) := \sup_{\substack{\tau\in[0,t]\\ j\in\{1,\dots,n\}}} \|\partial_j \widetilde{X}(Z(\tau))\|.$$

For $z \in V$ and $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, we have

$$\|d\widetilde{X}_z(v)\| \le \max_{1\le j\le n} \|\partial_j \widetilde{X}(z)\| \sum_{j=1}^n |v_j| \le \sqrt{n} \max_{1\le j\le n} \|\partial_j \widetilde{X}(z)\| \|v\|.$$

Thus the operator norm of $d\widetilde{X}_{Z(\tau)}$ is at most $\sqrt{n} M_1(t)$ for $0 \le \tau \le t$. Fixing t,

$$\|Y_k(s)\| \le 1 + \int_0^s \sqrt{n} M_1(t) \|Y_k(\tau)\|\, d\tau, \qquad 0 \le s \le t.$$

Applying Lemma 1.8.19 and evaluating at $s = t$,

$$\|Y_k(t)\| \le e^{\sqrt{n} M_1(t) t}.$$

Inductive step. Suppose $|\alpha| = m \ge 2$ and that the result holds for every multi-index of strictly smaller order. Apply $\partial_{x_0}^\alpha$ to

$$\frac{\partial}{\partial t} \widetilde{\Phi}(t, x_0) = \widetilde{X}(\widetilde{\Phi}(t, x_0)).$$

Commuting derivatives,

$$\frac{\partial}{\partial t} \left(\partial_{x_0}^\alpha \widetilde{\Phi}(t, x_0) \right) = \partial_{x_0}^\alpha (\widetilde{X}(\widetilde{\Phi}(t, x_0))).$$

Applying the Faà di Bruno formula (Theorem 1.1.19), we obtain

$$\partial_{x_0}^{\alpha}(\widetilde{X}(\widetilde{\Phi}(t,x_0))) = \sum_{\substack{\beta\in\mathbb{N}_0^n \\ 1\le|\beta|\le|\alpha|}} \sum_{\substack{\gamma=(\gamma_1,\dots,\gamma_{|\beta|}) \\ |\gamma_i|>0,\ \sum_{i=1}^{|\beta|}\gamma_i=\alpha}} \sum_{l\in\{1,\dots,n\}^{|\beta|}} c_{\alpha,\beta,\gamma,l}\,\partial_x^{\beta}\widetilde{X}(Z(t))\prod_{i=1}^{|\beta|}\partial_{x_0}^{\gamma_i}\widetilde{\Phi}_{l_i}(t,x_0).$$

If $|\beta|=1$, then $\gamma=(\gamma_1)$ has only one component and $\gamma_1=\alpha$, giving

$$d\widetilde{X}_{Z(t)}(\partial_{x_0}^{\alpha}\widetilde{\Phi}(t,x_0)).$$

We therefore define explicitly

$$G_{\alpha}(t) := \sum_{\substack{\beta\in\mathbb{N}_0^n \\ 2\le|\beta|\le|\alpha|}} \sum_{\substack{\gamma=(\gamma_1,\dots,\gamma_{|\beta|}) \\ |\gamma_i|>0,\ \sum_{i=1}^{|\beta|}\gamma_i=\alpha}} \sum_{l\in\{1,\dots,n\}^{|\beta|}} c_{\alpha,\beta,\gamma,l}\,\partial_x^{\beta}\widetilde{X}(Z(t))\prod_{i=1}^{|\beta|}\partial_{x_0}^{\gamma_i}\widetilde{\Phi}_{l_i}(t,x_0).$$

Thus

$$\frac{\partial}{\partial t}(\partial_{x_0}^{\alpha}\widetilde{\Phi}) = d\widetilde{X}_{Z(t)}(\partial_{x_0}^{\alpha}\widetilde{\Phi}) + G_{\alpha}(t), \qquad \partial_{x_0}^{\alpha}\widetilde{\Phi}(0,x_0)=0.$$

Define

$$M_{\alpha}(t) := \sup_{\substack{\tau\in[0,t],\ \beta\in\mathbb{N}_0^n \\ 1\le|\beta|\le|\alpha|}} \|\partial_x^{\beta}\widetilde{X}(Z(\tau))\|.$$

By the induction hypothesis,

$$\|\partial_{x_0}^{\gamma}\widetilde{\Phi}(\tau,x_0)\| \le \mathrm{Expr}_{\gamma}(M_{\alpha}(t),t).$$

Taking norms in G_{α} and using these bounds shows that there exists a polynomial P_{α} with nonnegative coefficients such that

$$\|G_{\alpha}(\tau)\| \le P_{\alpha}\big(M_{\alpha}(t),\mathrm{Expr}_{\gamma^{(1)}}(M_{\alpha}(t),t),\dots\big).$$

Define

$$B_{\alpha}(M,t) := P_{\alpha}\big(M,\mathrm{Expr}_{\gamma^{(1)}}(M,t),\dots\big).$$

Let

$$u(s) := \|\partial_{x_0}^{\alpha}\widetilde{\Phi}(s,x_0)\|.$$

The norm comparison in the first-order case gives $\|d\widetilde{X}_{Z(\tau)}\|_{\mathrm{op}} \le \sqrt{n}\,M_{\alpha}(t)$ for $0\le\tau\le t$. Integrating the equation for $\partial_{x_0}^{\alpha}\widetilde{\Phi}$, whose initial value is zero, gives

$$u(s) \le tB_{\alpha}(M_{\alpha}(t),t) + \int_0^s \sqrt{n}\,M_{\alpha}(t)u(\tau)\,d\tau, \qquad 0\le s\le t.$$

Applying Lemma 1.8.19,

$$u(s) \le tB_{\alpha}(M_{\alpha}(t),t)e^{\sqrt{n}M_{\alpha}(t)s}.$$

Evaluating at $s = t$,

$$\|\partial_{x_0}^{\alpha}\widetilde{\Phi}(t,x_0)\| \le tB_{\alpha}(M_{\alpha}(t),t)e^{\sqrt{n}M_{\alpha}(t)t}.$$

Define

$$\mathrm{Expr}_{\alpha}(M,t) := tB_{\alpha}(M,t)e^{\sqrt{n}Mt}.$$

Finally, the function Expr_{α} is nondecreasing in each variable. By the induction hypothesis, the functions Expr_{γ} have this property; since P_{α} has nonnegative coefficients, B_{α} is nondecreasing in each variable; and the function $(M,t) \mapsto te^{\sqrt{n}Mt}$ is nondecreasing in $[0,\infty)\times[0,\infty)$. The assertion follows.
Finally,

$$\partial_x^{\alpha}(\varphi \circ \Phi(t,\cdot)\circ\varphi^{-1})(\varphi(p)) = \partial_{x_0}^{\alpha}\widetilde{\Phi}(t,x_0),$$

and

$$\partial_x^{\beta}\widetilde{X}(Z(\tau)) = \partial_x^{\beta}(\varphi_*\mathbf{X})(\varphi(\Phi(\tau,p))),$$

which completes the proof. □

1.9 Lie groups and smooth actions

Many geometric symmetries depend on continuous parameters. Translations of $\mathbb{R}^n$, rotations of the circle, and changes of frame can be composed and inverted, but they can also be differentiated with respect to those parameters. Lie groups bring these two structures together: the group law expresses composition of symmetries, while the smooth structure allows us to study how they vary.

Basic examples include $(\mathbb{R}^n,+)$, the circle S^1, $\mathrm{GL}(n,\mathbb{R})$, and its orthogonal subgroups; in the complex case, unitary groups arise in the same way. Isometry groups provide further examples when the necessary geometric structure is available. Near the identity, the tangent space T_eG and its bracket form the Lie algebra of G. This infinitesimal linearization allows us to study curves of symmetries through equations in a vector space.

The relation between the group and its algebra will appear in flows and one-parameter subgroups. Smooth actions will later lead to homogeneous spaces and principal bundles; in infinite dimensions, analogous ideas will arise in gauge groups and diffeomorphism groups.

In this section, we specify the Lie group conventions used later. If G is a group and $g \in G$, we write

$$L_g(h) := gh, \qquad R_{\mathbf{g}}(h) := hg, \qquad C_g(h) := ghg^{-1}.$$

For the standard finite-dimensional theory and a comparison of conventions, see [94, Chs. 7 and 20]. We will explicitly indicate external results whose proofs are not reproduced.

Definition 1.9.1. (Lie group) A *finite-dimensional real Lie group* is a smooth manifold G without boundary, equipped with a group structure for which the maps

$$\mu\colon G\times G \longrightarrow G, \qquad \mu(g,h)=gh, \qquad \iota\colon G\longrightarrow G, \qquad \iota(g)=g^{-1},$$

are smooth. Denote the identity element of G by e. A *Lie group homomorphism* is a group homomorphism that is also smooth.

Proposition 1.9.2. Let G and H be finite-dimensional real Lie groups whose underlying smooth manifolds have no boundary. For every $g \in G$, the maps L_g, $R_{\mathbf{g}}$, and C_g are diffeomorphisms, and

$$L_g^{-1} = L_{g^{-1}}, \qquad R_{\mathbf{g}}^{-1} = R_{g^{-1}}, \qquad C_g^{-1} = C_{g^{-1}}.$$

Moreover,

$$C_{gh} = C_g \circ C_h.$$

If $\Phi\colon G \to H$ is a Lie group homomorphism, then

$$\Phi \circ L_g = L_{\Phi(g)} \circ \Phi, \qquad \Phi \circ R_{\mathbf{g}} = R_{\Phi(g)} \circ \Phi, \qquad \Phi \circ C_g = C_{\Phi(g)} \circ \Phi.$$

Proof. All identities follow directly from the group laws. Smoothness of the maps and their inverses follows from smoothness of multiplication and inversion. □

Proposition 1.9.3. (Classical matrix groups) The groups

$$\mathrm{GL}(n,\mathbb{R}), \quad \mathrm{GL}(n,\mathbb{C}), \quad \mathrm{O}(n), \quad \mathrm{SO}(n), \quad \mathrm{U}(n)$$

are finite-dimensional real Lie groups with the smooth structures induced by their inclusions into matrix spaces. In particular, $\mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$ is the circle viewed as a Lie group.

Proof. The set $\mathrm{GL}(n,\mathbb{K}) = \{A : \det A \neq 0\}$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, is open in the real vector space of matrices. Multiplication is polynomial, and the formula $A^{-1} = \mathrm{adj}(A)/\det A$ proves that inversion is smooth.
For the orthogonal group, consider

$$F\colon M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}_n(\mathbb{R}), \qquad F(A) = A^{\mathsf{T}}A.$$

If $A \in F^{-1}(I)$, then

$$dF_A(H) = A^{\mathsf{T}}H + H^{\mathsf{T}}A.$$

This differential is surjective: given a symmetric matrix S, simply take $H = AS/2$. By the regular value theorem, $\mathrm{O}(n) = F^{-1}(I)$ is a submanifold. The group operations are restrictions of smooth maps. Since the determinant takes only the values ± 1 on $\mathrm{O}(n)$, the subset $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}$ is open and closed in $\mathrm{O}(n)$, and is therefore also a Lie group.
The same argument, regarding complex spaces as real spaces and using

$$F_{\mathbb{C}}(A) = A^*A, \qquad d(F_{\mathbb{C}})_A(H) = A^*H + H^*A,$$

shows that $\mathrm{U}(n) = F_{\mathbb{C}}^{-1}(I)$ is a submanifold: for a Hermitian matrix S, the choice $H = AS/2$ proves surjectivity. The case $n = 1$ gives precisely the unit circle. □

Before defining the Lie algebra, we recall our convention for the bracket of vector fields.

Proposition 1.9.4. (Bracket of vector fields) Let M and N be smooth manifolds with or without boundary, and let $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$. There is a unique smooth vector field $[\mathbf{X}, \mathbf{Y}]$ characterized by

$$[\mathbf{X}, \mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f), \qquad f \in C^\infty(M). \tag{1.12}$$

The bracket is bilinear and antisymmetric and satisfies the Jacobi identity. If $F\colon M \to N$ is smooth, $\mathbf{X}_1, \mathbf{X}_2 \in \mathfrak{X}(M)$ and $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathfrak{X}(N)$, and $\mathbf{X}_i$ is F-related to $\mathbf{Y}_i$ for $i = 1, 2$, then $[\mathbf{X}_1, \mathbf{X}_2]$ is F-related to $[\mathbf{Y}_1, \mathbf{Y}_2]$. In particular, if F is a diffeomorphism,

$$F_*[\mathbf{X}, \mathbf{Y}] = [F_*\mathbf{X}, F_*\mathbf{Y}].$$

Proof. For $f, g \in C^\infty(M)$, the Leibniz rule gives

$$\big(\mathbf{XY} - \mathbf{YX}\big)(fg) = f\big(\mathbf{XY} - \mathbf{YX}\big)g + g\big(\mathbf{XY} - \mathbf{YX}\big)f.$$

Thus, at each point p, the operator $f \mapsto (\mathbf{XY} - \mathbf{YX})f(p)$ is a derivation and therefore a tangent vector at T_pM. In coordinates, if $\mathbf{X} = X^i\partial_i$ and $\mathbf{Y} = Y^j\partial_j$, then

$$[\mathbf{X}, \mathbf{Y}] = \left(X^i\partial_i Y^j - Y^i\partial_i X^j\right)\partial_j,$$

so the resulting field is smooth. Antisymmetry and the Jacobi identity are the corresponding identities for commutators of linear operators.
If $\mathbf{X}_i$ is F-related to $\mathbf{Y}_i$, then, for $f \in C^\infty(N)$,

$$\mathbf{X}_i(f \circ F) = (\mathbf{Y}_i f) \circ F.$$

Applying this equality twice gives

$$[\mathbf{X}_1, \mathbf{X}_2](f \circ F) = \big([\mathbf{Y}_1, \mathbf{Y}_2]f\big) \circ F,$$

which is precisely the required relation. □

Definition 1.9.5. (Invariant fields and the Lie algebra) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary. A field $\mathbf{X} \in \mathfrak{X}(G)$ is *left invariant* if

$$(L_g)_*\mathbf{X} = \mathbf{X} \qquad \text{for every } g \in G.$$

For $\xi \in T_eG$, define

$$\xi^L_g := (dL_g)_e\xi.$$

The vector space

$$\mathfrak{g} := T_eG$$

is called the *Lie algebra of* G. Its bracket is defined by

$$[\xi, \eta]_{\mathfrak{g}} := [\xi^L, \eta^L]_e. \tag{1.13}$$

We omit the subscript $\mathfrak{g}$ when there is no risk of confusion.

Proposition 1.9.6. Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary. The assignment

$$\mathfrak{g} \longrightarrow \{\mathbf{X} \in \mathfrak{X}(G) \mid \mathbf{X} \text{ is left invariant}\}, \qquad \xi \longmapsto \xi^L,$$

is a linear isomorphism with inverse $\mathbf{X} \mapsto \mathbf{X}_e$. Moreover,

$$[\xi^L, \eta^L] = [\xi, \eta]^L.$$

Therefore, (1.13) makes $\mathfrak{g}$ a Lie algebra.

Proof. Smoothness of ξ^L follows from smoothness of $(g, \xi) \mapsto (dL_g)_e \xi$. The identity $L_g \circ L_h = L_{gh}$ implies

$$(dL_g)_h(\xi^L_h) = (dL_g)_h (dL_h)_e \xi = (dL_{gh})_e \xi = \xi^L_{gh},$$

so ξ^L is left invariant. Conversely, if $\mathbf{X}$ is left invariant, then

$$\mathbf{X}_g = (dL_g)_e \mathbf{X}_e,$$

so $\mathbf{X} = (\mathbf{X}_e)^L$. Naturality of the bracket under L_g shows that the bracket of two left invariant fields is again left invariant. Evaluation at e gives $[\xi^L, \eta^L] = [\xi, \eta]^L$. The Lie algebra identities are then inherited from the bracket of vector fields. □

Proposition 1.9.7. (Lie algebras of the classical matrix groups) With the matrix bracket $[A, B] = AB - BA$, we have the identifications

$$\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}), \qquad \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$$

as real Lie algebras, and

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \{A \in M_n(\mathbb{R}) : A^{\mathsf{T}} + A = 0\}, \qquad \mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) : A^* + A = 0\}.$$

In particular, $\mathfrak{u}(1) = i\mathbb{R}$.

Proof. The tangent space at the identity of a general linear group is the entire ambient space. If $A(t)$ is a curve in $\mathrm{O}(n)$ with $A(0) = I$, differentiating $A(t)^{\mathsf{T}} A(t) = I$ gives $A'(0)^{\mathsf{T}} + A'(0) = 0$. Conversely, each skew-symmetric matrix belongs to the kernel of dF_I, which is the tangent space to the regular level set. Since $\mathrm{SO}(n)$ is open in $\mathrm{O}(n)$, both groups have the same tangent space at the identity. The identity $A(t)^* A(t) = I$ similarly gives the description of $\mathfrak{u}(n)$.

Finally, for a matrix A, the left invariant field is $\mathbf{X}_A(g) = gA$. A calculation in the matrix space gives $[\mathbf{X}_A, \mathbf{X}_B](g) = g(AB - BA)$, so the bracket defined in (1.13) agrees with the commutator. □

Proposition 1.9.8. (Differential of a homomorphism) Let G and H be finite-dimensional real Lie groups whose underlying smooth manifolds have no boundary, and let $\Phi \colon G \to H$ be a Lie group homomorphism.

Then

$$d\Phi_e \colon \mathfrak{g} \longrightarrow \mathfrak{h}$$

is a Lie algebra homomorphism:

$$d\Phi_e[\xi, \eta] = [d\Phi_e \xi, d\Phi_e \eta].$$

Proof. The identity $\Phi \circ L_g = L_{\Phi(g)} \circ \Phi$ implies

$$d\Phi_g(\xi^L_g) = (dL_{\Phi(g)})_e (d\Phi_e \xi) = (\boldsymbol{d\Phi_e \xi})^L_{\Phi(g)}.$$

Thus ξ^L and $(\boldsymbol{d\Phi_e \xi})^L$ are Φ-related. Proposition 1.9.4 shows that their brackets are also related. Evaluating at e gives the formula. □

Lemma 1.9.9. (Bracket and flow) Let M be a smooth manifold without boundary, let $\mathbf{X} \in \mathfrak{X}(M)$, and let Φ_t be its local flow. For every $\mathbf{Y} \in \mathfrak{X}(M)$,

$$[\mathbf{X}, \mathbf{Y}] = \left. \frac{d}{dt} \right|_{t=0} (\Phi_{-t})_* \mathbf{Y}, \tag{1.14}$$

on any open set where both sides are defined.

Proof. As operators on functions,

$$\big((\Phi_{-t})_* \mathbf{Y}\big) f = \Phi_t^* \big(\mathbf{Y}(\Phi_{-t}^* f)\big).$$

The derivative at $t = 0$ of the operator Φ_t^* is $\mathbf{X}$, while the derivative of Φ_{-t}^* is $-\mathbf{X}$. By the product rule,

$$\left. \frac{d}{dt} \right|_0 \Phi_t^* \mathbf{Y} \Phi_{-t}^* = \mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X} = [\mathbf{X}, \mathbf{Y}].$$

□

Theorem 1.9.10. (Exponential map of a Lie group) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary. For each $\xi \in \mathfrak{g}$, the field ξ^L is complete. If $\gamma_\xi \colon \mathbb{R} \to G$ is its integral curve with $\gamma_\xi(0) = e$, then

$$\gamma_\xi(t + s) = \gamma_\xi(t) \gamma_\xi(s).$$

The map

$$\exp_G \colon \mathfrak{g} \longrightarrow G, \qquad \exp_G(\xi) := \gamma_\xi(1),$$

is smooth and satisfies

$$\exp_G(t\xi) = \gamma_\xi(t), \qquad \exp_G((t + s)\xi) = \exp_G(t\xi) \exp_G(s\xi). \tag{1.15}$$

Moreover,

$$(d \exp_G)_0 = \mathrm{Id}_{\mathfrak{g}};$$

consequently, $\exp_G$ restricts to a diffeomorphism between an open neighborhood of $0 \in \mathfrak{g}$ and an open neighborhood of $e \in G$.

Proof. Let $\gamma_\xi : I \to G$ be the maximal integral curve of ξ^L starting at e. If $s \in I$, the curves

$$t \longmapsto \gamma_\xi(s+t), \qquad t \longmapsto \gamma_\xi(s)\gamma_\xi(t)$$

are integral curves of ξ^L and take the same value at $t = 0$. By uniqueness, they agree wherever both are defined.
The interval I contains some $(-\varepsilon, \varepsilon)$. If $b := \sup I < \infty$, choose $s \in I$ with $b - s < \varepsilon$. The formula

$$\widetilde{\gamma}(s+t) := \gamma_\xi(s)\gamma_\xi(t), \qquad |t| < \varepsilon,$$

agrees with γ_ξ on the overlap and extends the curve beyond b, contradicting maximality. The same argument at the left endpoint shows that $I = \mathbb{R}$. The group identity already obtained therefore holds for every $s, t \in \mathbb{R}$.
The rescaling lemma for integral curves gives

$$\gamma_{t\xi}(1) = \gamma_\xi(t),$$

which proves (1.15). To prove smoothness with respect to ξ, consider the smooth field on $G \times \mathfrak{g}$

$$\boldsymbol{\mathcal{X}}_{(g,\xi)} := \big((dL_g)_e \xi, 0\big).$$

Each integral curve of $\boldsymbol{\mathcal{X}}$ keeps its second component fixed, and its first component is an integral curve of a left invariant field; by what we have proved, $\boldsymbol{\mathcal{X}}$ is complete. The fundamental theorem of flows implies that its global flow depends smoothly on the initial point. Evaluating it at time 1 and at (e, ξ) shows that $\xi \mapsto \exp_G(\xi)$ is smooth.
Finally,

$$(d\exp_G)_0(\xi) = \left.\frac{d}{dt}\right|_0 \exp_G(t\xi) = \gamma'_\xi(0) = \xi.$$

Theorem 1.5.6 gives the local assertion. □

Corollary 1.9.11. (One-parameter subgroups) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary. For each $\xi \in \mathfrak{g}$, the map

$$t \longmapsto \exp_G(t\xi)$$

is a smooth homomorphism $(\mathbb{R}, +) \to G$. Conversely, if $\alpha : (\mathbb{R}, +) \to G$ is a smooth homomorphism and $\xi = \alpha'(0)$, then

$$\alpha(t) = \exp_G(t\xi) \qquad \text{for every } t \in \mathbb{R}.$$

Proof. The first assertion is (1.15). For the converse,

$$\alpha(t+s) = \alpha(t)\alpha(s)$$

implies, on differentiating with respect to s at 0,

$$\alpha'(t) = (dL_{\alpha(t)})_e \xi = \xi^L_{\alpha(t)}.$$

Thus α and $t \mapsto \exp_G(t\xi)$ are integral curves of ξ^L with the same initial data, and uniqueness proves the equality. □

Definition 1.9.12. (Adjoint and infinitesimal representations) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary. Define

$$\mathrm{Ad}_g := (dC_g)_e \in GL(\mathfrak{g}), \qquad \mathrm{Ad}\colon G \longrightarrow GL(\mathfrak{g}),$$

and

$$\mathrm{ad}_\xi \eta := \left.\frac{d}{dt}\right|_0 \mathrm{Ad}_{\exp_G(t\xi)}\eta.$$

Proposition 1.9.13. (Naturality and equivariance of the exponential map) Let G and H be finite-dimensional real Lie groups whose underlying smooth manifolds have no boundary. If $\Phi\colon G \to H$ is a Lie group homomorphism, then

$$\Phi(\exp_G \xi) = \exp_H(d\Phi_e \xi), \qquad \xi \in \mathfrak{g}.$$

In particular, for $g \in G$,

$$\exp_G(\mathrm{Ad}_g \xi) = g \exp_G(\xi) g^{-1}. \tag{1.16}$$

Proof. The two maps

$$t \longmapsto \Phi(\exp_G(t\xi)), \qquad t \longmapsto \exp_H(t\, d\Phi_e \xi)$$

are one-parameter subgroups of H with initial velocity $d\Phi_e\xi$. Uniqueness in Corollary 1.9.11 shows that they agree; evaluate at $t = 1$. For $\Phi = C_g$, its differential at e is Ad_g, giving (1.16). □

Proposition 1.9.14. (Identities for Ad and ad) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary. The following identities hold:

$$\mathrm{Ad}_{gh} = \mathrm{Ad}_g \mathrm{Ad}_h, \tag{1.17}$$

$$\mathrm{Ad}_g[\xi,\eta] = [\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta], \tag{1.18}$$

$$\mathrm{ad}_\xi \eta = [\xi,\eta], \tag{1.19}$$

$$\mathrm{Ad}_{\exp_G(t\xi)} = \exp\big(t\,\mathrm{ad}_\xi\big), \tag{1.20}$$

$$\mathrm{Ad}_g \mathrm{ad}_\xi \mathrm{Ad}_{g^{-1}} = \mathrm{ad}_{\mathrm{Ad}_g \xi}, \tag{1.21}$$

$$\mathrm{ad}_{[\xi,\eta]} = [\mathrm{ad}_\xi, \mathrm{ad}_\eta]. \tag{1.22}$$

Proof. The identity $C_{gh} = C_g \circ C_h$ implies (1.17). Since C_g is a Lie group automorphism, Proposition 1.9.8 gives (1.18).

The flow of ξ^L is

$$\Phi_t = R_{\exp_G(t\xi)},$$

since the curve starting at g is $g\exp_G(t\xi)$. A direct verification gives

$$(R_h)_*\eta^L = (\mathbf{Ad}_{h^{-1}}\,\eta)^L.$$

By Lemma 1.9.9,

$$[\xi^L, \eta^L] = \frac{d}{dt}\Big|_0 (R_{\exp_G(-t\xi)})_*\eta^L = \left(\frac{\boldsymbol{d}}{\boldsymbol{dt}}\Big|_{\mathbf{0}} \mathbf{Ad}_{\mathbf{exp}_G(t\xi)}\,\eta\right)^L.$$

Evaluating at e gives (1.19).
By (1.17), the map $B(t) := \mathrm{Ad}_{\exp_G(t\xi)}$ satisfies

$$B(t+s) = B(t)B(s), \qquad B(0) = I.$$

Differentiating with respect to s at 0 gives $B'(t) = B(t)\,\mathrm{ad}_\xi$, whose unique solution is (1.20). Identity (1.21) follows by differentiating

$$\mathrm{Ad}_g \mathrm{Ad}_{\exp(t\xi)} \mathrm{Ad}_{g^{-1}} = \mathrm{Ad}_{\exp(t\,\mathrm{Ad}_g\,\xi)}.$$

Finally, (1.22) is the Jacobi identity written as an identity of endomorphisms. □

Theorem 1.9.15. (Closed subgroup theorem) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary, and let $H \subseteq G$ be a subgroup closed in the topology of G. Then H has a unique smooth structure making it an embedded Lie subgroup of G. Its Lie algebra, identified with a subspace of $\mathfrak{g}$, is

$$\mathfrak{h} = \{\xi \in \mathfrak{g} \mid \exp_G(t\xi) \in H \text{ for every } t \in \mathbb{R}\}. \tag{1.23}$$

Existence and the characterization of the Lie algebra are [94, Theorem 20.12 and Proposition 20.9, pp. 521–523].

Proof. The existence assertion is the cited closed subgroup theorem. The structure is unique because an embedded Lie subgroup structure must be the submanifold structure induced by slice charts. Once it is constructed, (1.23) is the characterization given by the cited exponential characterization: the curves $t \mapsto \exp_G(t\xi)$ contained in H are precisely the one-parameter subgroups whose initial velocity belongs to T_eH. □

Proposition 1.9.16. (Centralizers) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary. For any subset $S \subseteq G$, its centralizer

$$Z_G(S) := \{g \in G \mid gs = sg \text{ for every } s \in S\}$$

is a closed subgroup, and hence an embedded Lie subgroup. If G is compact, $Z_G(S)$ is compact. Its Lie algebra is

$$\mathrm{Lie}(Z_G(S)) = \{\xi \in \mathfrak{g} \mid \mathrm{Ad}_s\,\xi = \xi \text{ for every } s \in S\}. \tag{1.24}$$

If $S = H$ is a connected Lie subgroup with algebra $\mathfrak{h}$, this expression is equivalent to

$$\operatorname{Lie}(Z_G(H)) = \{\xi \in \mathfrak{g} \mid [\xi, \eta] = 0 \text{ for every } \eta \in \mathfrak{h}\}.$$

Proof. The centralizer is a subgroup by the group laws, and

$$Z_G(S) = \bigcap_{s \in S} \{g \in G \mid C_g(s) = s\}.$$

Each set on the right is closed, since $g \mapsto C_g(s)$ is continuous and $\{s\}$ is closed. Theorem 1.9.15 gives it its embedded structure; if G is compact, the closed subset $Z_G(S)$ is also compact.
By (1.23), a vector ξ belongs to its algebra if and only if $\exp_G(t\xi)$ commutes with every $s \in S$ for every t. Using (1.16), this condition becomes

$$\exp_G(t \operatorname{Ad}_s \xi) = \exp_G(t\xi) \quad (s \in S,\ t \in \mathbb{R}).$$

Differentiating at $t = 0$ gives $\operatorname{Ad}_s \xi = \xi$; the converse follows from the same equivariance formula. This proves (1.24).
If H is connected, it is generated by any neighborhood of the identity, and such a neighborhood can be covered by exponentials. By $\operatorname{Ad}_{\exp(t\eta)} = \exp(t \operatorname{ad}_\eta)$, a vector is fixed by every H if and only if $\operatorname{ad}_\eta \xi = [\eta, \xi] = 0$ for every $\eta \in \mathfrak{h}$. □

Definition 1.9.17. (Smooth actions and positive fundamental fields) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary, and let M be a smooth manifold with or without boundary. A *smooth left action* of G on M is a smooth map

$$a \colon G \times M \longrightarrow M, \qquad (g, x) \longmapsto g \cdot x,$$

such that

$$e \cdot x = x, \qquad (gh) \cdot x = g \cdot (h \cdot x).$$

A *smooth right action* is a smooth map

$$r \colon M \times G \longrightarrow M, \qquad (x, g) \longmapsto x \cdot g,$$

such that

$$x \cdot e = x, \qquad (x \cdot g) \cdot h = x \cdot (gh).$$

We always use the *positive fundamental field*. For a left action, define

$$\xi_M(x) := \left.\frac{d}{dt}\right|_0 \exp_G(t\xi) \cdot x,$$

and for a right action, define

$$\xi_M(x) := \left.\frac{d}{dt}\right|_0 x \cdot \exp_G(t\xi).$$

Proposition 1.9.18. (Signs and transformation of fundamental fields) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary, let M be a smooth manifold with or without boundary, consider a smooth left or right action of G on M, and let $\xi, \eta \in \mathfrak{g}$.

(a) For a left action, the flow of ξ_M is

$$\Phi_t(x) = \exp_G(t\xi) \cdot x,$$

and

$$[\xi_M, \eta_M] = -[\xi, \eta]_M, \tag{1.25}$$
$$(a_g)_* \xi_M = (\mathbf{Ad}_g\, \xi)_M, \qquad a_g(x) := g \cdot x. \tag{1.26}$$

(b) For a right action, the flow of ξ_M is

$$\Psi_t(x) = x \cdot \exp_G(t\xi),$$

and

$$[\xi_M, \eta_M] = [\xi, \eta]_M, \tag{1.27}$$
$$(r_g)_* \xi_M = (\mathbf{Ad}_{g^{-1}}\, \xi)_M, \qquad r_g(x) := x \cdot g. \tag{1.28}$$

Proof. The flow formulas follow immediately from the action law and the group law for $t \mapsto \exp_G(t\xi)$. For a left action,

$$g \exp(t\xi) \cdot x = \exp(t \operatorname{Ad}_g \xi) \cdot (g \cdot x),$$

and differentiating at $t = 0$ gives (1.26). Applying this identity with $g = \exp(-t\xi)$ and using Lemma 1.9.9,

$$[\xi_M, \eta_M] = \left.\frac{d}{dt}\right|_0 (a_{\exp(-t\xi)})_* \eta_M = \left.\frac{d}{dt}\right|_0 (\mathbf{Ad}_{\mathbf{exp}(-t\xi)}\, \eta)_M = -[\xi, \eta]_M.$$

For a right action,

$$(x \cdot \exp(t\xi)) \cdot g = (x \cdot g) \cdot \exp(t \operatorname{Ad}_{g^{-1}} \xi),$$

which proves (1.28). Applying it to $g = \exp(-t\xi)$,

$$[\xi_M, \eta_M] = \left.\frac{d}{dt}\right|_0 (r_{\exp(-t\xi)})_* \eta_M = \left.\frac{d}{dt}\right|_0 (\mathbf{Ad}_{\mathbf{exp}(t\xi)}\, \eta)_M = [\xi, \eta]_M.$$

□

Definition 1.9.19. (Orbit, stabilizer, and orbit map) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary, let M be a smooth manifold with or without boundary, and consider a smooth left or right action of G on M. For a left action and $x \in M$, define

$$\mathcal{O}_x := G \cdot x, \qquad G_x := \{g \in G \mid g \cdot x = x\}, \qquad \vartheta_x \colon G \to M, \qquad \vartheta_x(g) = g \cdot x.$$

For a right action, use $\mathcal{O}_x = x \cdot G$, $G_x = \{g \mid x \cdot g = x\}$, and $\vartheta_x(g) = x \cdot g$. An action is *free* if all its stabilizers are trivial, and *transitive* if it has a single orbit.

Proposition 1.9.20. (Elementary properties of orbits) Let G be a finite-dimensional real Lie group whose underlying smooth manifold has no boundary, let M be a smooth manifold with or without boundary, and consider a smooth left or right action of G on M. The orbit map is smooth, G_x is a closed embedded Lie subgroup of G, and

$$(d\vartheta_x)_e\xi = \xi_M(x). \tag{1.29}$$

For a left action, $G_{g\cdot x} = gG_xg^{-1}$; for a right action, $G_{x\cdot g} = g^{-1}G_xg$. In particular, freeness and the dimension of the kernel of $(d\vartheta_x)_e$ are constant along each orbit.

Proof. Smoothness follows from that of the action with x fixed. The action laws show that G_x contains e and is closed under products and inverses. Moreover, $G_x = \vartheta_x^{-1}(\{x\})$ is closed because manifolds are Hausdorff; Theorem 1.9.15 gives it its embedded Lie subgroup structure. The identity (1.29) is exactly the definition of the fundamental field. Finally,

$$h\cdot(g\cdot x) = g\cdot x \iff g^{-1}hg\cdot x = x$$

proves the formula for a left action; the right-action formula follows from $(x\cdot g)\cdot h = x\cdot g$ by multiplying on the right by g^{-1}. The transformation of the kernels also follows from Proposition 1.9.18. □

Theorem 1.9.21. (Haar measure on compact groups) Let G be a compact finite-dimensional real Lie group whose underlying smooth manifold has no boundary. There exists a unique regular probability measure μ_G on G invariant under left and right translations:

$$(L_g)_*\mu_G = \mu_G = (R_g)_*\mu_G \qquad (g\in G).$$

For the construction of the normalized Haar volume form on a compact Lie group, see [94, Proposition 16.10]. Existence and uniqueness of Haar measure is the external result used here; right invariance follows by applying uniqueness of the left invariant probability measure to each right translation.

Proposition 1.9.22. (Averaging an inner product) Let G be a compact finite-dimensional real Lie group whose underlying smooth manifold has no boundary, and let $\rho\colon G\to GL(V)$ be a smooth representation on a finite-dimensional real vector space. Then V admits a G-invariant inner product.

Proof. Start with any inner product $\langle\cdot,\cdot\rangle_0$ and define

$$\langle v,w\rangle_G := \int_G \langle\rho(g)v,\rho(g)w\rangle_0\, d\mu_G(g).$$

The resulting form is bilinear, symmetric, and positive definite. For $h\in G$, right invariance of μ_G and $\rho(gh)=\rho(g)\rho(h)$ give

$$\langle\rho(h)v,\rho(h)w\rangle_G = \int_G \langle\rho(gh)v,\rho(gh)w\rangle_0\, d\mu_G(g) = \langle v,w\rangle_G.$$

□

Corollary 1.9.23. (Ad-invariant inner product) Let G be a compact finite-dimensional real Lie group whose underlying smooth manifold has no boundary. Its Lie algebra $\mathfrak{g}$ admits an inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ such that

$$\langle \mathrm{Ad}_g\,\xi, \mathrm{Ad}_g\,\eta\rangle_{\mathfrak{g}} = \langle \xi,\eta\rangle_{\mathfrak{g}} \qquad (g\in G).$$

Proof. Apply Proposition 1.9.22 to the representation $\rho = \mathrm{Ad}$. □

1.10 Tensor fields and differential forms

Tangent and cotangent spaces allow us to construct tensors pointwise. Assembling them over M gives tensor bundles, whose sections describe metrics, differential forms, curvature, and many other geometric fields. This final section of the chapter first recalls the necessary multilinear algebra and then organizes it smoothly over the manifold.

Definition 1.10.1. If $V_1,\dots,V_k$ and W are vector spaces, a function $F\colon V_1\times\cdots\times V_k\longrightarrow W$ is called **multilinear** if it is linear in each argument when the others are fixed. The space of multilinear functions is denoted by $L(V_1,\dots,V_k;W)$.

Definition 1.10.2. If $V_1,\dots,V_k$ and $W_1,\dots,W_l$ are vector spaces, and $\omega\in L(V_1,\dots,V_k;\mathbb{R})$ and $\eta\in L(W_1,\dots,W_l;\mathbb{R})$, define a function $\omega\otimes\eta\colon V_1\times\cdots\times V_k\times W_1\times\cdots\times W_l\longrightarrow\mathbb{R}$, called the **tensor product of ω and η**, by

$$\omega\otimes\eta(v_1,\dots,v_k,w_1,\dots,w_l)=\omega(v_1,\dots,v_k)\eta(w_1,\dots,w_l).$$

The space of multilinear functions is a vector space, and if the underlying vector spaces are finite dimensional, it is also finite dimensional:

Proposition 1.10.3. Let $V_1,\dots,V_k$ and W be real vector spaces. Then $L(V_1,\dots,V_k;W)$ is a real vector space under the operations

$$(F+G)(v_1,\dots,v_k)=F(v_1,\dots,v_k)+G(v_1,\dots,v_k),$$

$$(aF)(v_1,\dots,v_k)=a(F(v_1,\dots,v_k))$$

for each $F,G\in L(V_1,\dots,V_k;W)$ and each $a\in\mathbb{R}$.

Proof. Fixing all variables except the jth, the functions $F+G$ and aF are, respectively, a sum and a scalar multiple of linear maps $V_j\to W$; hence they are again linear in that variable. Since this holds for each j, both functions are multilinear. The vector space identities are checked pointwise and reduce to the corresponding identities in W. The zero function is the identity element, and the additive inverse of F is the function $-F$. □

Proposition 1.10.4. Let $V_1,\dots,V_k$ be vector spaces of dimensions $n_1,\dots,n_k$, respectively. For each $j\in\{1,\dots,k\},$ let $\{\mathbf{E}_1^{(j)},\dots,\mathbf{E}_{n_j}^{(j)}\}$ be a basis for V_j and let $\{\varepsilon^1_{(j)},\dots,\varepsilon^{n_j}_{(j)}\}$ be the corresponding dual basis for V_j^*. Then

$$\mathscr{B}=\{\varepsilon^{i_1}_{(1)}\otimes\cdots\otimes\varepsilon^{i_k}_{(k)}\mid 1\le i_1\le n_1,\dots,1\le i_k\le n_k\}$$

is a basis for $L(V_1,\dots,V_k;\mathbb{R})$, so this vector space has dimension $n_1\cdots n_k$.

Proof. Let $F\in L(V_1,\dots,V_k;\mathbb{R})$ and write

$$v_j=\sum_{i_j=1}^{n_j} v_j^{i_j}\mathbf{E}_{i_j}^{(j)}\qquad (1\le j\le k).$$

Successive application of linearity in each argument gives

$$F(v_1,\dots,v_k)=\sum_{i_1=1}^{n_1}\cdots\sum_{i_k=1}^{n_k}F(\mathbf{E}_{i_1}^{(1)},\dots,\mathbf{E}_{i_k}^{(k)})v_1^{i_1}\cdots v_k^{i_k}.$$

Since $v_j^{i_j}=\varepsilon_{(j)}^{i_j}(v_j)$, this equality expresses F as a linear combination of the elements of $\mathscr{B}$. To prove independence, suppose a combination with coefficients $c_{i_1\dots i_k}$ is zero. Evaluating it at $(\mathbf{E}_{r_1}^{(1)},\dots,\mathbf{E}_{r_k}^{(k)})$, the identities $\varepsilon_{(j)}^{i_j}(\mathbf{E}_{r_j}^{(j)})=\delta_{r_j}^{i_j}$ give $c_{r_1\dots r_k}=0$. All coefficients vanish, so $\mathscr{B}$ is a basis, and its cardinality is $n_1\cdots n_k$. □

Finally, the preceding definition of the tensor product of two multilinear functions allows us to define the tensor product of vector spaces as a vector space satisfying the following *universal property*:

Definition 1.10.5. (Universal property of the tensor product) Let $V_1,\dots,V_k$ be vector spaces. The **tensor product** $V_1\otimes\cdots\otimes V_k$ is a vector space equipped with a canonical multilinear map $\pi\colon V_1\times\cdots\times V_k\to V_1\otimes\cdots\otimes V_k$ having the following property: if $F\colon V_1\times\cdots\times V_k\longrightarrow X$ is any multilinear function taking values in a vector space X, there exists a unique linear function $\widetilde{F}\colon V_1\otimes\cdots\otimes V_k\longrightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} V_1\times\cdots\times V_k & \xrightarrow{F} & X \\ \downarrow{\scriptstyle\pi} & \nearrow{\scriptstyle \exists!\widetilde{F}} & \\ V_1\otimes\cdots\otimes V_k & & \end{array}$$

Remark 1.10.6. (Existence and uniqueness of the tensor product) Let $\mathbb{R}^{(V_1\times\cdots\times V_k)}$ be the free vector space generated by the symbols $[v_1,\dots,v_k]$. Take the quotient by the subspace generated, for each argument, by the relations

$$[v_1,\dots,av_j+bw_j,\dots,v_k]-a[v_1,\dots,v_j,\dots,v_k]-b[v_1,\dots,w_j,\dots,v_k].$$

The class of $[v_1,\dots,v_k]$ defines the map π, and the quotient relations ensure that it is multilinear. If F is multilinear, the assignment $[v_1,\dots,v_k]\mapsto F(v_1,\dots,v_k)$ vanishes on the subspace of relations and therefore induces a unique linear map $\widetilde{F}$. This proves existence. If two pairs (T,π_T) and (S,π_S) satisfy the universal property, it yields unique linear maps $T\to S$ and $S\to T$ preserving simple tensors; their compositions are the identity by the same uniqueness. Thus the tensor product is unique up to a unique isomorphism compatible with the canonical maps.

The following proposition relates the space of multilinear functions to the tensor product of vector spaces:

Proposition 1.10.7. Let $V_1,\dots,V_k$ be finite-dimensional real vector spaces. Then there is a canonical isomorphism

$$V_1^*\otimes\cdots\otimes V_k^* \cong L(V_1,\dots,V_k;\mathbb{R}).$$

Proof. The map

$$(\lambda_1,\dots,\lambda_k)\longmapsto\big[(v_1,\dots,v_k)\longmapsto\lambda_1(v_1)\cdots\lambda_k(v_k)\big]$$

is multilinear in $\lambda_1,\dots,\lambda_k$. The universal property therefore induces a canonical linear map

$$\Phi\colon V_1^*\otimes\cdots\otimes V_k^*\longrightarrow L(V_1,\dots,V_k;\mathbb{R}).$$

In the dual bases of the preceding proposition, Φ sends the tensor $\varepsilon^{i_1}_{(1)}\otimes\cdots\otimes\varepsilon^{i_k}_{(k)}$ to the element of the basis $\mathscr{B}$ with the same indices. Thus Φ sends a basis to a basis and is an isomorphism. The construction is independent of the bases, since Φ was defined solely by evaluation. □

We define tensors on a given vector space as follows:

Definition 1.10.8. Let V be a finite-dimensional vector space. Define the vector space of mixed tensors of type (k,l), k times contravariant and l times covariant, on V by

$$T^{(k,l)}(V) := \underbrace{V\otimes\cdots\otimes V}_{k\text{ times}}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{l\text{ times}}.$$

We are ready to define tensor bundles on a smooth manifold:

Definition 1.10.9. Let M be a smooth manifold with or without boundary. Define the **bundle of mixed tensors of type** (k,l) by

$$T^{(k,l)}(TM) := \bigsqcup_{p\in M} T^{(k,l)}(T_pM).$$

Remark 1.10.10. We have the following identifications:

(a) $T^{(0,0)}(TM) = M\times\mathbb{R}$.

(b) $T^{(0,1)}(TM) = T^*M$.

(c) $T^{(1,0)}(TM) = TM$.

We can equip tensor bundles with a smooth structure as follows:

Proposition 1.10.11. Let M be a smooth manifold with or without boundary and let $\dim M = n$. Then, for every pair of integers $k,l\geq 0$, the tensor bundle

$$T^{(k,l)}(TM)\longrightarrow M$$

admits a unique topology and a unique smooth structure making it a vector bundle of rank n^{k+l} over M for which the component trivializations induced by smooth charts of M are smooth.

Proof. We apply Lemma 1.7.8, first constructing the maps that will serve as local trivializations.
If $p \in U$ and (U, φ) is a smooth chart of M such that $p \in U$, every $A_p \in T^{(k,l)}(T_pM)$ can be written uniquely as

$$A_p = A^{i_1 \dots i_k}_{j_1 \dots j_l} \partial_{i_1}\Big|_p \otimes \cdots \otimes \partial_{i_k}\Big|_p \otimes \mathbf{d}x^{j_1}|_p \otimes \cdots \otimes \mathbf{d}x^{j_l}|_p,$$

where $\varphi = (x^1, \dots, x^n)$ and each $A^{i_1 \dots i_k}_{j_1 \dots j_l} \in \mathbb{R}$ is a component of A_p with respect to the tensor frame induced by the chart. When A_p varies as a section $\mathbf{A}$ over U, these components become functions $A^{i_1 \dots i_k}_{j_1 \dots j_l} \colon U \to \mathbb{R}$.
Define $\pi \colon T^{(k,l)}(TM) \longrightarrow M$ by $\pi(A_p) = p$.
Take an open cover $\{U_\alpha\}_{\alpha \in J}$ by smooth coordinate chart domains for M. For each $\alpha \in J$, define $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^{n^{k+l}}$ by $\Phi_\alpha(A_p) = (p, A^{i_1 \dots i_k}_{j_1 \dots j_l})$. The map Φ_α is bijective because the function $\widetilde{\Phi}_\alpha \colon U_\alpha \times \mathbb{R}^{n^{k+l}} \longrightarrow \pi^{-1}(U_\alpha)$ given by

$$\widetilde{\Phi}_\alpha(p, A^{i_1 \dots i_k}_{j_1 \dots j_l}) = A_p$$

is the inverse of Φ_α.
For each $p \in U_\alpha$, $\Phi_\alpha|_{T^{(k,l)}(T_pM)} \colon T^{(k,l)}(T_pM) \longrightarrow \{p\} \times \mathbb{R}^{n^{k+l}}$ is bijective because

$$\Phi_\alpha|_{T^{(k,l)}(T_pM)}(T^{(k,l)}(T_pM)) = \{p\} \times \mathbb{R}^{n^{k+l}}$$

and it is the restriction of a bijective function to its image. Identify the fiber $\{p\} \times \mathbb{R}^{n^{k+l}}$ with $\mathbb{R}^{n^{k+l}}$ by the second projection. It suffices to prove that the second component of Φ_α is linear:
Let $a \in \mathbb{R}$ and $A_p, B_p \in T^{(k,l)}(T_pM)$, with

$$A_p = A^{i_1 \dots i_k}_{j_1 \dots j_l}(p) \partial_{i_1}\Big|_p \otimes \cdots \otimes \partial_{i_k}\Big|_p \otimes \mathbf{d}x^{j_1}|_p \otimes \cdots \otimes \mathbf{d}x^{j_l}|_p$$

and

$$B_p = B^{i_1 \dots i_k}_{j_1 \dots j_l}(p) \partial_{i_1}\Big|_p \otimes \cdots \otimes \partial_{i_k}\Big|_p \otimes \mathbf{d}x^{j_1}|_p \otimes \cdots \otimes \mathbf{d}x^{j_l}|_p,$$

We have

$$\Phi_\alpha(aA_p + B_p) = \left(p, aA^{i_1 \dots i_k}_{j_1 \dots j_l} + B^{i_1 \dots i_k}_{j_1 \dots j_l}\right).$$

Consequently,

$$\operatorname{pr}_2 \Phi_\alpha(aA_p + B_p) = a \operatorname{pr}_2 \Phi_\alpha(A_p) + \operatorname{pr}_2 \Phi_\alpha(B_p),$$

which is the required linearity.
Therefore, $\Phi_\alpha|_{T^{(k,l)}(T_pM)} \colon T^{(k,l)}(T_pM) \longrightarrow \{p\} \times \mathbb{R}^{n^{k+l}}$ is a vector space isomorphism.
Finally, if $\alpha, \beta \in J$ are such that $U_\alpha \cap U_\beta \neq \varnothing$, we must compute the map $\Phi_\alpha \circ \Phi_\beta^{-1}$.
Suppose that $\varphi_\alpha = (x^1, \dots, x^n)$ and $\varphi_\beta = (y^1, \dots, y^n)$.
If $(p, B^{a_1 \dots a_k}_{b_1 \dots b_l}) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}}$, we have

$$(\Phi_\alpha \circ \Phi_\beta^{-1})(p, B^{a_1 \dots a_k}_{b_1 \dots b_l}) = \Phi_\alpha \left(B^{a_1 \dots a_k}_{b_1 \dots b_l} \partial_{a_1}\Big|_p \otimes \cdots \otimes \partial_{a_k}\Big|_p \otimes \mathbf{d}y^{b_1}|_p \otimes \cdots \otimes \mathbf{d}y^{b_l}|_p \right)$$

$$= \left(p, B^{a_1 \dots a_k}_{b_1 \dots b_l} \left(\frac{\partial x^{i_1}}{\partial y^{a_1}}\Big|_p \right) \dots \left(\frac{\partial x^{i_k}}{\partial y^{a_k}}\Big|_p \right) \left(\frac{\partial y^{b_1}}{\partial x^{j_1}}\Big|_p \right) \dots \left(\frac{\partial y^{b_l}}{\partial x^{j_l}}\Big|_p \right) \right)$$

where we have used

$$B^{a_1\dots a_k}_{b_1\dots b_l}\left(\frac{\partial x^{i_1}}{\partial y^{a_1}}\Big|_p\right)\dots\left(\frac{\partial x^{i_k}}{\partial y^{a_k}}\Big|_p\right)\left(\frac{\partial y^{b_1}}{\partial x^{j_1}}\Big|_p\right)\dots\left(\frac{\partial y^{b_l}}{\partial x^{j_l}}\Big|_p\right)\partial_{i_1}\Big|_p\otimes\dots\otimes\partial_{i_k}\Big|_p\otimes\mathbf{d}x^{j_1}|_p\otimes\dots\otimes\mathbf{d}x^{j_l}|_p$$

$$= B^{a_1\dots a_k}_{b_1\dots b_l}\partial_{a_1}\Big|_p\otimes\dots\otimes\partial_{a_k}\Big|_p\otimes\mathbf{d}y^{b_1}|_p\otimes\dots\otimes\mathbf{d}y^{b_l}|_p$$

which follows from the chain rule.
Now note that

$$\left(p, B^{a_1\dots a_k}_{b_1\dots b_l}\left(\frac{\partial x^{i_1}}{\partial y^{a_1}}\Big|_p\right)\dots\left(\frac{\partial x^{i_k}}{\partial y^{a_k}}\Big|_p\right)\left(\frac{\partial y^{b_1}}{\partial x^{j_1}}\Big|_p\right)\dots\left(\frac{\partial y^{b_l}}{\partial x^{j_l}}\Big|_p\right)\right) = (p, \tau_{\alpha\beta}(p)B^{a_1\dots a_k}_{b_1\dots b_l})$$

where

$$\tau_{\alpha\beta}(p) = \left(\left(\frac{\partial x^{i_1}}{\partial y^{a_1}}\Big|_p\right)\dots\left(\frac{\partial x^{i_k}}{\partial y^{a_k}}\Big|_p\right)\left(\frac{\partial y^{b_1}}{\partial x^{j_1}}\Big|_p\right)\dots\left(\frac{\partial y^{b_l}}{\partial x^{j_l}}\Big|_p\right)\right) \in GL(n^{k+l},\mathbb{R})$$

is smooth because its component functions are smooth.
By Lemma 1.7.8, there exist a unique topology and a unique smooth structure making $\pi\colon T^{(k,l)}(TM)\longrightarrow M$ a smooth vector bundle of rank n^{k+l} with Φ_α as smooth local trivializations for each $\alpha\in J$. The transition formula also holds when comparing a chart from this cover with any other smooth chart of M. Thus all component trivializations induced by smooth charts are smooth, and this property characterizes the constructed structure. □

Corollary 1.10.12. Let M be a smooth manifold with or without boundary of dimension n. If (U,φ) is a smooth chart of M with $\varphi=(x^1,\dots,x^n)$, then $\widetilde{\varphi}\colon \pi^{-1}(U)\longrightarrow\varphi(U)\times\mathbb{R}^{n^{k+l}}$, given by

$$\widetilde{\varphi}\left(A^{i_1\dots i_k}_{j_1\dots j_l}\partial_{i_1}\Big|_p\otimes\dots\otimes\partial_{i_k}\Big|_p\otimes\mathbf{d}x^{j_1}|_p\otimes\dots\otimes\mathbf{d}x^{j_l}|_p\right) = (x^1(p),\dots,x^n(p),A^{i_1\dots i_k}_{j_1\dots j_l})$$

makes $(\pi^{-1}(U),\widetilde{\varphi})$ a smooth chart for $T^{(k,l)}(TM)$.

Proof. By Proposition 1.10.11, $\Phi\colon\pi^{-1}(U)\longrightarrow U\times\mathbb{R}^{n^{k+l}}$, given by

$$\Phi\left(A^{i_1\dots i_k}_{j_1\dots j_l}\partial_{i_1}\Big|_p\otimes\dots\otimes\partial_{i_k}\Big|_p\otimes\mathbf{d}x^{j_1}|_p\otimes\dots\otimes\mathbf{d}x^{j_l}|_p\right) = (p,A^{i_1\dots i_k}_{j_1\dots j_l})$$

is a smooth local trivialization of the tensor bundle $T^{(k,l)}(TM)$. Moreover, Proposition 1.7.18 says that

$$\left(\partial_{i_1}\otimes\dots\otimes\partial_{i_k}\otimes\mathbf{d}x^{j_1}\otimes\dots\otimes\mathbf{d}x^{j_l}\right)$$

is a smooth local frame defined on U and associated with the local trivialization Φ. Thus Corollary 1.7.20 shows that the map $\widetilde{\varphi}\colon\pi^{-1}(U)\longrightarrow\varphi(U)\times\mathbb{R}^{n^{k+l}}$ makes $(\pi^{-1}(U),\widetilde{\varphi})$ a smooth chart of $T^{(k,l)}(TM)$. □

This construction of the product of TM and T^*M extends to the product of two or more vector bundles, as the reader can prove.

Exercise 1.10.13. Let M be a smooth manifold with or without boundary and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be two smooth vector bundles of ranks k and l, respectively. Prove that the fiberwise tensor product

$$\mathbf{E} \otimes \mathbf{F} \longrightarrow M, \qquad (\mathbf{E} \otimes \mathbf{F})_p := \mathbf{E}_p \otimes \mathbf{F}_p,$$

admits a unique topology and a unique smooth structure making it a smooth vector bundle of rank kl over M for which the tensor trivializations induced by smooth local frames of $\mathbf{E}$ and $\mathbf{F}$ are smooth.

There is a relation between Hom bundles and tensor products.

Proposition 1.10.14. Let M be a smooth manifold with or without boundary and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles of ranks k and l, respectively. Then there is a natural vector bundle isomorphism over M

$$\operatorname{Hom}(\mathbf{E}, \mathbf{F}) \cong \mathbf{E}^* \otimes \mathbf{F}.$$

Proof. Define a bundle map $\Phi\colon \operatorname{Hom}(\mathbf{E}, \mathbf{F}) \longrightarrow \mathbf{E}^* \otimes \mathbf{F}$ fiberwise by the canonical identification

$$\Phi_p\colon \operatorname{Hom}(\mathbf{E}_p, \mathbf{F}_p) \longrightarrow \mathbf{E}_p^* \otimes \mathbf{F}_p, \qquad \Phi_p(T_p) = \sum_{i=1}^{k} e^i \otimes T_p(e_i),$$

where $(e_i)_{i=1}^k$ is a basis for $\mathbf{E}_p$ and $(e^i)_{i=1}^k$ is the dual basis in $\mathbf{E}_p^*$. This definition is independent of the chosen basis, is linear on each fiber, and covers the identity of M.
To verify that Φ is smooth, work in local trivializations. Let $U_\alpha \subset M$ be an open set with smooth local frames $(\boldsymbol{\sigma}_{\alpha,i})_{i=1}^k$ of $\mathbf{E}$ and $(\boldsymbol{\tau}_{\alpha,j})_{j=1}^l$ of $\mathbf{F}$; denote the dual frame of $\mathbf{E}^*$ on U_α by $(\boldsymbol{\sigma}_\alpha^i)_{i=1}^k$.
In these trivializations, every $T_p \in \operatorname{Hom}(\mathbf{E}_p, \mathbf{F}_p)$ is determined by an associated matrix

$$T_\alpha(p) = (T_\alpha(p)_i^j) \in M_{l\times k}(\mathbb{R}), \qquad T_p(\boldsymbol{\sigma}_{\alpha,i}(p)) = \sum_{j=1}^{l} T_\alpha(p)_i^j \boldsymbol{\tau}_{\alpha,j}(p).$$

The trivialization of $\operatorname{Hom}(\mathbf{E}, \mathbf{F})$ on U_α is given by

$$\Theta_\alpha^{\operatorname{Hom}}(T_p) = (p, T_\alpha(p)).$$

On the other hand, in $\mathbf{E}^* \otimes \mathbf{F}$, any element of the fiber $(\mathbf{E}^* \otimes \mathbf{F})_p$ can be written as

$$\sum_{i=1}^{k}\sum_{j=1}^{l} S_i^j \boldsymbol{\sigma}_\alpha^i(p) \otimes \boldsymbol{\tau}_{\alpha,j}(p),$$

and the associated trivialization is

$$\Theta_\alpha^{\mathbf{E}^*\otimes\mathbf{F}}\Big(\sum_{\substack{1\le i\le k\\ 1\le j\le l}} S_i^j \boldsymbol{\sigma}_\alpha^i(p) \otimes \boldsymbol{\tau}_{\alpha,j}(p)\Big) = (p, (S_i^j)).$$

Now apply Φ_p to T_p. By definition,

$$\Phi_p(T_p) = \sum_{i=1}^{k} \boldsymbol{\sigma}_\alpha^i(p) \otimes T_p(\boldsymbol{\sigma}_{\alpha,i}(p)) = \sum_{i=1}^{k} \boldsymbol{\sigma}_\alpha^i(p) \otimes \Big(\sum_{j=1}^{l} T_\alpha(p)_i^j \boldsymbol{\tau}_{\alpha,j}(p) \Big).$$

Reordering the sum,

$$\Phi_p(T_p) = \sum_{i=1}^{k} \sum_{j=1}^{l} T_\alpha(p)_i^j \boldsymbol{\sigma}_\alpha^i(p) \otimes \boldsymbol{\tau}_{\alpha,j}(p).$$

In the trivialization of $\mathbf{E}^* \otimes \mathbf{F}$, this corresponds exactly to

$$\Theta_\alpha^{\mathbf{E}^*\otimes\mathbf{F}}\big(\Phi_p(T_p)\big) = (p, (T_\alpha(p)_i^j)).$$

In other words, passing through the two local trivializations, the map

$$\Theta_\alpha^{\mathbf{E}^*\otimes\mathbf{F}} \circ \Phi \circ \big(\Theta_\alpha^{\mathrm{Hom}}\big)^{-1} : U_\alpha \times M_{l\times k}(\mathbb{R}) \to U_\alpha \times M_{l\times k}(\mathbb{R})$$

is simply given by

$$(p, (T_i^j)) \longmapsto (p, (T_i^j)),$$

which is the identity, showing that Φ is smooth.
Thus Φ is smooth and locally an isomorphism of trivializations. Since each Φ_p is linear and bijective, Φ is a bijective bundle homomorphism. By Exercise 1.7.26, we conclude that

$$\Phi : \mathrm{Hom}(\mathbf{E}, \mathbf{F}) \xrightarrow{\cong} \mathbf{E}^* \otimes \mathbf{F}$$

is a smooth bundle isomorphism of rank kl.
The naturality of the construction follows directly from the fact that everything was defined fiberwise using the natural isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$. □

Sections of tensor bundles are tensor fields. They describe, among other objects, metrics, differential forms, curvature, and tensors arising in physical models.

Definition 1.10.15. Let M be a smooth manifold with or without boundary. A section

$$\mathbf{A} \in \Gamma\big(T^{(k,l)}(TM)\big)$$

is called a **tensor field of type** (k, l) on M.
In particular:

- If $k = 0$, we say that $\mathbf{A}$ is a l-**covariant tensor field**.
- If $l = 0$, we say that $\mathbf{A}$ is a k-**contravariant tensor field**.

Remark 1.10.16. We use the following classical notation:

- We denote the space of **vector fields** on M by $\mathfrak{X}(M) := \Gamma(T^{(1,0)}(TM))$.

- We denote the space of **1-forms** on M by $\Omega^1(M) := \Gamma(T^{(0,1)}(TM))$.

The notation $\Omega^1(M)$ will be explained in greater detail below.

The following criteria determine when a tensor field is smooth and when a map is induced by a smooth tensor field.

Proposition 1.10.17. (Smoothness criterion for tensor fields) Let M be a smooth manifold with or without boundary, and let $\mathbf{F}\colon M \longrightarrow T^{(k,l)}(TM)$ be a rough (k,l)-tensor field. Then $\mathbf{F}$ is smooth on an open set $U \subseteq M$ if and only if, for any smooth covector fields $\omega^1,\dots,\omega^k$ and smooth vector fields $\mathbf{X}_1,\dots,\mathbf{X}_l$ defined on U, the real-valued function

$$\mathbf{F}(\omega^1,\dots,\omega^k,\mathbf{X}_1,\dots,\mathbf{X}_l)(p) = \mathbf{F}_p(\omega^1|_p,\dots,\omega^k|_p,\mathbf{X}_1|_p,\dots,\mathbf{X}_l|_p)$$

is smooth on U.

Proof. If $\mathbf{F}$ is smooth, its evaluation on smooth fields is a smooth function, since in coordinates it is a finite sum of products of the smooth components of $\mathbf{F}$ and of the fields on which it is evaluated.
Conversely, suppose that every indicated evaluation is smooth, and fix $p \in U$. Choose a chart $(V, x^1,\dots,x^n)$ with $p \in V \subseteq U$, and an open set W such that $p \in W \Subset V$. Take $\rho \in C^\infty_c(V)$ equal to one on W. The fields

$$\rho\,\mathbf{d}x^1,\dots,\rho\,\mathbf{d}x^n, \qquad \rho\,\partial_1,\dots,\rho\,\partial_n$$

extend by zero to smooth fields defined on U. For indices $i_1,\dots,i_k,j_1,\dots,j_l$, the hypothesis implies that

$$\mathbf{F}\big(\rho\,\mathbf{d}x^{i_1},\dots,\rho\,\mathbf{d}x^{i_k},\rho\,\partial_{j_1},\dots,\rho\,\partial_{j_l}\big)$$

is smooth on U. On W, where $\rho = 1$, this function is precisely the coordinate component

$$F^{i_1\dots i_k}_{j_1\dots j_l} := \mathbf{F}\big(\mathbf{d}x^{i_1},\dots,\mathbf{d}x^{i_k},\partial_{j_1},\dots,\partial_{j_l}\big).$$

Thus all components of $\mathbf{F}$ are smooth in a neighborhood of p. Since p was arbitrary, $\mathbf{F}$ is smooth on U. □

Lemma 1.10.18. (Tensor characterization lemma) Let M be a smooth manifold with or without boundary. A map

$$F : \underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{k\text{ times}}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{l\text{ times}} \longrightarrow C^\infty(M)$$

is induced by a smooth (k,l)-tensor field if and only if it is multilinear over $C^\infty(M)$.
For the vector-valued statement, a map

$$F : \underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{k\text{ times}}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{l\text{ times}} \longrightarrow \mathfrak{X}(M)$$

is induced by a smooth $(k+1,l)$-tensor field if and only if it is multilinear over $C^\infty(M)$.

Proof. We first prove a locality observation valid for smooth sections of any finite-rank vector bundle. Let $\mathcal{F}$ be a map that is $C^\infty(M)$-linear in one of its arguments, and let $\mathbf{s}$ be a section that vanishes in a neighborhood of p. By Proposition 1.3.6, there exists $\vartheta \in C^\infty(M)$ with $\vartheta(p) = 1$ and support contained in that neighborhood. Then

$$\mathcal{F}(\dots,\mathbf{s},\dots)(p) = \vartheta(p)\mathcal{F}(\dots,\mathbf{s},\dots)(p) = \mathcal{F}(\dots,\vartheta\mathbf{s},\dots)(p) = 0.$$

If two sections $\mathbf{s}$ and $\widetilde{\mathbf{s}}$ have the same value at p, choose a trivialization around p and a cutoff function ρ that equals one in a neighborhood of p and has support within the trivialization. The section $\mathbf{r} := \mathbf{s} - \widetilde{\mathbf{s}}$ satisfies $\mathbf{r}(p) = 0$. The term $(1-\rho)\mathbf{r}$ vanishes near p, whereas, after extending the cutoff coefficients and the elements of a local frame by zero, we can write

$$\rho\mathbf{r} = \sum_{i=1}^{m} a_i \mathbf{e}_i, \qquad a_i \in C^\infty(M), \qquad a_i(p) = 0.$$

The preceding locality observation and $C^\infty(M)$-linearity give $\mathcal{F}(\dots,\mathbf{r},\dots)(p) = 0$. Thus the value at p depends only on the values at p of all the arguments.

Now suppose that

$$F : \Omega^1(M)^k \times \mathfrak{X}(M)^l \longrightarrow C^\infty(M)$$

is $C^\infty(M)$-multilinear. Given $\alpha^1,\dots,\alpha^k \in T_p^*M$ and $v_1,\dots,v_l \in T_pM$, choose smooth extensions and define

$$\mathbf{T}_p(\alpha^1,\dots,\alpha^k,v_1,\dots,v_l) := F(\widetilde{\boldsymbol{\alpha}}^1,\dots,\widetilde{\boldsymbol{\alpha}}^k,\widetilde{\mathbf{v}}_1,\dots,\widetilde{\mathbf{v}}_l)(p).$$

The preceding observation shows that this definition is independent of the extensions; the multilinearity of $\mathbf{T}_p$ is inherited from that of F. In a chart, choose global extensions of the coordinate coframe and frame that agree with them on a smaller open set. The components of $\mathbf{T}$ on that open set are the restrictions of the smooth functions obtained by evaluating F on those extensions. Hence $p \mapsto \mathbf{T}_p$ is a smooth (k,l)-tensor field and, by construction, induces F. Uniqueness follows by evaluating the field on arbitrary fiber values. Conversely, evaluating a smooth tensor field on smooth sections produces a smooth function and is $C^\infty(M)$-multilinear.

If the codomain of F is $\mathfrak{X}(M)$, we instead define

$$\mathbf{T}_p(\alpha^1,\dots,\alpha^k,v_1,\dots,v_l) := F(\widetilde{\boldsymbol{\alpha}}^1,\dots,\widetilde{\boldsymbol{\alpha}}^k,\widetilde{\mathbf{v}}_1,\dots,\widetilde{\mathbf{v}}_l)(p) \in T_pM.$$

The same locality argument gives independence of the extensions. The components of the vector field on the right-hand side are smooth; evaluating on local coframes and frames shows that $\mathbf{T}$ is a smooth $(k+1,l)$-tensor field. The converse implication and uniqueness follow by evaluation, exactly as in the scalar case. □

We now define the pullback of a tensor field.

Definition 1.10.19. Let M and N be smooth manifolds with or without boundary, and suppose that $F\colon M \longrightarrow N$ is a smooth map. For each point $p \in M$ and each k-covariant tensor $\alpha \in T^{(0,k)}(T_{F(p)}N)$, we define a tensor

$$dF_p^*(\alpha) \in T^{(0,k)}(T_pM),$$

called the **pullback of α by F at p**, by

$$dF_p^*(\alpha)(v_1,\dots,v_k) = \alpha\big(dF_p(v_1),\dots,dF_p(v_k)\big),$$

for any $v_1,\dots,v_k \in T_pM$.
If $\mathbf{A}$ is a k-covariant tensor field on N, that is, $\mathbf{A} \in \Gamma(T^{(0,k)}(TN))$, we define the tensor field on M

$$F^*\mathbf{A} \in \Gamma\big(T^{(0,k)}(TM)\big),$$

called the **pullback of A by** F, by

$$(F^*\mathbf{A})_p = dF_p^*(\mathbf{A}_{F(p)}),$$

that is,

$$(F^*\mathbf{A})_p(v_1,\dots,v_k) = \mathbf{A}_{F(p)}\big(dF_p(v_1),\dots,dF_p(v_k)\big),$$

for each $v_1,\dots,v_k \in T_pM$.

The pullback has the following properties:

Proposition 1.10.20. Let M, N, and P be smooth manifolds with or without boundary. Suppose that $F\colon M \longrightarrow N$ and $G\colon N \longrightarrow P$ are smooth maps. Let $k,l,r \in \mathbb{N}_0$, $\mathbf{A},\mathbf{B} \in \Gamma(T^{(0,k)}(TN))$, $\mathbf{C} \in \Gamma(T^{(0,l)}(TN))$, $\mathbf{D} \in \Gamma(T^{(0,r)}(TP))$, and $f \in C^\infty(N)$.

(a) $F^*(f\mathbf{A}) = (f\circ F)F^*\mathbf{A}$.

(b) $F^*(\mathbf{A}\otimes\mathbf{C}) = F^*\mathbf{A}\otimes F^*\mathbf{C}$.

(c) $F^*(\mathbf{A}+\mathbf{B}) = F^*\mathbf{A} + F^*\mathbf{B}$.

(d) $F^*\mathbf{A}$ is a smooth tensor field.

(e) $(G\circ F)^*\mathbf{D} = F^*(G^*\mathbf{D})$.

(f) $(\mathrm{Id}_N)^*\mathbf{A} = \mathbf{A}$.

Proof. The first equality follows by evaluating on $p \in M$:

$$\big(F^*(f\mathbf{A})\big)_p(v_1,\dots,v_k) = f(F(p))\,\mathbf{A}_{F(p)}(dF_pv_1,\dots,dF_pv_k).$$

The definition of the tensor product proves (b), and the linearity of dF_p^* proves (c). For (d), if in coordinates

$$\mathbf{A} = A_{i_1\dots i_k}\,\mathbf{d}y^{i_1}\otimes\cdots\otimes\mathbf{d}y^{i_k},$$

then

$$F^*\mathbf{A} = (A_{i_1\dots i_k}\circ F)\,d(y^{i_1}\circ F)\otimes\cdots\otimes d(y^{i_k}\circ F),$$

whose coefficients are smooth.
Substituting the chain rule $d(G\circ F)_p = dG_{F(p)}\circ dF_p$ into the pointwise definition gives $(G\circ F)^*\mathbf{D} = F^*(G^*\mathbf{D})$.
Finally, $d(\mathrm{Id}_N)_q = \mathrm{Id}_{T_qN}$, which proves (f). □

A related concept, the pushforward, requires F to be a diffeomorphism as well.
To introduce it, we first define the pushforward for vector fields:

Definition 1.10.21. Let M and N be smooth manifolds with or without boundary, and let $F\colon M \longrightarrow N$ be a diffeomorphism. For each $\mathbf{X} \in \mathfrak{X}(M)$, we define $F_*\mathbf{X} := dF \circ \mathbf{X} \circ F^{-1}$ and call it the **pushforward** of $\mathbf{X}$ by F. Pointwise, we have

$$(F_*\mathbf{X})_q = dF_{F^{-1}(q)}(\mathbf{X}_{F^{-1}(q)})$$

for each $q \in N$.

The most general form of pullback and pushforward is given by the following proposition:

Proposition 1.10.22. Let M, N, and P be smooth manifolds with or without boundary, and let $F\colon M \longrightarrow N$ and $G\colon N \longrightarrow P$ be diffeomorphisms. For each $k, l \in \mathbb{N}_0$, there are isomorphisms associated with F, called the **pushforward** and **pullback**,

$$F_*\colon \Gamma(T^{(k,l)}(TM)) \longrightarrow \Gamma(T^{(k,l)}(TN)), \qquad F^*\colon \Gamma(T^{(k,l)}(TN)) \longrightarrow \Gamma(T^{(k,l)}(TM)),$$

such that F^* agrees with the pullback of covariant tensor fields, F_* agrees with the pushforward of vector fields, and the following conditions hold:

(a) $F_* = (F^*)^{-1}$.

(b) $F^*(\mathbf{A} \otimes \mathbf{B}) = F^*\mathbf{A} \otimes F^*\mathbf{B}$ for any tensor fields $\mathbf{A}$ and $\mathbf{B}$ on N.

(c) $(G \circ F)_* = G_* \circ F_*$.

(d) $(G \circ F)^* = F^* \circ G^*$.

(e) $(\mathrm{Id}_M)^* = (\mathrm{Id}_M)_* = \mathrm{Id}\colon \Gamma(T^{(k,l)}(TM)) \longrightarrow \Gamma(T^{(k,l)}(TM))$.

(f) $F^*\big(\mathbf{A}(\mathbf{X}_1, \ldots, \mathbf{X}_k)\big) = (F^*\mathbf{A})\big((F^{-1})_*\mathbf{X}_1, \ldots, (F^{-1})_*\mathbf{X}_k\big)$ for each $\mathbf{A} \in \Gamma(T^{(0,k)}(TN))$ and any $\mathbf{X}_1, \ldots, \mathbf{X}_k \in \mathfrak{X}(N)$.

Proof. Let $q \in N$, set $p = F^{-1}(q)$, and let $\mathbf{T} \in \Gamma(T^{(k,l)}(TM))$. For $\beta^1, \ldots, \beta^k \in T^*_qN$ and $w_1, \ldots, w_l \in T_qN$, define

$$(F_*\mathbf{T})_q(\beta^1, \ldots, \beta^k, w_1, \ldots, w_l) := \mathbf{T}_p\big(dF_p^*\beta^1, \ldots, dF_p^*\beta^k, (dF_p)^{-1}w_1, \ldots, (dF_p)^{-1}w_l\big).$$

In charts, its components are finite sums of products of components of $\mathbf{T}$, dF, and dF^{-1}; it therefore defines a smooth field. For a vector field, it agrees with the pushforward already defined. Set $F^* := (F^{-1})_*$; on covariant tensors, this is the usual definition of the pullback.
Substituting the identity $d(F^{-1})_q = (dF_p)^{-1}$ into the formula shows that F_* and F^* are inverses. The same formula and the definition of the tensor product prove (b). The chain rule and

$$\big(d(G \circ F)_p\big)^{-1} = (dF_p)^{-1} \circ (dG_{F(p)})^{-1}$$

give $(G\circ F)_* = G_*\circ F_*$. Applying this equality to the inverses yields

$$(G\circ F)^* = (F^{-1}\circ G^{-1})_* = F^*\circ G^*.$$

The identity case is immediate from the formula.
Finally, if $p\in M$, the right-hand side of (f), evaluated on p, is

$$\mathbf{A}_{F(p)}\big(dF_p(dF_p)^{-1}\mathbf{X}_1(F(p)),\dots,dF_p(dF_p)^{-1}\mathbf{X}_k(F(p))\big),$$

which agrees with $\mathbf{A}(\mathbf{X}_1,\dots,\mathbf{X}_k)(F(p))$, that is, with the left-hand side evaluated on p. □

Definition 1.10.23. (Interior multiplication) Let M be a smooth manifold with or without boundary, and let $k,l\geq 0$ be nonnegative integers.

(a) Let $\mathbf{T}_p\in T^{(k,l+1)}(T_pM)$ and $\mathbf{V}\in T_pM$. We define the *interior multiplication* of $\mathbf{V}$ into $\mathbf{T}_p$ to be the tensor

$$\iota_{\mathbf{V}}\mathbf{T}_p\in T^{(k,l)}(T_pM),$$

given by

$$(\iota_{\mathbf{V}}\mathbf{T}_p)(\omega^1,\dots,\omega^k,X_1,\dots,X_l) := \mathbf{T}_p(\omega^1,\dots,\omega^k,\mathbf{V},X_1,\dots,X_l),$$

for all $\omega^1,\dots,\omega^k\in\mathbf{T}^*_pM$ and $X_1,\dots,X_l\in T_pM$.

(b) If $\mathbf{T}\in\Gamma\big(T^{(k,l+1)}(TM)\big)$ and $\mathbf{V}\in\mathfrak{X}(M)$, we define the section

$$\iota_{\mathbf{V}}\mathbf{T}\in\Gamma\big(T^{(k,l)}(TM)\big),$$

by

$$(\iota_{\mathbf{V}}\mathbf{T})(p) := \iota_{\mathbf{V}(p)}\big(\mathbf{T}(p)\big),\qquad p\in M.$$

Proposition 1.10.24. (Interior multiplication defines a tensor field) Let M be a smooth manifold with or without boundary, $\mathbf{T}\in\Gamma\big(T^{(k,l+1)}(TM)\big)$, and $\mathbf{V}\in\mathfrak{X}(M)$. Then the assignment

$$(\iota_{\mathbf{V}}\mathbf{T})(\omega^1,\dots,\omega^k,X_1,\dots,X_l) := \mathbf{T}(\omega^1,\dots,\omega^k,\mathbf{V},X_1,\dots,X_l)$$

defines a smooth (k,l)-tensor field $\iota_{\mathbf{V}}\mathbf{T}\in\Gamma\big(T^{(k,l)}(TM)\big)$.

Proof. Consider the map

$$F_{\mathbf{V},\mathbf{T}} : \underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{k\text{ times}}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{l\text{ times}}\longrightarrow C^\infty(M).$$

defined by

$$F_{\mathbf{V},\mathbf{T}}(\omega^1,\dots,\omega^k,X_1,\dots,X_l) = \mathbf{T}(\omega^1,\dots,\omega^k,\mathbf{V},X_1,\dots,X_l).$$

Since $\mathbf{T}$ is a smooth $(k,l+1)$-tensor field, the map

$$(\omega^1,\dots,\omega^k,Y,X_1,\dots,X_l)\longmapsto \mathbf{T}(\omega^1,\dots,\omega^k,Y,X_1,\dots,X_l)$$

is multilinear over $C^\infty(M)$ in its arguments. Fixing $Y=\mathbf{V}$, it follows that $F_{\mathbf{V},\mathbf{T}}$ is multilinear over $C^\infty(M)$. By Lemma 1.10.18, there exists a smooth (k,l)-tensor field $\mathbf{S}\in\Gamma\big(T^{(k,l)}(TM)\big)$ such that

$$\mathbf{S}(\omega^1,\dots,\omega^k,X_1,\dots,X_l)=F_{\mathbf{V},\mathbf{T}}(\omega^1,\dots,\omega^k,X_1,\dots,X_l).$$

Pointwise, we have

$$\mathbf{S}(p)(\omega^1_p,\dots,\omega^k_p,X_{1,p},\dots,X_{l,p})=\mathbf{T}_p(\omega^1_p,\dots,\omega^k_p,\mathbf{V}_p,X_{1,p},\dots,X_{l,p})=(\iota_{\mathbf{V}(p)}\mathbf{T}_p)(\omega^1_p,\dots,X_{l,p}),$$

so $\mathbf{S}=\iota_{\mathbf{V}}\mathbf{T}$ as sections. Therefore $\iota_{\mathbf{V}}\mathbf{T}$ is a smooth (k,l)-tensor field. □

Remark 1.10.25. If $\mathbf{T}\in\Gamma\big(T^{(k,l+1)}(TM)\big)$ and $\mathbf{V}\in\mathfrak{X}(M)$ with $\mathbf{V}=V^a\mathbf{E}_a$, then the interior multiplication $\iota_{\mathbf{V}}\mathbf{T}$ has local components

$$\begin{aligned}(\iota_{\mathbf{V}}\mathbf{T})^{i_1\dots i_k}_{j_1\dots j_l}&=(\iota_{\mathbf{V}}\mathbf{T})\big(\boldsymbol{\varepsilon}^{i_1},\dots,\boldsymbol{\varepsilon}^{i_k},\mathbf{E}_{j_1},\dots,\mathbf{E}_{j_l}\big)\\&=\mathbf{T}\big(\boldsymbol{\varepsilon}^{i_1},\dots,\boldsymbol{\varepsilon}^{i_k},\mathbf{V},\mathbf{E}_{j_1},\dots,\mathbf{E}_{j_l}\big)\\&=\mathbf{T}\big(\boldsymbol{\varepsilon}^{i_1},\dots,\boldsymbol{\varepsilon}^{i_k},V^a\mathbf{E}_a,\mathbf{E}_{j_1},\dots,\mathbf{E}_{j_l}\big)\\&=V^a\mathbf{T}\big(\boldsymbol{\varepsilon}^{i_1},\dots,\boldsymbol{\varepsilon}^{i_k},\mathbf{E}_a,\mathbf{E}_{j_1},\dots,\mathbf{E}_{j_l}\big)\\&=V^aT^{i_1\dots i_k}_{aj_1\dots j_l}.\end{aligned}$$

Thus, in any local frame, the operation $\iota_{\mathbf{V}}$ is obtained by contracting the contravariant index of $\mathbf{V}$ with the first covariant index of $\mathbf{T}$.

We can introduce two classes of tensors that will be particularly useful and that are used to define Riemannian metrics and differential forms. The first is the class of symmetric tensors:

Definition 1.10.26. Let V be a finite-dimensional vector space. A k-covariant tensor α on V is said to be **symmetric** if its value is unchanged when any two of its arguments are interchanged:

$$\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k)=\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k)$$

for any $1\le i<j\le k$. We denote the space of symmetric k-covariant tensors by $\Sigma^k(V^*)\subseteq T^k(V^*)$.

Definition 1.10.27. Let V be a finite-dimensional vector space. If $\alpha\in\Sigma^k(V^*)$ and $\beta\in\Sigma^l(V^*)$, we define their **symmetric product** to be the $(k+l)$-covector given by

$$\alpha\beta=\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}{}^\sigma(\alpha\otimes\beta)$$

where ${}^\sigma(\alpha\otimes\beta)(v_1,\dots,v_{k+l})=\alpha\otimes\beta(v_{\sigma(1)},\dots,v_{\sigma(k+l)})$.

Definition 1.10.28. A **symmetric tensor field** on a smooth manifold with or without boundary is a covariant tensor field whose value at each point is a symmetric tensor.

The second class consists of alternating tensors:

Definition 1.10.29. If V is a finite-dimensional real vector space, a k-covariant tensor α is said to be **alternating** or **antisymmetric** if, for any $v_1,\dots,v_k \in V$ and each $i,j \in \{1,\dots,k\}$ with $i \neq j$, we have

$$\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k) = -\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k).$$

The space of alternating k-covariant tensors is denoted by $\Lambda^k(V^*) \subseteq T^{(0,k)}(V)$.

Like the tensor spaces above, the space of alternating k-covariant tensors is a finite-dimensional vector space. Alternating k-covariant tensors are often called k-***covectors***.

Definition 1.10.30. Let V be a finite-dimensional vector space, and suppose that $(\varepsilon^1,\dots,\varepsilon^n)$ is any basis for V^*. For each multi-index $I = (i_1,\dots,i_k)$ of length k such that $1 \le i_1,\dots,i_k \le n$, define $\varepsilon^I = \varepsilon^{i_1\dots i_k}$ by

$$\varepsilon^I(v_1,\dots,v_k) = \det\begin{bmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{bmatrix} = \det\begin{bmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{bmatrix}$$

Remark 1.10.31. For each multi-index I, $\varepsilon^I \in \Lambda^k(V^*)$.

Proposition 1.10.32. Let V be a vector space of finite dimension n. If (ε^i) is any basis for V^*, then, for each $k \le n$, the collection of k-covectors

$$\mathscr{B} = \{\varepsilon^I \mid I \text{ is an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Therefore $\dim \Lambda^k(V^*) = \binom{n}{k}$.

Proof. Let $(e_1,\dots,e_n)$ be the basis of V dual to $(\varepsilon^1,\dots,\varepsilon^n)$. If $I = (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_k)$, the definition by determinants gives

$$\varepsilon^I(e_{j_1},\dots,e_{j_k}) = \begin{cases} 1, & I = J, \\ 0, & I \neq J. \end{cases}$$

Consequently, the elements of $\mathscr{B}$ are linearly independent.
Now let $\alpha \in \Lambda^k(V^*)$ and define

$$\widetilde{\alpha} := \sum_{1 \le i_1 < \cdots < i_k \le n} \alpha(e_{i_1},\dots,e_{i_k})\,\varepsilon^{i_1\dots i_k}.$$

If $j_1,\dots,j_k$ are distinct, there exists a permutation $\sigma \in S_k$ that arranges them in increasing order. The alternating property of α and $\widetilde{\alpha}$, together with the preceding equality, shows that

$$\widetilde{\alpha}(e_{j_1},\dots,e_{j_k}) = \alpha(e_{j_1},\dots,e_{j_k}).$$

If two indices coincide, both sides are zero. By multilinearity, $\widetilde{\alpha} = \alpha$ on all of V^k, so $\mathscr{B}$ spans $\Lambda^k(V^*)$. Finally, the number of strictly increasing multi-indices of length k is $\binom{n}{k}$, which proves the dimension formula. □

We define an operation on $\Lambda^k(V^*)$ as follows:

Definition 1.10.33. Let V be a finite-dimensional vector space. If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, we define their **wedge product** or **exterior product** to be the $(k+l)$-covector given by

$$\omega \wedge \eta = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sign}\sigma)({}^{\sigma}(\omega \otimes \eta)).$$

Here the combinatorial factor is included for normalization.

The wedge product has the following properties:

Proposition 1.10.34. Suppose that $\omega, \omega', \eta, \eta'$ and ξ are multicovectors on a finite-dimensional vector space V.

(a) The wedge product is bilinear over $\mathbb{R}$ and associative.

(b) If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,
$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

(c) If $\omega^1, \dots, \omega^k \in V^*$ and $v_1, \dots, v_k \in V$, then
$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \dots, v_k) = \det(\omega^j(v_i)).$$

Proof. In the sum defining $\omega \wedge \eta$, internal permutations of the first k arguments and the last l arguments yield equal terms because ω and η are alternating. Grouping these $k!l!$ terms gives the shuffle formula

$$(\omega \wedge \eta)(v_1, \dots, v_{k+l}) = \sum_{\sigma \in \mathrm{Sh}(k,l)} \operatorname{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}),$$

where $\mathrm{Sh}(k,l)$ is the set of permutations that preserve the relative order within the two blocks. This expression immediately proves bilinearity.

For associativity, let the degrees of ω, η, ξ be k, l, m, respectively. Applying the shuffle formula twice turns both $(\omega \wedge \eta) \wedge \xi$ and $\omega \wedge (\eta \wedge \xi)$ into

$$\sum_{\sigma \in \mathrm{Sh}(k,l,m)} \operatorname{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \xi(v_{\sigma(k+l+1)}, \dots, v_{\sigma(k+l+m)}),$$

where $\mathrm{Sh}(k,l,m)$ preserves the order within each of the three blocks. Thus the product is associative.

Interchanging the blocks of lengths k and l establishes a bijection between $\mathrm{Sh}(k,l)$ and $\mathrm{Sh}(l,k)$. This interchange requires kl transpositions, so the sign changes by the factor $(-1)^{kl}$. This proves

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

Finally, repeatedly applying the shuffle formula to 1-covectors gives

$$(\omega^1 \wedge \cdots \wedge \omega^k)(v_1,\dots,v_k) = \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} \omega^j(v_{\sigma(j)}),$$

which is the permutation expansion of the determinant $\det(\omega^j(v_i))$. □

Definition 1.10.35. Let M be a smooth manifold with or without boundary. We define the bundle of alternating k-tensors by

$$\Lambda^k(T^*M) := \coprod_{p\in M} \Lambda^k(T_p^*M).$$

Proposition 1.10.36. Let M be a smooth manifold of dimension n with or without boundary. Then $\Lambda^k(T^*M)$ is a smooth vector bundle of rank $\binom{n}{k}$ over M.

Proof. Let $(U, x^1,\dots,x^n)$ be a smooth chart on M. For each strictly increasing multi-index $I = (i_1 < \cdots < i_k)$, set $\mathbf{d}x^I := \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k}$. Proposition 1.10.32, applied at each point $p \in U$, shows that

$$\left(\mathbf{d}x^I|_p\right)_{|I|=k}$$

is a basis for $\Lambda^k(T_p^*M)$. Thus the map

$$\Lambda^k(T^*M)|_U \longrightarrow U \times \mathbb{R}^{\binom{n}{k}}, \qquad \sum_{|I|=k} a_I\, \mathbf{d}x^I|_p \longmapsto \left(p, (a_I)_{|I|=k}\right),$$

is a local trivialization that is linear on the fibers.
If $(V, y^1,\dots,y^n)$ is another chart, then on $U \cap V$ each $\mathbf{d}y^J$ can be expressed as

$$\mathbf{d}y^J = \sum_{|I|=k} \det\left(\frac{\partial y^{j_a}}{\partial x^{i_b}}\right)_{a,b=1}^{k} \mathbf{d}x^I.$$

The coefficients are smooth; consequently, the transitions between these trivializations are smooth and linear on the fibers. This gives $\Lambda^k(T^*M)$ the structure of a smooth vector bundle, and the number of increasing indices I is $\binom{n}{k}$. □

Definition 1.10.37. Let M be a smooth manifold with or without boundary. A section of $\Lambda^k(T^*M)$ is called a **differential k-form** or simply a **k-form**. The integer k is called the **degree** of the form. We denote the vector space of smooth k-forms by

$$\Omega^k(M) := \Gamma(\Lambda^k(T^*M)).$$

Remark 1.10.38. Every 1-covariant tensor is alternating, so $\Omega^1(M) = \Gamma(T^{(0,1)}(TM))$.

Proposition 1.10.39. Let $F\colon M \longrightarrow N$ be a smooth map between n-manifolds with or without boundary. If (x^i) and (y^i) are smooth coordinates on $U \subseteq M$ and $V \subseteq N$, respectively, and $u \in C^\infty(V)$, then the following

holds on $U \cap F^{-1}(V)$:

$$F^*(u\,\mathbf{d}y^1 \wedge \cdots \wedge \mathbf{d}y^n) = (u \circ F)(\det DF)\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n,$$

where DF denotes the Jacobian matrix of F in these coordinates.

Proof. By the multiplicativity of pullback,

$$F^*(u\,\mathbf{d}y^1 \wedge \cdots \wedge \mathbf{d}y^n) = (u \circ F)\, d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F).$$

In the coordinates x,

$$d(y^a \circ F) = \sum_{b=1}^{n} \frac{\partial (y^a \circ F)}{\partial x^b}\,\mathbf{d}x^b.$$

Expanding the exterior product, the terms with repeated indices vanish. The sum of the remaining terms, indexed by $\sigma \in S_n$, is

$$\det\left(\frac{\partial (y^a \circ F)}{\partial x^b}\right)_{a,b=1}^{n} \mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n,$$

by the Leibniz formula for the determinant. The matrix in parentheses is DF in the chosen charts. □

The exterior derivative extends the differential of functions while preserving its intrinsic character.

Theorem 1.10.40. (Existence and uniqueness of the exterior derivative) Let M be a smooth manifold with or without boundary. For each $k \in \mathbb{N}_0$, there exists a unique linear map

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

that, together with the operators in the other degrees, satisfies the following properties.

(a) If $f \in \Omega^0(M) = C^\infty(M)$, then df is the differential of f, that is, $df(X) = X(f)$.

(b) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d = 0$.

(d) If $F\colon M \longrightarrow N$ is a smooth map and $\omega \in \Omega^k(N)$, then

$$d(F^*\omega) = F^*(d\omega).$$

In a smooth chart with coordinates $(x^1, \ldots, x^n)$, its expression is

$$d\left(\sum_{i_1 < \cdots < i_k} \omega_{i_1 \ldots i_k}\,\mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k}\right) = \sum_{i_1 < \cdots < i_k} d\omega_{i_1 \ldots i_k} \wedge \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k}. \tag{1.30}$$

Proof. We first construct the operator by an intrinsic formula. For $\omega \in \Omega^k(M)$ and local vector fields $X_0, \dots, X_k$, define

$$\begin{aligned}(d\omega)(X_0,\dots,X_k) = &\sum_{i=0}^{k} (-1)^i X_i\big(\omega(X_0,\dots,\widehat{X}_i,\dots,X_k)\big) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\,\omega\big([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_k\big).\end{aligned} \tag{1.31}$$

In the second sum, the remaining fields appear after the bracket in their original order. For $k = 0$, the second sum is empty and the formula reduces to $(df)(X) = X(f)$.

The expression is linear over $\mathbb{R}$ in ω and in each field. The apparent differential dependence on the X_i cancels, and the result is in fact linear over $C^\infty(M)$ in each argument. Let us verify this for the argument with index r. Replacing X_r by fX_r multiplies every term that does not differentiate f by a factor of f. For $i < r$, the additional term from the first sum is

$$(-1)^i X_i(f)\,\omega(X_0,\dots,\widehat{X}_i,\dots,X_r,\dots,X_k).$$

In the second sum, the term corresponding to the pair (i,r) contains

$$[X_i, fX_r] = f[X_i,X_r] + X_i(f)X_r.$$

Moving X_r from the first argument of ω to the position it occupied in the preceding expression requires $r-1$ transpositions. Its additional contribution therefore has sign $(-1)^{i+r}(-1)^{r-1} = -(-1)^i$ and cancels the preceding term. If $j > r$, the identity

$$[fX_r, X_j] = f[X_r,X_j] - X_j(f)X_r$$

yields, after moving X_r through r arguments, the sign $(-1)^{r+j+1}(-1)^r = -(-1)^j$, which cancels the term in which X_j differentiates f in the first sum. No other derivatives of f appear. This proves $C^\infty(M)$-linearity in the argument r, and r was arbitrary.

Consequently, the value of (1.31) at a point depends only on the values of the fields at that point. In particular, it is independent of the chosen local extensions. Thus the formula defines a well-defined $(k+1)$-covariant tensor field.

We now show that this tensor is alternating and smooth. In a chart with coordinates $(x^1,\dots,x^n)$, write

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\dots i_k}\,\mathbf{d}x^{i_1}\wedge\cdots\wedge\mathbf{d}x^{i_k}.$$

Since coordinate vector fields commute, evaluating (1.31) on them gives

$$d\omega = \sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{a=1}^{n} \frac{\partial \omega_{i_1\dots i_k}}{\partial x^a}\,\mathbf{d}x^a\wedge\mathbf{d}x^{i_1}\wedge\cdots\wedge\mathbf{d}x^{i_k}.$$

Both sides are tensors and agree on every tuple of coordinate vector fields, so they agree throughout the chart. The right-hand side is a smooth differential form; hence the tensor constructed is alternating and

smooth. Moreover, this equality is precisely (1.30). Since it was derived from the intrinsic formula, the expressions obtained in different charts agree on their overlaps. This proves that the operator is globally well defined and independent of coordinates.

The graded Leibniz rule can be checked in a chart. For $d, q \in \mathbb{N}_0$, write $\mathcal{J}_d^q := \{(i_1, \dots, i_q) \in \{1, \dots, d\}^q \mid i_1 < \cdots < i_q\}$, with $\mathcal{J}_d^0 = \{\varnothing\}$. If $\omega = \sum_{I \in \mathcal{J}_n^k} a_I \mathbf{d}x^I$ has degree k and $\eta = \sum_{J \in \mathcal{J}_n^\ell} b_J \mathbf{d}x^J$ has degree ℓ, the local expression and the product rule for functions give

$$\begin{aligned} d(\omega \wedge \eta) &= \sum_{\substack{I \in \mathcal{J}_n^k \\ J \in \mathcal{J}_n^\ell}} d(a_I b_J) \wedge \mathbf{d}x^I \wedge \mathbf{d}x^J \\ &= \sum_{\substack{I \in \mathcal{J}_n^k \\ J \in \mathcal{J}_n^\ell}} da_I \wedge \mathbf{d}x^I \wedge b_J \mathbf{d}x^J + \sum_{\substack{I \in \mathcal{J}_n^k \\ J \in \mathcal{J}_n^\ell}} a_I \, db_J \wedge \mathbf{d}x^I \wedge \mathbf{d}x^J \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

The sign in the last equality arises from moving the 1-form db_J past the k factors of $\mathbf{d}x^I$.

Applying the local expression twice, we obtain

$$d^2\omega = \sum_{I \in \mathcal{J}_n^k} \sum_{a,b=1}^{n} \frac{\partial^2 a_I}{\partial x^b \partial x^a} \mathbf{d}x^b \wedge \mathbf{d}x^a \wedge \mathbf{d}x^I.$$

The terms with $a = b$ vanish. Terms with distinct indices cancel in pairs, since mixed partial derivatives commute and $\mathbf{d}x^b \wedge \mathbf{d}x^a = -\mathbf{d}x^a \wedge \mathbf{d}x^b$. Therefore $d^2 = 0$.

To prove naturality, let $F \colon M \to N$ be smooth and, in a chart on N, let $\omega = \sum_{I \in \mathcal{J}_{\dim N}^k} a_I \, \mathbf{d}y^I$. The chain rule gives

$$d(a_I \circ F) = F^*(da_I), \qquad F^*(\mathbf{d}y^j) = d(y^j \circ F).$$

Using the graded Leibniz rule and $d^2 = 0$, we obtain

$$\begin{aligned} d(F^*\omega) &= d\left(\sum_{I \in \mathcal{J}_{\dim N}^k} (a_I \circ F) \, d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F) \right) \\ &= \sum_{I \in \mathcal{J}_{\dim N}^k} F^*(da_I) \wedge F^*(\mathbf{d}y^I) = F^*(d\omega). \end{aligned}$$

The identity is local around each point and therefore holds on all of M.

If M has boundary, the charts used are relatively open in the half-space. Their functions, forms, and fields admit the local extensions specified in the manuscript's definition of smoothness. The intrinsic formula is independent of these extensions, and the coordinate expression above is obtained by restriction. All the arguments, including those using mixed derivatives, remain valid up to the boundary.

Only uniqueness remains. Let D be another family of operators satisfying the properties in the statement. Naturality applied to the inclusion of a coordinate domain into M allows us to restrict $D\omega$ and perform the calculation entirely within that domain. There, $D(\mathbf{d}x^i) = D^2 x^i = 0$, and the graded Leibniz rule implies

$D(\mathbf{d}x^I) = 0$. Therefore

$$D\left(\sum_{I\in\mathcal{J}_n^k} a_I \mathbf{d}x^I\right) = \sum_{I\in\mathcal{J}_n^k} da_I \wedge \mathbf{d}x^I,$$

which agrees with (1.30). Thus $D = d$ in every chart and hence on the entire manifold. □

2

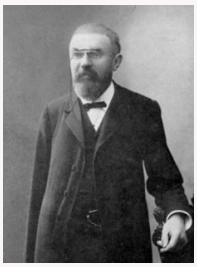
Riemannian manifolds

A smooth manifold has local coordinates, but by itself carries no intrinsic notion of length, angle, or volume. Introducing these quantities requires choosing an inner product on each tangent space that depends smoothly on the point. This structure is a Riemannian metric.

The metric defines the length of curves, the Riemannian distance, and the volume measure. Together with the Levi–Civita connection, it also allows us to construct the gradient, divergence, Laplacian, and covariant derivatives that will appear in the analytical chapters. Riemannian geometry therefore provides the intrinsic language in which we will formulate the Sobolev spaces, variational problems, and differential equations studied later.

Carl Friedrich Gauss
(1777–1855)

Bernhard Riemann
(1826–1866)



Henri Poincaré
(1854–1912)

Figure 2.1. Gauss, Riemann, and Poincaré were central figures in the development of modern geometry. Gauss's work on the intrinsic geometry of surfaces paved the way for Riemannian geometry, introduced by Riemann to study spaces of arbitrary dimension. Poincaré, in turn, advanced the global study of these spaces and laid some of the foundations of modern topology.

From Gauss's surfaces to Riemann's manifolds

Gauss discovered that the curvature of a surface can be calculated from the metric inherited by its coordinates, without making direct use of the ambient space. Riemann turned this observation into a much broader program: the study of spaces of arbitrary dimension from an infinitesimal rule for measuring lengths. Poincaré later showed that local geometry must coexist with global invariants. This chapter fol-

lows that evolution: it begins with the metric, constructs the connection and curvature, and ends with the tools of integration and the operators needed to carry out analysis on the manifold.

2.1 Riemannian metrics

A Riemannian metric adds an inner product on each tangent space to the smooth structure, allowing many constructions from Euclidean analysis to be transferred to the geometric setting.

Definition 2.1.1. Let M be a smooth manifold with or without boundary. A *Riemannian metric on M* is a smooth tensor field

$$\mathbf{g} \in \Gamma\big(T^{(0,2)}(TM)\big)$$

that is symmetric and positive definite at every point $p \in M$.

Definition 2.1.2. A **Riemannian manifold with or without boundary** is a pair $(M, \mathbf{g})$, where M is a smooth manifold with or without boundary and $\mathbf{g}$ is a Riemannian metric on M.

Example 2.1.3. $\mathbb{R}^n$ is a Riemannian manifold without boundary with the **Euclidean metric** $\overline{\mathbf{g}}$, given in standard coordinates by

$$\overline{\mathbf{g}} = \delta_{ij}\, \mathbf{d}x^i \otimes \mathbf{d}x^j$$

or by

$$\overline{\mathbf{g}} = (\mathbf{d}x^1)^2 + \cdots + (\mathbf{d}x^n)^2.$$

If $v, w \in T_p\mathbb{R}^n$, $\overline{\mathbf{g}}_p(v,w) = \delta_{ij} v^i w^j = \sum_{i=1}^{n} v^i w^i = v \cdot w$, which is the usual inner product on $\mathbb{R}^n \cong T_p\mathbb{R}^n$.

Remark 2.1.4. Given a Riemannian manifold $(M, \mathbf{g})$ with or without boundary, the Riemannian metric $\mathbf{g}$ defines an **inner product** on each tangent space T_pM for each $p \in M$. For $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$, we sometimes denote $\mathbf{g}(\mathbf{X}, \mathbf{Y}) \in C^\infty(M)$ by $\mathbf{g}(\mathbf{X}, \mathbf{Y}) := \langle \mathbf{X}, \mathbf{Y} \rangle$. Pointwise, we have $\langle \mathbf{X}_p, \mathbf{Y}_p \rangle = \mathbf{g}_p(\mathbf{X}_p, \mathbf{Y}_p) \in \mathbb{R}$. We can also define a **norm** on T_pM by $\|v\|_{\mathbf{g}} = \sqrt{\langle v, v \rangle_{\mathbf{g}}}$.
Finally, the concept of angle between two vectors extends from $\mathbb{R}^n$ to M as follows:
The angle between two vectors $v, w \in T_pM \setminus \{0\}$ is the unique $\theta \in [0, \pi]$ satisfying

$$\cos(\theta) = \frac{\langle v, w \rangle_{\mathbf{g}}}{\|v\|_{\mathbf{g}} \|w\|_{\mathbf{g}}}.$$

Two vectors $v, w \in T_pM$ are said to be **orthogonal** if $\langle v, w \rangle_{\mathbf{g}} = 0$.

Definition 2.1.5. (Pointwise order on symmetric forms and comparison of metrics) Let $\mathbf{a}$ and $\mathbf{b}$ be two symmetric tensor fields of type $(0,2)$, and let $A \subseteq M$. The notation $\mathbf{a} \leq \mathbf{b}$ on A means that

$$\mathbf{a}_p(v,v) \leq \mathbf{b}_p(v,v) \qquad \text{for every } p \in A \text{ and every } v \in T_pM.$$

Equivalently, $(\mathbf{b} - \mathbf{a})_p(v,v) \geq 0$ for the same p and v; that is, $\mathbf{b} - \mathbf{a}$ is positive semidefinite on each fiber over

A. This is the meaning of an inequality between symmetric tensors of type $(0,2)$ in this book.
In particular, if $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ are Riemannian metrics and $0 < c \le C < \infty$, writing

$$c\mathbf{g} \le \widetilde{\mathbf{g}} \le C\mathbf{g} \qquad \text{on } A$$

is shorthand for the numerical inequalities

$$c\,\mathbf{g}_p(v,v) \le \widetilde{\mathbf{g}}_p(v,v) \le C\,\mathbf{g}_p(v,v) \qquad (p \in A,\ v \in T_pM).$$

Since $\mathbf{g}_p(v,v) = |v|^2_{\mathbf{g}}$, this is exactly equivalent to

$$\sqrt{c}\,|v|_{\mathbf{g}} \le |v|_{\widetilde{\mathbf{g}}} \le \sqrt{C}\,|v|_{\mathbf{g}} \qquad (p \in A,\ v \in T_pM). \tag{2.1}$$

The constants are common to all points of A and all vectors in their fibers. When such constants exist, we say that the metrics are uniformly equivalent on A.

Remark 2.1.6. The comparison is made by evaluating both metrics on the same vector, repeated in both arguments. If $v = \sum_{i=1}^{n} v^i \partial_i$, its expression in a chart is

$$c \sum_{i,j=1}^{n} g_{ij}(p) v^i v^j \le \sum_{i,j=1}^{n} \widetilde{g}_{ij}(p) v^i v^j \le C \sum_{i,j=1}^{n} g_{ij}(p) v^i v^j \qquad \text{for every } (v^1,\dots,v^n) \in \mathbb{R}^n.$$

The matrix entries, considered individually, need not satisfy these inequalities. Nor is an inequality asserted for $\mathbf{g}_p(v,w)$ and $\widetilde{\mathbf{g}}_p(v,w)$ with arbitrary v and w. Diagonal evaluation is independent of the chart and determines the symmetric form through the polarization identity

$$\mathbf{g}_p(v,w) = \frac{1}{4}\big(\mathbf{g}_p(v+w,v+w) - \mathbf{g}_p(v-w,v-w)\big).$$

Comparison constants exist on each finite-dimensional fiber; uniformity in the base point is an additional assertion. On a compact set, it follows from continuity of the metrics, as will be proved in Lemma 7.1.8 and, for arbitrary vector bundles, in Lemma 9.1.43. The same diagonal evaluation convention is used to compare bundle metrics, with $v \in \mathbf{E}_p$; in the complex case, we compare the real values of the Hermitian forms on (v,v).

We can use the pullback of tensor fields, defined in Chapter 1, to obtain a Riemannian metric from smooth maps.

Proposition 2.1.7. Let M and N be smooth manifolds with or without boundary, let $F\colon M \longrightarrow N$ be a smooth map, and let $\mathbf{g}$ be a Riemannian metric on N. Then $F^*\mathbf{g}$ is a Riemannian metric on M if and only if F is an immersion.

Proof. By Proposition 1.10.20, we know that $F^*\mathbf{g} \in \Gamma(T^{(0,2)}(TM))$, that is, $F^*\mathbf{g}$ is a smooth 2-covariant tensor field. The definition of pullback shows that this tensor is symmetric; it only remains to determine when it is positive definite at every point.

First suppose that $F^*\mathbf{g}$ is a Riemannian metric on M. We show that F is an immersion. Take $p \in M$ and let $v \in T_pM$ with $v \neq 0$. By Definition 1.5.2, we must show that dF_p is injective. If $dF_p(v) = 0$, then

$$(F^*\mathbf{g})_p(v,v) = g_{F(p)}(dF_p(v), dF_p(v)) = 0$$

contradicting the positive definiteness of $F^*\mathbf{g}$ on p. It follows that $dF_p(v) \neq 0$ for every $v \neq 0$; thus dF_p is injective and F is an immersion.
Conversely, suppose that F is an immersion. If $p \in M$ and $v \in T_pM$ satisfy

$$0 = (F^*\mathbf{g})_p(v,v) = \mathbf{g}_{F(p)}(dF_p(v), dF_p(v)),$$

then, since $\mathbf{g}_{F(p)}$ is positive definite, the preceding equality implies $dF_p(v) = 0$. The injectivity of dF_p then gives $v = 0$, so $F^*\mathbf{g}$ is positive definite. □

Every smooth manifold, with or without boundary, admits at least one Riemannian metric.

Proposition 2.1.8. (Existence of Riemannian metrics) Every smooth manifold with or without boundary admits a Riemannian metric.

Proof. Let M be a smooth manifold with or without boundary. Let $(U_\alpha, \varphi_\alpha)_{\alpha\in J}$ be a family of charts such that $(U_\alpha)_{\alpha\in J}$ is an open cover of M. By Proposition 1.3.5, we can find a partition of unity $(\psi_\alpha)_{\alpha\in J}$ subordinate to the cover $(U_\alpha)_{\alpha\in J}$. By Proposition 2.1.7, in each coordinate chart, $\mathbf{g}_\alpha = \varphi_\alpha^*\overline{\mathbf{g}}$ defines a Riemannian metric on U_α whose coordinate expression is $\delta_{ij}\,\mathbf{d}x^i \otimes \mathbf{d}x^j$. Since $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$, the tensor $\psi_\alpha \mathbf{g}_\alpha$ extends smoothly by zero outside U_α. Define

$$\mathbf{g} := \sum_{\alpha\in J} \psi_\alpha \mathbf{g}_\alpha,$$

where each summand is understood to have this extension. The family of supports is locally finite, so the sum is finite around each point and defines a smooth tensor field.
The field $\mathbf{g}$ is symmetric because it is a finite sum of symmetric tensors in a neighborhood of each point.
To see that it is positive definite, let $p \in M$ and $v \in T_pM \setminus \{0\}$. For each index with $\psi_\alpha(p) > 0$, we have $p \in U_\alpha$ and $(\mathbf{g}_\alpha)_p(v,v) > 0$, while all the coefficients $\psi_\alpha(p)$ are nonnegative. Since their sum is one, there exists β such that $\psi_\beta(p) > 0$. Therefore,

$$\mathbf{g}_p(v,v) = \sum_{\alpha\in J} \psi_\alpha(p)(\mathbf{g}_\alpha)_p(v,v) > 0.$$

□

If $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ is a smooth local frame for TM defined on an open subset $U \subseteq M$ and $(\varepsilon^1, \dots, \varepsilon^n)$ is its dual coframe, we can write $\mathbf{g}$ locally on U as

$$\mathbf{g} = g_{ij}\,\varepsilon^i \otimes \varepsilon^j, \qquad g_{ij}(p) := \langle \mathbf{E}_i|_p, \mathbf{E}_j|_p \rangle_{\mathbf{g}}.$$

These functions are called the ***coefficients of the metric*** $\mathbf{g}$ ***with respect to the local frame*** $(\mathbf{E}_1, \dots, \mathbf{E}_n)$. We say that $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ is an ***orthonormal local frame*** if the vectors $\mathbf{E}_1|_p, \dots, \mathbf{E}_n|_p$ form an orthonormal basis of T_pM for every $p \in U$. Equivalently, a local frame is orthonormal if and only if $\langle \mathbf{E}_i, \mathbf{E}_j \rangle_{\mathbf{g}} = \delta_{ij}$.

Proposition 2.1.9. (Existence of orthonormal local frames) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. If $(\mathbf{X}_1,\dots,\mathbf{X}_n)$ is a smooth local frame for TM defined on $U \subseteq M$, then there exists a smooth orthonormal local frame $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ defined on U such that

$$\operatorname{span}\{\mathbf{E}_1|_p,\dots,\mathbf{E}_k|_p\} = \operatorname{span}\{\mathbf{X}_1|_p,\dots,\mathbf{X}_k|_p\}$$

for each $k \in \{1,\dots,n\}$ and each $p \in U$. In particular, for each $p \in M$, there exists an orthonormal local frame defined on some neighborhood of p.

Proof. By the Gram–Schmidt algorithm, for each $p \in U$, the vectors $\mathbf{E}_1|_p,\dots,\mathbf{E}_n|_p$ can be chosen recursively so that

$$\operatorname{span}\{\mathbf{E}_1|_p,\dots,\mathbf{E}_k|_p\} = \operatorname{span}\{\mathbf{X}_1|_p,\dots,\mathbf{X}_k|_p\}$$

for every k. Explicitly, $\mathbf{E}_1 = \frac{\mathbf{X}_1}{\|\mathbf{X}_1\|_{\mathbf{g}}}$ and

$$\mathbf{E}_j = \frac{\mathbf{X}_j - \sum_{i=1}^{j-1} \langle \mathbf{X}_j, \mathbf{E}_i \rangle_{\mathbf{g}} \mathbf{E}_i}{\left\| \mathbf{X}_j - \sum_{i=1}^{j-1} \langle \mathbf{X}_j, \mathbf{E}_i \rangle_{\mathbf{g}} \mathbf{E}_i \right\|_{\mathbf{g}}}, \qquad 2 \le j \le n.$$

Each denominator is a smooth, strictly positive function because $(\mathbf{X}_1,\dots,\mathbf{X}_n)$ is a frame. Therefore the fields $\mathbf{E}_i$ are smooth; the Gram–Schmidt identities show that they form an orthonormal frame and satisfy the stated equalities of subspaces. □

Riemannian metrics allow us to introduce a bundle isomorphism between TM and T^*M as follows:

Definition 2.1.10. (Musical isomorphisms) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. For each point $p \in M$, define the maps

$$\mathbf{g}_p^\flat \colon T_pM \longrightarrow T_p^*M, \qquad X_p \mapsto \mathbf{g}_p(X_p,\cdot),$$

and

$$\mathbf{g}_p^\sharp \colon T_p^*M \longrightarrow T_pM, \qquad \omega_p \mapsto \text{the unique } X_p \text{ such that } \mathbf{g}_p(X_p,\cdot) = \omega_p.$$

These maps are inverses of one another and define bundle isomorphisms

$$\mathbf{g}^\flat \colon TM \longrightarrow T^*M, \qquad \mathbf{g}^\sharp \colon T^*M \longrightarrow TM,$$

called the *musical isomorphisms*.

Remark 2.1.11. In local coordinates $(x^1,\dots,x^n)$, if $\mathbf{X} = X^i \partial_i$ is a vector field, then

$$\mathbf{X}^\flat = g_{ij} X^i \, \mathbf{d}x^j,$$

and if $\omega = \omega_i \, \mathbf{d}x^i$ is a 1-form,

$$\omega^\sharp = g^{ij}\omega_i \partial_j,$$

where (g^{ij}) is the inverse of the metric matrix (g_{ij}). Thus $^\flat$ is interpreted as "lowering indices" and $^\sharp$ as "raising indices."

2.2 Traces of tensors

Contraction removes one covariant and one contravariant index by means of a sum independent of the basis. This operation will appear repeatedly in the definitions of divergence, Laplacians, curvature, and energy.
Given a finite-dimensional vector space V and an endomorphism $T\colon V \longrightarrow V$, if $\beta = \{e_1, \dots, e_n\}$ is a basis for V and (A_{ij}) is the matrix of T with respect to the basis β, we can define $\operatorname{tr}(T) := \sum_{i=1}^{n} A_{i,i}$. This definition is independent of the chosen basis, and we can generalize the construction to mixed tensors:

Definition 2.2.1. For the $(1,1)$-tensor $F(\omega^1, \dots, \omega^k, \cdot, v_1, \dots, v_l, \cdot) \in T^{(1,1)}(V)$, we can calculate its trace (which is independent of the chosen basis) $\operatorname{tr}(F)(\omega^1, \dots, \omega^k, v_1, \dots, v_l)$. In coordinates, the components of $\operatorname{tr}(F)$ are

$$(\operatorname{tr}(F))^{i_1 \dots i_k}_{j_1 \dots j_l} = F^{i_1 \dots i_k m}_{j_1 \dots j_l m}.$$

This construction defines an operator, independent of the chosen basis, that we call the *trace operator* $\operatorname{tr}\colon T^{(k+1,l+1)}(V) \longrightarrow T^{(k,l)}(V)$.

The trace operator is also known as ***contraction***.

In constructing the trace operator, we set the last upper and lower indices equal, which, in Einstein notation, means summing all these terms. A similar construction can be performed for any pair of indices, provided that one is covariant and the other contravariant. In general, we do not distinguish this operation notationally from the operator tr, so we use only this notation and specify the contraction explicitly when necessary.

The operator tr can be applied to any pair consisting of one contravariant and one covariant index. It is a well-defined linear map from $T^{(k+1,l+1)}(V)$ to $T^{(k,l)}(V)$. If $\mathbf{F}$ is a tensor field of type $(k+1, l+1)$ on a manifold M, its contraction is defined pointwise by

$$\operatorname{tr}(\mathbf{F})|_p := \operatorname{tr}(\mathbf{F}|_p), \qquad p \in M.$$

Now let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{F}$ be a k-covariant tensor field. The metric allows us to contract two distinct covariant indices: when positions r and s are contracted, the coordinate expression is

$$g^{i_r i_s} F_{i_1 \dots i_r \dots i_s \dots i_k}, \qquad r \neq s.$$

This expression transforms tensorially and therefore defines a $(k-2)$-covariant field independent of coordinates. We denote it by $\operatorname{tr}_{\mathbf{g}}(\mathbf{F})$; when more than one pair is possible, the two indices being contracted will be specified explicitly.

2.3 Fiberwise inner products of tensor fields

Given a Riemannian manifold $(M, \mathbf{g})$ with or without boundary, we can define several useful concepts from $\mathbf{g}$. One is the inner product of tensor fields. This is a special case of a more general construction that does not necessarily require a Riemannian structure, known as a fiberwise inner product or bundle metric on a vector bundle $\mathbf{E}$.

Definition 2.3.1. Let M be a smooth manifold with or without boundary. If $\pi\colon \mathbf{E} \longrightarrow M$ is a smooth vector bundle, a **smooth bundle metric on E** is an inner product on each fiber $\mathbf{E}_p$ that varies smoothly in the sense that, for any smooth local sections σ, τ of $\mathbf{E}$, $\langle \sigma, \tau \rangle$ is a smooth function.

The following example constructs a bundle metric on a nontrivial vector bundle.

Example 2.3.2. (Bundle metric on the Möbius bundle) Return to the bundle $\mathbf{E} \to \mathbb{S}^1$ of Example 1.7.9. For $p \in \mathbb{S}^1$, define an inner product on the fiber $\pi^{-1}(p)$ as follows. If $x_0 \in \mathbb{R}$ satisfies $\varepsilon(x_0) = p$, then

$$\pi^{-1}(p) = \{[(x_0, t)] \mid t \in \mathbb{R}\},$$

and we consider the map

$$j_p\colon \pi^{-1}(p) \longrightarrow \mathbb{R}, \qquad j_p([(x_0, t)]) = t.$$

This map is well defined since, if $(x_0, t) \sim (x_0, t')$, then $t = t'$. We then define

$$\mathbf{h}_p(u, v) = j_p(u) j_p(v), \qquad u, v \in \pi^{-1}(p).$$

In this way, each fiber is equipped with an inner product. It remains to show that $\mathbf{h}$ is a smooth bundle metric in the sense of Definition 2.3.1.
Consider the cover $U_\pm = \mathbb{S}^1 \setminus \{(\pm 1, 0)\}$ with trivializations

$$\Phi_\pm\colon \pi^{-1}(U_\pm) \longrightarrow U_\pm \times \mathbb{R}, \qquad [(x, y)] \longmapsto (\varepsilon(x), y).$$

On the overlap $U_+ \cap U_-$, we have

$$(\Phi_+ \circ \Phi_-^{-1})(p, v) = (p, \tau_{+-}(p) v), \qquad \tau_{+-}(p) \in \{\pm 1\}.$$

This means that if a local section σ is written on U_- as $\Phi_-(\sigma(p)) = (p, s_-(p))$ with $s_-\colon U_- \longrightarrow \mathbb{R}$ smooth, then, for every $p \in U_+ \cap U_-$, the following holds:

$$\Phi_+(\sigma(p)) = \big(\Phi_+ \circ \Phi_-^{-1}\big)\big(\Phi_-(\sigma(p))\big) = \big(\Phi_+ \circ \Phi_-^{-1}\big)\big(p, s_-(p)\big) = (p, \tau_{+-}(p) s_-(p)).$$

Since $\Phi_+(\sigma(p)) = (p, s_+(p))$ by definition, comparing second coordinates yields

$$s_+(p) = \tau_{+-}(p) s_-(p).$$

Similarly, if τ is another section, then on the overlap

$$t_+(p) = \tau_{+-}(p) t_-(p),$$

where $s_\pm, t_\pm$ are the functions representing σ, τ in the trivializations $U_\pm$. Consequently,

$$s_+(p)t_+(p) = \big(\tau_{+-}(p)s_-(p)\big)\big(\tau_{+-}(p)t_-(p)\big) = s_-(p)t_-(p),$$

and the local expressions for the inner product agree on the intersection.
Finally, if σ, τ are smooth local sections over $U_\pm$, we have

$$\Phi_\pm(\sigma(p)) = (p, s_\pm(p)), \qquad \Phi_\pm(\tau(p)) = (p, t_\pm(p)),$$

and the definition of $\mathbf{h}$ gives

$$\mathbf{h}_p(\sigma(p), \tau(p)) = s_\pm(p)t_\pm(p),$$

which is a smooth function on p.
We conclude that $\mathbf{h}$ defines a smooth bundle metric on $(\mathbf{E}, \pi, \mathbb{S}^1)$, independent of the chosen trivialization and obtained naturally by identifying each fiber with $\mathbb{R}$ through j_p.

Proposition 2.3.3. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. Then there exists a unique smooth bundle metric on each tensor bundle $T^{(k,l)}(TM)$ with the property that, if $\alpha_1, \dots, \alpha_{k+l}, \beta_1, \dots, \beta_{k+l}$ are vector or covector fields as appropriate, then

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_{k+l}, \beta_1 \otimes \cdots \otimes \beta_{k+l} \rangle_{\mathbf{g}} = \langle \alpha_1, \beta_1 \rangle_{\mathbf{g}} \dots \langle \alpha_{k+l}, \beta_{k+l} \rangle_{\mathbf{g}}.$$

With this inner product, if $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ is an orthonormal local frame for TM and $(\varepsilon^1, \dots, \varepsilon^n)$ is the corresponding dual coframe, then the collection of tensor fields

$$(\mathbf{E}_{i_1} \otimes \cdots \otimes \mathbf{E}_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l})_{1 \le i_1, \dots, i_k, j_1, \dots, j_l \le n}$$

forms an orthonormal local frame on $T^{(k,l)}(TM)$ (orthonormal with respect to this bundle metric). In terms of any local frame, not necessarily orthonormal, this bundle metric satisfies the following: for any $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$, $\omega, \eta \in \Omega^1(M)$

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathbf{g}} = g_{ir} X^i Y^r, \ \langle \omega, \eta \rangle_{\mathbf{g}} = g^{js} \omega_j \eta_s.$$

Moreover, for $\mathbf{F}, \mathbf{G} \in \Gamma(T^{(k,l)}(TM))$,

$$\langle \mathbf{F}, \mathbf{G} \rangle_{\mathbf{g}} = g_{i_1 r_1} \cdots g_{i_k r_k} g^{j_1 s_1} \cdots g^{j_l s_l} F^{i_1 \dots i_k}_{j_1 \dots j_l} G^{r_1 \dots r_k}_{s_1 \dots s_l}.$$

We sometimes write $\mathbf{g}(\mathbf{F}, \mathbf{G}) := \langle \mathbf{F}, \mathbf{G} \rangle_{\mathbf{g}}$.

Proof. Consider $p \in M$, and let $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ be a smooth local frame for TM defined on an open set $U \subseteq M$ such that $p \in U$. We can define the bundle metric by

$$\langle \mathbf{F}, \mathbf{G} \rangle = g_{i_1 r_1} \cdots g_{i_k r_k} g^{j_1 s_1} \cdots g^{j_l s_l} F^{i_1 \dots i_k}_{j_1 \dots j_l} G^{r_1 \dots r_k}_{s_1 \dots s_l}$$

on U for any tensor fields $\mathbf{F}, \mathbf{G} \in \Gamma(T^{(k,l)}(TM))$. This definition is local. To show that it is well defined, we must first verify that it is independent of the chosen local frame; this will also prove uniqueness:
Suppose that $(\widetilde{\mathbf{E}}_1, \dots, \widetilde{\mathbf{E}}_n)$ is another local frame defined on an open set V such that $p \in V$. On $U \cap V$, the

local frames are related by

$$\mathbf{E}_i = A_i^j \widetilde{\mathbf{E}}_j$$

where (A_i^j) is a matrix of smooth functions defined on $U \cap V$. Let $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$ and $(\widetilde{\boldsymbol{\varepsilon}}^1, \dots, \widetilde{\boldsymbol{\varepsilon}}^n)$ be the local coframes associated with these frames. Let $(A^{-1})_i^j$ be the inverse matrix of (A_j^i), where the lower index denotes rows and the upper index columns. We have $\widetilde{\mathbf{E}}_i = (A^{-1})_i^j \mathbf{E}_j$, and we can also write $\boldsymbol{\varepsilon}^i = B_m^i \widetilde{\boldsymbol{\varepsilon}}^m$, so

$$(A^{-1})_j^i = (A^{-1})_j^k \delta_k^i = (A^{-1})_j^k \boldsymbol{\varepsilon}^i(\mathbf{E}_k) = \boldsymbol{\varepsilon}^i((A^{-1})_j^k \mathbf{E}_k) = \boldsymbol{\varepsilon}^i(\widetilde{\mathbf{E}}_j) = B_m^i \widetilde{\boldsymbol{\varepsilon}}^m(\widetilde{\mathbf{E}}_j) = B_m^i \delta_j^m = B_j^i.$$

We conclude that $\boldsymbol{\varepsilon}^i = (A^{-1})_j^i \widetilde{\boldsymbol{\varepsilon}}^j$. In turn, since $\mathbf{E}_i = A_i^j \widetilde{\mathbf{E}}_j$, we deduce that $\widetilde{\boldsymbol{\varepsilon}}^i = A_j^i \boldsymbol{\varepsilon}^j$.

We now verify that the expression is independent of the local frame. Let $\mathbf{F}, \mathbf{G} \in \Gamma(T^{(k,l)}(TM))$. On $U \cap V$, we have:

$$\begin{aligned}
&g_{i_1 r_1} \cdots g_{i_k r_k} g^{j_1 s_1} \cdots g^{j_l s_l} F_{j_1 \dots j_l}^{i_1 \dots i_k} G_{s_1 \dots s_l}^{r_1 \dots r_k} \\
&= (A_{i_1}^{a_1} A_{r_1}^{b_1} \widetilde{g}_{a_1 b_1}) \cdots (A_{i_k}^{a_k} A_{r_k}^{b_k} \widetilde{g}_{a_k b_k}) ((A^{-1})_{e_1}^{j_1} (A^{-1})_{f_1}^{s_1} \widetilde{g}^{e_1 f_1}) \cdots ((A^{-1})_{e_l}^{j_l} (A^{-1})_{f_l}^{s_l} \widetilde{g}^{e_l f_l}) \\
&((A^{-1})_{c_1}^{i_1} \cdots (A^{-1})_{c_k}^{i_k} A_{j_1}^{m_1} \cdots A_{j_l}^{m_l} \widetilde{F}_{m_1 \dots m_l}^{c_1 \dots c_k}) ((A^{-1})_{d_1}^{r_1} \cdots (A^{-1})_{d_k}^{r_k} A_{s_1}^{n_1} \cdots A_{s_l}^{n_l} \widetilde{G}_{n_1 \dots n_l}^{d_1 \dots d_k}) \\
&= A_{i_1}^{a_1} (A^{-1})_{c_1}^{i_1} \cdots A_{i_k}^{a_k} (A^{-1})_{c_k}^{i_k} A_{r_1}^{b_1} (A^{-1})_{d_1}^{r_1} \cdots A_{r_k}^{b_k} (A^{-1})_{d_k}^{r_k} \widetilde{g}_{a_1 b_1} \cdots \widetilde{g}_{a_k b_k} \\
&(A^{-1})_{e_1}^{j_1} A_{j_1}^{m_1} \cdots (A^{-1})_{e_l}^{j_l} A_{j_l}^{m_l} (A^{-1})_{f_1}^{s_1} A_{s_1}^{n_1} \cdots (A^{-1})_{f_l}^{s_l} A_{s_l}^{n_l} \widetilde{g}^{e_1 f_1} \dots \widetilde{g}^{e_l f_l} \widetilde{F}_{m_1 \dots m_l}^{c_1 \dots c_k} \widetilde{G}_{n_1 \dots n_l}^{d_1 \dots d_k} \\
&= \delta_{c_1}^{a_1} \cdots \delta_{c_k}^{a_k} \delta_{d_1}^{b_1} \cdots \delta_{d_k}^{b_k} \widetilde{g}_{a_1 b_1} \cdots \widetilde{g}_{a_k b_k} \delta_{e_1}^{m_1} \cdots \delta_{e_l}^{m_l} \delta_{f_1}^{n_1} \cdots \delta_{f_l}^{n_l} \widetilde{g}^{e_1 f_1} \dots \widetilde{g}^{e_l f_l} \widetilde{F}_{m_1 \dots m_l}^{c_1 \dots c_k} \widetilde{G}_{n_1 \dots n_l}^{d_1 \dots d_k} \\
&= \widetilde{g}_{a_1 b_1} \cdots \widetilde{g}_{a_k b_k} \widetilde{g}^{e_1 f_1} \cdots \widetilde{g}^{e_l f_l} \widetilde{F}_{e_1 \dots e_l}^{a_1 \dots a_k} \widetilde{G}_{f_1 \dots f_l}^{b_1 \dots b_k}.
\end{aligned}$$

Here the penultimate equality follows from the definitions of matrix multiplication and the inverse matrix, that is, $(A^{-1})_k^i A_j^k = \delta_j^i = A_j^m (A^{-1})_m^i$.

Therefore $\langle \mathbf{F}, \mathbf{G} \rangle_{\mathbf{g}}$ is independent of the chosen local frame.

Now that the inner product is well defined, we see that it is bilinear and symmetric, so it suffices to prove positive definiteness. If $k = l = 0$, it is the usual product on $\mathbb{R}$. If only one of the two indices is zero, the calculation below remains valid after omitting the absent index families and beginning the invertibility argument with the first family present. We may therefore assume that $k, l \geq 1$. Let $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ be a smooth local frame for TM defined on an open set $U \subseteq M$, and consider $\mathbf{F} \in \Gamma(T^{(k,l)}(TM))$. Since the matrix (g_{ij}) is positive definite, the spectral theorem for symmetric matrices gives a positive definite symmetric matrix R, hence invertible, such that $(g_{ij}) = R^2$.

We denote the entries of R by R_{ij} and those of R^{-1} by R^{ij}. In particular, we have $g_{ij} = \sum_{m=1}^{n} R_{im} R_{mj}$ and $g^{ij} = \sum_{m=1}^{n} R^{im} R^{mj}$.

In the following calculation, we write the sums in Einstein notation explicitly:

$$\langle \mathbf{F}, \mathbf{F} \rangle_{\mathbf{g}} = \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ 1 \leq j_1, \dots, j_l \leq n \\ 1 \leq r_1, \dots, r_k \leq n \\ 1 \leq s_1, \dots, s_l \leq n}} g_{i_1 r_1} \cdots g_{i_k r_k} g^{j_1 s_1} \cdots g^{j_l s_l} F_{j_1 \dots j_l}^{i_1 \dots i_k} F_{s_1 \dots s_l}^{r_1 \dots r_k} =$$

$$\sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n\\ 1\le r_1,\dots,r_k\le n\\ 1\le s_1,\dots,s_l\le n}} \left(\sum_{a_1=1}^{n} R_{i_1a_1}R_{a_1r_1}\right)\cdots\left(\sum_{a_k=1}^{n} R_{i_ka_k}R_{a_kr_k}\right)\left(\sum_{b_1=1}^{n} R^{j_1b_1}R^{b_1s_1}\right)\cdots\left(\sum_{b_l=1}^{n} R^{j_lb_l}R^{b_ls_l}\right) F^{i_1\dots i_k}_{j_1\dots j_l}F^{r_1\dots r_k}_{s_1\dots s_l}$$

$$= \sum_{\substack{1\le a_1,\dots,a_k\le n\\ 1\le b_1,\dots,b_l\le n}} \sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n\\ 1\le r_1,\dots,r_k\le n\\ 1\le s_1,\dots,s_l\le n}} R_{i_1a_1}R_{a_1r_1}\cdots R_{i_ka_k}R_{a_kr_k}R^{j_1b_1}R^{b_1s_1}\cdots R^{j_lb_l}R^{b_ls_l}F^{i_1\dots i_k}_{j_1\dots j_l}F^{r_1\dots r_k}_{s_1\dots s_l}$$

$$= \sum_{\substack{1\le a_1,\dots,a_k\le n\\ 1\le b_1,\dots,b_l\le n}} \left(\sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n}} R_{i_1a_1}\cdots R_{i_ka_k}R^{j_1b_1}\cdots R^{j_lb_l}F^{i_1\dots i_k}_{j_1\dots j_l}\right)\left(\sum_{\substack{1\le r_1,\dots,r_k\le n\\ 1\le s_1,\dots,s_l\le n}} R_{a_1r_1}\cdots R_{a_kr_k}R^{b_1s_1}\cdots R^{b_ls_l}F^{r_1\dots r_k}_{s_1\dots s_l}\right)$$

$$= \sum_{\substack{1\le a_1,\dots,a_k\le n\\ 1\le b_1,\dots,b_l\le n}} \left(\sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n}} R_{i_1a_1}\cdots R_{i_ka_k}R^{j_1b_1}\cdots R^{j_lb_l}F^{i_1\dots i_k}_{j_1\dots j_l}\right)^2 \ge 0,$$

which is nonnegative, and where we have used the symmetry of the matrices R and R^{-1}. Moreover, if $\langle \mathbf{F},\mathbf{F}\rangle_{\mathbf{g}} = 0$, then

$$\sum_{\substack{1\le a_1,\dots,a_k\le n\\ 1\le b_1,\dots,b_l\le n}} \left(\sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n}} R_{i_1a_1}\cdots R_{i_ka_k}R^{j_1b_1}\cdots R^{j_lb_l}F^{i_1\dots i_k}_{j_1\dots j_l}\right)^2 = 0,$$

so for any $a_1,\dots,a_k,b_1,\dots,b_l$,

$$\sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n}} R_{i_1a_1}\cdots R_{i_ka_k}R^{j_1b_1}\cdots R^{j_lb_l}F^{i_1\dots i_k}_{j_1\dots j_l} = 0,$$

This can be rewritten as

$$\sum_{i_1=1}^{n} R_{i_1a_1}\left(\sum_{\substack{1\le i_2,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n}} R_{i_2a_2}\cdots R_{i_ka_k}R^{j_1b_1}\cdots R^{j_lb_l}F^{i_1\dots i_k}_{j_1\dots j_l}\right) = 0,$$

and it holds for every $a_1\in\{1,\dots,n\}$. Varying $a_1\in\{1,\dots,n\}$ gives a system of equations in $n\times n$ with a unique solution because R is invertible, namely

$$\sum_{\substack{1\le i_2,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n}} R_{i_2a_2}\cdots R_{i_ka_k}R^{j_1b_1}\cdots R^{j_lb_l}F^{i_1\dots i_k}_{j_1\dots j_l} = 0$$

for every $i_1\in\{1,\dots,n\}$.

Using the invertibility of R and R^{-1} and proceeding in the same way iteratively and inductively from i_2 through j_{l-1}, we conclude that

$$\sum_{j_l=1}^{n} R^{j_lb_l}F^{i_1\dots i_k}_{j_1\dots j_l} = 0$$

for any $i_1,\dots,i_k,j_1,\dots,j_{l-1} \in \{1,\dots,n\}$. Repeating the argument and using the invertibility of R^{-1}, we conclude that, for any $i_1,\dots,i_k,j_1,\dots,j_l \in \{1,\dots,n\}$,

$$F^{i_1\dots i_k}_{j_1\dots j_l} = 0.$$

Therefore $\mathbf{F} = 0$ and $\langle\ ,\ \rangle_{\mathbf{g}}$ is positive definite. Its coefficients in any tensor frame are smooth. In an orthonormal frame, the formula reduces to products of Kronecker deltas and shows that the tensor frame in the statement is orthonormal. Finally, decomposable tensors span each fiber, so the required property determines the inner product uniquely. □

We prove a property that will be useful later.

Proposition 2.3.4. Let $(M,\mathbf{g})$ be a smooth Riemannian manifold with or without boundary. If $\mathbf{F} \in \Gamma(T^{(k_1,l_1)}(TM))$ and $\mathbf{G} \in \Gamma(T^{(k_2,l_2)}(TM))$, then

$$|\mathbf{F}\otimes\mathbf{G}|_{\mathbf{g}} = |\mathbf{F}|_{\mathbf{g}}|\mathbf{G}|_{\mathbf{g}}.$$

Proof. Let $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ be a smooth local frame defined on an open set $U \subseteq M$. Suppose that $F^{i_1\dots i_{k_1}}_{j_1\dots j_{l_1}}$ and $G^{\tilde{i}_1\dots\tilde{i}_{k_2}}_{\tilde{j}_1\dots\tilde{j}_{l_2}}$ are the coordinates of $\mathbf{F}$ and $\mathbf{G}$ on U with respect to this smooth local frame. We know that the coordinates of $\mathbf{F}\otimes\mathbf{G}$ are given by

$$(\mathbf{F}\otimes\mathbf{G})^{i_1\dots i_{k_1+k_2}}_{j_1\dots j_{l_1+l_2}} = F^{i_1\dots i_{k_1}}_{j_1\dots j_{l_1}} G^{i_{k_1+1}\dots i_{k_1+k_2}}_{j_{l_1+1}\dots j_{l_1+l_2}}.$$

Then:

$$|\mathbf{F}\otimes\mathbf{G}|^2_{\mathbf{g}} =$$

$$\sum_{\substack{1\le i_1,\dots,i_{k_1+k_2}\le n\\ 1\le j_1,\dots,j_{l_1+l_2}\le n\\ 1\le r_1,\dots,r_{k_1+k_2}\le n\\ 1\le s_1,\dots s_{l_1+l_2}\le n}} g_{i_1r_1}\cdots g_{i_{k_1}r_{k_1}} g_{i_{k_1+1}r_{k_1+1}}\cdots g_{i_{k_1+k_2}r_{k_1+k_2}} g^{j_1s_1}\cdots g^{j_{l_1}s_{l_1}} g^{j_{l_1+1}s_{l_1+1}}\cdots g^{j_{l_1+l_2}s_{l_1+l_2}}$$

$$F^{i_1\dots i_{k_1}}_{j_1\dots j_{l_1}} G^{i_{k_1+1}\dots i_{k_1+k_2}}_{j_{l_1+1}\dots j_{l_1+l_2}} F^{r_1\dots r_{k_1}}_{s_1\dots s_{l_1}} G^{r_{k_1+1}\dots r_{k_1+k_2}}_{s_{l_1+1}\dots s_{l_1+l_2}} = \left(\sum_{\substack{1\le i_1,\dots,i_{k_1}\le n\\ 1\le j_1,\dots,j_{l_1}\le n\\ 1\le r_1,\dots,r_{k_1}\le n\\ 1\le s_1,\dots s_{l_1}\le n}} g_{i_1r_1}\cdots g_{i_{k_1}r_{k_1}} g^{j_1s_1}\cdots g^{j_{l_1}s_{l_1}} F^{i_1\dots i_{k_1}}_{j_1\dots j_{l_1}} F^{r_1\dots r_{k_1}}_{s_1\dots s_{l_1}}\right)$$

$$\left(\sum_{\substack{1\le i_{k_1+1},\dots,i_{k_1+k_2}\le n\\ 1\le j_{l_1+1},\dots,j_{l_1+l_2}\le n\\ 1\le r_{k_1+1},\dots,r_{k_1+k_2}\le n\\ 1\le s_{l_1+1},\dots s_{l_1+l_2}\le n}} g_{i_{k_1+1}r_{k_1+1}}\cdots g_{i_{k_1+k_2}r_{k_1+k_2}} g^{j_{l_1+1}s_{l_1+1}}\cdots g^{j_{l_1+l_2}s_{l_1+l_2}} G^{i_{k_1+1}\dots i_{k_1+k_2}}_{j_{l_1+1}\dots j_{l_1+l_2}} G^{r_{k_1+1}\dots r_{k_1+k_2}}_{s_{l_1+1}\dots s_{l_1+l_2}}\right)$$

$$= |\mathbf{F}|_{\mathbf{g}}^2 |\mathbf{G}|_{\mathbf{g}}^2.$$

□

2.4 Connections

A vector field takes values in different tangent spaces, so directly subtracting its values at two points has no intrinsic meaning. A connection resolves this problem by specifying how to differentiate a field in a direction and how to compare vectors along a curve. We begin with the Euclidean case, where the directional derivative provides the model that will later be transferred to the manifold.

Example 2.4.1. Consider the smooth manifold without boundary $\mathbb{R}^n$ and fields $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\mathbb{R}^n)$. Recall that vector fields $\mathbf{X}$ and $\mathbf{Y}$ on $\mathbb{R}^n$ can be regarded as smooth functions $\mathbf{X}, \mathbf{Y} \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$. The concept of directional derivative tells us that, for each $p \in \mathbb{R}^n$,

$$(D_{\mathbf{X}_p}(\mathbf{Y}))(p) = \sum_{i=1}^{n} X^i(p) \frac{\partial \mathbf{Y}}{\partial x^i}(p).$$

This directional derivative gives the rate of change of $\mathbf{Y}$ at p in the direction $\mathbf{X}(p)$. Varying $p \in \mathbb{R}^n$ yields a smooth vector field whose pointwise action is $(\nabla_{\mathbf{X}}\mathbf{Y})_p := (D_{\mathbf{X}_p}\mathbf{Y})(p) = \sum_{i=1}^{n} X^i(p) \frac{\partial \mathbf{Y}}{\partial x^i}(p)$. This notion of the derivative of one vector field with respect to another relies heavily on the vector space structure of $\mathbb{R}^n$, since the definition of directional derivative involves

$$(D_{\mathbf{X}_p}(\mathbf{Y}))(p) := \lim_{t \to 0} \frac{\mathbf{Y}(p + t\mathbf{X}_p) - \mathbf{Y}(p)}{t}$$

and in general $\mathbf{Y}(p) \in T_p\mathbb{R}^n$ and $\mathbf{Y}(p + t\mathbf{X}_p) \in T_{p+t\mathbf{X}_p}\mathbb{R}^n$, with both tangent spaces canonically identified with $\mathbb{R}^n$. Although this cannot be interpreted directly on manifolds, we can abstract the properties of $\nabla_{\mathbf{X}}\mathbf{Y}$ inherited directly from directional derivatives:

(a) If $f \in C^\infty(\mathbb{R}^n)$, then $\nabla_{f\mathbf{X}}\mathbf{Y} = f\nabla_{\mathbf{X}}\mathbf{Y}$.

(b) $\nabla_{\mathbf{X}}\mathbf{Y}$ is $\mathbb{R}$-linear in $\mathbf{Y}$.

(c) For each $f \in C^\infty(\mathbb{R}^n)$,

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}.$$

The function $\nabla \colon \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \longrightarrow \mathfrak{X}(\mathbb{R}^n)$ defined in this way is called the *trivial connection* or *Euclidean connection*.

The preceding example leads to the following definition, which provides an intrinsic notion of directional derivative on manifolds:

Definition 2.4.2. Let $\pi \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle over a smooth manifold M with or without boundary. A **connection on E** is a map

$$\nabla \colon \mathfrak{X}(M) \times \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E}), \qquad (\mathbf{X}, \mathbf{u}) \longmapsto \nabla_{\mathbf{X}}\mathbf{u},$$

satisfying:

(a) $\nabla_{\mathbf{X}}\mathbf{u}$ is $C^\infty(M)$-linear in $\mathbf{X}$.

(b) $\nabla_{\mathbf{X}}\mathbf{u}$ is $\mathbb{R}$-linear in $\mathbf{u}$.

(c) For each $f \in C^\infty(M)$,

$$\nabla_{\mathbf{X}}(f\mathbf{u}) = f\nabla_{\mathbf{X}}\mathbf{u} + (\mathbf{X}f)\mathbf{u}.$$

Exercise 2.4.3. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let ∇ be a connection on TM. Let $\mathbf{A}\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ be a bilinear map over $C^\infty(M)$, that is, one satisfying

$$\mathbf{A}(f\mathbf{X},\mathbf{Y}) = f\mathbf{A}(\mathbf{X},\mathbf{Y}) \quad \text{and} \quad \mathbf{A}(\mathbf{X},f\mathbf{Y}) = f\mathbf{A}(\mathbf{X},\mathbf{Y}), \qquad \forall f \in C^\infty(M),\ \forall \mathbf{X},\mathbf{Y} \in \mathfrak{X}(M).$$

Show that the map

$$\widetilde{\nabla}_{\mathbf{X}}\mathbf{Y} = \nabla_{\mathbf{X}}\mathbf{Y} + \mathbf{A}(\mathbf{X},\mathbf{Y}), \qquad \mathbf{X},\mathbf{Y} \in \mathfrak{X}(M),$$

also defines a connection on TM.

Remark 2.4.4. If ∇ is a connection on TM and $(U,(x^i))$ is a coordinate chart on M, then ∇ determines n^3 smooth functions $\Gamma_{ij}^k\colon U \longrightarrow \mathbb{R}$ such that

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k.$$

These functions completely describe the local behavior of the connection. They are therefore particularly important and are called the ***Christoffel symbols***.

Example 2.4.5. Let $M = \mathbb{R}^n$ be regarded as a smooth manifold without boundary, and let ∇ be the Euclidean connection defined in the preceding example. Recall that, for each $\mathbf{X},\mathbf{Y} \in \mathfrak{X}(\mathbb{R}^n)$,

$$(\nabla_{\mathbf{X}}\mathbf{Y})_p = (D_{\mathbf{X}_p}\mathbf{Y})(p) = \sum_{i=1}^n X^i(p)\frac{\partial \mathbf{Y}}{\partial x^i}(p).$$

Now consider the canonical coordinate frame $\{\partial_1,\dots,\partial_n\}$, where ∂_i denotes the constant vector field whose ith component is 1 and whose other components are 0. Since the fields ∂_i are constant, their partial derivatives vanish:

$$\frac{\partial(\partial_j)}{\partial x^i} = 0 \quad \text{for all } i,j \in \{1,\dots,n\}.$$

The definition of ∇ then gives

$$\nabla_{\partial_i}\partial_j = 0 \quad \text{for all } i,j \in \{1,\dots,n\}.$$

Thus the coefficients of the Euclidean connection in Cartesian coordinates satisfy

$$\Gamma_{ij}^k = 0, \quad \text{for all } i,j,k \in \{1,\dots,n\}.$$

Remark 2.4.6. A connection on $\mathbf{E}$ is $C^\infty(M)$-linear in $\mathbf{X}$, so each $\sigma \in \Gamma(\mathbf{E})$ defines a bundle homomorphism

$$\nabla\sigma \in \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M),\Gamma(\mathbf{E})).$$

By Proposition 1.7.31, there is a canonical isomorphism

$$\mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M),\Gamma(\mathbf{E})) \cong \Gamma(\mathrm{Hom}(TM,\mathbf{E}))$$

and by Proposition 1.10.14, we know that $\mathrm{Hom}(TM,\mathbf{E}) \cong (TM)^*\otimes\mathbf{E} = T^{(0,1)}(TM)\otimes\mathbf{E}$. We therefore conclude that

$$\nabla\sigma \in \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M),\Gamma(\mathbf{E})) \cong \Gamma(T^{(0,1)}(TM)\otimes\mathbf{E}), \quad \forall\sigma\in\Gamma(\mathbf{E}).$$

The following additional properties uniquely characterize the Euclidean connection:

Proposition 2.4.7. Let $M=\mathbb{R}^n$, a smooth manifold without boundary, have its usual inner product $\langle\cdot,\cdot\rangle$, and let ∇ be the Euclidean connection defined, for $\mathbf{X},\mathbf{Y}\in\mathfrak{X}(\mathbb{R}^n)$, by

$$(\nabla_{\mathbf{X}}\mathbf{Y})_p = (D_{\mathbf{X}_p}\mathbf{Y})(p) = \sum_{i=1}^n X^i(p)\partial_i\mathbf{Y}(p).$$

Then ∇ satisfies the following properties:

(a) For any $\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathfrak{X}(\mathbb{R}^n)$,

$$\mathbf{X}(\langle\mathbf{Y},\mathbf{Z}\rangle) = \langle\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle + \langle\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\rangle.$$

(b) For any $\mathbf{X},\mathbf{Y}\in\mathfrak{X}(\mathbb{R}^n)$,

$$[\mathbf{X},\mathbf{Y}] = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X},$$

where $[\mathbf{X},\mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f))$ for every $f\in C^\infty(\mathbb{R}^n)$.

Moreover, ∇ is the unique connection on $\mathbb{R}^n$ satisfying both properties.

Proof. Let $\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathfrak{X}(\mathbb{R}^n)$.

To prove the first property, write $\mathbf{Y}=(Y^1,\dots,Y^n)$ and $\mathbf{Z}=(Z^1,\dots,Z^n)$. Since $\langle\mathbf{Y},\mathbf{Z}\rangle = \sum_{k=1}^n Y^kZ^k$, we have

$$\begin{aligned}\mathbf{X}(\langle\mathbf{Y},\mathbf{Z}\rangle) &= \sum_{i,k=1}^n X^i\partial_i(Y^kZ^k)\\ &= \sum_{i,k=1}^n X^i\big((\partial_iY^k)Z^k + Y^k(\partial_iZ^k)\big).\end{aligned}$$

By the definition of ∇, we have $(\nabla_{\mathbf{X}}\mathbf{Y})^k = \sum_{i=1}^n X^i\partial_iY^k$ and $(\nabla_{\mathbf{X}}\mathbf{Z})^k = \sum_{i=1}^n X^i\partial_iZ^k$. Substituting,

$$\mathbf{X}(\langle\mathbf{Y},\mathbf{Z}\rangle) = \sum_{k=1}^n(\nabla_{\mathbf{X}}\mathbf{Y})^kZ^k + \sum_{k=1}^n Y^k(\nabla_{\mathbf{X}}\mathbf{Z})^k = \langle\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle + \langle\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\rangle.$$

For the second property, recall that $[\mathbf{X},\mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f))$. Applying this to x^k,

$$\begin{aligned}[\mathbf{X},\mathbf{Y}](x^k) &= \mathbf{X}(Y^k) - \mathbf{Y}(X^k) \\ &= \sum_{i=1}^{n} \left(X^i \partial_i Y^k - Y^i \partial_i X^k\right),\end{aligned}$$

so $[\mathbf{X},\mathbf{Y}]^k = \sum_{i=1}^{n} \left(X^i \partial_i Y^k - Y^i \partial_i X^k\right)$. But also $(\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X})^k = \sum_{i=1}^{n} \left(X^i \partial_i Y^k - Y^i \partial_i X^k\right)$, and therefore $[\mathbf{X},\mathbf{Y}] = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X}$.

For uniqueness, let $\widetilde{\nabla}$ be another connection satisfying (a) and (b). In the coordinate frame $\{\partial_1,\dots,\partial_n\}$, write $\widetilde{\nabla}_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$. From (b), since $[\partial_i,\partial_j] = 0$, we obtain $\widetilde{\nabla}_{\partial_i}\partial_j = \widetilde{\nabla}_{\partial_j}\partial_i$, that is,

$$\Gamma_{ij}^k = \Gamma_{ji}^k. \tag{2.2}$$

By (a), using the fact that $\langle \partial_j, \partial_k \rangle = \delta_{jk}$ is constant,

$$0 = \partial_i \langle \partial_j, \partial_k \rangle = \langle \widetilde{\nabla}_{\partial_i}\partial_j, \partial_k \rangle + \langle \partial_j, \widetilde{\nabla}_{\partial_i}\partial_k \rangle = \Gamma_{ij}^k + \Gamma_{ik}^j, \tag{2.3}$$

whence

$$\Gamma_{ij}^k = -\Gamma_{ik}^j. \tag{2.4}$$

Let us show that (2.2) and (2.4) imply $\Gamma_{ij}^k = 0$. If $j = k$, then $\Gamma_{ij}^j = -\Gamma_{ij}^j$, hence $\Gamma_{ij}^j = 0$. If $j \neq k$, successively applying (2.2) and (2.4) gives

$$\Gamma_{ij}^k = -\Gamma_{ik}^j = -\Gamma_{ki}^j = \Gamma_{kj}^i = \Gamma_{jk}^i = -\Gamma_{ij}^k,$$

whence $\Gamma_{ij}^k = 0$.
Consequently, $\widetilde{\nabla}_{\partial_i}\partial_j = 0$, and $\widetilde{\nabla}$ agrees with the Euclidean connection. □

As the preceding example shows, these properties completely characterize the Euclidean connection and relate it to the standard Riemannian metric on $\mathbb{R}^n$. We formalize these two properties on an arbitrary Riemannian manifold as follows:

Definition 2.4.8. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let ∇ be a connection on TM. We say that:

(a) ∇ is **compatible with g** if, for any $\mathbf{X},\mathbf{Y},\mathbf{Z} \in \mathfrak{X}(M)$, $\mathbf{X}(\langle \mathbf{Y},\mathbf{Z} \rangle) = \langle \nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z} \rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle$.

(b) ∇ is **torsion-free** if, for any $\mathbf{X},\mathbf{Y} \in \mathfrak{X}(M)$, $[\mathbf{X},\mathbf{Y}] = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X}$, where

$$[\mathbf{X},\mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)).$$

These two conditions are independent, as the following example shows:

Example 2.4.9. Let ∇ be the Euclidean connection on the manifold without boundary $(\mathbb{R}^n,\overline{\mathbf{g}})$, defined by directional derivatives.
Consider two modifications of ∇ that alter torsion and metric compatibility, respectively. These modifica-

tions are obtained by adding a $C^\infty(M)$-bilinear term to the connection, as justified by Exercise 2.4.3.

(a) *A metric-compatible connection with torsion.*

On $\mathbb{R}^3$, define, for $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\mathbb{R}^3)$,

$$\widetilde{\nabla}_{\mathbf{X}}\mathbf{Y} = \nabla_{\mathbf{X}}\mathbf{Y} + \mathbf{X} \times \mathbf{Y},$$

where $\times$ denotes the usual cross product on $\mathbb{R}^3$. The additional term $(\mathbf{X}, \mathbf{Y}) \mapsto \mathbf{X} \times \mathbf{Y}$ is bilinear over $C^\infty(\mathbb{R}^3)$, so $\widetilde{\nabla}$ defines a connection.

To verify that $\widetilde{\nabla}$ preserves the metric, note that

$$\mathbf{X}\big(\langle \mathbf{Y}, \mathbf{Z} \rangle\big) = \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle,$$

and that the cross product satisfies

$$\langle \mathbf{X} \times \mathbf{Y}, \mathbf{Z} \rangle = -\langle \mathbf{Y}, \mathbf{X} \times \mathbf{Z} \rangle.$$

Thus,

$$\begin{aligned}\langle \widetilde{\nabla}_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle + \langle \mathbf{Y}, \widetilde{\nabla}_{\mathbf{X}}\mathbf{Z} \rangle &= \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle + \langle \mathbf{X} \times \mathbf{Y}, \mathbf{Z} \rangle + \langle \mathbf{Y}, \mathbf{X} \times \mathbf{Z} \rangle \\ &= \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle = \mathbf{X}\big(\langle \mathbf{Y}, \mathbf{Z} \rangle\big),\end{aligned}$$

so $\widetilde{\nabla}$ preserves the metric.

The torsion of $\widetilde{\nabla}$ is given by

$$\begin{aligned}T(\mathbf{X}, \mathbf{Y}) &= \widetilde{\nabla}_{\mathbf{X}}\mathbf{Y} - \widetilde{\nabla}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] \\ &= (\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]) + (\mathbf{X} \times \mathbf{Y} - \mathbf{Y} \times \mathbf{X}) \\ &= 2\mathbf{X} \times \mathbf{Y},\end{aligned}$$

which is nonzero in general. Consequently, $\widetilde{\nabla}$ preserves the metric but is not torsion-free.

(b) *A torsion-free connection that does not preserve the metric.*

Let $\mathbf{V} \in \mathfrak{X}(\mathbb{R}^n)$ be a smooth vector field, and define

$$\widehat{\nabla}_{\mathbf{X}}\mathbf{Y} = \nabla_{\mathbf{X}}\mathbf{Y} + \langle \mathbf{X}, \mathbf{Y} \rangle \mathbf{V}.$$

The additional term $(\mathbf{X}, \mathbf{Y}) \mapsto \langle \mathbf{X}, \mathbf{Y} \rangle \mathbf{V}$ is symmetric in $\mathbf{X}$ and $\mathbf{Y}$ and bilinear over $C^\infty(\mathbb{R}^n)$, so $\widehat{\nabla}$ is also a connection.

The torsion vanishes, since

$$\begin{aligned}\widehat{\nabla}_{\mathbf{X}}\mathbf{Y} - \widehat{\nabla}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] &= \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] + \langle \mathbf{X}, \mathbf{Y} \rangle \mathbf{V} - \langle \mathbf{Y}, \mathbf{X} \rangle \mathbf{V} \\ &= 0.\end{aligned}$$

We now check whether $\widehat{\nabla}$ preserves the metric. We have

$$\begin{aligned}\mathbf{X}\big(\langle \mathbf{Y},\mathbf{Z}\rangle\big) &= \langle \nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle + \langle \mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\rangle,\\ \langle \widehat{\nabla}_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle &= \langle \nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle + \langle \mathbf{X},\mathbf{Y}\rangle\langle \mathbf{V},\mathbf{Z}\rangle,\\ \langle \mathbf{Y},\widehat{\nabla}_{\mathbf{X}}\mathbf{Z}\rangle &= \langle \mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\rangle + \langle \mathbf{X},\mathbf{Z}\rangle\langle \mathbf{Y},\mathbf{V}\rangle.\end{aligned}$$

Substituting,

$$\mathbf{X}\big(\langle \mathbf{Y},\mathbf{Z}\rangle\big) - \langle \widehat{\nabla}_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle - \langle \mathbf{Y},\widehat{\nabla}_{\mathbf{X}}\mathbf{Z}\rangle = -\langle \mathbf{X},\mathbf{Y}\rangle\langle \mathbf{V},\mathbf{Z}\rangle - \langle \mathbf{X},\mathbf{Z}\rangle\langle \mathbf{Y},\mathbf{V}\rangle,$$

which does not vanish in general. Therefore $\widehat{\nabla}$ is torsion-free but does not preserve the Euclidean metric.

Like $\mathbb{R}^n$, every Riemannian manifold admits a unique metric-compatible, torsion-free connection, known as the Levi–Civita connection. This result is called the ***fundamental theorem of Riemannian geometry***.

Theorem 2.4.10. (Fundamental theorem of Riemannian geometry) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. Then there exists a unique connection ∇ on TM that is compatible with $\mathbf{g}$ and torsion-free. It is called the **Levi–Civita connection of g**.

Proof. If a connection is metric-compatible and torsion-free, then

$$\begin{aligned}\mathbf{X}\langle \mathbf{Y},\mathbf{Z}\rangle &= \langle \nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle + \langle \mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\rangle,\\ \mathbf{Y}\langle \mathbf{Z},\mathbf{X}\rangle &= \langle \nabla_{\mathbf{Y}}\mathbf{Z},\mathbf{X}\rangle + \langle \mathbf{Z},\nabla_{\mathbf{Y}}\mathbf{X}\rangle,\\ \mathbf{Z}\langle \mathbf{X},\mathbf{Y}\rangle &= \langle \nabla_{\mathbf{Z}}\mathbf{X},\mathbf{Y}\rangle + \langle \mathbf{X},\nabla_{\mathbf{Z}}\mathbf{Y}\rangle.\end{aligned}$$

Add the first two identities, subtract the third, and use $\nabla_{\mathbf{X}}\mathbf{Y}-\nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X},\mathbf{Y}]$. This necessarily gives the Koszul formula

$$\begin{aligned}2\langle \nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle = \mathbf{X}\langle \mathbf{Y},\mathbf{Z}\rangle + \mathbf{Y}\langle \mathbf{Z},\mathbf{X}\rangle - \mathbf{Z}\langle \mathbf{X},\mathbf{Y}\rangle\\ - \langle \mathbf{Y},[\mathbf{X},\mathbf{Z}]\rangle - \langle \mathbf{Z},[\mathbf{Y},\mathbf{X}]\rangle + \langle \mathbf{X},[\mathbf{Z},\mathbf{Y}]\rangle.\end{aligned}$$

Since the metric is nondegenerate, this identity determines $\nabla_{\mathbf{X}}\mathbf{Y}$ uniquely.
For existence, denote the right-hand side of the preceding formula by $\mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z})$. Substituting the identities

$$[f\mathbf{X},\mathbf{Z}] = f[\mathbf{X},\mathbf{Z}] - \mathbf{Z}(f)\mathbf{X}, \qquad [\mathbf{Z},f\mathbf{Y}] = f[\mathbf{Z},\mathbf{Y}] + \mathbf{Z}(f)\mathbf{Y}$$

and canceling the terms containing derivatives of f shows that

$$\mathcal{K}(f\mathbf{X},\mathbf{Y},\mathbf{Z}) = f\mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z}), \qquad \mathcal{K}(\mathbf{X},\mathbf{Y},f\mathbf{Z}) = f\mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z}).$$

Likewise,

$$\mathcal{K}(\mathbf{X},f\mathbf{Y},\mathbf{Z}) = f\mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z}) + 2\mathbf{X}(f)\langle \mathbf{Y},\mathbf{Z}\rangle.$$

Therefore, for each $\mathbf{X},\mathbf{Y}$, the function $\mathbf{Z}\mapsto \frac{1}{2}\mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is $C^\infty(M)$-linear. Nondegeneracy of the metric defines a unique vector field $\nabla_{\mathbf{X}}\mathbf{Y}$ by

$$2\langle \nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle = \mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z}) \quad \text{for every } \mathbf{Z}.$$

The preceding three identities prove the connection rules

$$\nabla_{f\mathbf{X}}\mathbf{Y} = f\nabla_{\mathbf{X}}\mathbf{Y}, \qquad \nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + \mathbf{X}(f)\mathbf{Y}.$$

Smoothness is checked by evaluating on a local frame, since the resulting components are combinations of smooth functions and their derivatives.
Finally, the Koszul formula itself gives

$$\mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z}) + \mathcal{K}(\mathbf{X},\mathbf{Z},\mathbf{Y}) = 2\mathbf{X}\langle \mathbf{Y},\mathbf{Z}\rangle$$

and

$$\mathcal{K}(\mathbf{X},\mathbf{Y},\mathbf{Z}) - \mathcal{K}(\mathbf{Y},\mathbf{X},\mathbf{Z}) = 2\langle [\mathbf{X},\mathbf{Y}],\mathbf{Z}\rangle.$$

The first identity proves metric compatibility; the second, together with nondegeneracy, proves that the torsion vanishes. □

The Levi–Civita connection will be fundamental in nonlinear analysis on manifolds, since most spaces we work with require a metric-compatible connection. The following corollary collects its local expressions and basic properties.

Corollary 2.4.11. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let ∇ be its Levi–Civita connection.

(a) For any $\mathbf{X},\mathbf{Y},\mathbf{Z} \in \mathfrak{X}(M)$, the ***Koszul formula*** holds:

$$\langle \nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle = \frac{1}{2}\left(\mathbf{X}\langle \mathbf{Y},\mathbf{Z}\rangle + \mathbf{Y}\langle \mathbf{Z},\mathbf{X}\rangle - \mathbf{Z}\langle \mathbf{X},\mathbf{Y}\rangle - \langle \mathbf{Y},[\mathbf{X},\mathbf{Z}]\rangle - \langle \mathbf{Z},[\mathbf{Y},\mathbf{X}]\rangle + \langle \mathbf{X},[\mathbf{Z},\mathbf{Y}]\rangle\right).$$

(b) In any coordinate chart (U,φ), we have

$$\Gamma_{ij}^{k} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

(c) In a local frame $(\mathbf{E}_i)$ defined on $U \subseteq M$, if $c_{ij}^{k}\colon U \longrightarrow \mathbb{R}$ are the n^3 smooth functions defined by

$$[\mathbf{E}_i,\mathbf{E}_j] = c_{ij}^{k}\mathbf{E}_k,$$

then the coefficients of the Levi–Civita connection with respect to this frame are

$$\Gamma_{ij}^{k} = \frac{1}{2}g^{kl}\left(\mathbf{E}_i g_{jl} + \mathbf{E}_j g_{il} - \mathbf{E}_l g_{ij} - g_{jm}c_{il}^{m} - g_{lm}c_{ji}^{m} + g_{im}c_{lj}^{m}\right).$$

(d) If $(\mathbf{E}_i)$ is orthonormal, the expression reduces to $\Gamma_{ij}^{k} = \frac{1}{2}\left(c_{ij}^{k} - c_{ik}^{j} - c_{jk}^{i}\right)$.

Proof. Part (1) is the formula obtained in the proof of the preceding theorem. In a coordinate frame, the brackets $[\partial_i,\partial_j]$ vanish. Taking $\mathbf{X} = \partial_i$, $\mathbf{Y} = \partial_j$, and $\mathbf{Z} = \partial_l$ gives

$$2g_{kl}\Gamma_{ij}^{k} = \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}.$$

Contracting with g^{lr} gives (2).
For an arbitrary frame $(\mathbf{E}_i)$, substitute $[\mathbf{E}_i, \mathbf{E}_j] = c_{ij}^k \mathbf{E}_k$ into the Koszul formula. We obtain

$$2g_{kl}\Gamma_{ij}^k = \mathbf{E}_i g_{jl} + \mathbf{E}_j g_{il} - \mathbf{E}_l g_{ij} - g_{jm}c_{il}^m - g_{lm}c_{ji}^m + g_{im}c_{lj}^m,$$

and contraction with the inverse metric proves (3). If the frame is orthonormal, $g_{ij} = \delta_{ij}$ and its derivatives vanish; the formula reduces to

$$\Gamma_{ij}^k = \frac{1}{2}\left(c_{ij}^k - c_{ik}^j - c_{jk}^i\right),$$

which is (4). □

The following lemma collects the transformation rules for the Christoffel symbols, the components of the metric, and the components of its inverse.

Lemma 2.4.12. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. Let (U, φ) and (V, ψ) be two charts on M such that $U \cap V \neq \varnothing$. Denote by g_{ij}, g^{ij}, and Γ_{ij}^k the components of the metric, the inverse metric, and the Christoffel symbols of the Levi–Civita connection in the coordinates $\varphi = (x^1, \dots, x^n)$, and by $\widetilde{g}_{ab}$, $\widetilde{g}^{ab}$, and $\widetilde{\Gamma}_{ab}^c$ the corresponding components in the coordinates $\psi = (y^1, \dots, y^n)$. Then, for every $p \in U \cap V$, the following formulas hold:

(a) The metric components transform according to

$$g_{ij}(p) = \frac{\partial y^a}{\partial x^i}(p)\frac{\partial y^b}{\partial x^j}(p)\,\widetilde{g}_{ab}(p).$$

(b) The inverse metric components transform according to

$$g^{ij}(p) = \frac{\partial x^i}{\partial y^a}(p)\frac{\partial x^j}{\partial y^b}(p)\,\widetilde{g}^{ab}(p).$$

(c) The Christoffel symbols transform according to

$$\Gamma_{ij}^k(p) = \frac{\partial x^k}{\partial y^c}(p)\frac{\partial y^a}{\partial x^i}(p)\frac{\partial y^b}{\partial x^j}(p)\,\widetilde{\Gamma}_{ab}^c(p) + \frac{\partial x^k}{\partial y^c}(p)\frac{\partial^2 y^c}{\partial x^i \partial x^j}(p).$$

Proof. Let $p \in U \cap V$. We begin with the metric. By the chain rule,

$$\partial_i\Big|_p = \frac{\partial y^a}{\partial x^i}(p)\,\partial_a\Big|_p.$$

Therefore,

$$g_{ij}(p) = \mathbf{g}_p\left(\partial_i\Big|_p, \partial_j\Big|_p\right) = \frac{\partial y^a}{\partial x^i}(p)\frac{\partial y^b}{\partial x^j}(p)\,\mathbf{g}_p\left(\partial_a\Big|_p, \partial_b\Big|_p\right).$$

Since $\mathbf{g}_p\left(\partial_a\big|_p, \partial_b\big|_p\right) = \widetilde{g}_{ab}(p)$, we obtain

$$g_{ij}(p) = \frac{\partial y^a}{\partial x^i}(p)\frac{\partial y^b}{\partial x^j}(p)\,\widetilde{g}_{ab}(p).$$

We turn to the inverse metric. The preceding identity can be written in matrix form as

$$G(p) = J(p)^T\,\widetilde{G}(p)\,J(p),$$

where $G(p) = \left(g_{ij}(p)\right)$, $\widetilde{G}(p) = \left(\widetilde{g}_{ab}(p)\right)$, and $J(p) = \left(\dfrac{\partial y^a}{\partial x^i}(p)\right)$. Taking inverses in this equality gives

$$G(p)^{-1} = J(p)^{-1}\,\widetilde{G}(p)^{-1}\left(J(p)^T\right)^{-1}.$$

Since

$$J(p)^{-1} = \left(\frac{\partial x^i}{\partial y^a}(p)\right),$$

it follows that

$$g^{ij}(p) = \frac{\partial x^i}{\partial y^a}(p)\frac{\partial x^j}{\partial y^b}(p)\,\widetilde{g}^{ab}(p).$$

Finally, we prove the transformation formula for the Christoffel symbols. By definition,

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k.$$

Using again the relation between the coordinate vector fields,

$$\partial_i = \frac{\partial y^a}{\partial x^i}\partial_a, \qquad \partial_j = \frac{\partial y^b}{\partial x^j}\partial_b,$$

and the Leibniz rule for the connection, we have

$$\nabla_{\partial_i}\partial_j = \nabla_{\frac{\partial y^a}{\partial x^i}\partial_a}\left(\frac{\partial y^b}{\partial x^j}\partial_b\right) = \frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}\nabla_{\partial_a}\partial_b + \frac{\partial y^a}{\partial x^i}\partial_a\left(\frac{\partial y^c}{\partial x^j}\right)\partial_c.$$

Since $\nabla_{\partial_a}\partial_b = \widetilde{\Gamma}_{ab}^c\partial_c$, this gives

$$\nabla_{\partial_i}\partial_j = \left(\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}\widetilde{\Gamma}_{ab}^c + \frac{\partial^2 y^c}{\partial x^i\partial x^j}\right)\partial_c.$$

On the other hand, $\partial_c = \dfrac{\partial x^k}{\partial y^c}\partial_k$. Substituting this identity into the preceding expression, we obtain

$$\nabla_{\partial_i}\partial_j = \left(\frac{\partial x^k}{\partial y^c}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}\widetilde{\Gamma}_{ab}^c + \frac{\partial x^k}{\partial y^c}\frac{\partial^2 y^c}{\partial x^i\partial x^j}\right)\partial_k.$$

Comparing with $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k$, we conclude that

$$\Gamma_{ij}^k(p) = \frac{\partial x^k}{\partial y^c}(p)\frac{\partial y^a}{\partial x^i}(p)\frac{\partial y^b}{\partial x^j}(p)\,\widetilde{\Gamma}_{ab}^c(p) + \frac{\partial x^k}{\partial y^c}(p)\frac{\partial^2 y^c}{\partial x^i\partial x^j}(p).$$

This completes the proof. □

The connection on TM induces connections on all tensor bundles. The construction begins with the dual and the tensor product.

Proposition 2.4.13. (Induced connections on tensor bundles) Let M be a smooth manifold with or without boundary, and let ∇ be a connection on TM. Then there exists a unique family of connections, also denoted by ∇, on

$$T^{(k,l)}(TM) = TM^{\otimes k} \otimes (T^*M)^{\otimes l}, \qquad k, l \in \mathbb{N}_0,$$

with the following properties.

(a) On TM it agrees with the original connection, and on $T^{(0,0)}(TM) = M \times \mathbb{R}$ it is given by $\nabla_{\mathbf{X}} f = \mathbf{X}(f)$.

(b) The dual connection satisfies

$$(\nabla_{\mathbf{X}} \omega)(\mathbf{Y}) = \mathbf{X}\big(\omega(\mathbf{Y})\big) - \omega(\nabla_{\mathbf{X}} \mathbf{Y}), \tag{2.5}$$

and hence the natural pairing is parallel:

$$\mathbf{X}\big(\omega(\mathbf{Y})\big) = (\nabla_{\mathbf{X}} \omega)(\mathbf{Y}) + \omega(\nabla_{\mathbf{X}} \mathbf{Y}).$$

(c) If $\mathbf{S} \in \Gamma(T^{(r,s)}(TM))$ and $\mathbf{T} \in \Gamma(T^{(p,q)}(TM))$, then

$$\nabla_{\mathbf{X}}(\mathbf{S} \otimes \mathbf{T}) = (\nabla_{\mathbf{X}} \mathbf{S}) \otimes \mathbf{T} + \mathbf{S} \otimes (\nabla_{\mathbf{X}} \mathbf{T}). \tag{2.6}$$

(d) If $\mathbf{T} \in \Gamma(T^{(k,l)}(TM))$, then

$$\begin{aligned} &(\nabla_{\mathbf{X}} \mathbf{T})(\omega^1, \dots, \omega^k, \mathbf{Y}_1, \dots, \mathbf{Y}_l) \\ &\quad = \mathbf{X}\big(\mathbf{T}(\omega^1, \dots, \omega^k, \mathbf{Y}_1, \dots, \mathbf{Y}_l)\big) \\ &\qquad - \sum_{a=1}^{k} \mathbf{T}(\omega^1, \dots, \nabla_{\mathbf{X}} \omega^a, \dots, \omega^k, \mathbf{Y}_1, \dots, \mathbf{Y}_l) \\ &\qquad - \sum_{b=1}^{l} \mathbf{T}(\omega^1, \dots, \omega^k, \mathbf{Y}_1, \dots, \nabla_{\mathbf{X}} \mathbf{Y}_b, \dots, \mathbf{Y}_l). \end{aligned} \tag{2.7}$$

(e) The connection commutes with every contraction between a contravariant and a covariant index. If C denotes one of these contractions,

$$\nabla_{\mathbf{X}}(CT) = C(\nabla_{\mathbf{X}} \mathbf{T}).$$

It also commutes with permutations of factors, symmetrization, and alternation. In particular, for differential forms,

$$\nabla_{\mathbf{X}}(\omega \wedge \eta) = (\nabla_{\mathbf{X}} \omega) \wedge \eta + \omega \wedge (\nabla_{\mathbf{X}} \eta).$$

If $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ is a local frame with dual coframe $(\varepsilon^1,\dots,\varepsilon^n)$ and $\nabla_{\mathbf{E}_a}\mathbf{E}_i = \Gamma_{ai}^{s}\mathbf{E}_s$, then

$$(\nabla_{\mathbf{E}_a}\mathbf{T})^{i_1\dots i_k}_{j_1\dots j_l} = \mathbf{E}_a\big(T^{i_1\dots i_k}_{j_1\dots j_l}\big) + \sum_{r=1}^{k}\Gamma_{as}^{i_r}T^{i_1\dots s\dots i_k}_{j_1\dots j_l} - \sum_{t=1}^{l}\Gamma_{aj_t}^{s}T^{i_1\dots i_k}_{j_1\dots s\dots j_l}. \tag{2.8}$$

Proof. We first construct the dual connection. For $\mathbf{X},\mathbf{Y}\in\mathfrak{X}(M)$ and $\omega\in\Omega^1(M)$, define $\nabla_{\mathbf{X}}\omega$ by (2.5). The right-hand side is linear over $C^\infty(M)$ in $\mathbf{Y}$. If $f\in C^\infty(M)$, then

$$\begin{aligned}\mathbf{X}\big(\omega(f\mathbf{Y})\big) - \omega(\nabla_{\mathbf{X}}(f\mathbf{Y})) &= \mathbf{X}(f)\omega(\mathbf{Y}) + f\mathbf{X}\big(\omega(\mathbf{Y})\big)\\ &\quad - \mathbf{X}(f)\omega(\mathbf{Y}) - f\,\omega(\nabla_{\mathbf{X}}\mathbf{Y})\\ &= f\left[\mathbf{X}(\omega(\mathbf{Y})) - \omega(\nabla_{\mathbf{X}}\mathbf{Y})\right].\end{aligned}$$

By the tensor criterion, this determines a 1-form. It is smooth because its evaluations on the elements of a local frame are smooth functions. Moreover,

$$\nabla_{f\mathbf{X}}\omega = f\nabla_{\mathbf{X}}\omega, \qquad \nabla_{\mathbf{X}}(f\omega) = \mathbf{X}(f)\omega + f\nabla_{\mathbf{X}}\omega,$$

as follows by substitution into (2.5). Thus it is a connection on T^*M. The pairing identity is the defining formula itself and uniquely determines $\nabla_{\mathbf{X}}\omega$, so the dual connection is unique.
Now consider, more generally, two smooth vector bundles $\mathbf{E},\mathbf{F}\to M$ equipped with connections $\nabla^{\mathbf{E}}$ and $\nabla^{\mathbf{F}}$. On decomposable local sections, we propose

$$\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}(\sigma\otimes\tau) = (\nabla^{\mathbf{E}}_{\mathbf{X}}\sigma)\otimes\tau + \sigma\otimes(\nabla^{\mathbf{F}}_{\mathbf{X}}\tau). \tag{2.9}$$

This rule respects the balancing relation. Indeed,

$$\nabla_{\mathbf{X}}((f\sigma)\otimes\tau) = \mathbf{X}(f)\sigma\otimes\tau + f(\nabla^{\mathbf{E}}_{\mathbf{X}}\sigma)\otimes\tau + f\sigma\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\tau,$$

whereas

$$\nabla_{\mathbf{X}}(\sigma\otimes(f\tau)) = f(\nabla^{\mathbf{E}}_{\mathbf{X}}\sigma)\otimes\tau + \mathbf{X}(f)\sigma\otimes\tau + f\sigma\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\tau.$$

Thus both representations of the same tensor yield the same result. On a trivializing open set, the isomorphism

$$\Gamma(\mathbf{E}|_U)\otimes_{C^\infty(U)}\Gamma(\mathbf{F}|_U)\longrightarrow\Gamma((\mathbf{E}\otimes\mathbf{F})|_U)$$

allows us to extend (2.9) by additivity to every section. This explicitly proves independence of the representation as a sum of decomposable tensors. In local frames, the formula contains only smooth coefficients and their derivatives, so it produces a smooth section. It also directly gives

$$\nabla_{f\mathbf{X}}\mathbf{S} = f\nabla_{\mathbf{X}}\mathbf{S}, \qquad \nabla_{\mathbf{X}}(f\mathbf{S}) = \mathbf{X}(f)\mathbf{S} + f\nabla_{\mathbf{X}}\mathbf{S}.$$

The local definitions agree on overlaps because they are characterized by the same rule on decomposable sections; they therefore glue to a global connection. Since these sections locally generate $\Gamma(\mathbf{E}\otimes\mathbf{F})$, the rule

determines the connection uniquely.

Apply this construction successively to TM and to the dual connection on T^*M. We obtain connections on $TM^{\otimes k} \otimes (T^*M)^{\otimes l}$. Uniqueness of the product connection shows that these connections are compatible with the canonical associative identifications, so they do not depend on how the tensor product is parenthesized. To obtain (2.7), contract $\mathbf{T} \otimes \omega^1 \otimes \cdots \otimes \omega^k \otimes \mathbf{Y}_1 \otimes \cdots \otimes \mathbf{Y}_l$ to a function. The tensor product rule and the parallel pairing identity give

$$\begin{aligned}\mathbf{X}\big(\mathbf{T}(\omega^1,\dots,\omega^k,\mathbf{Y}_1,\dots,\mathbf{Y}_l)\big) &= (\nabla_{\mathbf{X}}\mathbf{T})(\omega^1,\dots,\omega^k,\mathbf{Y}_1,\dots,\mathbf{Y}_l)\\ &\quad + \sum_{a=1}^{k} \mathbf{T}(\omega^1,\dots,\nabla_{\mathbf{X}}\omega^a,\dots,\omega^k,\mathbf{Y}_1,\dots,\mathbf{Y}_l)\\ &\quad + \sum_{b=1}^{l} \mathbf{T}(\omega^1,\dots,\omega^k,\mathbf{Y}_1,\dots,\nabla_{\mathbf{X}}\mathbf{Y}_b,\dots,\mathbf{Y}_l).\end{aligned}$$

Solving for the first term yields the intrinsic formula, including the negative signs of all the contravariant and covariant arguments on which $\mathbf{T}$ is evaluated.

An elementary contraction pairs a factor TM with a factor T^*M. When a decomposable tensor is differentiated, the two terms in which the connection acts on these factors combine, by (2.5), into the derivative of their pairing. The other factors remain unchanged. By linearity, it follows that $\nabla_{\mathbf{X}}(CT) = C(\nabla_{\mathbf{X}}\mathbf{T})$ for every contraction C.

If P_σ permutes factors of the same type, the product formula applied to a decomposable tensor shows term by term that $\nabla_{\mathbf{X}} P_\sigma = P_\sigma \nabla_{\mathbf{X}}$. Symmetrization and alternation are linear combinations of these operators and therefore also commute with the connection. Since the wedge product is the normalized alternation of the tensor product, we obtain

$$\nabla_{\mathbf{X}}(\boldsymbol{\omega} \wedge \eta) = (\nabla_{\mathbf{X}}\boldsymbol{\omega}) \wedge \eta + \boldsymbol{\omega} \wedge (\nabla_{\mathbf{X}}\eta).$$

Finally, the identity $\boldsymbol{\varepsilon}^j(\mathbf{E}_i) = \delta_i^j$ and compatibility with the pairing give

$$0 = \mathbf{E}_a(\delta_i^j) = (\nabla_{\mathbf{E}_a}\boldsymbol{\varepsilon}^j)(\mathbf{E}_i) + \boldsymbol{\varepsilon}^j(\nabla_{\mathbf{E}_a}\mathbf{E}_i),$$

and consequently

$$\nabla_{\mathbf{E}_a}\boldsymbol{\varepsilon}^j = -\Gamma_{as}^j \boldsymbol{\varepsilon}^s.$$

Differentiating

$$\mathbf{T} = T^{i_1\dots i_k}_{j_1\dots j_l}\, \mathbf{E}_{i_1} \otimes \cdots \otimes \mathbf{E}_{i_k} \otimes \boldsymbol{\varepsilon}^{j_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{j_l}$$

produces a positive sign for each factor $\mathbf{E}_{i_r}$ and a negative sign for each factor $\boldsymbol{\varepsilon}^{j_t}$. Reindexing these terms gives exactly (2.8).

The preceding rules determine the connection on all elements of a local tensor frame, and the Leibniz rule determines its action on coefficients. This also proves uniqueness of the entire family of induced connections. □

With this proposition, compatibility of the metric with the Levi–Civita connection can be rewritten as follows:

Theorem 2.4.14. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let ∇ be its Levi–Civita connection. The Levi–Civita connection is compatible with $\mathbf{g}$ in the sense that $\nabla_{\mathbf{X}}\mathbf{g} = 0$ for every

$\mathbf{X} \in \mathfrak{X}(M)$.

Proof. By Definition 2.4.8, since ∇ is a connection compatible with $\mathbf{g}$, we know that, for any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(M)$,

$$\nabla_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle = \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}\rangle,$$

so Proposition 2.4.13 gives

$$\nabla_{\mathbf{X}}\mathbf{g}(\mathbf{Y}, \mathbf{Z}) = \mathbf{X}(\mathbf{g}(\mathbf{Y}, \mathbf{Z})) - \mathbf{g}(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}) - \mathbf{g}(\mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z})$$

$$= \nabla_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle - \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle - \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}\rangle = 0.$$

□

In local coordinates, the covariant derivative of a tensor field takes the following form:

Lemma 2.4.15. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let ∇ be a connection on TM. Let $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ be a smooth local frame of TM over $U \subseteq M$, and let $(\varepsilon^1, \dots, \varepsilon^n)$ be its dual coframe. If $\mathbf{F}$ is a tensor field of type (k, l) written in this frame as

$$\mathbf{F} = \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ 1 \le j_1, \dots, j_l \le n}} F^{i_1 \dots i_k}_{j_1 \dots j_l}\, \mathbf{E}_{i_1} \otimes \cdots \otimes \mathbf{E}_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l},$$

then, for each $a \in \{1, \dots, n\}$,

$$\left(\nabla_{\mathbf{E}_a}\mathbf{F}\right)^{i_1 \dots i_k}_{j_1 \dots j_l} = \mathbf{E}_a\left(F^{i_1 \dots i_k}_{j_1 \dots j_l}\right) + \sum_{r=1}^{k} \Gamma^{i_r}_{as}\, F^{i_1 \dots s \dots i_k}_{j_1 \dots j_l} - \sum_{t=1}^{l} \Gamma^{s}_{a j_t}\, F^{i_1 \dots i_k}_{j_1 \dots s \dots j_l},$$

where Γ^c_{ai} are the connection coefficients of ∇ relative to the frame $(\mathbf{E}_1, \dots, \mathbf{E}_n)$, that is, $\nabla_{\mathbf{E}_a}\mathbf{E}_i = \Gamma^c_{ai}\mathbf{E}_c$.

Proof. Let $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ be a smooth local frame of TM over U, and let $(\varepsilon^1, \dots, \varepsilon^n)$ be the dual coframe, so that $\langle \varepsilon^j, \mathbf{E}_i\rangle = \delta^j_i$. Fix $a \in \{1, \dots, n\}$.
Since $\nabla_{\mathbf{E}_a}\mathbf{E}_i = \Gamma^s_{ai}\mathbf{E}_s$, the product rule for the pairing (Proposition 2.4.13) applied to $\langle \varepsilon^j, \mathbf{E}_i\rangle$ gives

$$0 = \mathbf{E}_a(\delta^j_i) = \nabla_{\mathbf{E}_a}\langle \varepsilon^j, \mathbf{E}_i\rangle = \langle \nabla_{\mathbf{E}_a}\varepsilon^j, \mathbf{E}_i\rangle + \langle \varepsilon^j, \nabla_{\mathbf{E}_a}\mathbf{E}_i\rangle = \langle \nabla_{\mathbf{E}_a}\varepsilon^j, \mathbf{E}_i\rangle + \Gamma^j_{ai},$$

and therefore $\langle \nabla_{\mathbf{E}_a}\varepsilon^j, \mathbf{E}_i\rangle = -\Gamma^j_{ai}$ for every i, which is equivalent to

$$\nabla_{\mathbf{E}_a}\varepsilon^j = -\Gamma^j_{as}\varepsilon^s.$$

Writing the tensor field $\mathbf{F}$ in the frame as

$$\mathbf{F} = \sum_{i_1, \dots, i_k = 1}^{n} \sum_{j_1, \dots, j_l = 1}^{n} F^{i_1 \dots i_k}_{j_1 \dots j_l}\, \mathbf{E}_{i_1} \otimes \cdots \otimes \mathbf{E}_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l},$$

and using the Leibniz rule for tensor products together with $\nabla_{\mathbf{E}_a} f = \mathbf{E}_a(f)$ yields a linear combination of

tensor fields in the frame, with terms arising from differentiating the coefficients and each factor $\mathbf{E}_{i_r}$ and $\boldsymbol{\varepsilon}^{j_t}$. Substituting $\nabla_{\mathbf{E}_a}\mathbf{E}_{i_r} = \Gamma^s_{ai_r}\mathbf{E}_s$ and $\nabla_{\mathbf{E}_a}\boldsymbol{\varepsilon}^{j_t} = -\Gamma^{j_t}_{as}\boldsymbol{\varepsilon}^s$ and rewriting in the basis of tensor fields associated with the frame, we conclude that

$$\left(\nabla_{\mathbf{E}_a}\mathbf{F}\right)^{i_1\dots i_k}_{j_1\dots j_l} = \mathbf{E}_a\left(F^{i_1\dots i_k}_{j_1\dots j_l}\right) + \sum_{r=1}^{k}\Gamma^{i_r}_{as}F^{i_1\dots s\dots i_k}_{j_1\dots j_l} - \sum_{t=1}^{l}\Gamma^{s}_{aj_t}F^{i_1\dots i_k}_{j_1\dots s\dots j_l}.$$

□

With this proposition, Theorem 2.4.14 takes the following form:

Corollary 2.4.16. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, with Levi–Civita connection ∇. In a local frame $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ with dual coframe $(\boldsymbol{\varepsilon}^1,\dots,\boldsymbol{\varepsilon}^n)$, we have

$$\mathbf{E}_a(g_{ij}) = \Gamma^s_{ai}g_{sj} + \Gamma^s_{aj}g_{is}.$$

Proof. Applying Theorem 2.4.14 and Lemma 2.4.15 to the $(0,2)$-tensor field $\mathbf{g}$ gives

$$0 = (\nabla_a\mathbf{g})_{ij} = \mathbf{E}_a(g_{ij}) - \Gamma^s_{ai}g_{sj} - \Gamma^s_{aj}g_{is}.$$

It follows directly that

$$\mathbf{E}_a(g_{ij}) = \Gamma^s_{ai}g_{sj} + \Gamma^s_{aj}g_{is}.$$

□

Corollary 2.4.17. Under the same hypotheses as Corollary 2.4.16, the inverse metric g^{ij} satisfies the relation

$$\mathbf{E}_a(g^{ij}) = -\left(\Gamma^i_{as}g^{sj} + \Gamma^j_{as}g^{is}\right).$$

Proof. Start from the identity $g^{ik}g_{kj} = \delta^i_j$. Applying E_a and using the Leibniz rule gives

$$E_a(g^{ik})g_{kj} + g^{ik}E_a(g_{kj}) = 0.$$

Substituting the expression for $E_a(g_{kj})$ given by Corollary 2.4.16, we have

$$E_a(g^{ik})g_{kj} + g^{ik}\left(\Gamma^s_{ak}g_{sj} + \Gamma^s_{aj}g_{ks}\right) = 0.$$

Expanding and using $g^{ik}g_{ks} = \delta^i_s$ yields

$$E_a(g^{ik})g_{kj} + g^{ik}\Gamma^s_{ak}g_{sj} + \Gamma^i_{aj} = 0.$$

Multiplying on the right by g^{jm} gives

$$E_a(g^{ik})\delta^m_k + g^{ik}\Gamma^s_{ak}\delta^m_s + \Gamma^i_{aj}g^{jm} = 0.$$

That is,

$$E_a(g^{im}) + g^{ik}\Gamma^m_{ak} + \Gamma^i_{aj}g^{jm} = 0.$$

Renaming dummy indices and rearranging, we conclude that

$$E_a(g^{ij}) = -\left(\Gamma^i_{as}g^{sj} + \Gamma^j_{as}g^{is}\right).$$

□

The connection on tensor bundles induced by the Levi–Civita connection is compatible with the induced inner product on tensors:

Proposition 2.4.18. (Compatibility of the Levi–Civita connection with the tensor inner product) Suppose that $(M, \mathbf{g})$ is a Riemannian manifold with or without boundary. The connection induced on each tensor bundle by the Levi–Civita connection is compatible with the inner product of tensor fields in the following sense:

$$\mathbf{X}\langle \mathbf{F}, \mathbf{G}\rangle = \langle \nabla_{\mathbf{X}}\mathbf{F}, \mathbf{G}\rangle + \langle \mathbf{F}, \nabla_{\mathbf{X}}\mathbf{G}\rangle$$

for any $\mathbf{X} \in \mathfrak{X}(M)$ and any $\mathbf{F}, \mathbf{G} \in \Gamma(T^{(k,l)}(TM))$.

Proof. The induced connection commutes with tensor products and contractions. It therefore suffices to verify the identity on simple tensors. Let

$$\mathbf{F} = \mathbf{Y}_1 \otimes \cdots \otimes \mathbf{Y}_k \otimes \boldsymbol{\alpha}^1 \otimes \cdots \otimes \boldsymbol{\alpha}^l$$

and use an analogous expression for $\mathbf{G}$, with factors $\mathbf{Z}_i$ and $\boldsymbol{\beta}^a$. Their inner product is the product of the pairings of their factors:

$$\langle \mathbf{F}, \mathbf{G}\rangle = \prod_{i=1}^{k}\langle \mathbf{Y}_i, \mathbf{Z}_i\rangle \prod_{a=1}^{l}\langle \boldsymbol{\alpha}^a, \boldsymbol{\beta}^a\rangle.$$

The ordinary Leibniz rule expresses $\mathbf{X}\langle \mathbf{F}, \mathbf{G}\rangle$ as a sum of terms in which one of these factors is differentiated. Compatibility of the Levi–Civita connection on TM gives

$$\mathbf{X}\langle \mathbf{Y}_i, \mathbf{Z}_i\rangle = \langle \nabla_{\mathbf{X}}\mathbf{Y}_i, \mathbf{Z}_i\rangle + \langle \mathbf{Y}_i, \nabla_{\mathbf{X}}\mathbf{Z}_i\rangle.$$

On the dual, the definition

$$(\nabla_{\mathbf{X}}\boldsymbol{\alpha})(\mathbf{Y}) = \mathbf{X}(\boldsymbol{\alpha}(\mathbf{Y})) - \boldsymbol{\alpha}(\nabla_{\mathbf{X}}\mathbf{Y})$$

and metric compatibility give the same identity for covectors. Grouping the terms in which a factor of $\mathbf{F}$ is differentiated and those in which a factor of $\mathbf{G}$ is differentiated yields exactly

$$\mathbf{X}\langle \mathbf{F}, \mathbf{G}\rangle = \langle \nabla_{\mathbf{X}}\mathbf{F}, \mathbf{G}\rangle + \langle \mathbf{F}, \nabla_{\mathbf{X}}\mathbf{G}\rangle.$$

Bilinearity extends the equality to arbitrary tensor fields. □

The total covariant derivative combines the derivatives in all directions into a single tensor and will serve as the geometric analogue of higher-order total derivatives.

Proposition 2.4.19. (The total covariant derivative) Let M be a smooth manifold with or without boundary, and let ∇ be a connection on TM. For each $\mathbf{F} \in \Gamma(T^{(k,l)}(TM))$, the map

$$\nabla\mathbf{F}\colon \underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{k \text{ times}}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{(l+1) \text{ times}} \longrightarrow C^\infty(M)$$

given by

$$(\nabla\mathbf{F})(\omega^1,\dots,\omega^k,\mathbf{X},\mathbf{Y}_1,\dots,\mathbf{Y}_l) = (\nabla_{\mathbf{X}}\mathbf{F})(\omega^1,\dots,\omega^k,\mathbf{Y}_1,\dots,\mathbf{Y}_l)$$

defines a smooth $(k,l+1)$-tensor field on M, called the **total covariant derivative of F**.

Proof. By Lemma 1.10.18, since $\nabla_{\mathbf{X}}\mathbf{F}$ is a (k,l)-tensor field, it is $C^\infty(M)$-multilinear in its $k+l$ arguments and is linear over $C^\infty(M)$ in $\mathbf{X}$ by the definition of a connection. Thus $\nabla\mathbf{F}$ defines a smooth $(k,l+1)$-tensor field on M. □

Lemma 2.4.15 can be expressed in terms of the total covariant derivative as follows:

Corollary 2.4.20. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, with a connection ∇, and let $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ be a smooth local frame of TM over $U \subseteq M$, with dual coframe $(\varepsilon^1,\dots,\varepsilon^n)$. If $\mathbf{F}$ is a tensor field of type (k,l), its total covariant derivative $\nabla\mathbf{F}$ takes the following form in this frame:

$$\nabla\mathbf{F} = \sum_{a=1}^{n}\sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n}} (\nabla_a\mathbf{F})^{i_1\dots i_k}_{j_1\dots j_l}\mathbf{E}_{i_1}\otimes\cdots\otimes\mathbf{E}_{i_k}\otimes\varepsilon^a\otimes\varepsilon^{j_1}\otimes\cdots\otimes\varepsilon^{j_l}.$$

In particular, its components are

$$(\nabla\mathbf{F})^{i_1\dots i_k}_{aj_1\dots j_l} = (\nabla_a\mathbf{F})^{i_1\dots i_k}_{j_1\dots j_l} = \mathbf{E}_a\left(F^{i_1\dots i_k}_{j_1\dots j_l}\right) + \sum_{r=1}^{k}\Gamma^{i_r}_{as}F^{i_1\dots s\dots i_k}_{j_1\dots j_l} - \sum_{t=1}^{l}\Gamma^{s}_{aj_t}F^{i_1\dots i_k}_{j_1\dots s\dots j_l},$$

where $\nabla_a := \nabla_{\mathbf{E}_a}$ and the connection coefficients Γ^c_{ab} are defined by $\nabla_{\mathbf{E}_a}\mathbf{E}_b = \Gamma^c_{ab}\mathbf{E}_c$.

Proof. The definition of the total covariant derivative implies

$$(\nabla\mathbf{F})(\varepsilon^{i_1},\dots,\varepsilon^{i_k},\mathbf{E}_a,\mathbf{E}_{j_1},\dots,\mathbf{E}_{j_l}) = (\nabla_{\mathbf{E}_a}\mathbf{F})(\varepsilon^{i_1},\dots,\varepsilon^{i_k},\mathbf{E}_{j_1},\dots,\mathbf{E}_{j_l}).$$

Lemma 2.4.15 computes the right-hand side as

$$\mathbf{E}_a\left(F^{i_1\dots i_k}_{j_1\dots j_l}\right) + \sum_{r=1}^{k}\Gamma^{i_r}_{as}F^{i_1\dots s\dots i_k}_{j_1\dots j_l} - \sum_{t=1}^{l}\Gamma^{s}_{aj_t}F^{i_1\dots i_k}_{j_1\dots s\dots j_l}.$$

These are precisely the components of $\nabla\mathbf{F}$ in the indicated frame and coframe; reconstructing the tensor from its components gives the first formula in the statement. □

Remark 2.4.21. Since $\nabla\mathbf{F}$ has been defined for a (k,l)-tensor field and is a $(k,l+1)$-tensor field, we can define $\nabla^2\mathbf{F} = \nabla(\nabla\mathbf{F})$, which is a $(k,l+2)$-tensor field. In general, we write $\nabla^s\mathbf{F} := \underbrace{\nabla(\nabla(\dots(\nabla\mathbf{F})))}_{s \text{ times}} \in \Gamma(T^{(k,l+s)}(TM))$.

For a function $u \in C^\infty(M)$, regarded as a tensor of type $(0,0)$, the same construction defines its iterated covariant derivatives $\nabla^s u$.

Connections on TM and its tensor bundles are related to the wedge product as follows:

Proposition 2.4.22. (Compatibility of the connection with the wedge product) Let M be a smooth manifold with or without boundary, and let ∇ be a connection on TM. Then, for any $k, l \geq 0$, if $\omega \in \Gamma(\Lambda^k T^*M)$ and $\eta \in \Gamma(\Lambda^l T^*M)$, we have

$$\nabla_X(\omega \wedge \eta) = (\nabla_X \omega) \wedge \eta + \omega \wedge (\nabla_X \eta) \qquad \text{for every } X \in \mathfrak{X}(M).$$

Proof. Let $\omega \in \Gamma(\Lambda^k T^*M)$ and $\eta \in \Gamma(\Lambda^l T^*M)$. By definition,

$$\omega \wedge \eta = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)^\sigma(\omega \otimes \eta),$$

where, for $\mathbf{T} \in \Gamma(T^{(0,k+l)}(TM))$ and $\sigma \in S_{k+l}$, we define

$$({}^\sigma\mathbf{T})(v_1, \dots, v_{k+l}) = \mathbf{T}(v_{\sigma(1)}, \dots, v_{\sigma(k+l)}).$$

For $\mathbf{T} \in \Gamma(T^{(0,k+l)}(TM))$, by (2.7),

$$(\nabla_X \mathbf{T})(w_1, \dots, w_{k+l}) = X(\mathbf{T}(w_1, \dots, w_{k+l})) - \sum_{j=1}^{k+l} \mathbf{T}(w_1, \dots, \nabla_X w_j, \dots, w_{k+l}).$$

Applying this to ${}^\sigma\mathbf{T}$ and evaluating on $(v_1, \dots, v_{k+l})$ gives

$$(\nabla_X({}^\sigma\mathbf{T}))(v_1, \dots, v_{k+l}) = X(\mathbf{T}(v_{\sigma(1)}, \dots, v_{\sigma(k+l)})) - \sum_{j=1}^{k+l} \mathbf{T}(v_{\sigma(1)}, \dots, \nabla_X v_{\sigma(j)}, \dots, v_{\sigma(k+l)})$$

$$= (\nabla_X \mathbf{T})(v_{\sigma(1)}, \dots, v_{\sigma(k+l)}) = ({}^\sigma(\nabla_X \mathbf{T}))(v_1, \dots, v_{k+l}),$$

whence $\nabla_X({}^\sigma\mathbf{T}) = {}^\sigma(\nabla_X \mathbf{T})$.
Differentiating the expression for $\omega \wedge \eta$ and using linearity,

$$\nabla_X(\omega \wedge \eta) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma) \nabla_X\big({}^\sigma(\omega \otimes \eta)\big).$$

From the preceding calculation,

$$\nabla_X(\omega \wedge \eta) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)^\sigma\big(\nabla_X(\omega \otimes \eta)\big).$$

By (2.6),

$$\nabla_X(\omega \otimes \eta) = (\nabla_X \omega) \otimes \eta + \omega \otimes (\nabla_X \eta).$$

Substituting,

$$\nabla_{\mathbf{X}}(\omega\wedge\eta)=\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)^{\sigma}\left((\nabla_{\mathbf{X}}\omega)\otimes\eta+\omega\otimes(\nabla_{\mathbf{X}}\eta)\right)$$

$$=\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)^{\sigma}((\nabla_{\mathbf{X}}\omega)\otimes\eta)+\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)^{\sigma}(\omega\otimes(\nabla_{\mathbf{X}}\eta)).$$

Recognizing the definition of the wedge product in each term, we conclude that

$$\nabla_{\mathbf{X}}(\omega\wedge\eta)=(\nabla_{\mathbf{X}}\omega)\wedge\eta+\omega\wedge(\nabla_{\mathbf{X}}\eta).$$

□

Another operation satisfying the Leibniz rule with respect to connections is interior multiplication:

Lemma 2.4.23. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, ∇ a connection on TM, and $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ a smooth local frame of TM with dual coframe $(\varepsilon^1,\dots,\varepsilon^n)$. For every tensor field $\mathbf{F}\in\Gamma(T^{(k,l+1)}(TM))$, we have

$$\nabla_a\left(\iota_{\mathbf{E}_a}\mathbf{F}\right)=\iota_{\nabla_a\mathbf{E}_a}\mathbf{F}+\iota_{\mathbf{E}_a}(\nabla_a\mathbf{F}).$$

Proof. By Proposition 1.10.24, if $\mathbf{F}$ has components $F^{i_1\dots i_k}_{j_1\dots j_{l+1}}$ in the frames $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ and $(\varepsilon^1,\dots,\varepsilon^n)$, then the components of $\iota_{\mathbf{E}_a}\mathbf{F}$ are $(\iota_{\mathbf{E}_a}\mathbf{F})^{i_1\dots i_k}_{j_1\dots j_l}=F^{i_1\dots i_k}_{aj_1\dots j_l}$. Applying Lemma 2.4.15 gives

$$(\nabla_a(\iota_{\mathbf{E}_a}\mathbf{F}))^{i_1\dots i_k}_{j_1\dots j_l}=\mathbf{E}_a\left(F^{i_1\dots i_k}_{aj_1\dots j_l}\right)+\sum_{r=1}^{k}\Gamma^{i_r}_{as}F^{i_1\dots s\dots i_k}_{aj_1\dots j_l}-\sum_{t=1}^{l}\Gamma^{s}_{aj_t}F^{i_1\dots i_k}_{aj_1\dots s\dots j_l}. \tag{2.10}$$

On the other hand, Remark 1.10.25 gives $\nabla_a\mathbf{E}_a=\Gamma^s_{aa}\mathbf{E}_s$, so we have

$$\left(\iota_{\nabla_a\mathbf{E}_a}\mathbf{F}\right)^{i_1\dots i_k}_{j_1\dots j_l}=\Gamma^s_{aa}F^{i_1\dots i_k}_{sj_1\dots j_l}.$$

Finally, by Lemma 2.4.15,

$$\left(\iota_{\mathbf{E}_a}(\nabla_a\mathbf{F})\right)^{i_1\dots i_k}_{j_1\dots j_l}=(\nabla_a\mathbf{F})^{i_1\dots i_k}_{aj_1\dots j_l}$$

$$=\mathbf{E}_a\left(F^{i_1\dots i_k}_{aj_1\dots j_l}\right)+\sum_{r=1}^{k}\Gamma^{i_r}_{as}F^{i_1\dots s\dots i_k}_{aj_1\dots j_l}-\Gamma^s_{aa}F^{i_1\dots i_k}_{sj_1\dots j_l}-\sum_{t=1}^{l}\Gamma^{s}_{aj_t}F^{i_1\dots i_k}_{aj_1\dots s\dots j_l}.$$

Therefore

$$(\nabla_a(\iota_{\mathbf{E}_a}\mathbf{F}))^{i_1\dots i_k}_{j_1\dots j_l}=\left(\iota_{\nabla_a\mathbf{E}_a}\mathbf{F}\right)^{i_1\dots i_k}_{j_1\dots j_l}+\left(\iota_{\mathbf{E}_a}(\nabla_a\mathbf{F})\right)^{i_1\dots i_k}_{j_1\dots j_l}.$$

□

The notion of a connection on tensor bundles can be generalized to the tensor product of two vector bundles as follows:

Proposition 2.4.24. Let $\mathbf{E},\mathbf{F}\longrightarrow M$ be smooth vector bundles over a smooth manifold M with or without boundary, equipped with connections $\nabla^{\mathbf{E}}$ and $\nabla^{\mathbf{F}}$ in the sense of Definition 2.4.2. Then:

(a) If $\mathbf{E}$ and $\mathbf{F}$ have connections $\nabla^{\mathbf{E}}$ and $\nabla^{\mathbf{F}}$, the tensor bundle $\mathbf{E}\otimes\mathbf{F}$ admits a unique connection

$\nabla^{\mathbf{E}\otimes\mathbf{F}}\colon \mathfrak{X}(M)\times\Gamma(\mathbf{E}\otimes\mathbf{F})\longrightarrow\Gamma(\mathbf{E}\otimes\mathbf{F})$ determined by the rule

$$\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}(\sigma\otimes\tau)=(\nabla^{\mathbf{E}}_{\mathbf{X}}\sigma)\otimes\tau+\sigma\otimes(\nabla^{\mathbf{F}}_{\mathbf{X}}\tau),\qquad \mathbf{X}\in\mathfrak{X}(M),\ \sigma\in\Gamma(\mathbf{E}),\ \tau\in\Gamma(\mathbf{F}),$$

and extended by bilinearity and the property

$$\nabla_{\mathbf{X}}(f\,\mathbf{T})=f\,\nabla_{\mathbf{X}}\mathbf{T}+(\mathbf{X}f)\,\mathbf{T},\qquad f\in C^\infty(M).$$

(b) The connection $\nabla^{\mathbf{E}}$ induces a unique connection $\nabla^{\mathbf{E}^*}\colon \mathfrak{X}(M)\times\Gamma(\mathbf{E}^*)\longrightarrow\Gamma(\mathbf{E}^*)$ characterized by the identity

$$\mathbf{X}\langle\mathbf{v},\mathbf{u}\rangle=\langle\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v},\mathbf{u}\rangle+\langle\mathbf{v},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}\rangle,\qquad \mathbf{u}\in\Gamma(\mathbf{E}),\ \mathbf{v}\in\Gamma(\mathbf{E}^*),\ \mathbf{X}\in\mathfrak{X}(M).$$

(c) The connection $\nabla^{\mathbf{E}}$ induces a connection $\nabla^{\mathrm{End}(\mathbf{E})}$ on $\mathrm{End}(\mathbf{E})\cong\mathbf{E}^*\otimes\mathbf{E}$, defined for every $\mathbf{T}\in\Gamma(\mathrm{End}(\mathbf{E}))$ by

$$(\nabla^{\mathrm{End}(\mathbf{E})}_{\mathbf{X}}\mathbf{T})(\mathbf{u})=\nabla^{\mathbf{E}}_{\mathbf{X}}(\mathbf{T}\mathbf{u})-\mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}),\qquad \mathbf{u}\in\Gamma(\mathbf{E}),\ \mathbf{X}\in\mathfrak{X}(M).$$

(d) The connections $\nabla^{\mathbf{E}}$ and $\nabla^{\mathbf{F}}$ induce a connection $\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})}$ on the bundle $\mathrm{Hom}(\mathbf{E},\mathbf{F})\cong\mathbf{E}^*\otimes\mathbf{F}$, determined by the rule

$$(\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})}_{\mathbf{X}}\mathbf{\Phi})(\mathbf{u})=\nabla^{\mathbf{F}}_{\mathbf{X}}(\mathbf{\Phi}(\mathbf{u}))-\mathbf{\Phi}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}),\qquad \mathbf{\Phi}\in\Gamma(\mathrm{Hom}(\mathbf{E},\mathbf{F})),\ \mathbf{u}\in\Gamma(\mathbf{E}),\ \mathbf{X}\in\mathfrak{X}(M).$$

Equivalently, in intrinsic terms, if $\mathbf{\Phi}\in\Gamma(\mathbf{E}^*\otimes\mathbf{F})$, then $\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})}$ is the connection induced by those on $\mathbf{E}^*$ and $\mathbf{F}$ through the tensor product.

Proof. For the tensor product, define the connection on decomposable local sections by the formula in part (a). If $f\in C^\infty(M)$, the two expansions

$$\nabla_{\mathbf{X}}((f\,\mathbf{u})\otimes\mathbf{v})=\mathbf{X}(f)\mathbf{u}\otimes\mathbf{v}+f(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v}+f\,\mathbf{u}\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v},$$
$$\nabla_{\mathbf{X}}(\mathbf{u}\otimes(f\,\mathbf{v}))=f(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v}+\mathbf{X}(f)\mathbf{u}\otimes\mathbf{v}+f\,\mathbf{u}\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v}$$

agree. The rule therefore respects the balancing relations and is independent of the representation of a section as a sum of decomposable tensors. In local frames, it produces smooth coefficients and satisfies the two rules for a connection. Uniqueness follows because decomposable sections locally generate $\mathbf{E}\otimes\mathbf{F}$.
For the dual, set

$$(\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v})(\mathbf{u}):=\mathbf{X}(\mathbf{v}(\mathbf{u}))-\mathbf{v}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}).$$

The same calculation

$$\mathbf{X}(\mathbf{v}(f\,\mathbf{u}))-\mathbf{v}(\nabla^{\mathbf{E}}_{\mathbf{X}}(f\,\mathbf{u}))=f\left[\mathbf{X}(\mathbf{v}(\mathbf{u}))-\mathbf{v}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\right]$$

shows that the right-hand side is linear over $C^\infty(M)$ in $\mathbf{u}$ and defines a smooth section of $\mathbf{E}^*$. Linearity over $C^\infty(M)$ in $\mathbf{X}$ and the rule $\nabla^{\mathbf{E}^*}_{\mathbf{X}}(f\,\mathbf{v})=\mathbf{X}(f)\mathbf{v}+f\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v}$ are verified directly. The pairing identity determines this connection uniquely.
The product connection on $\mathbf{E}^*\otimes\mathbf{E}$ induces the connection on $\mathrm{End}(\mathbf{E})$. Differentiating the identity $\mathbf{T}\mathbf{u}$ and

using the parallelism of the pairing gives

$$(\nabla_{\mathbf{X}}^{\mathrm{End}(\mathbf{E})}\mathbf{T})(\mathbf{u}) = \nabla_{\mathbf{X}}^{\mathbf{E}}(\mathbf{T}\mathbf{u}) - \mathbf{T}(\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{u}).$$

The same argument on $\mathbf{E}^* \otimes \mathbf{F}$ gives

$$(\nabla_{\mathbf{X}}^{\mathrm{Hom}(\mathbf{E},\mathbf{F})}\mathbf{\Phi})(\mathbf{u}) = \nabla_{\mathbf{X}}^{\mathbf{F}}(\mathbf{\Phi}(\mathbf{u})) - \mathbf{\Phi}(\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{u}).$$

These formulas also prove uniqueness in the last two cases. □

A connection can equivalently be defined as an operator between sections of certain vector bundles, as the following proposition shows. In global analysis, most authors use this equivalence in nearly every setting involving connections.

Proposition 2.4.25. Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle. Giving a connection on $\mathbf{E}$ in the sense of Definition 2.4.2 is equivalent to giving an operator

$$\nabla\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma\big(T^{(0,1)}(TM) \otimes \mathbf{E}\big),$$

that is linear over $\mathbb{R}$ and satisfies the Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \qquad f \in C^\infty(M), \sigma \in \Gamma(\mathbf{E}).$$

The relationship between the two formulations is described using the canonical isomorphism

$$\Upsilon : \Gamma\big(T^{(0,1)}(TM) \otimes \mathbf{E}\big) \xrightarrow{\cong} \mathrm{Hom}_{C^\infty(M)}\big(\mathfrak{X}(M), \Gamma(\mathbf{E})\big).$$

Under this identification, for $\mathbf{X} \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\mathbf{E})$, we have

$$\Upsilon(\nabla\sigma)(\mathbf{X}) = \nabla_{\mathbf{X}}\sigma.$$

Proof. By Proposition 1.10.14, applied to TM and $\mathbf{E}$, there is a canonical vector bundle isomorphism

$$\Phi : T^{(0,1)}(TM) \otimes \mathbf{E} \xrightarrow{\cong} \mathrm{Hom}(TM, \mathbf{E}), \qquad \Phi_p(\omega_p \otimes v)(\mathbf{X}_p) = \omega_p(\mathbf{X}_p)v,$$

for all $p \in M$, $\omega_p \in T_p^*M$, $v \in \mathbf{E}_p$, and $\mathbf{X}_p \in T_pM$.
Passing to sections gives

$$\Gamma(\Phi)\colon \Gamma\big(T^{(0,1)}(TM) \otimes \mathbf{E}\big) \longrightarrow \Gamma\big(\mathrm{Hom}(TM, \mathbf{E})\big), \qquad \big(\Gamma(\Phi)(\mathbf{s})\big)(p) = \Phi_p\big(\mathbf{s}(p)\big).$$

Composing with the identification of Proposition 1.7.31,

$$\Theta : \Gamma\big(\mathrm{Hom}(TM, \mathbf{E})\big) \xrightarrow{\cong} \mathrm{Hom}_{C^\infty(M)}\big(\mathfrak{X}(M), \Gamma(\mathbf{E})\big), \qquad \Theta(\mathbf{A})(\mathbf{X})(p) = \mathbf{A}(p)\big(\mathbf{X}_p\big),$$

we obtain the isomorphism of $C^\infty(M)$-modules

$$\Upsilon := \Theta \circ \Gamma(\Phi) : \Gamma\big(T^{(0,1)}(TM) \otimes \mathbf{E}\big) \xrightarrow{\cong} \mathrm{Hom}_{C^\infty(M)}\big(\mathfrak{X}(M), \Gamma(\mathbf{E})\big),$$

given explicitly by

$$\big(\Upsilon(\mathbf{s})\big)(\mathbf{X})(p) = \Theta\big(\Gamma(\Phi)(\mathbf{s})\big)(\mathbf{X})(p) = \Gamma(\Phi)(\mathbf{s})(p)\big(\mathbf{X}_p\big) = \Phi_p\big(\mathbf{s}(p)\big)\big(\mathbf{X}_p\big).$$

We denote its inverse by Υ^{-1}.
($\Rightarrow$) Suppose we have a connection in the sense of Definition 2.4.2. For fixed $\sigma \in \Gamma(\mathbf{E})$, the map

$$F_\sigma\colon\ \mathfrak{X}(M) \longrightarrow \Gamma(\mathbf{E}), \qquad \mathbf{X} \longmapsto \nabla_{\mathbf{X}}\sigma,$$

is $C^\infty(M)$-linear in $\mathbf{X}$. Therefore $F_\sigma \in \mathrm{Hom}_{C^\infty(M)}\big(\mathfrak{X}(M), \Gamma(\mathbf{E})\big)$ and, by the isomorphism Υ^{-1}, there exists a unique section $\mathbf{s} = \nabla\sigma \in \Gamma\big(T^{(0,1)}(TM)\otimes\mathbf{E}\big)$ such that

$$\Upsilon(\nabla\sigma)(\mathbf{X}) \ = \ \nabla_{\mathbf{X}}\sigma, \qquad \forall \mathbf{X} \in \mathfrak{X}(M).$$

In particular, for all $p \in M$ and $\mathbf{X}_p \in T_pM$,

$$\Phi_p\big((\nabla\sigma)(p)\big)(\mathbf{X}_p) \ = \ \nabla_{\mathbf{X}}\sigma(p).$$

Now let $f \in C^\infty(M)$. Using the Leibniz rule for the connection,

$$\Upsilon\big(\nabla(f\sigma)\big)(\mathbf{X}) = \nabla_{\mathbf{X}}(f\sigma) = (\mathbf{X}f)\sigma + f\nabla_{\mathbf{X}}\sigma.$$

On the other hand,

$$\Upsilon(df\otimes\sigma)(\mathbf{X})(p) = \Phi_p\big((df\otimes\sigma)(p)\big)(\mathbf{X}_p) = df_p(\mathbf{X}_p)\sigma(p) = (\mathbf{X}f)(p)\sigma(p),$$

and, since Υ is $C^\infty(M)$-linear,

$$\Upsilon(f\nabla\sigma)(\mathbf{X}) = f\cdot\Upsilon(\nabla\sigma)(\mathbf{X}) = f\nabla_{\mathbf{X}}\sigma.$$

Thus,

$$\Upsilon\big(\nabla(f\sigma)\big)(\mathbf{X}) = \Upsilon\big(df\otimes\sigma + f\nabla\sigma\big)(\mathbf{X}), \qquad \forall \mathbf{X} \in \mathfrak{X}(M).$$

Since Υ is an isomorphism, we conclude that

$$\nabla(f\sigma) = df\otimes\sigma + f\nabla\sigma.$$

($\Leftarrow$) Conversely, let $\nabla\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma\big(T^{(0,1)}(TM)\otimes\mathbf{E}\big)$ be an operator linear over $\mathbb{R}$ satisfying

$$\nabla(f\sigma) = df\otimes\sigma + f\nabla\sigma.$$

For $\mathbf{X} \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\mathbf{E})$, define

$$\nabla_{\mathbf{X}}\sigma \ := \ \Upsilon(\nabla\sigma)(\mathbf{X}), \quad \text{that is} \quad \nabla_{\mathbf{X}}\sigma(p) = \Phi_p\big((\nabla\sigma)(p)\big)(\mathbf{X}_p).$$

Then $\nabla_{\mathbf{X}}$ is linear in σ. Moreover, using the Leibniz identity on sections and the $C^\infty(M)$-linearity of Υ,

$$\nabla_{\mathbf{X}}(f\sigma) = \Upsilon\big(\nabla(f\sigma)\big)(\mathbf{X}) = \Upsilon\big(df\otimes\sigma + f\nabla\sigma\big)(\mathbf{X}) = \Upsilon(df\otimes\sigma)(\mathbf{X}) + f\Upsilon(\nabla\sigma)(\mathbf{X}) = (\mathbf{X}f)\sigma + f\nabla_{\mathbf{X}}\sigma.$$

Finally, since $\Upsilon(\nabla\sigma) \in \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), \Gamma(\mathbf{E}))$ is $C^\infty(M)$-linear in $\mathbf{X}$, we have

$$\nabla_{f\mathbf{X}}\sigma = \Upsilon(\nabla\sigma)(f\mathbf{X}) = f\Upsilon(\nabla\sigma)(\mathbf{X}) = f\nabla_{\mathbf{X}}\sigma.$$

Therefore $(\mathbf{X}, \sigma) \mapsto \nabla_{\mathbf{X}}\sigma$ satisfies the properties of Definition 2.4.2 and defines a connection.

□

Since the two approaches are equivalent, we will sometimes use the two descriptions of a connection interchangeably.

Example 2.4.26. (Trivial connection) Let M be a smooth manifold with or without boundary, and let $\underline{\mathbb{R}}^k_M = M \times \mathbb{R}^k$ be the trivial bundle of rank k over M. The space of sections $\Gamma(\underline{\mathbb{R}}^k_M)$ agrees with the space $C^\infty(M, \mathbb{R}^k)$ of smooth functions taking values in $\mathbb{R}^k$, and we have $(\underline{\mathbb{R}}^k_M)_p \cong \mathbb{R}^k$ (the fiber spaces of the trivial bundle).
Let $(e_1, \dots, e_k)$ be the canonical basis of $\mathbb{R}^k$, and denote the corresponding constant sections by $\mathbf{e}_i(p) := (p, e_i)$. If $\sigma = (\sigma_1, \dots, \sigma_k) \in C^\infty(M, \mathbb{R}^k)$, define

$$\nabla\sigma \;:=\; \sum_{i=1}^{k} d\sigma_i \otimes \mathbf{e}_i \;\in\; \Gamma\big(T^*M \otimes \underline{\mathbb{R}}^k_M\big).$$

In other words, for each $p \in M$,

$$(\nabla\sigma)(p) \;=\; \sum_{i=1}^{k} d\sigma_i(p) \otimes \mathbf{e}_i(p) \;\in\; T^*_pM \otimes \mathbb{R}^k.$$

If $\mathbf{X} \in \mathfrak{X}(M)$ is a vector field, then

$$(\nabla\sigma)_p(\mathbf{X}) \;=\; \sum_{i=1}^{k} d\sigma_i(p)(\mathbf{X}_p)\mathbf{e}_i(p) = \; (\mathbf{X}\sigma_1(p), \dots, \mathbf{X}\sigma_k(p)),$$

that is, the result is obtained by differentiating the section σ coordinate by coordinate.
The Leibniz rule is verified directly: for $f \in C^\infty(M)$,

$$\nabla(f\sigma) = \sum_{i=1}^{k} d(f\sigma_i) \otimes \mathbf{e}_i = \sum_{i=1}^{k} \big((df)\sigma_i + f\,d\sigma_i\big) \otimes \mathbf{e}_i = df \otimes \sigma + f\nabla\sigma.$$

Therefore ∇ defines a connection on $\underline{\mathbb{R}}^k_M$, called the *trivial connection*.

Connections and bundle metrics must be compatible with tensor operations. The following constructions allow us to differentiate tensors, duals, and homomorphisms, and will be needed to define Sobolev spaces of sections.

Proposition 2.4.27. (Induced connections on tensor products, duals, and homomorphisms) Let M be a smooth manifold with or without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles with connections $\nabla^{\mathbf{E}}$ and $\nabla^{\mathbf{F}}$.

(a) There exists a unique connection $\nabla^{\mathbf{E}\otimes\mathbf{F}}$ on $\mathbf{E}\otimes\mathbf{F}$ such that, for any $\mathbf{X}\in\mathfrak{X}(M)$, $\mathbf{u}\in\Gamma(\mathbf{E})$, and $\mathbf{v}\in\Gamma(\mathbf{F})$,

$$\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}(\mathbf{u}\otimes\mathbf{v}) = (\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v} + \mathbf{u}\otimes(\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v}). \tag{2.11}$$

(b) There exists a unique connection $\nabla^{\mathbf{E}^*}$ on the dual bundle $\mathbf{E}^*$ characterized by

$$\mathbf{X}\big(\boldsymbol{\alpha}(\mathbf{u})\big) = (\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\alpha})(\mathbf{u}) + \boldsymbol{\alpha}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}) \tag{2.12}$$

for any $\mathbf{X}\in\mathfrak{X}(M)$, $\boldsymbol{\alpha}\in\Gamma(\mathbf{E}^*)$, and $\mathbf{u}\in\Gamma(\mathbf{E})$.

(c) The bundle $\operatorname{Hom}(\mathbf{E},\mathbf{F})\cong\mathbf{E}^*\otimes\mathbf{F}$ has the induced connection determined by

$$(\nabla^{\operatorname{Hom}(\mathbf{E},\mathbf{F})}_{\mathbf{X}}\mathbf{T})(\mathbf{u}) = \nabla^{\mathbf{F}}_{\mathbf{X}}(\mathbf{T}\mathbf{u}) - \mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}). \tag{2.13}$$

In particular, taking $\mathbf{F}=\mathbf{E}$ gives the connection on $\operatorname{End}(\mathbf{E})$ and

$$(\nabla^{\operatorname{End}(\mathbf{E})}_{\mathbf{X}}\mathbf{T})(\mathbf{u}) = [\nabla^{\mathbf{E}}_{\mathbf{X}},\mathbf{T}]\mathbf{u}.$$

These constructions are compatible with iterated tensor products and with the canonical identifications between tensor bundles.

Proof. We begin with the tensor product. On a trivializing open set U, every section of $\mathbf{E}\otimes\mathbf{F}$ is a finite sum of simple tensor sections. Define the right-hand side of (2.11) on these sections and extend it by additivity. We must check that the definition respects the balancing relations of the tensor product. For $f\in C^\infty(U)$,

$$\begin{aligned}&\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}((f\mathbf{u})\otimes\mathbf{v})\\ &\quad= \mathbf{X}(f)\mathbf{u}\otimes\mathbf{v} + f(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v} + f\mathbf{u}\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v},\end{aligned}$$

whereas

$$\begin{aligned}&\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}(\mathbf{u}\otimes(f\mathbf{v}))\\ &\quad= (\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes f\mathbf{v} + \mathbf{u}\otimes\mathbf{X}(f)\mathbf{v} + \mathbf{u}\otimes f\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v}.\end{aligned}$$

The two expressions agree after using $(f\mathbf{u})\otimes\mathbf{v}=\mathbf{u}\otimes(f\mathbf{v})$. The definition is therefore independent of the representation of a section as a sum of simple tensors. The properties

$$\nabla^{\mathbf{E}\otimes\mathbf{F}}_{f\mathbf{X}}\mathbf{S} = f\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}\mathbf{S}$$

and

$$\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}(f\mathbf{S}) = \mathbf{X}(f)\mathbf{S} + f\nabla^{\mathbf{E}\otimes\mathbf{F}}_{\mathbf{X}}\mathbf{S}$$

are checked first on simple tensors and then by additivity. This proves existence of the connection. Uniqueness follows because simple tensor sections locally generate the module of sections of $\mathbf{E}\otimes\mathbf{F}$.
For the dual bundle, define, for $\boldsymbol{\alpha}\in\Gamma(\mathbf{E}^*)$,

$$(\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\alpha})(\mathbf{u}) := \mathbf{X}\big(\boldsymbol{\alpha}(\mathbf{u})\big) - \boldsymbol{\alpha}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}). \tag{2.14}$$

The right-hand side is $C^\infty(M)$-linear in $\mathbf{u}$. For $f \in C^\infty(M)$,

$$\begin{aligned}
&\mathbf{X}\big(\boldsymbol{\alpha}(f\mathbf{u})\big) - \boldsymbol{\alpha}(\nabla^{\mathbf{E}}_{\mathbf{X}}(f\mathbf{u})) \\
&\quad = \mathbf{X}(f)\boldsymbol{\alpha}(\mathbf{u}) + f\mathbf{X}\big(\boldsymbol{\alpha}(\mathbf{u})\big) - \mathbf{X}(f)\boldsymbol{\alpha}(\mathbf{u}) - f\,\boldsymbol{\alpha}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}) \\
&\quad = f\left[\mathbf{X}\big(\boldsymbol{\alpha}(\mathbf{u})\big) - \boldsymbol{\alpha}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\right].
\end{aligned}$$

Thus (2.14) defines a section of $\mathbf{E}^*$. The $C^\infty(M)$-linearity in $\mathbf{X}$ and the Leibniz rule in $\boldsymbol{\alpha}$ follow directly from the same formula. By construction, (2.12) holds, and this identity determines $\nabla^{\mathbf{E}^*}$ uniquely.

Now identify $\operatorname{Hom}(\mathbf{E},\mathbf{F})$ with $\mathbf{E}^* \otimes \mathbf{F}$ and use the product connection already constructed. If locally $\mathbf{T} = \sum_{a=1}^{N} \boldsymbol{\alpha}_a \otimes \mathbf{v}_a$, then

$$\mathbf{T}\mathbf{u} = \sum_{a=1}^{N} \boldsymbol{\alpha}_a(\mathbf{u})\mathbf{v}_a.$$

The Leibniz rules and (2.12) give

$$\begin{aligned}
\nabla^{\mathbf{F}}_{\mathbf{X}}(\mathbf{T}\mathbf{u}) = &\sum_{a=1}^{N} (\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\alpha}_a)(\mathbf{u})\mathbf{v}_a + \sum_{a=1}^{N} \boldsymbol{\alpha}_a(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\mathbf{v}_a \\
&+ \sum_{a=1}^{N} \boldsymbol{\alpha}_a(\mathbf{u})\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v}_a.
\end{aligned}$$

The first and third terms combine to give $(\nabla^{\operatorname{Hom}(\mathbf{E},\mathbf{F})}_{\mathbf{X}}\mathbf{T})(\mathbf{u})$, while the second is $\mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})$. This proves (2.13). The endomorphism case is the specialization $\mathbf{F} = \mathbf{E}$. Finally, iterations are obtained by successively applying the tensor product construction, and their compatibility with the canonical identifications follows from the uniqueness just proved. □

Lemma 2.4.28. Let M be a smooth manifold with or without boundary equipped with a connection ∇ on TM, and let $\pi_{\mathbf{E}}\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank r equipped with a connection $\nabla^{\mathbf{E}}$. Let $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ be a local frame of TM over an open set $U \subseteq M$, $(\boldsymbol{\varepsilon}^1,\dots,\boldsymbol{\varepsilon}^n)$ its dual coframe, and $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ a local frame of $\mathbf{E}\restriction_U$.

Let $\mathbf{F} \in \Gamma(T^{(k,l)}(TM)\otimes\mathbf{E})$ be written in these frames as

$$\mathbf{F} = \sum_{\substack{1\le i_1,\dots,i_k\le n\\ 1\le j_1,\dots,j_l\le n\\ 1\le a\le r}} F^{i_1\dots i_k\,a}_{j_1\dots j_l}\,\mathbf{E}_{i_1}\otimes\cdots\otimes\mathbf{E}_{i_k}\otimes\boldsymbol{\varepsilon}^{j_1}\otimes\cdots\otimes\boldsymbol{\varepsilon}^{j_l}\otimes\mathbf{e}_a.$$

Then, for each $c \in \{1,\dots,n\}$, the components of the covariant derivative

$$\nabla_{\partial_c}\mathbf{F} := \big(\nabla^{\mathbf{T}^{(k,l)}\otimes\mathbf{E}}\big)_{\partial_c}\mathbf{F}$$

are given by

$$(\nabla_{\partial_c}\mathbf{F})^{i_1\dots i_k\,a}_{j_1\dots j_l} = \partial_c\big(F^{i_1\dots i_k\,a}_{j_1\dots j_l}\big) + \sum_{r=1}^{k}\Gamma^{i_r}_{cs}\,F^{i_1\dots s\dots i_k\,a}_{j_1\dots j_l} - \sum_{t=1}^{l}\Gamma^{s}_{cj_t}\,F^{i_1\dots i_k\,a}_{j_1\dots s\dots j_l} + \sum_{b=1}^{r}(A_c)^a_b\,F^{i_1\dots i_k\,b}_{j_1\dots j_l},$$

where $\nabla_{\partial_c}\mathbf{E}_i = \Gamma_{ci}^s \mathbf{E}_s$ and $\nabla^{\mathbf{E}}_{\partial_c}\mathbf{e}_b = (A_c)_b^a \mathbf{e}_a$.

Proof. By Proposition 2.4.24, the connection ∇ on TM induces a connection on $T^{(k,l)}(TM)$ characterized by the product rule for the contravariant and covariant factors. In the frame

$$\left\{\mathbf{E}_{i_1}\otimes\cdots\otimes\mathbf{E}_{i_k}\otimes\boldsymbol{\varepsilon}^{j_1}\otimes\cdots\otimes\boldsymbol{\varepsilon}^{j_l}\right\}$$

we have, for every $\mathbf{T}\in\Gamma\left(T^{(k,l)}(TM)\right)$,

$$(\nabla_{\partial_c}\mathbf{T})^{i_1\ldots i_k}_{j_1\ldots j_l} = \partial_c\left(T^{i_1\ldots i_k}_{j_1\ldots j_l}\right) + \sum_{r=1}^{k}\Gamma^{i_r}_{cs}\,T^{i_1\ldots s\ldots i_k}_{j_1\ldots j_l} - \sum_{t=1}^{l}\Gamma^{s}_{cj_t}\,T^{i_1\ldots i_k}_{j_1\ldots s\ldots j_l}.$$

Now write $\mathbf{F} = \sum_{a=1}^{r}\mathbf{F}^{(a)}\otimes\mathbf{e}_a$, where, for each $a\in\{1,\ldots,r\}$, the tensor field $\mathbf{F}^{(a)}\in\Gamma\left(T^{(k,l)}(TM)\right)$ has components $\left(\mathbf{F}^{(a)}\right)^{i_1\ldots i_k}_{j_1\ldots j_l} = F^{i_1\ldots i_k\,a}_{j_1\ldots j_l}$.
The induced connection on $T^{(k,l)}(TM)\otimes\mathbf{E}$ is determined by

$$\nabla_{\partial_c}(\mathbf{S}\otimes\boldsymbol{\sigma}) = (\nabla_{\partial_c}\mathbf{S})\otimes\boldsymbol{\sigma} + \mathbf{S}\otimes\nabla^{\mathbf{E}}_{\partial_c}\boldsymbol{\sigma}, \qquad \mathbf{S}\in\Gamma\left(T^{(k,l)}(TM)\right),\ \boldsymbol{\sigma}\in\Gamma(\mathbf{E}),$$

so, for $\mathbf{F} = \sum_{a=1}^{r}\mathbf{F}^{(a)}\otimes\mathbf{e}_a$, we obtain

$$\nabla_{\partial_c}\mathbf{F} = \sum_{a=1}^{r}\left((\nabla_{\partial_c}\mathbf{F}^{(a)})\otimes\mathbf{e}_a + \mathbf{F}^{(a)}\otimes\nabla^{\mathbf{E}}_{\partial_c}\mathbf{e}_a\right).$$

In the frame $(\mathbf{E}_1,\ldots,\mathbf{E}_n)$, we have $\nabla_{\partial_c}\mathbf{E}_i = \Gamma_{ci}^s\mathbf{E}_s$, and, by the definition of the coefficients of the connection $\nabla^{\mathbf{E}}$ in the frame $(\mathbf{e}_1,\ldots,\mathbf{e}_r)$, $\nabla^{\mathbf{E}}_{\partial_c}\mathbf{e}_b = (A_c)_b^a\mathbf{e}_a$.
Applying Lemma 2.4.15 to each $\mathbf{F}^{(a)}$ gives

$$(\nabla_{\partial_c}\mathbf{F}^{(a)})^{i_1\ldots i_k}_{j_1\ldots j_l} = \partial_c\left(F^{i_1\ldots i_k\,a}_{j_1\ldots j_l}\right) + \sum_{r=1}^{k}\Gamma^{i_r}_{cs}\,F^{i_1\ldots s\ldots i_k\,a}_{j_1\ldots j_l} - \sum_{t=1}^{l}\Gamma^{s}_{cj_t}\,F^{i_1\ldots i_k\,a}_{j_1\ldots s\ldots j_l}.$$

On the other hand, using the local expression for the connection on $\mathbf{E}$, $\nabla^{\mathbf{E}}_{\partial_c}\mathbf{e}_a = \sum_{b=1}^{r}(A_c)_a^b\mathbf{e}_b$, we obtain

$$\mathbf{F}^{(a)}\otimes\nabla^{\mathbf{E}}_{\partial_c}\mathbf{e}_a = \sum_{b=1}^{r}(A_c)_a^b\left(\mathbf{F}^{(a)}\otimes\mathbf{e}_b\right).$$

Each of these terms is expressed in the basis

$$\left\{\mathbf{E}_{i_1}\otimes\cdots\otimes\mathbf{E}_{i_k}\otimes\boldsymbol{\varepsilon}^{j_1}\otimes\cdots\otimes\boldsymbol{\varepsilon}^{j_l}\otimes\mathbf{e}_a\right\},$$

since $\mathbf{F}^{(a)}\otimes\mathbf{e}_b = F^{i_1\ldots i_k\,a}_{j_1\ldots j_l}\,\mathbf{E}_{i_1}\otimes\cdots\otimes\mathbf{E}_{i_k}\otimes\boldsymbol{\varepsilon}^{j_1}\otimes\cdots\otimes\boldsymbol{\varepsilon}^{j_l}\otimes\mathbf{e}_b$.

Consequently, the term $\mathbf{F}^{(a)} \otimes \nabla^{\mathbf{E}}_{\partial_c} \mathbf{e}_a$ contributes the sum

$$\sum_{b=1}^{r} (A_c)^a_b \, F^{i_1 \dots i_k \, b}_{j_1 \dots j_l}.$$

to the components with output index a.
To obtain the components of $\nabla_{\partial_c} \mathbf{F}$, we now need only fix $a \in \{1, \dots, r\}$ and add the contributions of the two preceding blocks to the basis tensor with last factor $\mathbf{e}_a$:

- the block $(\nabla_{\partial_c} \mathbf{F}^{(a)}) \otimes \mathbf{e}_a$ contributes

$$\mathbf{E}_c \big(F^{i_1 \dots i_k \, a}_{j_1 \dots j_l} \big) + \sum_{r=1}^{k} \Gamma^{i_r}_{cs} \, F^{i_1 \dots s \dots i_k \, a}_{j_1 \dots j_l} - \sum_{t=1}^{l} \Gamma^{s}_{c j_t} \, F^{i_1 \dots i_k \, a}_{j_1 \dots s \dots j_l},$$

- and the block $\mathbf{F}^{(b)} \otimes \nabla^{\mathbf{E}}_{\partial_c} \mathbf{e}_b$ contributes

$$\sum_{b=1}^{r} (A_c)^a_b \, F^{i_1 \dots i_k \, b}_{j_1 \dots j_l}.$$

Grouping these contributions gives

$$(\nabla_{\partial_c} \mathbf{F})^{i_1 \dots i_k \, a}_{j_1 \dots j_l} = \partial_c \big(F^{i_1 \dots i_k \, a}_{j_1 \dots j_l} \big) + \sum_{r=1}^{k} \Gamma^{i_r}_{cs} \, F^{i_1 \dots s \dots i_k \, a}_{j_1 \dots j_l} - \sum_{t=1}^{l} \Gamma^{s}_{c j_t} \, F^{i_1 \dots i_k \, a}_{j_1 \dots s \dots j_l} + \sum_{b=1}^{r} (A_c)^a_b \, F^{i_1 \dots i_k \, b}_{j_1 \dots j_l},$$

which is the desired formula. □

Proposition 2.4.29. (Formulas for the second and third covariant derivatives) Let M be a smooth manifold with or without boundary equipped with a connection ∇^M on TM, and let $\mathbf{E} \to M$ be a smooth vector bundle with a connection $\nabla^{\mathbf{E}}$. On the bundles $T^{(0,q)}(TM) \otimes \mathbf{E}$, use the product connections induced by ∇^M and $\nabla^{\mathbf{E}}$. For $\mathbf{u} \in \Gamma(\mathbf{E})$, define

$$\nabla^0 \mathbf{u} := \mathbf{u}, \qquad \nabla^{q+1} \mathbf{u} := \nabla(\nabla^q \mathbf{u}), \qquad q \in \mathbb{N}_0.$$

Thus,

$$\nabla^q \mathbf{u} \in \Gamma\big(T^{(0,q)}(TM) \otimes \mathbf{E}\big),$$

and the new covariant index introduced at each step is placed in the first argument. Equivalently,

$$(\nabla^{q+1} \mathbf{u})(\mathbf{X}_0, \mathbf{X}_1, \dots, \mathbf{X}_q) = \big(\nabla_{\mathbf{X}_0}(\nabla^q \mathbf{u})\big)(\mathbf{X}_1, \dots, \mathbf{X}_q).$$

In particular, for any $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$,

$$(\nabla^2 \mathbf{u})(\mathbf{X}, \mathbf{Y}) = \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\mathbf{Y}} \mathbf{u} - \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{X}} \mathbf{Y}} \mathbf{u}. \tag{2.15}$$

Moreover, for any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(M)$,

$$\begin{aligned} (\nabla^3 \mathbf{u})(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = {} & \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\mathbf{Y}} \nabla^{\mathbf{E}}_{\mathbf{Z}} \mathbf{u} - \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{Y}} \mathbf{Z}} \mathbf{u} - \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{X}} \mathbf{Y}} \nabla^{\mathbf{E}}_{\mathbf{Z}} \mathbf{u} \\ & + \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{X}} \nabla^M_{\mathbf{Y}} \mathbf{Z}} \mathbf{u} - \nabla^{\mathbf{E}}_{\mathbf{Y}} \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{X}} \mathbf{Z}} \mathbf{u} + \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{Y}} \nabla^M_{\mathbf{X}} \mathbf{Z}} \mathbf{u}. \end{aligned} \tag{2.16}$$

Proof. The first covariant derivative is interpreted as the 1-form with values in $\mathbf{E}$ given by

$$(\nabla \mathbf{u})(\mathbf{X}) = \nabla^{\mathbf{E}}_{\mathbf{X}} \mathbf{u}.$$

The product connection on $T^*M \otimes \mathbf{E}$ satisfies the Leibniz rule for evaluation. Hence,

$$\begin{aligned}(\nabla^2 \mathbf{u})(\mathbf{X},\mathbf{Y}) &= \big(\nabla_{\mathbf{X}}(\nabla \mathbf{u})\big)(\mathbf{Y}) = \nabla^{\mathbf{E}}_{\mathbf{X}}\big((\nabla \mathbf{u})(\mathbf{Y})\big) - (\nabla \mathbf{u})(\nabla^M_{\mathbf{X}} \mathbf{Y}) \\ &= \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\mathbf{Y}} \mathbf{u} - \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{X}} \mathbf{Y}} \mathbf{u}.\end{aligned}$$

which proves (2.15).
For the third order, regard $\nabla^2 \mathbf{u}$ as a 2-covariant tensor with values in $\mathbf{E}$. The definition of the induced connection gives

$$(\nabla^3 \mathbf{u})(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \nabla^{\mathbf{E}}_{\mathbf{X}}\big((\nabla^2 \mathbf{u})(\mathbf{Y},\mathbf{Z})\big) - (\nabla^2 \mathbf{u})(\nabla^M_{\mathbf{X}} \mathbf{Y},\mathbf{Z}) - (\nabla^2 \mathbf{u})(\mathbf{Y},\nabla^M_{\mathbf{X}} \mathbf{Z}). \tag{2.17}$$

Expand the three terms separately. By (2.15),

$$\nabla^{\mathbf{E}}_{\mathbf{X}}\big((\nabla^2 \mathbf{u})(\mathbf{Y},\mathbf{Z})\big) = \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\mathbf{Y}} \nabla^{\mathbf{E}}_{\mathbf{Z}} \mathbf{u} - \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{Y}} \mathbf{Z}} \mathbf{u}.$$

Likewise,

$$(\nabla^2 \mathbf{u})(\nabla^M_{\mathbf{X}} \mathbf{Y},\mathbf{Z}) = \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{X}} \mathbf{Y}} \nabla^{\mathbf{E}}_{\mathbf{Z}} \mathbf{u} - \nabla^{\mathbf{E}}_{\nabla^M_{\nabla^M_{\mathbf{X}} \mathbf{Y}} \mathbf{Z}} \mathbf{u},$$

and

$$(\nabla^2 \mathbf{u})(\mathbf{Y},\nabla^M_{\mathbf{X}} \mathbf{Z}) = \nabla^{\mathbf{E}}_{\mathbf{Y}} \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{X}} \mathbf{Z}} \mathbf{u} - \nabla^{\mathbf{E}}_{\nabla^M_{\mathbf{Y}} \nabla^M_{\mathbf{X}} \mathbf{Z}} \mathbf{u}.$$

Substituting these three identities into (2.17), keeping the two negative signs in its second line, yields exactly (2.16). □

Remark 2.4.30. (On the use of normal coordinates at third order) Even if normal coordinates centered at a point p are chosen, the identity

$$(\nabla^2 \mathbf{u})(\mathbf{X},\mathbf{Y})(p) = \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\mathbf{Y}} \mathbf{u}(p)$$

holds only for extensions of $\mathbf{X}$ and $\mathbf{Y}$ whose first covariant derivatives vanish at p. It cannot be differentiated as though it were an equality valid in a neighborhood of p. Indeed, formula (2.16) contains terms with second covariant derivatives of the auxiliary fields, such as $\nabla^M_{\mathbf{Y}} \nabla^M_{\mathbf{X}} \mathbf{Z}$, which do not vanish merely because normal coordinates were chosen. For this reason, when contracting or commuting higher-order covariant derivatives, we will first establish tensor identities and only then evaluate them in a normal frame. The operators obtained by such contractions will be defined in the chapter on differential operators on bundles.

Lemma 2.4.31. (Leibniz rule for interior multiplication) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary equipped with a connection ∇ on TM. Let $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ be a smooth local frame of TM with dual coframe $(\varepsilon^1,\dots,\varepsilon^n)$. Let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$.
For every tensor

$$\mathbf{F} \in \Gamma\big(T^{(k,l+1)}(TM) \otimes \mathbf{E}\big),$$

we have

$$\nabla_a\big(\iota_{\mathbf{E}_a}\mathbf{F}\big) = \iota_{\nabla_a\mathbf{E}_a}\mathbf{F} + \iota_{\mathbf{E}_a}(\nabla_a\mathbf{F}),$$

where $\nabla_a = \nabla_{\mathbf{E}_a}$ is the total connection on $T^{(k,l+1)}(TM)\otimes\mathbf{E}$, and $\iota_{\mathbf{E}_a}$ acts only on the covariant block of the tensor.

Proof. The identity holds for each index a before Einstein summation is performed. Fix a point p, and evaluate both sides on arbitrary covectors and vectors in the remaining indices of $\mathbf{F}$. Extend these arguments locally so that their covariant derivatives in the direction $\mathbf{E}_a$ vanish at p. By the definitions of the induced connection and contraction,

$$\big(\iota_{\mathbf{E}_a}\mathbf{F}\big)(\omega^1,\dots,\omega^k,\mathbf{X}_1,\dots,\mathbf{X}_l) = \mathbf{F}(\omega^1,\dots,\omega^k,\mathbf{E}_a,\mathbf{X}_1,\dots,\mathbf{X}_l).$$

Differentiating this equality in the direction $\mathbf{E}_a$ and evaluating at p, the Leibniz rule for the connection produces exactly two terms:

$$\begin{aligned}\nabla_a\big(\iota_{\mathbf{E}_a}\mathbf{F}\big)(\omega^1,\dots,\mathbf{X}_l) = \mathbf{F}(\omega^1,\dots,\nabla_a\mathbf{E}_a,\mathbf{X}_1,\dots,\mathbf{X}_l)\\ + (\nabla_a\mathbf{F})(\omega^1,\dots,\mathbf{E}_a,\mathbf{X}_1,\dots,\mathbf{X}_l).\end{aligned}$$

The first summand is $\big(\iota_{\nabla_a\mathbf{E}_a}\mathbf{F}\big)(\omega^1,\dots,\mathbf{X}_l)$ and the second is $\big(\iota_{\mathbf{E}_a}\nabla_a\mathbf{F}\big)(\omega^1,\dots,\mathbf{X}_l)$. Since the point and arguments were arbitrary, we obtain the tensor identity; summing over a gives the formula in the statement. □

Consider the following Leibniz rule:

Proposition 2.4.32. (Leibniz rule for $(0,2)$ tensors) Let M be a smooth manifold with or without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. If $\mathbf{T}\in\Gamma(\mathbf{E}^*\otimes\mathbf{E}^*)$, then, for every $\mathbf{u},\mathbf{v}\in\Gamma(\mathbf{E})$ and every $\mathbf{X}\in\mathfrak{X}(M)$, we have

$$\mathbf{X}\big(\mathbf{T}(\mathbf{u},\mathbf{v})\big) = (\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}\mathbf{T})(\mathbf{u},\mathbf{v}) + \mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{v}) + \mathbf{T}(\mathbf{u},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v}).$$

Proof. By definition, for $\mathbf{u},\mathbf{v}\in\Gamma(\mathbf{E})$ the function $\mathbf{T}(\mathbf{u},\mathbf{v})\in C^\infty(M)$ is given by

$$\mathbf{T}(\mathbf{u},\mathbf{v})(p) = \mathbf{T}_p(\mathbf{u}_p,\mathbf{v}_p), \qquad p\in M.$$

Since $\Gamma(\mathbf{E}^*\otimes\mathbf{E}^*)$ is a $C^\infty(M)$-module generated by simple tensors, it suffices to prove the identity when $\mathbf{T}=\boldsymbol{\alpha}\otimes\boldsymbol{\beta}$ with $\boldsymbol{\alpha},\boldsymbol{\beta}\in\Gamma(\mathbf{E}^*)$ and then conclude by linearity. In this case,

$$\mathbf{T}(\mathbf{u},\mathbf{v}) = \langle\boldsymbol{\alpha},\mathbf{u}\rangle\langle\boldsymbol{\beta},\mathbf{v}\rangle.$$

Applying the product rule for functions and the characterization of the induced connection on the dual (Proposition 2.4.27),

$$\begin{aligned}\mathbf{X}\big(\mathbf{T}(\mathbf{u},\mathbf{v})\big) &= \mathbf{X}\big(\langle\boldsymbol{\alpha},\mathbf{u}\rangle\langle\boldsymbol{\beta},\mathbf{v}\rangle\big)\\ &= \mathbf{X}\langle\boldsymbol{\alpha},\mathbf{u}\rangle\langle\boldsymbol{\beta},\mathbf{v}\rangle + \langle\boldsymbol{\alpha},\mathbf{u}\rangle\mathbf{X}\langle\boldsymbol{\beta},\mathbf{v}\rangle\\ &= \big(\langle\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\alpha},\mathbf{u}\rangle + \langle\boldsymbol{\alpha},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}\rangle\big)\langle\boldsymbol{\beta},\mathbf{v}\rangle + \langle\boldsymbol{\alpha},\mathbf{u}\rangle\big(\langle\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\beta},\mathbf{v}\rangle + \langle\boldsymbol{\beta},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v}\rangle\big)\end{aligned}$$

$$= \underbrace{\langle \nabla^{\mathbf{E}^*}_{\mathbf{X}} \boldsymbol{\alpha}, \mathbf{u}\rangle\langle \boldsymbol{\beta}, \mathbf{v}\rangle + \langle \boldsymbol{\alpha}, \mathbf{u}\rangle\langle \nabla^{\mathbf{E}^*}_{\mathbf{X}} \boldsymbol{\beta}, \mathbf{v}\rangle}_{(\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}(\boldsymbol{\alpha}\otimes\boldsymbol{\beta}))(\mathbf{u},\mathbf{v}) \text{ by Proposition 2.4.27.}} + \underbrace{\langle \boldsymbol{\alpha}, \nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}\rangle\langle \boldsymbol{\beta}, \mathbf{v}\rangle}_{\mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{v})} + \underbrace{\langle \boldsymbol{\alpha}, \mathbf{u}\rangle\langle \boldsymbol{\beta}, \nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v}\rangle}_{\mathbf{T}(\mathbf{u},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v})}.$$

Consequently,

$$\mathbf{X}\big(\mathbf{T}(\mathbf{u},\mathbf{v})\big) = (\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}\mathbf{T})(\mathbf{u},\mathbf{v}) + \mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{v}) + \mathbf{T}(\mathbf{u},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v})$$

for $\mathbf{T} = \boldsymbol{\alpha}\otimes\boldsymbol{\beta}$. By linearity of every term, the equality extends to all $\mathbf{T} \in \Gamma(\mathbf{E}^*\otimes\mathbf{E}^*)$. □

This Leibniz rule motivates the following more general concept, which describes when a bundle homomorphism respects the structures given by the connection.

Definition 2.4.33. (Parallel smooth bundle homomorphisms) Let M be a smooth manifold with or without boundary, and let $\mathbf{E},\mathbf{F},\mathbf{G} \to M$ be smooth vector bundles with connections $\nabla^{\mathbf{E}},\nabla^{\mathbf{F}},\nabla^{\mathbf{G}}$. A smooth bundle homomorphism $\Phi\colon \mathbf{E}\otimes\mathbf{F} \longrightarrow \mathbf{G}$ is said to be parallel if, for all $\mathbf{u} \in \Gamma(\mathbf{E})$, $\mathbf{v} \in \Gamma(\mathbf{F})$, and $\mathbf{X} \in \mathfrak{X}(M)$, we have

$$\nabla^{\mathbf{G}}_{\mathbf{X}}\big(\Phi(\mathbf{u}\otimes\mathbf{v})\big) = \Phi\big((\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v}\big) + \Phi\big(\mathbf{u}\otimes(\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v})\big).$$

This can be written equivalently as shown in the following proposition:

Proposition 2.4.34. Let M be a smooth manifold with or without boundary, and let $\mathbf{E},\mathbf{F},\mathbf{G} \to M$ be smooth vector bundles with connections $\nabla^{\mathbf{E}},\nabla^{\mathbf{F}},\nabla^{\mathbf{G}}$. Let

$$\Phi\colon \mathbf{E}\otimes\mathbf{F} \longrightarrow \mathbf{G}$$

be a smooth bundle homomorphism. Then Φ is a parallel smooth bundle homomorphism in the sense of Definition 2.4.33 if and only if, when regarded as a section of $\mathrm{Hom}(\mathbf{E}\otimes\mathbf{F},\mathbf{G})$, it satisfies $\nabla_{\mathbf{X}}\Phi = 0$, $\forall \mathbf{X} \in \mathfrak{X}(M)$, where ∇ is the induced connection on $\mathrm{Hom}(\mathbf{E}\otimes\mathbf{F},\mathbf{G})$.

Proof. By the definition of the connection on $\mathrm{Hom}(\mathbf{E}\otimes\mathbf{F},\mathbf{G})$, we have

$$(\nabla_{\mathbf{X}}\Phi)(\mathbf{u}\otimes\mathbf{v}) = \nabla^{\mathbf{G}}_{\mathbf{X}}\big(\Phi(\mathbf{u}\otimes\mathbf{v})\big) - \Phi\big(\nabla_{\mathbf{X}}(\mathbf{u}\otimes\mathbf{v})\big),$$

for all $\mathbf{u} \in \Gamma(\mathbf{E})$, $\mathbf{v} \in \Gamma(\mathbf{F})$, and $\mathbf{X} \in \mathfrak{X}(M)$.
Since $\nabla_{\mathbf{X}}(\mathbf{u}\otimes\mathbf{v}) = (\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v} + \mathbf{u}\otimes(\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v})$, we obtain

$$(\nabla_{\mathbf{X}}\Phi)(\mathbf{u}\otimes\mathbf{v}) = \nabla^{\mathbf{G}}_{\mathbf{X}}\big(\Phi(\mathbf{u}\otimes\mathbf{v})\big) - \Phi\big((\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v}\big) - \Phi\big(\mathbf{u}\otimes(\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v})\big).$$

Therefore $(\nabla_{\mathbf{X}}\Phi)(\mathbf{u}\otimes\mathbf{v}) = 0$ for every $\mathbf{u},\mathbf{v}$ if and only if

$$\nabla^{\mathbf{G}}_{\mathbf{X}}\big(\Phi(\mathbf{u}\otimes\mathbf{v})\big) = \Phi\big((\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v}\big) + \Phi\big(\mathbf{u}\otimes(\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{v})\big),$$

that is, if and only if Φ is a parallel smooth bundle homomorphism in the sense of Definition 2.4.33.
Since these simple tensor fields generate $\Gamma(\mathbf{E}\otimes\mathbf{F})$, the equivalence holds for every section, completing the proof. □

Example 2.4.35. (Evaluation) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, and let $\nabla^{\mathbf{E}^*}$ be the induced connection on the dual.
Define the smooth bundle homomorphism called the *evaluation homomorphism*

$$\mathrm{ev}\colon \mathbf{E}^* \otimes \mathbf{E} \longrightarrow M \times \mathbb{R}$$

pointwise by

$$\mathrm{ev}_p(\mathbf{v}_p \otimes \mathbf{u}_p) = \langle \mathbf{v}_p, \mathbf{u}_p \rangle, \qquad p \in M, \mathbf{v}_p \in \mathbf{E}^*_p, \mathbf{u}_p \in \mathbf{E}_p.$$

The trivial bundle $M \times \mathbb{R}$ carries the trivial connection; that is, regarding $f \in C^\infty(M)$ as a section of $M \times \mathbb{R}$, we have

$$\nabla^{M\times\mathbb{R}}_{\mathbf{X}} f = \mathbf{X}(f).$$

Let $\mathbf{v} \in \Gamma(\mathbf{E}^*)$, $\mathbf{u} \in \Gamma(\mathbf{E})$, and $\mathbf{X} \in \mathfrak{X}(M)$. On the left-hand side:

$$\nabla^{M\times\mathbb{R}}_{\mathbf{X}}(\mathrm{ev}(\mathbf{v} \otimes \mathbf{u})) = \nabla^{M\times\mathbb{R}}_{\mathbf{X}}\big(p \mapsto \langle \mathbf{v}_p, \mathbf{u}_p \rangle\big) = \mathbf{X}(\langle \mathbf{v}, \mathbf{u} \rangle).$$

On the right-hand side:

$$\mathrm{ev}\big((\nabla^{\mathbf{E}^*}_{\mathbf{X}} \mathbf{v}) \otimes \mathbf{u} + \mathbf{v} \otimes \nabla^{\mathbf{E}}_{\mathbf{X}} \mathbf{u}\big) = \langle \nabla^{\mathbf{E}^*}_{\mathbf{X}} \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \nabla^{\mathbf{E}}_{\mathbf{X}} \mathbf{u} \rangle.$$

By the definition of the induced connection on $\mathbf{E}^*$,

$$\mathbf{X}(\langle \mathbf{v}, \mathbf{u} \rangle) = \langle \nabla^{\mathbf{E}^*}_{\mathbf{X}} \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \nabla^{\mathbf{E}}_{\mathbf{X}} \mathbf{u} \rangle.$$

Thus,

$$\nabla^{M\times\mathbb{R}}_{\mathbf{X}}\big(\mathrm{ev}(\mathbf{v} \otimes \mathbf{u})\big) = \mathrm{ev}\big((\nabla^{\mathbf{E}^*}_{\mathbf{X}} \mathbf{v}) \otimes \mathbf{u} + \mathbf{v} \otimes \nabla^{\mathbf{E}}_{\mathbf{X}} \mathbf{u}\big),$$

and we conclude that evaluation is a parallel smooth bundle homomorphism.

Example 2.4.36. (Contraction) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, and let $\mathbf{F} \to M$ be another bundle with connection $\nabla^{\mathbf{F}}$.
Define the smooth homomorphism called the *contraction homomorphism*

$$\mathrm{contr}\colon \mathbf{E}^* \otimes \mathbf{E} \otimes \mathbf{F} \longrightarrow \mathbf{F}$$

pointwise by

$$\mathrm{contr}_p(\mathbf{v}_p \otimes \mathbf{u}_p \otimes \mathbf{w}_p) = \langle \mathbf{v}_p, \mathbf{u}_p \rangle \mathbf{w}_p, \qquad p \in M, \mathbf{v}_p \in \mathbf{E}^*_p, \mathbf{u}_p \in \mathbf{E}_p, \mathbf{w}_p \in \mathbf{F}_p.$$

The notion of a parallel homomorphism extends naturally to three factors: a homomorphism

$$\Phi\colon \mathbf{E}^* \otimes \mathbf{E} \otimes \mathbf{F} \longrightarrow \mathbf{G}$$

is parallel if, for all $\mathbf{v} \in \Gamma(\mathbf{E}^*)$, $\mathbf{u} \in \Gamma(\mathbf{E})$, $\mathbf{w} \in \Gamma(\mathbf{F})$, and $\mathbf{X} \in \mathfrak{X}(M)$, we have

$$\nabla^{\mathbf{G}}_{\mathbf{X}}(\Phi(\mathbf{v} \otimes \mathbf{u} \otimes \mathbf{w})) = \Phi\big((\nabla^{\mathbf{E}^*}_{\mathbf{X}} \mathbf{v}) \otimes \mathbf{u} \otimes \mathbf{w}\big) + \Phi\big(\mathbf{v} \otimes (\nabla^{\mathbf{E}}_{\mathbf{X}} \mathbf{u}) \otimes \mathbf{w}\big) + \Phi\big(\mathbf{v} \otimes \mathbf{u} \otimes \nabla^{\mathbf{F}}_{\mathbf{X}} \mathbf{w}\big).$$

Let us verify that contraction satisfies this property.

For the left-hand side, we have

$$\begin{aligned}\nabla^{\mathbf{F}}_{\mathbf{X}}(\operatorname{contr}(\mathbf{v}\otimes\mathbf{u}\otimes\mathbf{w})) &= \nabla^{\mathbf{F}}_{\mathbf{X}}\big(p\mapsto\langle\mathbf{v}_p,\mathbf{u}_p\rangle\mathbf{w}_p\big)\\ &= \nabla^{\mathbf{F}}_{\mathbf{X}}(\langle\mathbf{v},\mathbf{u}\rangle\cdot\mathbf{w}).\end{aligned}$$

The Leibniz rule on $\mathbf{F}$ gives

$$\nabla^{\mathbf{F}}_{\mathbf{X}}(\langle\mathbf{v},\mathbf{u}\rangle\cdot\mathbf{w}) = \mathbf{X}(\langle\mathbf{v},\mathbf{u}\rangle)\mathbf{w} + \langle\mathbf{v},\mathbf{u}\rangle\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w},$$

so

$$\nabla^{\mathbf{F}}_{\mathbf{X}}(\operatorname{contr}(\mathbf{v}\otimes\mathbf{u}\otimes\mathbf{w})) = \mathbf{X}(\langle\mathbf{v},\mathbf{u}\rangle)\mathbf{w} + \langle\mathbf{v},\mathbf{u}\rangle\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w}.$$

We wish to calculate

$$\operatorname{contr}\big((\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v})\otimes\mathbf{u}\otimes\mathbf{w}\big) + \operatorname{contr}\big(\mathbf{v}\otimes(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{w}\big) + \operatorname{contr}\big(\mathbf{v}\otimes\mathbf{u}\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w}\big).$$

Evaluating each term using the definition of contraction gives

$$\begin{aligned}\operatorname{contr}\big((\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v})\otimes\mathbf{u}\otimes\mathbf{w}\big) &= \langle\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v},\mathbf{u}\rangle\mathbf{w},\\ \operatorname{contr}\big(\mathbf{v}\otimes(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{w}\big) &= \langle\mathbf{v},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}\rangle\mathbf{w},\\ \operatorname{contr}\big(\mathbf{v}\otimes\mathbf{u}\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w}\big) &= \langle\mathbf{v},\mathbf{u}\rangle\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w}.\end{aligned}$$

Thus,

$$\operatorname{contr}\big((\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v})\otimes\mathbf{u}\otimes\mathbf{w}\big) + \operatorname{contr}\big(\mathbf{v}\otimes(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{w}\big) + \operatorname{contr}\big(\mathbf{v}\otimes\mathbf{u}\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w}\big) = \langle\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v},\mathbf{u}\rangle\mathbf{w} + \langle\mathbf{v},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}\rangle\mathbf{w} + \langle\mathbf{v},\mathbf{u}\rangle\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w}.$$

Finally, by the definition of the connection on the dual $\mathbf{E}^*$, we have

$$\mathbf{X}(\langle\mathbf{v},\mathbf{u}\rangle) = \langle\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v},\mathbf{u}\rangle + \langle\mathbf{v},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}\rangle.$$

Equating the two terms, we obtain

$$\nabla^{\mathbf{F}}_{\mathbf{X}}(\operatorname{contr}(\mathbf{v}\otimes\mathbf{u}\otimes\mathbf{w})) = \operatorname{contr}\big((\nabla^{\mathbf{E}^*}_{\mathbf{X}}\mathbf{v})\otimes\mathbf{u}\otimes\mathbf{w}\big) + \operatorname{contr}\big(\mathbf{v}\otimes(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{w}\big) + \operatorname{contr}\big(\mathbf{v}\otimes\mathbf{u}\otimes\nabla^{\mathbf{F}}_{\mathbf{X}}\mathbf{w}\big).$$

Therefore contraction is a parallel smooth bundle homomorphism.

Proposition 2.4.37. (Compatibility of the connection with contractions) Let M be a smooth manifold with or without boundary, let $\mathbf{E}_1,\dots,\mathbf{E}_N,\mathbf{F}\to M$ be smooth vector bundles equipped with connections, and let

$$C\colon \mathbf{E}_1\otimes\cdots\otimes\mathbf{E}_N \longrightarrow \mathbf{F}$$

be a parallel smooth bundle homomorphism. Then, for every section $\mathbf{T}\in\Gamma(\mathbf{E}_1\otimes\cdots\otimes\mathbf{E}_N)$ and every $\mathbf{X}\in\mathfrak{X}(M)$,

$$\nabla^{\mathbf{F}}_{\mathbf{X}}\big(C(\mathbf{T})\big) = C(\nabla_{\mathbf{X}}\mathbf{T}). \tag{2.18}$$

In particular, induced connections commute with evaluation, permutations of factors, and all canonical contractions.

If $(M, \mathbf{g})$ is Riemannian and ∇^M is the Levi–Civita connection, metric contraction is parallel. Therefore, for every tensor S with two covariant indices that can be contracted,

$$\nabla\big(\operatorname{tr}_{\mathbf{g}} S\big) = \operatorname{tr}_{\mathbf{g}}(\nabla S). \tag{2.19}$$

The same identity holds for any finite composition of metric contractions.

Proof. By Proposition 2.4.34, the condition that C be parallel is equivalent to $\nabla C = 0$ when C is regarded as a section of the corresponding homomorphism bundle. The Leibniz rule for evaluation gives

$$\nabla^{\mathbf{F}}_{\mathbf{X}}(C(\mathbf{T})) = (\nabla_{\mathbf{X}} C)(\mathbf{T}) + C(\nabla_{\mathbf{X}}\mathbf{T}) = C(\nabla_{\mathbf{X}}\mathbf{T}),$$

which proves (2.18).
Evaluation and contraction are parallel by the preceding examples. Permutations of factors are parallel because the product connection differentiates each factor, while the permutation only changes their order. In the Riemannian case, $\nabla^M g = 0$ and also $\nabla^M g^{-1} = 0$; hence raising indices and contracting them with g^{-1} are parallel homomorphisms. Identity (2.19) is therefore an instance of (2.18). The assertion for a finite composition follows by applying the same argument successively. □

Finally, we give an example showing why parallelism of homomorphisms is more general than the Leibniz rule for tensor fields of type $(0,2)$:

Example 2.4.38. (Evaluation and the Leibniz rule) Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. Define the map

$$\widetilde{\mathrm{ev}}\colon (\mathbf{E}^*\otimes\mathbf{E}^*)\otimes(\mathbf{E}\otimes\mathbf{E}) \longrightarrow M\times\mathbb{R}, \qquad \widetilde{\mathrm{ev}}_p(\beta_p\otimes(\mathbf{u}_p\otimes\mathbf{v}_p)) = \langle\beta_p, \mathbf{u}_p\otimes\mathbf{v}_p\rangle = \beta_p(\mathbf{u}_p,\mathbf{v}_p).$$

Then, for all $\mathbf{T}\in\Gamma(\mathbf{E}^*\otimes\mathbf{E}^*)$, $\mathbf{u},\mathbf{v}\in\Gamma(\mathbf{E})$, and $\mathbf{X}\in\mathfrak{X}(M)$, the Leibniz rule

$$\mathbf{X}\big(\mathbf{T}(\mathbf{u},\mathbf{v})\big) = (\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}\mathbf{T})(\mathbf{u},\mathbf{v}) + \mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{v}) + \mathbf{T}(\mathbf{u},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v})$$

is equivalent to $\widetilde{\mathrm{ev}}$ being a parallel smooth bundle homomorphism, that is,

$$\nabla^{M\times\mathbb{R}}_{\mathbf{X}}\big(\widetilde{\mathrm{ev}}(\mathbf{T}\otimes(\mathbf{u}\otimes\mathbf{v}))\big) = \widetilde{\mathrm{ev}}\big((\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}\mathbf{T})\otimes(\mathbf{u}\otimes\mathbf{v})\big) + \widetilde{\mathrm{ev}}\big(\mathbf{T}\otimes((\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v})\big) + \widetilde{\mathrm{ev}}\big(\mathbf{T}\otimes(\mathbf{u}\otimes\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v})\big).$$

Proof. By the pointwise definition,

$$\widetilde{\mathrm{ev}}(\mathbf{T}\otimes(\mathbf{u}\otimes\mathbf{v})) = \langle\mathbf{T},\mathbf{u}\otimes\mathbf{v}\rangle = \mathbf{T}(\mathbf{u},\mathbf{v}), \qquad \nabla^{M\times\mathbb{R}}_{\mathbf{X}} f = \mathbf{X}(f) \text{ for } f\in C^\infty(M).$$

If $\widetilde{\mathrm{ev}}$ is parallel, then

$$\begin{aligned}
\mathbf{X}\big(\mathbf{T}(\mathbf{u},\mathbf{v})\big) &= \nabla^{M\times\mathbb{R}}_{\mathbf{X}}\big(\widetilde{\mathrm{ev}}(\mathbf{T}\otimes(\mathbf{u}\otimes\mathbf{v}))\big)\\
&= \widetilde{\mathrm{ev}}\big((\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}\mathbf{T})\otimes(\mathbf{u}\otimes\mathbf{v})\big) + \widetilde{\mathrm{ev}}\big(\mathbf{T}\otimes((\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v})\big) + \widetilde{\mathrm{ev}}\big(\mathbf{T}\otimes(\mathbf{u}\otimes\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v})\big)\\
&= \langle\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}\mathbf{T},\mathbf{u}\otimes\mathbf{v}\rangle + \langle\mathbf{T},(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})\otimes\mathbf{v}\rangle + \langle\mathbf{T},\mathbf{u}\otimes\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v}\rangle\\
&= (\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}_{\mathbf{X}}\mathbf{T})(\mathbf{u},\mathbf{v}) + \mathbf{T}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{v}) + \mathbf{T}(\mathbf{u},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v}).
\end{aligned}$$

This recovers the Leibniz rule.
Conversely, if the Leibniz rule holds for every $\mathbf{T}, \mathbf{u}, \mathbf{v}$, then

$$\nabla_{\mathbf{X}}^{M\times\mathbb{R}}\big(\widetilde{\mathrm{ev}}(\mathbf{T}\otimes(\mathbf{u}\otimes\mathbf{v}))\big) = \mathbf{X}\big(\mathbf{T}(\mathbf{u},\mathbf{v})\big)$$

$$= (\nabla_{\mathbf{X}}^{\mathbf{E}^*\otimes\mathbf{E}^*}\mathbf{T})(\mathbf{u},\mathbf{v}) + \mathbf{T}(\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{u},\mathbf{v}) + \mathbf{T}(\mathbf{u},\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{v})$$

$$= \langle\nabla_{\mathbf{X}}^{\mathbf{E}^*\otimes\mathbf{E}^*}\mathbf{T}, \mathbf{u}\otimes\mathbf{v}\rangle + \langle\mathbf{T}, (\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{u})\otimes\mathbf{v}\rangle + \langle\mathbf{T}, \mathbf{u}\otimes\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{v}\rangle$$

$$= \widetilde{\mathrm{ev}}\big((\nabla_{\mathbf{X}}^{\mathbf{E}^*\otimes\mathbf{E}^*}\mathbf{T})\otimes(\mathbf{u}\otimes\mathbf{v})\big) + \widetilde{\mathrm{ev}}\big(\mathbf{T}\otimes((\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{u})\otimes\mathbf{v})\big) + \widetilde{\mathrm{ev}}\big(\mathbf{T}\otimes(\mathbf{u}\otimes\nabla_{\mathbf{X}}^{\mathbf{E}}\mathbf{v})\big).$$

This proves that the Leibniz rule is equivalent to $\widetilde{\mathrm{ev}}$ being parallel. □

The condition that a connection ∇ on a smooth vector bundle $\mathbf{E}$ be compatible with a bundle metric $\mathbf{h}$ can be stated compactly as follows:

Definition 2.4.39. Let M be a smooth manifold with or without boundary, let $\mathbf{E}\to M$ be a smooth vector bundle, let ∇ be a connection on $\mathbf{E}$, and let $\mathbf{h}$ be a bundle metric on $\mathbf{E}$. We say that ∇ is compatible with $\mathbf{h}$ if

$$\nabla^{\mathbf{E}^*\otimes\mathbf{E}^*}\mathbf{h} = 0.$$

In this case, the metric $\mathbf{h}$ is also said to be *parallel* with respect to ∇.

This condition is equivalent to the one given in Proposition 2.4.18:

Proposition 2.4.40. Let M be a smooth manifold with or without boundary, let $\mathbf{E}\longrightarrow M$ be a smooth vector bundle, let ∇ be a connection on $\mathbf{E}$, and let $\mathbf{h}$ be a bundle metric on $\mathbf{E}$. Then ∇ is compatible with $\mathbf{h}$ if and only if, for any $\mathbf{u},\mathbf{v}\in\Gamma(\mathbf{E})$ and any $\mathbf{X}\in\mathfrak{X}(M)$,

$$\mathbf{X}\big(\mathbf{h}(\mathbf{u},\mathbf{v})\big) = \mathbf{h}(\nabla_{\mathbf{X}}\mathbf{u},\mathbf{v}) + \mathbf{h}(\mathbf{u},\nabla_{\mathbf{X}}\mathbf{v}).$$

Proof. By Proposition 2.4.32, the covariant derivative of the section $\mathbf{h}\in\Gamma(T^{(0,2)}(\mathbf{E}))$ satisfies

$$(\nabla_{\mathbf{X}}\mathbf{h})(\mathbf{u},\mathbf{v}) = \mathbf{X}\big(\mathbf{h}(\mathbf{u},\mathbf{v})\big) - \mathbf{h}(\nabla_{\mathbf{X}}\mathbf{u},\mathbf{v}) - \mathbf{h}(\mathbf{u},\nabla_{\mathbf{X}}\mathbf{v}).$$

Metric compatibility is the identity $\nabla\mathbf{h} = 0$. Thus this identity is exactly equivalent to the formula in the statement for all $\mathbf{X}$, $\mathbf{u}$, and $\mathbf{v}$. □

2.5 Hermitian bundles and conjugation of connections

The bundle metrics introduced at the beginning of the chapter are real. We now fix their complex version and the precise relationship between conjugation, duals, and connections. Throughout this section, vector fields on the base are real, and Hermitian products are linear in the first variable.

Definition 2.5.1. (Hermitian bundle metric) Let M be a smooth manifold with or without boundary, and let $\mathbf{E}\to M$ be a smooth complex vector bundle. A *Hermitian bundle metric* is a smooth family $\mathbf{h}_{\mathbf{E}}$ of positive definite Hermitian forms on the fibers, linear in the first variable and antilinear in the second. Equivalently,

the bilinear map

$$\widetilde{\mathbf{h}}_{\mathbf{E}} \colon \mathbf{E} \times_M \overline{\mathbf{E}} \longrightarrow \underline{\mathbb{C}}_M, \qquad \widetilde{\mathbf{h}}_{\mathbf{E}}(v, \overline{w}) = \mathbf{h}_{\mathbf{E}}(v, w),$$

is a smooth section of $\mathbf{E}^* \otimes (\overline{\mathbf{E}})^*$ and is positive definite in the sense that $\widetilde{\mathbf{h}}_{\mathbf{E}}(v, \overline{v}) > 0$ for $v \neq 0$.

Proposition 2.5.2. (Metric on the conjugate bundle) The formula

$$\mathbf{h}_{\overline{\mathbf{E}}}(\overline{v}, \overline{w}) := \overline{\mathbf{h}_{\mathbf{E}}(v, w)} \tag{2.20}$$

defines a smooth Hermitian metric on $\overline{\mathbf{E}}$, linear in the first variable and antilinear in the second. Under the identification $\overline{\overline{\mathbf{E}}} \cong \mathbf{E}$, the twice-conjugated metric is $\mathbf{h}_{\mathbf{E}}$.

Proof. If $\lambda, \mu \in \mathbb{C}$, the scalar structure of $\overline{\mathbf{E}}$ and the convention for $\mathbf{h}_{\mathbf{E}}$ give

$$\mathbf{h}_{\overline{\mathbf{E}}}(\lambda \overline{v}, \mu \overline{w}) = \overline{\mathbf{h}_{\mathbf{E}}(\overline{\lambda} v, \overline{\mu} w)} = \lambda \overline{\mu} \, \overline{\mathbf{h}_{\mathbf{E}}(v, w)}.$$

Hermitian symmetry, positivity, and smoothness follow by conjugating the corresponding properties of $\mathbf{h}_{\mathbf{E}}$. Conjugating twice recovers the original formula. □

Proposition 2.5.3. (Fiberwise Riesz isomorphism) The map

$$\mathcal{R}_{\mathbf{E}} \colon \overline{\mathbf{E}} \longrightarrow \mathbf{E}^*, \qquad \mathcal{R}_{\mathbf{E}}(\overline{v})(w) := \mathbf{h}_{\mathbf{E}}(w, v), \tag{2.21}$$

is a smooth complex-linear bundle isomorphism compatible with restrictions. For each $\overline{v}$, the functional $w \mapsto \mathbf{h}_{\mathbf{E}}(w, v)$ is complex-linear. In a unitary local frame $(\mathbf{e}_1, \dots, \mathbf{e}_r)$, if $\ell = \sum_{a=1}^{r} \ell_a \mathbf{e}^a$, then

$$\mathcal{R}_{\mathbf{E}}^{-1}(\ell) = \overline{\sum_{a=1}^{r} \overline{\ell_a} \, \mathbf{e}_a}. \tag{2.22}$$

In particular, the map $\mathbf{E} \to \mathbf{E}^*$, $v \mapsto \mathbf{h}_{\mathbf{E}}(\cdot, v)$, is antilinear, whereas $\mathcal{R}_{\mathbf{E}} \colon \overline{\mathbf{E}} \to \mathbf{E}^*$ is complex-linear. In the real case, conjugation disappears and we recover the usual real identification.

Proof. Linearity in w is part of the Hermitian convention. For linearity in $\overline{v}$, using $\lambda \overline{v} = \overline{\overline{\lambda} v}$, we obtain

$$\mathcal{R}_{\mathbf{E}}(\lambda \overline{v})(w) = \mathbf{h}_{\mathbf{E}}(w, \overline{\lambda} v) = \lambda \mathbf{h}_{\mathbf{E}}(w, v).$$

In a unitary frame, $\mathbf{h}_{\mathbf{E}}(\sum_{a=1}^{r} w^a \mathbf{e}_a, \sum_{a=1}^{r} v^a \mathbf{e}_a) = \sum_{a=1}^{r} w^a \overline{v^a}$; this directly gives (2.22). The formula proves both bijectivity and local smoothness of $\mathcal{R}_{\mathbf{E}}$ and its inverse. Since the definition is intrinsic, the local formulas glue and commute with every restriction to an open set. The assertion about $\mathbf{E} \to \mathbf{E}^*$ follows from $\mathbf{h}_{\mathbf{E}}(w, \lambda v) = \overline{\lambda} \mathbf{h}_{\mathbf{E}}(w, v)$. □

Definition 2.5.4. (Complex, dual, and conjugate connections) Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth complex vector bundle. A *complex connection* on $\mathbf{E}$ is a map

$$\nabla^{\mathbf{E}} \colon \mathfrak{X}(M) \times \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E})$$

that is $C^\infty(M,\mathbb{R})$-linear in the real vector field, additive in the section, satisfies

$$\nabla^{\mathbf{E}}_{\mathbf{X}}(f\mathbf{u}) = \mathbf{X}(f)\mathbf{u} + f\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}, \qquad f \in C^\infty(M,\mathbb{C}),$$

and is complex-linear in $\mathbf{u}$ for fixed $\mathbf{X}$.
The dual connection $\nabla^{\mathbf{E}^*}$ and the conjugate connection $\nabla^{\overline{\mathbf{E}}}$ are defined, respectively, by

$$(\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\ell})(\mathbf{u}) := \mathbf{X}(\boldsymbol{\ell}(\mathbf{u})) - \boldsymbol{\ell}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}), \tag{2.23}$$

$$\nabla^{\overline{\mathbf{E}}}_{\mathbf{X}}\overline{\mathbf{u}} := \overline{\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}}. \tag{2.24}$$

Proposition 2.5.5. (Well-definedness and naturality) Let M be a smooth manifold with or without boundary, and let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth complex vector bundles equipped with complex connections. Formulas (2.23) and (2.24) define complex connections. They commute with restrictions, and double conjugation recovers $\nabla^{\mathbf{E}}$. If $\mathbf{A}\colon \mathbf{E} \to \mathbf{F}$ is parallel, that is, $\nabla^{\mathbf{F}}_{\mathbf{X}}(\mathbf{A}\mathbf{u}) = \mathbf{A}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u})$, then $\overline{\mathbf{A}}$ is parallel for the conjugate connections. In conjugate frames, the matrix of $\nabla^{\overline{\mathbf{E}}}$ is the conjugate of the matrix of $\nabla^{\mathbf{E}}$.

Proof. Only the Leibniz rule for the conjugate connection requires attention. If $f \in C^\infty(M,\mathbb{C})$, then $f\overline{\mathbf{u}} = \overline{\overline{f}\,\mathbf{u}}$, and, since $\mathbf{X}$ is real, $\mathbf{X}(\overline{f}) = \overline{\mathbf{X}(f)}$. Hence

$$\begin{aligned}\nabla^{\overline{\mathbf{E}}}_{\mathbf{X}}(f\overline{\mathbf{u}}) &= \overline{\nabla^{\mathbf{E}}_{\mathbf{X}}(\overline{f}\,\mathbf{u})} = \overline{\overline{\mathbf{X}(f)}\mathbf{u} + \overline{f}\,\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u}} \\ &= \mathbf{X}(f)\overline{\mathbf{u}} + f\nabla^{\overline{\mathbf{E}}}_{\mathbf{X}}\overline{\mathbf{u}}.\end{aligned}$$

The properties of the dual connection follow by expanding $\mathbf{X}(\boldsymbol{\ell}(f\mathbf{u}))$ and using the Leibniz rule for $\nabla^{\mathbf{E}}$; canceling $\mathbf{X}(f)\boldsymbol{\ell}(\mathbf{u})$ leaves $(\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\ell})(f\mathbf{u}) = f(\nabla^{\mathbf{E}^*}_{\mathbf{X}}\boldsymbol{\ell})(\mathbf{u})$. The remaining assertions are verified directly on local sections. □

Definition 2.5.6. (Hermitian connection) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \longrightarrow M$ be a smooth complex vector bundle with Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$, and let $\nabla^{\mathbf{E}}$ be a complex connection. The connection $\nabla^{\mathbf{E}}$ is *compatible with* $\mathbf{h}_{\mathbf{E}}$, or *Hermitian*, if, for all $\mathbf{X} \in \mathfrak{X}(M)$ and $\mathbf{u},\mathbf{v} \in \Gamma(\mathbf{E})$,

$$\mathbf{X}\big(\mathbf{h}_{\mathbf{E}}(\mathbf{u},\mathbf{v})\big) = \mathbf{h}_{\mathbf{E}}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{v}) + \mathbf{h}_{\mathbf{E}}(\mathbf{u},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v}). \tag{2.25}$$

Proposition 2.5.7. (Parallelism of the Riesz isomorphism) Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth complex vector bundle with Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and complex connection $\nabla^{\mathbf{E}}$. The connection $\nabla^{\mathbf{E}}$ is Hermitian if and only if the Riesz isomorphism is parallel:

$$\nabla^{\mathbf{E}^*}_{\mathbf{X}}\big(\mathcal{R}_{\mathbf{E}}(\overline{\mathbf{v}})\big) = \mathcal{R}_{\mathbf{E}}\big(\nabla^{\overline{\mathbf{E}}}_{\mathbf{X}}\overline{\mathbf{v}}\big), \qquad \mathbf{X} \in \mathfrak{X}(M), \quad \mathbf{v} \in \Gamma(\mathbf{E}). \tag{2.26}$$

In this case, $\nabla^{\overline{\mathbf{E}}}$ is compatible with $\mathbf{h}_{\overline{\mathbf{E}}}$, and $\mathcal{R}_{\mathbf{E}}^{-1}$ is also parallel.

Proof. Evaluating the left-hand side of (2.26) on $\mathbf{u}$ and using the dual connection gives

$$\mathbf{X}(\mathbf{h}_{\mathbf{E}}(\mathbf{u},\mathbf{v})) - \mathbf{h}_{\mathbf{E}}(\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{v}).$$

The right-hand side evaluated on $\mathbf{u}$ is $\mathbf{h}_{\mathbf{E}}(\mathbf{u},\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{v})$. Consequently, equality for all $\mathbf{u},\mathbf{v},\mathbf{X}$ is exactly equivalent to (2.25). Conjugating this identity proves compatibility of the conjugate connection with $\mathbf{h}_{\overline{\mathbf{E}}}$. Finally, differentiating $\mathcal{R}^{-1}_{\mathbf{E}}\mathcal{R}_{\mathbf{E}} = \mathrm{id}_{\overline{\mathbf{E}}}$ and using (2.26) proves parallelism of the inverse. □

2.6 Principal bundles and principal connections

A vector bundle describes vectors varying over the base, but a local trivialization also requires choosing a frame in each fiber. The various frames are related by the structure group, and this information is not represented by any particular vector in the bundle. A principal bundle records all admissible frames simultaneously, together with the group action that transforms them.

This description also clarifies why a connection is needed. To compare data in different fibers, one must specify which displacements in the total space are regarded as horizontal. The resulting parallel transport generally depends on the path; curvature measures this dependence infinitesimally. Later, principal connections and their curvature will be the natural variables of Yang–Mills theory, while gauge transformations will express changes of frame.

Throughout this section, all principal bundles carry a right action. The conventions are chosen so that, if $\mathbf{s}_\beta = \mathbf{s}_\alpha g_{\alpha\beta}$ are two local sections, the connection potentials satisfy

$$\mathbf{A}_\beta = \mathrm{Ad}_{g^{-1}_{\alpha\beta}}\mathbf{A}_\alpha + g^{-1}_{\alpha\beta}dg_{\alpha\beta}.$$

This is also the convention that will later produce the right gauge action $\mathbf{A}\triangleleft\mathbf{u} = \mathbf{u}^{-1}\mathbf{A}\mathbf{u} + \mathbf{u}^{-1}d\mathbf{u}$. For a compatible external treatment of principal bundles and their connections, see [21, Ch. 15]; the transition, curvature, and gauge identities needed in this book are verified below using the right-action convention fixed above.

Definition 2.6.1. (Principal bundle) Let G be a Lie group and M a smooth manifold with or without boundary. A *right principal bundle with structure group* G consists of a smooth manifold $\mathbf{P}$, a smooth surjective map $\pi\colon \mathbf{P}\to M$, and a smooth right action

$$R\colon \mathbf{P}\times G\longrightarrow \mathbf{P}, \qquad (p,g)\longmapsto p\cdot g,$$

such that:

(a) $\pi(p\cdot g) = \pi(p)$ for every $p\in\mathbf{P}$ and $g\in G$;

(b) the action is free;

(c) each $x\in M$ has an open neighborhood $U\ni x$ and a G-equivariant diffeomorphism

$$\Phi\colon \pi^{-1}(U)\longrightarrow U\times G, \qquad \Phi(p\cdot g) = \big(\pi(p),\mathrm{pr}_2(\Phi(p))g\big),$$

such that $\mathrm{pr}_1\circ\Phi = \pi$.

Every trivialization determines a local section $\mathbf{s}\colon U \to \mathbf{P}$, $\mathbf{s}(x) = \Phi^{-1}(x, e)$, and each $p \in \pi^{-1}(U)$ can be written uniquely as

$$p = \mathbf{s}(x) \cdot g, \qquad x = \pi(p).$$

If $\mathbf{s}_\alpha$ and $\mathbf{s}_\beta$ are local sections, there exists a unique smooth map

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$$

such that

$$\mathbf{s}_\beta = \mathbf{s}_\alpha g_{\alpha\beta}. \tag{2.27}$$

Proposition 2.6.2. (Cocycles, reconstruction, and triviality) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a right principal bundle. The transition functions satisfy

$$g_{\alpha\alpha} = e, \qquad g_{\beta\alpha} = g_{\alpha\beta}^{-1}, \qquad g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$$

on their respective domains. Conversely, a cover (U_α) and smooth maps satisfying these identities determine a right principal bundle up to principal bundle isomorphism: glue the $U_\alpha \times G$ by

$$(x, h)_\beta \sim (x, g_{\alpha\beta}(x)h)_\alpha. \tag{2.28}$$

Moreover, a principal bundle is trivial if and only if it admits a smooth global section.

Proof. The first two identities follow from uniqueness in (2.27); on a triple intersection, $\mathbf{s}_\gamma = \mathbf{s}_\beta g_{\beta\gamma} = \mathbf{s}_\alpha g_{\alpha\beta} g_{\beta\gamma}$ proves the third. These same identities make (2.28) an equivalence relation. The local quotients glue smoothly, the action $[x, h]_\alpha \cdot k = [x, hk]_\alpha$ is well defined, and the charts $[x, h]_\alpha \mapsto (x, h)$ are principal trivializations. Two realizations obtained in this way admit a principal bundle isomorphism that is the identity in the prescribed charts. Uniqueness is understood relative to these charts, since a bundle may have nontrivial gauge automorphisms.
If $\mathbf{s}\colon M \to \mathbf{P}$ is global, every $p \in \mathbf{P}_x$ can be written uniquely as $p = \mathbf{s}(x)g$, and $\mathbf{P} \to M \times G$, $\mathbf{s}(x)g \mapsto (x, g)$, is a principal trivialization. The section $x \mapsto (x, e)$ proves the converse. □

Definition 2.6.3. (Principal bundle morphisms and pullbacks) Let M and N be smooth manifolds with or without boundary, let $\mathbf{P} \to M$ and $\mathbf{Q} \to N$ be principal bundles with the same group G, and let $f\colon M \to N$ be a smooth map. A morphism between principal bundles with the same group is a smooth map $\Phi\colon \mathbf{P} \to \mathbf{Q}$ covering f and satisfying $\Phi(pg) = \Phi(p)g$. The pullback of $\mathbf{Q} \to N$ by f is

$$f^*\mathbf{Q} := \{(x, q) \in M \times \mathbf{Q} \mid f(x) = \pi_{\mathbf{Q}}(q)\}, \qquad (x, q)g = (x, qg).$$

With the projection onto M, it is a principal bundle, and $(x, q) \mapsto q$ is the universal morphism covering f.

Proof. A trivialization of $\mathbf{Q}$ over $V \subseteq N$ induces $f^{-1}(V) \times G \cong (f^*\mathbf{Q})|_{f^{-1}(V)}$, proving the local assertion. If $\Psi\colon \mathbf{R} \to \mathbf{Q}$ covers $f \circ \pi_{\mathbf{R}}$, the unique morphism $\mathbf{R} \to f^*\mathbf{Q}$ over M projecting to Ψ is $r \mapsto (\pi_{\mathbf{R}}(r), \Psi(r))$. □

Definition 2.6.4. (Associated bundle) Let M be a smooth manifold with or without boundary, let $\mathbf{P} \to M$ be a right principal bundle with group G, and let $\rho\colon G \to GL(V)$ be a finite-dimensional real or complex linear representation. Define

$$\mathbf{P} \times_\rho V := (\mathbf{P} \times V)/\!\sim,$$

where

$$(p \cdot g, v) \sim (p, \rho(g)v).$$

We write $[p, v]_\rho$ for the class of (p, v). The projection

$$[p, v]_\rho \longmapsto \pi(p)$$

makes $\mathbf{P} \times_\rho V$ a smooth vector bundle over M.

Proposition 2.6.5. (Associated trivializations and cocycles) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a right principal bundle. If $\mathbf{s}_\alpha$ is a local principal section, then

$$\Psi_\alpha\colon (\mathbf{P} \times_\rho V)|_{U_\alpha} \longrightarrow U_\alpha \times V, \qquad \Psi_\alpha([\mathbf{s}_\alpha(x)g, v]) = (x, \rho(g)v),$$

is well defined and is a vector bundle trivialization. If $\mathbf{s}_\beta = \mathbf{s}_\alpha g_{\alpha\beta}$, the coordinates are related by

$$v_\beta = \rho(g_{\alpha\beta}^{-1})v_\alpha.$$

Thus the associated bundle depends only on the principal bundle and the representation, and is independent of the chosen local sections.

Proof. The relation $(pg, v) \sim (p, \rho(g)v)$ shows that $[\mathbf{s}_\alpha gh, v] = [\mathbf{s}_\alpha g, \rho(h)v]$; both representatives have the same image under Ψ_α. The inverse is $(x, v) \mapsto [\mathbf{s}_\alpha(x), v]$. Finally, $[\mathbf{s}_\beta, v_\beta] = [\mathbf{s}_\alpha, \rho(g_{\alpha\beta})v_\beta]$, so $v_\alpha = \rho(g_{\alpha\beta})v_\beta$, which is equivalent to the stated formula. □

Proposition 2.6.6. Let M be a smooth manifold with or without boundary, and let $\mathbf{E} = \mathbf{P} \times_\rho V \to M$ be an associated bundle. Smooth sections of $\mathbf{E}$ are in bijective correspondence with smooth maps $\widetilde{u}\colon \mathbf{P} \to V$ satisfying

$$\widetilde{u}(p \cdot g) = \rho(g^{-1})\widetilde{u}(p). \tag{2.29}$$

The correspondence is given by

$$\mathbf{u}(x) = [p, \widetilde{u}(p)]_\rho, \qquad p \in \mathbf{P}_x.$$

Proof. If p is replaced by $p \cdot g$, then

$$[p \cdot g, \widetilde{u}(p \cdot g)]_\rho = [p, \rho(g)\rho(g^{-1})\widetilde{u}(p)]_\rho = [p, \widetilde{u}(p)]_\rho.$$

Thus the formula defines a section. Conversely, given $\mathbf{u}(x)$ and $p \in \mathbf{P}_x$, there is a unique $v \in V$ such that $\mathbf{u}(x) = [p, v]_\rho$; set $\widetilde{u}(p) = v$. Uniqueness of v gives (2.29). Smoothness in both directions is checked in a principal trivialization. □

Definition 2.6.7. (Adjoint and adjoint group bundles) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a right principal bundle with group G. Define two distinct bundles:

(a) The *adjoint group bundle*

$$\mathrm{Ad}(\mathbf{P}) := \mathbf{P} \times_{\mathrm{Conj}} G, \qquad \mathrm{Conj}_g(h) = ghg^{-1}.$$

Its fibers are groups; in general, it is not a vector bundle.

(b) The *adjoint bundle*

$$\mathrm{ad}(\mathbf{P}) := \mathbf{P} \times_{\mathrm{Ad}} \mathfrak{g}.$$

This is a vector bundle, of rank $\dim \mathfrak{g}$.

Fiberwise multiplication on $\mathrm{Ad}(\mathbf{P})$ is well defined. Its sections $\mathbf{u}$ are identified with smooth maps $\widehat{u}\colon \mathbf{P} \to G$ satisfying

$$\widehat{u}(p \cdot g) = g^{-1}\widehat{u}(p)g. \tag{2.30}$$

By contrast, sections ξ of $\mathrm{ad}(\mathbf{P})$ are identified with maps $\widehat{\xi}\colon \mathbf{P} \to \mathfrak{g}$ such that

$$\widehat{\xi}(p \cdot g) = \mathrm{Ad}_{g^{-1}}\,\widehat{\xi}(p).$$

Definition 2.6.8. (Horizontal and equivariant forms) Let M be a smooth manifold with or without boundary, let $\mathbf{P} \to M$ be a right principal bundle with group G, and let $\rho\colon G \to GL(V)$ be a representation. A form $\boldsymbol{\alpha} \in \Omega^k(\mathbf{P}; V)$ is *horizontal* if it vanishes whenever at least one of its arguments belongs to

$$\mathbf{V}_p\mathbf{P} := \ker(d\pi_p).$$

It is *ρ-equivariant* if

$$R_{\mathbf{g}}^*\boldsymbol{\alpha} = \rho(g^{-1})\boldsymbol{\alpha} \qquad (g \in G).$$

Proposition 2.6.9. (Forms with values in an associated bundle) Let M be a smooth manifold with or without boundary, and let $\mathbf{E} = \mathbf{P} \times_\rho V \to M$ be an associated bundle. There is a natural isomorphism

$$\Omega^k(M; \mathbf{E}) \cong \{\boldsymbol{\alpha} \in \Omega^k(\mathbf{P}; V) \mid \boldsymbol{\alpha} \text{ is horizontal and } \rho\text{-equivariant}\}.$$

If $\boldsymbol{\alpha}$ belongs to the second space, the associated form $\underline{\boldsymbol{\alpha}}$ is defined by

$$\underline{\boldsymbol{\alpha}}_x(X_1, \dots, X_k) := [p, \boldsymbol{\alpha}_p(\widetilde{X}_1, \dots, \widetilde{X}_k)]_\rho, \tag{2.31}$$

where $p \in \mathbf{P}_x$ and $d\pi_p \widetilde{X}_i = X_i$.

Proof. Horizontality shows that (2.31) is independent of the lifts $\widetilde{X}_i$. If p is replaced by $p \cdot g$, we may choose the lifts $(dR_{\mathbf{g}})_p \widetilde{X}_i$; equivariance then gives

$$\boldsymbol{\alpha}_{p\cdot g}\big((dR_{\mathbf{g}})\widetilde{X}_1, \dots, (dR_{\mathbf{g}})\widetilde{X}_k\big) = \rho(g^{-1})\boldsymbol{\alpha}_p(\widetilde{X}_1, \dots, \widetilde{X}_k),$$

and the equivalence relation of the associated bundle shows that the class in (2.31) is unchanged.

Conversely, given $\beta \in \Omega^k(M;\mathbf{E})$ and $p \in \mathbf{P}_x$, there is a unique $v \in V$ such that

$$\beta_x(d\pi\,\widetilde{X}_1,\dots,d\pi\,\widetilde{X}_k) = [p,v]_\rho.$$

Define $\alpha_p(\widetilde{X}_1,\dots,\widetilde{X}_k) := v$. This form is horizontal, and uniqueness of v proves equivariance. In a principal trivialization, both constructions reduce to the usual identifications with V-valued forms, proving smoothness and that the two operations are inverses. □

For $\xi \in \mathfrak{g}$, denote by

$$\xi_{\mathbf{P}}(p) := \left.\frac{d}{dt}\right|_0 p \cdot \exp_G(t\xi)$$

the positive fundamental vector field of the right action. By (1.27),

$$[\xi_{\mathbf{P}}, \eta_{\mathbf{P}}] = [\xi,\eta]_{\mathbf{P}}.$$

For each p, the map $\xi \mapsto \xi_{\mathbf{P}}(p)$ is an isomorphism $\mathfrak{g} \to \mathbf{V}_p\mathbf{P}$.

Definition 2.6.10. (Principal connection) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a right principal bundle. A *principal connection form* on $\mathbf{P}$ is a form $\omega \in \Omega^1(\mathbf{P};\mathfrak{g})$ such that

$$\omega(\xi_{\mathbf{P}}) = \xi, \tag{2.32}$$

$$R_{\mathbf{g}}^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega. \tag{2.33}$$

The corresponding horizontal distribution is

$$\mathbf{H}_p\mathbf{P} := \ker \omega_p.$$

Then

$$T_p\mathbf{P} = \mathbf{H}_p\mathbf{P} \oplus \mathbf{V}_p\mathbf{P}, \qquad (dR_{\mathbf{g}})_p\mathbf{H}_p\mathbf{P} = \mathbf{H}_{p\cdot g}\mathbf{P}.$$

Proposition 2.6.11. (Horizontal distributions and connection forms) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a right principal bundle. The assignment $\mathbf{H}_p\mathbf{P} = \ker\omega_p$ establishes a bijection between principal connection forms and smooth distributions $\mathbf{H} \subset T\mathbf{P}$ such that

$$T_p\mathbf{P} = \mathbf{H}_p \oplus \mathbf{V}_p\mathbf{P}, \qquad (dR_{\mathbf{g}})_p\mathbf{H}_p = \mathbf{H}_{pg}.$$

The form corresponding to $\mathbf{H}$ is obtained by writing uniquely $Z = Z^{\mathbf{H}} + \xi_{\mathbf{P}}(p)$ and setting $\omega_p(Z) = \xi$.

Proof. For a connection form, the first property implies that $\omega_p|_{\mathbf{V}_p\mathbf{P}}\colon \mathbf{V}_p\mathbf{P} \to \mathfrak{g}$ is the inverse of $\xi \mapsto \xi_{\mathbf{P}}(p)$, giving the direct sum. The second property implies invariance of the kernels.
Conversely, the smooth isomorphism $\mathfrak{g} \to \mathbf{V}_p\mathbf{P}$, $\xi \mapsto \xi_{\mathbf{P}}(p)$, and the smooth projection onto $\mathbf{V}_p\mathbf{P}$ along $\mathbf{H}_p\mathbf{P}$ define the indicated form. We have $\omega(\xi_{\mathbf{P}}) = \xi$. Invariance of $\mathbf{H}$ and the identity $(dR_{\mathbf{g}})_p\xi_{\mathbf{P}}(p) = (\mathrm{Ad}_{g^{-1}}\xi)_{\mathbf{P}}(pg)$ imply $R_{\mathbf{g}}^*\omega = \mathrm{Ad}_{g^{-1}}\omega$. The two constructions are inverses by uniqueness of the horizontal–vertical decomposition. □

Definition 2.6.12. (Wedge bracket) Let N be a smooth manifold with or without boundary. For $\alpha \in \Omega^k(N;\mathfrak{g})$ and $\beta \in \Omega^\ell(N;\mathfrak{g})$, define

$$[\alpha \wedge \beta](\mathbf{X}_1,\dots,\mathbf{X}_{k+\ell}) := \sum_{\sigma \in \mathrm{Sh}(k,\ell)} \operatorname{sgn}(\sigma)\left[\alpha(\mathbf{X}_{\sigma(1)},\dots,\mathbf{X}_{\sigma(k)}),\beta(\mathbf{X}_{\sigma(k+1)},\dots,\mathbf{X}_{\sigma(k+\ell)})\right],$$

where $\mathrm{Sh}(k,\ell)$ is the set of (k,ℓ)-shuffle permutations. In particular,

$$[\alpha \wedge \beta](\mathbf{X},\mathbf{Y}) = [\alpha(\mathbf{X}),\beta(\mathbf{Y})] - [\alpha(\mathbf{Y}),\beta(\mathbf{X})]$$

for $\alpha,\beta \in \Omega^1(N;\mathfrak{g})$.

Lemma 2.6.13. (Graded algebra of the wedge bracket) Let N be a smooth manifold with or without boundary. Homogeneous forms on N satisfy

$$[\beta \wedge \alpha] = -(-1)^{k\ell}[\alpha \wedge \beta], \tag{2.34}$$

$$d[\alpha \wedge \beta] = [d\alpha \wedge \beta] + (-1)^k[\alpha \wedge d\beta], \tag{2.35}$$

and the graded Jacobi identity. In particular, if $\mathbf{A} \in \Omega^1(N;\mathfrak{g})$, then

$$[\mathbf{A} \wedge [\mathbf{A} \wedge \mathbf{A}]] = 0. \tag{2.36}$$

Proof. The first two identities follow by reordering the finite sets of shuffles and using, respectively, antisymmetry of the bracket on $\mathfrak{g}$ and the Leibniz rule for the exterior product. For each ordered partition of the arguments, the cyclic sum defining the graded Jacobi identity groups into a Jacobi identity in $\mathfrak{g}$. Taking the degree-one form $\mathbf{A}$ three times makes the three terms of the graded Jacobi identity equal, giving (2.36). □

Definition 2.6.14. (Principal curvature) Let M be a smooth manifold with or without boundary, and let ω be a principal connection on $\mathbf{P} \to M$. The curvature of ω is

$$\mathbf{\Omega}_\omega := d\omega + \frac{1}{2}[\omega \wedge \omega] \in \Omega^2(\mathbf{P};\mathfrak{g}). \tag{2.37}$$

We will use an elementary identity for forms and flows.

Lemma 2.6.15. (Cartan's formula) Let N be a smooth manifold with or without boundary, let $\mathbf{X}$ be a vector field whose local flow remains in N, and let α be a differential form. Then

$$\mathcal{L}_{\mathbf{X}}\alpha := \left.\frac{d}{dt}\right|_0 \Phi_t^*\alpha = d(\iota_{\mathbf{X}}\alpha) + \iota_{\mathbf{X}}(d\alpha).$$

Proof. Both sides are graded derivations of degree zero for the exterior product and commute with restrictions. For a function f, both sides equal $\mathbf{X}f$. In coordinates, every form is a sum of products of functions and differentials $\mathbf{d}x^i$; on the latter,

$$\mathcal{L}_{\mathbf{X}}(\mathbf{d}x^i) = d(\mathbf{X}x^i) = d(\iota_{\mathbf{X}}\mathbf{d}x^i).$$

The Leibniz rule establishes the identity for every form. □

Proposition 2.6.16. Let M be a smooth manifold with or without boundary, and let ω be a principal connection on $\mathbf{P} \to M$. The curvature $\mathbf{\Omega}_\omega$ is horizontal and Ad-equivariant:

$$R_{\mathbf{g}}^* \mathbf{\Omega}_\omega = \mathrm{Ad}_{g^{-1}} \mathbf{\Omega}_\omega.$$

Thus, by Proposition 2.6.9, it determines a global form

$$\mathbf{F}_\omega \in \Omega^2(M; \mathrm{ad}(\mathbf{P})).$$

Proof. Equivariance follows from $R_{\mathbf{g}}^* d\omega = d(R_{\mathbf{g}}^* \omega)$ and from the fact that $\mathrm{Ad}_{g^{-1}}$ preserves the bracket. The flow of $\xi_{\mathbf{P}}$ is $R_{\exp(t\xi)}$. Differentiating the equivariance of ω gives

$$\mathcal{L}_{\xi_{\mathbf{P}}} \omega = -\mathrm{ad}_\xi \, \omega = -[\xi, \omega].$$

Since $\iota_{\xi_{\mathbf{P}}} \omega = \xi$ is constant, Cartan's formula gives

$$\iota_{\xi_{\mathbf{P}}} d\omega = -[\xi, \omega].$$

On the other hand, the definition of the wedge bracket implies

$$\frac{1}{2} \iota_{\xi_{\mathbf{P}}} [\omega \wedge \omega] = [\xi, \omega].$$

The two terms cancel, so $\iota_{\xi_{\mathbf{P}}} \mathbf{\Omega}_\omega = 0$. Every vertical vector has the form $\xi_{\mathbf{P}}(p)$, and hence the curvature is horizontal. □

Proposition 2.6.17. (Local potentials and curvatures) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and let $\mathbf{s}_\alpha : U_\alpha \to \mathbf{P}$ be a local section. Define

$$\mathbf{A}_\alpha := \mathbf{s}_\alpha^* \omega \in \Omega^1(U_\alpha; \mathfrak{g}), \qquad \mathbf{F}_\alpha := \mathbf{s}_\alpha^* \mathbf{\Omega}_\omega.$$

Then

$$\mathbf{F}_\alpha = d\mathbf{A}_\alpha + \frac{1}{2}[\mathbf{A}_\alpha \wedge \mathbf{A}_\alpha]. \tag{2.38}$$

If $\mathbf{s}_\beta = \mathbf{s}_\alpha g_{\alpha\beta}$, then

$$\mathbf{A}_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \mathbf{A}_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}, \tag{2.39}$$

$$\mathbf{F}_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \mathbf{F}_\alpha. \tag{2.40}$$

Here $g^{-1}dg$ denotes the pullback of the left Maurer–Cartan form of G.

Proof. The first identity is the pullback of (2.37). Let $X \in T_x M$ and write $g = g_{\alpha\beta}(x)$. Differentiating $\mathbf{s}_\beta = \mathbf{s}_\alpha g_{\alpha\beta}$ gives

$$d(\mathbf{s}_\beta)_x X = (dR_{\mathbf{g}})_{\mathbf{s}_\alpha(x)} d(\mathbf{s}_\alpha)_x X + \big(g^{-1} dg(X)\big)_{\mathbf{P}}\big|_{\mathbf{s}_\beta(x)}.$$

Applying ω and using (2.32)–(2.33) yields (2.39).
For curvature, the second summand of the preceding expression is vertical, while $\mathbf{\Omega}_\omega$ is horizontal. The

horizontal summands transform by $dR_{\mathbf{g}}$, and equivariance of $\mathbf{\Omega}_\omega$ gives (2.40). □

Definition 2.6.18. (Exterior covariant derivative) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, let $\rho\colon G \to GL(V)$ be a representation, and let $\rho_*\colon \mathfrak{g} \to \mathrm{End}(V)$ be its differential. For a horizontal, ρ-equivariant form $\boldsymbol{\alpha} \in \Omega^k(\mathbf{P}; V)$, define

$$D_\omega \boldsymbol{\alpha} := d\boldsymbol{\alpha} + \rho_*(\omega) \wedge \boldsymbol{\alpha}. \tag{2.41}$$

The second operation is given by

$$\big(\rho_*(\omega) \wedge \boldsymbol{\alpha}\big)(\mathbf{X}_0, \dots, \mathbf{X}_k) = \sum_{i=0}^{k} (-1)^i \rho_*(\omega(\mathbf{X}_i))\boldsymbol{\alpha}(\mathbf{X}_0, \dots, \widehat{\mathbf{X}}_i, \dots, \mathbf{X}_k).$$

Proposition 2.6.19. Let M be a smooth manifold with or without boundary, and let ω, ρ, and $\boldsymbol{\alpha}$ be as in the preceding definition. The form $D_\omega \boldsymbol{\alpha}$ is again horizontal and ρ-equivariant. Consequently, it descends to an operator

$$d_\omega\colon \Omega^k(M; \mathbf{P} \times_\rho V) \longrightarrow \Omega^{k+1}(M; \mathbf{P} \times_\rho V).$$

In a local section $\mathbf{s}_\alpha$, if $\mathbf{a}_\alpha := \mathbf{s}_\alpha^* \boldsymbol{\alpha}$, then

$$(d_\omega \mathbf{a})_\alpha = d_{\mathbf{A}_\alpha} \mathbf{a}_\alpha := d\mathbf{a}_\alpha + \rho_*(\mathbf{A}_\alpha) \wedge \mathbf{a}_\alpha. \tag{2.42}$$

If $\mathbf{a}_\beta = \rho(g_{\alpha\beta}^{-1})\mathbf{a}_\alpha$, then

$$d_{\mathbf{A}_\beta} \mathbf{a}_\beta = \rho(g_{\alpha\beta}^{-1}) d_{\mathbf{A}_\alpha} \mathbf{a}_\alpha. \tag{2.43}$$

For the adjoint representation,

$$d_{\mathbf{A}} \mathbf{a} = d\mathbf{a} + [\mathbf{A} \wedge \mathbf{a}].$$

Proof. Equivariance of $D_\omega \boldsymbol{\alpha}$ follows by applying $R_{\mathbf{g}}^*$ to (2.41) and using

$$\rho_*(\mathrm{Ad}_{g^{-1}} \xi) = \rho(g^{-1}) \rho_*(\xi) \rho(g).$$

To prove horizontality, let $\boldsymbol{\xi}_{\mathbf{P}}$ be a fundamental vertical vector field. Infinitesimal equivariance of $\boldsymbol{\alpha}$ gives

$$\mathcal{L}_{\boldsymbol{\xi}_{\mathbf{P}}} \boldsymbol{\alpha} = -\rho_*(\xi) \boldsymbol{\alpha}.$$

Since $\iota_{\boldsymbol{\xi}_{\mathbf{P}}} \boldsymbol{\alpha} = 0$, Cartan's formula implies

$$\iota_{\boldsymbol{\xi}_{\mathbf{P}}} d\boldsymbol{\alpha} = -\rho_*(\xi) \boldsymbol{\alpha}.$$

On the other hand,

$$\iota_{\boldsymbol{\xi}_{\mathbf{P}}} \big(\rho_*(\omega) \wedge \boldsymbol{\alpha}\big) = \rho_*(\xi) \boldsymbol{\alpha},$$

and the terms cancel. Thus $D_\omega \boldsymbol{\alpha}$ descends by Proposition 2.6.9.
The local formula follows by pulling back by $\mathbf{s}_\alpha$. The covariance identity expresses that two local pullbacks represent the same global form. It can also be checked directly by substituting (2.39) and $\mathbf{a}_\beta = \rho(g_{\alpha\beta}^{-1})\mathbf{a}_\alpha$. For $\rho = \mathrm{Ad}$, we have $\rho_*(\xi)\eta = [\xi, \eta]$. □

Corollary 2.6.20. (Induced connection on an associated bundle) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and let $\rho\colon G \to GL(V)$ be a representation. For $\mathbf{E} = \mathbf{P} \times_\rho V$, the degree-zero operator

$$\nabla^{\mathbf{E},\omega} := d_\omega\colon \Gamma(\mathbf{E}) = \Omega^0(M;\mathbf{E}) \longrightarrow \Omega^1(M;\mathbf{E})$$

is a complex or real connection according to V. If u_α are the local components of a section $\mathbf{u}$, then

$$(\nabla^{\mathbf{E},\omega}\mathbf{u})_\alpha = du_\alpha + \rho_*(\mathbf{A}_\alpha)u_\alpha. \tag{2.44}$$

Equivalently, if $\widetilde{u}\colon \mathbf{P} \to V$ is the associated equivariant map and $\widetilde{\mathbf{X}}$ is the horizontal lift of a vector field $\mathbf{X}$, then

$$(\nabla^{\mathbf{E},\omega}_{\mathbf{X}}\mathbf{u})(x) = [p, d\widetilde{u}_p(\widetilde{\mathbf{X}}_p)]_\rho.$$

Proof. The local formula (2.42) with $k = 0$ is (2.44). It gives linearity and $\nabla(f\mathbf{u}) = df \otimes \mathbf{u} + f\nabla\mathbf{u}$. The covariance identity (2.43) shows that the expressions glue. In the equivariant model, $\omega(\widetilde{\mathbf{X}}) = 0$, so $D_\omega\widetilde{u}(\widetilde{\mathbf{X}}) = d\widetilde{u}(\widetilde{\mathbf{X}})$; the descent of forms proves the final formula. □

Theorem 2.6.21. (Square of the covariant derivative and the Bianchi identity) Let M be a smooth manifold with or without boundary, and let $\mathbf{A}$ be a local potential for a principal connection on $\mathbf{P} \to M$. For every representation ρ and every local form $\mathbf{a}$ with values in V,

$$d^2_{\mathbf{A}}\mathbf{a} = \rho_*(\mathbf{F}_\mathbf{A}) \wedge \mathbf{a}. \tag{2.45}$$

In particular, for forms with values in $\mathrm{ad}(\mathbf{P})$,

$$d^2_{\mathbf{A}}\mathbf{a} = [\mathbf{F}_\mathbf{A} \wedge \mathbf{a}]. \tag{2.46}$$

The curvature satisfies the Bianchi identity

$$d_\mathbf{A}\mathbf{F}_\mathbf{A} = d\mathbf{F}_\mathbf{A} + [\mathbf{A} \wedge \mathbf{F}_\mathbf{A}] = 0. \tag{2.47}$$

These identities are compatible with changes of local section and are therefore global.

Proof. Write $\mathbf{B} = \rho_*(\mathbf{A})$. Using the exterior Leibniz rule,

$$(d + \mathbf{B}\wedge)^2\mathbf{a} = d\mathbf{B} \wedge \mathbf{a} + \mathbf{B} \wedge \mathbf{B} \wedge \mathbf{a}.$$

Since ρ_* is a Lie algebra homomorphism,

$$\mathbf{B} \wedge \mathbf{B} = \frac{1}{2}\rho_*([\mathbf{A} \wedge \mathbf{A}]).$$

This proves (2.45), and the adjoint representation gives (2.46).

For the Bianchi identity, by $\mathbf{F_A} = d\mathbf{A} + \frac{1}{2}[\mathbf{A} \wedge \mathbf{A}]$ and Lemma 2.6.13,

$$d\mathbf{F_A} = \frac{1}{2}\big([d\mathbf{A} \wedge \mathbf{A}] - [\mathbf{A} \wedge d\mathbf{A}]\big) = -[\mathbf{A} \wedge d\mathbf{A}].$$

Moreover,

$$[\mathbf{A} \wedge \mathbf{F_A}] = [\mathbf{A} \wedge d\mathbf{A}] + \frac{1}{2}[\mathbf{A} \wedge [\mathbf{A} \wedge \mathbf{A}]] = [\mathbf{A} \wedge d\mathbf{A}].$$

The sum is zero. The covariance identities (2.43) and (2.40) show that all these identities glue. □

Theorem 2.6.22. (Existence of principal connections) Let M be a smooth manifold with or without boundary. Every smooth principal bundle $\mathbf{P} \to M$ admits a principal connection.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be a trivializing cover. On $\pi^{-1}(U_\alpha) \cong U_\alpha \times G$, define

$$\omega_\alpha(X, \zeta) := \theta^L_g(\zeta), \qquad (X, \zeta) \in T_x U_\alpha \oplus T_g G,$$

where $\theta^L_g = (dL_{g^{-1}})_g$ is the left Maurer–Cartan form. A direct verification gives

$$\omega_\alpha(\boldsymbol{\xi}_\mathbf{P}) = \xi, \qquad R_h^* \omega_\alpha = \mathrm{Ad}_{h^{-1}}\, \omega_\alpha;$$

therefore ω_α is a local principal connection.
Let $\{\chi_\alpha\}_{\alpha \in I}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$, whose existence is given by Theorem 1.3.5. Each form $(\pi^* \chi_\alpha)\omega_\alpha$, extended by zero outside $\pi^{-1}(U_\alpha)$, is smooth, and the family is locally finite. Define

$$\omega := \sum_{\alpha \in I} (\pi^* \chi_\alpha)\omega_\alpha.$$

Since $\sum_{\alpha \in I} \chi_\alpha = 1$,

$$\omega(\boldsymbol{\xi}_\mathbf{P}) = \sum_{\alpha \in I} (\pi^* \chi_\alpha)\xi = \xi.$$

The functions $\pi^* \chi_\alpha$ are invariant under the right action and each ω_α is equivariant, so $R_\mathbf{g}^* \omega = \mathrm{Ad}_{g^{-1}}\, \omega$. Therefore ω is a global principal connection. □

Theorem 2.6.23. (Affine structure of the space of connections) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a principal bundle. Let $\mathrm{Conn}(\mathbf{P})$ be the set of principal connections on $\mathbf{P}$. If $\omega_0, \omega_1 \in \mathrm{Conn}(\mathbf{P})$, then

$$\mathbf{a} := \omega_1 - \omega_0$$

is horizontal and Ad-equivariant, and therefore corresponds to a unique element of

$$\Omega^1(M; \mathrm{ad}(\mathbf{P})).$$

Conversely, if ω_0 is a connection and $\mathbf{a} \in \Omega^1(M; \mathrm{ad}(\mathbf{P}))$, the corresponding horizontal equivariant form $\widetilde{\mathbf{a}}$ satisfies the condition that

$$\omega_0 + \widetilde{\mathbf{a}}$$

is a principal connection. Consequently, $\mathrm{Conn}(\mathbf{P})$ is an affine space modeled on $\Omega^1(M;\mathrm{ad}(\mathbf{P}))$.

Proof. For every $\xi \in \mathfrak{g}$,

$$\mathbf{a}(\xi_{\mathbf{P}}) = \omega_1(\xi_{\mathbf{P}}) - \omega_0(\xi_{\mathbf{P}}) = \xi - \xi = 0,$$

so $\mathbf{a}$ is horizontal. Equivariance follows by subtracting the equivariance identities for ω_1 and ω_0. Proposition 2.6.9 then identifies $\mathbf{a}$ with a form in $\Omega^1(M;\mathrm{ad}(\mathbf{P}))$.
Conversely, a horizontal equivariant form vanishes on fundamental vector fields and satisfies the same transformation law as a connection form. Therefore,

$$(\omega_0 + \widetilde{\mathbf{a}})(\xi_{\mathbf{P}}) = \xi, \qquad R_{\mathbf{g}}^*(\omega_0 + \widetilde{\mathbf{a}}) = \mathrm{Ad}_{g^{-1}}(\omega_0 + \widetilde{\mathbf{a}}).$$

□

Corollary 2.6.24. Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a principal bundle with compact group G. Any Ad-invariant inner product given by Corollary 1.9.23 induces a well-defined bundle metric on $\mathrm{ad}(\mathbf{P})$ by

$$\langle [p,\xi],[p,\eta]\rangle_{\mathrm{ad}(\mathbf{P})} := \langle \xi,\eta\rangle_{\mathfrak{g}}.$$

Proof. Replacing p by $p \cdot g$ replaces the representatives by $\mathrm{Ad}_{g^{-1}}\xi$ and $\mathrm{Ad}_{g^{-1}}\eta$. Invariance of the inner product shows that the value does not change. □

2.6.1 Parallel transport and holonomy

Theorem 2.6.25. (Horizontal lift) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve, and let $p_0 \in \mathbf{P}_{\gamma(a)}$. There exists a unique piecewise smooth curve $\widetilde{\gamma}_{p_0}\colon [a,b] \to \mathbf{P}$ such that

$$\widetilde{\gamma}_{p_0}(a) = p_0, \qquad \pi \circ \widetilde{\gamma}_{p_0} = \gamma, \qquad \dot{\widetilde{\gamma}}_{p_0}(t) \in \mathbf{H}_{\widetilde{\gamma}_{p_0}(t)}\mathbf{P}$$

on each interval of smoothness. Moreover,

$$\widetilde{\gamma}_{p_0 g}(t) = \widetilde{\gamma}_{p_0}(t)g.$$

Proof. In a trivialization, write $\widetilde{\gamma}(t) = \mathbf{s}(\gamma(t))g(t)$ and $\mathbf{A} = \mathbf{s}^*\omega$. The horizontality condition is the differential equation on G

$$\theta^L_{g(t)}\dot{g}(t) = -\mathrm{Ad}_{g(t)^{-1}}\mathbf{A}_{\gamma(t)}(\dot{\gamma}(t)),$$

with prescribed initial data. Existence and uniqueness for differential equations gives a solution while the curve remains in the trivialization. A finite subdivision of $[a,b]$ into trivializing domains glues the unique solutions. The invariance $(dR_{\mathbf{g}})\mathbf{H} = \mathbf{H}$ shows that $\widetilde{\gamma}_{p_0}g$ is the horizontal lift with initial data $p_0 g$, and uniqueness proves the final formula. □

Definition 2.6.26. (Principal parallel transport) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve. The **principal parallel transport** of γ is

$$\mathcal{P}^{\omega}_{\gamma} : \mathbf{P}_{\gamma(a)} \longrightarrow \mathbf{P}_{\gamma(b)}, \qquad \mathcal{P}^{\omega}_{\gamma}(p) := \widetilde{\gamma}_p(b).$$

Proposition 2.6.27. (Laws of principal transport) Let M be a smooth manifold with or without boundary, and let ω be a principal connection on $\mathbf{P} \to M$. Principal transport is G-equivariant and satisfies

$$\mathcal{P}^{\omega}_{\gamma^{-1}} = (\mathcal{P}^{\omega}_{\gamma})^{-1}, \qquad \mathcal{P}^{\omega}_{\gamma_2 * \gamma_1} = \mathcal{P}^{\omega}_{\gamma_2} \circ \mathcal{P}^{\omega}_{\gamma_1},$$

where $\gamma_2 * \gamma_1$ traverses γ_1 first and then γ_2. It is also invariant under orientation-preserving reparametrizations.

Proof. Equivariance is the final assertion of the preceding theorem. Reversing the parameter of a horizontal lift produces a horizontal lift of the reversed curve. For a concatenation, glue the lift of γ_1 to the lift of γ_2 whose initial point is the endpoint of the first; uniqueness gives the formula. The same argument proves invariance under reparametrization. □

Definition 2.6.28. (Principal holonomy group) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and let $x = \pi(p)$. The **holonomy group of ω based at p** is

$$\mathrm{Hol}_p(\omega) := \{g \in G \mid \mathcal{P}^{\omega}_{\gamma}(p) = pg \text{ for some piecewise smooth loop } \gamma \text{ based at } x\}.$$

Proposition 2.6.29. (Structure and change of base point for holonomy) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and let $p \in \mathbf{P}$. The set $\mathrm{Hol}_p(\omega)$ is a subgroup of G. If $h \in G$, then

$$\mathrm{Hol}_{ph}(\omega) = h^{-1}\,\mathrm{Hol}_p(\omega)\,h.$$

If λ is a curve from x to y and $q = \mathcal{P}^{\omega}_{\lambda}(p)$, then $\mathrm{Hol}_q(\omega) = \mathrm{Hol}_p(\omega)$. For another choice $qh \in \mathbf{P}_y$, the groups are conjugate by h.

Proof. The constant loop represents e. If the loops α, β represent g, h, respectively, traversing β and then α takes us, by equivariance, from p to pgh; reversing a loop representing g produces g^{-1}. This proves the subgroup property. Starting at ph, the lift of a loop that starts at p and ends at pg instead ends at $pgh = (ph)(h^{-1}gh)$, giving the conjugation formula. Finally, conjugating a loop based at y by λ and λ^{-1} preserves the element of G when q is the transport of p. The final assertion combines this fact with the formula within a single fiber. □

Proposition 2.6.30. (Holonomy of an associated bundle) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and let $\mathbf{E} = \mathbf{P} \times_{\rho} V$. The transport induced by ω is

$$\mathcal{P}^{\mathbf{E},\omega}_{\gamma}[p, v] := [\mathcal{P}^{\omega}_{\gamma}(p), v].$$

It is well defined, linear, and satisfies the inversion and concatenation laws. Therefore,

$$\mathrm{Hol}_x(\mathbf{E},\omega) := \{\mathcal{P}_\gamma^{\mathbf{E},\omega} \mid \gamma \text{ is a loop based at } x\} \subseteq GL(\mathbf{E}_x)$$

is a subgroup. If $\mathcal{P}_\gamma^\omega(p) = pg$, the identification $\mathbf{E}_x \cong V$ determined by p represents $\mathcal{P}_\gamma^{\mathbf{E},\omega}$ by $\rho(g)$. When the base point changes, the vector bundle holonomy groups are conjugate through parallel transport along a curve joining the points.

Proof. If $[pg, v] = [p, \rho(g)v]$, principal equivariance gives

$$[\mathcal{P}_\gamma^\omega(pg), v] = [\mathcal{P}_\gamma^\omega(p)g, v] = [\mathcal{P}_\gamma^\omega(p), \rho(g)v],$$

so the definition is independent of the representative. The remaining assertions follow from Proposition 2.6.27. □

Definition 2.6.31. (Parallel section of the adjoint group bundle) Let M be a smooth manifold with or without boundary, and let ω be a principal connection on $\mathbf{P} \to M$. A section $\mathbf{u} \in \Gamma(\mathrm{Ad}(\mathbf{P}))$ is **ω-parallel** if its equivariant map $\widehat{u}\colon \mathbf{P} \to G$ is constant along every horizontal curve. Denote the group of these sections by $\Gamma^\omega_{\mathrm{par}}(\mathrm{Ad}(\mathbf{P}))$.

Theorem 2.6.32. (Parallel sections and the centralizer of holonomy) Let M be a connected smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and fix $p \in \mathbf{P}$. Evaluation

$$\mathrm{ev}_p\colon \Gamma^\omega_{\mathrm{par}}(\mathrm{Ad}(\mathbf{P})) \longrightarrow G, \qquad \mathbf{u} \longmapsto \widehat{u}(p),$$

is a group isomorphism onto $Z_G(\mathrm{Hol}_p(\omega))$.

Proof. Let $\mathbf{u}$ be parallel, let $\widehat{u}$ be its equivariant representative, and set $c = \widehat{u}(p)$. If a loop has a horizontal lift from p to pg, constancy along horizontal curves and equivariance give

$$c = \widehat{u}(p) = \widehat{u}(pg) = g^{-1}\widehat{u}(p)g = g^{-1}cg.$$

Therefore c centralizes $\mathrm{Hol}_p(\omega)$.
Conversely, let $c \in Z_G(\mathrm{Hol}_p(\omega))$. For $q \in \mathbf{P}_y$, choose a curve γ from $\pi(p)$ to y and write uniquely

$$q = \mathcal{P}_\gamma^\omega(p)h.$$

Define $\widehat{u}(q) := h^{-1}ch$. If another curve produces $q = \mathcal{P}_{\gamma'}^\omega(p)h'$, then $\mathcal{P}_{\gamma'}^\omega(p) = \mathcal{P}_\gamma^\omega(p)a$ for some $a \in \mathrm{Hol}_p(\omega)$ and $h' = a^{-1}h$. Since $ac = ca$, the two formulas for $\widehat{u}(q)$ agree. The construction satisfies $\widehat{u}(qk) = k^{-1}\widehat{u}(q)k$ and is constant along horizontal lifts. In a trivializing neighborhood, choose paths depending smoothly on the point; smooth dependence of transport shows that $\widehat{u}$ is smooth.
The construction is inverse to evaluation. It also shows that a parallel section equal to e at p is the identity section, so evaluation is injective. Since multiplication is pointwise, it is a group isomorphism. □

Definition 2.6.33. (Gauge group and right action on connections) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a right principal bundle. A *gauge transformation* $\mathbf{u}$ is a section of $\mathrm{Ad}(\mathbf{P})$ or, equivalently, is represented by a smooth map $\widehat{u}\colon \mathbf{P} \to G$ satisfying

$$\widehat{u}(p \cdot g) = g^{-1}\widehat{u}(p)g.$$

The corresponding principal automorphism is

$$\Phi_{\mathbf{u}}\colon \mathbf{P} \longrightarrow \mathbf{P}, \qquad \Phi_{\mathbf{u}}(p) := p \cdot \widehat{u}(p).$$

Define the right gauge action on connections by

$$\omega \triangleleft \mathbf{u} := \Phi_{\mathbf{u}}^{*}\omega.$$

Proposition 2.6.34. (Three models of the gauge group) Let M be a smooth manifold with or without boundary, and let $\mathbf{P} \to M$ be a right principal bundle. There is a group correspondence between:

(a) principal automorphisms $\Phi\colon \mathbf{P} \to \mathbf{P}$ covering id_M;

(b) smooth equivariant maps $\widehat{u}\colon \mathbf{P} \to G$ with $\widehat{u}(pg) = g^{-1}\widehat{u}(p)g$;

(c) smooth sections $\mathbf{u}$ of $\mathrm{Ad}(\mathbf{P})$.

The correspondence between the first two models is $\Phi_{\mathbf{u}}(p) = p \cdot \widehat{u}(p)$. For a connection ω, the stabilizer is

$$\mathrm{Stab}(\omega) := \{\mathbf{u} \in \mathcal{G}(\mathbf{P}) \mid \omega \triangleleft \mathbf{u} = \omega\}.$$

If $\xi \in \Gamma(\mathrm{ad}(\mathbf{P}))$ and $\mathbf{u}_t$ is the gauge transformation with equivariant representative $\widehat{u_t}(p) = \exp_G(t\widehat{\xi}(p))$, the differential at the identity of the right gauge action is

$$\left.\frac{d}{dt}\right|_0 (\omega \triangleleft \mathbf{u}_t) = d_\omega \xi; \tag{2.48}$$

therefore the space of infinitesimal directions annihilating the orbit map at ω is $\ker d_\omega$. The identification of this kernel with the Lie algebra of the Sobolev stabilizer will be proved later, once that Lie group structure has been constructed.

Proof. If Φ covers the identity, freeness and transitivity of the action on each fiber give a unique $\widehat{u}(p)$ with $\Phi(p) = p \cdot \widehat{u}(p)$. From $\Phi(pg) = \Phi(p)g$, we obtain $(p \cdot g) \cdot \widehat{u}(p \cdot g) = (p \cdot \widehat{u}(p)) \cdot g$ and, by freeness, $\widehat{u}(pg) = g^{-1}\widehat{u}(p)g$. In trivializations, the function $\widehat{u}$ is smooth. The converse is checked by substituting the equivariance law, and Proposition 2.6.6, applied to the conjugation action, gives the third model.
In a local section, let $\mathbf{A}$ be the connection potential and ξ_U the local representative of ξ. Differentiating $u_t^{-1}\mathbf{A}u_t + u_t^{-1}du_t$ for $u_t = \exp_G(t\xi_U)$ gives $d\xi_U + [\mathbf{A}, \xi_U] = d_{\mathbf{A}}\xi_U$. Thus the kernel of the differential of the orbit map at the identity is exactly $\ker d_\omega$, which is the infinitesimal assertion in the statement. □

Proposition 2.6.35. (Gauge formulas) Let M be a smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and let $\mathbf{u}, \mathbf{v} \in \mathcal{G}(\mathbf{P})$. With pointwise multiplication of gauge transforma-

tions,

$$\Phi_{\mathbf{u}} \circ \Phi_{\mathbf{v}} = \Phi_{\mathbf{uv}}, \qquad (\omega \triangleleft \mathbf{u}) \triangleleft \mathbf{v} = \omega \triangleleft (\mathbf{uv}).$$

If $\mathbf{s}\colon U \to \mathbf{P}$ is a local section, $\mathbf{A} = \mathbf{s}^*\omega$, and $u_U = \widehat{u} \circ \mathbf{s}$, then

$$\mathbf{A} \triangleleft u_U := \mathbf{s}^*(\omega \triangleleft \mathbf{u}) = \mathrm{Ad}_{u_U^{-1}} \mathbf{A} + u_U^{-1} du_U, \tag{2.49}$$

$$\mathbf{F}_{\mathbf{A} \triangleleft u_U} = \mathrm{Ad}_{u_U^{-1}} \mathbf{F}_{\mathbf{A}}. \tag{2.50}$$

For a matrix group,

$$\mathbf{A} \triangleleft u = u^{-1}\mathbf{A}u + u^{-1}du, \qquad \mathbf{F}_{\mathbf{A} \triangleleft u} = u^{-1}\mathbf{F}_{\mathbf{A}}u.$$

Proof. Equivariance of the representatives $\widehat{u}$ and $\widehat{v}$ gives

$$\Phi_{\mathbf{u}}(\Phi_{\mathbf{v}}(p)) = p\,\widehat{v}(p)\,\widehat{u}(p\widehat{v}(p)) = p\,\widehat{v}(p)\,\widehat{v}(p)^{-1}\widehat{u}(p)\widehat{v}(p) = p\,\widehat{u}(p)\widehat{v}(p) = \Phi_{\mathbf{uv}}(p).$$

By contravariance of pullback,

$$\Phi_{\mathbf{v}}^*\Phi_{\mathbf{u}}^*\omega = (\Phi_{\mathbf{u}} \circ \Phi_{\mathbf{v}})^*\omega = \Phi_{\mathbf{uv}}^*\omega,$$

which proves the right action law.
Moreover,

$$\Phi_{\mathbf{u}} \circ \mathbf{s} = \mathbf{s}\,u_U.$$

The local transformation formula (2.39), applied to the sections $\mathbf{s}$ and $\mathbf{s}\,u_U$, gives (2.49). Horizontality and equivariance of curvature, or equivalently (2.40), give (2.50). □

Theorem 2.6.36. (Smooth stabilizer and holonomy) Let M be a connected smooth manifold with or without boundary, let ω be a principal connection on $\mathbf{P} \to M$, and fix $p \in \mathbf{P}$. For a smooth gauge transformation $\mathbf{u}$ with equivariant representative $\widehat{u}$, the following are equivalent:

(a) $\omega \triangleleft \mathbf{u} = \omega$;

(b) the equivariant map $\widehat{u}\colon \mathbf{P} \to G$ is constant along every horizontal curve;

(c) the section $\mathbf{u}$ of $\mathrm{Ad}(\mathbf{P})$ is ω-parallel.

Consequently, evaluation at p induces an isomorphism

$$\mathrm{Stab}_{C^\infty}(\omega) \xrightarrow{\cong} Z_G(\mathrm{Hol}_p(\omega)). \tag{2.51}$$

Moreover, evaluation identifies the infinitesimal directions:

$$\ker\big(d_\omega\colon \Omega^0(M;\mathrm{ad}(\mathbf{P})) \to \Omega^1(M;\mathrm{ad}(\mathbf{P}))\big) \cong \mathrm{Lie}\, Z_G(\mathrm{Hol}_p(\omega)). \tag{2.52}$$

Proof. On a horizontal vector $X \in \mathbf{H}_q\mathbf{P}$, the global transformation formula for a connection reduces to

$$(\Phi_{\mathbf{u}}^*\omega)_q(X) = \theta^L_{\widehat{u}(q)}(d\widehat{u}_q X),$$

because $\omega_q(X) = 0$. Therefore $\Phi_{\mathbf{u}}^*\omega = \omega$ implies $d\widehat{u}(X) = 0$ for every horizontal vector. Conversely, this last

condition makes both connection forms vanish on $\mathbf{H}$; on vertical vectors, both reproduce the infinitesimal generator. Since $T\mathbf{P} = \mathbf{H} \oplus \mathbf{V}$, the forms agree. This proves the equivalence of (a)–(c).
The isomorphism (2.51) is now Theorem 2.6.32. For the infinitesimal statement, the formula for the induced connection on $\mathrm{ad}(\mathbf{P})$ shows that $d_\omega \xi = 0$ if and only if the equivariant map $\widehat{\xi}\colon \mathbf{P} \to \mathfrak{g}$ is constant along horizontal curves. The same evaluation argument identifies it with the vectors fixed by $\mathrm{Ad}(\mathrm{Hol}_p(\omega))$, which form the Lie algebra of the centralizer by (1.24). □

2.7 Covariant differentiation along curves and parallel transport

In many geometric constructions, differentiating sections defined on open sets of the manifold is not enough. We often need to study objects defined only along a curve. This occurs, for example, when comparing elements belonging to different fibers, studying the variation of a section along a path, or, in the particular case of the tangent bundle, defining the covariant acceleration of a curve.

The difficulty is that, if $\pi\colon \mathbf{E} \longrightarrow M$ is a smooth vector bundle and $\gamma\colon I \longrightarrow M$ is a smooth curve, a section of $\mathbf{E}$ along γ assigns to each $t \in I$ a vector in the fiber $\mathbf{E}_{\gamma(t)}$. Since these fibers are, in principle, different vector spaces, differentiating such a section by directly subtracting its values makes no sense. A connection resolves this problem: it gives an intrinsic way of differentiating sections in tangent directions and thereby induces a derivative for sections defined only along the curve.

Before formalizing this construction, we recall the pullback of a vector bundle.

Definition 2.7.1. (Pullbacks of vector bundles) Let M and N be smooth manifolds with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank k, and let $f\colon N \longrightarrow M$ be a smooth map. Define the *pullback* of $\mathbf{E}$ by f to be the set

$$f^*\mathbf{E} := \{(y, e) \in N \times \mathbf{E} \mid \pi(e) = f(y)\}.$$

Equip $f^*\mathbf{E}$ with the projection $\pi_{f^*\mathbf{E}}\colon f^*\mathbf{E} \longrightarrow N$ given by $\pi_{f^*\mathbf{E}}(y, e) := y$. For each $y \in N$, the fiber of $f^*\mathbf{E}$ over y is given by $(f^*\mathbf{E})_y = \{(y, e) \in f^*\mathbf{E} \mid e \in \mathbf{E}_{f(y)}\}$. Using the natural identification $(y, e) \mapsto e$, we identify $(f^*\mathbf{E})_y$ with $\mathbf{E}_{f(y)}$.

Proposition 2.7.2. Let M and N be smooth manifolds with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank k, and let $f\colon N \longrightarrow M$ be a smooth map. Then $f^*\mathbf{E} \longrightarrow N$ admits a natural structure of a smooth vector bundle of rank k.

Proof. Recall that $f^*\mathbf{E} = \{(y, e) \in N \times \mathbf{E} \mid \pi(e) = f(y)\}$, with projection $\pi_{f^*\mathbf{E}}(y, e) = y$. For each $y \in N$, the fiber of $f^*\mathbf{E}$ over y is given by $(f^*\mathbf{E})_y = \{(y, e) \mid e \in \mathbf{E}_{f(y)}\}$, which we identify with $\mathbf{E}_{f(y)}$ through $(y, e) \mapsto e$.
Let $\{U_\alpha\}_{\alpha \in A}$ be a trivializing cover of $\mathbf{E}$, with trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{K}^k$. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of N because f is continuous.
For each α, define

$$\widetilde{\Phi}_\alpha : \pi^{-1}_{f^*\mathbf{E}}(f^{-1}(U_\alpha)) \longrightarrow f^{-1}(U_\alpha) \times \mathbb{K}^k$$

by

$$\widetilde{\Phi}_\alpha(y, e) := \big(y, \mathrm{pr}_2(\Phi_\alpha(e))\big),$$

where $\mathrm{pr}_2\colon U_\alpha \times \mathbb{K}^k \longrightarrow \mathbb{K}^k$ is the second projection. This expression is well defined because, if $(y, e) \in f^*\mathbf{E}$ and $y \in f^{-1}(U_\alpha)$, then $f(y) \in U_\alpha$ and, since $\pi(e) = f(y)$, we have $e \in \pi^{-1}(U_\alpha)$.

Let us show that $\widetilde{\Phi}_\alpha$ is bijective. Let $(y,v) \in f^{-1}(U_\alpha) \times \mathbb{K}^k$. Since $f(y) \in U_\alpha$, there is a unique $e \in \mathbf{E}_{f(y)}$ such that $\Phi_\alpha(e) = (f(y), v)$. Therefore $(y,e) \in f^*\mathbf{E}$ and $\widetilde{\Phi}_\alpha(y,e) = (y,v)$. This proves surjectivity.
If $\widetilde{\Phi}_\alpha(y,e) = \widetilde{\Phi}_\alpha(y',e')$, then $y = y'$ and $\mathrm{pr}_2(\Phi_\alpha(e)) = \mathrm{pr}_2(\Phi_\alpha(e'))$. Moreover, since (y,e) and (y,e') belong to $f^*\mathbf{E}$, we have $\pi(e) = f(y) = \pi(e')$. Therefore the first components of $\Phi_\alpha(e)$ and $\Phi_\alpha(e')$ also agree. Thus $\Phi_\alpha(e) = \Phi_\alpha(e')$. Since Φ_α is injective, we obtain $e = e'$. Hence $\widetilde{\Phi}_\alpha$ is injective. In fact, its inverse is given by

$$\widetilde{\Phi}_\alpha^{-1}(y,v) = \big(y, \Phi_\alpha^{-1}(f(y),v)\big).$$

Moreover, the restriction of $\widetilde{\Phi}_\alpha$ to each fiber $(f^*\mathbf{E})_y$ is a linear isomorphism onto $\{y\} \times \mathbb{K}^k$, since it agrees with the trivialization of $\mathbf{E}$ on the fiber $\mathbf{E}_{f(y)}$ under the natural identification $(f^*\mathbf{E})_y \cong \mathbf{E}_{f(y)}$.
Now consider α, β such that $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \neq \varnothing$. On $U_\alpha \cap U_\beta$, the trivializations of $\mathbf{E}$ satisfy

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p,v) = (p, \tau_{\alpha\beta}(p)v),$$

where $\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow GL(k,\mathbb{K})$ is smooth. Then, for $y \in f^{-1}(U_\alpha \cap U_\beta)$, we have

$$\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1}(y,v) = \big(y, \tau_{\alpha\beta}(f(y))v\big).$$

Thus the transition functions of the system $\{\widetilde{\Phi}_\alpha\}$ are given by $\widetilde{\tau}_{\alpha\beta}(y) = \tau_{\alpha\beta}(f(y))$. Since $\tau_{\alpha\beta}$ and f are smooth, $\widetilde{\tau}_{\alpha\beta}$ is also smooth.
By Lemma 1.7.8, there exist a unique topology and smooth structure on $f^*\mathbf{E}$ making it a smooth vector bundle of rank k over N, with projection $\pi_{f^*\mathbf{E}}$ and local trivializations $\widetilde{\Phi}_\alpha$. □

Definition 2.7.3. (Sections along a curve) Let M be a smooth manifold with or without boundary, let $\pi \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle, and let $\gamma \colon I \longrightarrow M$ be a smooth curve. A section of $\mathbf{E}$ along γ is a section of the pullback bundle $\gamma^*\mathbf{E} \longrightarrow I$.
Using the natural identification $(\gamma^*\mathbf{E})_t \cong \mathbf{E}_{\gamma(t)}$, we regard a section of $\mathbf{E}$ along γ as a map $\mathbf{u} \colon I \longrightarrow \mathbf{E}$ such that $\mathbf{u}(t) \in \mathbf{E}_{\gamma(t)}$ for every $t \in I$. Under this identification, we write $\Gamma^k(\gamma^*\mathbf{E})$ for the space of sections of class C^k of $\mathbf{E}$ along γ, and we write $\Gamma(\gamma^*\mathbf{E}) = \Gamma^\infty(\gamma^*\mathbf{E})$.
We say that $\mathbf{u} \in \Gamma^k(\gamma^*\mathbf{E})$ is extendible if there exists a section $\widetilde{\mathbf{u}} \in \Gamma^k(\mathbf{E})$, defined on a neighborhood of $\gamma(I)$, such that $\mathbf{u}(t) = \widetilde{\mathbf{u}}(\gamma(t))$ for every $t \in I$.

Remark 2.7.4. When $\mathbf{E} = TM$, the elements of $\Gamma^k(\gamma^*TM)$ are precisely vector fields of class C^k along γ. More generally, if $\mathbf{E} = T^{(r,s)}(TM)$ carries the connection induced by a connection on TM, we obtain the notion of a tensor field of type (r,s) along γ.

The following result introduces covariant differentiation along a curve. The essential condition is that, if a section along γ is obtained by restricting a section $\widetilde{\mathbf{u}}$ of $\mathbf{E}$, its derivative along the curve must agree with $\nabla^{\mathbf{E}}_{\dot\gamma}\widetilde{\mathbf{u}}$. The result states that this condition uniquely determines a derivative for every section of class C^1 of $\gamma^*\mathbf{E}$.

Theorem 2.7.5. (Covariant derivative along a curve) Let M be a smooth manifold with or without boundary, let $\pi \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, and let $\gamma \colon I \longrightarrow M$ be a smooth curve. There exists a unique operator

$$\frac{D}{dt} \colon \Gamma^1(\gamma^*\mathbf{E}) \longrightarrow \Gamma^0(\gamma^*\mathbf{E})$$

satisfying the following properties:

(a) $\dfrac{D}{dt}(a\mathbf{u}+b\mathbf{v}) = a\dfrac{D\mathbf{u}}{dt} + b\dfrac{D\mathbf{v}}{dt}$ for every $a,b\in\mathbb{K}$ and every $\mathbf{u},\mathbf{v}\in\Gamma^1(\gamma^*\mathbf{E})$.

(b) $\dfrac{D}{dt}(f\mathbf{u}) = \dfrac{df}{dt}\mathbf{u} + f\dfrac{D\mathbf{u}}{dt}$ for every $f\in C^1(I)$ and every $\mathbf{u}\in\Gamma^1(\gamma^*\mathbf{E})$.

(c) If $\mathbf{u}$ is extendible and $\widetilde{\mathbf{u}}$ is an extension of class C^1 of $\mathbf{u}$ to a neighborhood of $\gamma(I)$, then $\dfrac{D\mathbf{u}}{dt}(t) = \nabla^{\mathbf{E}}_{\dot\gamma(t)}\widetilde{\mathbf{u}}$ for every $t\in I$.

If $\mathbf{u}\in\Gamma^k(\gamma^*\mathbf{E})$ with $k\geq 1$, then $\dfrac{D\mathbf{u}}{dt}\in\Gamma^{k-1}(\gamma^*\mathbf{E})$. In particular, this operator restricts to an operator $\dfrac{D}{dt}\colon\Gamma(\gamma^*\mathbf{E})\longrightarrow\Gamma(\gamma^*\mathbf{E})$.

Proof. We first construct the operator in a local trivialization and then prove independence of the choices. Let $t_0\in I$. Choose a smooth chart (U,φ) on M, with local coordinates $(x^1,\dots,x^n)$, such that $\gamma(t_0)\in U$, and choose a smooth local frame $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ of $\mathbf{E}$ over U. Since $\gamma^{-1}(U)$ is open in I and contains t_0, there is an interval $J\subseteq I$, open in I, such that $t_0\in J$ and $\gamma(J)\subseteq U$.
On J, every section $\mathbf{u}\in\Gamma^1(\gamma^*\mathbf{E})$ can be written uniquely as $\mathbf{u}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t))$, where $u^1,\dots,u^r$ are functions of class C^1 on J. Define the local connection coefficients by $\nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_\beta = \Gamma^\alpha_{i\beta}\mathbf{e}_\alpha$. Then define locally

$$\frac{D\mathbf{u}}{dt}(t) = \left(\frac{du^\alpha}{dt}(t) + \frac{d\gamma^i}{dt}(t)\Gamma^\alpha_{i\beta}(\gamma(t))u^\beta(t)\right)\mathbf{e}_\alpha(\gamma(t)). \tag{2.53}$$

We show that this definition is independent of the local frame. Let $(\widetilde{\mathbf{e}}_1,\dots,\widetilde{\mathbf{e}}_r)$ be another local frame of $\mathbf{E}$ over the same open set, and write $\widetilde{\mathbf{e}}_a = B^\beta_a\mathbf{e}_\beta$, where $B=(B^\beta_a)$ is a matrix of smooth functions with values in $GL(r,\mathbb{K})$. If

$$\mathbf{u}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t)) = \widetilde{u}^a(t)\widetilde{\mathbf{e}}_a(\gamma(t)),$$

then $u^\alpha(t) = B^\alpha_a(\gamma(t))\widetilde{u}^a(t)$. On the other hand, if $\nabla^{\mathbf{E}}_{\partial_i}\widetilde{\mathbf{e}}_b = \widetilde{\Gamma}^a_{ib}\widetilde{\mathbf{e}}_a$, then, using $\widetilde{\mathbf{e}}_b = B^\beta_b\mathbf{e}_\beta$, we obtain

$$\nabla^{\mathbf{E}}_{\partial_i}\widetilde{\mathbf{e}}_b = (\partial_i B^\alpha_b)\mathbf{e}_\alpha + B^\beta_b\nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_\beta = (\partial_i B^\alpha_b + B^\beta_b\Gamma^\alpha_{i\beta})\mathbf{e}_\alpha.$$

Since also

$$\nabla^{\mathbf{E}}_{\partial_i}\widetilde{\mathbf{e}}_b = \widetilde{\Gamma}^a_{ib}\widetilde{\mathbf{e}}_a = \widetilde{\Gamma}^a_{ib}B^\alpha_a\mathbf{e}_\alpha,$$

we have

$$\partial_i B^\alpha_b + B^\beta_b\Gamma^\alpha_{i\beta} = B^\alpha_a\widetilde{\Gamma}^a_{ib}.$$

Differentiating $u^\alpha = B^\alpha_a(\gamma)\widetilde{u}^a$ with respect to t gives

$$\frac{du^\alpha}{dt} = \frac{d\gamma^i}{dt}(\partial_i B^\alpha_a)(\gamma)\widetilde{u}^a + B^\alpha_a(\gamma)\frac{d\widetilde{u}^a}{dt}.$$

Therefore,

$$\frac{du^\alpha}{dt} + \frac{d\gamma^i}{dt}\Gamma^\alpha_{i\beta}(\gamma)u^\beta = B^\alpha_a(\gamma)\frac{d\widetilde{u}^a}{dt} + \frac{d\gamma^i}{dt}\big(\partial_i B^\alpha_b + B^\beta_b\Gamma^\alpha_{i\beta}\big)(\gamma)\widetilde{u}^b.$$

Using the preceding relation between the connection coefficients yields

$$\frac{du^\alpha}{dt} + \frac{d\gamma^i}{dt}\Gamma^\alpha_{i\beta}(\gamma)u^\beta = B^\alpha_a(\gamma)\left(\frac{d\widetilde{u}^a}{dt} + \frac{d\gamma^i}{dt}\widetilde{\Gamma}^a_{ib}(\gamma)\widetilde{u}^b\right).$$

Multiplying by $\mathbf{e}_\alpha(\gamma(t))$ and using $B^\alpha_a\mathbf{e}_\alpha = \widetilde{\mathbf{e}}_a$ gives exactly the expression for (2.53) in the frame $\widetilde{\mathbf{e}}_a$. Thus the definition is independent of the local frame.

Since the resulting expression is intrinsic with respect to the frame, it also agrees on overlaps of local trivializations. It therefore defines a global section $\dfrac{D\mathbf{u}}{dt} \in \Gamma^0(\gamma^*\mathbf{E})$. If $\mathbf{u} \in \Gamma^k(\gamma^*\mathbf{E})$, the local formula shows that $\dfrac{D\mathbf{u}}{dt} \in \Gamma^{k-1}(\gamma^*\mathbf{E})$.

Linearity over $\mathbb{K}$ is checked directly in the local formula. For the Leibniz rule, let $f \in C^1(I)$. The components of $f\mathbf{u}$ are fu^α, and then

$$\frac{D(f\mathbf{u})}{dt} = \left(\frac{d(fu^\alpha)}{dt} + \frac{d\gamma^i}{dt}\Gamma^\alpha_{i\beta}(\gamma)fu^\beta\right)\mathbf{e}_\alpha(\gamma(t)).$$

Since $\dfrac{d(fu^\alpha)}{dt} = \dfrac{df}{dt}u^\alpha + f\dfrac{du^\alpha}{dt}$, we obtain $\dfrac{D(f\mathbf{u})}{dt} = \dfrac{df}{dt}\mathbf{u} + f\dfrac{D\mathbf{u}}{dt}$.

We now prove compatibility with extensions. Suppose that $\mathbf{u}(t) = \widetilde{\mathbf{u}}(\gamma(t))$, where $\widetilde{\mathbf{u}}$ is a section of class C^1 of $\mathbf{E}$ defined on a neighborhood of $\gamma(I)$. In a local frame, write $\widetilde{\mathbf{u}} = \widetilde{u}^\alpha\mathbf{e}_\alpha$. The components of $\mathbf{u}$ along γ are then $\widetilde{u}^\alpha(\gamma(t))$. By the chain rule,

$$\frac{d}{dt}\big(\widetilde{u}^\alpha(\gamma(t))\big) = \frac{d\gamma^i}{dt}(t)\partial_i\widetilde{u}^\alpha(\gamma(t)).$$

Substituting into (2.53) gives

$$\frac{D\mathbf{u}}{dt}(t) = \frac{d\gamma^i}{dt}(t)\left(\partial_i\widetilde{u}^\alpha(\gamma(t)) + \Gamma^\alpha_{i\beta}(\gamma(t))\widetilde{u}^\beta(\gamma(t))\right)\mathbf{e}_\alpha(\gamma(t)).$$

The expression in parentheses is the local expression for $\nabla^{\mathbf{E}}_{\partial_i}\widetilde{\mathbf{u}}$ in the frame $\mathbf{e}_\alpha$. Therefore $\dfrac{D\mathbf{u}}{dt}(t) = \nabla^{\mathbf{E}}_{\dot\gamma(t)}\widetilde{\mathbf{u}}$.

Finally, we prove uniqueness. Suppose that $\mathcal{D}$ is another operator satisfying the three properties in the statement. On an interval J with $\gamma(J) \subseteq U$, write $\mathbf{u}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t))$. Since each $\mathbf{e}_\alpha \circ \gamma$ is extendible, the properties of $\mathcal{D}$ imply

$$\mathcal{D}\mathbf{u} = \frac{du^\alpha}{dt}\mathbf{e}_\alpha(\gamma(t)) + u^\alpha(t)\mathcal{D}(\mathbf{e}_\alpha \circ \gamma).$$

By compatibility with extensions, $\mathcal{D}(\mathbf{e}_\alpha \circ \gamma)(t) = \nabla^{\mathbf{E}}_{\dot\gamma(t)}\mathbf{e}_\alpha$. Using $\dot\gamma(t) = \frac{d\gamma^i}{dt}(t)\partial_i|_{\gamma(t)}$ and $\nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_\beta = \Gamma^\alpha_{i\beta}\mathbf{e}_\alpha$ gives precisely formula (2.53). Therefore $\mathcal{D} = \frac{D}{dt}$ locally. Since t_0 was arbitrary, $\mathcal{D} = \frac{D}{dt}$ on all of I. □

Remark 2.7.6. For the induced connections on $\mathbf{E}^*$, on tensor products, and in particular on tensor bundles constructed from TM, covariant differentiation along curves commutes with contractions and satisfies the Leibniz rule for the tensor product. In particular, taking $\mathbf{E} = T^{(r,s)}(TM)$ gives covariant differentiation of tensor fields of type (r,s) along a curve.

Proposition 2.7.7. (Local expression for the covariant derivative along a curve) Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank r with connection $\nabla^{\mathbf{E}}$, and let $\gamma\colon I \longrightarrow M$ be a smooth curve. Suppose that $\gamma(J) \subseteq U$, where (U,φ) is a smooth chart on M with coordinates $(x^1,\dots,x^n)$, and let $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ be a smooth local frame of $\mathbf{E}$ over U.

If $\mathbf{u}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t))$, with $t \in J$, and the local connection coefficients are given by $\nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_\beta = \Gamma^\alpha_{i\beta}\mathbf{e}_\alpha$, then

$$\frac{D\mathbf{u}}{dt}(t) = \left(\frac{du^\alpha}{dt}(t) + \frac{d\gamma^i}{dt}(t)\Gamma^\alpha_{i\beta}(\gamma(t))u^\beta(t)\right)\mathbf{e}_\alpha(\gamma(t)).$$

Proof. Write $\mathbf{u} = u^\alpha\mathbf{e}_\alpha \circ \gamma$. By the Leibniz rule,

$$\frac{D\mathbf{u}}{dt} = \frac{du^\alpha}{dt}\mathbf{e}_\alpha(\gamma(t)) + u^\alpha\frac{D(\mathbf{e}_\alpha \circ \gamma)}{dt}.$$

Since $\mathbf{e}_\alpha \circ \gamma$ is extendible, we have $\dfrac{D(\mathbf{e}_\alpha \circ \gamma)}{dt}(t) = \nabla^{\mathbf{E}}_{\dot\gamma(t)}\mathbf{e}_\alpha$. Using $\dot\gamma(t) = \frac{d\gamma^i}{dt}(t)\partial_i|_{\gamma(t)}$ and $\nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_\beta = \Gamma^\alpha_{i\beta}\mathbf{e}_\alpha$, we obtain

$$\frac{D(\mathbf{e}_\beta \circ \gamma)}{dt}(t) = \frac{d\gamma^i}{dt}(t)\Gamma^\alpha_{i\beta}(\gamma(t))\mathbf{e}_\alpha(\gamma(t)).$$

Substituting this expression into the formula for $\frac{D\mathbf{u}}{dt}$ gives

$$\frac{D\mathbf{u}}{dt}(t) = \frac{du^\alpha}{dt}(t)\mathbf{e}_\alpha(\gamma(t)) + u^\beta(t)\frac{d\gamma^i}{dt}(t)\Gamma^\alpha_{i\beta}(\gamma(t))\mathbf{e}_\alpha(\gamma(t)).$$

Grouping the coefficients of $\mathbf{e}_\alpha(\gamma(t))$ yields the desired formula. □

The preceding result shows that the covariant derivative of a section along a curve is calculated using a local system of ordinary functions. In particular, if a global section is restricted to a curve, its derivative along the curve depends only on the values of the section along that curve and on the curve's velocity.

Proposition 2.7.8. Let M be a smooth manifold with or without boundary, and let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. Let $p \in M$, $v \in T_pM$, and let $\mathbf{s}, \widetilde{\mathbf{s}}$ be sections of class C^1 of $\mathbf{E}$ defined on a neighborhood of p. Suppose that there exist a smooth curve $\gamma\colon I \longrightarrow M$ and a point $t_0 \in I$ such that $\gamma(t_0) = p$, $\dot\gamma(t_0) = v$, and $\mathbf{s}(\gamma(t)) = \widetilde{\mathbf{s}}(\gamma(t))$ for every t in a neighborhood of t_0. Then $\nabla^{\mathbf{E}}_v\mathbf{s} = \nabla^{\mathbf{E}}_v\widetilde{\mathbf{s}}$.

Proof. Define a section $\mathbf{u}$ of $\gamma^*\mathbf{E}$ near t_0 by $\mathbf{u}(t) = \mathbf{s}(\gamma(t)) = \widetilde{\mathbf{s}}(\gamma(t))$. Then $\mathbf{s}$ and $\widetilde{\mathbf{s}}$ are two local extensions of $\mathbf{u}$. By compatibility with extensions in Theorem 2.7.5, we have

$$\nabla^{\mathbf{E}}_{\dot\gamma(t_0)}\mathbf{s} = \frac{D\mathbf{u}}{dt}(t_0) = \nabla^{\mathbf{E}}_{\dot\gamma(t_0)}\widetilde{\mathbf{s}}.$$

Since $\dot\gamma(t_0) = v$, we conclude that $\nabla^{\mathbf{E}}_v\mathbf{s} = \nabla^{\mathbf{E}}_v\widetilde{\mathbf{s}}$. □

Definition 2.7.9. (Parallel sections along a curve) Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, and let $\gamma\colon I \longrightarrow M$ be a smooth curve. A section $\mathbf{u} \in \Gamma^1(\gamma^*\mathbf{E})$ is said to be parallel along γ if $\dfrac{D\mathbf{u}}{dt} \equiv 0$.

In a local frame, the parallelism condition becomes a linear system of ordinary differential equations. If $\mathbf{u}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t))$, then $\mathbf{u}$ is parallel along γ if and only if

$$\frac{du^\alpha}{dt}(t) = -\frac{d\gamma^i}{dt}(t)\Gamma^\alpha_{i\beta}(\gamma(t))u^\beta(t), \qquad \alpha \in \{1,\dots,r\}. \tag{2.54}$$

This observation reduces existence and uniqueness of parallel sections to a classical existence and uniqueness theorem for linear systems of ordinary differential equations.

Proposition 2.7.10. Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, and let $\mathbf{s} \in \Gamma^1(\mathbf{E})$. Then $\nabla^{\mathbf{E}}\mathbf{s} = 0$ if and only if, for every smooth curve $\gamma\colon I \longrightarrow M$, the section $\mathbf{s} \circ \gamma$ of $\gamma^*\mathbf{E}$ is parallel along γ.

Proof. First suppose that $\nabla^{\mathbf{E}}\mathbf{s} = 0$. For every smooth curve $\gamma\colon I \longrightarrow M$, the section $\mathbf{s} \circ \gamma$ is extendible by $\mathbf{s}$. By Theorem 2.7.5, we have $\dfrac{D(\mathbf{s} \circ \gamma)}{dt}(t) = \nabla^{\mathbf{E}}_{\dot\gamma(t)}\mathbf{s} = 0$. Thus $\mathbf{s} \circ \gamma$ is parallel along γ.
Conversely, suppose that $\mathbf{s} \circ \gamma$ is parallel along every smooth curve γ. Let $p \in M$ and $v \in T_pM$. Choose a smooth curve γ defined near 0 such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then

$$\nabla^{\mathbf{E}}_v\mathbf{s} = \nabla^{\mathbf{E}}_{\dot\gamma(0)}\mathbf{s} = \frac{D(\mathbf{s} \circ \gamma)}{dt}(0) = 0.$$

Since p and v were arbitrary, we conclude that $\nabla^{\mathbf{E}}\mathbf{s} = 0$. □

Theorem 2.7.11. (Existence, uniqueness, and smoothness for linear ODEs) Let $I \subseteq \mathbb{R}$ be an open interval and, for $1 \le \beta, \alpha \le r$, let $A^\alpha_\beta\colon I \longrightarrow \mathbb{K}$ be a smooth function. For every $t_0 \in I$ and every initial vector $c = (c^1, \dots, c^r) \in \mathbb{K}^r$, the linear initial value problem

$$\begin{cases} \dfrac{dU^\alpha}{dt}(t) = A^\alpha_\beta(t)U^\beta(t), \\ U^\alpha(t_0) = c^\alpha, \end{cases} \tag{2.55}$$

has a unique smooth solution defined on all of I. Moreover, for fixed t_0, the solution depends smoothly on $(t, c) \in I \times \mathbb{K}^r$.

Proof. We first prove the result when $t_0 = 0$ and $0 \in I$. To simplify notation, we present the argument over $\mathbb{R}$. In the complex case, apply the same reasoning to the real and imaginary parts, identifying $\mathbb{C}^r$ with $\mathbb{R}^{2r}$. Let $(x^0, x^1, \dots, x^r)$ be the standard coordinates on $I \times \mathbb{R}^r$. Consider the smooth vector field

$$\mathbf{Y} = \partial_0 + A^\alpha_\beta(x^0)x^\beta\partial_\alpha$$

on $I \times \mathbb{R}^r$.
If $U(t) = (U^1(t), \dots, U^r(t))$ is a solution of (2.55) defined on an interval $I_0 \subseteq I$, then $\eta(t) = (t, U^1(t), \dots, U^r(t))$ is an integral curve of $\mathbf{Y}$. The first component satisfies $\dot\eta^0(t) = 1$, and the others satisfy

$$\dot\eta^\alpha(t) = \frac{dU^\alpha}{dt}(t) = A^\alpha_\beta(t)U^\beta(t) = A^\alpha_\beta(\eta^0(t))\eta^\beta(t).$$

Conversely, if $\eta(t) = (\eta^0(t), \eta^1(t), \dots, \eta^r(t))$ is an integral curve of Y with $\eta(0) = (0, c^1, \dots, c^r)$, then $\dot\eta^0(t) = 1$ and $\eta^0(0) = 0$. Therefore $\eta^0(t) = t$. Defining $U(t) = (\eta^1(t), \dots, \eta^r(t))$ gives a solution of the system (2.55) with initial condition $U(0) = c$. Thus solutions of the linear system correspond to integral curves of Y passing through $(0, c)$.
By Theorem 1.8.15, there exists a unique maximal integral curve of Y with initial condition $(0, c)$, defined

on a maximal open interval $I_0 \subseteq I$ containing 0. By the preceding argument, this is equivalent to existence and uniqueness of a maximal solution $U\colon I_0 \longrightarrow \mathbb{R}^r$ of the linear system. It remains to show that $I_0 = I$.
Write $I = (a,b)$ and $I_0 = (a_0,b_0)$, with $a,b,a_0,b_0 \in \mathbb{R}\cup\{-\infty,\infty\}$. Suppose, for a contradiction, that $b_0 < b$. Then b_0 is finite and $[0,b_0] \subseteq I$. For $t \in [0,b_0)$, we have

$$\frac{d}{dt}\|U(t)\|^2 = 2\frac{dU^\alpha}{dt}(t)U^\alpha(t) = 2A^\alpha_\beta(t)U^\beta(t)U^\alpha(t).$$

Using the Cauchy–Schwarz inequality for the Frobenius norm, we obtain $\frac{d}{dt}\|U(t)\|^2 \le 2\|A(t)\|_F\|U(t)\|^2$. Since the functions A^α_β are continuous, there exists $C > 0$ such that $\|A(t)\|_F \le C$ for every $t \in [0,b_0]$. Therefore,

$$\frac{d}{dt}\left(e^{-2Ct}\|U(t)\|^2\right) = e^{-2Ct}\left(\frac{d}{dt}\|U(t)\|^2 - 2C\|U(t)\|^2\right) \le 0.$$

Thus $e^{-2Ct}\|U(t)\|^2 \le \|U(0)\|^2$ for every $t \in [0,b_0)$, and hence $\|U(t)\| \le e^{Cb_0}\|U(0)\|$. Consequently, the corresponding integral curve remains in the compact set

$$[0,b_0]\times\overline{B}_{\mathrm{euc}}(0,R) \subseteq I\times\mathbb{R}^r, \qquad R = e^{Cb_0}\|U(0)\|.$$

This contradicts Lemma 1.8.18. Therefore $b_0 = b$.
To prove that $a_0 = a$, consider the reparametrized solution $W(s) = U(-s)$, which satisfies the linear system $\frac{dW^\alpha}{ds}(s) = -A^\alpha_\beta(-s)W^\beta(s)$. The preceding argument, applied to W, shows that the solution cannot cease to exist before reaching the left endpoint of I either. We conclude that $I_0 = I$.
Smooth dependence on (t,c) follows from smooth dependence of the flow of Y on time and initial conditions, since the solution $U(t)$ is the $\mathbb{R}^r$-component of the integral curve of Y starting at $(0,c)$.
Finally, consider the general case $t_0 \in I$. Define $\widetilde{I} = I - t_0$ and $\widetilde{A}^\alpha_\beta(s) = A^\alpha_\beta(s+t_0)$. The translated system

$$\frac{d\widetilde{U}^\alpha}{ds}(s) = \widetilde{A}^\alpha_\beta(s)\widetilde{U}^\beta(s), \qquad \widetilde{U}^\alpha(0) = c^\alpha,$$

has a unique smooth solution on all of $\widetilde{I}$. Defining $U(t) = \widetilde{U}(t-t_0)$ gives the solution of the original system on all of I. Uniqueness and smooth dependence are preserved under the change of variable $s = t - t_0$. □

Theorem 2.7.12. (Existence and uniqueness of parallel transport) Let M be a smooth manifold with or without boundary, and let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. Let $\gamma\colon I \longrightarrow M$ be a smooth curve, let $t_0 \in I$, and let $v \in \mathbf{E}_{\gamma(t_0)}$. Then there exists a unique parallel section $\mathbf{u} \in \Gamma(\gamma^*\mathbf{E})$ such that $\mathbf{u}(t_0) = v$.

Proof. We first solve the problem locally. Suppose that $\gamma(J) \subseteq U$, where $J \subseteq I$ is an interval and $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ is a smooth local frame of $\mathbf{E}$ over U. If $\mathbf{u}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t))$, then, by (2.54), the condition $\frac{D\mathbf{u}}{dt} = 0$ is equivalent to the linear system

$$\frac{du^\alpha}{dt}(t) = -\frac{d\gamma^i}{dt}(t)\Gamma^\alpha_{i\beta}(\gamma(t))u^\beta(t).$$

Defining $A^\alpha_\beta(t) = -\frac{d\gamma^i}{dt}(t)\Gamma^\alpha_{i\beta}(\gamma(t))$ gives a system of the form $\frac{du^\alpha}{dt}(t) = A^\alpha_\beta(t)u^\beta(t)$. Since γ and the coefficients $\Gamma^\alpha_{i\beta}$ are smooth, the functions A^α_β are smooth on J. By Theorem 2.7.11, for every initial condition at a point of J, there exists a unique smooth solution on all of J. Thus, on any parameter interval whose image lies in

a single local trivialization of $\mathbf{E}$, there exists a unique parallel section with prescribed initial value.
We now prove global existence and uniqueness to the right of t_0; the left endpoint is treated by reversing the parameter. Let $b = \sup I$ and consider the set

$$\mathcal{S} = \{s \in I \mid s > t_0 \text{ and there is a unique parallel section on } [t_0, s] \text{ with initial value } v\}.$$

This set is nonempty. Choose a local trivialization of $\mathbf{E}$ on a neighborhood U of $\gamma(t_0)$. Since $\gamma^{-1}(U)$ is open in I and contains t_0, there exists $\varepsilon > 0$ such that $\gamma((t_0 - \varepsilon, t_0 + \varepsilon) \cap I) \subseteq U$. By the local case, there is a unique parallel section on $[t_0, s]$ for every $s \in I$ with $t_0 < s < t_0 + \varepsilon$.
Let $\beta = \sup \mathcal{S}$. For each $s \in \mathcal{S}$, denote by $\mathbf{u}_s$ the unique parallel section on $[t_0, s]$ with $\mathbf{u}_s(t_0) = v$. If $s_1, s_2 \in \mathcal{S}$ and $s_1 < s_2$, then $\mathbf{u}_{s_2}|_{[t_0,s_1]}$ is a parallel section on $[t_0, s_1]$ with initial value v. By the uniqueness defining $s_1 \in \mathcal{S}$, we have $\mathbf{u}_{s_2}|_{[t_0,s_1]} = \mathbf{u}_{s_1}$. Thus the solutions $\mathbf{u}_s$ are mutually compatible.
Using this compatibility, define a section $\mathbf{u}$ on $[t_0, \beta)$ as follows. If $t \in [t_0, \beta)$, choose $s \in \mathcal{S}$ such that $t \le s < \beta$ and set $\mathbf{u}(t) = \mathbf{u}_s(t)$. The definition is independent of the choice of s precisely because of the preceding compatibility. Moreover, $\mathbf{u}$ is smooth and parallel on $[t_0, \beta)$ because it locally agrees with one of the smooth parallel sections $\mathbf{u}_s$.
Let us show that $\beta = b$. Suppose, for a contradiction, that $\beta < b$. Then $\beta \in I$. Choose a local trivialization of $\mathbf{E}$ over a neighborhood U of $\gamma(\beta)$. Since $\gamma^{-1}(U)$ is open in I and contains β, there exists $\delta > 0$ such that $\gamma((\beta - \delta, \beta + \delta) \cap I) \subseteq U$. Choose $\tau \in \mathcal{S}$ with $\beta - \frac{\delta}{2} < \tau < \beta$. By the local case, there is a unique parallel section $\widetilde{\mathbf{u}}$ along γ on $(\beta - \delta, \beta + \delta) \cap I$ such that $\widetilde{\mathbf{u}}(\tau) = \mathbf{u}(\tau)$.
We claim that $\widetilde{\mathbf{u}} = \mathbf{u}$ on $[\tau, \beta)$. Let $t \in [\tau, \beta)$. Choose $s \in \mathcal{S}$ such that $t \le s < \beta$. On the interval $[\tau, s]$, both $\widetilde{\mathbf{u}}$ and $\mathbf{u}_s$ are parallel sections with the same value at τ. By local uniqueness within the trivialization over U, we have $\widetilde{\mathbf{u}} = \mathbf{u}_s$ on $[\tau, s]$. Since $\mathbf{u} = \mathbf{u}_s$ on $[\tau, s]$, we conclude that $\widetilde{\mathbf{u}}(t) = \mathbf{u}(t)$. Since t was arbitrary, $\widetilde{\mathbf{u}} = \mathbf{u}$ on $[\tau, \beta)$.
Now choose $\varepsilon > 0$ such that $\beta + \varepsilon \in I$ and $\varepsilon < \delta$. Define a section $\widehat{\mathbf{u}}$ on $[t_0, \beta + \varepsilon]$ by

$$\widehat{\mathbf{u}}(t) = \begin{cases} \mathbf{u}(t), & t \in [t_0, \tau], \\ \widetilde{\mathbf{u}}(t), & t \in [\tau, \beta + \varepsilon]. \end{cases}$$

This definition is smooth because $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ agree on an interval to the right of τ. Moreover, $\widehat{\mathbf{u}}$ is parallel because it locally agrees with parallel sections.
We also prove uniqueness of $\widehat{\mathbf{u}}$. Let $\mathbf{w}$ be another parallel section on $[t_0, \beta + \varepsilon]$ with $\mathbf{w}(t_0) = v$. Then $\mathbf{w}|_{[t_0,\tau]}$ is a parallel section with the same initial value as $\mathbf{u}_\tau$. By uniqueness on $[t_0, \tau]$, we have $\mathbf{w} = \mathbf{u}$ on $[t_0, \tau]$. In particular, $\mathbf{w}(\tau) = \mathbf{u}(\tau) = \widetilde{\mathbf{u}}(\tau)$. By local uniqueness in the trivialization over U, we obtain $\mathbf{w} = \widetilde{\mathbf{u}}$ on $[\tau, \beta + \varepsilon]$. Therefore, $\mathbf{w} = \widehat{\mathbf{u}}$ on all of $[t_0, \beta + \varepsilon]$.
This shows that $\beta + \varepsilon \in \mathcal{S}$, contradicting the definition of $\beta = \sup \mathcal{S}$. Therefore $\beta = b$, and there is a unique parallel section to the right of t_0.
The same argument to the left, using intervals of the form $[s, t_0]$ with $s < t_0$, shows that there is a unique parallel section to the left of t_0. Gluing the two sections gives a unique parallel section on all of I with initial value v. □

Definition 2.7.13. (Parallel transport) Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, and let $\gamma\colon I \longrightarrow M$ be a smooth curve. Given $t_0, t_1 \in I$,

define parallel transport from t_0 to t_1 along γ to be the map

$$P^{\mathbf{E}}_{\gamma,t_0\to t_1} : \mathbf{E}_{\gamma(t_0)} \longrightarrow \mathbf{E}_{\gamma(t_1)}$$

given by $P^{\mathbf{E}}_{\gamma,t_0\to t_1}(v) = \mathbf{u}(t_1)$, where $\mathbf{u}$ is the unique parallel section along γ such that $\mathbf{u}(t_0) = v$.
When the bundle is clear from context, we omit the superscript and write $P_{\gamma,t_0\to t_1}$.

Proposition 2.7.14. Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, and let $\gamma\colon I \longrightarrow M$ be a smooth curve. For any $t_0, t_1 \in I$, parallel transport $P^{\mathbf{E}}_{\gamma,t_0\to t_1} : \mathbf{E}_{\gamma(t_0)} \longrightarrow \mathbf{E}_{\gamma(t_1)}$ is a linear isomorphism. Moreover, $P^{\mathbf{E}}_{\gamma,t_1\to t_0} = (P^{\mathbf{E}}_{\gamma,t_0\to t_1})^{-1}$. More generally, if $t_0, t_1, t_2 \in I$, then

$$P^{\mathbf{E}}_{\gamma,t_1\to t_2} \circ P^{\mathbf{E}}_{\gamma,t_0\to t_1} = P^{\mathbf{E}}_{\gamma,t_0\to t_2}.$$

Proof. Linearity follows from linearity of the parallelism equation. If $\mathbf{u}$ and $\mathbf{w}$ are the parallel sections with initial conditions v and z at t_0, then $a\mathbf{u} + b\mathbf{w}$ is parallel and satisfies $(a\mathbf{u} + b\mathbf{w})(t_0) = av + bz$. By uniqueness, $a\mathbf{u} + b\mathbf{w}$ is the parallel section corresponding to the initial data $av + bz$. Evaluating at t_1 gives

$$P^{\mathbf{E}}_{\gamma,t_0\to t_1}(av + bz) = aP^{\mathbf{E}}_{\gamma,t_0\to t_1}(v) + bP^{\mathbf{E}}_{\gamma,t_0\to t_1}(z).$$

To prove the inverse formula, let $v \in \mathbf{E}_{\gamma(t_0)}$ and let $\mathbf{u}$ be the parallel section with $\mathbf{u}(t_0) = v$. Then $\mathbf{u}(t_1) = P^{\mathbf{E}}_{\gamma,t_0\to t_1}(v)$. If we transport $\mathbf{u}(t_1)$ from t_1 to t_0, the resulting parallel section must again be $\mathbf{u}$, by uniqueness. Therefore,

$$P^{\mathbf{E}}_{\gamma,t_1\to t_0}(P^{\mathbf{E}}_{\gamma,t_0\to t_1}(v)) = v.$$

The same argument, interchanging t_0 and t_1, shows that composition in the other order is also the identity. Thus $P^{\mathbf{E}}_{\gamma,t_1\to t_0} = (P^{\mathbf{E}}_{\gamma,t_0\to t_1})^{-1}$.
Finally, let $\mathbf{u}$ be the parallel section with $\mathbf{u}(t_0) = v$. Then $P^{\mathbf{E}}_{\gamma,t_0\to t_1}(v) = \mathbf{u}(t_1)$. Transporting this vector from t_1 to t_2 again gives the value $\mathbf{u}(t_2)$, by uniqueness. Therefore,

$$P^{\mathbf{E}}_{\gamma,t_1\to t_2}(P^{\mathbf{E}}_{\gamma,t_0\to t_1}(v)) = \mathbf{u}(t_2) = P^{\mathbf{E}}_{\gamma,t_0\to t_2}(v).$$

Since v was arbitrary, the composition identity follows. □

Remark 2.7.15. (Parallel frames along a curve) Let $(b_1, \dots, b_r)$ be a basis of $\mathbf{E}_{\gamma(t_0)}$. For each $\alpha \in \{1, \dots, r\}$, let $\mathbf{E}_\alpha$ be the parallel section along γ such that $\mathbf{E}_\alpha(t_0) = b_\alpha$. Since parallel transport is a linear isomorphism, for each $t \in I$ the family $(\mathbf{E}_1(t), \dots, \mathbf{E}_r(t))$ is a basis of $\mathbf{E}_{\gamma(t)}$. This family of parallel sections is called a parallel frame along γ.
If $\mathbf{u} \in \Gamma^1(\gamma^*\mathbf{E})$, then there is a unique family of functions $u^1, \dots, u^r$ of class C^1 such that $\mathbf{u}(t) = u^\alpha(t)\mathbf{E}_\alpha(t)$. Since $\frac{D\mathbf{E}_\alpha}{dt} = 0$, we obtain

$$\frac{D\mathbf{u}}{dt}(t) = \frac{du^\alpha}{dt}(t)\mathbf{E}_\alpha(t).$$

Consequently, a section $\mathbf{u}$ is parallel along γ if and only if its components in a parallel frame are constant.

Parallel transport along a single curve produces a frame along that curve. In a normal neighborhood, it can be performed simultaneously along all radial geodesics; this construction will provide the natural trivialization for

spaces of sections on manifolds of bounded geometry.

Definition 2.7.16. (Synchronous frame) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $p \in M$, and let U be a normal neighborhood of p. For $q \in U$, let

$$\gamma_q(t) := \exp_p\bigl(t \exp_p^{-1}(q)\bigr), \qquad 0 \le t \le 1.$$

If $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ is a vector bundle with connection $\nabla^{\mathbf{E}}$ and $(b_1,\dots,b_r)$ is a basis of $\mathbf{E}_p$, the frame

$$\mathbf{e}_a(q) := P^{\mathbf{E}}_{\gamma_q,0\to 1}(b_a), \qquad a \in \{1,\dots,r\},$$

is called the *synchronous frame centered at* p.

Proposition 2.7.17. (Properties of synchronous frames) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $p \in M$, let U be a normal neighborhood of p, and let $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ be a vector bundle with connection $\nabla^{\mathbf{E}}$. The synchronous frame centered at p is smooth, and its restriction to every radial geodesic is parallel. If the connection is compatible with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and the initial basis is orthonormal, the frame is orthonormal throughout U. In normal coordinates centered at p, if $\nabla^{\mathbf{E}} = d + \mathbf{A}$ and $\mathbf{A} = A_i\,\mathbf{d}x^i$, then

$$\sum_{i=1}^{n} x^i A_i(x) = 0, \qquad A_i(0) = 0 \quad (i \in \{1,\dots,n\}).$$

Proof. In a fixed trivialization, transport of b_a satisfies

$$\frac{dv_a}{dt}(t,q) = -\Gamma_i\bigl(\gamma_q(t)\bigr)\dot\gamma_q^i(t) v_a(t,q), \qquad v_a(0,q) = b_a.$$

Smooth dependence of solutions on parameters proves smoothness of the frame, and uniqueness proves radial parallelism. Metric compatibility gives $\frac{d}{dt}\mathbf{h}_{\mathbf{E}}(\mathbf{e}_a,\mathbf{e}_b) = 0$. In normal coordinates, the ray ending at x is $t \mapsto tx$; parallelism implies

$$0 = \nabla^{\mathbf{E}}_{\sum_{i=1}^{n} x^i \partial_i} \mathbf{e}_a = \left(\sum_{i=1}^{n} x^i A_i(x)\right)\mathbf{e}_a.$$

The radial identity follows because the $\mathbf{e}_a$ form a basis. Substituting $x = tv$, dividing by $t > 0$, and letting $t \to 0$ gives $A_i(0) = 0$. □

Proposition 2.7.18. (Local equation for parallel transport) Let M be a smooth manifold with or without boundary, let $\pi \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, let $\gamma \colon I \longrightarrow M$ be a smooth curve, and suppose that $\gamma(J) \subseteq U$, where $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ is a local frame of $\mathbf{E}$ over U. Fix $t_0 \in J$, and represent parallel transport $R(t) := P^{\mathbf{E}}_{\gamma,t_0\to t} \colon \mathbf{E}_{\gamma(t_0)} \longrightarrow \mathbf{E}_{\gamma(t)}$ by a matrix $r(t)$ in the frame $\mathbf{e}_1,\dots,\mathbf{e}_r$. That is, if $v = v^\beta \mathbf{e}_\beta(\gamma(t_0))$, then

$$R(t)v = r^\alpha_\beta(t) v^\beta \mathbf{e}_\alpha(\gamma(t)).$$

Then $r(t)$ satisfies

$$\frac{dr}{dt}(t) + \dot\gamma^i(t)\Gamma_i(\gamma(t))r(t) = 0, \qquad r(t_0) = I,$$

where $\Gamma_i = (\Gamma_{i\beta}^{\alpha})_{\alpha,\beta}$. In particular, if $\dot\gamma(t_0) = \partial_i|_{\gamma(t_0)}$, then $r'(t_0) = -\Gamma_i(\gamma(t_0))$.
Moreover, if $Q(t) := R(t)^{-1} = P^{\mathbf{E}}_{\gamma,t\to t_0}$ and $q(t) = r(t)^{-1}$ is its matrix, then $q(t)$ depends smoothly on t and satisfies

$$\frac{dq}{dt}(t) = q(t)\dot\gamma^i(t)\Gamma_i(\gamma(t)).$$

Proof. Let $v = v^\beta \mathbf{e}_\beta(\gamma(t_0))$. By definition, $\mathbf{u}(t) := R(t)v$ is the parallel section along γ with $\mathbf{u}(t_0) = v$. In the local frame, write $\mathbf{u}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t))$, where $u^\alpha(t) = r^\alpha_\beta(t)v^\beta$.
Since $\mathbf{u}$ is parallel, the local equation (2.54) gives

$$\frac{du^\alpha}{dt}(t) = -\dot\gamma^i(t)\Gamma^\alpha_{i\mu}(\gamma(t))u^\mu(t).$$

Substituting $u^\alpha(t) = r^\alpha_\beta(t)v^\beta$, we obtain

$$\frac{dr^\alpha_\beta}{dt}(t)v^\beta = -\dot\gamma^i(t)\Gamma^\alpha_{i\mu}(\gamma(t))r^\mu_\beta(t)v^\beta.$$

Since v is arbitrary, it follows that

$$\frac{dr^\alpha_\beta}{dt}(t) + \dot\gamma^i(t)\Gamma^\alpha_{i\mu}(\gamma(t))r^\mu_\beta(t) = 0.$$

In matrix form, this is $\dfrac{dr}{dt}(t) + \dot\gamma^i(t)\Gamma_i(\gamma(t))r(t) = 0$. Moreover, since $R(t_0)$ is the identity on $\mathbf{E}_{\gamma(t_0)}$, we have $r(t_0) = I$. If $\dot\gamma(t_0) = \partial_i|_{\gamma(t_0)}$, then $\dot\gamma^j(t_0) = \delta^j_i$, and evaluating the equation at t_0 gives $r'(t_0) = -\Gamma_i(\gamma(t_0))$.
Finally, $q(t) = r(t)^{-1}$ is smooth because $r(t)$ is smooth and takes values in $GL(r,\mathbb{K})$. Differentiating the identity $q(t)r(t) = I$ gives $q'(t)r(t) + q(t)r'(t) = 0$. Therefore $q'(t) = -q(t)r'(t)r(t)^{-1}$. Using $r'(t) = -\dot\gamma^i(t)\Gamma_i(\gamma(t))r(t)$ yields $q'(t) = q(t)\dot\gamma^i(t)\Gamma_i(\gamma(t))$. □

Theorem 2.7.19. (Covariant derivative through parallel transport) Let M be a smooth manifold with or without boundary, let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$, let $\gamma\colon I \longrightarrow M$ be a smooth curve, and let $\mathbf{u} \in \Gamma^1(\gamma^*\mathbf{E})$. Then, for every $t_0 \in I$,

$$\frac{D\mathbf{u}}{dt}(t_0) = \lim_{t\to t_0} \frac{P^{\mathbf{E}}_{\gamma,t\to t_0}\mathbf{u}(t) - \mathbf{u}(t_0)}{t - t_0}.$$

The limit is taken in the fixed vector space $\mathbf{E}_{\gamma(t_0)}$. If t_0 is an endpoint of the interval of definition, the limit is understood to be one-sided.

Proof. Choose a parallel frame $(\mathbf{E}_1,\dots,\mathbf{E}_r)$ along γ. Write $\mathbf{u}(t) = u^\alpha(t)\mathbf{E}_\alpha(t)$. By the preceding remark,

$$\frac{D\mathbf{u}}{dt}(t_0) = \frac{du^\alpha}{dt}(t_0)\mathbf{E}_\alpha(t_0).$$

On the other hand, since $\mathbf{E}_\alpha$ is parallel, parallel transport of $\mathbf{E}_\alpha(t)$ from t to t_0 is $\mathbf{E}_\alpha(t_0)$. By linearity of parallel transport,

$$P^{\mathbf{E}}_{\gamma,t\to t_0}\mathbf{u}(t) = u^\alpha(t)\mathbf{E}_\alpha(t_0).$$

Then

$$\frac{P^{\mathbf{E}}_{\gamma,t\to t_0}\mathbf{u}(t)-\mathbf{u}(t_0)}{t-t_0}=\frac{u^\alpha(t)-u^\alpha(t_0)}{t-t_0}\mathbf{E}_\alpha(t_0).$$

Taking the limit as $t\to t_0$ gives

$$\lim_{t\to t_0}\frac{P^{\mathbf{E}}_{\gamma,t\to t_0}\mathbf{u}(t)-\mathbf{u}(t_0)}{t-t_0}=\frac{du^\alpha}{dt}(t_0)\mathbf{E}_\alpha(t_0)=\frac{D\mathbf{u}}{dt}(t_0).$$

□

Corollary 2.7.20. Let M be a smooth manifold with or without boundary, and let $\pi\colon \mathbf{E}\longrightarrow M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. Let $\mathbf{s}\in\Gamma^1(\mathbf{E})$, let $p\in M$, and let $\mathbf{X}\in\mathfrak{X}(M)$. If γ is a smooth curve defined near 0 such that $\gamma(0)=p$ and $\dot\gamma(0)=\mathbf{X}_p$, then

$$\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{s}(p)=\lim_{h\to 0}\frac{P^{\mathbf{E}}_{\gamma,h\to 0}\mathbf{s}(\gamma(h))-\mathbf{s}(p)}{h}.$$

Proof. Define a section $\mathbf{u}$ of $\gamma^*\mathbf{E}$ by $\mathbf{u}(h)=\mathbf{s}(\gamma(h))$. Since $\mathbf{u}$ is the restriction of the section $\mathbf{s}$ to the curve γ, Theorem 2.7.5 implies that

$$\frac{D\mathbf{u}}{dt}(0)=\nabla^{\mathbf{E}}_{\dot\gamma(0)}\mathbf{s}=\nabla^{\mathbf{E}}_{\mathbf{X}_p}\mathbf{s}=\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{s}(p).$$

Using Theorem 2.7.19 at $t_0=0$, we obtain

$$\frac{D\mathbf{u}}{dt}(0)=\lim_{h\to 0}\frac{P^{\mathbf{E}}_{\gamma,h\to 0}\mathbf{u}(h)-\mathbf{u}(0)}{h}.$$

Since $\mathbf{u}(h)=\mathbf{s}(\gamma(h))$ and $\mathbf{u}(0)=\mathbf{s}(p)$, the formula follows. □

Lemma 2.7.21. (Fundamental theorem of covariant calculus) Let M be a smooth manifold with or without boundary, and let $\pi_{\mathbf{F}}\colon \mathbf{F}\longrightarrow M$ be a smooth vector bundle equipped with a connection $\nabla^{\mathbf{F}}$. Let $\gamma\colon [a,b]\longrightarrow M$ be a smooth curve, and let $\mathbf{s}\in\Gamma^1(\gamma^*\mathbf{F})$. For $t,t_0\in[a,b]$, denote by $P^{\mathbf{F}}_{\gamma,t\to t_0}\colon \mathbf{F}_{\gamma(t)}\longrightarrow\mathbf{F}_{\gamma(t_0)}$ parallel transport along γ from $\gamma(t)$ to $\gamma(t_0)$. Then

$$P^{\mathbf{F}}_{\gamma,t\to t_0}(\mathbf{s}(t))-\mathbf{s}(t_0)=\int_{t_0}^{t}P^{\mathbf{F}}_{\gamma,\tau\to t_0}\left(\frac{D\mathbf{s}}{d\tau}(\tau)\right)d\tau.$$

The integral is taken in the finite-dimensional vector space $\mathbf{F}_{\gamma(t_0)}$.

Proof. Fix $t_0\in[a,b]$. Choose a basis $v_1,\dots,v_r$ of the fiber $\mathbf{F}_{\gamma(t_0)}$. For each $\mu\in\{1,\dots,r\}$, let $\mathbf{e}_\mu$ be the parallel section of $\mathbf{F}$ along γ satisfying $\mathbf{e}_\mu(t_0)=v_\mu$. Then $\mathbf{e}_1(t),\dots,\mathbf{e}_r(t)$ is a basis of $\mathbf{F}_{\gamma(t)}$ for every $t\in[a,b]$.

Since $\mathbf{s}$ is a section of $\mathbf{F}$ along γ, there exist functions $a^1,\dots,a^r\in C^1([a,b])$ such that $\mathbf{s}(t)=\sum_{\mu=1}^{r}a^\mu(t)\mathbf{e}_\mu(t)$. Each $\mathbf{e}_\mu$ is parallel, so its covariant derivative along γ vanishes, and therefore $\frac{D\mathbf{s}}{dt}(t)=\sum_{\mu=1}^{r}\frac{da^\mu}{dt}(t)\mathbf{e}_\mu(t)$.

Moreover, by the definition of parallel transport, $P^{\mathbf{F}}_{\gamma,t\to t_0}(\mathbf{e}_\mu(t))=\mathbf{e}_\mu(t_0)=v_\mu$. Consequently, $P^{\mathbf{F}}_{\gamma,t\to t_0}(\mathbf{s}(t))=$

$\sum_{\mu=1}^{r} a^\mu(t)v_\mu$ and $P^{\mathbf{F}}_{\gamma,t\to t_0}\left(\frac{D\mathbf{s}}{dt}(t)\right) = \sum_{\mu=1}^{r}\frac{da^\mu}{dt}(t)v_\mu$.
Applying the fundamental theorem of calculus componentwise in the basis $(v_1,\dots,v_r)$ of $\mathbf{F}_{\gamma(t_0)}$, we obtain

$$P^{\mathbf{F}}_{\gamma,t\to t_0}(\mathbf{s}(t)) - \mathbf{s}(t_0) = \sum_{\mu=1}^{r}(a^\mu(t) - a^\mu(t_0))\,v_\mu.$$

Since $a^\mu(t) - a^\mu(t_0) = \int_{t_0}^{t}\frac{da^\mu}{d\tau}(\tau)\,d\tau$, it follows that

$$P^{\mathbf{F}}_{\gamma,t\to t_0}(\mathbf{s}(t)) - \mathbf{s}(t_0) = \int_{t_0}^{t} P^{\mathbf{F}}_{\gamma,\tau\to t_0}\left(\frac{D\mathbf{s}}{d\tau}(\tau)\right)d\tau,$$

which proves the identity. □

Lemma 2.7.22. (Integral criterion for the covariant derivative) Let M be a smooth manifold with or without boundary, and let $\pi_{\mathbf{F}}\colon \mathbf{F}\longrightarrow M$ be a smooth vector bundle equipped with a connection $\nabla^{\mathbf{F}}$. Let $\mathbf{s}\in\Gamma^0(\mathbf{F})$ and $\mathbf{A}\in\Gamma^0(T^*M\otimes\mathbf{F})$.
Suppose that, for every smooth curve $\gamma\colon[a,b]\longrightarrow M$ and any $t,t_0\in[a,b]$, we have

$$P^{\mathbf{F}}_{\gamma,t\to t_0}(\mathbf{s}(\gamma(t))) - \mathbf{s}(\gamma(t_0)) = \int_{t_0}^{t} P^{\mathbf{F}}_{\gamma,\tau\to t_0}\left(\mathbf{A}_{\gamma(\tau)}(\dot\gamma(\tau))\right)d\tau.$$

Then $\mathbf{s}\in\Gamma^1(\mathbf{F})$ and $\nabla^{\mathbf{F}}\mathbf{s} = \mathbf{A}$.

Proof. Fix $p\in M$. Choose a chart (U,φ) around p, with $\varphi = (x^1,\dots,x^d)$, and a smooth local frame $\mathbf{e}_1,\dots,\mathbf{e}_r$ of $\mathbf{F}$ over U.
Write $\mathbf{s} = \sum_{\mu=1}^{r} s^\mu\mathbf{e}_\mu$, where the functions $s^\mu\colon U\longrightarrow\mathbb{K}$ are continuous. Likewise, write

$$\mathbf{A} = \sum_{i=1}^{d}\sum_{\mu=1}^{r} A_i^\mu\,\mathbf{d}x^i\otimes\mathbf{e}_\mu,$$

where the functions $A_i^\mu\colon U\longrightarrow\mathbb{K}$ are continuous.
Denote by $\Gamma_{i\nu}^{\mu}$ the coefficients of the connection $\nabla^{\mathbf{F}}$ in the chosen frame, that is,

$$\nabla^{\mathbf{F}}_{\partial_i}\mathbf{e}_\nu = \sum_{\mu=1}^{r}\Gamma_{i\nu}^{\mu}\mathbf{e}_\mu, \qquad \partial_i := \partial_i.$$

Fix $i\in\{1,\dots,d\}$. We work with the coordinate curve in the direction i. If p is an interior point, choose $\delta>0$ so that

$$\gamma_i\colon(-\delta,\delta)\longrightarrow U, \qquad \gamma_i(t) := \varphi^{-1}\left(\varphi(p)+te_i\right),$$

is well defined. If $p\in\partial M$ and a coordinate direction is defined only on one side within the boundary chart, take the corresponding one-sided interval. The following argument applies unchanged, replacing ordinary

limits by one-sided limits in that direction. In all cases, $\gamma_i(0) = p$ and $\dot\gamma_i(t) = \partial_i|_{\gamma_i(t)}$ for the admissible values of t.

Let $R_i(t)\colon \mathbf{F}_p \longrightarrow \mathbf{F}_{\gamma_i(t)}$ be parallel transport from $p = \gamma_i(0)$ to $\gamma_i(t)$, and let $Q_i(t) := R_i(t)^{-1} = P^{\mathbf{F}}_{\gamma_i,t\to 0}$. In the local frame $\mathbf{e}_1,\dots,\mathbf{e}_r$, represent $R_i(t)$ and $Q_i(t)$ by matrices denoted by $r_i(t)$ and $q_i(t)$, respectively. Thus $q_i(t) = r_i(t)^{-1}$ and $r_i(0) = q_i(0) = I$.

By Proposition 2.7.18, the local differential equation for parallel transport gives

$$r_i'(t) + \Gamma_i(\gamma_i(t))\, r_i(t) = 0,$$

where $\Gamma_i := \left(\Gamma_{i\nu}^{\mu}\right)_{\mu,\nu}$. In particular, $r_i'(0) = -\Gamma_i(p)$.

Use the hypothesis of the lemma with the curve γ_i and with $t_0 = 0$. Since $\dot\gamma_i(t) = \partial_i|_{\gamma_i(t)}$, we obtain

$$P^{\mathbf{F}}_{\gamma_i,t\to 0}(\mathbf{s}(\gamma_i(t))) - \mathbf{s}(p) = \int_0^t P^{\mathbf{F}}_{\gamma_i,\tau\to 0}\left(\mathbf{A}_{\gamma_i(\tau)}(\partial_i)\right) d\tau.$$

Express this identity in the local frame. Define

$$\widehat{s} := \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix}, \qquad \widehat{A}_i := \begin{pmatrix} A_i^1 \\ \vdots \\ A_i^r \end{pmatrix}.$$

Then

$$q_i(t)\,\widehat{s}(\gamma_i(t)) - \widehat{s}(p) = \int_0^t q_i(\tau)\,\widehat{A}_i(\gamma_i(\tau))\, d\tau.$$

Multiplying by $r_i(t)$ gives

$$\widehat{s}(\gamma_i(t)) = r_i(t)\widehat{s}(p) + r_i(t)\int_0^t q_i(\tau)\,\widehat{A}_i(\gamma_i(\tau))\, d\tau.$$

Subtract $\widehat{s}(p)$ and divide by $t \neq 0$:

$$\frac{\widehat{s}(\gamma_i(t)) - \widehat{s}(p)}{t} = \frac{r_i(t) - I}{t}\,\widehat{s}(p) + r_i(t)\,\frac{1}{t}\int_0^t q_i(\tau)\,\widehat{A}_i(\gamma_i(\tau))\, d\tau.$$

Let $t \to 0$ through the admissible parameter values of the curve. Since r_i is differentiable and $r_i'(0) = -\Gamma_i(p)$, we have

$$\frac{r_i(t) - I}{t} \longrightarrow -\Gamma_i(p).$$

Moreover, $r_i(t) \to I$, $q_i(t) \to I$, and $\widehat{A}_i(\gamma_i(t)) \to \widehat{A}_i(p)$ by continuity. Therefore the function $\tau \mapsto q_i(\tau)\widehat{A}_i(\gamma_i(\tau))$ is continuous at 0, and consequently

$$\frac{1}{t}\int_0^t q_i(\tau)\,\widehat{A}_i(\gamma_i(\tau))\, d\tau \longrightarrow \widehat{A}_i(p).$$

We conclude that

$$\frac{\widehat{s}(\gamma_i(t)) - \widehat{s}(p)}{t} \longrightarrow -\Gamma_i(p)\widehat{s}(p) + \widehat{A}_i(p).$$

The left-hand side is the difference quotient of the components of $\mathbf{s}$ in the coordinate direction ∂_i. Therefore $\partial_i\widehat{s}(p)$ exists, in the usual sense if p is interior and in the corresponding one-sided sense if p lies in a

boundary chart, and satisfies

$$\partial_i \widehat{s}(p) = \widehat{A}_i(p) - \Gamma_i(p)\widehat{s}(p).$$

Since $p \in U$ was arbitrary, we obtain the identity $\partial_i \widehat{s} = \widehat{A}_i - \Gamma_i \widehat{s}$ throughout U. The right-hand side is continuous because $\widehat{A}_i$, Γ_i, and $\widehat{s}$ are continuous. Thus the partial derivatives of the components s^μ exist and are continuous in the corresponding charts. Therefore $\mathbf{s}$ is of class C^1 on U.
Finally, the local formula for the covariant derivative gives

$$\nabla^{\mathbf{F}}_{\partial_i} \mathbf{s} = \sum_{\mu=1}^{r} \left(\partial_i s^\mu + \sum_{\nu=1}^{r} \Gamma_{i\nu}^{\mu} s^\nu \right) \mathbf{e}_\mu.$$

The identity $\partial_i \widehat{s} = \widehat{A}_i - \Gamma_i \widehat{s}$ implies

$$\nabla^{\mathbf{F}}_{\partial_i} \mathbf{s} = \sum_{\mu=1}^{r} A_i^\mu \mathbf{e}_\mu = \mathbf{A}(\partial_i).$$

Since this holds for every $i \in \{1, \dots, d\}$, we conclude that $\nabla^{\mathbf{F}} \mathbf{s} = \mathbf{A}$ on U.
Since $p \in M$ was arbitrary, it follows that $\mathbf{s} \in \Gamma^1(\mathbf{F})$ and $\nabla^{\mathbf{F}} \mathbf{s} = \mathbf{A}$ on all of M. □

Remark 2.7.23. In the particular case $\mathbf{E} = TM$, parallel transport allows us to compare tangent vectors at different points of a curve. If ∇ is a connection on TM, the velocity $\dot\gamma$ of a smooth curve γ is a section of $\gamma^* TM$, so it makes sense to consider $\dfrac{D\dot\gamma}{dt}$. This leads to the notion of a geodesic: a curve whose velocity is parallel along itself, that is, a curve satisfying $\dfrac{D\dot\gamma}{dt} = 0$.

2.7.1 Horizontal–vertical decomposition induced by a connection

Covariant differentiation along curves also describes the geometry of the total space of a vector bundle. The following construction is first carried out for an arbitrary bundle; the cotangent decomposition used later in symbolic calculus will be a specialization.

Let M be a finite-dimensional smooth manifold with or without boundary, let $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of finite rank over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\nabla^{\mathbf{E}}$ be a linear connection on $\mathbf{E}$. For $e \in \mathbf{E}_x$, define the **vertical space** at e by

$$V_e \mathbf{E} := \ker(d\pi_{\mathbf{E}})_e \subseteq T_e \mathbf{E}. \tag{2.56}$$

If $w \in \mathbf{E}_x$, the curve $t \mapsto e + tw$ remains in $\mathbf{E}_x$ and determines the **vertical lift**

$$w_e^{\mathrm{V}} := \left. \frac{d}{dt} \right|_{t=0} (e + tw) \in V_e \mathbf{E}. \tag{2.57}$$

Here, including in the complex case, the real tangent space is used; the complex structure of $\mathbf{E}_x$ canonically induces the corresponding complex structure on $V_e \mathbf{E}$.

Proposition 2.7.24. (Vertical subbundle and vertical lift) Let M be a finite-dimensional smooth manifold with or without boundary, and let $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of finite rank. The set

$$V\mathbf{E} := \coprod_{e \in \mathbf{E}} V_e \mathbf{E}$$

is a smooth vector subbundle of $T\mathbf{E} \to \mathbf{E}$. For each $e \in \mathbf{E}_x$, the map

$$\mathbf{E}_x \longrightarrow V_e\mathbf{E}, \qquad w \longmapsto w_e^{\mathrm{V}}, \tag{2.58}$$

is a canonical linear isomorphism.

Proof. Let $U \subseteq M$ be a trivializing open set, and write a point of $\mathbf{E}|_U$ as $(x,u) \in U \times \mathbb{K}^r$. The differential of the projection is

$$(d\pi_{\mathbf{E}})_{(x,u)}(X,W) = X, \qquad (X,W) \in T_xM \times \mathbb{K}^r.$$

Therefore,

$$V_{(x,u)}\mathbf{E} = \{(0,W) \mid W \in \mathbb{K}^r\}.$$

These local descriptions show that $V\mathbf{E}$ has constant rank and smooth trivializations. In the same coordinates, the curve $t \mapsto (x, u+tw)$ gives

$$w^{\mathrm{V}}_{(x,u)} = (0,w).$$

Thus (2.58) is a linear isomorphism. The definition in terms of addition and scalar multiplication in the fiber uses no trivialization; hence the isomorphism is canonical. □

The connection selects a complement to the vertical subbundle. To construct it without choosing coordinates, we associate to each tangent vector of the total space the covariant derivative of the fiber vector representing it.

Definition 2.7.25. (Connection map and horizontal subbundle) Let M be a finite-dimensional smooth manifold with or without boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a connection $\nabla^{\mathbf{E}}$. Let $v \in T_e\mathbf{E}$, with $e \in \mathbf{E}_x$. In local bundle coordinates associated with a frame $(\mathbf{e}_1, \dots, \mathbf{e}_r)$, write

$$e = u^\beta \mathbf{e}_\beta(x), \qquad v = X^i \partial_i\Big|_e + W^\alpha \partial_{u^\alpha}\Big|_e, \qquad \nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_\beta = \Gamma^\alpha_{i\beta}\mathbf{e}_\alpha.$$

The **connection map** is defined by

$$K^{\nabla^{\mathbf{E}}}(v) := \big(W^\alpha + \Gamma^\alpha_{i\beta}(x) X^i u^\beta\big)\mathbf{e}_\alpha(x) \in \mathbf{E}_x. \tag{2.59}$$

The transformation law for the connection coefficients makes this expression independent of the coordinates and frame. If $\mathbf{s}(t) \in \mathbf{E}_{\gamma(t)}$ is a smooth curve with $\mathbf{s}(0) = e$ and $\dot{\mathbf{s}}(0) = v$, then $K^{\nabla^{\mathbf{E}}}(v) = \frac{D\mathbf{s}}{dt}\Big|_{t=0}$. The **horizontal space** at e is

$$H^{\nabla^{\mathbf{E}}}_e\mathbf{E} := \ker K^{\nabla^{\mathbf{E}}}_e, \tag{2.60}$$

where $K^{\nabla^{\mathbf{E}}}_e$ denotes the restriction to $T_e\mathbf{E}$.

Proposition 2.7.26. (Horizontal–vertical decomposition) Let M be a finite-dimensional smooth manifold with or without boundary, and let $\pi_{\mathbf{E}}\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of finite rank equipped with a connection $\nabla^{\mathbf{E}}$. The local expressions (2.59) are compatible on overlaps and define a smooth map $K^{\nabla^{\mathbf{E}}}\colon T\mathbf{E} \to \mathbf{E}$ covering $\pi_{\mathbf{E}}$, and each

$$K^{\nabla^{\mathbf{E}}}_e\colon T_e\mathbf{E} \longrightarrow \mathbf{E}_x$$

is linear over $\mathbb{R}$; its restriction to the vertical subspace is linear over $\mathbb{K}$ under the isomorphism (2.58).

Moreover,

$$K_e^{\nabla^{\mathbf{E}}}(w_e^{\mathrm{V}}) = w, \qquad w \in \mathbf{E}_x,$$

and we have the smooth decompositions

$$T_e\mathbf{E} = H_e^{\nabla^{\mathbf{E}}}\mathbf{E} \oplus V_e\mathbf{E}, \qquad T\mathbf{E} = H^{\nabla^{\mathbf{E}}}\mathbf{E} \oplus V\mathbf{E}. \tag{2.61}$$

The restriction

$$(d\pi_{\mathbf{E}})_e\big|_{H_e^{\nabla^{\mathbf{E}}}\mathbf{E}} \colon H_e^{\nabla^{\mathbf{E}}}\mathbf{E} \longrightarrow T_xM \tag{2.62}$$

is an isomorphism. Its inverse sends $X \in T_xM$ to the **horizontal lift** X_e^{H}.

Proof. Choose coordinates $(x^1, \dots, x^n)$ on an open set $U \subseteq M$ and a local frame $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ of $\mathbf{E}|_U$. Write

$$\nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_\beta = \Gamma_{i\beta}^{\alpha}\mathbf{e}_\alpha.$$

If $\mathbf{s}(t) = u^\alpha(t)\mathbf{e}_\alpha(\gamma(t))$, the local formula for covariant differentiation along a curve, obtained in (2.53), gives

$$K^{\nabla^{\mathbf{E}}}(x, u; X, W) = \left(x, W^\alpha + \Gamma_{i\beta}^{\alpha}(x)X^i u^\beta\right). \tag{2.63}$$

The right-hand side depends only on $(x, u; X, W) = \dot{\mathbf{s}}(0)$ and not on the higher-order terms of the curve. This proves independence of the representing curve. The same formula proves smoothness and linearity of $K_e^{\nabla^{\mathbf{E}}}$ in (X, W).

For a vertical vector, $X = 0$; therefore,

$$K^{\nabla^{\mathbf{E}}}(x, u; 0, W) = (x, W).$$

By (2.58), the restriction of $K_e^{\nabla^{\mathbf{E}}}$ to $V_e\mathbf{E}$ is precisely the inverse of the vertical lift. In particular, $H_e^{\nabla^{\mathbf{E}}}\mathbf{E} \cap V_e\mathbf{E} = \{0\}$. For $v \in T_e\mathbf{E}$, set

$$v^{\mathrm{V}} := \left(K^{\nabla^{\mathbf{E}}}(v)\right)_e^{\mathrm{V}}.$$

Then $v - v^{\mathrm{V}} \in H_e^{\nabla^{\mathbf{E}}}\mathbf{E}$ and $v = (v - v^{\mathrm{V}}) + v^{\mathrm{V}}$. This proves the first equality in (2.61). The local formula (2.63) also shows that the kernels $H_e^{\nabla^{\mathbf{E}}}\mathbf{E}$ vary smoothly, giving the equality of subbundles.

If a horizontal vector belongs to the kernel of $(d\pi_{\mathbf{E}})_e$, it is also vertical and hence, by the direct sum, zero. The restriction (2.62) is therefore injective. For $X = X^i\partial_i|_x$, the formula

$$X_e^{\mathrm{H}} = X^i\left(\partial_i - \Gamma_{i\beta}^{\alpha}(x)u^\beta\partial_{u^\alpha}\right)_{(x,u)} \tag{2.64}$$

defines a horizontal vector projecting to X. This proves surjectivity and completes the proof. □

A curve $\mathbf{s}(t) \in \mathbf{E}$ is horizontal if $\dot{\mathbf{s}}(t)$ belongs to $H_{\mathbf{s}(t)}^{\nabla^{\mathbf{E}}}\mathbf{E}$ for every t. By the definition of the connection map, this is equivalent to $\frac{D\mathbf{s}}{dt} = 0$; that is, $\mathbf{s}$ is horizontal if and only if the fiber vector it represents is parallel along $\pi_{\mathbf{E}} \circ \mathbf{s}$. Consequently, $H^{\nabla^{\mathbf{E}}}\mathbf{E}$ is the Ehresmann connection induced by the linear connection $\nabla^{\mathbf{E}}$. The subbundle $V\mathbf{E}$ is canonical, whereas $H^{\nabla^{\mathbf{E}}}\mathbf{E}$ depends on the chosen connection. This construction should not be confused with the nonlinear connections introduced in Finsler geometry: here the horizontal distribution comes from a linear connection on a vector bundle.

We now specialize the construction to the cotangent bundle. Let ∇ be a linear connection on TM, and equip T^*M with the dual connection characterized in Proposition 2.4.27. Let $\pi\colon T^*M \to M$ be the projection. Then

$$V(T^*M) = \ker d\pi, \qquad H^{\nabla}(T^*M) := H^{\nabla^{T^*M}}(T^*M).$$

Proposition 2.7.27. (Horizontal lift in the cotangent bundle) Let M be a finite-dimensional smooth manifold with or without boundary, let ∇ be a linear connection on TM, and let $(x^1,\dots,x^n)$ be local coordinates. Write

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k, \qquad \boldsymbol{\xi} = \xi_j\, \mathbf{d}x^j.$$

With these conventions, the horizontal lift of ∂_i to T^*M is

$$\mathbf{H}_i = \partial_i + \Gamma_{ij}^k(x)\xi_k \frac{\partial}{\partial \xi_j}. \tag{2.65}$$

Proof. Let $x(t)$ be a curve, and let $\boldsymbol{\xi}(t) = \xi_j(t)\mathbf{d}x^j|_{x(t)}$ be a section of the cotangent bundle along it. The dual connection satisfies

$$(\nabla_{\dot x}\boldsymbol{\xi})_j = \dot\xi_j - \Gamma_{ij}^k(x(t))\dot x^i \xi_k.$$

The curve $(x(t),\boldsymbol{\xi}(t))$ is horizontal if and only if the left-hand side vanishes. Therefore,

$$\dot\xi_j = \Gamma_{ij}^k(x(t))\dot x^i \xi_k.$$

If $\dot x(0) = \partial_i|_x$, the vertical component of the velocity is $\Gamma_{ij}^k(x)\xi_k\partial_{\xi_j}$, proving (2.65) and, in particular, fixing its sign. By contrast, for $v = v^j\partial_j \in T_xM$, the equation $\nabla_{\dot x}v = 0$ gives $\dot v^k = -\Gamma_{ij}^k \dot x^i v^j$; according to (2.64), the lift in TM has the opposite sign. □

It remains to specify how this decomposition acts on sections of the pullback of a homomorphism bundle. Let $\mathbf{E},\mathbf{F} \to M$ be smooth vector bundles of finite rank with connections $\nabla^{\mathbf{E}}$ and $\nabla^{\mathbf{F}}$. The induced connection $\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})}$ is characterized by (2.13). The pullback connection on $\pi^*\mathrm{Hom}(\mathbf{E},\mathbf{F}) \to T^*M$ is the unique connection that, for a local section $\mathbf{A}$ of $\mathrm{Hom}(\mathbf{E},\mathbf{F})$ and $\mathbf{Z} \in \mathfrak{X}(T^*M)$, satisfies

$$(\pi^*\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})})_{\mathbf{Z}}(\pi^*\mathbf{A}) = \pi^*\left(\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})}_{d\pi(\mathbf{Z})}\mathbf{A}\right), \tag{2.66}$$

and extends to all sections by the Leibniz rule. Existence and uniqueness are verified in the pullback frames associated with the trivializations of Proposition 2.7.2; the local connection forms are the forms of $\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})}$ composed with π.

For

$$\mathbf{a} \in \Gamma(\pi^*\mathrm{Hom}(\mathbf{E},\mathbf{F})), \quad X \in T_xM, \quad \eta \in T_x^*M, \quad \xi \in T_x^*M,$$

define, at the point (x,ξ),

$$\nabla^{\mathrm{H}}_X\mathbf{a} := (\pi^*\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})})_{X^{\mathrm{H}}_{(x,\xi)}}\mathbf{a}, \qquad \nabla^{\mathrm{V}}_\eta\mathbf{a} := (\pi^*\nabla^{\mathrm{Hom}(\mathbf{E},\mathbf{F})})_{\eta^{\mathrm{V}}_{(x,\xi)}}\mathbf{a}. \tag{2.67}$$

These are, respectively, the **horizontal covariant derivative** and the **vertical covariant derivative** of $\mathbf{a}$.

Proposition 2.7.28. (Local formulas in the cotangent bundle) Let M be a smooth manifold with or without boundary. In the preceding coordinates, choose local frames of $\mathbf{E}$ and $\mathbf{F}$, and denote their connection matrices by $\omega_i^{\mathbf{E}}$ and $\omega_i^{\mathbf{F}}$. If $a(x,\xi)$ is the local matrix of a section of $\pi^*\operatorname{Hom}(\mathbf{E},\mathbf{F})$, then

$$\nabla_j^{\mathrm{V}} a = \frac{\partial a}{\partial \xi_j}, \qquad \nabla_i^{\mathrm{H}} a = \frac{\partial a}{\partial x^i} + \Gamma_{ij}^k \xi_k \frac{\partial a}{\partial \xi_j} + \omega_i^{\mathbf{F}} a - a\omega_i^{\mathbf{E}}. \tag{2.68}$$

The two derivatives defined in (2.67) are independent of the coordinates and frames used to calculate these expressions.

Proof. The vertical vector ∂_{ξ_j} projects to zero. By (2.66), the pullback connection therefore contributes no connection term, leaving $\nabla_j^{\mathrm{V}} a = \frac{\partial a}{\partial \xi_j}$. For the horizontal direction, (2.65) differentiates the components of a by

$$\partial_i + \Gamma_{ij}^k \xi_k \frac{\partial}{\partial \xi_j}.$$

The connection on $\operatorname{Hom}(\mathbf{E},\mathbf{F})$, given by (2.13), adds exactly $\omega_i^{\mathbf{F}} a - a\omega_i^{\mathbf{E}}$. This proves (2.68). Finally, X^{H} and η^{V} were defined as intrinsic lifts, and the pullback connection is also intrinsic. Thus a change of coordinates or frames merely transforms the matrices in the formula according to the laws for these objects and does not alter the resulting derivatives. □

2.8 Geodesics

Straight lines in Euclidean space have several equivalent characterizations. On the one hand, they are the curves that locally realize the shortest length between sufficiently close points. On the other, they can be described as curves whose velocity remains constant, in the sense that their acceleration vanishes, that is, they satisfy $\gamma''(t) = 0$.

On a Riemannian manifold, it is natural to seek an analogue of these curves. The Euclidean notion of a straight line no longer makes sense directly, but the metric structure allows us to measure lengths and formulate the problem of finding curves that connect two points with minimal length. These curves are known as *geodesics*.

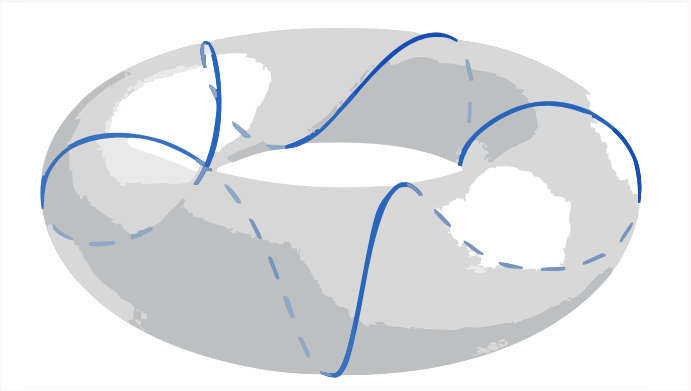


Figure 2.2. Geodesics on the torus. They may close up periodically or fill the torus densely.

However, to obtain an intrinsic characterization analogous to the condition $\gamma''(t) = 0$ in $\mathbb{R}^n$, we cannot simply

differentiate the curve as though all its velocities belonged to the same vector space. If $\gamma\colon I \longrightarrow M$ is a smooth curve, its velocity $\dot\gamma(t)$ belongs to the tangent space $T_{\gamma(t)}M$, which depends on the point $\gamma(t)$. Thus it makes no sense to compare $\dot\gamma(t)$ directly with $\dot\gamma(t+h)$ by ordinary subtraction.

The preceding section resolves precisely this difficulty: a connection on TM induces a covariant derivative on sections of γ^*TM. Since the velocity $\dot\gamma$ is a section of γ^*TM, we may consider its covariant derivative $\frac{D\dot\gamma}{dt}$. This allows us to formulate the condition of vanishing covariant acceleration and thereby define the geodesics associated with a connection.

Definition 2.8.1. Let M be a smooth manifold with or without boundary and let ∇ be a connection on TM. We say that a smooth curve $\gamma\colon I \longrightarrow M$ is a **geodesic** if

$$\frac{D\dot\gamma}{dt} \equiv 0.$$

Equivalently, if (U,φ) is a smooth chart such that $\gamma(I)\subset U$ and

$$x(t) = (\varphi\circ\gamma)(t) = (x^1(t),\dots,x^n(t)),$$

then γ is a geodesic if and only if it satisfies the geodesic equations

$$\frac{d^2x^k}{dt^2}(t) + \frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t)\Gamma^k_{ij}(x(t)) = 0. \tag{2.69}$$

We have defined geodesics, which in coordinates are described by a system of second-order ordinary differential equations. Before studying them in greater depth, we must ask whether such curves actually exist for every initial condition. Geodesics exist locally, as guaranteed by the fundamental theorem on flows. To see this geometrically, we first introduce the *geodesic vector field*.

Proposition 2.8.2. (Geodesic vector field) Let M be a smooth manifold without boundary and let ∇ be a connection on TM. Then there exists a unique smooth vector field $\mathbf{G}\in\mathfrak{X}(TM)$ with the following property: for every smooth curve

$$\zeta\colon I\longrightarrow TM, \qquad \zeta(t) = \big(\gamma(t),\mathbf{V}(t)\big),$$

ζ is an integral curve of $\mathbf{G}$ if and only if

$$\mathbf{V}(t) = \dot\gamma(t) \quad\text{and}\quad \frac{D\mathbf{V}}{dt}(t) = 0$$

for every $t\in I$.
Moreover, if (U,φ) is a smooth chart of M and $(x^1,\dots,x^n,v^1,\dots,v^n)$ are the induced coordinates on $\pi^{-1}(U)\subset TM$, then $\mathbf{G}$ is given by

$$\mathbf{G} = v^k\partial_{x^k} - v^iv^j\Gamma^k_{ij}(x)\partial_{v^k}.$$

In particular, if $\zeta\colon I\longrightarrow TM$ is given by $\zeta(t) = \big(\gamma(t),\dot\gamma(t)\big)$, then ζ is an integral curve of $\mathbf{G}$ if and only if γ is a geodesic.

Proof. Let (U,φ) be a smooth chart of M. On $\pi^{-1}(U)$, consider the vector field given, in the induced coor-

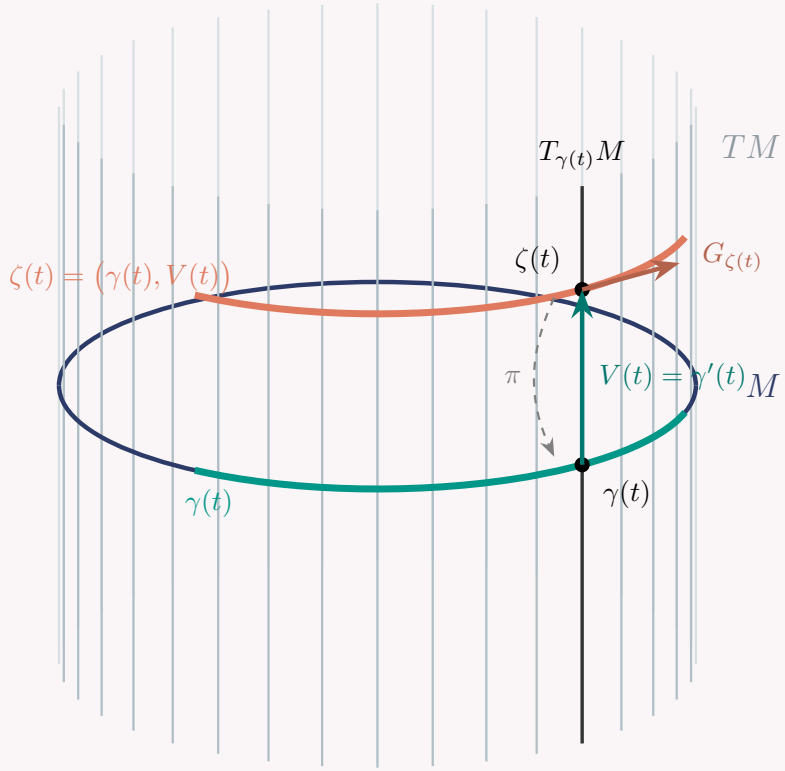


Figure 2.3. Schematic representation of the geodesic vector field on TM. The curve $\zeta(t) = (\gamma(t), \mathbf{V}(t))$ is an integral curve of $\mathbf{G}$ if and only if $\mathbf{V}(t) = \dot\gamma(t)$ and $\frac{D\mathbf{V}}{dt}(t) = 0$; in particular, when $\mathbf{V}(t) = \dot\gamma(t)$, the integral curves of $\mathbf{G}$ correspond to the geodesics of M.

dinates $(x^1,\dots,x^n,v^1,\dots,v^n)$, by

$$\mathbf{G}_U = v^k \partial_{x^k} - v^i v^j \Gamma_{ij}^k(x)\partial_{v^k}.$$

Let us first determine which curves are integral curves of $\mathbf{G}_U$. Let $\zeta\colon I \longrightarrow \pi^{-1}(U)$ be a smooth curve and write $\zeta(t) = \big(\gamma(t), \mathbf{V}(t)\big)$, where $\gamma = \pi\circ\zeta$ and $\mathbf{V}(t) \in T_{\gamma(t)}M$ for every $t \in I$. Let $\widetilde{\varphi}\colon \pi^{-1}(U) \longrightarrow \varphi(U)\times\mathbb{R}^n$ be the diffeomorphism defining the induced coordinates on TM. Write the coordinate representation of ζ as $\widetilde{\varphi}\circ\zeta = (\xi^1,\dots,\xi^n,\eta^1,\dots,\eta^n)\colon I \longrightarrow \varphi(U)\times\mathbb{R}^n$. Then $\varphi\circ\gamma = (\xi^1,\dots,\xi^n)$ and $\mathbf{V}(t) = \eta^k(t)\partial_k|_{\gamma(t)}$.
By the definition of an integral curve, ζ is an integral curve of $\mathbf{G}_U$ if and only if

$$\frac{d\xi^k}{dt}(t) = \eta^k(t), \qquad \frac{d\eta^k}{dt}(t) = -\Gamma_{ij}^k(\xi(t))\eta^i(t)\eta^j(t)$$

for every $t \in I$.
The first equation is equivalent to $\mathbf{V}(t) = \dot\gamma(t)$. Since $\varphi\circ\gamma = (\xi^1,\dots,\xi^n)$, we have $\dot\gamma(t) = \frac{d\xi^k}{dt}(t)\partial_k|_{\gamma(t)}$, so $\dot\gamma(t) = \mathbf{V}(t)$ if and only if $\frac{d\xi^k}{dt}(t) = \eta^k(t)$ for every k.

On the other hand, by Proposition 2.7.7,

$$\frac{D\mathbf{V}}{dt}(t) = \left(\frac{d\eta^k}{dt}(t) + \frac{d\xi^i}{dt}(t)\eta^j(t)\Gamma_{ij}^k(\xi(t)) \right) \partial_k\big|_{\gamma(t)}.$$

If, in addition, $\mathbf{V}(t) = \dot{\gamma}(t)$, then $\frac{d\xi^i}{dt}(t) = \eta^i(t)$, and consequently $\frac{D\mathbf{V}}{dt}(t) = 0$ is equivalent to

$$\frac{d\eta^k}{dt}(t) = -\Gamma_{ij}^k(\xi(t))\eta^i(t)\eta^j(t).$$

We have therefore proved that ζ is an integral curve of $\mathbf{G}_U$ if and only if $\mathbf{V}(t) = \dot{\gamma}(t)$ and $\frac{D\mathbf{V}}{dt}(t) = 0$ for every $t \in I$.

We now prove that these local vector fields agree on overlaps. Let (U, φ) and $(\widetilde{U}, \widetilde{\varphi})$ be two smooth charts with $U \cap \widetilde{U} \neq \varnothing$, and let $F = \widetilde{\varphi} \circ \varphi^{-1}$. If $(x^1, \dots, x^n, v^1, \dots, v^n)$ and $(\widetilde{x}^1, \dots, \widetilde{x}^n, \widetilde{v}^1, \dots, \widetilde{v}^n)$ denote the corresponding induced coordinates, then $\widetilde{x}^a = F^a(x)$ and $\widetilde{v}^a = \frac{\partial F^a}{\partial x^i}(x)v^i$.

Let ζ be an integral curve of $\mathbf{G}_U$ contained in $\pi^{-1}(U \cap \widetilde{U})$. If we write its coordinate representation in the system (x, v) as $\widetilde{\varphi}_U \circ \zeta = (\xi^1, \dots, \xi^n, \eta^1, \dots, \eta^n)$, then

$$\frac{d\xi^k}{dt}(t) = \eta^k(t), \qquad \frac{d\eta^k}{dt}(t) = -\Gamma_{ij}^k(\xi(t))\eta^i(t)\eta^j(t).$$

The components of the same curve in the system $(\widetilde{x}, \widetilde{v})$ are $\widetilde{\xi}^a(t) = F^a(\xi(t))$ and $\widetilde{\eta}^a(t) = \frac{\partial F^a}{\partial x^i}(\xi(t))\eta^i(t)$. By the chain rule,

$$\frac{d\widetilde{\xi}^a}{dt}(t) = \widetilde{\eta}^a(t).$$

Moreover,

$$\frac{d\widetilde{\eta}^a}{dt}(t) = \frac{\partial^2 F^a}{\partial x^i \partial x^j}(\xi(t))\eta^i(t)\eta^j(t) + \frac{\partial F^a}{\partial x^k}(\xi(t))\frac{d\eta^k}{dt}(t).$$

Substituting the equation for $\dfrac{d\eta^k}{dt}$ gives

$$\frac{d\widetilde{\eta}^a}{dt}(t) = \left(\frac{\partial^2 F^a}{\partial x^i \partial x^j} - \frac{\partial F^a}{\partial x^k}\Gamma_{ij}^k \right)(\xi(t))\eta^i(t)\eta^j(t).$$

By the transformation law for the Christoffel symbols (Lemma 2.4.12),

$$\widetilde{\Gamma}_{bc}^a(F(x))\frac{\partial F^b}{\partial x^i}(x)\frac{\partial F^c}{\partial x^j}(x) = \frac{\partial F^a}{\partial x^k}(x)\Gamma_{ij}^k(x) - \frac{\partial^2 F^a}{\partial x^i \partial x^j}(x).$$

Applying this identity in $x = \xi(t)$, we conclude that

$$\frac{d\widetilde{\eta}^a}{dt}(t) = -\widetilde{\Gamma}_{bc}^a(\widetilde{\xi}(t))\widetilde{\eta}^b(t)\widetilde{\eta}^c(t).$$

This shows that, in the induced coordinates associated with $(\widetilde{U}, \widetilde{\varphi})$, the curve ζ satisfies the differential system corresponding to the vector field $\mathbf{G}_{\widetilde{U}}$. Therefore, the local vector fields $\mathbf{G}_U$ and $\mathbf{G}_{\widetilde{U}}$ agree on overlaps and define a global smooth vector field $\mathbf{G} \in \mathfrak{X}(TM)$.

Let us now verify that this global vector field has the stated property. Let $\zeta\colon I \longrightarrow TM$ be a smooth curve

and write $\zeta(t) = \big(\gamma(t), \mathbf{V}(t)\big)$. First suppose that ζ is an integral curve of $\mathbf{G}$. For fixed $t_0 \in I$, there exist a chart (U, φ) of M and an open interval $J \subseteq I$ with $t_0 \in J$ such that $\zeta(J) \subseteq \pi^{-1}(U)$. Then $\zeta|_J$ is an integral curve of $\mathbf{G}_U$ and, by what we have proved, $\mathbf{V}(t) = \dot\gamma(t)$ and $\frac{D\mathbf{V}}{dt}(t) = 0$ for every $t \in J$. Since t_0 is arbitrary, we conclude that these equalities hold for every $t \in I$.
Conversely, suppose that $\mathbf{V}(t) = \dot\gamma(t)$ and $\frac{D\mathbf{V}}{dt}(t) = 0$ for every $t \in I$. Given $t_0 \in I$, choose again a chart (U, φ) and an open interval $J \subseteq I$ with $t_0 \in J$ such that $\zeta(J) \subseteq \pi^{-1}(U)$. By the local result already proved, the restriction $\zeta|_J$ is an integral curve of $\mathbf{G}_U$ and hence of $\mathbf{G}$. Since this holds for every $t_0 \in I$, it follows that ζ is an integral curve of $\mathbf{G}$ on all of I.
This proves the characterizing property of $\mathbf{G}$.
For uniqueness, let $\mathbf{H} \in \mathfrak{X}(TM)$ be another smooth vector field with the same property. Let (U, φ) be a smooth chart of M and let $q \in \pi^{-1}(U)$. Consider the integral curve ζ of $\mathbf{G}$ such that $\zeta(0) = q$. By the characterizing property, ζ satisfies $\mathbf{V}(t) = \dot\gamma(t)$ and $\frac{D\mathbf{V}}{dt}(t) = 0$ and is therefore also an integral curve of $\mathbf{H}$. Consequently, $\mathbf{G}_q = \zeta'(0) = \mathbf{H}_q$. Since q is arbitrary, we conclude that $\mathbf{G} = \mathbf{H}$.
The last assertion follows immediately: if $\zeta(t) = \big(\gamma(t), \dot\gamma(t)\big)$, then the condition $\mathbf{V}(t) = \dot\gamma(t)$ and $\frac{D\mathbf{V}}{dt}(t) = 0$ is equivalent to $\frac{D\dot\gamma}{dt}(t) = 0$, which is precisely the geodesic equation. □

A geodesic $\gamma\colon I \longrightarrow M$ that cannot be extended to a geodesic on a larger interval is called a ***maximal geodesic***. A ***geodesic segment*** is a geodesic whose domain is a compact interval. The existence of maximal geodesics is established in the following theorem:

Theorem 2.8.3. (Existence, uniqueness, and maximality of geodesics) Let M be a smooth manifold without boundary and let ∇ be a connection on TM. For each $p \in M$, $v \in T_pM$, and $t_0 \in \mathbb{R}$, there exists a unique maximal geodesic

$$\gamma_{(p,v,t_0)}\colon I_{(p,v,t_0)} \longrightarrow M$$

such that $t_0 \in I_{(p,v,t_0)}$ and $\gamma_{(p,v,t_0)}(t_0) = p$, $\dot\gamma_{(p,v,t_0)}(t_0) = v$.
Moreover, if $\gamma_1\colon I_1 \longrightarrow M$ and $\gamma_2\colon I_2 \longrightarrow M$ are geodesics such that $\gamma_1(t_0) = \gamma_2(t_0)$ and $\dot\gamma_1(t_0) = \dot\gamma_2(t_0)$, then $\gamma_1(t) = \gamma_2(t)$ for every $t \in I_1 \cap I_2$.

Proof. Let $\mathbf{G} \in \mathfrak{X}(TM)$ be the geodesic vector field given by Proposition 2.8.2. By Theorem 1.8.15, for each $(p, v) \in TM$ there exists a unique maximal integral curve $\zeta_{(p,v,t_0)}\colon I_{(p,v,t_0)} \longrightarrow TM$ such that $\zeta_{(p,v,t_0)}(t_0) = (p, v)$. Define $\gamma_{(p,v,t_0)} := \pi \circ \zeta_{(p,v,t_0)}$. Then $\gamma_{(p,v,t_0)}$ is a smooth curve in M and $\gamma_{(p,v,t_0)}(t_0) = \pi(\zeta_{(p,v,t_0)}(t_0)) = p$.
Write $\zeta_{(p,v,t_0)}(t) = \big(\gamma_{(p,v,t_0)}(t), \mathbf{V}(t)\big)$ for $t \in I_{(p,v,t_0)}$, where $\mathbf{V}(t) \in T_{\gamma_{(p,v,t_0)}(t)}M$. By Proposition 2.8.2, we have $\mathbf{V}(t) = \dot\gamma_{(p,v,t_0)}(t)$ for every $t \in I_{(p,v,t_0)}$, and moreover

$$\frac{D\dot\gamma_{(p,v,t_0)}}{dt}(t) = 0.$$

Therefore, $\gamma_{(p,v,t_0)}$ is a geodesic. Evaluating at $t = t_0$ gives $\mathbf{V}(t_0) = v$, that is, $\dot\gamma_{(p,v,t_0)}(t_0) = v$.
This proves existence.
We now prove uniqueness. Let $\widetilde\gamma\colon J \longrightarrow M$ be a geodesic such that $\widetilde\gamma(t_0) = p$ and $\dot{\widetilde\gamma}(t_0) = v$. Define $\widetilde\zeta\colon J \longrightarrow TM$ by $\widetilde\zeta(t) := \dot{\widetilde\gamma}(t)$. Since $\widetilde\gamma$ is a geodesic, Proposition 2.8.2 implies that $\widetilde\zeta$ is an integral curve of $\mathbf{G}$. Moreover, $\widetilde\zeta(t_0) = (p, v) = \zeta_{(p,v,t_0)}(t_0)$. Uniqueness of integral curves of the vector field $\mathbf{G}$ gives $\widetilde\zeta(t) = \zeta_{(p,v,t_0)}(t)$ for every $t \in J \cap I_{(p,v,t_0)}$. Applying the projection π gives $\widetilde\gamma(t) = \gamma_{(p,v,t_0)}(t)$ for every $t \in J \cap I_{(p,v,t_0)}$. This proves uniqueness.

Finally, let us verify maximality. Suppose that $\gamma_{(p,v,t_0)}$ is not maximal. Then there exists a geodesic $\overline{\gamma}\colon \widetilde{I} \longrightarrow M$ such that $I_{(p,v,t_0)} \subsetneq \widetilde{I}$ and $\overline{\gamma}(t) = \gamma_{(p,v,t_0)}(t)$ for every $t \in I_{(p,v,t_0)}$. Define $\overline{\zeta}\colon \widetilde{I} \longrightarrow TM$ by $\overline{\zeta}(t) := \dot{\overline{\gamma}}(t)$. Since $\overline{\gamma}$ is a geodesic, Proposition 2.8.2 implies that $\overline{\zeta}$ is an integral curve of $\mathbf{G}$. Moreover, $\overline{\zeta}(t) = \dot{\overline{\gamma}}(t) = \dot{\gamma}_{(p,v,t_0)}(t) = \zeta_{(p,v,t_0)}(t)$ for every $t \in I_{(p,v,t_0)}$. Therefore, $\overline{\zeta}$ is a proper extension of the integral curve $\zeta_{(p,v,t_0)}$, contradicting its maximality. We conclude that $\gamma_{(p,v,t_0)}$ is maximal. □

From now on, we denote this unique maximal geodesic by γ_v and assume $t_0 = 0$ to simplify the notation.

Remark 2.8.4. By Theorem 2.8.3, there exists a unique maximal flow Φ_t associated with the geodesic vector field $\mathbf{G}$, given by

$$\Phi_t(p,v) = \big(\gamma_v(t), \dot{\gamma}_v(t)\big).$$

By the definition of the vector field as the derivative of its flow, we have

$$\mathbf{G}_{(p,v)} = \frac{d}{dt}\Phi_t(p,v)\Big|_{t=0}.$$

Therefore, for every $f \in C^\infty(TM)$,

$$\mathbf{G}f(p,v) = df_{(p,v)}\big(\mathbf{G}_{(p,v)}\big) = \frac{d}{dt}\Big(f(\Phi_t(p,v))\Big)\Big|_{t=0}.$$

Substituting the expression for the flow gives

$$\mathbf{G}f(p,v) = \frac{d}{dt}\Big(f(\gamma_v(t), \dot{\gamma}_v(t))\Big)\Big|_{t=0}.$$

On the other hand, in induced coordinates $(x^1,\dots,x^n,v^1,\dots,v^n)$ on $\pi^{-1}(U)$, we have

$$\mathbf{G} = v^k\partial_k - v^i v^j \Gamma_{ij}^k(x)\partial_{v^k},$$

and consequently, for every $f \in C^\infty(\pi^{-1}(U))$,

$$\mathbf{G}f = v^k\frac{\partial f}{\partial x^k} - v^i v^j \Gamma_{ij}^k(x)\frac{\partial f}{\partial v^k}.$$

In particular, if $f \in C^\infty(M)$, then its lift $\pi^* f = f \circ \pi$ satisfies

$$\mathbf{G}(\pi^* f)(p,v) = \frac{d}{dt}\Big(f(\gamma_v(t))\Big)\Big|_{t=0} = df_p(v).$$

The exponential map compares a neighborhood of the origin in T_pM with a neighborhood of p in the manifold. Before constructing it, we should observe how a geodesic changes when its initial velocity is rescaled.

Lemma 2.8.5. (Rescaling lemma) Let M be a smooth manifold without boundary and let ∇ be a connection on TM. For every $p \in M$, $v \in T_pM$, and $c \in \mathbb{R}$, we have

$$\gamma_{cv}(t) = \gamma_v(ct)$$

for every t for which both sides are defined.

Proof. If $c = 0$, both curves are constant and equal to p, so the assertion is trivial. Suppose $c \neq 0$.
Let $I \subseteq \mathbb{R}$ be the maximal domain of γ_v and define

$$\widetilde{\gamma}_v \colon c^{-1}I \longrightarrow M, \qquad \widetilde{\gamma}_v(t) = \gamma_v(ct).$$

We will show that $\widetilde{\gamma}_v$ is a geodesic with initial point p and initial velocity cv. By uniqueness and maximality of γ_{cv}, it will follow that $\widetilde{\gamma}_v = \gamma_{cv}$.
Both curves start at p, since $\widetilde{\gamma}_v(0) = \gamma_v(0) = p$. Consider a chart (U, φ) and suppose that

$$(\varphi \circ \gamma_v)(t) = (x^1(t), \ldots, x^n(t)).$$

Then

$$(\varphi \circ \widetilde{\gamma}_v)(t) = (x^1(ct), \ldots, x^n(ct)).$$

By the chain rule,

$$\dot{\widetilde{\gamma}}_v^i(t) = \frac{d}{dt} x^i(ct) = c\, \dot{x}^i(ct).$$

In particular,

$$\dot{\widetilde{\gamma}}_v(0) = c\, \dot{\gamma}_v(0) = cv.$$

To see that $\widetilde{\gamma}_v$ is a geodesic, we compute its covariant derivative. Using the local expression,

$$\frac{D\dot{\widetilde{\gamma}}_v}{dt}(t) = \left(\frac{d}{dt} \dot{\widetilde{\gamma}}_v^k(t) + \Gamma_{ij}^k(\widetilde{\gamma}_v(t)) \dot{\widetilde{\gamma}}_v^i(t) \dot{\widetilde{\gamma}}_v^j(t) \right) \partial_k.$$

Substituting the preceding expression for $\dot{\widetilde{\gamma}}_v^i(t)$ gives

$$\frac{d}{dt} \dot{\widetilde{\gamma}}_v^k(t) = c^2 \ddot{x}^k(ct),$$

and

$$\dot{\widetilde{\gamma}}_v^i(t) \dot{\widetilde{\gamma}}_v^j(t) = c^2 \dot{x}^i(ct) \dot{x}^j(ct).$$

Therefore,

$$\frac{D\dot{\widetilde{\gamma}}_v}{dt}(t) = c^2 \left(\ddot{x}^k(ct) + \Gamma_{ij}^k(\gamma_v(ct)) \dot{x}^i(ct) \dot{x}^j(ct) \right) \partial_k.$$

Since γ_v is a geodesic, the term in parentheses is zero, and consequently

$$\frac{D\dot{\widetilde{\gamma}}_v}{dt}(t) = 0.$$

This shows that $\widetilde{\gamma}_v$ is a geodesic with initial conditions (p, cv), so $\widetilde{\gamma}_v = \gamma_{cv}$. □

2.8.1 The exponential map

The assignment $v \mapsto \gamma_v$ defines a map from TM to the set of geodesics of M. Moreover, by the rescaling lemma, we may define a function from a subset of TM to M that sends each line through the origin of T_pM to a geodesic:

Definition 2.8.6. Let M be a smooth manifold without boundary and let ∇ be a connection on TM. By Theorem 2.8.3, each $p \in M$ and $v \in T_pM$ determine a unique maximal geodesic γ_v, defining a map $v \mapsto \gamma_v$ from TM to the set of geodesics in M.
Let

$$\mathcal{E} = \{v \in TM \mid \gamma_v \text{ is defined on an interval containing } [0,1]\}.$$

Define the **exponential map** $\exp\colon \mathcal{E} \longrightarrow M$ by $\exp(v) = \gamma_v(1)$. For each $p \in M$, the **exponential map restricted to** p, denoted by $\exp_p$, is the restriction of the exponential map to the set $\mathcal{E}_p = \mathcal{E} \cap T_pM$.

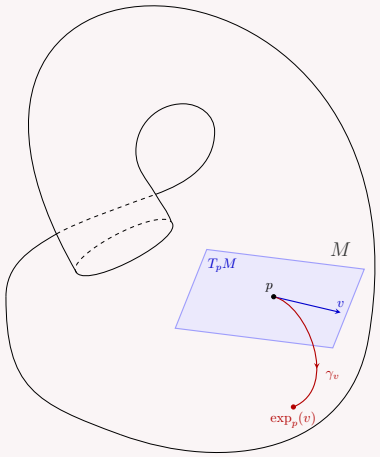


Figure 2.4. Exponential map on a Riemannian manifold.

From this point onward, when $(M, \mathbf{g})$ is a Riemannian manifold, geodesics and the exponential map are understood to be associated with the Levi–Civita connection of $\mathbf{g}$, unless otherwise stated. The preceding results formulated for an arbitrary connection retain their original meaning.

The following proposition collects the basic properties of the exponential map that will be used to construct normal coordinates and study the local geometry.

Proposition 2.8.7. (Properties of the exponential map) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $\exp\colon \mathcal{E} \longrightarrow M$ be the exponential map of its Levi–Civita connection.

(a) $\mathcal{E}$ is an open subset of TM containing $0_p \in T_pM$ for every $p \in M$, and each $\mathcal{E}_p \subseteq T_pM$ is star-shaped with respect to 0.

(b) For each $v \in TM$, the geodesic γ_v is given by

$$\gamma_v(t) = \exp(tv)$$

for every t for which both sides are defined.

(c) exp is smooth.

(d) For each $p \in M$, the differential

$$d(\exp_p)_0 \colon T_0(T_pM) \cong T_pM \longrightarrow T_pM$$

is the identity under the usual identification of $T_0(T_pM)$ with T_pM.

Proof. By Lemma 2.8.5, for every $v \in TM$ and every $t \in \mathbb{R}$ for which both sides are defined, we have

$$\gamma_v(t) = \gamma_{tv}(1) = \exp(tv).$$

This proves (b).
Moreover, for each $p \in M$, the constant geodesic at p has initial velocity 0_p, so $0_p \in \mathcal{E}_p$. Now, if $v \in \mathcal{E}_p$, then γ_v is defined on $t = 1$. By Lemma 2.8.5, for every $t \in [0,1]$ we have

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t),$$

and in particular $tv \in \mathcal{E}_p$. This shows that $\mathcal{E}_p$ is star-shaped with respect to 0.
We now show that $\mathcal{E}$ is open and exp is smooth. Let $\mathbf{G}$ be the geodesic vector field and let

$$\Phi \colon \mathcal{D}(\Phi) \longrightarrow TM$$

be its maximal flow. By Theorem 1.8.15, the domain $\mathcal{D}(\Phi) \subseteq \mathbb{R} \times TM$ is open and the map Φ is smooth. By Remark 2.8.4, we have

$$\Phi_t(v) = \big(\gamma_v(t), \dot{\gamma}_v(t)\big)$$

whenever both sides are defined.
By the definition of $\mathcal{E}$,

$$\mathcal{E} = \{v \in TM \mid \gamma_v(1) \text{ is defined}\}.$$

Since the maximal domain of γ_v contains 0, it follows that $\gamma_v(1)$ is defined if and only if $(1, v) \in \mathcal{D}(\Phi)$. Therefore,

$$\mathcal{E} = \{v \in TM \mid (1, v) \in \mathcal{D}(\Phi)\}.$$

If we define $\iota \colon TM \longrightarrow \mathbb{R} \times TM$ by $\iota(v) = (1, v)$, then

$$\mathcal{E} = \iota^{-1}(\mathcal{D}(\Phi)).$$

Since ι is continuous and $\mathcal{D}(\Phi)$ is open, we conclude that $\mathcal{E}$ is open.
Moreover, for every $v \in \mathcal{E}$,

$$\exp(v) = \gamma_v(1) = \pi\big(\Phi_1(v)\big).$$

Consequently,

$$\exp = \pi \circ \Phi_1\big|_{\mathcal{E}},$$

and since Φ and π are smooth, it follows that exp is smooth. This proves (c) and also completes the proof of (a).

Finally, fix $p \in M$ and let $v \in T_pM$. Consider the curve $\alpha(t) = tv \in T_pM$. By (b),

$$(\exp_p \circ \alpha)(t) = \exp_p(tv) = \gamma_v(t)$$

for every t in an open interval around 0 on which the expression is defined. Differentiating at $t = 0$ gives

$$d(\exp_p)_0(v) = \frac{d}{dt}\Big(\exp_p(tv)\Big)\Big|_{t=0} = \dot{\gamma}_v(0) = v.$$

Under the usual identification of $T_0(T_pM)$ with T_pM, this shows that $d(\exp_p)_0$ is the identity. This proves (d). □

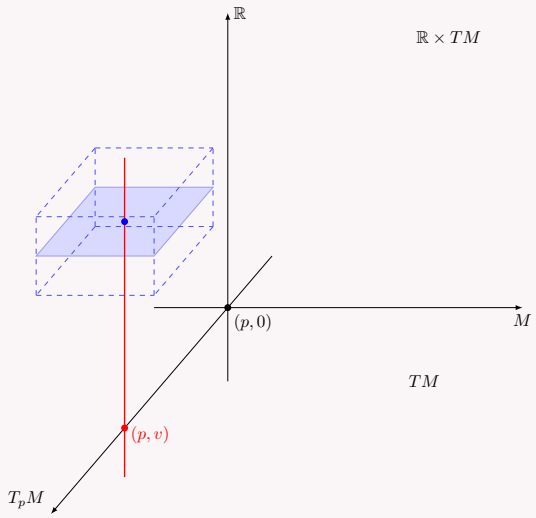


Figure 2.5. $\mathcal{E}$ is open.

2.8.2 Geodesic normal coordinates

Among the properties of the exponential map, we proved that $d(\exp_p)_0$ is invertible, so the inverse function theorem for manifolds (Theorem 1.5.6) guarantees the existence of open sets $V \subseteq T_pM$ and $U \subseteq M$ such that $0 \in V$, $p \in U$, and

$$\exp_p : V \longrightarrow U$$

is a diffeomorphism.

If V is star-shaped with respect to the origin, we say that U is a ***normal neighborhood of*** p. Every orthonormal basis $\{\beta_1, \ldots, \beta_n\}$ of T_pM determines a canonical isomorphism $B : \mathbb{R}^n \longrightarrow T_pM$ by $B(x^1, \ldots, x^n) = x^i \beta_i$. If $U = \exp_p(V)$ is a normal neighborhood of p, we may combine this isomorphism with the exponential map to obtain a smooth chart (U, φ) of M containing p, where $\varphi = B^{-1} \circ (\exp_p |_V)^{-1} : U \longrightarrow \mathbb{R}^n$. These coordinates are called ***normal coordinates centered at*** p. They are unique in the following sense:

Proposition 2.8.8. Let $(M,\mathbf{g})$ be a Riemannian manifold of dimension n without boundary, let $p \in M$, and let U be a normal neighborhood of p. For each normal coordinate chart (U,φ) centered at p, the coordinate basis $\{\partial_1|_p,\dots,\partial_n|_p\} \subseteq T_pM$ is orthonormal. Conversely, for each orthonormal basis $\{\beta_1,\dots,\beta_n\}$ of T_pM, there exists a unique normal coordinate chart (x^i) on U such that $\partial_i|_p = \beta_i$ for every $i \in \{1,\dots,n\}$. Moreover, any two normal coordinate charts (x^i) and $(\widetilde{x}^i)$ are related by

$$\widetilde{x}^j = A^j_i x^i$$

for some matrix $(A^j_i) \in O(n)$.

Proof. Let (U,φ) be a normal coordinate chart centered at p, with coordinate functions $\varphi = (x^1,\dots,x^n)$. By definition,

$$\varphi = B^{-1} \circ \exp_p^{-1},$$

where $B\colon \mathbb{R}^n \longrightarrow T_pM$ is an isomorphism determined by an orthonormal basis $\{b_1,\dots,b_n\}$ of T_pM. Then

$$\varphi^{-1} = \exp_p \circ B,$$

and therefore

$$d(\varphi^{-1})_0 = d(\exp_p)_0 \circ dB_0 = B,$$

since $d(\exp_p)_0 = \mathrm{Id}_{T_pM}$ and B is linear. Consequently,

$$\partial_i|_p = d(\varphi^{-1})_0(\partial_i|_0) = B(\partial_i|_0) = b_i,$$

which shows that the basis $\{\partial_i|_p\}$ is orthonormal.
Conversely, an orthonormal basis $\{\beta_1,\dots,\beta_n\}$ of T_pM determines an isomorphism $B\colon \mathbb{R}^n \longrightarrow T_pM$ and hence a normal coordinate chart $\varphi = B^{-1} \circ \exp_p^{-1}$ satisfying $\partial_i|_p = \beta_i$.
If $\widetilde{\varphi} = \widetilde{B}^{-1} \circ \exp_p^{-1}$ is another normal coordinate chart centered at p, then the change of coordinates is given by

$$\widetilde{\varphi} \circ \varphi^{-1} = \widetilde{B}^{-1} \circ B,$$

which is a linear isometry of $\mathbb{R}^n$. Therefore, there exists a constant matrix $(A^j_i) \in O(n)$ such that

$$\widetilde{x}^j = A^j_i x^i.$$

This proves uniqueness. □

Other properties of normal coordinates include the following:

Proposition 2.8.9. (Properties of normal coordinates) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let (U,φ) be a normal coordinate chart centered at $p \in M$.

(a) The coordinates of p are $(0,\dots,0)$.

(b) The metric components at p are $g_{ij}(p) = \delta_{ij}$.

(c) For $v = v^i\partial_i|_p \in T_pM$, the geodesic γ_v starting at p with $\dot\gamma(0) = v$ is represented in normal coordinates

by

$$(\varphi \circ \gamma_v)(t) = (tv^1, \dots, tv^n)$$

if $t \in I$, where I is an interval such that $0 \in I$ and $\gamma_v(I) \subseteq U$.

(d) The Christoffel symbols in these coordinates vanish at p.

(e) The first derivatives of the metric components vanish at p:

$$\partial_k g_{ij}(p) = 0 \quad \text{for all } i, j, k.$$

Proof. Part (a) follows directly from the definition of normal coordinates, since $\varphi(p) = 0$.
For (b), Proposition 2.8.8 shows that the coordinate basis $\{\partial_i|_p\}$ is orthonormal, giving $g_{ij}(p) = \delta_{ij}$.
Part (c) follows from Proposition 2.8.7(b): since $\gamma_v(t) = \exp_p(tv)$, composing with $\varphi = B^{-1} \circ \exp_p^{-1}$ gives

$$(\varphi \circ \gamma_v)(t) = B^{-1}(tv) = (tv^1, \dots, tv^n).$$

To prove (d), let $v = v^i \partial_i|_p \in T_pM$. By (c), writing $x^i(t) = (\varphi \circ \gamma_v)^i(t)$ gives $x^i(t) = tv^i$, so $\dot{x}^i(t) = v^i$ and $\ddot{x}^k(t) = 0$. Substituting into the equation for the geodesic (2.69), we obtain

$$\Gamma_{ij}^k(x(t))\, v^i v^j = 0.$$

Evaluating at $t = 0$, that is, at p, we conclude that

$$\Gamma_{ij}^k(p)\, v^i v^j = 0$$

for every k and every $v \in T_pM$.
The Levi–Civita connection is torsion-free; therefore, in a coordinate frame,

$$\Gamma_{ab}^k = \Gamma_{ba}^k.$$

Taking $v = \partial_a|_p$ gives $\Gamma_{aa}^k(p) = 0$. If $a \neq b$, taking $v = \partial_a|_p + \partial_b|_p$ gives

$$0 = \Gamma_{aa}^k(p) + \Gamma_{ab}^k(p) + \Gamma_{ba}^k(p) + \Gamma_{bb}^k(p) = 2\Gamma_{ab}^k(p).$$

Thus, $\Gamma_{ab}^k(p) = 0$ for all a, b, k, proving (d).
Finally, the Levi–Civita connection is compatible with $\mathbf{g}$. Corollary 2.4.16, applied to $\mathbf{E}_k = \partial_k$ and evaluated at p, gives

$$\partial_k g_{ij}(p) = \Gamma_{ki}^a(p) g_{aj}(p) + \Gamma_{kj}^a(p) g_{ia}(p) = 0,$$

by (d). This proves (e). □

Continuity of the metric allows us to make the identity $g_{ij}(p) = \delta_{ij}$ quantitative on a sufficiently small ball. The following result specifies the type of normal coordinates we will use in approximation arguments.

Proposition 2.8.10. (Normal coordinate balls with Euclidean control) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $p \in \mathrm{Int}(M)$, and let $0 < \theta < 1$. There exist numbers $0 < r < R$ and a linear

isometry $B_p\colon \mathbb{R}^n \longrightarrow (T_pM, \mathbf{g}_p)$ such that $\exp_p$ is a diffeomorphism from $B_{\mathbf{g}_p}(0,R)$ onto an open set $U_p' \subseteq \mathrm{Int}(M)$ and, setting

$$U_p := \exp_p\big(B_{\mathbf{g}_p}(0,r)\big), \qquad \kappa_p := B_p^{-1} \circ \exp_p^{-1}\colon U_p \longrightarrow B_{\mathrm{euc}}(0,r),$$

the pair (U_p, κ_p) is a regular normal coordinate ball centered at p, with $\overline{U}_p \subseteq U_p'$, and satisfies

$$(1-\theta)\|w\| \le \left|d(\kappa_p^{-1})_z w\right|_{\mathbf{g}} \le (1+\theta)\|w\|$$

for all $z \in B_{\mathrm{euc}}(0,r)$ and $w \in \mathbb{R}^n$.

Proof. Since $d(\exp_p)_0 = \mathrm{Id}_{T_pM}$, the inverse function theorem gives an open neighborhood $W \subseteq T_pM$ of 0 on which $\exp_p$ is a diffeomorphism. Since p is an interior point, we may shrink W so that its image is contained in $\mathrm{Int}(M)$. Choose $R > 0$ so that $\overline{B}_{\mathbf{g}_p}(0,R) \subseteq W$ and fix a linear isometry $B_p\colon \mathbb{R}^n \longrightarrow (T_pM, \mathbf{g}_p)$. The normal chart

$$\kappa_p' := B_p^{-1} \circ \exp_p^{-1} : U_p' := \exp_p\big(B_{\mathbf{g}_p}(0,R)\big) \longrightarrow B_{\mathrm{euc}}(0,R)$$

is centered at p. If $G(z)$ denotes the matrix of the metric in these coordinates, Proposition 2.8.9 implies $G(0) = I_n$. Since $z \mapsto G(z)$ is also continuous in the operator norm, there exists $0 < r < R$ such that

$$(1-\theta)^2\|w\|^2 \le \langle G(z)w, w\rangle_{\mathbb{R}^n} \le (1+\theta)^2\|w\|^2$$

for every $z \in \overline{B}_{\mathrm{euc}}(0,r)$ and every $w \in \mathbb{R}^n$. Indeed, it suffices to make $\|G(z) - I_n\|_{\mathrm{op}}$ smaller than the minimum of $1-(1-\theta)^2$ and $(1+\theta)^2 - 1$.

The restriction of κ_p' to $U_p := \exp_p(B_{\mathbf{g}_p}(0,r))$ is the chart κ_p in the statement. Moreover, $\overline{U}_p = \exp_p(\overline{B}_{\mathbf{g}_p}(0,r))$ is compact and contained in U_p', so U_p is a regular normal coordinate ball in the sense of Definition 1.2.9. Finally,

$$\left|d(\kappa_p^{-1})_z w\right|_{\mathbf{g}}^2 = \langle G(z)w, w\rangle_{\mathbb{R}^n},$$

and taking square roots gives the required inequalities. □

2.9 The Riemannian distance function

The metric first acts infinitesimally, assigning a length to each tangent vector. Integrating these lengths along curves gives a global notion of distance between points. This step is more than a formality: the resulting distance reflects both the local geometry and the ways in which curves can travel through the manifold.

Definition 2.9.1. Let M be a Riemannian manifold with or without boundary.

(a) A **curve segment** is a smooth curve whose domain is a compact interval.

(b) A smooth curve $\gamma\colon I \longrightarrow M$ is called a **regular curve** if $\gamma'(t) \neq 0$ for every $t \in I$ (which implies that γ is an immersion).

(c) A continuous curve segment $\gamma\colon [a,b] \longrightarrow M$ is said to be **piecewise regular** if there exists a partition $\{t_0, \dots, t_k\}$ of $[a,b]$ such that $\gamma|_{[t_{i-1},t_i]}$ is a regular (smooth) curve segment for each $i \in \{1,\dots,k\}$. We also

call these **admissible curves**, and partitions satisfying this condition are called **admissible partitions for** γ. For convenience, we regard a curve $\gamma\colon \{a\} \longrightarrow M$ as an admissible curve.

(d) If $\gamma\colon I \longrightarrow M$ is a smooth curve, a **reparametrization of** γ is a curve of the form $\widetilde{\gamma} = \gamma \circ \varphi\colon I' \longrightarrow M$, where I' is an interval and $\varphi\colon I' \longrightarrow I$ is a diffeomorphism. If φ is strictly increasing, we say that $\widetilde{\gamma}$ is a **forward parametrization** or an **orientation-preserving parametrization**. If φ is strictly decreasing, we say that $\widetilde{\gamma}$ is a **backward parametrization** or an **orientation-reversing parametrization**.

(e) For an admissible curve $\gamma\colon [a,b] \longrightarrow M$, a **reparametrization of** γ is a curve of the form $\widetilde{\gamma} = \gamma \circ \varphi$, where $\varphi\colon [c,d] \longrightarrow [a,b]$ is a homeomorphism for which there exists a partition $\{c_0,\dots,c_k\}$ of $[c,d]$ such that the restriction $\varphi|_{[c_{i-1},c_i]}$ is a diffeomorphism onto its image for each $i \in \{1,\dots,k\}$.

(f) If $\gamma\colon [a,b] \longrightarrow M$ is an admissible curve, we define the **length of** γ to be

$$L_{\mathbf{g}}(\gamma) := \int_a^b \|\gamma'(t)\|_{\mathbf{g}}\, dt.$$

We may define the distance between two points on a connected Riemannian manifold as follows:

Definition 2.9.2. Let $(M,\mathbf{g})$ be a connected Riemannian manifold with or without boundary. For each pair of points $p,q \in M$, define the **Riemannian distance from** p **to** q, denoted by $d_{\mathbf{g}}(p,q)$, to be

$$d_{\mathbf{g}}(p,q) := \inf\{L_{\mathbf{g}}(\gamma) \mid \gamma \text{ is an admissible curve from } p \text{ to } q\}.$$

The infimum is finite on each connected component by the following result.

Proposition 2.9.3. If M is a connected smooth manifold (with or without boundary), then any two points of M can be joined by an admissible curve.

Proof. Fix $p \in M$ and let A be the set of points that can be joined to p by an admissible curve. The set A is nonempty. If $q \in A$, choose a chart around q whose image contains a Euclidean ball, or a relative ball in $\mathbb{H}^n$, centered at the image of q. After shrinking it, this set is convex. Straight segments in the chart, transported to M, join q to all the points in a neighborhood of it; concatenating them with a curve from p to q gives an admissible curve. Therefore, A is open. The same argument shows that $M \setminus A$ is open: if one of its points had a point of A in this neighborhood, the local segment would also join it to p. Since M is connected, it follows that $A = M$. □

Every Riemannian manifold is metrizable by its Riemannian distance.

Theorem 2.9.4. Let $(M,\mathbf{g})$ be a connected Riemannian manifold with or without boundary. With the distance function $d_{\mathbf{g}}$, M is a metric space whose metric topology agrees with its original topology as a smooth manifold.

Proof. The preceding proposition shows that $d_{\mathbf{g}}(p,q)$ is finite. Nonnegativity is immediate. Reversing the parameter of a curve proves that $d_{\mathbf{g}}(p,q) = d_{\mathbf{g}}(q,p)$, and concatenating admissible curves, then taking infima, gives

$$d_{\mathbf{g}}(p,r) \le d_{\mathbf{g}}(p,q) + d_{\mathbf{g}}(q,r).$$

It remains to prove that distinct points have positive distance and to compare the topologies. Fix $p \in M$. In a chart (U, x) around p, choose $r > 0$ so that the relative ball

$$Q := B_{\mathrm{euc}}(x(p), 2r) \cap \mathbb{H}^n$$

is contained in $x(U)$ and has closure contained in $x(U)$; in an interior chart, omit the intersection with $\mathbb{H}^n$. Continuity and positive definiteness of the metric give constants $0 < c \le C$ such that

$$c|v| \le \left| d(x^{-1})_z v \right|_{\mathbf{g}} \le C|v| \tag{2.70}$$

for $z \in \overline{Q}$ and $v \in \mathbb{R}^n$ tangent to the half-space. Set

$$V := x^{-1}\big(B_{\mathrm{euc}}(x(p), r) \cap \mathbb{H}^n\big).$$

If $q \in V$, the straight segment from $x(p)$ to $x(q)$ stays in the relative ball, and the second inequality in (2.70) shows that

$$d_{\mathbf{g}}(p, q) \le C|x(q) - x(p)|.$$

On the other hand, every admissible curve from p to a point outside V must first reach the coordinate sphere of radius r. On the portion before this first exit point, the first inequality in (2.70) gives a length of at least cr. If $q \in V$ and a curve from p to q does not leave V, the same inequality and the Euclidean inequality between length and displacement give a length of at least $c|x(q) - x(p)|$; if it leaves V, its length is at least cr and the same bound remains valid. Consequently,

$$c|x(q) - x(p)| \le d_{\mathbf{g}}(p, q) \le C|x(q) - x(p)|, \qquad q \in V. \tag{2.71}$$

These bounds first prove that $d_{\mathbf{g}}(p, q) > 0$ when $p \neq q$: if $q \notin V$, use the bound cr, and if $q \in V$, use the first inequality in (2.71). Therefore, $d_{\mathbf{g}}$ is a metric. The second inequality shows that convergence in the manifold topology implies convergence with respect to $d_{\mathbf{g}}$ near each point; the first, together with the barrier cr for leaving V, gives the converse. The two topologies agree. □

Lemma 2.9.5. (Gauss lemma) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, let $p \in M$, and let $v, w \in T_pM$ be such that $\exp_p$ is defined on a neighborhood of v. Then

$$\mathbf{g}_{\exp_p(v)}\big(d(\exp_p)_v(v), d(\exp_p)_v(w)\big) = \mathbf{g}_p(v, w).$$

Proof. For small s and $0 \le t \le 1$, consider the variation

$$\alpha(s, t) := \exp_p\big(t(v + sw)\big).$$

Along the curve $t \mapsto \alpha(0, t)$, set $\mathbf{T} = \partial_t \alpha$ and $\mathbf{J} = \partial_s \alpha|_{s=0}$. The Levi–Civita connection is torsion-free, so $D\mathbf{J}/dt = D\mathbf{T}/ds$, and each curve $t \mapsto \alpha(s, t)$ is a geodesic. Since this connection is also compatible with $\mathbf{g}$, we

can differentiate inner products along the variation. Therefore,

$$\begin{aligned}\frac{d}{dt}\langle \mathbf{J}, \mathbf{T}\rangle &= \left\langle \frac{D\mathbf{J}}{dt}, \mathbf{T}\right\rangle + \left\langle \mathbf{J}, \frac{D\mathbf{T}}{dt}\right\rangle \\ &= \left\langle \frac{D\mathbf{T}}{ds}, \mathbf{T}\right\rangle = \frac{1}{2}\frac{\partial}{\partial s}\Big|_{s=0} |\partial_t \alpha(s,t)|_{\mathbf{g}}^2.\end{aligned}$$

For each s, metric compatibility and the geodesic equation give

$$\frac{d}{dt}|\partial_t \alpha(s,t)|_{\mathbf{g}}^2 = 2\left\langle \frac{D}{dt}\partial_t \alpha(s,t), \partial_t \alpha(s,t)\right\rangle_{\mathbf{g}} = 0.$$

The speed of $t \mapsto \alpha(s,t)$ is therefore constant and equal to $|v + sw|_{\mathbf{g}}$. Thus the last member of the preceding equality is $\langle v, w\rangle_{\mathbf{g}}$. Since $\mathbf{J}(0) = 0$, integration between 0 and 1 gives

$$\langle \mathbf{J}(1), \mathbf{T}(1)\rangle = \langle v, w\rangle_{\mathbf{g}}.$$

Finally, $\mathbf{J}(1) = d(\exp_p)_v(w)$ and $\mathbf{T}(1) = d(\exp_p)_v(v)$. □

Proposition 2.9.6. (Radial minimization in a normal neighborhood) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $p \in M$. There exists $\varepsilon > 0$ such that, for every $v \in B_{T_pM}(0, \varepsilon)$, the radial geodesic

$$\gamma_v(t) = \exp_p(tv), \qquad 0 \le t \le 1,$$

realizes the distance between p and $\exp_p(v)$. Up to monotone reparametrizations, it is the unique minimizing curve between these points that stays in the normal neighborhood.

Proof. Choose $\varepsilon > 0$ so that $\exp_p$ is a diffeomorphism on the tangent ball of radius ε. In the corresponding normal neighborhood, define

$$\rho(\exp_p z) := |z|_{\mathbf{g}}.$$

The Gauss lemma shows that the radial unit vector is the gradient of ρ. Indeed, if $z \neq 0$, decompose any $w \in T_pM$ as $w = az + w^{\perp}$, and the lemma gives

$$d\rho_{\exp_p z}\big(d(\exp_p)_z w\big) = \left\langle d(\exp_p)_z w, \frac{d(\exp_p)_z z}{|z|_{\mathbf{g}}}\right\rangle = a|z|_{\mathbf{g}}.$$

In particular, $|\operatorname{grad}\rho|_{\mathbf{g}} = 1$. For every admissible curve c contained in the normal neighborhood,

$$\left|\frac{d}{dt}(\rho \circ c)\right| \le |\dot{c}|_{\mathbf{g}}$$

on each smooth portion. If it joins p to $\exp_p(v)$, integration gives $L_{\mathbf{g}}(c) \ge |v|_{\mathbf{g}}$. The radial curve has constant speed $|v|_{\mathbf{g}}$ and attains equality.

If a curve c leaves the normal neighborhood before reaching $\exp_p(v)$, let t_* be its first exit time. For $t < t_*$, we may write $c(t) = \exp_p z(t)$, with $|z(t)|_{\mathbf{g}} = \rho(c(t)) < \varepsilon$. As t approaches t_*, necessarily $|z(t)|_{\mathbf{g}} \to \varepsilon$: otherwise, a subsequence of $z(t)$ would converge to a point in the interior of the tangent ball, and $c(t_*)$ would still belong to the normal neighborhood. Consequently, for each ε' with $|v|_{\mathbf{g}} < \varepsilon' < \varepsilon$, continuity gives a first time before

t_* at which $\rho = \varepsilon'$. The preceding inequality, applied up to this time, shows that the length is already at least $\varepsilon' > |v|_{\mathbf{g}}$. This proves minimization among all curves.
Now suppose equality is attained. In the chain

$$L_{\mathbf{g}}(c) = \int |\dot{c}| \geq \int |(\rho \circ c)'| \geq \left| \int (\rho \circ c)' \right| = |v|_{\mathbf{g}}$$

both inequalities are equalities. Hence, $|(\rho \circ c)'| = |\dot{c}|$ and $(\rho \circ c)' \geq 0$ almost everywhere. Equality in Cauchy–Schwarz implies that $\dot{c}$ is a nonnegative multiple of the radial gradient; it has no component tangent to the spheres $\rho = \text{const}$. Therefore, up to a monotone choice of parameter, the image of c is the radial geodesic. □

Corollary 2.9.7. Every minimizing admissible curve parametrized with constant speed is a geodesic.

Proof. Every subarc of a minimizing curve is again minimizing; otherwise, replacing it by a shorter curve would reduce the total length. Fix an interior parameter t_0 and choose a normal neighborhood centered at $c(t_0)$. For sufficiently small $\delta > 0$, the subarcs $c|_{[t_0, t_0+\delta]}$ and $c|_{[t_0-\delta, t_0]}$ stay in this neighborhood. The preceding proposition shows that both are radial segments, up to monotone reparametrization. Since c has constant speed, these reparametrizations are affine.
The entire subarc $c|_{[t_0-\delta, t_0+\delta]}$ is also minimizing. If the two radial segments formed a corner at $c(t_0)$, their concatenation could not be the unique minimizing radial curve between $c(t_0 - \delta)$ and $c(t_0 + \delta)$ in a sufficiently small normal neighborhood of the first of these points. Therefore, the one-sided velocities agree. Uniqueness for the geodesic equation shows that both segments are restrictions of a single geodesic. Since t_0 was arbitrary, c is a geodesic on its entire domain. □

Riemannian distance and geodesics are closely related: curves realizing the distance between their endpoints must satisfy the geodesic equation. The following notion allows us to formulate this global property.

Definition 2.9.8. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary. An admissible curve $\gamma : [a, b] \longrightarrow M$ in M is called a **minimizing curve** if $L_{\mathbf{g}}(\gamma) \leq L_{\mathbf{g}}(\widetilde{\gamma})$ for every admissible curve $\widetilde{\gamma}$ with the same endpoints ($\gamma(a) = \widetilde{\gamma}(a)$ and $\gamma(b) = \widetilde{\gamma}(b)$).

Recall that a Riemannian manifold is *geodesically complete* if every maximal geodesic is defined on $\mathbb{R}$.

Theorem 2.9.9. (Hopf–Rinow) Let $(M, \mathbf{g})$ be a connected Riemannian manifold without boundary. The following are equivalent:

(a) $(M, d_{\mathbf{g}})$ is a complete metric space;

(b) every closed bounded subset of $(M, d_{\mathbf{g}})$ is compact;

(c) M is geodesically complete;

(d) for every $p \in M$, $\exp_p$ is defined on all of T_pM;

(e) there exists $p_0 \in M$ such that $\exp_{p_0}$ is defined on all of $T_{p_0}M$.

If any of these conditions holds, any two points of M are joined by a geodesic segment that realizes their distance.

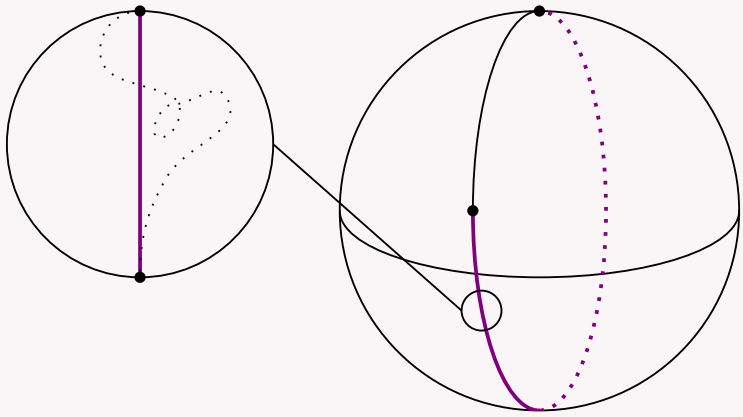


Figure 2.6. Although every minimizing curve is a geodesic, a geodesic minimizes distance only locally between sufficiently close points.

The proof follows Lee's strategy [95].

Proof. Conditions (c) and (d) are equivalent by the definition of $\exp_p(v) = \gamma_v(1)$ and affine changes of parameter for geodesics; moreover, (d) implies (e).
Assume (e), fix $q \in M$, and set $r = d_{\mathbf{g}}(p_0, q)$. If $r = 0$, there is nothing to prove. If $r > 0$, consider

$$A := \big\{ s \in [0,r] \;\big|\; \text{there exists } v_s \in T_{p_0}M, |v_s|_{\mathbf{g}} = 1, \text{ such that} \\ x_s := \exp_{p_0}(sv_s), \quad d_{\mathbf{g}}(p_0, x_s) = s, \quad d_{\mathbf{g}}(x_s, q) = r - s \big\}.$$

We have $0 \in A$. The set A is closed: if $s_j \to s$ and $s_j \in A$, the unit sphere of $T_{p_0}M$ is compact, so a subsequence of v_{s_j} converges to a unit vector v. Continuity of $\exp_{p_0}$ and of the distance preserves the three equalities defining A.
Let $s \in A$ with $s < r$ and write $x = x_s$. First choose a normal neighborhood V of x with compact closure. This is possible by local compactness and because normal neighborhoods form a basis at x. Since the Riemannian distance induces the manifold topology, there exists $\delta > 0$, smaller than $r - s$, such that $\overline{B}_{\mathbf{g}}(x, \delta) \subseteq V$. This ball is closed in M and contained in the compact set $\overline{V}$; therefore, it is compact. By the definition of distance, there exist curves c_j from x to q, parametrized by arc length, with

$$L_{\mathbf{g}}(c_j) \longrightarrow d_{\mathbf{g}}(x, q) = r - s.$$

Let $y_j = c_j(\delta)$. Then

$$d_{\mathbf{g}}(x, y_j) \le \delta, \qquad d_{\mathbf{g}}(y_j, q) \le L_{\mathbf{g}}(c_j) - \delta.$$

By compactness, we may assume that $y_j \to y$. Passing to the limit in the triangle inequality

$$r - s = d_{\mathbf{g}}(x, q) \le d_{\mathbf{g}}(x, y_j) + d_{\mathbf{g}}(y_j, q) \le d_{\mathbf{g}}(x, y_j) + L_{\mathbf{g}}(c_j) - \delta$$

gives

$$d_{\mathbf{g}}(x,y)=\delta, \qquad d_{\mathbf{g}}(y,q)=r-s-\delta.$$

Proposition 2.9.6 gives a minimizing geodesic from x to y. Concatenating it with the minimizing radial segment from p_0 to x gives a curve of length $s+\delta$. Moreover,

$$r \le d_{\mathbf{g}}(p_0,y)+d_{\mathbf{g}}(y,q) \le s+\delta+r-s-\delta=r,$$

so the concatenation is minimizing. If $s=0$, choose v_0 to be the initial unit velocity of the segment from $p_0=x$ to y; the preceding equalities give $\delta\in A$ directly. If $s>0$, Corollary 2.9.7 shows that it has no corner at x; by uniqueness for the geodesic equation, it is the extension of $t\mapsto \exp_{p_0}(tv_s)$. Therefore, $s+\delta\in A$.
Since A is closed, it contains its supremum. The preceding extension property prevents this supremum from being less than r, so $r\in A$. Thus there exists a vector $v\in T_{p_0}M$ with

$$\exp_{p_0}(v)=q, \qquad |v|_{\mathbf{g}}=r,$$

and the corresponding radial geodesic is minimizing.
Now let (q_j) be a Cauchy sequence. It is bounded, so $d_{\mathbf{g}}(p_0,q_j)\le R$ for some R. By the preceding argument, we may write $q_j=\exp_{p_0}(v_j)$ with $|v_j|_{\mathbf{g}}\le R$. The closed ball of radius R in the finite-dimensional space $T_{p_0}M$ is compact; a subsequence v_{j_ℓ} converges to v, and then $q_{j_\ell}\to\exp_{p_0}(v)$. Since the original sequence is Cauchy, the entire sequence converges to the same point. This proves $(e)\Rightarrow(a)$.
Now assume (a). If a maximal geodesic $\gamma\colon[0,b)\to M$ had $b<\infty$, its speed would be constant and

$$d_{\mathbf{g}}(\gamma(s),\gamma(t))\le L_{\mathbf{g}}(\gamma\restriction_{[s,t]})=|t-s|\,|\dot\gamma(0)|_{\mathbf{g}}.$$

Thus, $\gamma(t)$ would have a limit $p\in M$ as $t\to b^-$. Choose a chart around p and a smaller neighborhood whose compact closure is contained in that chart. A tail of γ stays in the smaller neighborhood. On its closure, the metric is uniformly equivalent to the Euclidean metric and the Christoffel symbols are bounded. If $x^i(t)$ are the coordinates of $\gamma(t)$ and $v^i(t)=\dot x^i(t)$, the constant Riemannian speed bounds all the components $v^i(t)$. The geodesic equation

$$\dot v^k(t)=-\Gamma^k_{ij}(x(t))v^i(t)v^j(t)$$

then bounds their derivatives. Hence, $v(t)$ is uniformly Lipschitz on this tail and has a limit v_b as $t\to b^-$. Extend $(x(t),v(t))$ continuously to the data $(x(p),v_b)$. The integral equation for the geodesic system passes to the limit at b; local existence and uniqueness for this system, with these initial data, extend the solution beyond b, a contradiction. The same argument at the left endpoint proves (c). We now also have (d); applying the radial argument of the preceding paragraph with any point as center gives a minimizing geodesic segment between two arbitrary points.
Under these conditions, for $p\in M$ and $R>0$, the existence of minimizers gives

$$\overline{B}_{\mathbf{g}}(p,R)=\exp_p\big(\overline{B}_{\mathbf{g}_p}(0,R)\big).$$

The right-hand side is the continuous image of a compact finite-dimensional set, so every closed ball is compact. Every closed bounded set is contained as a closed subset in one of these balls, giving (b). Finally, (b) implies (a): a Cauchy sequence is bounded, has a convergent subsequence in a compact closed ball, and,

being Cauchy, converges to the same limit. This proves all the equivalences and consequences. □

We record separately three consequences that will be used later:

Corollary 2.9.10. If $(M, \mathbf{g})$ is a connected Riemannian manifold without boundary and there exists $p \in M$ such that $\exp_p$ is defined on all of T_pM, then M is complete.

Proof. This is implication $(e) \Rightarrow (a)$ of Theorem 2.9.9, taking $p_0 = p$. □

Corollary 2.9.11. If M is a connected complete Riemannian manifold without boundary, then any two points of M can be joined by a minimizing geodesic segment.

Proof. Metric completeness is condition (a) of Hopf–Rinow; the last assertion of the theorem gives the minimizing segment. □

Corollary 2.9.12. If M is a connected complete Riemannian manifold without boundary, then every maximal geodesic in M is defined on all of $\mathbb{R}$.

Proof. By Hopf–Rinow, metric completeness is equivalent to geodesic completeness, which is precisely the assertion. □

2.10 Curvature

Curvature is the central notion in the study of Riemannian geometry. Intuitively, it measures the failure of covariant derivatives to commute: transporting a vector in different directions and then comparing the resulting vectors reveals the presence of curvature.

Definition 2.10.1. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. The $(1,3)$-***Riemann curvature tensor*** of M is the tensor field $\mathbf{R} \in \Gamma(T^{(1,3)}(TM))$ of type $(1,3)$ given by

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})(\mathbf{Z}) := \nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{Z} - \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}} \mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]} \mathbf{Z}.$$

Proposition 2.10.2. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. The $(1,3)$-Riemann curvature tensor $\mathbf{R}$ defines a tensor field of type $(1,3)$ on M.

Proof. By Lemma 1.10.18, it suffices to verify that $\mathbf{R}$ is $C^\infty(M)$-multilinear in each argument. Additivity in all three arguments follows from the linearity of the connection and the bracket; it remains to check compatibility with multiplication by a function. Take $f \in C^\infty(M)$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(M)$.
Let us verify this for the first argument:

$$\begin{aligned}
\mathbf{R}(f\mathbf{X}, \mathbf{Y})\mathbf{Z} &= \nabla_{f\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{Z} - \nabla_{\mathbf{Y}} \nabla_{f\mathbf{X}} \mathbf{Z} - \nabla_{[f\mathbf{X},\mathbf{Y}]} \mathbf{Z} \\
&= f \nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{Z} - \nabla_{\mathbf{Y}} (f \nabla_{\mathbf{X}} \mathbf{Z}) - \nabla_{f[\mathbf{X},\mathbf{Y}] - (\mathbf{Y}f)\mathbf{X}} \mathbf{Z} \\
&= f \nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{Z} - (\mathbf{Y}f) \nabla_{\mathbf{X}} \mathbf{Z} - f \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}} \mathbf{Z} - f \nabla_{[\mathbf{X},\mathbf{Y}]} \mathbf{Z} + (\mathbf{Y}f) \nabla_{\mathbf{X}} \mathbf{Z} \\
&= f \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}.
\end{aligned}$$

For the second argument:

$$\begin{aligned}\mathbf{R}(\mathbf{X}, f\mathbf{Y})\mathbf{Z} &= \nabla_{\mathbf{X}}\nabla_{f\mathbf{Y}}\mathbf{Z} - \nabla_{f\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},f\mathbf{Y}]}\mathbf{Z}\\ &= \nabla_{\mathbf{X}}(f\nabla_{\mathbf{Y}}\mathbf{Z}) - f\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{f[\mathbf{X},\mathbf{Y}]+(\mathbf{X}f)\mathbf{Y}}\mathbf{Z}\\ &= (\mathbf{X}f)\nabla_{\mathbf{Y}}\mathbf{Z} + f\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - f\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - f\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} - (\mathbf{X}f)\nabla_{\mathbf{Y}}\mathbf{Z}\\ &= f\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}.\end{aligned}$$

Finally, for the third and last argument:

$$\begin{aligned}\mathbf{R}(\mathbf{X},\mathbf{Y})(f\mathbf{Z}) &= \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}(f\mathbf{Z}) - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}(f\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}(f\mathbf{Z})\\ &= \nabla_{\mathbf{X}}\big((\mathbf{Y}f)\mathbf{Z} + f\nabla_{\mathbf{Y}}\mathbf{Z}\big) - \nabla_{\mathbf{Y}}\big((\mathbf{X}f)\mathbf{Z} + f\nabla_{\mathbf{X}}\mathbf{Z}\big) - ([\mathbf{X},\mathbf{Y}]f)\mathbf{Z} - f\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}\\ &= (\mathbf{X}(\mathbf{Y}f))\mathbf{Z} + (\mathbf{Y}f)\nabla_{\mathbf{X}}\mathbf{Z} + (\mathbf{X}f)\nabla_{\mathbf{Y}}\mathbf{Z} + f\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}\\ &\quad - (\mathbf{Y}(\mathbf{X}f))\mathbf{Z} - (\mathbf{X}f)\nabla_{\mathbf{Y}}\mathbf{Z} - (\mathbf{Y}f)\nabla_{\mathbf{X}}\mathbf{Z} - f\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}\\ &\quad - ([\mathbf{X},\mathbf{Y}]f)\mathbf{Z} - f\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}.\end{aligned}$$

The terms containing first derivatives of f cancel. Moreover,

$$\mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f) - [\mathbf{X},\mathbf{Y}]f = 0,$$

by the definition of the Lie bracket acting on functions. Consequently,

$$\mathbf{R}(\mathbf{X},\mathbf{Y})(f\mathbf{Z}) = f\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}.$$

We have verified multilinearity in all three arguments. Therefore, $\mathbf{R}$ is a tensor field of type $(1,3)$ on M. □

Since $\mathbf{R}$ is a tensor field of type $(1,3)$, we may write it in local coordinates (x^i) as

$$\mathbf{R} = R_{ijk}{}^{l}\,\mathbf{d}x^i \otimes \mathbf{d}x^j \otimes \mathbf{d}x^k \otimes \partial_l.$$

The small space before the index l visually separates the three covariant indices from the contravariant index.

The coefficients satisfy the relation

$$\mathbf{R}(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{l}\partial_l.$$

The following proposition shows how to compute the components of R in coordinates:

Proposition 2.10.3. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. The components of the $(1,3)$-curvature tensor in coordinates (x^i) are given by

$$R_{ijk}{}^{l} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}.$$

Proof. Let (x^i) be local coordinates on M and let $\{\partial_i\}$ be the associated coordinate frame. By the definition of the Christoffel symbols,

$$\nabla_{\partial_i}\partial_j = \Gamma^m_{ij}\partial_m.$$

Using the linearity and Leibniz rule of the connection, we have

$$\nabla_{\partial_i}\nabla_{\partial_j}\partial_k = \nabla_{\partial_i}\left(\Gamma^l_{jk}\partial_l\right) = \partial_i\Gamma^l_{jk}\partial_l + \Gamma^l_{jk}\nabla_{\partial_i}\partial_l = \partial_i\Gamma^l_{jk}\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m.$$

Interchanging the indices i and j in the preceding calculation gives

$$\nabla_{\partial_j}\nabla_{\partial_i}\partial_k = \partial_j\Gamma^l_{ik}\partial_l + \Gamma^l_{ik}\Gamma^m_{jl}\partial_m.$$

Since $[\partial_i, \partial_j] = 0$ in our coordinate frame, the third term in the definition vanishes:

$$\nabla_{[\partial_i,\partial_j]}\partial_k = 0.$$

Therefore, recalling that $\mathbf{R}(\partial_i,\partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{[\partial_i,\partial_j]}\partial_k$ and interchanging l and m in $\Gamma^l_{jk}\Gamma^m_{il}\partial_m - \Gamma^l_{ik}\Gamma^m_{jl}\partial_m$,

$$\mathbf{R}(\partial_i,\partial_j)\partial_k = \left(\partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}\right)\partial_l.$$

Comparing coefficients in the basis $\{\partial_l\}$, we conclude that

$$R_{ijk}{}^l = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm},$$

which is the desired formula. □

For any vector fields $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$, the map $\mathbf{R}(\mathbf{X},\mathbf{Y})\colon \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ given by $\mathbf{Z} \mapsto \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}$ is a smooth homomorphism of the tangent bundle; in particular, $\mathbf{R}(\mathbf{X},\mathbf{Y})$ is $C^\infty(M)$-linear. It is called the ***curvature endomorphism determined by*** $\mathbf{X}$ ***and*** $\mathbf{Y}$. Lowering the contravariant index with the metric gives the covariant tensor field defined next.

Definition 2.10.4. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. Define the ***Riemann curvature tensor*** to be the tensor field of type $(0,4)$ given by

$$\mathbf{Rm}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \langle \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}, \mathbf{W}\rangle_{\mathbf{g}}.$$

As a corollary of Proposition 2.10.3, we obtain a local expression for the Riemann curvature tensor:

Corollary 2.10.5. If $(M, \mathbf{g})$ is a Riemannian manifold with or without boundary and

$$\mathbf{Rm} = R_{ijkl}\,\mathbf{d}x^i \otimes \mathbf{d}x^j \otimes \mathbf{d}x^k \otimes \mathbf{d}x^l,$$

then

$$R_{ijkl} = g_{lm}\left(\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}\right).$$

Proof. By the definition of $\mathbf{Rm}$ and the formula in Proposition 2.10.3,

$$R_{ijkl} = \mathbf{Rm}(\partial_i,\partial_j,\partial_k,\partial_l) = g_{lm}R^m{}_{ijk}.$$

Substituting

$$R^m{}_{ijk} = \partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}$$

gives the stated expression. □

The symmetries and antisymmetries of the Riemann curvature tensor relate its components and simplify calculations involving it.

Proposition 2.10.6. (Symmetries of the curvature tensor) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. The Riemann curvature tensor has the following properties for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W} \in \mathfrak{X}(M)$.

(a) $\mathbf{Rm}(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -\mathbf{Rm}(\mathbf{X}, \mathbf{W}, \mathbf{Y}, \mathbf{Z})$.

(b) $\mathbf{Rm}(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -\mathbf{Rm}(\mathbf{W}, \mathbf{X}, \mathbf{Z}, \mathbf{Y})$.

(c) $\mathbf{Rm}(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{Rm}(\mathbf{Y}, \mathbf{Z}, \mathbf{W}, \mathbf{X})$.

(d) $\mathbf{Rm}(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) + \mathbf{Rm}(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}) + \mathbf{Rm}(\mathbf{Y}, \mathbf{W}, \mathbf{X}, \mathbf{Z}) = 0$. This identity is known as the ***Bianchi identity***.

Proof. Write

$$R(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) := \langle \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \mathbf{W} \rangle.$$

The definition of $\mathbf{R}$ immediately gives $\mathbf{R}(\mathbf{X}, \mathbf{Y}) = -\mathbf{R}(\mathbf{Y}, \mathbf{X})$, proving (1).
Metric compatibility of the connection implies

$$\begin{aligned} &R(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) + R(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}) \\ &\quad = \mathbf{X}\mathbf{Y}\langle \mathbf{Z}, \mathbf{W} \rangle - \mathbf{Y}\mathbf{X}\langle \mathbf{Z}, \mathbf{W} \rangle - [\mathbf{X}, \mathbf{Y}]\langle \mathbf{Z}, \mathbf{W} \rangle = 0. \end{aligned}$$

In the first equality, the three covariant derivatives in each curvature term are expanded and grouped using the compatibility rule; the last equality is the definition of the bracket acting on functions. This proves (2).
Since the connection is torsion-free, expanding the cyclic sum gives

$$\begin{aligned} &\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} + \mathbf{R}(\mathbf{Y}, \mathbf{Z})\mathbf{X} + \mathbf{R}(\mathbf{Z}, \mathbf{X})\mathbf{Y} \\ &= \nabla_{\mathbf{X}}(\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Z}}\mathbf{Y}) + \nabla_{\mathbf{Y}}(\nabla_{\mathbf{Z}}\mathbf{X} - \nabla_{\mathbf{X}}\mathbf{Z}) + \nabla_{\mathbf{Z}}(\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X}) \\ &\quad - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} - \nabla_{[\mathbf{Y},\mathbf{Z}]}\mathbf{X} - \nabla_{[\mathbf{Z},\mathbf{X}]}\mathbf{Y} \\ &= \nabla_{\mathbf{X}}[\mathbf{Y}, \mathbf{Z}] + \nabla_{\mathbf{Y}}[\mathbf{Z}, \mathbf{X}] + \nabla_{\mathbf{Z}}[\mathbf{X}, \mathbf{Y}] \\ &\quad - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} - \nabla_{[\mathbf{Y},\mathbf{Z}]}\mathbf{X} - \nabla_{[\mathbf{Z},\mathbf{X}]}\mathbf{Y} = 0. \end{aligned}$$

In the last equality, terms are grouped, for example, as $\nabla_{\mathbf{X}}[\mathbf{Y}, \mathbf{Z}] - \nabla_{[\mathbf{Y},\mathbf{Z}]}\mathbf{X} = [\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]]$, and the three resulting expressions sum to zero by the Jacobi identity. Pairing with $\mathbf{W}$ gives (4).
It remains to prove (3). Fix four vector fields and abbreviate their names by a, b, c, d. Using only the antisymmetries already proved, the Bianchi identities corresponding to the relevant permutations can be written as

$$\begin{aligned} R_{bcad} &= R_{acbd} - R_{abcd}, \\ R_{bdac} &= R_{abcd} + R_{adbc}, \\ R_{cdab} &= R_{acbd} - R_{adbc}, \\ R_{bcad} &= R_{bdac} - R_{cdab}, \end{aligned}$$

where $R_{abcd} := R(a,b,c,d)$. Substituting the first three equalities into the fourth gives

$$R_{abcd} = R_{acbd} - R_{adbc} = R_{cdab}.$$

This is the pair-interchange symmetry in (3). □

We may express the symmetries of the curvature tensor in coordinates as follows:

Corollary 2.10.7. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. In any local coordinates, the components of $\mathbf{Rm}$ satisfy:

(a) $R_{ijkl} = -R_{jikl}$.

(b) $R_{ijkl} = -R_{ijlk}$.

(c) $R_{ijkl} = R_{klij}$.

(d) $R_{ijkl} + R_{jkil} + R_{kijl} = 0$.

Proof. Apply the four parts of the preceding proposition to the coordinate vector fields $\partial_i, \partial_j, \partial_k, \partial_l$. □

Definition 2.10.8. (Ricci tensor and Ricci endomorphism) Let $(M,\mathbf{g})$ be a Riemannian manifold of dimension n with or without boundary. For $X, Y \in T_pM$, define

$$\mathbf{Ric}_p(X,Y) := \sum_{i=1}^{n} \mathbf{g}_p\big(\mathbf{R}(\mathbf{e}_i, X)Y, \mathbf{e}_i\big), \tag{2.72}$$

where $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ is an orthonormal basis of T_pM. This expression is independent of the chosen orthonormal basis and defines a symmetric tensor field of type $(0,2)$, called the *Ricci tensor*.
The *Ricci endomorphism* is the field $\mathbf{Ric}^\sharp \in \Gamma(\mathrm{End}(TM))$ determined by

$$\mathbf{g}(\mathbf{Ric}^\sharp X, Y) = \mathbf{Ric}(X,Y). \tag{2.73}$$

Proposition 2.10.9. (Curvature contraction and the Ricci endomorphism) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary of dimension n. For every $p \in M$, every orthonormal basis $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ of T_pM, and every $X \in T_pM$,

$$-\sum_{i=1}^{n} \mathbf{R}(\mathbf{e}_i, X)\mathbf{e}_i = \mathbf{Ric}^\sharp X. \tag{2.74}$$

In particular, $\mathbf{Ric}^\sharp$ is self-adjoint with respect to $\mathbf{g}$.

Proof. Independence of the basis in (2.72) follows from basis independence of the trace of an endomorphism. Let us verify symmetry. If $X, Y \in T_pM$, the pair symmetries and antisymmetries of the curvature

tensor give

$$\begin{aligned}\mathbf{Ric}(Y,X) &= \sum_{i=1}^{n} \mathbf{Rm}(\mathbf{e}_i, Y, X, \mathbf{e}_i)\\ &= \sum_{i=1}^{n} \mathbf{Rm}(X, \mathbf{e}_i, \mathbf{e}_i, Y)\\ &= -\sum_{i=1}^{n} \mathbf{Rm}(\mathbf{e}_i, X, \mathbf{e}_i, Y)\\ &= \sum_{i=1}^{n} \mathbf{Rm}(\mathbf{e}_i, X, Y, \mathbf{e}_i) = \mathbf{Ric}(X,Y).\end{aligned}$$

To prove (2.74), take $Y \in T_pM$ and use the antisymmetry of $\mathbf{Rm}$ in its last two arguments:

$$\begin{aligned}\mathbf{g}\left(-\sum_{i=1}^{n} \mathbf{R}(\mathbf{e}_i, X)\mathbf{e}_i, Y\right) &= -\sum_{i=1}^{n} \mathbf{Rm}(\mathbf{e}_i, X, \mathbf{e}_i, Y)\\ &= \sum_{i=1}^{n} \mathbf{Rm}(\mathbf{e}_i, X, Y, \mathbf{e}_i) = \mathbf{Ric}(X,Y).\end{aligned}$$

Nondegeneracy of $\mathbf{g}$ gives the identity. Finally, symmetry of $\mathbf{Ric}$ implies

$$\mathbf{g}(\mathbf{Ric}^\sharp X, Y) = \mathbf{Ric}(X,Y) = \mathbf{Ric}(Y,X) = \mathbf{g}(X, \mathbf{Ric}^\sharp Y),$$

so $\mathbf{Ric}^\sharp$ is self-adjoint. □

Definition 2.10.10. (Scalar curvature) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. The *scalar curvature* is the smooth function

$$R_{\mathbf{g}} := \operatorname{tr}_{\mathbf{g}}(\mathbf{Ric}_{\mathbf{g}}).$$

If $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is a local orthonormal frame, then

$$R_{\mathbf{g}} = \sum_{i=1}^{n} \mathbf{Ric}_{\mathbf{g}}(\mathbf{e}_i, \mathbf{e}_i),$$

and in coordinates, $R_{\mathbf{g}} = g^{ij}\,\mathrm{Ric}_{ij}$. In the analytic chapters, we retain the notation $R_{\mathbf{g}}$ to distinguish the scalar from the curvature tensor or endomorphism.

Definition 2.10.11. (Sectional curvature) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let $p \in M$. Given a two-dimensional plane $\Pi \subseteq T_pM$ spanned by two linearly independent vectors $u, v \in T_pM$, the *sectional curvature* of Π is defined by

$$K(\Pi) = \frac{\mathbf{g}(\mathbf{R}(u,v)v, u)}{\mathbf{g}(u,u)\mathbf{g}(v,v) - \mathbf{g}(u,v)^2},$$

where $\mathbf{R}$ is the Riemann curvature tensor. This value depends only on the plane Π and is independent of the basis $\{u, v\}$ chosen to span it.

Theorem 2.10.12. (Curvature tensor for spaces of constant curvature) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. If $(M, \mathbf{g})$ has constant sectional curvature K, then, for every $p \in M$ and all $u, v, w \in T_pM$, we have

$$\mathbf{R}(u,v)w = K\big(\mathbf{g}(v,w)u - \mathbf{g}(u,w)v\big).$$

Proof. By hypothesis, the sectional curvature is a constant K. From the definition of sectional curvature, we rewrite the equation to obtain

$$\mathbf{g}(\mathbf{R}(u,v)v,u) = K\big(\mathbf{g}(u,u)\mathbf{g}(v,v) - \mathbf{g}(u,v)^2\big).$$

This equality holds for every pair of linearly independent vectors u, v and extends trivially to the case in which they are linearly dependent (since both sides vanish).
Given arbitrary u, v, w, we compute the expression $\mathbf{g}(\mathbf{R}(u+w,v)v,u+w)$ in two different ways.
First, applying the preceding formula directly to the pair $(u+w, v)$, we obtain:

$$\begin{aligned}\mathbf{g}(\mathbf{R}(u+w,v)v,u+w) &= K\big(\mathbf{g}(v,v)\mathbf{g}(u+w,u+w) - \mathbf{g}(u+w,v)^2\big)\\ &= K\big(\mathbf{g}(v,v)[\mathbf{g}(u,u)+\mathbf{g}(w,w)+2\mathbf{g}(u,w)]\\ &\qquad - [\mathbf{g}(u,v)^2+\mathbf{g}(w,v)^2+2\mathbf{g}(u,v)\mathbf{g}(w,v)]\big).\end{aligned}$$

Second, using multilinearity of the Riemann tensor on the left-hand side:

$$\begin{aligned}\mathbf{g}(\mathbf{R}(u+w,v)v,u+w) &= \mathbf{g}(\mathbf{R}(u,v)v+\mathbf{R}(w,v)v,u+w)\\ &= \mathbf{g}(\mathbf{R}(u,v)v,u)+\mathbf{g}(\mathbf{R}(w,v)v,w)+\mathbf{g}(\mathbf{R}(u,v)v,w)+\mathbf{g}(\mathbf{R}(w,v)v,u).\end{aligned}$$

By Proposition 2.10.6 (parts 1, 2, and 3), we know that $\mathbf{Rm}(u,v,v,w) = \mathbf{Rm}(w,v,v,u)$, which is equivalent to $\mathbf{g}(\mathbf{R}(u,v)v,w) = \mathbf{g}(\mathbf{R}(w,v)v,u)$. Moreover, applying our basic formula to the first two terms simplifies the expression to

$$K\big(\mathbf{g}(u,u)\mathbf{g}(v,v) - \mathbf{g}(u,v)^2\big) + K\big(\mathbf{g}(w,w)\mathbf{g}(v,v) - \mathbf{g}(w,v)^2\big) + 2\mathbf{g}(\mathbf{R}(u,v)v,w).$$

Equating the two expansions of $\mathbf{g}(\mathbf{R}(u+w,v)v,u+w)$ and canceling the common terms gives

$$2\mathbf{g}(\mathbf{R}(u,v)v,w) = 2K\big(\mathbf{g}(v,v)\mathbf{g}(u,w) - \mathbf{g}(u,v)\mathbf{g}(w,v)\big).$$

After division by 2, this equality can be written as

$$\mathbf{g}\big(\mathbf{R}(u,v)v - K(\mathbf{g}(v,v)u - \mathbf{g}(u,v)v), w\big) = 0 \qquad \text{for every } w \in T_pM.$$

Nondegeneracy of $\mathbf{g}_p$ forces the first argument to be zero. Therefore,

$$\mathbf{R}(u,v)v = K\big(\mathbf{g}(v,v)u - \mathbf{g}(u,v)v\big). \tag{2.75}$$

We now polarize in the second and third arguments. Take arbitrary u, v, w and compute $\mathbf{R}(u, v+w)(v+w)$

in two ways. First, using (2.75):

$$\begin{aligned}\mathbf{R}(u,v+w)(v+w) &= K\big(\mathbf{g}(v+w,v+w)u-\mathbf{g}(u,v+w)(v+w)\big)\\ &= K\big([\mathbf{g}(v,v)+\mathbf{g}(w,w)+2\mathbf{g}(v,w)]u\\ &\qquad -\mathbf{g}(u,v)v-\mathbf{g}(u,w)v-\mathbf{g}(u,v)w-\mathbf{g}(u,w)w\big).\end{aligned}$$

Second, by multilinearity of $\mathbf{R}$:

$$\mathbf{R}(u,v+w)(v+w)=\mathbf{R}(u,v)v+\mathbf{R}(u,w)w+\mathbf{R}(u,v)w+\mathbf{R}(u,w)v.$$

Applying (2.75) to the first two terms of this sum:

$$K\big(\mathbf{g}(v,v)u-\mathbf{g}(u,v)v\big)+K\big(\mathbf{g}(w,w)u-\mathbf{g}(u,w)w\big)+\mathbf{R}(u,v)w+\mathbf{R}(u,w)v.$$

Equating these two new expansions and canceling corresponding terms, we arrive at:

$$\mathbf{R}(u,v)w+\mathbf{R}(u,w)v=K\big(2\mathbf{g}(v,w)u-\mathbf{g}(u,w)v-\mathbf{g}(u,v)w\big). \tag{2.76}$$

The final step uses the Bianchi identity, which states that $\mathbf{R}(v,u)w+\mathbf{R}(u,w)v+\mathbf{R}(w,v)u=0$. Interchanging the roles of u and v in equation (2.76), we obtain:

$$\mathbf{R}(v,u)w+\mathbf{R}(v,w)u=K\big(2\mathbf{g}(u,w)v-\mathbf{g}(v,w)u-\mathbf{g}(u,v)w\big).$$

By the Bianchi identity and antisymmetry in the first two arguments, $\mathbf{R}(v,w)u=-\mathbf{R}(w,v)u=\mathbf{R}(v,u)w+\mathbf{R}(u,w)v$. Substituting gives

$$\mathbf{R}(v,u)w+\big(\mathbf{R}(v,u)w+\mathbf{R}(u,w)v\big)=2\mathbf{R}(v,u)w+\mathbf{R}(u,w)v.$$

Using $\mathbf{R}(v,u)w=-\mathbf{R}(u,v)w$, this is equivalent to $-2\mathbf{R}(u,v)w+\mathbf{R}(u,w)v$. Therefore,

$$-2\mathbf{R}(u,v)w+\mathbf{R}(u,w)v=K\big(2\mathbf{g}(u,w)v-\mathbf{g}(v,w)u-\mathbf{g}(u,v)w\big). \tag{2.77}$$

Finally, subtract equation (2.77) from equation (2.76):

$$\begin{aligned}&\big(\mathbf{R}(u,v)w+\mathbf{R}(u,w)v\big)-\big(-2\mathbf{R}(u,v)w+\mathbf{R}(u,w)v\big)=\\ &K\big(2\mathbf{g}(v,w)u-\mathbf{g}(u,w)v-\mathbf{g}(u,v)w\big)-K\big(2\mathbf{g}(u,w)v-\mathbf{g}(v,w)u-\mathbf{g}(u,v)w\big).\end{aligned}$$

Simplifying both sides, the terms involving $\mathbf{R}(u,w)v$ and $\mathbf{g}(u,v)w$ cancel, leaving

$$3\mathbf{R}(u,v)w=K\big(3\mathbf{g}(v,w)u-3\mathbf{g}(u,w)v\big).$$

Dividing by 3 gives the desired formula:

$$\mathbf{R}(u,v)w=K\big(\mathbf{g}(v,w)u-\mathbf{g}(u,w)v\big).$$

□

2.11 Riemannian submanifolds

Trace conditions and many variational problems are formulated on a submanifold. To compare intrinsic derivatives with derivatives in M, we must separate the tangential and normal components of the Levi–Civita connection.

Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. The induced metric is $\mathbf{g}_N = \mathbf{g}\restriction_{TN}$, and

$$T_xM = T_xN \oplus T_x^{\perp}N, \qquad T_x^{\perp}N = \{v \in T_xM \mid \mathbf{g}(v,w) = 0 \text{ for every } w \in T_xN\}.$$

The spaces $T_x^{\perp}N$ form the normal bundle $T^{\perp}N$. For a section $\mathbf{V}$ of $TM|_N$, we write $\mathbf{V}^{\top}$ and $\mathbf{V}^{\perp}$ for its orthogonal projections.

Lemma 2.11.1. (Ambient connection along a submanifold) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. Let $\iota\colon N \hookrightarrow M$ be the inclusion. There exists a unique connection $\overline{\nabla}$ on the restricted bundle $\iota^*TM \to N$ such that, if $\mathbf{X} \in \mathfrak{X}(N)$ and $\mathbf{V} \in \Gamma(\iota^*TM)$ admit local extensions $\widetilde{\mathbf{X}}, \widetilde{\mathbf{V}} \in \mathfrak{X}(M)$, then

$$\overline{\nabla}_{\mathbf{X}}\mathbf{V} = (\nabla^M_{\widetilde{\mathbf{X}}}\widetilde{\mathbf{V}})\restriction_N .$$

This expression does not depend on the extensions. Moreover,

$$\mathbf{X}\mathbf{g}(\mathbf{V},\mathbf{W}) = \mathbf{g}(\overline{\nabla}_{\mathbf{X}}\mathbf{V},\mathbf{W}) + \mathbf{g}(\mathbf{V},\overline{\nabla}_{\mathbf{X}}\mathbf{W})$$

for $\mathbf{V},\mathbf{W} \in \Gamma(\iota^*TM)$ and, if $\mathbf{X},\mathbf{Y} \in \mathfrak{X}(N)$, then

$$\overline{\nabla}_{\mathbf{X}}\mathbf{Y} - \overline{\nabla}_{\mathbf{Y}}\mathbf{X} = [\mathbf{X},\mathbf{Y}].$$

Proof. In coordinates $(x^1,\dots,x^m)$ on M, if $\mathbf{V} = V^j\partial_j\restriction_N$ and $d\iota(\mathbf{X}) = X^i\partial_i\restriction_N$, necessarily

$$\overline{\nabla}_{\mathbf{X}}\mathbf{V} = \Big(\mathbf{X}(V^k) + X^iV^j\Gamma^k_{ij}\restriction_N\Big)\partial_k\restriction_N .$$

The right-hand side contains only the values of $\mathbf{X}$, $\mathbf{V}$, and the tangential derivatives $\mathbf{X}(V^k)$ on N; it therefore does not depend on any extension. The transformation rule for the Christoffel symbols shows that the local expressions glue together, and the connection rules are verified directly from the formula. The last two identities are the restrictions, respectively, of metric compatibility in Theorem 2.4.14 and of the torsion-free property of the Levi–Civita connection. □

Definition 2.11.2. (Second fundamental form and normal connection) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. For $\mathbf{X},\mathbf{Y} \in \mathfrak{X}(N)$ and $\eta \in \Gamma(T^{\perp}N)$, define

$$\mathbf{II}(\mathbf{X},\mathbf{Y}) = (\overline{\nabla}_{\mathbf{X}}\mathbf{Y})^{\perp}, \qquad \nabla^{\perp}_{\mathbf{X}}\eta = (\overline{\nabla}_{\mathbf{X}}\eta)^{\perp}.$$

For $\eta \in \Gamma(T^{\perp}N)$, the *Weingarten map* or *shape operator* $\mathbf{A}_\eta\colon TN \longrightarrow TN$ is the unique endomorphism satis-

fying

$$\mathbf{g}_N(\mathbf{A}_\eta \mathbf{X}, \mathbf{Y}) = \mathbf{g}(\mathbf{II}(\mathbf{X}, \mathbf{Y}), \eta)$$

for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(N)$.

Proposition 2.11.3. (Gauss and Weingarten formulas) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. The second fundamental form $\mathbf{II}$ is a symmetric tensor, and $\nabla^\perp$ is a metric connection on the normal bundle. We have

$$\overline{\nabla}_{\mathbf{X}} \mathbf{Y} = \nabla^N_{\mathbf{X}} \mathbf{Y} + \mathbf{II}(\mathbf{X}, \mathbf{Y}), \tag{2.78}$$

$$\overline{\nabla}_{\mathbf{X}} \eta = -\mathbf{A}_\eta \mathbf{X} + \nabla^\perp_{\mathbf{X}} \eta. \tag{2.79}$$

The Weingarten map depends $C^\infty(N)$–linearly on η and is self-adjoint with respect to $\mathbf{g}_N$.

Proof. Lemma 2.11.1 eliminates all dependence on extensions. Linearity of a connection in its first argument gives $\mathbf{II}(f\mathbf{X}, \mathbf{Y}) = f\,\mathbf{II}(\mathbf{X}, \mathbf{Y})$. In the second argument,

$$\overline{\nabla}_{\mathbf{X}}(f\mathbf{Y}) = \mathbf{X}(f)\mathbf{Y} + f\overline{\nabla}_{\mathbf{X}}\mathbf{Y};$$

the first summand is tangent, so $\mathbf{II}(\mathbf{X}, f\mathbf{Y}) = f\,\mathbf{II}(\mathbf{X}, \mathbf{Y})$. Thus, $\mathbf{II}$ is tensorial. Since $\overline{\nabla}$ is torsion-free on tangent vector fields,

$$\mathbf{II}(\mathbf{X}, \mathbf{Y}) - \mathbf{II}(\mathbf{Y}, \mathbf{X}) = [\mathbf{X}, \mathbf{Y}]^\perp = 0.$$

Define $\mathbf{D}_{\mathbf{X}}\mathbf{Y} := (\overline{\nabla}_{\mathbf{X}}\mathbf{Y})^\top$. Metric compatibility of $\overline{\nabla}$ and orthogonality of the decomposition show that $\mathbf{D}$ is metric; the preceding torsion identity shows that it is torsion-free. By uniqueness of the Levi–Civita connection, $\mathbf{D} = \nabla^N$, and the tangential–normal decomposition gives the Gauss formula (2.78).
Differentiating the identity $\mathbf{g}(\eta, \mathbf{Y}) = 0$ gives

$$\mathbf{g}((\overline{\nabla}_{\mathbf{X}}\eta)^\top, \mathbf{Y}) = -\mathbf{g}(\eta, \mathbf{II}(\mathbf{X}, \mathbf{Y})) = -\mathbf{g}_N(\mathbf{A}_\eta \mathbf{X}, \mathbf{Y}).$$

Nondegeneracy of $\mathbf{g}_N$ gives $(\overline{\nabla}_{\mathbf{X}}\eta)^\top = -\mathbf{A}_\eta \mathbf{X}$, proving (2.79). The connection rules and metric compatibility of $\nabla^\perp$ follow by projecting those of $\overline{\nabla}$. Finally, symmetry of $\mathbf{II}$ implies

$$\mathbf{g}_N(\mathbf{A}_\eta \mathbf{X}, \mathbf{Y}) = \mathbf{g}_N(\mathbf{X}, \mathbf{A}_\eta \mathbf{Y}),$$

and linearity in η follows from the identity defining $\mathbf{A}_\eta$. □

Corollary 2.11.4. (Weingarten map of a Euclidean hypersurface) Let $N \subseteq \mathbb{R}^{n+1}$ be a smooth oriented hypersurface without boundary, of dimension n, and let $\boldsymbol{\nu} \colon N \longrightarrow \mathbb{S}^n$ be its unit normal field. Then

$$\mathbf{A}_{\boldsymbol{\nu}} \mathbf{X} = -d\boldsymbol{\nu}(\mathbf{X}), \qquad \mathbf{X} \in \mathfrak{X}(N).$$

In particular, $d\boldsymbol{\nu}$ is self-adjoint with the opposite sign; the eigenvalues of $\mathbf{A}_{\boldsymbol{\nu}}$ are the principal curvatures, and $H := \frac{1}{n}\operatorname{tr}(\mathbf{A}_{\boldsymbol{\nu}})$ is the scalar mean curvature with this convention.

Proof. Identify $T_{\boldsymbol{\nu}(x)}\mathbb{S}^n$ with the subspace $\boldsymbol{\nu}(x)^\perp \subseteq \mathbb{R}^{n+1}$. Since $\|\boldsymbol{\nu}\| = 1$, we have $\mathbf{g}(d\boldsymbol{\nu}(\mathbf{X}), \boldsymbol{\nu}) = \frac{1}{2}\mathbf{X}\mathbf{g}(\boldsymbol{\nu}, \boldsymbol{\nu}) = 0$, so $d\boldsymbol{\nu}(\mathbf{X})$ is tangent. The Euclidean connection along N satisfies $\overline{\nabla}_{\mathbf{X}}\boldsymbol{\nu} = d\boldsymbol{\nu}(\mathbf{X})$. The Weingarten formula (2.79) and vanishing of the normal component give $d\boldsymbol{\nu}(\mathbf{X}) = -\mathbf{A}_{\boldsymbol{\nu}}\mathbf{X}$. The remaining assertions are the corresponding spectral definitions; reality of the eigenvalues follows from self-adjointness proved in Proposition 2.11.3. □

Definition 2.11.5. (Covariant derivatives of the second fundamental form) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. The connections ∇^N on TN and $\nabla^\perp$ on $T^\perp N$ induce a connection, denoted by $\nabla^{\mathbf{II}}$, on the bundle

$$T^*N \otimes T^*N \otimes T^\perp N.$$

Explicitly, if $\mathbf{B} \in \Gamma(T^*N \otimes T^*N \otimes T^\perp N)$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(N)$, then

$$(\nabla^{\mathbf{II}}_{\mathbf{X}}\mathbf{B})(\mathbf{Y}, \mathbf{Z}) = \nabla^\perp_{\mathbf{X}}(\mathbf{B}(\mathbf{Y}, \mathbf{Z})) - \mathbf{B}(\nabla^N_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}) - \mathbf{B}(\mathbf{Y}, \nabla^N_{\mathbf{X}}\mathbf{Z}).$$

Define recursively

$$(\nabla^{\mathbf{II}})^0\mathbf{II} := \mathbf{II}, \qquad (\nabla^{\mathbf{II}})^{m+1}\mathbf{II} := \nabla^{\mathbf{II}}\big((\nabla^{\mathbf{II}})^m\mathbf{II}\big), \qquad m \in \mathbb{N}_0,$$

where the right-hand side uses the induced connection on each additional covariant factor. In particular,

$$(\nabla^{\mathbf{II}})^m\mathbf{II} \in \Gamma\big((T^*N)^{\otimes(m+2)} \otimes T^\perp N\big).$$

The dual metric of $\mathbf{g}_N$ on the factors T^*N and the restriction of $\mathbf{g}$ to the normal bundle induce the pointwise norm $|(\nabla^{\mathbf{II}})^m\mathbf{II}|_{\mathbf{g}}$. We say that this derivative is uniformly bounded if

$$\sup_{x \in N} \big|(\nabla^{\mathbf{II}})^m\mathbf{II}(x)\big|_{\mathbf{g}} < \infty.$$

Definition 2.11.6. (Normal curvature) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. The curvature of the normal connection is the field $\mathbf{R}^\perp$ defined by

$$\mathbf{R}^\perp(\mathbf{X}, \mathbf{Y})\boldsymbol{\eta} := \nabla^\perp_{\mathbf{X}}\nabla^\perp_{\mathbf{Y}}\boldsymbol{\eta} - \nabla^\perp_{\mathbf{Y}}\nabla^\perp_{\mathbf{X}}\boldsymbol{\eta} - \nabla^\perp_{[\mathbf{X},\mathbf{Y}]}\boldsymbol{\eta}.$$

Theorem 2.11.7. (Gauss, Codazzi, and Ricci equations) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. For tangent vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$ and normal fields $\boldsymbol{\eta}, \boldsymbol{\zeta}$, we have

$$\mathbf{g}(\mathbf{R}^N(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \mathbf{W}) = \mathbf{g}(\mathbf{R}^M(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \mathbf{W}) + \mathbf{g}(\mathbf{II}(\mathbf{X}, \mathbf{W}), \mathbf{II}(\mathbf{Y}, \mathbf{Z})) - \mathbf{g}(\mathbf{II}(\mathbf{X}, \mathbf{Z}), \mathbf{II}(\mathbf{Y}, \mathbf{W})), \tag{2.80}$$

$$(\mathbf{R}^M(\mathbf{X}, \mathbf{Y})\mathbf{Z})^\perp = (\nabla^{\mathbf{II}}_{\mathbf{X}}\mathbf{II})(\mathbf{Y}, \mathbf{Z}) - (\nabla^{\mathbf{II}}_{\mathbf{Y}}\mathbf{II})(\mathbf{X}, \mathbf{Z}), \tag{2.81}$$

$$\mathbf{g}(\mathbf{R}^\perp(\mathbf{X}, \mathbf{Y})\boldsymbol{\eta}, \boldsymbol{\zeta}) = \mathbf{g}(\mathbf{R}^M(\mathbf{X}, \mathbf{Y})\boldsymbol{\eta}, \boldsymbol{\zeta}) + \mathbf{g}_N([\mathbf{A}_{\boldsymbol{\eta}}, \mathbf{A}_{\boldsymbol{\zeta}}]\mathbf{X}, \mathbf{Y}). \tag{2.82}$$

We use the convention $\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}$ from Definition 2.10.1.

Proof. Lemma 2.11.1 allows us to compute $\mathbf{R}^M(\mathbf{X},\mathbf{Y})$ using fields along N. Substituting the Gauss and Weingarten formulas into each of the three terms in the definition of $\mathbf{R}^M$ gives

$$(\mathbf{R}^M(\mathbf{X},\mathbf{Y})\mathbf{Z})^{\top} = \mathbf{R}^N(\mathbf{X},\mathbf{Y})\mathbf{Z} - \mathbf{A}_{\mathbf{II}(\mathbf{Y},\mathbf{Z})}\mathbf{X} + \mathbf{A}_{\mathbf{II}(\mathbf{X},\mathbf{Z})}\mathbf{Y}.$$

Pairing with $\mathbf{W}$ and using $\mathbf{g}_N(\mathbf{A}_\eta\mathbf{X},\mathbf{W}) = \mathbf{g}(\mathbf{II}(\mathbf{X},\mathbf{W}),\eta)$ proves the Gauss equation. The normal component is

$$(\mathbf{R}^M(\mathbf{X},\mathbf{Y})\mathbf{Z})^{\perp} = (\nabla^{\mathbf{II}}_{\mathbf{X}}\mathbf{II})(\mathbf{Y},\mathbf{Z}) - (\nabla^{\mathbf{II}}_{\mathbf{Y}}\mathbf{II})(\mathbf{X},\mathbf{Z}),$$

because the terms containing $\nabla^N_{\mathbf{X}}\mathbf{Y} - \nabla^N_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}]$ vanish by the torsion-free property. This proves Codazzi. Expanding $\mathbf{R}^M(\mathbf{X},\mathbf{Y})\eta$ gives

$$(\mathbf{R}^M(\mathbf{X},\mathbf{Y})\eta)^{\perp} = \mathbf{R}^{\perp}(\mathbf{X},\mathbf{Y})\eta - \mathbf{II}(\mathbf{X},\mathbf{A}_\eta\mathbf{Y}) + \mathbf{II}(\mathbf{Y},\mathbf{A}_\eta\mathbf{X}),$$

and pairing with ζ proves Ricci. □

2.12 Tubular neighborhoods and Fermi coordinates

The exponential map applied to the normal bundle turns the preceding tangential–normal decomposition into coordinates adapted to N.

We may adapt the exponential map to obtain information about the normal bundle as follows, defining what is known as the *normal exponential map*.

Definition 2.12.1. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. Define the normal exponential map $\exp^{\perp}\colon \mathcal{E}_N \longrightarrow M$, with $\mathcal{E}_N := \mathcal{E} \cap T^{\perp}N$, by $\exp^{\perp} = \exp|_{\mathcal{E}_N}$.

We will be interested in "normal neighborhoods around N," which we call *tubular neighborhoods*, as in the following figure:

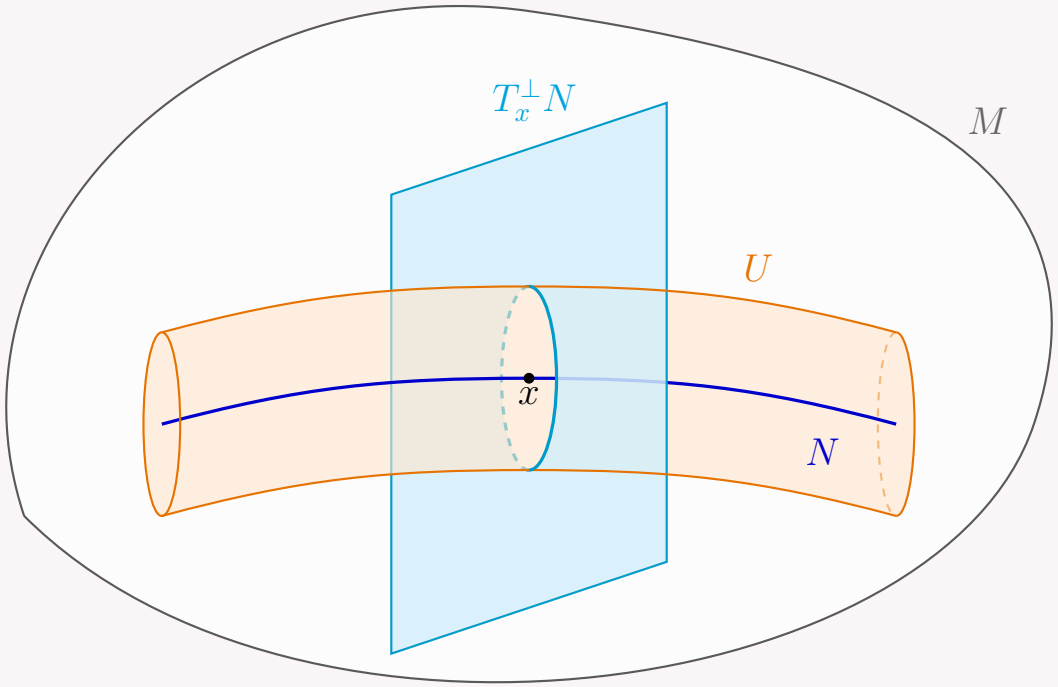


Figure 2.7. Tubular neighborhood around the embedded submanifold N.

Definition 2.12.2. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. We say that $U \subseteq M$ is a *normal neighborhood* of N in M if there exists an open set $V \subseteq \mathcal{E}_N$ such that:

(a) V intersects each fiber $T_x^\perp N$ in a set star-shaped with respect to 0,

(b) the restriction

$$\exp^\perp \restriction_V : V \longrightarrow U$$

is a diffeomorphism.

We say that U is a *tubular neighborhood* if it is a normal neighborhood and, in addition, V can be chosen in the form

$$V := \{(x, v) \in T^\perp N \mid |v|_{\mathbf{g}} < \delta(x)\},$$

for some continuous function $\delta : N \longrightarrow (0, \infty)$.
If V can be chosen with $\delta(x) \equiv \varepsilon$, we say that U is a *uniform tubular neighborhood* of radius ε, or an ε-tubular neighborhood.

Every embedded submanifold admits a tubular neighborhood, and if the submanifold is compact, this tubular

neighborhood can be chosen to be uniform:

Theorem 2.12.3. (Existence of tubular neighborhoods) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $N \subseteq M$ be an embedded submanifold without boundary. Then N has a tubular neighborhood in M. If N is compact, it has a uniform tubular neighborhood.

Proof. Let $N_0 := \{(x,0) \mid x \in N\} \subseteq T^\perp N$. At a point of the zero section, there is a canonical decomposition

$$T_{(x,0)}(T^\perp N) = T_{(x,0)}N_0 \oplus T_{(x,0)}(T^\perp_x N) \cong T_x N \oplus T^\perp_x N. \tag{2.83}$$

The restriction of $\exp^\perp$ to N_0 is the inclusion $N \hookrightarrow M$; its differential maps the first summand onto $T_x N$. On the fiber $T^\perp_x N$, the normal exponential map equals $\exp_x$, and $d(\exp_x)_0 = I$ by Proposition 2.8.7. Therefore, $d(\exp^\perp)_{(x,0)}$ is the isomorphism

$$T_x N \oplus T^\perp_x N \longrightarrow T_x M, \qquad (X, V) \longmapsto X + V.$$

The inverse function theorem, Theorem 1.5.6, shows that $\exp^\perp$ is a local diffeomorphism near each point of N_0.

To obtain a single neighborhood of the entire zero section, we must prevent normal geodesics with different initial points from intersecting. Denote the ambient distance restricted to $N \times N$ by $d_{\mathbf{g}}$ and set

$$V_\delta(x) := \{(y, v) \in T^\perp N \mid d_{\mathbf{g}}(x, y) < \delta,\ |v|_{\mathbf{g}} < \delta\}, \qquad 0 < \delta \leq 1.$$

For each x, the preceding local result allows us to choose $\delta > 0$ so that $\exp^\perp$ is a diffeomorphism from $V_\delta(x)$ onto its image. Define

$$\rho(x) := \sup\{0 < \delta \leq 1 \mid \exp^\perp \restriction_{V_\delta(x)} \text{ is a diffeomorphism onto its image}\}. \tag{2.84}$$

Then $\rho(x) > 0$. The property defining the supremum is monotone with respect to δ. Hence, given two points of $V_{\rho(x)}(x)$, there exist $\delta < \rho(x)$ containing both and an δ', with $\delta < \delta' < \rho(x)$, for which the restriction is a diffeomorphism. Thus, $\exp^\perp$ is injective and a local diffeomorphism on $V_{\rho(x)}(x)$, and therefore is a diffeomorphism from this set onto its image.

The function ρ is continuous. If $d_{\mathbf{g}}(x, x') < \rho(x)$, the triangle inequality gives

$$V_{\rho(x) - d_{\mathbf{g}}(x,x')}(x') \subseteq V_{\rho(x)}(x),$$

and hence $\rho(x') \geq \rho(x) - d_{\mathbf{g}}(x, x')$. If $d_{\mathbf{g}}(x, x') \geq \rho(x)$, this same inequality is trivial. Interchanging x and x' gives

$$|\rho(x) - \rho(x')| \leq d_{\mathbf{g}}(x, x'). \tag{2.85}$$

The ambient distance induces the topology of the embedded submanifold, so ρ is continuous.

Consider the open set

$$V := \{(x, v) \in T^\perp N \mid |v|_{\mathbf{g}} < \tfrac{1}{2}\rho(x)\}. \tag{2.86}$$

Let us prove that $\exp^\perp$ is injective on V. Suppose that $\exp_x(v) = \exp_{x'}(v')$ and, without loss of generality, $\rho(x') \leq \rho(x)$. The two radial geodesics, the second traversed in reverse, form an admissible curve from x to

x' of length $|v|_{\mathbf{g}} + |v'|_{\mathbf{g}}$. Consequently,

$$d_{\mathbf{g}}(x,x') \leq |v|_{\mathbf{g}} + |v'|_{\mathbf{g}} < \tfrac{1}{2}\rho(x) + \tfrac{1}{2}\rho(x') \leq \rho(x).$$

Both (x,v) and (x',v') then belong to $V_{\rho(x)}(x)$, where $\exp^\perp$ is injective; the two points are equal.
Every point of V belongs to an open set on which $\exp^\perp$ is a local diffeomorphism. Consequently, $\exp^\perp \restriction_V$ is an injective local diffeomorphism, hence a diffeomorphism onto the open set $U := \exp^\perp(V)$. The formulas (2.86) and continuity of $\frac{\rho}{2}$ show that U is a tubular neighborhood. If N is compact, $\frac{\rho}{2}$ attains a positive minimum ε, and restriction to $\{|v|_{\mathbf{g}} < \varepsilon\}$ gives the uniform tubular neighborhood. □

2.12.1 Fermi coordinates

The tubular neighborhood identifies a neighborhood of N with a neighborhood of the zero section of its normal bundle. Local trivializations of this bundle give coordinates adapted simultaneously to the tangential and normal directions.

Definition 2.12.4. (Fermi coordinates) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary of dimension n, let $N \subseteq M$ be an embedded submanifold without boundary of dimension k, and let $U = \exp^\perp(V)$ be a normal neighborhood of N. Let $\theta = (y^1,\dots,y^k)\colon W \longrightarrow \theta(W) \subseteq \mathbb{R}^k$ be a chart of N and let $(\mathbf{E}_1,\dots,\mathbf{E}_c)$, $c = n-k$, be a local orthonormal frame of $T^\perp N$ over W. On any open product $W' \times Z \subseteq \theta(W) \times \mathbb{R}^c$ on which the following expression takes values in V, define

$$\Phi(u,z) := \exp^\perp_{\theta^{-1}(u)} \left(\sum_{\alpha=1}^{c} z^\alpha \mathbf{E}_\alpha \big|_{\theta^{-1}(u)} \right). \tag{2.87}$$

The inverse chart of Φ is called a *Fermi chart*; its coordinates are denoted by $(y^1,\dots,y^k,z^1,\dots,z^c)$.

The parametrization (2.87) is a diffeomorphism onto its image: it is the composition of the local trivialization of the normal bundle with $\exp^\perp \restriction_V$. Thus the definition presupposes no additional property of the coordinates.

Proposition 2.12.5. (Local properties of Fermi coordinates) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary of dimension n, let $N \subseteq M$ be an embedded submanifold without boundary of dimension k, and let $(y^1,\dots,y^k,z^1,\dots,z^c)$ be a Fermi chart associated with the local normal frame $(\mathbf{E}_1,\dots,\mathbf{E}_c)$. With indices $1 \leq a,b \leq k$ and $1 \leq \alpha,\beta,\gamma \leq c$, the following hold:

(a) N is described by $z^1 = \cdots = z^c = 0$.

(b) On N,

$$g_{a,k+\alpha} = 0, \qquad g_{k+\alpha,k+\beta} = \delta_{\alpha\beta}. \tag{2.88}$$

(c) For $q \in W$ and $v = z^\alpha \mathbf{E}_\alpha|_q$, the geodesic with initial data (q,v) has the expression

$$t \longmapsto (y^1(q),\dots,y^k(q),tz^1,\dots,tz^c). \tag{2.89}$$

(d) On N,

$$\Gamma^i_{k+\alpha,k+\beta} = 0 \quad (1 \leq i \leq n,\ 1 \leq \alpha,\beta \leq c). \tag{2.90}$$

(e) On N,

$$\partial_{z^\alpha} g_{k+\beta,k+\gamma} = 0. \tag{2.91}$$

The local identities agree with Proposition 5.26 in [95].

Proof. Part (a) follows from $\exp_q(0) = q$ and injectivity of the normal exponential map on the chosen neighborhood. At $(q,0)$, the differential of Φ sends ∂_{y^a} to a vector in T_qN and ∂_{z^α} to $\mathbf{E}_\alpha|_q \in T_q^\perp N$. Orthogonality and orthonormality of the frame prove (2.88).
The rescaling lemma for geodesics, Lemma 2.8.5, gives $\exp_q(tv) = \gamma_v(t)$ and proves (2.89). Substituting this curve into the geodesic equation and evaluating at $t = 0$ gives, for each i,

$$\Gamma^i_{k+\alpha,k+\beta}(q,0) z^\alpha z^\beta = 0 \quad \text{for every } z \in \mathbb{R}^c.$$

The Levi–Civita connection is torsion-free, so the two lower indices of Γ are symmetric. Polarizing the preceding quadratic form proves (2.90). Finally, metric compatibility in coordinates, Corollary 2.4.16, gives

$$\partial_{z^\alpha} g_{k+\beta,k+\gamma} = \Gamma^m_{k+\alpha,k+\beta} g_{m,k+\gamma} + \Gamma^m_{k+\alpha,k+\gamma} g_{k+\beta,m},$$

and both terms vanish by (2.90). This proves (2.91). □

Proposition 2.12.6. (Boundary normal coordinates) Let $(M,\mathbf{g})$ be a smooth Riemannian manifold of dimension n with nonempty boundary and let $p \in \partial M$. There exist a chart $\theta = (y^1,\dots,y^{n-1})\colon W \longrightarrow \theta(W)$ of ∂M around p, an $\varepsilon > 0$, and a chart of M

$$\Phi(u,t) = \exp_{\theta^{-1}(u)}\big(t\boldsymbol{\nu}_{\theta^{-1}(u)}\big), \qquad 0 \le t < \varepsilon, \tag{2.92}$$

where $\boldsymbol{\nu}$ is the inward unit normal. Throughout the chart,

$$\mathbf{g} = dt^2 + \mathbf{g}_t, \qquad g_{tt} = 1, \qquad g_{at} = 0. \tag{2.93}$$

If $\mathbf{II}^{\boldsymbol{\nu}}(\mathbf{X},\mathbf{Y}) := \mathbf{g}(\mathbf{II}(\mathbf{X},\mathbf{Y}),\boldsymbol{\nu})$ and we use the convention $\nabla_{\mathbf{X}}\boldsymbol{\nu} = -\mathbf{A}_{\boldsymbol{\nu}}\mathbf{X}$, then

$$\partial_t g_{ab}|_{t=0} = -2\mathbf{II}^{\boldsymbol{\nu}}_{ab}. \tag{2.94}$$

These semigeodesic coordinates are also described in [95, Example 6.44].

Proof. By Corollary 1.6.13, a smooth manifold with boundary is included as one half of its double, and the metric $\mathbf{g}$ extends to a Riemannian metric on a neighborhood of this half. There, apply Theorem 2.12.3 to the hypersurface ∂M and choose the sign of the normal coordinate to point inward. This gives (2.92).
Let $\mathbf{T} = \partial_t$ and let $\mathbf{J}_a = \partial_{y^a}$ be the variational field obtained by varying the initial point of the normal geodesics. Each curve $t \mapsto \Phi(y,t)$ is a unit-speed geodesic, so $\mathbf{g}(\mathbf{T},\mathbf{T}) = 1$. The coordinate vector fields commute and the connection is torsion-free; therefore,

$$\frac{d}{dt}\mathbf{g}(\mathbf{J}_a,\mathbf{T}) = \mathbf{g}(\nabla_{\mathbf{T}}\mathbf{J}_a,\mathbf{T}) + \mathbf{g}(\mathbf{J}_a,\nabla_{\mathbf{T}}\mathbf{T}) = \mathbf{g}(\nabla_{\mathbf{J}_a}\mathbf{T},\mathbf{T}) = \frac{1}{2}\mathbf{J}_a\mathbf{g}(\mathbf{T},\mathbf{T}) = 0.$$

At $t = 0$, $\mathbf{J}_a$ is tangent to ∂M and $\mathbf{T} = \boldsymbol{\nu}$, so $\mathbf{g}(\mathbf{J}_a,\mathbf{T}) = 0$. This proves (2.93) throughout the collar.

Using $[\mathbf{T}, \mathbf{J}_a] = 0$ and metric compatibility again,

$$\partial_t g_{ab} = \mathbf{g}(\nabla_{\mathbf{T}} \mathbf{J}_a, \mathbf{J}_b) + \mathbf{g}(\mathbf{J}_a, \nabla_{\mathbf{T}} \mathbf{J}_b) = 2\mathbf{g}(\nabla_{\mathbf{J}_a} \mathbf{T}, \mathbf{J}_b).$$

At $t = 0$, the Weingarten identity $\nabla_{\mathbf{J}_a} \boldsymbol{\nu} = -\mathbf{A}_{\boldsymbol{\nu}} \mathbf{J}_a$ gives (2.94). □

2.13 Jacobi fields and comparison theory

Jacobi fields describe the differential of the exponential map and therefore the infinitesimal separation of nearby geodesics.

Definition 2.13.1. (Variation and variational field) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary. A **smooth variation** of a curve $\gamma \colon I \to M$ is a smooth map

$$\alpha \colon (-\varepsilon, \varepsilon) \times I \longrightarrow M, \qquad \alpha(0, t) = \gamma(t).$$

We write

$$\mathbf{S} := d\alpha(\partial_s), \qquad \mathbf{T} := d\alpha(\partial_t),$$

and call $\mathbf{J}(t) := \mathbf{S}(0, t)$ the variational field along γ. The variation is **through geodesics** if $t \mapsto \alpha(s, t)$ is a geodesic for each s.

Proposition 2.13.2. (Commutation in a variation) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, let α be a smooth variation, and let $\mathbf{Z}$ be a vector field along α. Then

$$D_s \mathbf{T} = D_t \mathbf{S}, \qquad (D_s D_t - D_t D_s)\mathbf{Z} = \mathbf{R}(\mathbf{S}, \mathbf{T})\mathbf{Z}.$$

Proof. The coordinate vector fields ∂_s and ∂_t commute. The first identity is the torsion formula applied to $\mathbf{S}$ and $\mathbf{T}$, since the Levi–Civita connection is torsion-free. The second is directly the definition of $\mathbf{R}$ applied to the coordinate vector fields of the parametrized surface; the term corresponding to $[\partial_s, \partial_t]$ is zero. □

Theorem 2.13.3. (Jacobi equation) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary. The variational field $\mathbf{J}$ of a variation through geodesics satisfies

$$\frac{D^2 \mathbf{J}}{dt^2} + \mathbf{R}(\mathbf{J}, \dot\gamma)\dot\gamma = 0. \tag{2.95}$$

A smooth field along a geodesic satisfying (2.95) is called a **Jacobi field**.

Proof. For each s, $D_t \mathbf{T} = 0$. Differentiating with respect to s and using the preceding proposition,

$$0 = D_s D_t \mathbf{T} = D_t D_s \mathbf{T} + \mathbf{R}(\mathbf{S}, \mathbf{T})\mathbf{T} = D_t^2 \mathbf{S} + \mathbf{R}(\mathbf{S}, \mathbf{T})\mathbf{T}.$$

Restriction to $s = 0$ gives the stated equation. The sign agrees with the convention $\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}$ fixed in Definition 2.10.1. □

Proposition 2.13.4. (Existence and uniqueness) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, let $\gamma\colon I \to M$ be a geodesic, and let $t_0 \in I$ and $a, b \in T_{\gamma(t_0)}M$. There exists a unique Jacobi field $\mathbf{J}$ along γ such that

$$\mathbf{J}(t_0) = a, \qquad \frac{D\mathbf{J}}{dt}(t_0) = b.$$

In particular, the space of Jacobi fields along γ has dimension $2\dim M$.

Proof. Choose a parallel frame along γ. In this frame, (2.95) is a homogeneous second-order linear system whose coefficients are the smooth components of the curvature. The existence and uniqueness theorem for linear systems gives a unique solution with the prescribed data on all of I. The map $\mathbf{J} \mapsto (\mathbf{J}(t_0), D_t\mathbf{J}(t_0))$ is then an isomorphism with $T_{\gamma(t_0)}M \oplus T_{\gamma(t_0)}M$. □

Proposition 2.13.5. (Representation by geodesic variations) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, let $\gamma\colon I \to M$ be a geodesic, and let $\mathbf{J}$ be a Jacobi field along γ. Locally in the parameter, $\mathbf{J}$ is the variational field of a variation through geodesics. More precisely, this representation can be realized simultaneously on any compact subinterval contained in the domain of γ.

Proof. Fix t_0 and set $a = \mathbf{J}(t_0)$ and $b = D_t\mathbf{J}(t_0)$. Choose a curve $\sigma(s)$ with $\sigma(0) = \gamma(t_0)$ and $\sigma'(0) = a$, and a field $\mathbf{V}(s) \in T_{\sigma(s)}M$ such that

$$\mathbf{V}(0) = \dot\gamma(t_0), \qquad D_s\mathbf{V}(0) = b.$$

Smooth dependence of solutions of the geodesic equation on their initial data allows us to define, for small s,

$$\alpha(s,t) = \exp_{\sigma(s)}\big((t - t_0)\mathbf{V}(s)\big).$$

Each curve with s fixed is a geodesic. Its variational field has the same initial data a, b at t_0 as $\mathbf{J}$; the preceding uniqueness result shows that it equals $\mathbf{J}$. Smooth dependence of the geodesic flow allows us to choose a single interval of values of s on any compact subinterval of I. □

Proposition 2.13.6. (Differential of the exponential map) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, let $p \in M$, let v be a vector in the domain of $\exp_p$, and let $w \in T_pM$. Along $\gamma_v(t) = \exp_p(tv)$, define

$$\mathbf{J}_w(t) := d(\exp_p)_{tv}(tw).$$

Then $\mathbf{J}_w$ is the unique Jacobi field with

$$\mathbf{J}_w(0) = 0, \qquad \frac{D\mathbf{J}_w}{dt}(0) = w,$$

and, in particular,

$$d(\exp_p)_v(w) = \mathbf{J}_w(1).$$

Proof. The family $\alpha(s,t) = \exp_p(t(v+sw))$ is a variation through geodesics and its variational field is precisely $\mathbf{J}_w(t)$. At $t = 0$, all the curves start at p, so $\mathbf{J}_w(0) = 0$; the initial derivative is w. The last equality follows by taking $t = 1$. □

Definition 2.13.7. (Conjugate points) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $\gamma\colon I\to M$ be a geodesic, and let $t_0,t_1\in I$ be distinct. The points $\gamma(t_0)$ and $\gamma(t_1)$ are **conjugate along** γ if there exists a nonzero Jacobi field $\mathbf{J}$ such that $\mathbf{J}(t_0)=\mathbf{J}(t_1)=0$. Their multiplicity is the dimension of the space of such fields.

Proposition 2.13.8. (Conjugacy and critical values of the exponential map) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $p\in M$, and let $v\neq 0$ be a vector in the domain of $\exp_p$. Then $\exp_p(v)$ is conjugate to p along $t\mapsto\exp_p(tv)$ if and only if v is a critical point of $\exp_p$. The multiplicity of the conjugate point is $\dim\ker d(\exp_p)_v$.

Proof. By the preceding proposition, the map $w\mapsto \mathbf{J}_w$ identifies $\ker d(\exp_p)_v$ with the Jacobi fields that vanish at $t=0$ and $t=1$. Since the domain and codomain of $d(\exp_p)_v$ have the same dimension, it is singular exactly when its kernel is nontrivial. □

To formulate the constant-curvature models, define

$$s_c(t):=\begin{cases}\dfrac{\sin(\sqrt{c}\,t)}{\sqrt{c}}, & c>0,\\[2mm] t, & c=0,\\[2mm] \dfrac{\sinh(\sqrt{-c}\,t)}{\sqrt{-c}}, & c<0,\end{cases}\qquad c_c(t):=s_c'(t).$$

These functions satisfy $s_c''+cs_c=0$, $s_c(0)=0$, and $s_c'(0)=1$.

Proposition 2.13.9. (Jacobi fields in constant curvature) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary of constant sectional curvature c, let γ be a unit-speed geodesic, and let $\mathbf{J}$ be a Jacobi field along γ. Let P_t be parallel transport from $\gamma(0)$ to $\gamma(t)$. If $a=\mathbf{J}(0)$ and $b=D_t\mathbf{J}(0)$, decompose $a=a^\top+a^\perp$ and $b=b^\top+b^\perp$ with respect to $\mathbb{R}\dot\gamma(0)\oplus\dot\gamma(0)^\perp$. Then

$$\mathbf{J}(t)=P_t\big(a^\top+tb^\top+c_c(t)a^\perp+s_c(t)b^\perp\big).$$

In particular, if $\mathbf{J}(0)=0$ and $D_t\mathbf{J}(0)=w\perp\dot\gamma(0)$, then $\mathbf{J}(t)=s_c(t)P_tw$.

Proof. In constant curvature, $\mathbf{R}(\mathbf{X},\dot\gamma)\dot\gamma=c\mathbf{X}$ for $\mathbf{X}\perp\dot\gamma$, while the curvature term vanishes in the direction of $\dot\gamma$. In a parallel frame, the Jacobi equation thus splits into $u''=0$ for the tangential component and $u''+cu=0$ for each normal component. The solutions with the given initial data are exactly those in the formula. □

To formulate the variational argument underlying comparison, we introduce the index form. A partition $a=t_0<t_1<\cdots<t_N=b$ is *admissible* for a field $\mathbf{V}$ along γ if $\mathbf{V}$ is continuous on $[a,b]$ and its restriction to each $[t_{r-1},t_r]$ is smooth up to the endpoints. Denote the space of these piecewise smooth fields by $\mathcal{V}_{\mathrm{ps}}(\gamma)$ and write

$$\mathcal{V}_0(\gamma):=\{\mathbf{V}\in\mathcal{V}_{\mathrm{ps}}(\gamma)\mid \mathbf{V}(a)=\mathbf{V}(b)=0\}.$$

Definition 2.13.10. (Index form) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\gamma\colon [a,b]\longrightarrow M$ be a geodesic. For $\mathbf{V},\mathbf{W}\in\mathcal{V}_{\mathrm{ps}}(\gamma)$, define

$$I_\gamma(\mathbf{V},\mathbf{W}) := \int_a^b \left[\left\langle \frac{D\mathbf{V}}{dt},\frac{D\mathbf{W}}{dt}\right\rangle_{\mathbf{g}} - \left\langle \mathbf{R}(\mathbf{V},\dot\gamma)\dot\gamma,\mathbf{W}\right\rangle_{\mathbf{g}}\right] dt.$$

Covariant derivatives are taken on each subinterval of an admissible partition.

Proposition 2.13.11. (Properties and integration by parts for the index form) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\gamma\colon [a,b]\to M$ be a geodesic. The form I_γ is well-defined, bilinear, and symmetric. If $a=t_0<\cdots<t_N=b$ is a common admissible partition for $\mathbf{V}$ and $\mathbf{W}$, and

$$\left[\frac{D\mathbf{V}}{dt}\right]_{t_r} := \frac{D\mathbf{V}}{dt}(t_r^+) - \frac{D\mathbf{V}}{dt}(t_r^-), \qquad 1\le r\le N-1,$$

then

$$\begin{aligned} I_\gamma(\mathbf{V},\mathbf{W}) = &\left[\left\langle \frac{D\mathbf{V}}{dt},\mathbf{W}\right\rangle_{\mathbf{g}}\right]_a^b - \sum_{r=1}^{N-1}\left\langle \left[\frac{D\mathbf{V}}{dt}\right]_{t_r},\mathbf{W}(t_r)\right\rangle_{\mathbf{g}} \\ &- \int_a^b \left\langle \frac{D^2\mathbf{V}}{dt^2}+\mathbf{R}(\mathbf{V},\dot\gamma)\dot\gamma,\mathbf{W}\right\rangle_{\mathbf{g}} dt. \end{aligned} \tag{2.96}$$

The values at a and b are the limits from the interior. In particular, if $\mathbf{J}$ is a smooth Jacobi field, then

$$I_\gamma(\mathbf{J},\mathbf{W}) = \left[\left\langle \frac{D\mathbf{J}}{dt},\mathbf{W}\right\rangle_{\mathbf{g}}\right]_a^b. \tag{2.97}$$

Proof. On a common admissible partition, the integrand is continuous in the interior of each subinterval and has one-sided limits at its endpoints. Refining the partition merely subdivides existing integrals, and the partition points have measure zero; the value is therefore independent of the partition. Bilinearity is immediate. The symmetries of the curvature tensor imply

$$\langle \mathbf{R}(\mathbf{V},\dot\gamma)\dot\gamma,\mathbf{W}\rangle_{\mathbf{g}} = \langle \mathbf{R}(\mathbf{W},\dot\gamma)\dot\gamma,\mathbf{V}\rangle_{\mathbf{g}},$$

so I_γ is symmetric.
On each interval $[t_{r-1},t_r]$, we have

$$\frac{d}{dt}\left\langle \frac{D\mathbf{V}}{dt},\mathbf{W}\right\rangle_{\mathbf{g}} = \left\langle \frac{D^2\mathbf{V}}{dt^2},\mathbf{W}\right\rangle_{\mathbf{g}} + \left\langle \frac{D\mathbf{V}}{dt},\frac{D\mathbf{W}}{dt}\right\rangle_{\mathbf{g}}.$$

Integrate this equality and sum over r. At an interior point t_r, the boundary terms contribute

$$\left\langle \frac{D\mathbf{V}}{dt}(t_r^-),\mathbf{W}(t_r)\right\rangle_{\mathbf{g}} - \left\langle \frac{D\mathbf{V}}{dt}(t_r^+),\mathbf{W}(t_r)\right\rangle_{\mathbf{g}} = -\left\langle \left[\frac{D\mathbf{V}}{dt}\right]_{t_r},\mathbf{W}(t_r)\right\rangle_{\mathbf{g}},$$

because $\mathbf{W}$ is continuous. Including the curvature term gives (2.96). If $\mathbf{J}$ is a Jacobi field, it has no jumps and the integrand in the second line vanishes, giving (2.97). □

The index form is the second variation of energy at a geodesic with fixed endpoints.

Proposition 2.13.12. (Second variation of energy) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $\alpha\colon (-\varepsilon,\varepsilon)\times[a,b]\to M$ be a smooth variation of a geodesic $\gamma = \alpha(0,\cdot)$ with fixed endpoints, and let $\mathbf{V}$ be its variational field. If

$$E(\alpha_s) := \frac{1}{2}\int_a^b \left|d\alpha_{(s,t)}(\partial_t)\right|_{\mathbf{g}}^2 dt,$$

then

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\alpha_s) = I_\gamma(\mathbf{V},\mathbf{V}).$$

Without the fixed-endpoint hypothesis, the formula also contains the boundary term

$$\Big[\langle D_s\mathbf{S},\mathbf{T}\rangle_{\mathbf{g}}\Big]_a^b\Big|_{s=0}, \qquad \mathbf{S} = d\alpha(\partial_s),\quad \mathbf{T} = d\alpha(\partial_t).$$

Proof. Write $\mathbf{S} = d\alpha(\partial_s)$ and $\mathbf{T} = d\alpha(\partial_t)$. Metric compatibility and the identity $D_s\mathbf{T} = D_t\mathbf{S}$ give

$$\frac{d}{ds}E(\alpha_s) = \int_a^b \langle D_t\mathbf{S},\mathbf{T}\rangle_{\mathbf{g}}\,dt.$$

Differentiating again and restricting to $s=0$, use $D_sD_t\mathbf{S} = D_tD_s\mathbf{S} + \mathbf{R}(\mathbf{S},\mathbf{T})\mathbf{S}$ and $D_s\mathbf{T} = D_t\mathbf{S}$ to obtain

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\alpha_s) = \int_a^b \left[\langle D_tD_s\mathbf{S},\dot\gamma\rangle_{\mathbf{g}} + \langle \mathbf{R}(\mathbf{V},\dot\gamma)\mathbf{V},\dot\gamma\rangle_{\mathbf{g}} + \left|\frac{D\mathbf{V}}{dt}\right|_{\mathbf{g}}^2\right] dt.$$

After integration by parts, the first integral reduces to the boundary term in the statement, since the geodesic satisfies $D_t\dot\gamma = 0$. Because the endpoints of the variation are fixed, $D_s\mathbf{S} = 0$ there, and this term vanishes. Finally,

$$\langle \mathbf{R}(\mathbf{V},\dot\gamma)\mathbf{V},\dot\gamma\rangle_{\mathbf{g}} = -\langle \mathbf{R}(\mathbf{V},\dot\gamma)\dot\gamma,\mathbf{V}\rangle_{\mathbf{g}},$$

which gives the asserted formula. □

Lemma 2.13.13. (Index lemma) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $\gamma\colon[a,b]\to M$ be a geodesic, and suppose that $\gamma(a)$ is not conjugate to $\gamma(t)$ along γ for any $t\in(a,b]$. Let $\mathbf{J}$ be a Jacobi field along γ with $\mathbf{J}(a) = 0$. If $\mathbf{V}\in\mathcal{V}_{\mathrm{ps}}(\gamma)$ satisfies

$$\mathbf{V}(a) = 0, \qquad \mathbf{V}(b) = \mathbf{J}(b),$$

then

$$I_\gamma(\mathbf{J},\mathbf{J}) \le I_\gamma(\mathbf{V},\mathbf{V}).$$

Equality holds if and only if $\mathbf{V} = \mathbf{J}$.

Proof. Choose an orthonormal basis $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ of $T_{\gamma(a)}M$ and extend it to a parallel frame along γ. For each i, let $\mathbf{J}_i$ be the unique Jacobi field determined by

$$\mathbf{J}_i(a) = 0, \qquad \frac{D\mathbf{J}_i}{dt}(a) = \mathbf{e}_i.$$

For $t > a$, the vectors $\mathbf{J}_1(t),\dots,\mathbf{J}_n(t)$ form a basis of $T_{\gamma(t)}M$. Indeed, a linear combination vanishing at t would give a Jacobi field vanishing at a and t; by the absence of conjugate points, it would have to be the zero field, and its initial data would force all the coefficients to vanish.

We first prove that

$$I_\gamma(\mathbf{W},\mathbf{W}) \geq 0 \qquad \text{for every } \mathbf{W} \in \mathcal{V}_0(\gamma), \tag{2.98}$$

and determine the equality case. For $t > a$, write uniquely

$$\mathbf{W}(t) = f^i(t)\mathbf{J}_i(t).$$

The functions f^i are piecewise smooth. They also extend smoothly from the right to a on the first admissible subinterval. To see this, express the fields in the parallel frame and denote by $A(t)$ the matrix whose columns are the components of the $\mathbf{J}_i(t)$. Since $A(a) = 0$ and $A'(a) = I$, the integral formula

$$A(t) = (t-a)B(t), \qquad B(t) := \int_0^1 A'\big(a + s(t-a)\big)\,ds,$$

defines a smooth matrix with $B(a) = I$. Similarly, $\mathbf{W}(a) = 0$ allows us to write $\mathbf{W}(t) = (t-a)w(t)$ with w smooth on the first subinterval. For t close to a, $B(t)$ is invertible and

$$(f^1(t),\dots,f^n(t))^{\mathsf{T}} = B(t)^{-1}w(t),$$

which proves the extension and controls the endpoint where all the $\mathbf{J}_i$ vanish.

For each pair i, j, the function

$$\left\langle \frac{D\mathbf{J}_i}{dt}, \mathbf{J}_j \right\rangle_{\mathbf{g}} - \left\langle \mathbf{J}_i, \frac{D\mathbf{J}_j}{dt} \right\rangle_{\mathbf{g}}$$

is constant. Its derivative vanishes by the Jacobi equation and symmetry of the endomorphism $\mathbf{Z} \mapsto \mathbf{R}(\mathbf{Z},\dot\gamma)\dot\gamma$; its value at a is zero. Therefore,

$$\left\langle \frac{D\mathbf{J}_i}{dt}, \mathbf{J}_j \right\rangle_{\mathbf{g}} = \left\langle \mathbf{J}_i, \frac{D\mathbf{J}_j}{dt} \right\rangle_{\mathbf{g}}. \tag{2.99}$$

On each subinterval where $\mathbf{W}$ is smooth, set

$$\mathbf{X} := f^i \frac{D\mathbf{J}_i}{dt}, \qquad \mathbf{Y} := \frac{df^i}{dt}\mathbf{J}_i.$$

Then $\frac{D\mathbf{W}}{dt} = \mathbf{X} + \mathbf{Y}$. The Jacobi equation and (2.99) give

$$\frac{D\mathbf{X}}{dt} = \frac{df^i}{dt}\frac{D\mathbf{J}_i}{dt} - \mathbf{R}(\mathbf{W},\dot\gamma)\dot\gamma, \qquad \left\langle \mathbf{W}, \frac{df^i}{dt}\frac{D\mathbf{J}_i}{dt} \right\rangle_{\mathbf{g}} = \langle \mathbf{Y}, \mathbf{X}\rangle_{\mathbf{g}}.$$

Consequently,

$$\left|\frac{D\mathbf{W}}{dt}\right|^2_{\mathbf{g}} - \langle \mathbf{R}(\mathbf{W},\dot\gamma)\dot\gamma, \mathbf{W}\rangle_{\mathbf{g}} = \frac{d}{dt}\langle \mathbf{W}, \mathbf{X}\rangle_{\mathbf{g}} + |\mathbf{Y}|^2_{\mathbf{g}}. \tag{2.100}$$

The coefficients f^i are continuous at the partition points because $\mathbf{W}$ is continuous and $(\mathbf{J}_i(t))$ is a basis;

therefore, $\mathbf{X}$ is also continuous there. Integrating (2.100) over all the subintervals, the interior boundary terms cancel. At a, they vanish because $\mathbf{W}(a) = 0$ and $\mathbf{X}$ has a limit, and at b they vanish because $\mathbf{W}(b) = 0$. Thus,

$$I_\gamma(\mathbf{W}, \mathbf{W}) = \int_a^b |\mathbf{Y}|_{\mathbf{g}}^2 dt \geq 0.$$

If equality holds, $\mathbf{Y} = 0$ on each subinterval. Since the $\mathbf{J}_i(t)$ are linearly independent for $t > a$, all the functions f^i are constant on each subinterval; their continuity shows that they are constant on all of $[a, b]$. Then $\mathbf{W}$ is a Jacobi field. Since $\mathbf{W}(a) = \mathbf{W}(b) = 0$ and the endpoints are not conjugate, it follows that $\mathbf{W} = 0$. Conversely, $\mathbf{W} = 0$ gives equality. This proves (2.98) and its equality case.

Return to the fields in the statement and set $\mathbf{W} := \mathbf{V} - \mathbf{J}$. Then $\mathbf{W} \in \mathcal{V}_0(\gamma)$. The integration-by-parts formula and the Jacobi equation imply

$$I_\gamma(\mathbf{J}, \mathbf{W}) = \left[\left\langle \frac{D\mathbf{J}}{dt}, \mathbf{W} \right\rangle_{\mathbf{g}} \right]_a^b = 0.$$

By symmetry and bilinearity,

$$I_\gamma(\mathbf{V}, \mathbf{V}) = I_\gamma(\mathbf{J}, \mathbf{J}) + 2I_\gamma(\mathbf{J}, \mathbf{W}) + I_\gamma(\mathbf{W}, \mathbf{W}) = I_\gamma(\mathbf{J}, \mathbf{J}) + I_\gamma(\mathbf{W}, \mathbf{W}).$$

The inequality already proved gives the conclusion, and equality holds exactly when $\mathbf{W} = 0$, that is, when $\mathbf{V} = \mathbf{J}$. □

Theorem 2.13.14. (Comparison of Jacobi fields) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary of dimension at least two, let $\gamma\colon [0, b] \to M$ be a unit-speed geodesic, and let $\mathbf{J}$ be a normal Jacobi field, that is, $\langle \mathbf{J}, \dot{\gamma} \rangle_{\mathbf{g}} = 0$, with $\mathbf{J}(0) = 0$. Define

$$\rho_c := \begin{cases} +\infty, & c \leq 0, \\ \dfrac{\pi}{\sqrt{c}}, & c > 0, \end{cases}$$

and let τ be the first time conjugate to 0 along γ, with the convention $\tau = +\infty$ if there is none.

(a) If every sectional curvature of M is less than or equal to c, then

$$|\mathbf{J}(t)|_{\mathbf{g}} \geq s_c(t) \left| \frac{D\mathbf{J}}{dt}(0) \right|_{\mathbf{g}}.$$

The inequality holds for $0 \leq t \leq \min\{b, \rho_c\}$. Moreover, $\gamma(0)$ has no conjugate points along γ as long as $0 < t < \min\{b, \rho_c\}$.

(b) If every sectional curvature of M is greater than or equal to c, then

$$|\mathbf{J}(t)|_{\mathbf{g}} \leq s_c(t) \left| \frac{D\mathbf{J}}{dt}(0) \right|_{\mathbf{g}}$$

for $0 \leq t \leq \min\{b, \tau, \rho_c\}$. If $c > 0$ and $b \geq \rho_c$, then $\tau \leq \rho_c$.

If $\mathbf{J}$ is not the zero field and $0 < t_0 < \min\{b, \tau, \rho_c\}$ in the case involving τ, or $0 < t_0 < \min\{b, \rho_c\}$ in the other

case, equality in the corresponding comparison at t_0 holds if and only if

$$\mathbf{J}(t) = s_c(t)P_t\left(\frac{D\mathbf{J}}{dt}(0)\right) \quad \text{and} \quad \mathbf{R}(P_t w, \dot\gamma(t))\dot\gamma(t) = cP_t w$$

for $0 < t \le t_0$, where $w = \frac{D\mathbf{J}}{dt}(0)$ and P_t is parallel transport from $\gamma(0)$ to $\gamma(t)$.

Proof. Set $w = \frac{D\mathbf{J}}{dt}(0)$. If $w = 0$, uniqueness for the Jacobi equation implies $\mathbf{J} = 0$, and both inequalities are immediate. Thus, suppose that $w \neq 0$. Choose a parallel orthonormal frame $(\mathbf{E}_2, \dots, \mathbf{E}_n)$ of the orthogonal complement of $\dot\gamma$ and write

$$\mathbf{J}(t) = u^\alpha(t)\mathbf{E}_\alpha(t), \qquad u(t) = (u^2(t), \dots, u^n(t)) \in \mathbb{R}^{n-1}.$$

In particular, $|\mathbf{J}|_{\mathbf{g}} = |u|$, $\left|\frac{D\mathbf{J}}{dt}\right|_{\mathbf{g}} = |u'|$, and $u'(0)$ represents w.
For $t < \rho_c$, consider on $[0, t]$ the model index form

$$I_c(v, v) := \int_0^t \left(|v'(s)|^2 - c|v(s)|^2\right) ds$$

on fields with values in $\mathbb{R}^{n-1}$. The model Jacobi equation is $v'' + cv = 0$. Since $s_c(s) > 0$ for $0 < s < \rho_c$, this equation has no nonzero solution vanishing at 0 and t. The field

$$\overline{u}_t(s) := \frac{s_c(s)}{s_c(t)}u(t)$$

minimizes I_c among fields taking the value 0 at $s = 0$ and $u(t)$ at $s = t$. Indeed, for such a field, write $v(s) = s_c(s)f(s)$ on $(0, t]$. The factorization at the initial endpoint used in the proof of Lemma 2.13.13 shows that f extends regularly to 0. Since $s_c'' = -cs_c$,

$$|v'|^2 - c|v|^2 = \frac{d}{ds}\left(s_c c_c |f|^2\right) + s_c^2|f'|^2.$$

The boundary term at 0 vanishes. The value at t is fixed and the remaining integral is nonnegative, with equality exactly when f is constant. This proves the minimizing property and, for $v = \overline{u}_t$, gives

$$I_c(\overline{u}_t, \overline{u}_t) = \frac{c_c(t)}{s_c(t)}|u(t)|^2. \tag{2.101}$$

First suppose that $K \le c$. Since $\mathbf{J}$ is normal and γ has unit speed,

$$\langle \mathbf{R}(\mathbf{J}, \dot\gamma)\dot\gamma, \mathbf{J}\rangle_{\mathbf{g}} \le c|\mathbf{J}|_{\mathbf{g}}^2.$$

The index lemma for the model and this inequality imply

$$\begin{aligned}\frac{c_c(t)}{s_c(t)}|\mathbf{J}(t)|_{\mathbf{g}}^2 &= I_c(\overline{u}_t, \overline{u}_t)\\ &\le I_c(u, u)\\ &\le I_{\gamma|_{[0,t]}}(\mathbf{J}, \mathbf{J})\end{aligned}$$

$$= \left\langle \frac{D\mathbf{J}}{dt}(t), \mathbf{J}(t) \right\rangle_{\mathbf{g}}.$$

The last equality is (2.97), since $\mathbf{J}(0) = 0$. On every interval where $\mathbf{J}(t) \neq 0$, it follows that

$$\frac{d}{dt} \log \left(\frac{|\mathbf{J}(t)|_{\mathbf{g}}}{s_c(t)} \right) = \frac{\left\langle \frac{D\mathbf{J}}{dt}, \mathbf{J} \right\rangle_{\mathbf{g}}}{|\mathbf{J}|^2_{\mathbf{g}}} - \frac{c_c}{s_c} \geq 0.$$

In the parallel frame, $u(0) = 0$ and $u'(0) = w$; hence,

$$\lim_{t \to 0^+} \frac{|\mathbf{J}(t)|_{\mathbf{g}}}{s_c(t)} = \lim_{t \to 0^+} \frac{\left| \frac{u(t)}{t} \right|}{\frac{s_c(t)}{t}} = |w|_{\mathbf{g}}.$$

The quotient is nondecreasing, giving

$$|\mathbf{J}(t)|_{\mathbf{g}} \geq s_c(t)|w|_{\mathbf{g}}$$

as long as $s_c(t) > 0$. In particular, $\mathbf{J}$ cannot vanish there. If an arbitrary Jacobi field vanishes at 0 and at some $t > 0$, its tangential component has the form $at\dot{\gamma}$ and must vanish; the field is therefore normal. The preceding inequality excludes such a nonzero field for $t < \rho_c$. The comparison extends to the endpoint $t = \min\{b, \rho_c\}$ by continuity. This proves (a).

Now suppose that $K \geq c$ and fix $0 < t < \min\{b, \tau, \rho_c\}$. Along $\gamma|_{[0,t]}$, define the field

$$\mathbf{V}_t(s) := \frac{s_c(s)}{s_c(t)} P_{t,s} \mathbf{J}(t),$$

where $P_{t,s}$ is parallel transport from $\gamma(t)$ to $\gamma(s)$. The fields $\mathbf{V}_t$ and $\mathbf{J}$ have the same values at the endpoints. Since there are no conjugate points before t, Lemma 2.13.13 and the lower curvature bound give

$$\begin{aligned} \left\langle \frac{D\mathbf{J}}{dt}(t), \mathbf{J}(t) \right\rangle_{\mathbf{g}} &= I_{\gamma|_{[0,t]}}(\mathbf{J}, \mathbf{J}) \\ &\leq I_{\gamma|_{[0,t]}}(\mathbf{V}_t, \mathbf{V}_t) \\ &\leq I_c(\overline{u}_t, \overline{u}_t) \\ &= \frac{c_c(t)}{s_c(t)} |\mathbf{J}(t)|^2_{\mathbf{g}}. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \log \left(\frac{|\mathbf{J}(t)|_{\mathbf{g}}}{s_c(t)} \right) \leq 0,$$

and the same limit at 0 shows that

$$|\mathbf{J}(t)|_{\mathbf{g}} \leq s_c(t)|w|_{\mathbf{g}}.$$

Continuity allows us to include the first conjugate time and the endpoint of the indicated interval. If $c > 0$, $b \geq \rho_c$, and there were no conjugate point before ρ_c, the inequality, applied for $t < \rho_c$, would give $\mathbf{J}(\rho_c) = 0$ for every normal Jacobi field with $\mathbf{J}(0) = 0$. Taking $w \neq 0$ would produce a conjugate point at ρ_c. Consequently, $\tau \leq \rho_c$, completing (b).

It remains to study equality. In either case, the quotient $\frac{|\mathbf{J}|_{\mathbf{g}}}{s_c}$ is monotone and its limit at zero is $|w|_{\mathbf{g}}$. If

equality holds at t_0, this quotient is constant on $[0, t_0]$. In the preceding chains of inequalities, equality must hold both in the comparison of the index forms and in the curvature estimate. The equality condition for the model identity in the first case and that of Lemma 2.13.13 in the second give

$$\mathbf{J}(t) = s_c(t) P_t w.$$

Moreover, the self-adjoint endomorphism $\mathbf{Z} \mapsto \mathbf{R}(\mathbf{Z}, \dot\gamma)\dot\gamma$ is bounded above by cI in the first case and bounded below by cI in the second. Equality of their quadratic forms along $P_t w$ implies

$$\mathbf{R}(P_t w, \dot\gamma)\dot\gamma = c P_t w.$$

Conversely, these two identities turn all the preceding steps into equalities and yield equality in the comparison. □

Proposition 2.13.15. (Continuity of the injectivity radius) Let $(M, \mathbf{g})$ be a complete connected Riemannian manifold without boundary. Then

$$\operatorname{inj} \colon M \longrightarrow (0, \infty]$$

is continuous, where $(0, \infty]$ carries the extended order topology.

We use continuity of the cut time on the unit tangent bundle, proved by Lee [95].

Proof. For $v \in SM$, let

$$c(v) := \sup\{t > 0 \mid d_{\mathbf{g}}(\pi(v), \exp_{\pi(v)}(tv)) = t\}$$

be the cut time of the unit-speed geodesic determined by v. On a complete manifold, the cut time is a continuous function $c \colon SM \to (0, \infty]$. Moreover,

$$\operatorname{inj}(p) = \inf_{v \in S_pM} c(v).$$

In a local trivialization of SM, the fibers S_pM are identified with the compact sphere $\mathbb{S}^{n-1}$. Continuity of c and compactness of this sphere imply that the infimum depends continuously on p (including when its value is infinite). The assertion follows. □

2.14 Integration on Riemannian manifolds

Integration on a manifold requires replacing the Euclidean volume element by an object that transforms correctly under changes of coordinates. On an oriented manifold, differential forms of top degree provide such an object; the Riemannian metric will later select a canonical volume form. We begin with orientation and integration of forms in charts.

Definition 2.14.1. Let V be a finite-dimensional vector space. We say that two ordered bases $(E_1, \dots, E_n)$ and $(\widetilde E_1, \dots, \widetilde E_n)$ are **consistently oriented** if the change-of-basis matrix (B_i^j) satisfying $E_i = B_i^j \widetilde E_j$ has positive determinant.

One can show that consistent orientation is an equivalence relation on the set of ordered bases of a finite-

dimensional vector space. Moreover, this equivalence relation has exactly two classes, allowing us to give the following definition:

Definition 2.14.2. Let V be a finite-dimensional vector space. An **orientation of** V is an equivalence class of ordered bases. Given an ordered basis $(E_1,\dots,E_n)$, denote the orientation it determines by $[E_1,\dots,E_n]$. An **oriented vector space** is a vector space V together with an orientation determined by an ordered basis $[E_1,\dots,E_n]$. Once we fix an orientation of V, a basis with this orientation is said to be **positively oriented**. An ordered basis that is not positively oriented is said to be **negatively oriented**.

Definition 2.14.3. Let M be a smooth manifold with or without boundary. A **pointwise orientation** on M is a choice of orientation on each tangent space. If $(\mathbf{E}_i)$ is a local frame of TM, we say that $(\mathbf{E}_i)$ **is positively oriented** if $(\mathbf{E}_1|_p,\dots,\mathbf{E}_n|_p)$ is a positively oriented ordered basis of T_pM for each $p \in U$, where U is the domain of the local frame $(\mathbf{E}_i)$. If $(\mathbf{E}_1|_p,\dots,\mathbf{E}_n|_p)$ is negatively oriented for each $p \in U$, we say that the local frame is **negatively oriented**.

A **continuous orientation of** M is a pointwise orientation such that, for each $p \in M$, there exists an open set $U \subseteq M$ such that U is the domain of a positively oriented local frame. We say that M is **orientable** if it has a continuous pointwise orientation. We denote this orientation by $\mathscr{O}$ and say that $(M,\mathscr{O})$ is an **oriented smooth manifold**.

Finally, if (U,φ) is a smooth chart of an oriented manifold M, with $\varphi = (x^1,\dots,x^n)$, we say that it is a **positively oriented chart** (or simply an **oriented chart**) if the coordinate frame $\left(\partial_i\right)$ is positively oriented. If this coordinate frame is negatively oriented, we say that the chart is a **negatively oriented chart**.

Proposition 2.14.4. (The Riemannian volume form) Let $(M,\mathbf{g})$ be an oriented Riemannian manifold of dimension n with or without boundary. There exists a unique n-form $dV_{\mathbf{g}} \in \Omega^n(M)$, called the **Riemannian volume form**, characterized by any of the following properties:

(a) If $(\varepsilon^1,\dots,\varepsilon^n)$ is any positively oriented orthonormal local coframe of T^*M, then

$$dV_{\mathbf{g}} = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n.$$

(b) If $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ is any positively oriented orthonormal local frame of TM, then

$$dV_{\mathbf{g}}(\mathbf{E}_1,\dots,\mathbf{E}_n) = 1.$$

(c) If (U,φ), $\varphi = (x^1,\dots,x^n)$ is an oriented chart of M, then

$$dV_{\mathbf{g}} = \sqrt{\det(\mathbf{g})}\,\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n.$$

Proof. Let $(U,x^1,\dots,x^n)$ be a positively oriented chart. The matrix $G_x = (g_{ij})$ is symmetric and positive definite, so $\det G_x > 0$. On U, define

$$\omega_x := \sqrt{\det G_x}\,\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n. \tag{2.102}$$

This form is smooth. We will show that the forms obtained in this way agree on overlaps.

Let $(x^1,\dots,x^n)$ and $(y^1,\dots,y^n)$ be two positively oriented coordinate systems and write $G_y = (\widetilde{g}_{ab})$ for the matrix of the metric in the coordinates y. On the overlap,

$$\widetilde{g}_{ab} = g_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b},$$

that is,

$$G_y = \left(\frac{\partial x}{\partial y}\right)^T G_x \left(\frac{\partial x}{\partial y}\right).$$

Taking determinants gives

$$\det G_y = \det\left(\frac{\partial x}{\partial y}\right)^2 \det G_x. \tag{2.103}$$

Both coordinate systems induce the fixed orientation; therefore,

$$\det\left(\frac{\partial x}{\partial y}\right) > 0.$$

The transformation law for the wedge product, in turn, gives

$$\mathbf{d}y^1 \wedge \cdots \wedge \mathbf{d}y^n = \det\left(\frac{\partial y}{\partial x}\right)\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n.$$

Combining this equality with (2.103), we conclude that

$$\begin{aligned}\sqrt{\det G_y}\,\mathbf{d}y^1 \wedge \cdots \wedge \mathbf{d}y^n &= \det\left(\frac{\partial x}{\partial y}\right)\sqrt{\det G_x}\det\left(\frac{\partial y}{\partial x}\right)\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n \\ &= \sqrt{\det G_x}\,\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n.\end{aligned}$$

Thus the local forms glue together to define a smooth global form $dV_{\mathbf{g}}$. Property (c) holds by construction. Now let $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ be a positively oriented orthonormal local frame. In an oriented chart, write

$$\mathbf{E}_a = B_a^i \partial_i$$

and denote by B the matrix whose columns are the components of the $\mathbf{E}_a$. Orthonormality is equivalent to

$$B^T G_x B = I.$$

Therefore, $(\det B)^2 \det G_x = 1$. Since the frame and the chart have the same orientation, $\det B > 0$, and hence

$$\det B = \frac{1}{\sqrt{\det G_x}}.$$

It follows from (2.102) that

$$dV_{\mathbf{g}}(\mathbf{E}_1,\dots,\mathbf{E}_n) = \sqrt{\det G_x}\,\det B = 1.$$

This proves (b). If $(\varepsilon^1,\dots,\varepsilon^n)$ is the dual coframe, then $(\varepsilon^1 \wedge \cdots \wedge \varepsilon^n)(\mathbf{E}_1,\dots,\mathbf{E}_n) = 1$. Since the space of n-covectors at each point is one-dimensional, (b) implies

$$dV_{\mathbf{g}} = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n,$$

which is (a). Conversely, (a) immediately implies (b), and the calculation with the coordinate frame shows that (b) implies (c). Thus the three characterizations are equivalent.
The construction does not depend on the orthonormal frame. Two positively oriented orthonormal frames are related by a matrix with values in $SO(n)$, whose determinant is one; the wedge products of their dual coframes agree. If another n-form satisfies any of the characterizations, its value on each positively oriented orthonormal frame is one, so it agrees pointwise with $dV_{\mathbf{g}}$. This proves uniqueness.

□

Remark 2.14.5. The notation $dV_{\mathbf{g}}$ here denotes an n-form and requires a fixed orientation. The Riemannian density $\boldsymbol{\mu}_{\mathbf{g}}$, defined later, exists even when M is not orientable and transforms with the absolute value of the Jacobian. The Riemann–Lebesgue measure $\lambda_{\mathbf{g}}$ will be constructed by integrating this density; the notation $d\lambda_{\mathbf{g}}$ indicates only integration with respect to the measure $\lambda_{\mathbf{g}}$ and does not represent a differential form or an exterior derivative.

The volume form has the important property of compatibility with the Levi–Civita connection. To prove this, we first recall an important result from differential calculus:

Proposition 2.14.6. (Jacobi's formula) If $A\colon I \subseteq \mathbb{R} \longrightarrow GL(n,\mathbb{R})$ is differentiable, then

$$\frac{d}{dt}\det\big(A(t)\big) = \det\big(A(t)\big)\,\operatorname{tr}\left(A(t)^{-1}\frac{d}{dt}A(t)\right).$$

Proof. Fix $t \in I$ and denote the columns of $A(t)$ by $\mathbf{a}_1(t),\dots,\mathbf{a}_n(t)$. Multilinearity of the determinant gives

$$\frac{d}{dt}\det A(t) = \sum_{j=1}^{n}\det\big(\mathbf{a}_1(t),\dots,\mathbf{a}_j'(t),\dots,\mathbf{a}_n(t)\big).$$

Set $B(t) := A(t)^{-1}A'(t)$. Then $A' = AB$, and

$$\mathbf{a}_j' = \sum_{i=1}^{n} B_{ij}\mathbf{a}_i.$$

Substituting this sum into the determinant indexed by j, all the terms with $i \neq j$ vanish because they contain two equal columns. The remaining term is $B_{jj}\det A$. Therefore,

$$\frac{d}{dt}\det A(t) = \det A(t)\sum_{j=1}^{n} B_{jj}(t) = \det A(t)\operatorname{tr}\big(A(t)^{-1}A'(t)\big).$$

□

Remark 2.14.7. Proposition 2.14.6 allows us to compute the partial derivatives of $\sqrt{\det(\mathbf{g})}$ and relate them to the Christoffel symbols. Consider a chart (U,φ) with induced coordinates

$$\varphi = (x^1,\dots,x^n)\colon U \longrightarrow \mathbb{R}^n.$$

Denote by $G = (g_{ij})$ the matrix of metric components, where

$$g_{ij}(p) = \mathbf{g}_p\left(\partial_i\big|_p, \partial_j\big|_p\right), \qquad p \in U.$$

Fix a point $p \in U$ and a coordinate x^a. Define a curve in U by

$$\gamma(t) = \varphi^{-1}(x^1(p), \ldots, x^a(p) + t, \ldots, x^n(p)).$$

In coordinates, we have

$$(\varphi \circ \gamma)(t) = (x^1(p), \ldots, x^a(p) + t, \ldots, x^n(p)).$$

Let $f : U \longrightarrow \mathbb{R}$ be a smooth function and write $F = f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. Then

$$f(\gamma(t)) = F((\varphi \circ \gamma)(t)).$$

Applying the chain rule,

$$\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = dF_{\varphi(p)}\left(\frac{d}{dt}\bigg|_{t=0} (\varphi \circ \gamma)(t)\right).$$

Since

$$\frac{d}{dt}\bigg|_{t=0} (\varphi \circ \gamma)(t) = (0, \ldots, 1, \ldots, 0) = e_a \in \mathbb{R}^n,$$

it follows that

$$\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = dF_{\varphi(p)}(e_a) = \frac{\partial F}{\partial u^a}(\varphi(p)),$$

where $(u^1, \ldots, u^n)$ are the standard coordinates of $\mathbb{R}^n$. By definition, this agrees with the action of $\partial_a|_p$ on f. Consequently,

$$\gamma'(0) = \partial_a\big|_p.$$

Applying this to each coordinate component of the metric, which is a smooth function $g_{ij} : U \longrightarrow \mathbb{R}$, we conclude that

$$\frac{d}{dt}\bigg|_{t=0} g_{ij}(\gamma(t)) = \partial_a g_{ij}(p).$$

Defining the curve of matrices

$$A(t) = (g_{ij}(\gamma(t))),$$

and applying Proposition 2.14.6 gives

$$\frac{d}{dt}\det\left(A(t)\right) = \det\left(A(t)\right)\operatorname{tr}\left(A(t)^{-1}\frac{d}{dt}A(t)\right).$$

Evaluating at $t = 0$ and using $\gamma(0) = p$ together with the preceding calculation, we conclude that

$$\partial_a(\det G)(p) = (\det G(p))g^{ij}(p)\partial_a g_{ij}(p).$$

Thus Jacobi's formula in one real variable becomes, in local coordinates on the manifold, the identity

$$\partial_a(\det G) = (\det G)g^{ij}\partial_a g_{ij}.$$

It follows that

$$\partial_a \sqrt{\det G} = \frac{1}{2\sqrt{\det G}} \partial_a (\det G).$$

Substituting the preceding identity gives

$$\partial_a \sqrt{\det G} = \frac{1}{2\sqrt{\det G}} \left((\det G) g^{ij} \partial_a g_{ij} \right),$$

and therefore

$$\partial_a \sqrt{\det G} = \frac{1}{2} \sqrt{\det G}\, g^{ij} \partial_a g_{ij}.$$

In particular, using Corollary 2.4.11, this formula gives the identity

$$\Gamma^i_{ik} = \frac{1}{\sqrt{\det G}} \partial_k \left(\sqrt{\det G} \right),$$

where Γ^i_{jk} denote the Christoffel symbols of the Levi–Civita connection.

Proposition 2.14.8. Let $(M, \mathbf{g})$ be an oriented Riemannian manifold with or without boundary and with Levi–Civita connection ∇. Then the Riemannian volume form is parallel with respect to the connection, that is,

$$\nabla dV_{\mathbf{g}} = 0.$$

Proof. In an oriented chart $(x^1, \dots, x^n)$, we have

$$dV_{\mathbf{g}} = \sqrt{\det G}\, \mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n.$$

Using Proposition 2.4.22 and linearity of the connection gives

$$\nabla_a dV_{\mathbf{g}} = \nabla_a (\sqrt{\det G})\, \mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n + \sqrt{\det G}\, \nabla_a (\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n).$$

For the first term, since $\sqrt{\det G}$ is a function, we have

$$\nabla_a (\sqrt{\det G}) = \partial_a (\sqrt{\det G}).$$

For the second term, applying Proposition 2.4.22 again,

$$\nabla_a (\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n) = \sum_{s=1}^{n} \mathbf{d}x^1 \wedge \cdots \wedge \nabla_a \mathbf{d}x^s \wedge \cdots \wedge \mathbf{d}x^n.$$

In coordinates,

$$\nabla_a \mathbf{d}x^s = -\Gamma^s_{ar} \mathbf{d}x^r.$$

If $r \neq s$, the term

$$\mathbf{d}x^1 \wedge \cdots \wedge (-\Gamma^s_{ar} \mathbf{d}x^r) \wedge \cdots \wedge \mathbf{d}x^n$$

contains two equal factors ($\mathbf{d}x^r$ appears twice), so it vanishes. The only nontrivial contribution occurs when

$r = s$, giving

$$\nabla_a(\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n) = -\left(\Gamma^1_{a1} + \cdots + \Gamma^n_{an}\right)\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n.$$

Combining both results,

$$\nabla_a dV_{\mathbf{g}} = \left(\partial_a(\sqrt{\det G}) - \sqrt{\det G}(\Gamma^1_{a1} + \cdots + \Gamma^n_{an})\right)\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n.$$

By Remark 2.14.7,

$$\partial_a(\sqrt{\det G}) = \frac{1}{2}\sqrt{\det G}\, g^{ij}\partial_a g_{ij}.$$

Since the Levi–Civita connection is compatible with $\mathbf{g}$, we have

$$\Gamma^r_{ra} = \frac{1}{2} g^{ij}\partial_a g_{ij}.$$

Moreover, symmetry in the lower indices implies $\Gamma^r_{ar} = \Gamma^r_{ra}$, so

$$\Gamma^1_{a1} + \cdots + \Gamma^n_{an} = \Gamma^r_{ra}.$$

Substituting these identities,

$$\nabla_a dV_{\mathbf{g}} = \left(\Gamma^r_{ra}\sqrt{\det G} - \Gamma^r_{ra}\sqrt{\det G}\right)\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n = 0.$$

Thus, $dV_{\mathbf{g}}$ is parallel with respect to the Levi–Civita connection. □

We now have the ingredients needed to define integrals of n-forms on smooth manifolds. To do so, we use Theorem 1.3.5.

Definition 2.14.9. Let $D \subseteq \mathbb{H}^n$ be a Jordan-measurable set and let ω be a continuous n-form defined on $\overline{D}$. We have $\omega = f\,\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n$ for some continuous function $f : \overline{D} \longrightarrow \mathbb{R}$. Define the **integral of ω over D** to be

$$\int_D \omega = \int_D f\, dx^1 \cdots dx^n$$

where the integral on the right is the usual Riemann integral. More generally, if $U \subseteq \mathbb{H}^n$ is open and ω is a continuous n-form with compact support contained in U, define

$$\int_U \omega = \int_D \omega$$

where D is a Jordan-measurable set such that $\operatorname{supp}(\omega) \subseteq D \subseteq U$, regarding ω as extended by zero outside its support. One can show that such a Jordan-measurable set always exists and that the definition is independent of the chosen set D.

Definition 2.14.10. Let M be an oriented smooth manifold with or without boundary and let ω be a continuous n-form on M with compact support contained in the domain of a smooth chart (U, φ) that is

positively or negatively oriented. Define the integral of ω over M to be

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega,$$

where the sign is positive if the chart is positively oriented and negative if it is negatively oriented.

Definition 2.14.11. Let M be an oriented smooth manifold with or without boundary and let ω be a continuous n-form with compact support. Let $\{U_i\}_{i=1}^k$ be an open cover of $\operatorname{supp}(\omega)$ by domains of positively or negatively oriented smooth charts. Let $(\psi_i)_{i=1}^k$ be a partition of unity subordinate to this open cover. Define the integral of ω over M to be

$$\int_M \omega = \sum_{i=1}^k \int_M \psi_i \omega.$$

It is proved in [94] that this definition is independent of the choice of oriented charts and partition of unity.

We extend this definition to Riemannian manifolds as follows:

Definition 2.14.12. Let $(M, \mathbf{g})$ be an oriented Riemannian manifold with or without boundary and let $f : M \longrightarrow \mathbb{R}$ be continuous with compact support. Define the integral of f over M to be

$$\int_M f = \int_M f \, dV_{\mathbf{g}}.$$

The nonorientable case is handled by densities, which incorporate the absolute value of the Jacobian and require no choice of orientation.

Definition 2.14.13. (Density of weight w on a vector space) Let V be a vector space of dimension n and let $w \in \mathbb{R}$. Write

$$\operatorname{Fr}(V) := \{(v_1, \ldots, v_n) \in V^n \mid (v_1, \ldots, v_n) \text{ is an ordered basis of } V\}.$$

A **density of weight w on** V is a function

$$\mu : \operatorname{Fr}(V) \longrightarrow \mathbb{R}$$

such that, for every invertible linear operator $T : V \longrightarrow V$ and every $(v_1, \ldots, v_n) \in \operatorname{Fr}(V)$,

$$\mu(T(v_1), \ldots, T(v_n)) = |\det(T)|^w \mu(v_1, \ldots, v_n).$$

A density μ is called **positive** if $\mu(v_1, \ldots, v_n) > 0$ for every ordered basis $(v_1, \ldots, v_n)$ of V.
Denote the set of densities of weight w on V by $\mathcal{D}_w(V)$.

With pointwise operations, $\mathcal{D}_w(V)$ is a one-dimensional vector space. Indeed, fixing an ordered basis $\mathcal{E} = (e_1, \ldots, e_n)$, every ordered basis $\mathcal{V} = (v_1, \ldots, v_n)$ can be written uniquely as $\mathcal{V} = T\mathcal{E}$ with $T \in \mathrm{GL}(V)$. Therefore,

$$\mu(\mathcal{V}) = |\det T|^w \mu(\mathcal{E}).$$

Thus the value of μ at $\mathcal{E}$ determines all its values. Conversely, for each $c \in \mathbb{R}$, the formula $\mu_c(T\mathcal{E}) := |\det T|^w c$ is well-defined and determines a density. The map $\mu \mapsto \mu(\mathcal{E})$ is consequently a linear isomorphism from $\mathcal{D}_w(V)$ onto

$\mathbb{R}$.

Definition 2.14.14. (Bundle of densities of weight w on a manifold) Let M be a smooth manifold with or without boundary. The **bundle of densities of weight w on M**, denoted by $\boldsymbol{\mathcal{D}}_w(M)$, is the set

$$\boldsymbol{\mathcal{D}}_w(M) := \coprod_{p\in M} \mathcal{D}_w(T_pM).$$

$\boldsymbol{\mathcal{D}}_w(M)$ is a smooth vector bundle of rank 1. Over any coordinate chart (U, x^i), a local basis for this bundle is given by $|\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n|^w$.
A **density of weight w on M** is a smooth section of $\boldsymbol{\mathcal{D}}_w(M)$.

We will define the integral of a density on a smooth manifold. For this, we use a construction analogous to the one for differential forms.

Definition 2.14.15. If $D \subseteq \mathbb{H}^n$ is a Jordan-measurable set and $\boldsymbol{\mu}$ is a continuous density of weight one on $\overline{D}$, we may write $\boldsymbol{\mu} = f|\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n|$ for a unique continuous function $f : \overline{D} \longrightarrow \mathbb{R}$. Define the integral of $\boldsymbol{\mu}$ over D by

$$\int_D \boldsymbol{\mu} = \int_D f\, dx^1 \cdots dx^n.$$

Similarly, if $U \subseteq \mathbb{H}^n$ is open and $\boldsymbol{\mu}$ is a continuous density with compact support contained in U, define

$$\int_U \boldsymbol{\mu} = \int_D \boldsymbol{\mu}$$

where D is a Jordan-measurable set such that

$$\operatorname{supp}(\boldsymbol{\mu}) \subseteq D \quad \text{and} \quad \overline{D} \subseteq U.$$

Such a set exists: cover the support by finitely many sufficiently small closed rectangular boxes contained in U and take their union.

Definition 2.14.16. Let M be a smooth manifold with or without boundary (not necessarily orientable). If $\boldsymbol{\mu}$ is a compactly supported density on M whose support is contained in the domain of a smooth chart (U, φ), define the integral of $\boldsymbol{\mu}$ over M to be

$$\int_M \boldsymbol{\mu} = \int_{\varphi(U)} (\varphi^{-1})^* \boldsymbol{\mu}.$$

We may extend this definition to arbitrary compactly supported densities by defining the **integral of $\boldsymbol{\mu}$ over M** by

$$\int_M \boldsymbol{\mu} = \sum_{i=1}^{k} \int_M \psi_i \boldsymbol{\mu},$$

where the chart domains $U_1, \ldots, U_k$ cover $\operatorname{supp}(\boldsymbol{\mu})$, each $\operatorname{supp}(\psi_i)$ is contained in U_i, and $\sum_{i=1}^{k} \psi_i = 1$ on a neighborhood of $\operatorname{supp}(\boldsymbol{\mu})$. These functions are obtained by applying a partition of unity to the cover

$U_1, \dots, U_k, M \setminus \operatorname{supp}(\boldsymbol{\mu})$.

Proposition 2.14.17. (Well-definedness of the integral of densities) The preceding integral is independent of the chosen Jordan-measurable set, chart, finite cover, and partition of unity.

Proof. Begin with an open set $U \subseteq \mathbb{H}^n$ and a continuous density $\boldsymbol{\mu} = f\,|\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n|$ whose support K is compact and contained in U. The extension of f by zero outside U is continuous on a neighborhood of any Jordan-measurable set D such that $K \subseteq D$ and $\overline{D} \subseteq U$. If D_1 and D_2 are two such sets, the extended function vanishes on both $(D_1 \cup D_2) \setminus D_1$ and $(D_1 \cup D_2) \setminus D_2$. Since $D_1 \cup D_2$ is Jordan-measurable, additivity of the Riemann integral gives

$$\int_{D_1} f \, d\lambda_n = \int_{D_1 \cup D_2} f \, d\lambda_n = \int_{D_2} f \, d\lambda_n.$$

This proves independence of D.

Now let (U, φ) and (V, ψ) be two charts whose domains contain $\operatorname{supp}(\boldsymbol{\mu})$, and set $\Theta := \psi \circ \varphi^{-1}$ on $\varphi(U \cap V)$. If

$$(\varphi^{-1})^* \boldsymbol{\mu} = f_\varphi \, |\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n|, \qquad (\psi^{-1})^* \boldsymbol{\mu} = f_\psi \, |\mathbf{d}y^1 \wedge \cdots \wedge \mathbf{d}y^n|,$$

the transformation law for densities gives

$$f_\varphi(x) = f_\psi(\Theta(x)) \, |\det D\Theta(x)|.$$

The change-of-variables theorem B.6.15, applied on $\varphi(U \cap V)$, then implies

$$\int_{\varphi(U)} f_\varphi \, d\lambda_n = \int_{\psi(V)} f_\psi \, d\lambda_n,$$

since both integrands vanish outside the image of the support. Thus the local integral is independent of the chart.

Finally, let $(\psi_i)_{i=1}^k$ and $(\eta_j)_{j=1}^\ell$ be two partitions of unity subordinate to respective finite covers of the support of $\boldsymbol{\mu}$ by chart domains. Additivity of the local integral and the chart independence already proved allow us to integrate each $\psi_i \eta_j \boldsymbol{\mu}$ in either of the two charts containing its support. Since the sums are finite,

$$\begin{aligned} \sum_{i=1}^{k} \int_M \psi_i \boldsymbol{\mu} &= \sum_{i=1}^{k} \sum_{j=1}^{\ell} \int_M \psi_i \eta_j \boldsymbol{\mu} \\ &= \sum_{j=1}^{\ell} \int_M \eta_j \boldsymbol{\mu}. \end{aligned}$$

This proves independence of both the partition and the cover. □

Proposition 2.14.18. (The Riemannian density) If $(M, \mathbf{g})$ is a Riemannian manifold with or without boundary, then there exists a unique smooth positive density $\boldsymbol{\mu}_{\mathbf{g}}$ defined on M, called the **Riemannian density**, with the property that $\boldsymbol{\mu}_{\mathbf{g}}(\mathbf{E}_1, \dots, \mathbf{E}_n) = 1$ for every local orthonormal frame $(\mathbf{E}_i)$.

Proof. In a chart $(U, x^1, \dots, x^n)$, set

$$\boldsymbol{\mu}_{\mathbf{g}} = \sqrt{\det(\mathbf{g})}\,|\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n|.$$

We have the identity

$$\det \widetilde{G} = \det\left(\frac{\partial x}{\partial y}\right)^2 \det G,$$

where $G = (g_{ij})$ and $\widetilde{G} = (\widetilde{g}_{ij})$ are the matrices of components in the coordinates x and y, respectively. This identity, together with the transformation law for a density, shows that these expressions agree on overlaps, regardless of the sign of the Jacobian. Therefore, they define a global smooth positive density. If $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ is an orthonormal frame and B is its matrix of components in the coordinate frame, then $B^T(g_{ij})B = I$ and $|\det B| = (\det(\mathbf{g}))^{-\frac{1}{2}}$; this gives $\boldsymbol{\mu}_{\mathbf{g}}(\mathbf{E}_1, \dots, \mathbf{E}_n) = 1$. Every density with this property agrees with $\boldsymbol{\mu}_{\mathbf{g}}$ on each frame and consequently equals it. □

We define the integral of a compactly supported function on a Riemannian manifold, not necessarily orientable, as follows:

Definition 2.14.19. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. If $f : M \longrightarrow \mathbb{R}$ is a continuous compactly supported function, define the **integral of f over M** to be

$$\int_M f = \int_M f\,\boldsymbol{\mu}_{\mathbf{g}}.$$

We now state the most important integration theorems on manifolds, whose proofs can be found in [94].

Theorem 2.14.20. (Stokes' theorem) Let M be an oriented smooth manifold of dimension n with or without boundary, and let ω be a smooth $(n-1)$-form on M with compact support. Equip ∂M with the orientation induced by the outward-normal-first convention. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

If $\partial M = \varnothing$, the right-hand side is interpreted as zero.

Theorem 2.14.21. (Divergence theorem) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, not necessarily orientable, and let $\mathbf{X} \in \mathfrak{X}(M)$ be a smooth vector field with compact support. If $\partial M \neq \varnothing$, denote by $\boldsymbol{\nu}$ the outward unit normal field and by $\widetilde{\mathbf{g}}$ the induced metric on ∂M. Then

$$\int_M \operatorname{div}(\mathbf{X})\, d\lambda_{\mathbf{g}} = \int_{\partial M} \langle \mathbf{X}, \boldsymbol{\nu} \rangle_{\mathbf{g}}\, d\lambda_{\widetilde{\mathbf{g}}}.$$

If $\partial M = \varnothing$, the right-hand side is zero.

Proof. First suppose that M is oriented and let $dV_{\mathbf{g}}$ be the positive volume form. In an oriented chart, write $\mathbf{X} = X^i \partial_i$ and

$$dV_{\mathbf{g}} = \sqrt{\det(\mathbf{g})}\,\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n.$$

Then

$$\iota_{\mathbf{X}} dV_{\mathbf{g}} = \sqrt{\det(\mathbf{g})} \sum_{i=1}^{n} (-1)^{i-1} X^i \, \mathbf{d}x^1 \wedge \cdots \wedge \widehat{\mathbf{d}x^i} \wedge \cdots \wedge \mathbf{d}x^n.$$

Differentiating each summand, the components of $\mathbf{d}(\sqrt{\det(\mathbf{g})} X^i)$ other than $\mathbf{d}x^i$ vanish when taking the exterior product, because the corresponding differential already occurs in the $(n-1)$-form. The remaining component produces the sign $(-1)^{i-1}$ twice, once from contraction and once from ordering the differentials. Hence

$$d\left(\iota_{\mathbf{X}} dV_{\mathbf{g}}\right) = \sum_{i=1}^{n} \partial_i \left(\sqrt{\det(\mathbf{g})} \, X^i \right) \mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n = \operatorname{div}(\mathbf{X}) \, dV_{\mathbf{g}}.$$

Apply Stokes' theorem to the compactly supported form $\iota_{\mathbf{X}} dV_{\mathbf{g}}$. If $(\mathbf{e}_1, \ldots, \mathbf{e}_{n-1})$ is a positively oriented orthonormal frame of $T(\partial M)$, the boundary orientation convention makes $(\boldsymbol{\nu}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1})$ positively oriented, and

$$\begin{aligned} \left(\iota_{\mathbf{X}} dV_{\mathbf{g}}\right)(\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}) &= dV_{\mathbf{g}}(\mathbf{X}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}) \\ &= \mathbf{g}(\mathbf{X}, \boldsymbol{\nu}). \end{aligned}$$

Therefore,

$$\int_M \operatorname{div}(\mathbf{X}) \, d\lambda_{\mathbf{g}} = \int_{\partial M} \mathbf{g}(\mathbf{X}, \boldsymbol{\nu}) \, d\lambda_{\tilde{\mathbf{g}}}$$

in the orientable case.

Now consider an arbitrary M. The compact set $K := \operatorname{supp}(\mathbf{X})$ admits a finite cover by chart domains $U_1, \ldots, U_N$, each of which is orientable. Add $M \setminus K$ to the cover and choose a subordinate smooth partition of unity. Denote the functions corresponding to U_i by χ_i. The function corresponding to $M \setminus K$ has zero product with $\mathbf{X}$, and

$$\sum_{i=1}^{N} \chi_i = 1, \qquad \sum_{i=1}^{N} \mathbf{d}\chi_i = 0$$

on a neighborhood of K. Each field $\chi_i \mathbf{X}$ has compact support contained in U_i. Since U_i is open in a manifold with boundary, its boundary as a manifold is $U_i \cap \partial M$; no artificial boundary appears inside M. The orientable case, applied to U_i with either of its two orientations, gives

$$\int_{U_i} \operatorname{div}(\chi_i \mathbf{X}) \, d\lambda_{\mathbf{g}} = \int_{\partial M \cap U_i} \chi_i \mathbf{g}(\mathbf{X}, \boldsymbol{\nu}) \, d\lambda_{\tilde{\mathbf{g}}}.$$

The Leibniz rule for divergence implies

$$\sum_{i=1}^{N} \operatorname{div}(\chi_i \mathbf{X}) = \left(\sum_{i=1}^{N} \chi_i \right) \operatorname{div}(\mathbf{X}) + \left\langle \sum_{i=1}^{N} \operatorname{grad} \chi_i, \mathbf{X} \right\rangle_{\mathbf{g}} = \operatorname{div}(\mathbf{X}).$$

Summing the local identities and again using $\sum_{i=1}^{N} \chi_i = 1$ on $K \cap \partial M$ gives the global formula. The assertion for empty boundary is the same equality with an integral over the empty set. $\square$

2.15 The Riemann–Lebesgue measure

Integration of smooth forms suffices for elementary differential geometry, but analysis requires integrating measurable functions and working with convergence, L^p spaces, and weak derivatives. The Riemann–Lebesgue measure extends geometric integration to this setting while retaining, in each chart, the density determined by the metric. It allows us to handle much less regular functions while preserving the intrinsic structure of the manifold.

Definition 2.15.1. Let M be a smooth manifold with or without boundary. A set $A \subseteq M$ is **Lebesgue measurable** if, for every smooth chart (U, φ) of M, the set $\varphi(A \cap U)$ is Lebesgue measurable in $\mathbb{R}^n$. Write

$$\mathcal{L}_M := \{A \subseteq M \mid A \text{ is Lebesgue measurable}\}.$$

Proposition 2.15.2. Let M be a smooth manifold with or without boundary. Then $\mathcal{L}_M$ is a σ-algebra on M containing the Borel sets.

Proof. Let (U, φ) be a chart of M. If $A \in \mathcal{L}_M$, then $\varphi(A \cap U) \in \mathcal{L}(n)$. Since $\mathcal{L}(n)$ is a σ-algebra, we have $\varphi((M \setminus A) \cap U) = \varphi(U) \setminus \varphi(A \cap U) \in \mathcal{L}(n)$, and thus $M \setminus A \in \mathcal{L}_M$. If $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}_M$, then $\varphi(A_k \cap U) \in \mathcal{L}(n)$, so

$$\varphi\left(\left(\bigcup_{k\in\mathbb{N}} A_k\right) \cap U\right) = \varphi\left(\bigcup_{k\in\mathbb{N}} (A_k \cap U)\right) = \bigcup_{k\in\mathbb{N}} \varphi(A_k \cap U) \in \mathcal{L}(n),$$

which implies $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{L}_M$. Now $\varphi(M \cap U) = \varphi(U) \in \mathcal{L}(n)$ because it is open, so $M \in \mathcal{L}_M$, and we conclude that $\mathcal{L}_M$ is a σ-algebra on M.
Finally, if $O \subseteq M$ is open, then $\varphi(O \cap U) \in \mathcal{L}(n)$ because it is open in $\mathbb{H}^n$. Therefore, $\mathcal{B}(M) \subseteq \mathcal{L}_M$. □

Definition 2.15.3. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let (U, φ) be a chart of M. If $A \in \mathcal{L}_M$ and $A \subseteq U$, define $\lambda_{\mathbf{g},U}(A) := \displaystyle\int_{\varphi(A)} \varphi_*\left(\sqrt{\det(\mathbf{g})}\right) d\lambda_n$, where $\varphi_*\left(\sqrt{\det(\mathbf{g})}\right) = \left(\sqrt{\det(\mathbf{g})}\right) \circ \varphi^{-1}$.

Proposition 2.15.4. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let $A \in \mathcal{L}_M$.

(a) The definition of $\lambda_{\mathbf{g},U}$ is independent of the chart (U, φ) when $A \subseteq U$.

(b) $\lambda_{\mathbf{g},U}$ defines a measure on the measurable space $(U, (\mathcal{L}_M)\restriction_U)$.

Proof. Let (U, φ) be a chart and write

$$G_\varphi = (g_{ij}), \qquad \rho_\varphi := \sqrt{\det G_\varphi} \circ \varphi^{-1}.$$

The function ρ_φ is continuous and strictly positive on $\varphi(U)$.

(a) Let (U, φ) and (V, ψ) be two charts with $A \subseteq U \cap V$, and let

$$f := \psi \circ \varphi^{-1} \colon \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

be the change of coordinates. If $G_\psi = (\widetilde{g}_{ab})$ is the matrix of components of $\mathbf{g}$ in the chart ψ, the transformation law for a covariant tensor of order two gives

$$G_\psi(f(x)) = Df(x)^{-\mathsf{T}} G_\varphi(x) Df(x)^{-1}.$$

Taking determinants and square roots gives

$$\rho_\varphi(x) = \rho_\psi(f(x))\,|\det Df(x)|. \tag{2.104}$$

The change-of-variables formula applied to the nonnegative function $\rho_\psi \mathbf{1}_{\psi(A)}$ implies

$$\begin{aligned}\lambda_{\mathbf{g},V}(A) &= \int_{\psi(A)} \rho_\psi(y)\, d\lambda_n(y) \\ &= \int_{\varphi(A)} \rho_\psi(f(x))|\det Df(x)|\, d\lambda_n(x) \\ &= \int_{\varphi(A)} \rho_\varphi(x)\, d\lambda_n(x) = \lambda_{\mathbf{g},U}(A),\end{aligned}$$

where the last equality uses (2.104). Thus the local measure is independent of the chart.

(b) For every Lebesgue measurable set $B \subseteq \varphi(U)$, define

$$\nu_\varphi(B) := \int_B \rho_\varphi\, d\lambda_n.$$

Since $\rho_\varphi \geq 0$ is measurable, ν_φ is a measure: σ-additivity follows by applying the monotone convergence theorem to sums of indicator functions of pairwise disjoint sets. Moreover,

$$\lambda_{\mathbf{g},U}(A) = \nu_\varphi(\varphi(A)), \qquad A \in (\mathcal{L}_M)\restriction_U .$$

Since φ is a measurable bijection with measurable inverse, this identity transfers σ-additivity from ν_φ to $\lambda_{\mathbf{g},U}$. Thus, $\lambda_{\mathbf{g},U}$ is a measure on $(U, (\mathcal{L}_M)\restriction_U)$.

□

Definition 2.15.5. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. For each $A \in \mathcal{L}_M$, choose a countable atlas $\mathcal{A} = \{(U_k, \varphi_k) \mid k \in \mathbb{N}\}$ of M and define

$$A_0 := A \cap U_0, \qquad A_{k+1} := (A \cap U_{k+1}) \setminus \bigcup_{j=0}^{k} A_j.$$

The sets A_k are measurable, pairwise disjoint, satisfy $A_k \subseteq U_k$, and cover A. Define

$$\lambda_{\mathbf{g}}(A) := \sum_{k=0}^{\infty} \lambda_{\mathbf{g},U_k}(A_k).$$

This function is called the **Riemann–Lebesgue measure** associated with $\mathbf{g}$.

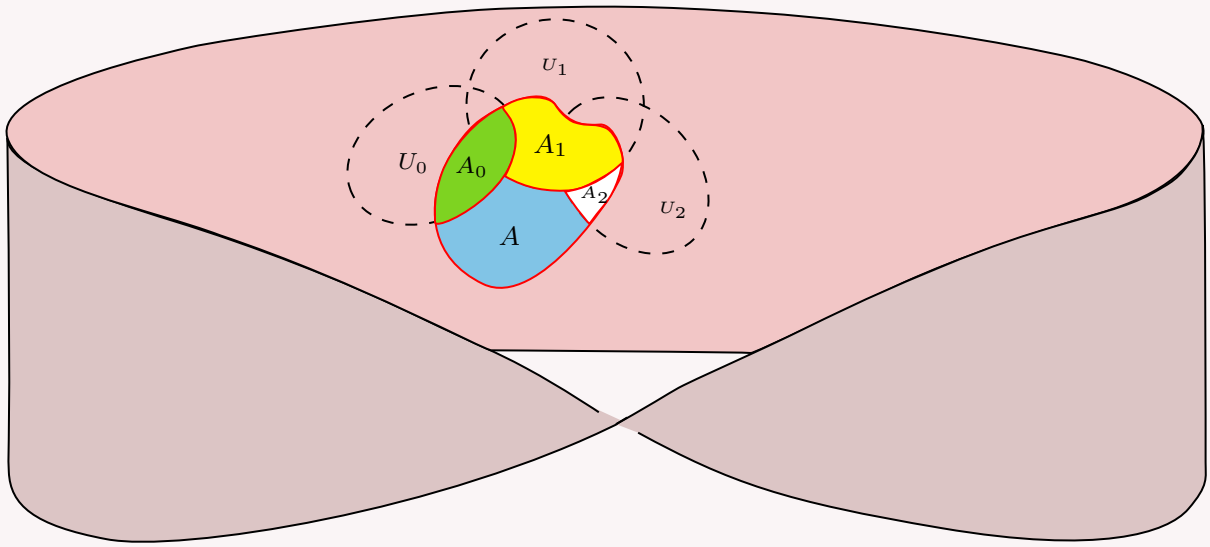


Figure 2.8. The definition does not depend on the orientability of the manifold.

Proposition 2.15.6. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. The definition of $\lambda_{\mathbf{g}}$ is independent of the choice of countable atlas.

Proof. Let $\widetilde{\mathcal{A}}$ be another countable atlas of M. Let $\widetilde{A}_0 := A \cap \widetilde{U}_0$; for each $i \in \mathbb{N}$ with $i \geq 0$, define $\widetilde{A}_{i+1} := (A \cap \widetilde{U}_{i+1}) \setminus \bigcup_{j=0}^{i} \widetilde{A}_j$. By Proposition 2.15.4, $\lambda_{\mathbf{g},U_k}$ and $\lambda_{\mathbf{g},\widetilde{U}_i}$ are measures on U_k and $\widetilde{U}_i$, respectively, so they are σ-additive, and using $\bigcup_{k\in\mathbb{N}} A_k = \bigcup_{i\in\mathbb{N}} \widetilde{A}_i = A$,

$$\lambda_{\mathbf{g},U_k}(A_k) = \lambda_{\mathbf{g},U_k}(A_k \cap A) = \lambda_{\mathbf{g},U_k}\left(A_k \cap \bigcup_{i\in\mathbb{N}} \widetilde{A}_i\right) = \lambda_{\mathbf{g},U_k}\left(\bigcup_{i\in\mathbb{N}} (A_k \cap \widetilde{A}_i)\right) = \sum_{i\in\mathbb{N}} \lambda_{\mathbf{g},U_k}(A_k \cap \widetilde{A}_i).$$

Apply the same decomposition to $\lambda_{\mathbf{g},\widetilde{U}_i}(\widetilde{A}_i) = \sum_{k\in\mathbb{N}} \lambda_{\mathbf{g},\widetilde{U}_i}(\widetilde{A}_i \cap A_k)$. By Proposition 2.15.4, we know that $\lambda_{\mathbf{g},U_k}(A_k \cap \widetilde{A}_i) = \lambda_{\mathbf{g},\widetilde{U}_i}(\widetilde{A}_i \cap A_k)$ since $A_k \cap \widetilde{A}_i \subseteq U_k \cap \widetilde{U}_i$.
Therefore,

$$\sum_{k=0}^{\infty} \lambda_{\mathbf{g},U_k}(A_k) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \lambda_{\mathbf{g},U_k}(A_k \cap \widetilde{A}_i)$$

$$= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \lambda_{\mathbf{g},\widetilde{U}_i}(\widetilde{A}_i \cap A_k) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{\mathbf{g},\widetilde{U}_i}(\widetilde{A}_i \cap A_k) = \sum_{i=0}^{\infty} \lambda_{\mathbf{g},\widetilde{U}_i}(\widetilde{A}_i).$$

□

The measure $\lambda_{\mathbf{g}}$ agrees with $\lambda_{\mathbf{g},U}$ on measurable sets contained in U:

Proposition 2.15.7. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary and let (U,φ) be a smooth chart. If $A \in \mathcal{L}_M$ and $A \subseteq U$, then $\lambda_{\mathbf{g}}(A) = \lambda_{\mathbf{g},U}(A)$.

Proof. Let $(U_k,\varphi_k)_{k\in\mathbb{N}}$ be a countable atlas of M such that $U_0 = U$. Consider $A_0 = A \cap U_0 = A \cap U = A$ and $A_{k+1} = (A \cap U_{k+1}) \setminus \bigcup_{j=0}^{k} A_j$. We know that $(A_k)_{k\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable sets whose union is A. Since $A_0 = A$, observe that this implies $A_k = \varnothing$ for every $k \geq 1$. Therefore,

$$\lambda_{\mathbf{g}}(A) = \sum_{k=0}^{\infty} \lambda_{\mathbf{g},U_k}(A_k) = \lambda_{\mathbf{g},U_0}(A_0) = \lambda_{\mathbf{g},U}(A).$$

□

Theorem 2.15.8. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. Then $\lambda_{\mathbf{g}}$ is a complete Radon measure on $(M,\mathcal{L}_M)$.

Proof. Fix a countable atlas $\{(U_j,\varphi_j) \mid j \in \mathbb{N}\}$. For each $B \in \mathcal{L}_M$, set

$$B^{(0)} = B \cap U_0, \qquad B^{(j+1)} = (B \cap U_{j+1}) \setminus \bigcup_{l=0}^{j} B^{(l)}.$$

These pieces are measurable, pairwise disjoint, cover B, and satisfy $B^{(j)} \subseteq U_j$. If (A_k) is a countable family of pairwise disjoint measurable sets and $A = \bigcup_k A_k$, then, for each j,

$$A^{(j)} = \bigcup_k A_k^{(j)},$$

and the union on the right is also disjoint. Countable additivity of the local measures and interchange of two series of nonnegative terms give

$$\begin{aligned}\lambda_{\mathbf{g}}(A) &= \sum_{j=1}^{\infty} \lambda_{\mathbf{g},U_j}(A^{(j)}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \lambda_{\mathbf{g},U_j}(A_k^{(j)}) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \lambda_{\mathbf{g},U_j}(A_k^{(j)}) = \sum_{k=1}^{\infty} \lambda_{\mathbf{g}}(A_k).\end{aligned}$$

Moreover, $\lambda_{\mathbf{g}}(\varnothing) = 0$; therefore, $\lambda_{\mathbf{g}}$ is a measure.
We first establish local regularity. In a chart (U,φ), the local measure is the transport by φ of the measure

$$\nu(B) = \int_B w \, d\lambda_n, \qquad w = \varphi_* \sqrt{\det(\mathbf{g})},$$

on the relatively open set $\varphi(U) \subseteq \mathbb{H}^n$. The function w is continuous and strictly positive. Take an increasing sequence of relatively compact open sets W_m whose union is $\varphi(U)$. On each $\overline{W}_m$, the function w has a finite upper bound and a positive lower bound. Regularity of Lebesgue measure, applied on each W_m, then shows

that for every measurable set $B \subseteq \varphi(U)$,

$$\nu(B) = \sup\{\nu(K) \mid K \subseteq B, K \text{ compact}\} = \inf\{\nu(O) \mid B \subseteq O, O \text{ open in } \varphi(U)\}.$$

Indeed, for inner approximation, first choose m so that $\nu(B \setminus W_m)$ is small and then a compact set $K \subseteq B \cap W_m$ whose complement has small Lebesgue measure; the upper bound for w turns this last estimate into an estimate for ν. If $\nu(B) = \infty$, the same argument applied to $B \cap W_m$ gives compact sets of arbitrarily large measure. For outer approximation, decompose B into the measurable sets $B \cap (W_m \setminus W_{m-1})$ and, again using the upper bound for w on $\overline{W}_m$, choose for the mth piece an open set whose excess has measure less than $\frac{\varepsilon}{2^{m+1}}$. The union of these open sets contains B and its total excess has measure less than ε.

We may choose the countable atlas so that each U_j is a precompact coordinate domain with bounded image and $\lambda_{\mathbf{g}}(U_j) < \infty$. For $A \in \mathcal{L}_M$, let (A_j) be the disjoint pieces associated with this atlas. If $\lambda_{\mathbf{g}}(A) < \infty$ and $\varepsilon > 0$, choose N so that

$$\sum_{j>N} \lambda_{\mathbf{g}}(A_j) < \frac{\varepsilon}{2}.$$

Local regularity gives compact sets $K_j \subseteq A_j$, for $0 \le j \le N$, such that

$$\sum_{j=0}^{N} \lambda_{\mathbf{g}}(A_j \setminus K_j) < \frac{\varepsilon}{2}.$$

The finite union $K = \bigcup_{j=0}^{N} K_j$ is compact, is contained in A, and satisfies $\lambda_{\mathbf{g}}(A \setminus K) < \varepsilon$. If $\lambda_{\mathbf{g}}(A) = \infty$, first take a partial sum greater than a given number and approximate its terms from within; this gives compact sets contained in A of arbitrarily large measure. This proves inner regularity.

For outer regularity, we need only consider $\lambda_{\mathbf{g}}(A) < \infty$. For each j, use local regularity to choose an open subset O_j of M such that

$$A_j \subseteq O_j \subseteq U_j, \qquad \lambda_{\mathbf{g}}(O_j \setminus A_j) < \frac{\varepsilon}{2^{j+1}}.$$

Then $O = \bigcup_j O_j$ is open, contains A, and

$$\lambda_{\mathbf{g}}(O \setminus A) \le \sum_{j=1}^{\infty} \lambda_{\mathbf{g}}(O_j \setminus A_j) < \varepsilon.$$

If $\lambda_{\mathbf{g}}(A) = \infty$, every open set containing A has infinite measure. This proves outer regularity.

Finally, every point has a coordinate domain V with compact closure contained in another coordinate domain. The function w is bounded on the image of this closure, and $\varphi(V)$ can be chosen bounded; therefore, $\lambda_{\mathbf{g}}(V) < \infty$. The measure is locally finite. It is also complete: if $N \subseteq Z$ and $\lambda_{\mathbf{g}}(Z) = 0$, positivity of w and its lower bounds on compact sets imply that $\varphi(Z \cap U)$ has Lebesgue measure zero in every chart. Every subset of this set is Lebesgue measurable and has local measure zero, so $N \in \mathcal{L}_M$ and $\lambda_{\mathbf{g}}(N) = 0$. We conclude that $\lambda_{\mathbf{g}}$ is a Radon measure on the indicated measurable space. □

Remark 2.15.9. The expression $\int_M f \, d\lambda_{\mathbf{g}}$ denotes the Lebesgue integral with respect to the measure $\lambda_{\mathbf{g}}$. For every continuous compactly supported function,

$$\int_M f \, d\lambda_{\mathbf{g}} = \int_M f \, \boldsymbol{\mu}_{\mathbf{g}}.$$

If M is oriented and $dV_{\mathbf{g}}$ corresponds to the chosen orientation, $\boldsymbol{\mu}_{\mathbf{g}}$ is the density associated with $dV_{\mathbf{g}}$, and therefore

$$\int_M f \, d\lambda_{\mathbf{g}} = \int_M f \, dV_{\mathbf{g}}.$$

Thus, $dV_{\mathbf{g}}$, $\boldsymbol{\mu}_{\mathbf{g}}$, and $d\lambda_{\mathbf{g}}$ denote, respectively, a form, a density, and the differential notation for a measure.

Local finiteness of the Riemann–Lebesgue measure has the following consequence:

Remark 2.15.10. Let $(M, \mathbf{g})$ be a compact Riemannian manifold with or without boundary. Then $\lambda_{\mathbf{g}}(M) < \infty$.

Proof. Since M is compact and $\lambda_{\mathbf{g}}$ is locally finite, we can find a finite open cover of M, say $M = \bigcup_{j=0}^{N} U_j$, such that $\lambda_{\mathbf{g}}(U_j) < \infty$ for every $j \in \{0, \dots, N\}$.
Then

$$\lambda_{\mathbf{g}}(M) \leq \sum_{j=0}^{N} \lambda_{\mathbf{g}}(U_j) < \infty.$$

□

For later analytic applications, we will explicitly need the Borel completion and σ-finiteness of the Riemann–Lebesgue measure.

Lemma 2.15.11. (Borel completion and σ-finiteness of the Riemann–Lebesgue measure) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let $\lambda_{\mathbf{g}}$ be the associated Riemann–Lebesgue measure. Then

$$\mathcal{L}_M = \overline{\mathcal{B}(M)}^{\lambda_{\mathbf{g}}},$$

where the right-hand side denotes the completion of the Borel σ-algebra with respect to $\lambda_{\mathbf{g}}$. Moreover, $(M, \mathcal{L}_M, \lambda_{\mathbf{g}})$ is σ-finite.

Proof. By Proposition 1.2.11, we may choose a countable cover $(U_j, \varphi_j)_{j \in \mathbb{N}}$ by regular coordinate balls and, in the boundary case, by the corresponding regular coordinate half-balls, so that $\overline{U_j}$ is compact for every j.
We first prove equality of the σ-algebras. The inclusion $\overline{\mathcal{B}(M)}^{\lambda_{\mathbf{g}}} \subseteq \mathcal{L}_M$ follows from Theorem 2.15.8: $\mathcal{B}(M) \subseteq \mathcal{L}_M$ and $\lambda_{\mathbf{g}}$ is complete on $\mathcal{L}_M$.
For the reverse inclusion, take $A \in \mathcal{L}_M$ and form the disjoint decomposition

$$A_0 := A \cap U_0, \qquad A_j := (A \cap U_j) \setminus \bigcup_{i=0}^{j-1} A_i, \qquad j \geq 1.$$

Then $A = \bigcup_{j=0}^{\infty} A_j$, each A_j belongs to $\mathcal{L}_M$, and $A_j \subseteq U_j$. By the definition of $\mathcal{L}_M$, $\varphi_j(A_j)$ is Lebesgue measurable in $\varphi_j(U_j)$. Since the Lebesgue σ-algebra of a Euclidean open set or a relatively open subset of the half-space is the completion of its Borel σ-algebra, there exist a Borel set $B_j \subseteq \varphi_j(U_j)$ and a set of Lebesgue measure zero $N_j \subseteq \varphi_j(U_j)$ such that

$$\varphi_j(A_j) \triangle B_j \subseteq N_j.$$

Since $\overline{U_j}$ is compact, the coordinate density $\sqrt{\det(\mathbf{g})} \circ \varphi_j^{-1}$ is bounded above and, on the compact sets exhausting $\varphi_j(U_j)$, is also bounded away from zero. In particular, every set of Lebesgue measure zero in $\varphi_j(U_j)$ has preimage of $\lambda_{\mathbf{g}}$-measure zero. Therefore,

$$Z_j := \varphi_j^{-1}(N_j)$$

is $\lambda_{\mathbf{g}}$-null. Moreover, $C_j := \varphi_j^{-1}(B_j)$ is Borel in U_j and, since U_j is open, it is Borel in M. Defining

$$C := \bigcup_{j=0}^{\infty} C_j, \qquad Z := \bigcup_{j=0}^{\infty} Z_j,$$

gives $C \in \mathcal{B}(M)$, $\lambda_{\mathbf{g}}(Z) = 0$, and

$$A \triangle C \subseteq Z.$$

Thus, A belongs to the completion of $\mathcal{B}(M)$, proving

$$\mathcal{L}_M \subseteq \overline{\mathcal{B}(M)}^{\lambda_{\mathbf{g}}}.$$

It remains to prove σ-finiteness. For each j, compactness of $\overline{U_j}$ and local finiteness of $\lambda_{\mathbf{g}}$ imply

$$\lambda_{\mathbf{g}}(U_j) \leq \lambda_{\mathbf{g}}(\overline{U_j}) < \infty.$$

Since $M = \bigcup_{j=0}^{\infty} U_j$, the measure $\lambda_{\mathbf{g}}$ is σ-finite. □

The remaining properties of the Riemann–Lebesgue measure are left as an exercise:

Exercise 2.15.12. (Basic properties of the Riemann–Lebesgue measure) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let $\lambda_{\mathbf{g}}$ be the associated Riemann–Lebesgue measure.

(a) Prove that $\lambda_{\mathbf{g}}$ is complete, that is, if $A \subset M$ is measurable and $\lambda_{\mathbf{g}}(A) = 0$, then every subset of A is measurable.

(b) Let (U, φ) be a coordinate chart. Prove that if $A \subset U$ satisfies $\lambda_{\mathbf{g}}(A) = 0$, then $\lambda_n\big(\varphi(A)\big) = 0$ in $\mathbb{R}^n$, where λ_n denotes Lebesgue measure on $\mathbb{R}^n$.

(c) Prove that a function $f : M \longrightarrow \mathbb{R}$ is $\mathcal{L}_M$-measurable if and only if, for every chart (U, φ), the function

$$f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}$$

is λ_n-measurable on $\varphi(U) \subset \mathbb{R}^n$.

(d) Prove that if $U \subset M$ is nonempty and open, then $\lambda_{\mathbf{g}}(U) > 0$.

(e) Let $(M, \mathbf{g})$ be orientable and let $dV_{\mathbf{g}}$ be the Riemannian volume form associated with an orientation. Prove that, for every continuous compactly supported function,

$$\int_M f \, d\lambda_{\mathbf{g}} = \int_M f \, dV_{\mathbf{g}}.$$

2.16 Gradient, divergence, and the Laplace–Beltrami operator

The gradient, divergence, and Laplace–Beltrami operator form the basic differential vocabulary of Riemannian analysis. They generalize familiar Euclidean operations, but their intrinsic definitions show exactly where the metric, connection, and measure enter. In Chapter 8, after defining the general notion of a differential operator on bundles, we will verify that these constructions are examples of it. These operations will be used to formulate the elliptic equations, energy identities, and geometric flows of later chapters.

We first define the gradient of a scalar function $f : M \longrightarrow \mathbb{R}$.

The covariant derivative of f equals the covector df. To obtain a vector field, we must raise this index with the metric. In the Euclidean case, this identification is expressed by

$$\langle \operatorname{grad} f, \mathbf{X} \rangle_{\mathbf{g}} = \mathbf{X}(f),$$

and leads to the following intrinsic definition.

Definition 2.16.1. Let $(M, \mathbf{g})$ be a Riemannian manifold (with or without boundary). For a function $f : M \longrightarrow \mathbb{R}$ of class C^1, the *gradient* of f is the unique continuous vector field $\operatorname{grad}(f) \in \Gamma^0(TM)$ such that

$$\langle \operatorname{grad}(f), \mathbf{X} \rangle_{\mathbf{g}} = df(\mathbf{X}) \quad \text{for every } \mathbf{X} \in \mathfrak{X}(M).$$

That is, $\operatorname{grad}(f) = (df)^\sharp$, where $^\sharp : T^*M \longrightarrow TM$ is the musical isomorphism induced by $\mathbf{g}$ given in Definition 2.1.10.

Remark 2.16.2. (Local expression for the gradient) Let (U, φ) be a chart with coordinates $(x^1, \ldots, x^n)$. In this chart, the differential of f is written as

$$df = \partial_j f \, \mathbf{d}x^j, \qquad \partial_j f = \frac{\partial (f \circ \varphi^{-1})}{\partial x^j}.$$

Applying the musical isomorphism $^\sharp : T^*M \longrightarrow TM$ gives

$$(df)^\sharp = g^{ij} \partial_j f \, \partial_i.$$

The gradient satisfies the Leibniz rule, just as the differential of a function does:

Proposition 2.16.3. (Leibniz rule for the gradient) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. If $f,g\in C^1(M)$, then

$$\operatorname{grad}(fg)=f\operatorname{grad}(g)+g\operatorname{grad}(f).$$

Proof. For every smooth vector field $\mathbf{X}$,

$$\begin{aligned}\langle\operatorname{grad}(fg),\mathbf{X}\rangle &= d(fg)(\mathbf{X})=f\,dg(\mathbf{X})+g\,df(\mathbf{X})\\ &=\langle f\operatorname{grad}g+g\operatorname{grad}f,\mathbf{X}\rangle.\end{aligned}$$

Nondegeneracy of the metric gives equality of the fields. □

Divergence of a vector field is a central notion in vector calculus. We introduce its analogue on Riemannian manifolds:

Definition 2.16.4. Let $(M,\mathbf{g})$ be a Riemannian manifold (with or without boundary) and let $\mathbf{X}$ be a vector field of class C^1. Define the *divergence* of $\mathbf{X}$ by

$$\operatorname{div}(\mathbf{X})=\operatorname{tr}(\nabla\mathbf{X}),$$

where tr denotes the natural contraction of a tensor of type $(1,1)$ in its two indices. In a local orthonormal frame $(\mathbf{e}_1,\dots,\mathbf{e}_n)$, we recover the classical expression

$$\operatorname{div}(\mathbf{X})=\sum_{i=1}^{n}\mathbf{g}(\nabla_{\mathbf{e}_i}\mathbf{X},\mathbf{e}_i).$$

In local coordinates, we have the following:

Proposition 2.16.5. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary and let $\mathbf{X}=X^i\partial_i$ be a vector field of class C^1. In local coordinates $(x^1,\dots,x^n)$, we have

$$\operatorname{div}(\mathbf{X})=\frac{1}{\sqrt{\det(\mathbf{g})}}\partial_i\left(\sqrt{\det(\mathbf{g})}X^i\right).$$

Proof. By definition, divergence is the trace of $\nabla\mathbf{X}$:

$$\operatorname{div}(\mathbf{X})=\operatorname{tr}(\nabla\mathbf{X}).$$

In local coordinates $(x^1,\dots,x^n)$ with coordinate frame $\partial_j=\partial_j$, we have

$$\nabla_{\partial_j}\mathbf{X}=(\partial_jX^i+\Gamma^i_{jk}X^k)\partial_i,\quad\text{that is}\quad(\nabla_j\mathbf{X})^i=\partial_jX^i+\Gamma^i_{jk}X^k.$$

Taking the trace of $\nabla\mathbf{X}$ means contracting the covariant differentiation index j with the contravariant index i, which in index notation is written as

$$\operatorname{div}(\mathbf{X})=\nabla_iX^i.$$

Substituting the expression for $(\nabla_j \mathbf{X})^i$ gives

$$\operatorname{div}(\mathbf{X}) = \partial_i X^i + \Gamma^i_{ik} X^k.$$

We now use the identity

$$\Gamma^i_{ik} = \frac{1}{\sqrt{\det(\mathbf{g})}} \partial_k \big(\sqrt{\det(\mathbf{g})}\big),$$

obtained in Remark 2.14.7.
Substituting into the expression for $\operatorname{div}(\mathbf{X})$ gives

$$\operatorname{div}(\mathbf{X}) = \partial_i X^i + \frac{1}{\sqrt{\det(\mathbf{g})}} \partial_k \big(\sqrt{\det(\mathbf{g})}\big) X^k.$$

Applying the Leibniz rule to the product $\sqrt{\det(\mathbf{g})} X^i$,

$$\partial_i \big(\sqrt{\det(\mathbf{g})} X^i\big) = \sqrt{\det(\mathbf{g})} \partial_i X^i + X^i \partial_i \sqrt{\det(\mathbf{g})},$$

and dividing by $\sqrt{\det(\mathbf{g})}$ gives

$$\frac{1}{\sqrt{\det(\mathbf{g})}} \partial_i \big(\sqrt{\det(\mathbf{g})} X^i\big) = \partial_i X^i + \frac{1}{\sqrt{\det(\mathbf{g})}} \partial_i \big(\sqrt{\det(\mathbf{g})}\big) X^i.$$

Comparing with the preceding expression yields the compact formula

$$\operatorname{div}(\mathbf{X}) = \frac{1}{\sqrt{\det(\mathbf{g})}} \partial_i \big(\sqrt{\det(\mathbf{g})} X^i\big),$$

as desired. □

Divergence satisfies the following Leibniz rule:

Proposition 2.16.6. (Leibniz rule for divergence) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. Then, for every $\mathbf{X} \in \mathfrak{X}(M)$ and every $f \in C^\infty(M)$,

$$\operatorname{div}(f\mathbf{X}) = f \operatorname{div}(\mathbf{X}) + \langle \operatorname{grad}(f), \mathbf{X} \rangle_{\mathbf{g}}.$$

Proof. In an orthonormal frame $(\mathbf{e}_i)$,

$$\begin{aligned} \operatorname{div}(f\mathbf{X}) &= \sum_{i=1}^n \langle \nabla_{\mathbf{e}_i}(f\mathbf{X}), \mathbf{e}_i \rangle \\ &= \sum_{i=1}^n \mathbf{e}_i(f) \langle \mathbf{X}, \mathbf{e}_i \rangle + f \sum_{i=1}^n \langle \nabla_{\mathbf{e}_i} \mathbf{X}, \mathbf{e}_i \rangle \\ &= \langle \operatorname{grad} f, \mathbf{X} \rangle + f \operatorname{div} \mathbf{X}. \end{aligned}$$

□

The definition extends naturally to tensor fields of type (k, l):

Definition 2.16.7. Let $(M,\mathbf{g})$ be a Riemannian manifold (with or without boundary) and let $\mathbf{F} \in \Gamma(T^{(k,l)}(TM))$ with $k \geq 1$. Define

$$\operatorname{div}(\mathbf{F}) \in \Gamma(T^{(k-1,l)}(TM))$$

to be the contraction of the differentiation index of $\nabla\mathbf{F}$ with the first contravariant index of $\mathbf{F}$. In index notation,

$$(\operatorname{div}\mathbf{F})^{i_2\ldots i_k}_{j_1\ldots j_l} = \nabla_a F^{ai_2\ldots i_k}_{j_1\ldots j_l}.$$

Proposition 2.16.8. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. In local coordinates $(x^1,\ldots,x^n)$, if

$$\mathbf{F} = F^{i_1\ldots i_k}_{j_1\ldots j_l}\partial_{i_1}\otimes\cdots\otimes\partial_{i_k}\otimes \mathbf{d}x^{j_1}\otimes\cdots\otimes\mathbf{d}x^{j_l},$$

then its divergence is given by

$$(\operatorname{div}\mathbf{F})^{i_2\ldots i_k}_{j_1\ldots j_l} = \partial_a F^{ai_2\ldots i_k}_{j_1\ldots j_l} + \sum_{r=2}^{k}\Gamma^{i_r}_{am}F^{ai_2\ldots m\ldots i_k}_{j_1\ldots j_l} - \sum_{s=1}^{l}\Gamma^{m}_{aj_s}F^{ai_2\ldots i_k}_{j_1\ldots m\ldots j_l} + \Gamma^{a}_{am}F^{mi_2\ldots i_k}_{j_1\ldots j_l}.$$

Proof. The expression follows by applying the definition

$$(\operatorname{div}\mathbf{F})^{i_2\ldots i_k}_{j_1\ldots j_l} = \nabla_a F^{ai_2\ldots i_k}_{j_1\ldots j_l}.$$

By Corollary 2.4.20, in local coordinates,

$$(\nabla_a\mathbf{F})^{i_1\ldots i_k}_{j_1\ldots j_l} = \partial_a F^{i_1\ldots i_k}_{j_1\ldots j_l} + \sum_{r=1}^{k}\Gamma^{i_r}_{am}F^{i_1\ldots m\ldots i_k}_{j_1\ldots j_l} - \sum_{s=1}^{l}\Gamma^{m}_{aj_s}F^{i_1\ldots i_k}_{j_1\ldots m\ldots j_l}.$$

Contracting the differentiation index a with the first contravariant index i_1 gives

$$(\operatorname{div}\mathbf{F})^{i_2\ldots i_k}_{j_1\ldots j_l} = \nabla_a F^{ai_2\ldots i_k}_{j_1\ldots j_l}.$$

Substituting into the preceding expression gives

$$(\operatorname{div}\mathbf{F})^{i_2\ldots i_k}_{j_1\ldots j_l} = \partial_a F^{ai_2\ldots i_k}_{j_1\ldots j_l} + \sum_{r=1}^{k}\Gamma^{i_r}_{am}F^{ai_2\ldots m\ldots i_k}_{j_1\ldots j_l} - \sum_{s=1}^{l}\Gamma^{m}_{aj_s}F^{ai_2\ldots i_k}_{j_1\ldots m\ldots j_l}.$$

In the summand with $r=1$, we have

$$\Gamma^{i_1}_{am}F^{mi_2\ldots i_k}_{j_1\ldots j_l} = \Gamma^{a}_{am}F^{mi_2\ldots i_k}_{j_1\ldots j_l},$$

which produces the last term. Rearranging, we finally obtain

$$(\operatorname{div}\mathbf{F})^{i_2\ldots i_k}_{j_1\ldots j_l} = \partial_a F^{ai_2\ldots i_k}_{j_1\ldots j_l} + \sum_{r=2}^{k}\Gamma^{i_r}_{am}F^{ai_2\ldots m\ldots i_k}_{j_1\ldots j_l} - \sum_{s=1}^{l}\Gamma^{m}_{aj_s}F^{ai_2\ldots i_k}_{j_1\ldots m\ldots j_l} + \Gamma^{a}_{am}F^{mi_2\ldots i_k}_{j_1\ldots j_l}.$$

□

The Laplacian is defined by the property that characterizes it in Euclidean space: it is the divergence of the gradient of a function.

Definition 2.16.9. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary and let $f : M \longrightarrow \mathbb{R}$ be a function of class C^2. The *Laplace–Beltrami operator* is defined by

$$\Delta_{\mathbf{g}} f := \operatorname{div}(\operatorname{grad} f).$$

Remark 2.16.10. First recall that the gradient of f is the vector field characterized by

$$\mathbf{g}(\operatorname{grad} f, \mathbf{X}) = df(\mathbf{X}), \qquad \mathbf{X} \in \mathfrak{X}(M).$$

In local coordinates $(x^1,\dots,x^n)$, this becomes

$$\operatorname{grad} f = g^{ij}\partial_j f\, \partial_i,$$

where (g^{ij}) are the entries of the inverse metric matrix.
Applying the definition of divergence now gives, in coordinates,

$$\operatorname{div}(\mathbf{X}) = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_i\big(\sqrt{\det(\mathbf{g})}X^i\big).$$

Substituting $\mathbf{X} = \operatorname{grad} f$ with components $X^i = g^{ij}\partial_j f$, we obtain

$$\Delta_{\mathbf{g}} f = \operatorname{div}(\operatorname{grad} f) = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_i\big(\sqrt{\det(\mathbf{g})}g^{ij}\partial_j f\big).$$

Thus the local coordinate expression for the Laplace–Beltrami operator is

$$\Delta_{\mathbf{g}} f = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_i\left(\sqrt{\det(\mathbf{g})}g^{ij}\partial_j f\right).$$

Combining the Leibniz rules for the gradient and divergence gives the following rule for the Laplacian of the product of two functions:

Proposition 2.16.11. (Leibniz rule for the Laplacian) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. If $f, g \in C^\infty(M)$, then

$$\Delta_{\mathbf{g}}(fg) = f\Delta_{\mathbf{g}} g + 2\langle \operatorname{grad}(f), \operatorname{grad}(g)\rangle_{\mathbf{g}} + g\Delta_{\mathbf{g}} f.$$

Proof. By the two preceding Leibniz rules,

$$\begin{aligned}\Delta_{\mathbf{g}}(fg) &= \operatorname{div}\big(f\operatorname{grad} g + g\operatorname{grad} f\big)\\ &= f\Delta_{\mathbf{g}} g + \langle \operatorname{grad} f, \operatorname{grad} g\rangle + g\Delta_{\mathbf{g}} f + \langle \operatorname{grad} g, \operatorname{grad} f\rangle.\end{aligned}$$

Symmetry of $\mathbf{g}$ combines the two cross terms. □

2.17 Clifford geometry and spin structures

At this point, we have developed Lie groups and Lie algebras, vector bundles, principal bundles, associated bundles, and principal connections. These are precisely the geometric ingredients needed to construct spinor bundles and their connection. We will not yet define an operator associated with these data: contraction of a Clifford connection will be studied after developing the general theory of differential operators; its symbol will be computed only after symbols have been defined, and its pseudodifferential interpretation is reserved for the corresponding chapter.

Throughout this section, we use the convention

$$vw + wv = -2\mathbf{g}(v,w)1.$$

The constructions and conventions may be compared with [92, Chapter I, §§1–6; Chapter II, §§1 and 3–5] and [21, Chapter 17].

Clifford and spinors

In the nineteenth century, William Kingdon Clifford introduced an algebra combining the metric with exterior multiplication. Representations of this algebra give rise to spinors, objects that encode the geometry of the spin group linearly. On a Riemannian manifold, a spin structure allows these representations to be globalized and the Levi–Civita connection to be lifted to the spinor bundle. This geometric construction will underlie the operators appearing in later chapters.

2.17.1 Clifford algebras

Definition 2.17.1. (Clifford algebra) Let $(V,\mathbf{g})$ be a real Euclidean space of dimension n. Its *real Clifford algebra* is the unital quotient

$$\mathrm{Cl}(V,\mathbf{g}) := T(V)\big/\langle v\otimes v + \mathbf{g}(v,v)1 \mid v\in V\rangle.$$

Identify V with its image in the quotient. The complexification is

$$\mathrm{Cl}_{\mathbb{C}}(V,\mathbf{g}) := \mathrm{Cl}(V,\mathbf{g})\otimes_{\mathbb{R}}\mathbb{C}.$$

For the standard Euclidean space, we write Cl_n and $\mathrm{Cl}_n^{\mathbb{C}}$.

Polarization of $v^2 = -|v|_{\mathbf{g}}^2 1$ gives

$$vw + wv = -2\mathbf{g}(v,w)1, \qquad v^2 = -|v|_{\mathbf{g}}^2 1. \tag{2.105}$$

Proposition 2.17.2. (Universal property) Let A be a real unital associative algebra and let $q\colon V\to A$ be a linear map such that

$$q(v)^2 = -|v|_{\mathbf{g}}^2 1_A \qquad (v\in V).$$

There exists a unique unital algebra homomorphism $\widehat{q}\colon \mathrm{Cl}(V,\mathbf{g})\to A$ whose restriction to V is q.

Proof. The universal property of the tensor algebra extends q uniquely to a unital homomorphism $T(V) \to A$. Every generator $v \otimes v + |v|^2 1$ of the defining ideal belongs to its kernel. The homomorphism therefore factors through the quotient. Since $\mathrm{Cl}(V, \mathbf{g})$ is generated by V and the unit, this factorization is unique. □

Proposition 2.17.3. (Basis, grading, and reversion) If $(e_1, \dots, e_n)$ is an orthonormal basis of V, then

$$1, \qquad e_{i_1} \cdots e_{i_k}, \qquad 1 \le i_1 < \cdots < i_k \le n,$$

form a basis of $\mathrm{Cl}(V, \mathbf{g})$. In particular, $\dim_{\mathbb{R}} \mathrm{Cl}(V, \mathbf{g}) = 2^n$. Moreover:

(a) the automorphism α determined by $\alpha(v) = -v$ induces the intrinsic grading

$$\mathrm{Cl}(V, \mathbf{g}) = \mathrm{Cl}^0(V, \mathbf{g}) \oplus \mathrm{Cl}^1(V, \mathbf{g}),$$

given by the eigenspaces of α with eigenvalues 1 and -1;

(b) there exists a unique antiautomorphism τ, called *reversion*, such that $\tau(v) = v$ for every $v \in V$, and

$$\tau(v_1 \cdots v_k) = v_k \cdots v_1.$$

Proof. The relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$ allow us to order any word and show that the indicated monomials span the algebra. To prove their independence, define on $\Lambda^\bullet V$

$$c_\Lambda(v) := \varepsilon(v) - \iota_v,$$

where $\varepsilon(v)$ is exterior multiplication and ι_v is contraction using $\mathbf{g}$. The identities between contraction and exterior product give $c_\Lambda(v)^2 = -|v|^2 I$; the universal property yields a Clifford representation. If $I = (i_1 < \cdots < i_k)$, then

$$c_\Lambda(e_{i_1}) \cdots c_\Lambda(e_{i_k}) 1 = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

The forms on the right constitute a basis of $\Lambda^\bullet V$, so the Clifford monomials are linearly independent.
The automorphism of $T(V)$ multiplying a tensor of degree k by $(-1)^k$ preserves the defining ideal and descends to α. Its two eigenspaces give the stated direct sum. Similarly, reversing the order of factors in $T(V)$ preserves the ideal and descends to the antiautomorphism τ. In both cases, uniqueness follows because V generates the algebra. □

2.17.2 The groups $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$

Definition 2.17.4. (Pin and spin groups) Let $S(V) = \{v \in V : |v|_{\mathbf{g}} = 1\}$. Define

$$\mathrm{Pin}(V, \mathbf{g}) := \{v_1 \cdots v_k : k \ge 0,\ v_j \in S(V)\} \subseteq \mathrm{Cl}(V, \mathbf{g})^\times$$

and

$$\mathrm{Spin}(V, \mathbf{g}) := \mathrm{Pin}(V, \mathbf{g}) \cap \mathrm{Cl}^0(V, \mathbf{g}).$$

The empty product is 1. For $V = \mathbb{R}^n$, write $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$.

Every unit vector satisfies $v^{-1} = -v$. In particular, the preceding sets are groups. The parity of a product of unit vectors is well-defined because the Clifford grading is a direct sum.

Theorem 2.17.5. (Cartan–Dieudonné) Every transformation $A \in \mathrm{O}(V, \mathbf{g})$ is a composition of at most n reflections in hyperplanes through the origin. The parity of the number of reflections agrees with the sign of $\det A$.

Proof. Proceed by induction on n. The result is immediate if $n = 0$. Let x be a unit vector. If $Ax = x$, then A preserves $x^{\perp}$ and the induction hypothesis applies to $A|_{x^{\perp}}$. If $Ax \neq x$, set

$$u := \frac{Ax - x}{|Ax - x|}.$$

For the reflection $r_u(y) = y - 2\mathbf{g}(y, u)u$, one verifies directly that $r_u(Ax) = x$. Hence $r_u A$ preserves $x^{\perp}$ and is a product of at most $n - 1$ reflections in that subspace. Since $r_u^2 = I$, this gives a factorization of A with at most n reflections. Finally, each reflection has determinant -1, which determines the parity. □

For a homogeneous element a of the Clifford algebra, define its twisted adjoint action on V by

$$\widetilde{\rho}(a)w := \alpha(a) w a^{-1}.$$

If v is a unit vector, the Clifford relations give

$$\widetilde{\rho}(v)w = (-v)w(-v) = w - 2\mathbf{g}(v, w)v = r_v(w).$$

Lemma 2.17.6. (Group of units of a finite-dimensional algebra) If A is a finite-dimensional real unital associative algebra, its group of units $A^{\times}$ is open in A and, with this manifold structure, is a Lie group.

Proof. For $a \in A$, let $L_a \in \mathrm{End}_{\mathbb{R}}(A)$ be left multiplication. The element a is a unit if and only if L_a is invertible. Indeed, if L_a is bijective, there exists b with $ab = 1$; moreover, $L_a(ba - 1) = (ab)a - a = 0$, and injectivity gives $ba = 1$. In a basis of A, the entries of L_a depend linearly on a, so

$$A^{\times} = \{a \in A : \det L_a \neq 0\}$$

is open. Multiplication is bilinear and, since $a^{-1} = L_a^{-1}(1)$, the matrix formula for the inverse proves that $a \mapsto a^{-1}$ is smooth. □

Proposition 2.17.7. (The pin group) For $n \geq 1$, the preceding formula defines a surjective homomorphism

$$\widetilde{\rho}\colon \mathrm{Pin}(n) \longrightarrow \mathrm{O}(n)$$

with kernel $\{\pm 1\}$. The group $\mathrm{Pin}(n)$ is a compact subgroup of $\mathrm{Cl}_n^{\times}$ and hence a Lie group. $\mathrm{Spin}(n)$ is a compact Lie subgroup.

Proof. Multiplicativity of α shows that $\widetilde{\rho}$ is a homomorphism. Its surjectivity follows from Theorem 2.17.5 and the calculation of reflections.

Let a be a homogeneous element of the kernel. If a is even, it commutes with every vector; if it is odd, it

anticommutes with every vector. Write a in the basis of Proposition 2.17.3. For a monomial e_I of degree k, we have

$$e_I e_j = (-1)^k e_j e_I \quad (j \notin I), \qquad e_I e_j = (-1)^{k-1} e_j e_I \quad (j \in I).$$

Comparing coefficients shows that the only even element commuting with all the e_j is a scalar, and that no nonzero odd element anticommutes with all of them. Thus, $a = \lambda 1$. Since an element of the kernel is even, it can be written as a product of an even number of unit vectors; consequently, $a^{-1} = \tau(a)$. Reversion fixes scalars, so $\lambda^{-1} = \lambda$ and $\lambda = \pm 1$.

If $a \in \mathrm{Pin}(n)$, Cartan–Dieudonné gives a product $b = v_1 \cdots v_k$, with $k \le n$, having the same orthogonal image. The kernel calculation gives $a = \pm b$. Consequently, $\mathrm{Pin}(n)$ is the finite union of the images of the compact sets

$$\{\pm 1\} \times S(\mathbb{R}^n)^k, \qquad 0 \le k \le n,$$

under Clifford multiplication. It is compact and, in particular, closed in the open Lie group $\mathrm{Cl}_n^\times$. The closed subgroup theorem gives it a Lie group structure. Finally, $\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cl}_n^0$ is a closed subgroup of $\mathrm{Pin}(n)$ and is also a compact Lie group. □

On the even part, $\alpha(a) = a$, so the twisted adjoint action is ordinary conjugation.

Theorem 2.17.8. (Spin covering map) For $n \ge 1$, the formula

$$\rho\colon \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n), \qquad \rho(a)v = ava^{-1},$$

defines a surjective Lie group homomorphism with kernel $\{\pm 1\}$. Its Lie algebra is

$$\mathfrak{spin}(n) = \mathrm{span}_{\mathbb{R}} \left\{ \frac{1}{2} e_i e_j : 1 \le i < j \le n \right\},$$

and

$$d\rho\left(\frac{1}{2} e_i e_j\right) e_i = e_j, \qquad d\rho\left(\frac{1}{2} e_i e_j\right) e_j = -e_i,$$

while the action vanishes on the remaining basis vectors. In particular, $d\rho\colon \mathfrak{spin}(n) \to \mathfrak{so}(n)$ is an isomorphism and ρ is a two-sheeted covering map:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \longrightarrow 1.$$

Proof. Proposition 2.17.7 shows that the restriction to the even part takes values in $\mathrm{SO}(n)$, is surjective, and has kernel $\{\pm 1\}$. For $i < j$, the curve

$$\gamma_{ij}(t) = \exp\left(\frac{t}{2} e_i e_j\right) = \cos\frac{t}{2} + \sin\frac{t}{2}\, e_i e_j$$

belongs to $\mathrm{Spin}(n)$; indeed,

$$\cos\frac{t}{2} + \sin\frac{t}{2}\, e_i e_j = (-e_i)\left(\cos\frac{t}{2}\, e_i - \sin\frac{t}{2}\, e_j\right)$$

is a product of two unit vectors. Differentiating $\rho(\gamma_{ij}(t))v = \gamma_{ij}(t)v\gamma_{ij}(t)^{-1}$ at $t=0$ gives

$$d\rho\left(\frac{1}{2}e_ie_j\right)v = \left[\frac{1}{2}e_ie_j, v\right],$$

from which the stated formulas follow. Their images form the usual basis of $\mathfrak{so}(n)$.
On the other hand, $d\rho$ is injective: if X is in its kernel, then $\rho(\exp(tX)) = \exp(t\,d\rho(X)) = I$ for every t, and discreteness of $\{\pm 1\}$ forces $X = 0$. Thus the preceding bivectors span all of $\mathfrak{spin}(n)$, and $d\rho$ is an isomorphism. The inverse function theorem gives a neighborhood W of 1 in $\mathrm{Spin}(n)$ on which ρ is a diffeomorphism onto a neighborhood U of I. Shrinking W if necessary, W and $-W$ are disjoint. Since each fiber is a coset of $\{\pm 1\}$, $\rho^{-1}(U) = W \sqcup (-W)$, and both restrictions are diffeomorphisms onto U. Translations give the same description around every point of $\mathrm{SO}(n)$. This is precisely the property of being a two-sheeted covering map. □

2.17.3 Complex spinor modules

The following classification is included to fix the dimensions, grading, and Hermitian conventions.

Proposition 2.17.9. (Classification of complex Clifford algebras) For $m \geq 0$, there exist complex algebra isomorphisms

$$\mathrm{Cl}^{\mathbb{C}}_{2m} \cong \mathrm{End}(\mathbb{C}^{2^m}), \qquad \mathrm{Cl}^{\mathbb{C}}_{2m+1} \cong \mathrm{End}(\mathbb{C}^{2^m}) \oplus \mathrm{End}(\mathbb{C}^{2^m}).$$

Proof. The base cases are $\mathrm{Cl}^{\mathbb{C}}_0 = \mathbb{C}$ and

$$\mathrm{Cl}^{\mathbb{C}}_1 \cong \mathbb{C}[t]/(t^2+1) \cong \mathbb{C} \oplus \mathbb{C}.$$

For the induction step, consider

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We have $A^2 = B^2 = -I$, $J^2 = I$, and every pair of distinct matrices anticommutes. If $e_1, \dots, e_n$ are the generators of $\mathrm{Cl}^{\mathbb{C}}_n$, the assignments

$$e_j \longmapsto e_j \otimes J, \qquad e_{n+1} \longmapsto 1 \otimes A, \qquad e_{n+2} \longmapsto 1 \otimes B$$

satisfy the Clifford relations and define a homomorphism

$$\mathrm{Cl}^{\mathbb{C}}_{n+2} \longrightarrow \mathrm{Cl}^{\mathbb{C}}_n \otimes M_2(\mathbb{C}).$$

The matrices I, A, B, J generate $M_2(\mathbb{C})$; moreover, once $1 \otimes J$ has been obtained, $e_j \otimes 1$ can be recovered. Thus the homomorphism is surjective. Both sides have complex dimension 2^{n+2} by Proposition 2.17.3, so it is an isomorphism. Induction gives the two formulas. □

Proposition 2.17.10. (Complex spinor modules) If $n = 2m$, there exists, up to equivalence, a unique irreducible module Σ_n over $\mathrm{Cl}_n^{\mathbb{C}}$, and $\dim_{\mathbb{C}} \Sigma_n = 2^m$. If $n = 2m+1$, there are two irreducible modules of dimension 2^m; their restrictions to $\mathrm{Spin}(n)$ are equivalent, and we fix one of the two Clifford extensions as Σ_n.

In even dimension, for an oriented orthonormal basis, define

$$\omega_{\mathbb{C}} := i^m c(e_1)\cdots c(e_{2m}). \tag{2.106}$$

This endomorphism is independent of the chosen basis, satisfies $\omega_{\mathbb{C}}^2 = I$, and determines the chiral decomposition

$$\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-, \qquad \Sigma_n^{\pm} = \ker(\omega_{\mathbb{C}} \mp I).$$

Multiplication by a vector interchanges the two summands. In every dimension, a Hermitian inner product can be chosen for which

$$c(v)^* = -c(v), \qquad v \in \mathbb{R}^n. \tag{2.107}$$

The spin representation is

$$\rho_\Sigma \colon \mathrm{Spin}(n) \longrightarrow \mathrm{U}(\Sigma_n), \qquad \rho_\Sigma(a) = c(a).$$

Proof. Proposition 2.17.9 and the elementary classification of irreducible modules over a matrix algebra prove the assertions on existence, uniqueness, and dimension. In odd dimension, the even algebra is identified with the complex Clifford algebra in dimension $2m$; this is why the two representations restricted to $\mathrm{Spin}(2m+1)$ are equivalent.

The Clifford relation allows us to reverse the order of the $2m$ factors in (2.106) and gives $\omega_{\mathbb{C}}^2 = I$. An oriented orthonormal change of basis multiplies the alternating product by its determinant, which is one; hence the element is independent of the basis. Moving a vector past the other $2m-1$ factors introduces a negative sign, so $\omega_{\mathbb{C}}$ anticommutes with $c(v)$, and the latter interchanges the two eigenspaces.

By Proposition 2.17.7, we may average any Hermitian inner product over $\mathrm{Pin}(n)$. For the resulting inner product, $c(v)$ is unitary when v is a unit vector. Since $c(v)^{-1} = -c(v)$, we obtain $c(v)^* = -c(v)$; homogeneity extends the identity to every vector. The restriction to $\mathrm{Spin}(n)$ is unitary and defines ρ_Σ. □

Conjugation in the algebra gives the fundamental equivariance identity

$$\rho_\Sigma(a)c(v)\rho_\Sigma(a)^{-1} = c(\rho(a)v), \qquad a \in \mathrm{Spin}(n),\ v \in \mathbb{R}^n. \tag{2.108}$$

2.17.4 The orthonormal frame bundle and spin structures

Let $(M^n, \mathbf{g})$ be a smooth oriented Riemannian manifold with or without boundary. Define

$$\mathbf{P}_{\mathrm{SO}}(M)_x := \{p \colon \mathbb{R}^n \to T_xM : p \text{ is an orientation-preserving linear isometry}\}.$$

The right action is $p \cdot A = p \circ A$. A local oriented orthonormal frame $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ identifies p with the matrix $A \in \mathrm{SO}(n)$ determined by $p(e_j^{\mathrm{est}}) = \sum_{i=1}^{n} A_i{}^j \mathbf{e}_i$. These charts make $\mathbf{P}_{\mathrm{SO}}(M) \to M$ a principal bundle with group $\mathrm{SO}(n)$, called

the *oriented orthonormal frame bundle*. Evaluation induces a bundle isomorphism

$$\mathbf{P}_{\mathrm{SO}}(M)\times_{\mathrm{SO}(n)}\mathbb{R}^n \longrightarrow TM, \qquad [p,v]\longmapsto p(v). \tag{2.109}$$

Indeed, the relation $[pA,v]=[p,Av]$ proves that the map is well-defined, and on each fiber it is a bijective linear isometry.

Definition 2.17.11. (Spinorial structure, or spin structure) A *spinorial structure*, also called a *spin structure*, on $(M,\mathbf{g})$ consists of a right principal bundle $\mathbf{P}_{\mathrm{Spin}}(M)\to M$ with group $\mathrm{Spin}(n)$ and a smooth map

$$\Lambda\colon \mathbf{P}_{\mathrm{Spin}}(M)\longrightarrow \mathbf{P}_{\mathrm{SO}}(M)$$

covering id_M and satisfying

$$\Lambda(p\cdot a)=\Lambda(p)\cdot\rho(a). \tag{2.110}$$

Two structures are equivalent if there exists a principal bundle isomorphism commuting with their maps Λ. An oriented Riemannian manifold is called *spin* if it admits a spin structure; when speaking of its spinors, an equivalence class of such structures is assumed to have been chosen.

In principal trivializations, the identity (2.110) shows that Λ has the form $(x,a)\mapsto(x,b(x)\rho(a))$. By Theorem 2.17.8, Λ is a two-sheeted covering map restricting on each fiber to the spin map. Existence of these data is not automatic. Appendix A, after introducing the necessary characteristic classes, proves the topological existence criterion and classifies spin structures.

Definition 2.17.12. (Spinor bundle and Clifford multiplication) Let $(M,\mathbf{g})$ be a spin manifold with a chosen structure. Its *complex spinor bundle* is the associated bundle

$$\Sigma M := \mathbf{P}_{\mathrm{Spin}}(M)\times_{\rho_\Sigma}\Sigma_n.$$

For $\xi\in T^*_xM$, $p\in\mathbf{P}_{\mathrm{Spin}}(M)_x$, and $\psi\in\Sigma_n$, let $v=\Lambda(p)^{-1}(\xi^\sharp)\in\mathbb{R}^n$. *Clifford multiplication* is defined by

$$c(\xi)[p,\psi]:=[p,c(v)\psi]. \tag{2.111}$$

The definition is independent of the representative. Indeed, replacing p by pa changes the coordinates of $\xi^\sharp$ and the spinor to $\rho(a)^{-1}v$ and $\rho_\Sigma(a)^{-1}\psi$; the identity (2.108) gives the same class. Therefore, (2.111) defines a smooth bundle homomorphism

$$c\colon \mathrm{Cl}_{\mathbb{C}}(T^*M,\mathbf{g}^{-1})\longrightarrow \mathrm{End}(\Sigma M)$$

and, for real covectors,

$$c(\xi)c(\eta)+c(\eta)c(\xi)=-2\mathbf{g}^{-1}(\xi,\eta)I, \qquad c(\xi)^*=-c(\xi). \tag{2.112}$$

If $n=2m$, chirality is invariant under $\mathrm{Spin}(n)$ and gives

$$\Sigma M=\Sigma^+M\oplus\Sigma^-M;$$

multiplication by a covector is odd with respect to this grading.

2.17.5 The spin connection

Denote by ω^{SO} the Levi–Civita principal connection form on $\mathbf{P}_{\mathrm{SO}}(M)$. Since $d\rho$ is an isomorphism, we may define

$$\omega^{\mathrm{Spin}} := (d\rho)^{-1}\Lambda^{*}\omega^{\mathrm{SO}}. \tag{2.113}$$

Proposition 2.17.13. (Spin connection) The form (2.113) is the unique principal connection on $\mathbf{P}_{\mathrm{Spin}}(M)$ that projects to the Levi–Civita connection. The associated connection ∇^{Σ} is Hermitian, preserves the grading in even dimension, and satisfies

$$\nabla^{\Sigma}_{\mathbf{X}}\big(c(\xi)\psi\big) = c(\nabla_{\mathbf{X}}\xi)\psi + c(\xi)\nabla^{\Sigma}_{\mathbf{X}}\psi. \tag{2.114}$$

If $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ is an oriented orthonormal frame admitting a local spin lift and

$$\omega_{ij}(\mathbf{X}) := \mathbf{g}(\nabla_{\mathbf{X}}\mathbf{e}_i,\mathbf{e}_j),$$

then, in this trivialization,

$$\big(\nabla^{\Sigma}_{\mathbf{X}}\psi\big)^{\wedge} = \mathbf{X}(\widehat{\psi}) + \frac{1}{4}\sum_{i,j=1}^{n}\omega_{ij}(\mathbf{X})c(e_i)c(e_j)\widehat{\psi}. \tag{2.115}$$

Proof. The pullback $\Lambda^{*}\omega^{\mathrm{SO}}$ reproduces vertical fields after applying $d\rho$ and is equivariant under the adjoint action. Since $d\rho$ intertwines the adjoint representations, its composition with $(d\rho)^{-1}$ satisfies the two axioms of a principal connection form. Any connection projecting to Levi–Civita must have this form, which proves uniqueness.

The calculation in Theorem 2.17.8 identifies the element of $\mathfrak{spin}(n)$ corresponding to the skew-symmetric matrix (ω_{ij}) with $\frac{1}{4}\sum_{i,j=1}^{n}\omega_{ij}e_ie_j$. The local formula for associated connections gives (2.115). Each $c(e_i)c(e_j)$, with $i \neq j$, is skew-adjoint, so the connection is Hermitian. The complex volume element is invariant under $\mathrm{Spin}(n)$ and parallel; hence the grading is preserved. Finally, differentiating the equivariance identity (2.108), or substituting the local formula directly, gives (2.114). □

2.17.6 Clifford modules and connections

The later analytic construction requires only the structure just obtained, without necessarily requiring a global spin structure.

Definition 2.17.14. (Hermitian Clifford module) A *Hermitian Clifford module* over $(M,\mathbf{g})$ is a complex Hermitian bundle $\mathbf{S} \to M$ with a smooth unital homomorphism of algebra bundles

$$c\colon \mathrm{Cl}_{\mathbb{C}}(T^{*}M,\mathbf{g}^{-1}) \longrightarrow \mathrm{End}(\mathbf{S})$$

such that $c(\xi)^{*} = -c(\xi)$ for every real covector ξ. Equivalently, it suffices to give a smooth linear map $c\colon T^{*}M \to \mathrm{End}(\mathbf{S})$ satisfying (2.112); the equivalence is the Clifford universal property applied fiberwise.

A Hermitian connection $\nabla^{\mathbf{S}}$ is a *Clifford connection* if

$$\nabla^{\mathbf{S}}_X(c(\xi)\mathbf{s}) = c(\nabla_X\xi)\mathbf{s} + c(\xi)\nabla^{\mathbf{S}}_X\mathbf{s}. \tag{2.116}$$

If $\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-$ is graded, we also require the connection to preserve the grading and $c(\xi)$ to be odd.

The bundle ΣM with ∇^{Σ} is the fundamental spinorial example.

Example 2.17.15. (Differential forms) Every Riemannian manifold, whether spin or not, has the graded module

$$\mathbf{S} = \Lambda^\bullet T^*M \otimes \mathbb{C}, \qquad c(\xi) = \varepsilon(\xi) - \iota_{\xi^\sharp}.$$

The identities

$$\iota_{\xi^\sharp}\varepsilon(\eta) + \varepsilon(\eta)\iota_{\xi^\sharp} = \mathbf{g}^{-1}(\xi,\eta)I$$

prove the Clifford relation. For the Hermitian inner product induced by $\mathbf{g}$, we have $\varepsilon(\xi)^* = \iota_{\xi^\sharp}$, so $c(\xi)^* = -c(\xi)$. The connection induced by Levi–Civita commutes with exterior product and contraction in the sense of the Leibniz rule; it is therefore a Clifford connection. The grading is the parity of forms.

2.17.7 Spin^c structures

Definition 2.17.16. (The group $\mathrm{Spin}^c(n)$) Define the compact Lie group

$$\mathrm{Spin}^c(n) := \big(\mathrm{Spin}(n) \times \mathrm{U}(1)\big)/\{(1,1),(-1,-1)\}.$$

The homomorphisms

$$\rho_c[a,z] = \rho(a), \qquad \det_c[a,z] = z^2$$

take values in $\mathrm{SO}(n)$ and $\mathrm{U}(1)$, respectively. The complex spin representation is

$$\rho^c_\Sigma[a,z] = z\rho_\Sigma(a). \tag{2.117}$$

All three formulas are well-defined on the quotient: replacing (a,z) by $(-a,-z)$ changes none of them, since $\rho(-a) = \rho(a)$ and $\rho_\Sigma(-1) = -I$. The quotient by the finite central subgroup inherits a Lie group structure. Indeed, the action of $\{(1,1),(-1,-1)\}$ by translations is free and, since the group is finite, properly discontinuous. A sufficiently small neighborhood of each point is disjoint from its nontrivial translate; its projection to the quotient is a chart, and in these charts the induced multiplication and inversion are smooth.

Definition 2.17.17. (Spin^c structure) A Spin^c *structure* on $(M,\mathbf{g})$ consists of a principal bundle $\mathbf{P}_{\mathrm{Spin}^c}(M) \to M$ and a map

$$\Lambda_c : \mathbf{P}_{\mathrm{Spin}^c}(M) \longrightarrow \mathbf{P}_{\mathrm{SO}}(M)$$

covering the identity and satisfying $\Lambda_c(pa) = \Lambda_c(p)\rho_c(a)$. Its *determinant line* and *spinor bundle* are, respectively,

$$\mathbf{L} := \mathbf{P}_{\mathrm{Spin}^c}(M) \times_{\det_c} \mathbb{C}, \qquad \Sigma^c M := \mathbf{P}_{\mathrm{Spin}^c}(M) \times_{\rho^c_\Sigma} \Sigma_n.$$

The equivariance identity (2.108) proves, exactly as in the spin case, that

$$c(\xi)[p,\psi] := [p, c(\Lambda_c(p)^{-1}\xi^\sharp)\psi]$$

defines Clifford multiplication on $\Sigma^c M$. In even dimension, it gives the grading $\Sigma^{c,+}M \oplus \Sigma^{c,-}M$.

A unitary connection on $\mathbf{L}$, together with the Levi–Civita connection, uniquely determines a principal connection on $\mathbf{P}_{\mathrm{Spin}^c}(M)$. Indeed, the Lie algebra map

$$(d\rho_c, d\det_c)\colon \mathfrak{spin}^c(n) \longrightarrow \mathfrak{so}(n) \oplus i\mathbb{R}$$

is an isomorphism: on the first factor it is $d\rho$, and on the second, the differential of $z \mapsto z^2$ is multiplication by 2. The pullbacks of the two connection forms, followed by the inverse of this isomorphism, give the desired principal connection. The associated connection on $\Sigma^c M$ is Hermitian and Clifford by the same argument as in Proposition 2.17.13.

The topological existence criterion is stated and proved, after the required characteristic classes have been introduced, in Theorem A.8.3 of Appendix A.

We have thus constructed the purely geometric data: the spin group and its covering map, spin structures, spinor bundles, Clifford multiplication, and their connections. In Chapter 8, a Clifford connection will be contracted to obtain a differential operator. Its symbol will be studied in Chapter 11, and its pseudodifferential symbol, with the appropriate Fourier convention, only in Chapter 15.

Part II

Nonlinear analysis in Euclidean spaces

Introduction to Part II

When an estimate on a manifold is proved within a chart, we are ultimately using a result formulated on an open subset of $\mathbb{R}^n$. Before returning to geometry, it is therefore useful to develop the Euclidean model carefully and determine which tools depend on the linear structure of the space, which survive a change of coordinates, and which can later be localized using cutoff functions.

Sobolev spaces arise when derivatives are allowed to be understood in the weak sense. Distributions extend this idea further, making it possible to treat singularities and data that are not ordinary functions. The Fourier transform allows differentiation and regularity to be studied from the frequency viewpoint. Interpolation theory then provides a common language for comparing different levels of regularity and leads naturally to the Bessel, Slobodeckij, Besov, and Triebel–Lizorkin scales. Orders of regularity are thus no longer restricted to integers.

This part supplies many of the tools that will later be used with almost no change to their mathematical substance: Sobolev inequalities, Rellich–Kondrashov compactness, multipliers, changes of variables, extension operators, convolutions, and localization arguments. On returning to a manifold, what changes is how these tools are transported and how the constants are controlled. The Euclidean work in these chapters makes it possible to distinguish clearly what belongs to analysis from what geometry must provide.

From classical solutions to weak solutions

For a long time, solving a differential equation meant finding a sufficiently differentiable function that satisfied it pointwise. Twentieth-century analysis decisively broadened this idea. Sobolev incorporated weak derivatives into normed spaces; Schwartz constructed a stable theory of distributions; Calderón and Zygmund developed fundamental tools of harmonic analysis; Gagliardo and Slobodeckij studied noninteger regularity; and Lions, Peetre, and Triebel organized broad scales of function spaces. These theories did not replace classical solutions: they made it possible to begin with less regularity and understand the conditions under which it could later be recovered.

3

Sobolev spaces on open subsets of $\mathbb{R}^n$

Sobolev spaces arise naturally in the Dirichlet problem for the Poisson equation. Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and consider

$$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Suppose for the moment that u and φ are smooth, that u satisfies the boundary condition, and that φ is a test function vanishing on the boundary. Integration by parts gives

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx.$$

This identity does not require classical second derivatives of u: it is enough for u and its gradient to have suitable integrability. This leads to weak derivatives, to $W^{1,2}(\Omega)$, and to the subspace $W_0^{1,2}(\Omega)$ that encodes the Dirichlet condition. The trace provides an interpretation of boundary values, while the Poincaré inequality, embeddings, and compactness theorems provide the estimates needed to solve the problem.

The same equation is obtained as the critical point condition for the functional

$$\mathcal{E}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega f u \, dx.$$

Indeed, its Euler–Lagrange equation is precisely the preceding weak identity. This relationship between a PDE and an energy will serve as the model for the variational methods developed at the end of the book.

The nonlinear problem

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = f$$

is formally associated with

$$\mathcal{E}_p(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega f u \, dx.$$

This example leads to $W^{1,p}(\Omega)$ and shows why convexity and reflexivity matter in the study of minimizing sequences. The theory of the p-Laplacian will not be developed here; it will guide the definitions and estimates in this chapter.

Sobolev and the shift toward weak regularity

In the 1930s, Sergei Sobolev showed that the regularity useful for an equation need not coincide with classical pointwise differentiability. By incorporating weak derivatives into normed spaces, he made it possible to combine integration by parts, completeness, and regularity estimates within a single framework. His embedding theorems also explain a central phenomenon in analysis: a certain degree of integrability of the derivatives forces a function to gain integrability, continuity, or even Hölder regularity. This chapter develops these ideas in the Euclidean setting before transferring them to manifolds and vector bundles [136].

3.1 Weak derivatives

We begin with the elementary formula that motivates the definition.

Proposition 3.1.1. (Integration by parts) If $f \in C^1(\Omega)$ and $\varphi \in C_c^1(\Omega)$, then

$$\int_\Omega \frac{\partial f}{\partial x_i}\varphi + \int_\Omega f\frac{\partial \varphi}{\partial x_i} = 0$$

for every $i \in \{1,\dots,n\}$.

Proof. The extension of φ by zero belongs to $C_c^1(\mathbb{R}^n)$, since the support of φ is a compact subset of Ω. We may therefore work in $\mathbb{R}^n$. Choose $M > 0$ such that $\operatorname{supp}(\varphi) \subset (-M,M)^n$. For $i = 1$ and each $\widehat{x} = (x_2,\dots,x_n) \in \mathbb{R}^{n-1}$, Theorem B.6.5 gives

$$\int_{-M}^{M} \frac{\partial \varphi}{\partial x_1}(x_1,\widehat{x})\,dx_1 = \varphi(M,\widehat{x}) - \varphi(-M,\widehat{x}) = 0.$$

Theorem B.6.18 allows us to integrate with respect to the remaining variables and conclude that $\int_\Omega \frac{\partial \varphi}{\partial x_1} = 0$. The other indices are treated in the same way. Applying this identity to $f\varphi \in C_c^1(\Omega)$ and using the product rule, we obtain

$$0 = \int_\Omega \frac{\partial (f\varphi)}{\partial x_i} = \int_\Omega \frac{\partial f}{\partial x_i}\varphi + \int_\Omega f\frac{\partial \varphi}{\partial x_i}.$$

□

The preceding identity suggests the following definition.

Definition 3.1.2. Let $u \in L^1_{\mathrm{loc}}(\Omega)$. We say that u is **weakly differentiable on** Ω if there exist $v_1,\dots,v_n \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u\frac{\partial \varphi}{\partial x_i} + \int_\Omega v_i\varphi = 0$$

for every $\varphi \in C_c^\infty(\Omega)$ and every $i \in \{1,\dots,n\}$. Each v_i is called the ***i*th weak derivative of** u **on** Ω and is denoted by $D_i u$.

Differential operators constructed from partial derivatives thus admit a weak version. In particular, one can define the weak gradient, the weak Laplacian, and, analogously, higher-order operators.

Remark 3.1.3. If $v, w \in L^1_{\mathrm{loc}}(\Omega)$ satisfy

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega v\varphi \quad \text{and} \quad \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega w\varphi \qquad \forall \varphi \in C_c^\infty(\Omega),$$

then $v = w$ a.e. (by Proposition B.6.20).

The weak derivative agrees with the classical derivative whenever the latter exists. If $u \in C^1(\Omega)$, Proposition 3.1.1 gives

$$\int_\Omega \frac{\partial u}{\partial x_i}\varphi + \int_\Omega u \frac{\partial \varphi}{\partial x_i} = 0$$

so that u is weakly differentiable and

$$D_i u = \frac{\partial u}{\partial x_i} \ \forall i \in \{1, \dots, n\}.$$

Weak derivatives have the following elementary properties.

Proposition 3.1.4. (a) If $u, v \in L^1_{\mathrm{loc}}(\Omega)$ are weakly differentiable and $\lambda, \mu \in \mathbb{R}$, then $\lambda u + \mu v$ is weakly differentiable and

$$D_i(\lambda u + \mu v) = \lambda D_i u + \mu D_i v$$

for each $i \in \{1, \dots, n\}$.

(b) If $u \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable on Ω and $\zeta \in C_c^\infty(\Omega)$, then the extension of ζu by zero is weakly differentiable on $\mathbb{R}^n$ and

$$D_i(\zeta u) = \frac{\partial \zeta}{\partial x_i} u + \zeta D_i u$$

Proof. (a) For each $i \in \{1, \dots, n\}$ and $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \left[(\lambda u + \mu v) \frac{\partial \varphi}{\partial x_i} + (\lambda D_i u + \mu D_i v)\varphi \right] = \lambda \int_\Omega \left[u \frac{\partial \varphi}{\partial x_i} + (D_i u)\varphi \right] + \mu \int_\Omega \left[v \frac{\partial \varphi}{\partial x_i} + (D_i v)\varphi \right] = 0$$

where the second equality uses the weak differentiability of u and v. This proves that $\lambda u + \mu v$ is weakly differentiable and $D_i(\lambda u + \mu v) = \lambda D_i u + \mu D_i v$.

(b) Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then $\zeta\varphi \in C_c^\infty(\Omega)$ and

$$\int_{\mathbb{R}^n} \zeta u \frac{\partial \varphi}{\partial x_i} + \int_{\mathbb{R}^n} \left(\zeta D_i u + u \frac{\partial \zeta}{\partial x_i} \right) \varphi = \int_\Omega u \frac{\partial (\zeta\varphi)}{\partial x_i} + \int_\Omega (D_i u)(\zeta\varphi) = 0.$$

□

A function whose weak derivatives vanish retains the constancy property of differentiable functions with zero differential.

Lemma 3.1.5. (Functions with vanishing weak derivatives) Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and let $v \in L^1_{\mathrm{loc}}(\Omega)$ be weakly differentiable. If

$$D_j v = 0 \qquad \text{almost everywhere on } \Omega, \quad j \in \{1, \dots, n\}, \tag{3.1}$$

then v is constant almost everywhere on each connected component of Ω.

Proof. It suffices to consider the case in which Ω is connected and nonempty. Fix a ball $B = B_{\mathrm{euc}}(x_0, r)$ with $\overline{B} \subseteq \Omega$. Choose $R > r$ such that $\overline{B_{\mathrm{euc}}(x_0, R)} \subseteq \Omega$. Let ρ be the mollifier from Theorem B.9.1 and, for $0 < \varepsilon < R - r$, define

$$v_\varepsilon(x) := \int_\Omega v(y)\rho_\varepsilon(x-y)\,dy, \qquad x \in B.$$

For each $x \in B$, the function $y \mapsto \rho_\varepsilon(x-y)$ has compact support in $B_{\mathrm{euc}}(x_0, R)$. The local integrability of v and part (a) of that theorem allow differentiation under the integral. Applying the definition of weak derivative to this test function, we obtain

$$\begin{aligned}\partial_j v_\varepsilon(x) &= \int_\Omega v(y)\partial_{x_j}\rho_\varepsilon(x-y)\,dy \\ &= -\int_\Omega v(y)\partial_{y_j}\rho_\varepsilon(x-y)\,dy \\ &= \int_\Omega D_j v(y)\rho_\varepsilon(x-y)\,dy = 0.\end{aligned}$$

Therefore, v_ε is smooth and has zero differential on the connected ball B, so that $v_\varepsilon = c_\varepsilon$ on B for some constant $c_\varepsilon \in \mathbb{R}$.

Part (b) of Theorem B.9.1, applied with $p = 1$ and the compact set $\overline{B}$, gives $v_\varepsilon \to v$ in $L^1(B)$. Set

$$c_B := \frac{1}{\lambda_n(B)} \int_B v\,d\lambda_n.$$

Since $v_\varepsilon = c_\varepsilon$ on B,

$$|c_\varepsilon - c_B| \le \frac{1}{\lambda_n(B)} \|v_\varepsilon - v\|_{L^1(B)} \longrightarrow 0.$$

Consequently,

$$\|v - c_B\|_{L^1(B)} \le \|v - v_\varepsilon\|_{L^1(B)} + \lambda_n(B)|c_\varepsilon - c_B| \longrightarrow 0,$$

and $v = c_B$ almost everywhere on B.

If two relatively compact balls in Ω intersect, their intersection is a nonempty open set and has positive measure. The corresponding constants agree, since both equal v almost everywhere on that intersection. Thus, we can define $c(x) := c_B$ using any ball $B \Subset \Omega$ containing x. This function is well defined and locally constant. Since Ω is connected, it has a single value $c \in \mathbb{R}$.

Finally, second countability of Ω allows us to choose a countable cover by such balls. On each ball, $v = c$ outside a set of measure zero; the countable union of these sets also has measure zero. Therefore, $v = c$ almost everywhere on Ω. □

3.2 Sobolev spaces

The notion of weak derivative allows us to define Sobolev spaces on open subsets of $\mathbb{R}^n$.

Definition 3.2.1. Let $p \in [1, \infty]$. The **Sobolev space** $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) \mid u \text{ is weakly differentiable on } \Omega \text{ and } D_i u \in L^p(\Omega) \text{ for every } i \in \{1, \dots, n\}\}.$$

We introduce the normed structure as follows.

Definition 3.2.2. Let $p \in [1,\infty]$. For $u \in W^{1,p}(\Omega)$, define

$$\|u\|_{W^{1,p}(\Omega)} := \begin{cases} (\|u\|_{L^p(\Omega)}^p + \sum_{i=1}^n \|D_i u\|_{L^p(\Omega)}^p)^{\frac{1}{p}} & \text{if } p \in [1,\infty) \\ \max\{\|u\|_{L^\infty(\Omega)}, \|D_1 u\|_{L^\infty(\Omega)}, \dots, \|D_n u\|_{L^\infty(\Omega)}\} & \text{if } p = \infty \end{cases}$$

The preceding expression defines a norm on $W^{1,p}(\Omega)$.

Proposition 3.2.3. $W^{1,p}(\Omega)$ is a real vector space and $\|\cdot\|_{W^{1,p}(\Omega)}$ is a norm on $W^{1,p}(\Omega)$ for every $p \in [1,\infty]$.

Proof. By Proposition 3.1.4, $W^{1,p}(\Omega)$ is a vector subspace of $L^p(\Omega)$, and hence is a vector space. Now, given $u \in W^{1,p}(\Omega)$, if $u = 0$, then $D_i u = 0$ for every $i \in \{1,\dots,n\}$, so that $\|u\|_{W^{1,p}(\Omega)} = 0$. Conversely, if $\|u\|_{W^{1,p}(\Omega)} = 0$, since $\|u\|_{L^p(\Omega)} \le \|u\|_{W^{1,p}(\Omega)}$, we have $u = 0$ in $L^p(\Omega)$.
We have $\|\lambda u\|_{W^{1,p}(\Omega)} = |\lambda| \|u\|_{W^{1,p}(\Omega)}$ for any $u \in W^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$.
Finally, if $p \in [1,\infty)$, applying first Minkowski's inequality in $L^p(\Omega)$, given in Theorem B.7.3, and then the triangle inequality for the $\|\cdot\|_p$ norm on $\mathbb{R}^{n+1}$, we obtain, for any $u, v \in W^{1,p}(\Omega)$,

$$\begin{aligned} \|u+v\|_{W^{1,p}(\Omega)} &\le \left((\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)})^p + \sum_{i=1}^n (\|D_i u\|_{L^p(\Omega)} + \|D_i v\|_{L^p(\Omega)})^p \right)^{\frac{1}{p}} \\ &\le \left(\|u\|_{L^p(\Omega)}^p + \sum_{i=1}^n \|D_i u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} + \left(\|v\|_{L^p(\Omega)}^p + \sum_{i=1}^n \|D_i v\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \\ &= \|u\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,p}(\Omega)}. \end{aligned}$$

If $p = \infty$, the triangle inequality follows in exactly the same way from the triangle inequality for the $\|\cdot\|_\infty$ norm on $L^\infty(\Omega)$ and that for the $\|\cdot\|_\infty$ norm on $\mathbb{R}^{n+1}$. □

The following lemma will be particularly useful for working with Sobolev spaces, both in Euclidean spaces and later on Riemannian manifolds.

Lemma 3.2.4. Let $p \in [1,\infty]$ and let (u_k) be a sequence in $W^{1,p}(\Omega)$ such that $u_k \to u$ in $L^p(\Omega)$ and $D_i u_k \to v_i$ in $L^p(\Omega)$ for each $i \in \{1,\dots,n\}$. Then u is weakly differentiable on Ω, $v_i = D_i u$ for every $i \in \{1,\dots,n\}$, and $u_k \to u$ in $W^{1,p}(\Omega)$.

Proof. Let $\varphi \in C_c^\infty(\Omega)$. Then $\varphi, \dfrac{\partial \varphi}{\partial x_i} \in L^q(\Omega)$ for every $q \in [1,\infty]$. Let $q > 1$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ if $p \in (1,\infty)$, $q = 1$ if $p = \infty$, and $q = \infty$ if $p = 1$. By Hölder's inequality (Proposition B.7.9), we have

$$\left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} - \int_\Omega u_k \frac{\partial \varphi}{\partial x_i} \right| \le \|u - u_k\|_{L^p(\Omega)} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^q(\Omega)} \to 0 \text{ as } k \to \infty,$$

$$\left| \int_\Omega v_i \varphi - \int_\Omega (D_i u_k) \varphi \right| \le \|v_i - D_i u_k\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)} \to 0 \text{ as } k \to \infty.$$

Consequently,

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}+\int_\Omega v_i\varphi=\lim_{k\to\infty}\left(\int_\Omega u_k\frac{\partial\varphi}{\partial x_i}+\int_\Omega (D_iu_k)\varphi\right)=0$$

for each $i\in\{1,\dots,n\}$. This proves that u is weakly differentiable on Ω and that $v_i=D_iu$. Consequently, $u\in W^{1,p}(\Omega)$. Moreover, if $p\in[1,\infty)$, we have

$$\lim_{k\to\infty}\|u_k-u\|^p_{W^{1,p}(\Omega)}=\lim_{k\to\infty}\|u_k-u\|^p_{L^p(\Omega)}+\sum_{i=1}^n\lim_{k\to\infty}\|D_iu_k-D_iu\|^p_{L^p(\Omega)}=0.$$

If $p=\infty$, we have

$$\begin{aligned}\lim_{k\to\infty}\|u_k-u\|_{W^{1,\infty}(\Omega)}&=\lim_{k\to\infty}\max\left\{\|u_k-u\|_{L^\infty(\Omega)},\|D_1u_k-D_1u\|_{L^\infty(\Omega)},\dots,\|D_nu_k-D_nu\|_{L^\infty(\Omega)}\right\}\\&=0,\end{aligned}$$

by continuity of the function max and the assumed convergences in $L^\infty(\Omega)$. □

The preceding lemma helps us prove the following theorem.

Theorem 3.2.5. The space $W^{1,p}(\Omega)$ is a Banach space for every $p\in[1,\infty]$.

Proof. Let (u_k) be a Cauchy sequence in $W^{1,p}(\Omega)$. Then the sequences (u_k) and (D_iu_k) are Cauchy in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, we have $u_k\to u$ in $L^p(\Omega)$ and $D_iu_k\to v_i$ in $L^p(\Omega)$ for each $i\in\{1,\dots,n\}$. Lemma 3.2.4 gives $u\in W^{1,p}(\Omega)$ and $u_k\to u$ in $W^{1,p}(\Omega)$. This proves that $W^{1,p}(\Omega)$ is complete; since Proposition 3.2.3 already shows that it is a normed space, we conclude that it is a Banach space. □

In addition to being a Banach space, for $p=2$, $W^{1,2}(\Omega)$ is a Hilbert space, for which we use the notation $H^1(\Omega):=W^{1,2}(\Omega)$. We record this as a proposition.

Proposition 3.2.6. If $p=2$, $W^{1,p}(\Omega)$ is a Hilbert space with the inner product

$$\langle u,v\rangle_{H^1(\Omega)}:=\int_\Omega uv+\sum_{i=1}^n\int_\Omega (D_iu)(D_iv)=\int_\Omega uv+\int_\Omega\nabla u\cdot\nabla v.$$

Proof. The expression $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$ is bilinear by linearity of the integral, distributivity of the product of real-valued functions, and bilinearity of the inner product on R^n. It is symmetric because multiplication of real-valued functions is commutative and the dot product on $\mathbb{R}^n$ is symmetric. It remains to show that if $\langle v,v\rangle=0$, then $v=0$. Suppose that $\langle v,v\rangle=0$. Then

$$\int_\Omega v^2+\int_\Omega|\nabla v|^2=0.$$

Since both summands are nonnegative, both must equal 0, whence $\displaystyle\int_\Omega v^2=0$ and therefore $v=0$ almost everywhere; that is, $v=0$ in $L^2(\Omega)$, and hence $v=0$ in $W^{1,2}(\Omega)$. □

Remark 3.2.7. $C_c^\infty(\Omega) \subseteq W^{1,p}(\Omega)$ for every $p \in [1,\infty)$, since every function of class $C_c^\infty(\Omega)$ is weakly differentiable and its partial derivatives are continuous and compactly supported, and thus belong to $L^p(\Omega)$.

The space $C_c^\infty(\Omega)$ leads to a basic question in Sobolev analysis: under what conditions can a Sobolev function be approximated by smooth functions with compact support? We know that, in general, a Sobolev function need not be classically differentiable; nevertheless, under suitable assumptions on the domain, these functions can be approximated by elements of $C_c^\infty(\Omega)$. Questions of this kind are called *approximation problems* and provide an essential tool both in the abstract theory and in applications to partial differential equations. We therefore give the following definition.

Definition 3.2.8. Let $p \in [1,\infty)$. We define the Sobolev space $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. We write

$$H_0^1(\Omega) := W_0^{1,2}(\Omega).$$

Remark 3.2.9. This notation makes sense because $W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,2}(\Omega)}$ is a closed vector subspace of $W^{1,2}(\Omega)$, which is a Hilbert space, and therefore $H_0^1(\Omega)$ is also a Hilbert space.

In the case $\Omega = \mathbb{R}^n$, there are no boundary obstructions. The argument combines cutoffs moving out toward infinity with regularization by mollifiers, and will serve as the model for the proof on complete Riemannian manifolds.

Proposition 3.2.10. $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$ for every $p \in [1,\infty)$.

Proof. Let $u \in W^{1,p}(\mathbb{R}^n)$. By Proposition 1.3.6, there exists $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \chi \le 1$, $\chi = 1$ on $B_{\mathrm{euc}}(0,1)$, and $\operatorname{supp}\chi \subseteq B_{\mathrm{euc}}(0,2)$. For $R \ge 1$, set

$$\chi_R(x) := \chi\left(\frac{x}{R}\right), \qquad u_R := \chi_R u.$$

The weak Leibniz rule gives

$$D_i u_R = \chi_R D_i u + u D_i \chi_R.$$

Indeed, if $\varphi \in C_c^\infty(\mathbb{R}^n)$, we apply the definition of the weak derivative of u to $\chi_R\varphi$ and expand $D_i(\chi_R\varphi)$. Thus, we do not assume an unproved product rule.
Since $\chi_R(x) \to 1$ for every $x \in \mathbb{R}^n$ and $|1-\chi_R| \le 1$, the dominated convergence theorem B.6.14, applied to $|u|^p$ and $|D_i u|^p$, shows that

$$\|(1-\chi_R)u\|_{L^p(\mathbb{R}^n)} + \|(1-\chi_R)D_i u\|_{L^p(\mathbb{R}^n)} \longrightarrow 0.$$

Moreover,

$$\|u D_i \chi_R\|_{L^p(\mathbb{R}^n)} \le \frac{\|D_i\chi\|_{L^\infty(\mathbb{R}^n)}}{R}\|u\|_{L^p(\mathbb{R}^n)} \longrightarrow 0.$$

Therefore, $u_R \to u$ in $W^{1,p}(\mathbb{R}^n)$.
Now fix R. Let ρ be the mollifier from Theorem B.9.1 and define $u_{R,\varepsilon} := \rho_\varepsilon * u_R$. Then $u_{R,\varepsilon} \in C_c^\infty(\mathbb{R}^n)$ and, by Fubini's theorem B.6.18 and the definition of weak derivative,

$$D_i u_{R,\varepsilon} = \rho_\varepsilon * D_i u_R.$$

Continuity of translations from Proposition B.8.17, Minkowski's integral inequality from Theorem B.8.2, and the dominated convergence theorem B.6.14 imply that, for every $v \in L^p(\mathbb{R}^n)$,

$$\|\rho_\varepsilon * v - v\|_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \rho(z)\|v(\cdot - \varepsilon z) - v\|_{L^p(\mathbb{R}^n)}\, dz \longrightarrow 0.$$

We apply this estimate to $v = u_R$ and $v = D_i u_R$ to conclude that $u_{R,\varepsilon} \to u_R$ in $W^{1,p}(\mathbb{R}^n)$. Choosing first $R \to \infty$ and then $\varepsilon \to 0^+$, or taking a diagonal sequence, completes the proof. □

3.3 Sobolev embeddings

By definition, the Sobolev spaces $W_0^{1,p}(\Omega)$ are contained in $L^p(\Omega)$. An interesting question is: for which other values q is $W_0^{1,p}(\Omega)$ continuously embedded in $L^q(\Omega)$?

The fundamental tool for answering this question consists of inequalities that estimate the $L^q(\mathbb{R}^n)$ norm of a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ in terms of the $(L^p(\mathbb{R}^n))^n$ norm of its gradient. Such inequalities are usually called *Sobolev inequalities*.

We begin by giving a norm on the product space

$$(L^p(\mathbb{R}^n))^n := L^p(\mathbb{R}^n) \times \cdots \times L^p(\mathbb{R}^n)$$

with $p \in [1, \infty)$. This norm is given by

$$\|(f_1, \ldots, f_n)\|_{(L^p(\mathbb{R}^n))^n} := \left(\sum_{i=1}^n \|f_i\|_{L^p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.$$

Now consider $\varphi \in C_c^\infty(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$,

$$\nabla\varphi = \left(\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n} \right) \in (L^p(\mathbb{R}^n))^n$$

for every $p \in [1, \infty)$.

We ask for which numbers $p, q \in [1, \infty)$ it is possible to bound $\|\varphi\|_{L^q(\mathbb{R}^n)}$ uniformly in terms of $\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$. We first show that p and q cannot be arbitrary.

Proposition 3.3.1. Let $p, q \in [1, \infty)$. If there is a constant $C > 0$ depending only on n, p, and q such that

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, then $p < n$ and $q = \dfrac{np}{n-p}$.

Proof. For each $\lambda > 0$, $\psi(x) = \lambda x$ is a diffeomorphism of $\mathbb{R}^n$ onto itself of class C^∞ such that $|\det(D\psi)| = |\lambda^n| = \lambda^n$. By Theorem B.6.15,

$$\|\varphi_\lambda\|_{L^q(\mathbb{R}^n)}^q = \frac{1}{\lambda^n}\|\varphi\|_{L^q(\mathbb{R}^n)}^q,$$

and

$$\left\| \frac{\partial \varphi_\lambda}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}^p = \frac{\lambda^p}{\lambda^n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}^p.$$

Applying the hypothesis to φ_λ gives

$$\frac{1}{\lambda^{\frac{n}{q}}} \|\varphi\|_{L^q(\mathbb{R}^n)} \leq C \frac{\lambda}{\lambda^{\frac{n}{p}}} \|\nabla \varphi\|_{(L^p(\mathbb{R}^n))^n},$$

that is,

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|\nabla \varphi\|_{(L^p(\mathbb{R}^n))^n}.$$

We may take the limit as $\lambda \to 0^+$ if $1 - \frac{n}{p} + \frac{n}{q} > 0$, or as $\lambda \to \infty$ if $1 - \frac{n}{p} + \frac{n}{q} < 0$, and observe that if $1 - \frac{n}{p} + \frac{n}{q} \neq 0$, the preceding inequality cannot hold for any $\varphi \neq 0$, since we would have $\|\varphi\|_{L^q(\mathbb{R}^n)} = 0$.
We must therefore have $1 - \frac{n}{p} + \frac{n}{q} = 0$, which implies $\frac{n}{p} > 1$ and $q = \frac{np}{n-p}$.

□

This value of q plays an important role in Sobolev space theory and thus receives a special name.

Definition 3.3.2. If $p \in [1, n)$, the critical Sobolev exponent is defined by

$$p^* := \frac{np}{n-p}.$$

Moreover, $p^* > p$.

In fact, the inequality involving the norm of the gradient holds for $p \in [1, n)$ and $q = p^*$, as we shall now prove. We first need a lemma.

Lemma 3.3.3. Let $n \geq 2$ and $f_1, \ldots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, write $\hat{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$ and define

$$f(x) := \prod_{i=1}^n f_i(\hat{x}_i)$$

Then $f \in L^1(\mathbb{R}^n)$ and

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Proof. We argue by induction on n. If $n = 2$, we have

$$\int_{\mathbb{R}^2} |f(x_1, x_2)| dx_1 dx_2 = \int_{\mathbb{R}^2} |f_1(x_2) f_2(x_1)| dx_1 dx_2 = \int_{\mathbb{R}^2} |f_1(x_2)| |f_2(x_1)| dx_1 dx_2$$

$$= \int_{\mathbb{R}} |f_1(x_2)| dx_2 \int_{\mathbb{R}} |f_2(x_1)| dx_1.$$

Suppose that the assertion holds for some $n \geq 2$; we show that it holds for $n+1$.

Let $f_1, \dots, f_{n+1} \in L^n(\mathbb{R}^n)$. Fix the value of x_{n+1} and define

$$g_i(z_1, \dots, z_{n-1}) := |f_i(z_1, \dots, z_{n-1}, x_{n+1})|^{\frac{n}{n-1}}, i \in \{1, \dots, n\}.$$

By Theorem B.6.18, we have $g_i \in L^{n-1}(\mathbb{R}^{n-1})$ for almost every x_{n+1} (by Fubini's theorem with $m = n-1$). The induction hypothesis therefore shows that the function

$$g(y) := \prod_{i=1}^{n} g_i(\hat{y}_i) = \prod_{i=1}^{n} |f_i(\hat{y}_i, x_{n+1})|^{\frac{n}{n-1}} = \left(\prod_{i=1}^{n} |f_i(\hat{y}_i, x_{n+1})| \right)^{\frac{n}{n-1}}$$

$y \in \mathbb{R}^n$ belongs to $L^1(\mathbb{R}^n)$ and that

$$\|g\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \|g_i\|_{L^{n-1}(\mathbb{R}^{n-1})} = \prod_{i=1}^{n} \left(\int_{\mathbb{R}^{n-1}} |f_i(z, x_{n+1})|^n dz \right)^{\frac{1}{n-1}}.$$

We have

$$|f(y, x_{n+1})| = \left(\prod_{i=1}^{n} |f_i(\hat{y}_i, x_{n+1})| \right) |f_{n+1}(y)| = |g(y)|^{\frac{n-1}{n}} |f_{n+1}(y)|.$$

Using Proposition B.7.9 (Hölder's inequality) and the preceding inequality for $\|g\|_{L^1(\mathbb{R}^n)}$, we obtain the following:

$$\int_{\mathbb{R}^n} |f(y, x_{n+1})| dy \leq \|g\|_{L^1(\mathbb{R}^n)}^{\frac{n-1}{n}} \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^{n-1}} |f_i(z, x_{n+1})|^n dz \right)^{\frac{1}{n}} \|f_{n+1}\|_{L^n(\mathbb{R}^n)}.$$

The preceding argument holds for almost every x_{n+1}, so the functions

$$h_i(x_{n+1}) := \left(\int_{\mathbb{R}^{n-1}} |f_i(z, x_{n+1})|^n \, dz \right)^{\frac{1}{n}}, \qquad i \in \{1, \dots, n\},$$

belong to $L^n(\mathbb{R})$. Indeed, by Tonelli's theorem,

$$\begin{aligned} \|h_i\|_{L^n(\mathbb{R})}^n &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f_i(z, x_{n+1})|^n \, dz \, dx_{n+1} \\ &= \|f_i\|_{L^n(\mathbb{R}^n)}^n. \end{aligned}$$

Since each $h_i \in L^n(\mathbb{R})$, we may apply Proposition B.7.9 with n factors and all exponents equal to n. Thus,

$$\int_{\mathbb{R}} \prod_{i=1}^{n} |h_i(x_{n+1})| \, dx_{n+1} \leq \prod_{i=1}^{n} \|h_i\|_{L^n(\mathbb{R})}.$$

We previously showed that

$$\int_{\mathbb{R}^n} |f(y, x_{n+1})| dy \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^{n-1}} |f_i(z, x_{n+1})|^n dz \right)^{\frac{1}{n}} \|f_{n+1}\|_{L^n(\mathbb{R}^n)}.$$

Integrating this inequality with respect to x_{n+1} and using the preceding inequality gives

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} |f(x)|\,dx &\le \left(\int_{\mathbb{R}} \prod_{i=1}^{n} |h_i(x_{n+1})|\,dx_{n+1}\right) \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \\ &\le \left(\prod_{i=1}^{n} \|h_i\|_{L^n(\mathbb{R})}\right) \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \\ &= \prod_{i=1}^{n+1} \|f_i\|_{L^n(\mathbb{R}^n)},\end{aligned}$$

which is the desired conclusion. □

Theorem 3.3.4. There are constants $C > 0$, depending only on n and p, with the following properties:

(a) Gagliardo–Nirenberg–Sobolev inequality: If $p \in [1, n)$, then

$$\|\varphi\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$.

(b) If $p \in (n, \infty)$, then

$$\|\varphi\|_{L^\infty(\mathbb{R}^n)} \le C(\lambda_n(\operatorname{supp}(\varphi)))^{\frac{1}{n}-\frac{1}{p}} \|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, where $\lambda_n(\operatorname{supp}(\varphi))$ denotes the Lebesgue measure of the support of φ.

Proof. We begin by assuming $n \ge 2$; the case $n = 1$ in part (b) is treated separately at the end of the proof. Let $\varphi \in C_c^1(\mathbb{R}^n)$. Since φ has compact support, Theorem B.6.5 gives

$$\varphi(x) = \int_{-\infty}^{x_i} \frac{\partial\varphi}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)dt$$

for each $i \in \{1, \dots, n\}$, and consequently

$$|\varphi(x)| \le \int_{-\infty}^{\infty} \left|\frac{\partial\varphi}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\right| dt$$

Since $n \ge 2$, we can define

$$f_i(z_1, \dots, z_{n-1}) := \left(\int_{-\infty}^{\infty} \left|\frac{\partial\varphi}{\partial x_i}(z_1, \dots, z_{i-1}, t, z_i, \dots, z_{n-1})\right| dt\right)^{\frac{1}{n-1}}.$$

The function f_i is well defined because $\varphi \in C_c^1(\mathbb{R}^n)$.
Then $f_1, \dots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$ because

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |f_i(z)|^{n-1} dz &= \int_{\mathbb{R}^{n-1}} \left|\left(\int_{-\infty}^{\infty} \left|\frac{\partial\varphi}{\partial x_i}(z_1, \dots, z_{i-1}, t, z_i, \dots, z_{n-1})\right| dt\right)^{\frac{1}{n-1}}\right|^{n-1} dz \\ &= \int_{\mathbb{R}^n} \left|\frac{\partial\varphi}{\partial x_i}(x)\right| dx < \infty\end{aligned}$$

because $\dfrac{\partial\varphi}{\partial x_i} \in C_c^0(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$.
Moreover, the preceding argument tells us that

$$\|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})} = \left\| \frac{\partial\varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n-1}}.$$

Raising the inequalities

$$|\varphi(x)| \leq \int_{-\infty}^{\infty} \left| \frac{\partial\varphi}{\partial x_i}(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n) \right| dt$$

to the power $\dfrac{1}{n-1}$ and taking their product gives

$$|\varphi(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} f_i(\hat{x}_i)$$

where $\hat{x}_i$ omits the ith entry of x. Applying Lemma 3.3.3 gives

$$\int_{\mathbb{R}^n} |\varphi|^{\frac{n}{n-1}} \leq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} f_i(\hat{x}_i) \right) \leq \prod_{i=1}^{n} \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})} = \prod_{i=1}^{n} \left\| \frac{\partial\varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n-1}}$$

Therefore,

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \left\| \frac{\partial\varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}$$

Now the arithmetic–geometric mean inequality gives

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \left\| \frac{\partial\varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{\partial\varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)} = \frac{1}{n} \|\nabla\varphi\|_{(L^1(\mathbb{R}^n))^n}.$$

The preceding inequality also holds for compactly supported functions φ defined on $\mathbb{R}^n$ that are of class C^1 on a bounded open subset of $\mathbb{R}^n$.
On the other hand, for each $\gamma > 1$, consider the function $\widetilde{\varphi}(x) = |\varphi(x)|^{\gamma-1}\varphi(x)$. This function has compact support, is at least continuous, and is of class C^1 on $\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}$, which is open because φ is continuous, and bounded because $\operatorname{supp}(\varphi)$ is compact.
Its partial derivatives on this open subset are

$$\frac{\partial(|\varphi|^{\gamma-1}\varphi)}{\partial x_i} = \frac{\partial(|\varphi|^{\gamma-1})}{\partial x_i}\varphi + |\varphi|^{\gamma-1}\frac{\partial\varphi}{\partial x_i} = (\gamma-1)|\varphi|^{\gamma-2}\frac{\varphi}{|\varphi|}\frac{\partial\varphi}{\partial x_i}\varphi + |\varphi|^{\gamma-1}\frac{\partial\varphi}{\partial x_i} = \gamma\frac{\partial\varphi}{\partial x_i}|\varphi|^{\gamma-1}$$

, which are continuous on $\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}$, so that $\widetilde{\varphi}$ is of class C^1 at least on $\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}$.
Outside $\operatorname{supp}(\varphi)$, $\widetilde{\varphi} \equiv 0$, so it is also of class C^1 there. It remains to examine the boundary points.
In fact, the function $\psi\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $\psi(x) = x|x|^{\gamma-1}$ is of class C^1, since for $x < 0$, $\psi'(x) = \gamma(-x)^{\gamma-1}$, and for $x > 0$, $\psi'(x) = \gamma x^{\gamma-1}$. By Proposition B.1.1, ψ is differentiable at 0 and $\psi'(0) = 0$. Moreover, since the derivatives of ψ are continuous except possibly at 0, the expression for the derivative at 0 gives $\psi \in C^1(\mathbb{R})$.
Therefore, $\widetilde{\varphi}$ is of class $C^1(\mathbb{R}^n)$ as a composition of functions of class C^1, and hence $\widetilde{\varphi} \in C_0^1(\mathbb{R}^n)$.

On the other hand, the absolute value of this partial derivative, where it differs from 0, is

$$\left|\frac{\partial(|\varphi|^{\gamma-1}\varphi)}{\partial x_i}\right| = \left|(\gamma-1)|\varphi|^{\gamma-2}\frac{\varphi}{|\varphi|}\frac{\partial\varphi}{\partial x_i}\varphi + |\varphi|^{\gamma-1}\frac{\partial\varphi}{\partial x_i}\right| =$$

$$\left|\frac{\partial\varphi}{\partial x_i}\right|\left|(\gamma-1)|\varphi|^{\gamma-1} + |\varphi|^{\gamma-1}\right| = \gamma|\varphi|^{\gamma-1}\left|\frac{\partial\varphi}{\partial x_i}\right|$$

Replacing φ by $|\varphi|^{\gamma-1}\varphi$ in the inequality

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \prod_{i=1}^{n}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}$$

and, for $1 < p < \infty$, applying Hölder's inequality from Proposition B.7.9, we obtain

$$\left(\int_{\mathbb{R}^n}|\varphi|^{\frac{n\gamma}{n-1}}\right)^{\frac{n-1}{n}} \le \gamma\left(\int_{\mathbb{R}^n}|\varphi|^{\frac{p(\gamma-1)}{p-1}}\right)^{\frac{p-1}{p}}\prod_{i=1}^{n}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{n}}.$$

The geometric–arithmetic mean inequality, followed by the equivalence of finite-dimensional norms in Proposition B.1.2, gives

$$\prod_{i=1}^{n}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{n}} \le \left(\sum_{i=1}^{n}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)}^{p}\right)^{\frac{1}{p}} = \|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}.$$

We shall use these inequalities to prove the assertions of the theorem.

(a) Suppose that $p \in [1, n)$. If $p = 1$, then $p^* = \dfrac{n}{n-1}$, and the inequality

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac{1}{n}\|\nabla\varphi\|_{(L^1(\mathbb{R}^n))^n}$$

already proved is the desired one.

If $p \neq 1$, take $\gamma := \dfrac{n-1}{n}p^* = \dfrac{n-1}{n}\dfrac{np}{n-p} = \dfrac{np-p}{n-p} > 1$, since $1 < p < n$ implies $n < np < n^2$, whence $n < np$; thus $np - p > n - p$, as required.

We may therefore apply the inequality previously derived for $\gamma > 1$. To do so, we first compute $\dfrac{p(\gamma-1)}{p-1} = \dfrac{p(\frac{np-p}{n-p} - \frac{n-p}{n-p})}{p-1} = \dfrac{p(\frac{n(p-1)}{n-p})}{p-1} = \dfrac{pn}{n-p} =: p^*$, $\dfrac{n\gamma}{n-1} = p^*$, and $\dfrac{n-1}{n} - \dfrac{p-1}{p} = \dfrac{pn-p-np+n}{np} = \dfrac{n-p}{np} = \dfrac{1}{p^*}$, so that the inequality for $\gamma > 1$ gives the following:

$$\left(\int_{\mathbb{R}^n}|\varphi|^{p^*}\right)^{\frac{1}{p^*}} \le \gamma\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$$

which is the same as

$$\|\varphi\|_{L^{p^*}(\mathbb{R}^n)} \le \gamma\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$$

, as desired.

(b) Suppose that $p \in (n, \infty)$. If $n = 1$, the inequality

$$|\varphi(x)| \leq \int_{-\infty}^{\infty} |\varphi'(t)|\, dt$$

together with Proposition B.7.12, with $X = \operatorname{supp}(\varphi)$, implies

$$\|\varphi\|_{L^\infty(\mathbb{R})} \leq \|\varphi'\|_{L^1(\mathbb{R})} \leq \lambda_1(\operatorname{supp}(\varphi))^{\frac{p-1}{p}} \|\varphi'\|_{L^p(\mathbb{R})}$$

. This is the desired inequality once we observe that $\frac{1}{n} - \frac{1}{p} = 1 - \frac{1}{p} = \frac{p-1}{p}$.

If $n \geq 2$, first consider the case in which $\lambda_n(\operatorname{supp}(\varphi)) = 1$ and $\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n} = 1$. The inequality for $\gamma > 1$ and Hölder's inequality on the support of measure one, with exponents $\frac{\gamma}{\gamma - 1}$ and γ, give:

$$\|\varphi\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{R}^n)} \leq \gamma^{\frac{1}{\gamma}} \left(\int_{\mathbb{R}^n} |\varphi|^{\frac{p(\gamma-1)}{p-1}} \right)^{\frac{p-1}{p\gamma}} \leq \gamma^{\frac{1}{\gamma}} (\|\varphi\|_{L^{\frac{p\gamma}{p-1}}(\mathbb{R}^n)})^{\frac{\gamma-1}{\gamma}}$$

for every $\gamma > 1$.

Take $\gamma := \frac{n}{n-1}\frac{p-1}{p}$. Since $p > n$ and $n \geq 2$, we have $p > p - 1 > n - 1 \geq 1$, because $p > n$ and $n \geq 2$. On the other hand, $p > n$ implies $-n > -p$; adding np to this inequality gives $np - n > np - p$, which is equivalent to $n(p-1) > p(n-1)$. Since $n - 1 > 0$ and $p > 0$, we may divide by $p(n-1)$ without reversing the inequality to obtain $\frac{n(p-1)}{p(n-1)} > 1$, that is, $\gamma > 1$.

Now, for $k \in \mathbb{N}$, we have $\gamma^k = (\frac{n}{n-1}\frac{p-1}{p})^k = (\frac{n}{n-1})^k (\frac{p-1}{p})^k$ and also $\gamma^{k-1} = (\frac{n}{n-1}\frac{p-1}{p})^{k-1} = (\frac{n}{n-1})^{k-1}(\frac{p-1}{p})^{k-1}$, whence $\frac{p}{p-1}\gamma^k = \frac{n}{n-1}\gamma^{k-1}$.

Substituting γ^k for γ in the preceding inequality then gives the following:

$$\|\varphi\|_{L^{\frac{n}{n-1}\gamma^k}(\mathbb{R}^n)} \leq \gamma^{\frac{k}{\gamma^k}} (\|\varphi\|_{L^{\frac{n}{n-1}\gamma^{k-1}}(\mathbb{R}^n)})^{\frac{\gamma^k - 1}{\gamma^k}}.$$

Proposition B.7.13 and the inequality

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n}\|\nabla\varphi\|_{(L^1(\mathbb{R}^n))^n}$$

give

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n}\|\nabla\varphi\|_{(L^1(\mathbb{R}^n))^n} \leq \frac{1}{n}(n\lambda_n(\operatorname{supp}(\varphi)))^{\frac{p-1}{p}} \|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$$

, where the last expression equals the following, using $\lambda_n(\operatorname{supp}(\varphi)) = 1$ and $\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n} = 1$:

$$\frac{1}{n}(n\lambda_n(\operatorname{supp}(\varphi)))^{\frac{p-1}{p}} \|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n} = \frac{n^{\frac{p-1}{p}}}{n} = n^{-\frac{1}{p}} = \frac{1}{n^{\frac{1}{p}}} < 1.$$

Therefore,

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n}\|\nabla\varphi\|_{(L^1(\mathbb{R}^n))^n}$$

gives

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n}\|\nabla\varphi\|_{(L^1(\mathbb{R}^n))^n} \leq \frac{1}{n}(n\lambda_n(\operatorname{supp}(\varphi)))^{\frac{p-1}{p}}\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n} < 1.$$

If $\|\varphi\|_{L^{\frac{n}{n-1}\gamma^k}(\mathbb{R}^n)} \leq 1$ for infinitely many $k \in \mathbb{N}$, Proposition B.7.14 tells us that $\|\varphi\|_{L^\infty(\mathbb{R}^n)} \leq 1$. If, on the contrary, there exists $k_0 \in \mathbb{N}_0$ such that $\|\varphi\|_{L^{\frac{n}{n-1}\gamma^{k_0}}(\mathbb{R}^n)} \leq 1$ and $\|\varphi\|_{L^{\frac{n}{n-1}\gamma^k}(\mathbb{R}^n)} > 1$ for every $k > k_0$, then for every $k > k_0 + 1$ we have $\|\varphi\|_{L^{\frac{n}{n-1}\gamma^{k-1}}(\mathbb{R}^n)} > 1$; using $0 < \dfrac{\gamma^k - 1}{\gamma^k} < 1$, we obtain

$$(\|\varphi\|_{L^{\frac{n}{n-1}\gamma^{k-1}}(\mathbb{R}^n)})^{\frac{\gamma^k-1}{\gamma^k}} \leq \|\varphi\|_{L^{\frac{n}{n-1}\gamma^{k-1}}(\mathbb{R}^n)}$$

for every $k > k_0 + 1$.

Iterating the inequality obtained earlier and combining it with the preceding inequality yields the following inequalities:

$$\|\varphi\|_{L^{\frac{n}{n-1}\gamma^k}(\mathbb{R}^n)} \leq \gamma^{\left(\frac{k}{\gamma^k}+\cdots+\frac{k_0+1}{\gamma^{k_0+1}}\right)}(\|\varphi\|_{L^{\frac{n}{n-1}\gamma^{k_0}}(\mathbb{R}^n)})^{\frac{\gamma^{k_0+1}-1}{\gamma^{k_0+1}}} \leq \gamma^\alpha$$

where $\alpha := \displaystyle\sum_{i=1}^{\infty}\frac{i}{\gamma^i} < \infty$, since for each $i \geq 1$, $\dfrac{\left|\dfrac{i+1}{\gamma^{i+1}}\right|}{\left|\dfrac{i}{\gamma^i}\right|} = \dfrac{i+1}{i\gamma} = \dfrac{1}{\gamma} + \dfrac{1}{i\gamma} \to \dfrac{1}{\gamma}$ if $i \to \infty$, and $\dfrac{1}{\gamma} < 1$ because $\gamma > 1$, so the ratio test shows that the series converges.

By Proposition B.7.14, we obtain $\|\varphi\|_{L^\infty(\mathbb{R}^n)} \leq \gamma^\alpha$. Thus, in either case,

$$\|\varphi\|_{L^\infty(\mathbb{R}^n)} \leq \gamma^\alpha = \gamma^\alpha(\lambda_n(\operatorname{supp}(\varphi)))^{\frac{p-n}{np}}\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n}$$

provided that $\lambda_n(\operatorname{supp}(\varphi)) = 1$ and $\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n} = 1$, where we have used $\gamma^\alpha > 1$.

We now obtain the desired inequality in the general case.

If $\lambda_n(\operatorname{supp}(\varphi)) = 0$ or $\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n} = 0$, then $\varphi = 0$ almost everywhere and the inequality holds immediately.

If $\lambda_n(\operatorname{supp}(\varphi)) \neq 0$ and $\|\nabla\varphi\|_{(L^p(\mathbb{R}^n))^n} \neq 0$, we may consider the function $\theta := \dfrac{\psi}{\|\nabla\psi\|_{(L^p(\mathbb{R}^n))^n}}$, where $\psi(x) := \varphi(\lambda_n(\operatorname{supp}(\varphi))^{\frac{1}{n}}x)$. We have $\operatorname{supp}(\theta) = \operatorname{supp}(\psi)$ and $\operatorname{supp}(\psi) = \overline{\{x \in \mathbb{R}^n \mid \psi(x) \neq 0\}} = \overline{\{x \in \mathbb{R}^n \mid \varphi((\lambda_n(\operatorname{supp}(\varphi)))^{\frac{1}{n}}x) \neq 0\}} = \operatorname{supp}(\varphi \circ g)$, where $g\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, given by $g(x) = (\lambda_n(\operatorname{supp}(\varphi)))^{\frac{1}{n}}x$, is a diffeomorphism of class $C^\infty(\mathbb{R}^n)$. Since g is a homeomorphism, $\operatorname{supp}(\varphi \circ g) = g^{-1}(\operatorname{supp}(\varphi))$; by Theorem B.6.15, taking $|\det Dg^{-1}| = \dfrac{1}{\lambda_n(\operatorname{supp}(\varphi))}$ into account, we obtain

$$\lambda_n(g^{-1}(\operatorname{supp}(\varphi))) = \int_{g^{-1}(\operatorname{supp}(\varphi))} 1 = \frac{1}{\lambda_n(\operatorname{supp}(\varphi))}\int_{\operatorname{supp}(\varphi)} 1 = 1$$

and therefore $\lambda_n(\operatorname{supp}(\theta)) = \lambda_n(\operatorname{supp}(\psi)) = 1$.

Moreover,

$$\nabla\psi(x) = \lambda_n(\operatorname{supp}(\varphi))^{\frac{1}{n}}\nabla\varphi(g(x)).$$

Theorem B.6.15 gives

$$\left\| \frac{\partial \varphi}{\partial x_i} \circ g \right\|_{L^p(\mathbb{R}^n)}^p = \lambda_n(\operatorname{supp}(\varphi))^{-1} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}^p.$$

Consequently,

$$\|\nabla \varphi \circ g\|_{(L^p(\mathbb{R}^n))^n} = \lambda_n(\operatorname{supp}(\varphi))^{-\frac{1}{p}} \|\nabla \varphi\|_{(L^p(\mathbb{R}^n))^n},$$

and therefore

$$\|\nabla \psi\|_{(L^p(\mathbb{R}^n))^n} = \lambda_n(\operatorname{supp}(\varphi))^{\frac{1}{n}-\frac{1}{p}} \|\nabla \varphi\|_{(L^p(\mathbb{R}^n))^n} = \lambda_n(\operatorname{supp}(\varphi))^{\frac{p-n}{np}} \|\nabla \varphi\|_{(L^p(\mathbb{R}^n))^n}.$$

On the other hand, $\|\varphi\|_{L^\infty(\mathbb{R}^n)} = \|\psi\|_{L^\infty(\mathbb{R}^n)}$, so we may apply the preceding argument to $\theta \in C_c^\infty(\mathbb{R}^n)$ to obtain:

$$\|\varphi\|_{L^\infty(\mathbb{R}^n)} = \|\psi\|_{L^\infty(\mathbb{R}^n)} \leq \gamma^\alpha \|\nabla \psi\|_{(L^p(\mathbb{R}^n))^n} = \gamma^\alpha (\lambda_n(\operatorname{supp}(\varphi)))^{\frac{p-n}{np}} \|\nabla \varphi\|_{(L^p(\mathbb{R}^n))^n}$$

as desired.

□

For $u \in W_0^{1,p}(\Omega)$, we shall write $\nabla u = (D_1 u, \ldots, D_n u)$ and

$$\|\nabla u\|_{(L^p(\Omega))^n} := \left(\sum_{i=1}^n \|D_i u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

The extension of the preceding inequality to the completed space is recorded in the following corollary.

Corollary 3.3.5. If $\Omega \subseteq \mathbb{R}^n$ is open and $p \in [1, n)$, then $W_0^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$, and there is a constant $C > 0$, depending only on n and p, such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|\nabla u\|_{(L^p(\Omega))^n}$$

for every $u \in W_0^{1,p}(\Omega)$.

Proof. The Gagliardo–Nirenberg–Sobolev inequality shows that the inclusion operator

$$I_0 \colon C_c^\infty(\Omega) \longrightarrow L^{p^*}(\Omega)$$

is linear and satisfies $\|I_0 \varphi\|_{L^{p^*}(\Omega)} \leq C \|\varphi\|_{W^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ and $L^{p^*}(\Omega)$ is a Banach space, Theorem B.2.36 provides a unique continuous extension $I \colon W_0^{1,p}(\Omega) \longrightarrow L^{p^*}(\Omega)$. It remains to identify Iu with the usual representative of u and retain the estimate in terms of the gradient. Let $(\varphi_k) \subseteq C_c^\infty(\Omega)$ be a sequence converging to u in $W^{1,p}(\Omega)$. Then $\varphi_k \to u$ in $L^p(\Omega)$ and $\varphi_k \to Iu$ in $L^{p^*}(\Omega)$. Theorem B.7.15, applied to subsequences, shows that both limits agree almost everywhere. Moreover,

$$\|Iu\|_{L^{p^*}(\Omega)} = \lim_{k \to \infty} \|\varphi_k\|_{L^{p^*}(\Omega)} \leq C \lim_{k \to \infty} \|\nabla \varphi_k\|_{(L^p(\Omega))^n} = C \|\nabla u\|_{(L^p(\Omega))^n}.$$

Consequently, $u \in L^{p^*}(\Omega)$, the extension agrees with the canonical inclusion, and we obtain the stated inequality. □

The preceding corollary, combined with interpolation between L^p and L^{p^*}, yields the first continuous Sobolev embedding of this chapter.

Theorem 3.3.6. (Sobolev embedding) If Ω is an open subset of $\mathbb{R}^n$ and $p \in [1, n)$, then

$$W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$$

for every $q \in [p, p^*]$, and this inclusion is continuous.

Proof. Corollary 3.3.5 states that $W_0^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$ and that there is a constant $C > 0$ such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{(L^p(\Omega))^n} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for every $u \in W_0^{1,p}(\Omega)$.
Since we also have $W_0^{1,p}(\Omega) \subseteq L^p(\Omega)$ and $\|u\|_{L^p(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}$, Theorem B.7.16 ensures that $u \in L^q(\Omega)$ for every $u \in W_0^{1,p}(\Omega)$ and $q \in [p, p^*]$, and that

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^p(\Omega)}^{1-\alpha}\|u\|_{L^{p^*}(\Omega)}^{\alpha} \leq C^{\alpha}\|u\|_{W^{1,p}(\Omega)}^{1-\alpha}\|u\|_{W^{1,p}(\Omega)}^{\alpha} = C^{\alpha}\|u\|_{W^{1,p}(\Omega)}$$

, where $\alpha \in [0,1]$ satisfies $\dfrac{1}{q} = \dfrac{1-\alpha}{p} + \dfrac{\alpha}{p^*}$. This proves that $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$ and that this inclusion is continuous. □

When Ω is bounded, $\|u\|_{L^q(\Omega)}$ can be bounded in terms of $\|\nabla u\|_{(L^p(\Omega))^n}$ and the measure of Ω for a wider range of values of p and q. The following inequality is known as the Poincaré inequality.

Theorem 3.3.7. (Poincaré inequality) Let Ω be a bounded open subset of $\mathbb{R}^n$. There is a constant $C > 0$ depending only on n, p, and q such that

$$\|u\|_{L^q(\Omega)} \leq C(\lambda_n(\Omega))^{\frac{1}{q}+\frac{1}{n}-\frac{1}{p}}\|\nabla u\|_{(L^p(\Omega))^n}$$

for every $u \in W_0^{1,p}(\Omega)$ if any of the following three conditions holds:

(a) $p \in [1, n)$ and $q \in [1, p^*]$.

(b) $p = n$ and $q \in [1, \infty)$.

(c) $p \in (n, \infty)$ and $q \in [1, \infty]$.

Consequently:

- If $p \in [1, n)$, then $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$ for each $q \in [1, p^*]$, and the inclusion is continuous.
- $W_0^{1,n}(\Omega) \subseteq L^q(\Omega)$ for each $q \in [1, \infty)$, and the inclusion is continuous.
- Furthermore, if $p \in (n, \infty)$, then, after choosing a representative, $W_0^{1,p}(\Omega) \subseteq C^0(\overline{\Omega})$, and the inclusion is continuous.

Proof. (a) If $p \in [1,n)$ and $q \in [1,p^*]$, applying the inequality from Corollary 3.3.5 and Proposition B.7.17 gives

$$\|u\|_{L^q(\Omega)} \le (\lambda_n(\Omega))^{\frac{1}{q}-\frac{1}{p^*}} \|u\|_{L^{p^*}(\Omega)} \le C(\lambda_n(\Omega))^{\frac{1}{q}+\frac{1}{n}-\frac{1}{p}} \|\nabla u\|_{(L^p(\Omega))^n}$$

for every $u \in W_0^{1,p}(\Omega)$.

(b) If $n = 1$, then for $\varphi \in C_c^\infty(\Omega)$, extension by zero and the fundamental theorem of calculus give $\|\varphi\|_{L^\infty(\Omega)} \le \|\varphi'\|_{L^1(\Omega)}$. By Hölder's inequality,

$$\|\varphi\|_{L^q(\Omega)} \le \lambda_1(\Omega)^{\frac{1}{q}} \|\varphi'\|_{L^1(\Omega)}.$$

Approximation by test functions extends this estimate to $W_0^{1,1}(\Omega)$ and proves this part in dimension one. Now assume $n \ge 2$. Let $p = n$ and $q \in [1,\infty)$. If $n = 1$, the inequality

$$|\varphi(x)| \le \int_{-\infty}^{\infty} |\varphi'(t)|\, dt$$

gives

$$\|\varphi\|_{L^q(\Omega)} \le \lambda_1(\Omega)^{\frac{1}{q}} \|\varphi'\|_{L^1(\Omega)}$$

for every $\varphi \in C_c^\infty(\Omega)$. Arguing as in Corollary 3.3.5, we conclude that

$$\|u\|_{L^q(\Omega)} \le (\lambda_1(\Omega))^{\frac{1}{q}} \|Du\|_{L^1(\Omega)}$$

for every $u \in W_0^{1,1}(\Omega)$.

If $n \ge 2$, define $r := \max\{\frac{nq}{n+q}, 1\}$. The number r belongs to $[1,n)$ because $n^2 > 0$ implies $n^2 + nq > nq$, and hence $\frac{nq}{n+q} < n$. Moreover, $r^* = q$ if $r = \frac{nq}{n+q}$, and $q \le \frac{n}{n-1}$ if $r = 1$. Thus, by (a), Proposition B.7.17, and Proposition B.7.13, for every $u \in W_0^{1,p}(\Omega)$ we have

$$\|u\|_{L^q(\Omega)} \le C\|\nabla u\|_{(L^r(\Omega))^n} \le Cn^{\frac{1}{q}} (\lambda_n(\Omega))^{\frac{1}{q}} \|\nabla u\|_{(L^n(\Omega))^n}$$

if $r = \frac{nq}{n+q}$ and

$$\|u\|_{L^q(\Omega)} \le (\lambda_n(\Omega))^{\frac{1}{q}-\frac{n-1}{n}} \|u\|_{\frac{n}{n-1}} \le C(\lambda_n(\Omega))^{\frac{1}{q}-\frac{n-1}{n}} \|\nabla u\|_{(L^1(\Omega))^n} \le Cn^{\frac{n-1}{n}} (\lambda_n(\Omega))^{\frac{1}{q}} \|\nabla u\|_{(L^n(\Omega))^n}$$

if $r = 1$.

(c) Let $p \in (n,\infty)$ and $q \in [1,\infty]$. The second inequality in Theorem 3.3.4, applied to the extension by zero, gives

$$\|\varphi\|_{L^\infty(\Omega)} \le C(\lambda_n(\Omega))^{\frac{1}{n}-\frac{1}{p}} \|\nabla\varphi\|_{(L^p(\Omega))^n}$$

for every $\varphi \in C_c^\infty(\Omega)$.

Now recall that $(C^0(\overline{\Omega}), \|\cdot\|_{C^0(\overline{\Omega})})$ is a Banach space, since Ω is bounded and therefore $\overline{\Omega}$ is compact. Arguing as in Corollary 3.3.5 shows that each $u \in W_0^{1,p}(\Omega)$ agrees almost everywhere with a function belonging to $C^0(\overline{\Omega})$, and that the preceding inequality holds for every u. Combining that inequality

with Proposition B.7.17 gives

$$\|u\|_{L^q(\Omega)} \le (\lambda_n(\Omega))^{\frac{1}{q}} \|u\|_{C^0(\overline{\Omega})} \le C(\lambda_n(\Omega))^{\frac{1}{q}+\frac{1}{n}-\frac{1}{p}} \|\nabla u\|_{(L^p(\Omega))^n}$$

, which is the desired inequality.

□

Just as we defined the Sobolev spaces $W^{1,p}(\Omega)$, we can define the "higher-order Sobolev spaces" $W^{m,p}(\Omega)$:

Definition 3.3.8. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $m \in \mathbb{N}$, $m \ge 2$, and $p \in [1,\infty]$. Define recursively

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) \mid u \in W^{1,p}(\Omega) \text{ and } D_i u \in W^{m-1,p}(\Omega), \forall i \in \{1,\dots,n\}\}.$$

That is, $u \in W^{m,p}(\Omega)$ if its weak derivatives of order at most m, namely

$$D^\alpha u := D_1^{\alpha_1} \cdots D_n^{\alpha_n} u$$

with $\alpha = (\alpha_1,\dots,\alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n \le m$, exist and belong to $L^p(\Omega)$, where $D_i^0 u := u$ and $D_i^{\alpha_i} u := \underbrace{D_i \cdots D_i}_{\alpha_i\text{−times}} u$ if $\alpha_i \ge 1$. Note that all weak derivatives of order at most m have this form.
Using this notation, define

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, \quad \text{if } p < \infty,$$

and

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

Finally, define $W_0^{m,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{m,p}(\Omega)}$.

We have that $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ is a real normed vector space; the proof is analogous to that of Proposition 3.2.3. We shall give results analogous to those for the spaces $W^{1,p}(\Omega)$ and use the latter to establish the corresponding proofs.

Lemma 3.3.9. Let $p \in [1,\infty]$ and $m \ge 1$, and let (u_k) be a sequence in $W^{m,p}(\Omega)$ such that $u_k \to u$ in $L^p(\Omega)$ and $D^\alpha u_k \to v_\alpha$ in $L^p(\Omega)$ for each multi-index α with $|\alpha| \le m$. Then $u \in W^{m,p}(\Omega)$, $v_\alpha = D^\alpha u$ for every multi-index α with $|\alpha| \le m$, and $u_k \to u$ in $W^{m,p}(\Omega)$.

Proof. We prove the assertion by induction on m. The case $m = 1$ is Lemma 3.2.4. Assume $m \ge 2$ and fix a multi-index β with $|\beta| \le m-1$. For each $i \in \{1,\dots,n\}$,

$$D^\beta u_k \longrightarrow v_\beta, \qquad D_i(D^\beta u_k) = D^{\beta+e_i} u_k \longrightarrow v_{\beta+e_i} \quad \text{on } L^p(\Omega).$$

Lemma 3.2.4, applied to the sequence $(D^\beta u_k)$, shows that $v_\beta \in W^{1,p}(\Omega)$ and $D_i v_\beta = v_{\beta+e_i}$. For $\beta = 0$, uniqueness of the limit in $L^p(\Omega)$ gives $v_0 = u$. Iterating these identities according to the length of β yields $v_\alpha = D^\alpha u$ for every $|\alpha| \le m$; in particular, $u \in W^{m,p}(\Omega)$.

If $p < \infty$, then

$$\|u_k - u\|_{W^{m,p}(\Omega)}^p = \sum_{|\alpha| \le m} \|D^\alpha u_k - D^\alpha u\|_{L^p(\Omega)}^p \longrightarrow 0.$$

If $p = \infty$, the family of multi-indices with $|\alpha| \le m$ is finite, and therefore

$$\|u_k - u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \le m} \|D^\alpha u_k - D^\alpha u\|_{L^\infty(\Omega)} \longrightarrow 0.$$

This completes the proof. □

Theorem 3.3.10. $W^{m,p}(\Omega)$ is a Banach space for every $p \in [1, \infty]$.

Proof. Let (u_k) be a Cauchy sequence in $W^{m,p}(\Omega)$. Then the sequences (u_k) and $(D^\alpha u_k)$ are Cauchy in $L^p(\Omega)$ for every multi-index α with $|\alpha| \le m$. Since $L^p(\Omega)$ is a Banach space by Theorem B.7.3, we have $u_k \to u$ in $L^p(\Omega)$ and $D^\alpha u_k \to v_\alpha$ in $L^p(\Omega)$ for each multi-index α with $|\alpha| \le m$. By Lemma 3.3.9, $u \in W^{m,p}(\Omega)$ and $u_k \to u$ in $W^{m,p}(\Omega)$. Thus $W^{m,p}(\Omega)$ is complete. □

3.3.1 Density and extension by zero

Approximation by cutoffs and mollifiers also controls all derivatives of integer order. The following local construction is the Meyers–Serrin theorem; see [97, Section 11.2].

Lemma 3.3.11. (Restriction and extension by zero with interior support) Let $U \subseteq V \subseteq \mathbb{R}^n$ be open, and let $m \in \mathbb{N}_0$ and $1 \le p < \infty$. The restriction of $w \in W^{m,p}(V)$ belongs to $W^{m,p}(U)$ and satisfies

$$D^\alpha(w \restriction_U) = (D^\alpha w) \restriction_U, \quad |\alpha| \le m, \qquad \|w \restriction_U\|_{W^{m,p}(U)} \le \|w\|_{W^{m,p}(V)}.$$

If $u \in W^{m,p}(U)$ vanishes almost everywhere outside a compact set $K \subset U$, its extension by zero $\widetilde{u}$ to V belongs to $W^{m,p}(V)$ and

$$D^\alpha \widetilde{u} = \widetilde{D^\alpha u}, \qquad |\alpha| \le m, \qquad \|\widetilde{u}\|_{W^{m,p}(V)} = \|u\|_{W^{m,p}(U)}.$$

Moreover, $u \in W_0^{m,p}(U)$.

Proof. If $\varphi \in C_c^\infty(U)$, its extension by zero to V is a smooth test function. For this test function, the identity defining $D^\alpha w$ on V reduces to the identity on U. This identifies the derivatives of the restriction. The norm inequality follows by restricting each integral to U; no regularity of its boundary is required.
For extension by zero, the case $K = \varnothing$ is immediate. Choose $\eta \in C_c^\infty(U)$ equal to one on a neighborhood of K. Locality of weak derivatives gives $D^\alpha u = 0$ almost everywhere on $U \setminus K$: it suffices to test the weak identity against functions in $C_c^\infty(U \setminus K)$ and use uniqueness of the function representing the zero distribution. If

$\varphi \in C_c^\infty(V)$, then $\eta\varphi \restriction_U$ is an admissible test function and

$$\begin{aligned}\int_V \widetilde{u}\, D^\alpha \varphi \, dx &= \int_U u\, D^\alpha(\eta\varphi)\, dx \\ &= (-1)^{|\alpha|} \int_U D^\alpha u\, \eta\varphi \, dx \\ &= (-1)^{|\alpha|} \int_V \widetilde{D^\alpha u}\, \varphi \, dx.\end{aligned}$$

The first equality uses $\eta = 1$ near K and $u = 0$ outside K; the last uses the same property for $D^\alpha u$. Each extension preserves the L^p norm, and therefore also the Sobolev norm.
To prove the final assertion, apply the preceding argument with $V = \mathbb{R}^n$. If ρ_ε is a mollifier with support contained in $\overline{B}_{\mathrm{euc}}(0,\varepsilon)$, Fubini's theorem and the weak identity give

$$D^\alpha(\rho_\varepsilon * \widetilde{u}) = \rho_\varepsilon * \widetilde{D^\alpha u}.$$

Approximation by mollifiers in L^p implies convergence of each derivative in $L^p(\mathbb{R}^n)$. For sufficiently small ε, the support of the convolution is contained in $K + \overline{B}_{\mathrm{euc}}(0,\varepsilon) \Subset U$. Their restrictions belong to $C_c^\infty(U)$ and converge to u in $W^{m,p}(U)$, proving membership in $W_0^{m,p}(U)$. □

Proposition 3.3.12. (Density of smooth functions) Let $m \in \mathbb{N}_0$ and $1 \le p < \infty$. For every open subset $\Omega \subseteq \mathbb{R}^n$, the space $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$. Moreover,

$$W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n).$$

Here $W^{0,p}(\Omega) := L^p(\Omega)$ and $W_0^{0,p}(\Omega) = L^p(\Omega)$, by Proposition B.7.18.

Proof. Successive applications of the weak Leibniz rule in Proposition 3.1.4 give, for $|\alpha| \le m$,

$$D^\alpha(\chi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^\beta \chi) D^{\alpha-\beta} u,$$

where $\alpha, \beta \in \mathbb{N}_0^n$ and $\beta \le \alpha$ means $\beta_i \le \alpha_i$ for every $i \in \{1,\dots,n\}$. The induction producing this formula uses the first-order rule on each summand and Pascal's identity to combine the coefficients.
On $\mathbb{R}^n$, take the cutoffs χ_R from Proposition 3.2.10. The summand with $\beta = 0$ converges to $D^\alpha u$ in L^p by dominated convergence. If $\beta \neq 0$, then

$$\|(D^\beta \chi_R) D^{\alpha-\beta} u\|_{L^p(\mathbb{R}^n)} \le R^{-|\beta|} \|D^\beta \chi\|_{L^\infty(\mathbb{R}^n)} \|D^{\alpha-\beta} u\|_{L^p(\mathbb{R}^n)} \longrightarrow 0.$$

Thus, $\chi_R u \to u$ in $W^{m,p}(\mathbb{R}^n)$. For fixed R, Fubini's theorem and the definition of weak derivative give

$$D^\alpha(\rho_\varepsilon * (\chi_R u)) = \rho_\varepsilon * D^\alpha(\chi_R u).$$

Continuity of translations and the mollification estimate used in Proposition 3.2.10 prove convergence in L^p for each of the finitely many multi-indices. These mollifications are smooth and compactly supported, proving the stated equality.

For a general open subset, choose an exhaustion $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ with union Ω, and a smooth partition of unity $(\eta_j)_{j\in\mathbb{N}}$ whose supports are compact and contained in the open sets $V_j := \Omega_{j+1} \setminus \overline{\Omega_{j-1}}$; we may take $\Omega_0 = \Omega_{-1} = \varnothing$ and adjust the first two members. The existence of this partition follows from Theorem 1.3.5 by grouping the functions subordinate to each V_j. The family $(V_j)_{j\in\mathbb{N}}$ is locally finite. The Leibniz formula shows that $\eta_j u \in W^{m,p}(V_j)$, and its support is contained in the compact set $\operatorname{supp}\eta_j \Subset V_j$. Lemma 3.3.11 allows its extension by zero to $\mathbb{R}^n$ without changing either its norm or its weak derivatives.
Given $\delta > 0$, for each j choose $\varepsilon_j > 0$ sufficiently small that

$$v_j := \rho_{\varepsilon_j} * (\eta_j u) \in C_c^\infty(V_j), \qquad \|v_j - \eta_j u\|_{W^{m,p}(\mathbb{R}^n)} < \delta 2^{-j}.$$

The sum $v := \sum_{j=1}^{\infty} v_j$ is locally finite and therefore smooth on Ω. The series of errors $\sum_{j=1}^{\infty}(v_j - \eta_j u)$ converges absolutely in the Banach space $W^{m,p}(\mathbb{R}^n)$. Its limit, restricted to Ω, agrees with $v - u$: on each Ω_N, only finitely many terms occur, by local finiteness of the supports. In particular, $v - u \in W^{m,p}(\Omega)$ before we take its norm, and the triangle inequality gives

$$\|v - u\|_{W^{m,p}(\Omega)} \le \sum_{j=1}^{\infty} \|v_j - \eta_j u\|_{W^{m,p}(\mathbb{R}^n)} \le \delta.$$

This proves that $v \in W^{m,p}(\Omega)$ and establishes density. □

Proposition 3.3.13. (Extension by zero and derivatives of $W_0^{m,p}$) Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $m \in \mathbb{N}_0$ and $1 \le p < \infty$. Extension by zero Z_Ω is an isometry

$$Z_\Omega \colon W_0^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^n)$$

and, for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$,

$$D^\alpha(Z_\Omega u) = Z_\Omega(D^\alpha u), \qquad D^\alpha u \in W_0^{m-|\alpha|,p}(\Omega).$$

If $v \in W^{m,p}(\mathbb{R}^n)$ has support contained in a compact set $K \subseteq \Omega$, then $v \restriction_\Omega \in W_0^{m,p}(\Omega)$.

Proof. Choose $u_j \in C_c^\infty(\Omega)$ with $u_j \to u$ in $W^{m,p}(\Omega)$. For these functions, $D^\alpha(Z_\Omega u_j) = Z_\Omega(D^\alpha u_j)$ and

$$\|Z_\Omega u_j - Z_\Omega u_\ell\|_{W^{m,p}(\mathbb{R}^n)} = \|u_j - u_\ell\|_{W^{m,p}(\Omega)}.$$

Completeness and Lemma 3.3.9 identify the limit with $Z_\Omega u$ and all its derivatives with the corresponding extensions by zero. Equality of norms passes to the limit. Moreover, $D^\alpha u_j \to D^\alpha u$ in $W^{m-|\alpha|,p}(\Omega)$; since each $D^\alpha u_j \in C_c^\infty(\Omega)$, membership in the indicated closed subspace follows.
For the final assertion, the mollifications of v have support in $K + \overline{B}_{\mathrm{euc}}(0,\varepsilon) \Subset \Omega$ if ε is sufficiently small, and converge in $W^{m,p}$ by the proof of Proposition 3.3.12. Their restrictions approximate $v \restriction_\Omega$ by functions in $C_c^\infty(\Omega)$. □

The definition by closure requires no regularity of $\partial\Omega$. In particular, the preceding proposition does not identify $W_0^{m,p}(\Omega)$ with the kernel of a trace on an arbitrary open subset. For $W^{m,p}(\Omega)$, one must distinguish

smooth approximation in the interior from extension across the boundary.

Definition 3.3.14. (Sobolev extension domain) We say that an open subset Ω is an extension domain for $W^{m,p}$ if there is a continuous linear operator $\mathcal{E}\colon W^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^n)$ such that $(\mathcal{E}u)\restriction_\Omega = u$ almost everywhere.

If $p < \infty$ and such an operator exists, approximation of $\mathcal{E}u$ by functions in $C_c^\infty(\mathbb{R}^n)$ shows that their restrictions are dense in $W^{m,p}(\Omega)$. These restrictions need not vanish on the boundary; their density does not assert that $W_0^{m,p}(\Omega) = W^{m,p}(\Omega)$. The restriction to finite exponents in the equality on $\mathbb{R}^n$ is necessary: the constant function one belongs to $W^{m,\infty}(\mathbb{R}^n)$ and cannot be approximated uniformly by compactly supported functions.

Definition 3.3.15. If $m < n$ and $p \in [1, \frac{n}{m})$, the mth critical Sobolev exponent is defined by

$$p_m^* := \frac{np}{n - mp}.$$

Note that $p_1^* = p^*$.

The following formulation gathers the higher-order embeddings for finite exponents.

Theorem 3.3.16. (Sobolev embedding with finite orders and exponents) Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $m > k \geq 0$ be integers and $1 \leq p, q < \infty$ satisfy

$$m - \frac{n}{p} \geq k - \frac{n}{q}.$$

Then $W_0^{m,p}(\Omega) \hookrightarrow W_0^{k,q}(\Omega)$ continuously. If Ω is an extension domain for $W^{m,p}$, then also $W^{m,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$ continuously.

Proof. For $q \leq p$, Hölder's inequality gives, for $|\alpha| \leq k$,

$$\|D^\alpha u\|_{L^q(\Omega)} \leq \lambda_n(\Omega)^{\frac{1}{q} - \frac{1}{p}} \|D^\alpha u\|_{L^p(\Omega)}.$$

Summing over the finitely many multi-indices gives $\|u\|_{W^{k,q}(\Omega)} \leq C\|u\|_{W^{m,p}(\Omega)}$. If $u \in W_0^{m,p}(\Omega)$, applying this inequality to approximations of $C_c^\infty(\Omega)$ also shows that $u \in W_0^{k,q}(\Omega)$. Assume $q > p$ and set $d := m - k$ and

$$\frac{1}{r_i} := \frac{1}{p} - \frac{i}{d}\left(\frac{1}{p} - \frac{1}{q}\right), \qquad i \in \{0, \dots, d\}.$$

The exponents are finite, $r_0 = p$, $r_d = q$, and

$$\frac{1}{r_{i+1}} \geq \frac{1}{r_i} - \frac{1}{n} \qquad (0 \leq i < d).$$

If $r_i < n$, this means $r_{i+1} \leq r_i^*$; if $r_i = n$ or $r_i > n$, the next exponent is still finite. In all three cases, the corresponding part of Theorem 3.3.7 gives $W_0^{1,r_i}(\Omega) \hookrightarrow L^{r_{i+1}}(\Omega)$. Applying it to $D^\alpha \varphi$, with $|\alpha| \leq m - i - 1$ and $\varphi \in C_c^\infty(\Omega)$, yields

$$\|D^\alpha \varphi\|_{L^{r_{i+1}}(\Omega)} \leq C_i \|D^\alpha \varphi\|_{W^{1,r_i}(\Omega)}.$$

Summing over the multi-indices and using the equivalence of norms in finite dimensions, Proposition B.1.2, gives

$$\|\varphi\|_{W^{m-i-1,r_{i+1}}(\Omega)} \le C_i \left(\sum_{|\alpha|\le m-i-1} \|D^\alpha \varphi\|^{r_{i+1}}_{W^{1,r_i}(\Omega)} \right)^{\frac{1}{r_{i+1}}} \le C_i' \|\varphi\|_{W^{m-i,r_i}(\Omega)}.$$

In the last inequality, each derivative of order at most $m-i$ appears a finite number of times, depending only on n, m, i.
If $u \in W_0^{m-i,r_i}(\Omega)$, choose $\varphi_\nu \in C_c^\infty(\Omega)$ converging to u in $W^{m-i,r_i}(\Omega)$. The preceding estimate, applied to $\varphi_\nu - \varphi_\mu$, shows that this sequence is Cauchy in $W^{m-i-1,r_{i+1}}(\Omega)$. By completeness, it converges to an element v of this space, and $v \in W_0^{m-i-1,r_{i+1}}(\Omega)$ by the definition of closure. Since Ω has finite measure, convergence in L^{r_i} and in $L^{r_{i+1}}$ implies convergence in L^1; uniqueness of this limit gives $v = u$ almost everywhere. Passing to the limit in the estimate yields a continuous embedding

$$W_0^{m-i,r_i}(\Omega) \hookrightarrow W_0^{m-i-1,r_{i+1}}(\Omega).$$

The composition of these d embeddings starts at $W_0^{m,p}(\Omega)$ and ends at $W_0^{m-d,r_d}(\Omega) = W_0^{k,q}(\Omega)$, as required.
If there is an extension operator $\mathcal{E}$, choose a ball B containing $\overline{\Omega}$ and $\chi \in C_c^\infty(B)$ equal to one on a neighborhood of $\overline{\Omega}$. Set $Tu := (\chi \mathcal{E} u) \restriction_B$. The Leibniz rule and Proposition 3.3.13 give $Tu \in W_0^{m,p}(B)$, $Tu \restriction_\Omega = u$, and

$$\|Tu\|_{W^{m,p}(B)} \le \|\chi \mathcal{E} u\|_{W^{m,p}(\mathbb{R}^n)} \le C_\chi \|\mathcal{E} u\|_{W^{m,p}(\mathbb{R}^n)} \le C_\chi \|\mathcal{E}\| \|u\|_{W^{m,p}(\Omega)}.$$

The assertion already proved on B and continuity of restriction from Lemma 3.3.11 yield

$$\|u\|_{W^{k,q}(\Omega)} \le \|Tu\|_{W^{k,q}(B)} \le C_B C_\chi \|\mathcal{E}\| \|u\|_{W^{m,p}(\Omega)}.$$

This proves the second embedding. We have not required u to vanish on $\partial\Omega$: the cutoff vanishes near the boundary of the larger ball B. □

When $(m-k)p < n$, the Sobolev index is expressed through the critical exponent p^*_{m-k}. The usual formulation is a direct consequence of the preceding theorem.

Corollary 3.3.17. (Sobolev embedding and critical exponent) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset. Let $j \ge 0$ and $d \ge 1$ be integers, and let $1 \le p < n/d$. Then

$$W_0^{j+d,p}(\Omega) \hookrightarrow W_0^{j,q}(\Omega) \qquad \text{for every } 1 \le q \le p^*_d := \frac{np}{n-dp},$$

and these inclusions are continuous. In particular, $W_0^{d,p}(\Omega) \hookrightarrow L^q(\Omega)$ for the same exponents. If Ω is an extension domain for $W^{j+d,p}$, then also $W^{j+d,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$ continuously.

Proof. The condition $q \le p^*_d$ is equivalent to

$$j + d - \frac{n}{p} \ge j - \frac{n}{q}.$$

Apply Theorem 3.3.16 with total orders $m = j + d$ and $k = j$. For $j = 0$, use $W_0^{0,q}(\Omega) = L^q(\Omega)$. □

3.4 The Rellich–Kondrashov theorem in Euclidean spaces

The Poincaré inequality provides continuous embeddings of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ up to the critical exponent. Continuity, however, does not ensure that a bounded sequence has a convergent subsequence in the target space. The Rellich–Kondrashov theorem determines the exponents for which this gain in compactness occurs, a decisive feature in many existence arguments. The proof begins with an estimate for translations.

Lemma 3.4.1. (Translation estimate in L^p) Let $1 \le p < \infty$, $u \in W^{1,p}(\mathbb{R}^n)$, and $\xi \in \mathbb{R}^n$. If $T_\xi u(x) := u(x - \xi)$, then

$$\|T_\xi u - u\|_{L^p(\mathbb{R}^n)} \le \|\xi\| \sum_{i=1}^{n} \|D_i u\|_{L^p(\mathbb{R}^n)} \le n^{1-\frac{1}{p}} \|\xi\| \|\nabla u\|_{(L^p(\mathbb{R}^n))^n}.$$

Proof. For $\varphi \in C_c^\infty(\mathbb{R}^n)$, the fundamental theorem of calculus along the segment $t \mapsto x - t\xi$ gives

$$\varphi(x - \xi) - \varphi(x) = -\int_0^1 \nabla\varphi(x - t\xi) \cdot \xi \, dt.$$

Minkowski's integral inequality from Theorem B.8.2 and translation invariance of the measure imply

$$\begin{aligned} \|T_\xi \varphi - \varphi\|_{L^p(\mathbb{R}^n)} &\le \|\xi\| \int_0^1 \sum_{i=1}^{n} \|D_i \varphi(\cdot - t\xi)\|_{L^p(\mathbb{R}^n)} \, dt \\ &= \|\xi\| \sum_{i=1}^{n} \|D_i \varphi\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

For $p = 1$, this also follows directly by applying Tonelli's theorem to the pointwise estimate. For $u \in W^{1,p}(\mathbb{R}^n)$, choose $\varphi_j \to u$ in $W^{1,p}$ by Proposition 3.2.10. Since translations are isometries,

$$\|T_\xi u - u\|_{L^p(\mathbb{R}^n)} \le 2\|u - \varphi_j\|_{L^p(\mathbb{R}^n)} + \|T_\xi \varphi_j - \varphi_j\|_{L^p(\mathbb{R}^n)}.$$

Let $j \to \infty$. The second inequality in the statement is Hölder's inequality for the finite sum, with equality of the factors when $p = 1$. □

Theorem 3.4.2. (Rellich–Kondrashov with finite orders and exponents) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset. Let $m > k \ge 0$ be integers and $1 \le p, q < \infty$ satisfy

$$m - \frac{n}{p} > k - \frac{n}{q}. \tag{3.2}$$

Then the embedding

$$W_0^{m,p}(\Omega) \hookrightarrow W_0^{k,q}(\Omega)$$

is compact. If, in addition, Ω is an extension domain for $W^{m,p}$, the embedding

$$W^{m,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$$

is also compact. The continuity constants depend on n, m, k, p, q, Ω and, in the second assertion, on the norm of the chosen extension operator.

Proof. Let $(u_j)_{j\in\mathbb{N}}$ be a bounded sequence in $W_0^{m,p}(\Omega)$. We begin by proving convergence of a subsequence in $W^{m-1,p}$. Extend by zero and write $v_j := Z_\Omega u_j$. By Proposition 3.3.13, (v_j) is bounded in $W^{m,p}(\mathbb{R}^n)$ and, for each $|\alpha| \le m-1$,

$$D^\alpha v_j \in W^{1,p}(\mathbb{R}^n), \qquad \operatorname{supp}(D^\alpha v_j) \subseteq \overline{\Omega}.$$

If $M_0 := \sup_{j\in\mathbb{N}} \|u_j\|_{W^{m,p}(\Omega)} < \infty$, Lemma 3.4.1 gives

$$\sup_{j\in\mathbb{N}} \|T_\xi D^\alpha v_j - D^\alpha v_j\|_{L^p(\mathbb{R}^n)} \le C_{n,p} M_0 \|\xi\|, \qquad |\alpha| \le m-1.$$

Apply Proposition B.7.19 on the open subset $\mathbb{R}^n$. The translation condition holds on every relatively compact open subset by the global estimate; the tail condition holds with zero error upon choosing a ball containing $\overline{\Omega}$. Boundedness in L^p is already known. Therefore, $(D^\alpha v_j)_j$ has a subsequence converging in L^p.

Enumerate the finitely many multi-indices of order at most $m-1$, extract a subsequence for the first, then a subsequence of that one for the second, and so on. The last subsequence retains all the convergences. Restricting to Ω and applying Lemma 3.3.9 yields convergence in $W^{m-1,p}(\Omega)$; when $m = 1$, completeness of L^p suffices. Each term belongs to $W_0^{m-1,p}(\Omega)$ by its approximation in $W^{m,p}$ by test functions. This subspace is closed, so the limit belongs to it. The argument uses strong compactness in L^p and also works for $p = 1$.

If $q \le p$, the continuous inclusion $W^{m-1,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$, obtained from the finite measure of Ω and $k \le m-1$, proves the desired convergence. If $q > p$, the strict inequality (3.2) allows us to choose a finite exponent $r > q$ such that

$$m - \frac{n}{p} \ge k - \frac{n}{r}.$$

Theorem 3.3.16 shows that (u_j) is bounded in $W^{k,r}(\Omega)$. For each $|\alpha| \le k$, the interpolation inequality from Theorem B.7.16 gives

$$\|D^\alpha(u_j - u_\ell)\|_{L^q(\Omega)} \le \|D^\alpha(u_j - u_\ell)\|_{L^p(\Omega)}^{\theta} \|D^\alpha(u_j - u_\ell)\|_{L^r(\Omega)}^{1-\theta}, \quad \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r},$$

with $0 < \theta < 1$. Along the preceding subsequence, the first factor tends to zero and the second remains bounded. Summing over the multi-indices, we obtain a Cauchy sequence in $W^{k,q}(\Omega)$, which converges by completeness. In both cases, Theorem 3.3.16 places each term in $W_0^{k,q}(\Omega)$; closedness ensures that the limit also belongs to this space.

For $W^{m,p}(\Omega)$, use the operator $Tu = (\chi \mathcal{E} u)\restriction_B \in W_0^{m,p}(B)$ constructed in the proof of Theorem 3.3.16. For a bounded sequence $(u_j)_{j\in\mathbb{N}}$, with the same operator and the same cutoff, we have

$$\sup_{j\in\mathbb{N}} \|Tu_j\|_{W^{m,p}(B)} \le C_\chi \|\mathcal{E}\| \sup_{j\in\mathbb{N}} \|u_j\|_{W^{m,p}(\Omega)} < \infty.$$

The first part provides a subsequence $Tu_{j_\nu} \to z$ in $W_0^{k,q}(B)$. Since $Tu_{j_\nu}\restriction_\Omega = u_{j_\nu}$,

$$\|u_{j_\nu} - z\restriction_\Omega\|_{W^{k,q}(\Omega)} \le \|Tu_{j_\nu} - z\|_{W^{k,q}(B)} \longrightarrow 0.$$

Compactness is applied on the fixed ball B, not on all of $\mathbb{R}^n$. □

Corollary 3.4.3. (Loss of one derivative with the same exponent) For every bounded open subset $\Omega \subseteq \mathbb{R}^n$, every integer $m \geq 1$, and every $1 \leq p < \infty$, the embedding

$$W_0^{m,p}(\Omega) \hookrightarrow W_0^{m-1,p}(\Omega)$$

is compact. If Ω is an extension domain for $W^{m,p}$, then $W^{m,p}(\Omega) \hookrightarrow W^{m-1,p}(\Omega)$ is also compact.

Proof. Take $k = m-1$ and $q = p$ in the preceding theorem. The difference between the Sobolev indices is exactly one, in every dimension. □

The subcritical embeddings follow by choosing the corresponding exponents in the general theorem.

Corollary 3.4.4. (First-order Rellich–Kondrashov) If $\Omega \subseteq \mathbb{R}^n$ is open and bounded, $p \in [1, n)$, and $q \in [1, p^*)$, then the inclusion $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$ is compact.

Proof. Take $m = 1$ and $k = 0$ in Theorem 3.4.2. Since $p < n$, the condition $1 - \dfrac{n}{p} > -\dfrac{n}{q}$ is equivalent to $q < p^*$. □

At higher orders, the finite critical exponent appears when the product of the increase in order and p is less than n.

Corollary 3.4.5. (Rellich–Kondrashov and critical exponent) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset, let $j \in \mathbb{N}_0$ and $m \in \{1, \dots, n-1\}$, and let $1 \leq p < \frac{n}{m}$. Then, for every $1 \leq q < p_m^*$, the inclusion

$$W_0^{j+m,p}(\Omega) \hookrightarrow W_0^{j,q}(\Omega)$$

is compact. At the endpoint $q = p_m^*$, the embedding is continuous by Theorem 3.3.16.

Proof. Since $mp < n$, we have

$$q < p_m^* \quad \Longleftrightarrow \quad j + m - \frac{n}{p} > j - \frac{n}{q}.$$

Apply Theorem 3.4.2 with total orders $j + m$ and j. Continuity at the endpoint is Corollary 3.3.17. □

Remark 3.4.6. (The critical endpoint and assumptions on the domain) The strict inequality cannot in general be replaced by equality. Let $m > k \geq 0$, $(m-k)p < n$, $q = \frac{np}{n-(m-k)p}$, and $B_{\mathrm{euc}}(x_0, r) \Subset \Omega$. Choose a nonzero $\varphi \in C_c^\infty(B_{\mathrm{euc}}(0,1))$ and set, for $0 < \varepsilon < r$,

$$u_\varepsilon(x) := \varepsilon^{m-\frac{n}{p}} \varphi\left(\frac{x - x_0}{\varepsilon}\right).$$

For each $|\alpha| \leq m$,

$$\|D^\alpha u_\varepsilon\|_{L^p(\Omega)} = \varepsilon^{m-|\alpha|} \|D^\alpha \varphi\|_{L^p(B_{\mathrm{euc}}(0,1))}.$$

Thus, the family is bounded in $W_0^{m,p}(\Omega)$. For some $|\beta| = k$, the function $D^\beta \varphi$ is nonzero and

$$\|D^\beta u_\varepsilon\|_{L^q(\Omega)} = \|D^\beta \varphi\|_{L^q(B_{\mathrm{euc}}(0,1))} > 0.$$

However, $D^\beta u_\varepsilon \to 0$ almost everywhere as $\varepsilon \to 0^+$. No subsequence can converge in L^q. This prevents compactness at the critical exponent, although continuity does hold.

Nor does boundedness of an open subset suffice for compactness on $W^{m,p}$ without the zero subscript. Let Ω be the union of disjoint open balls $(B_j)_{j\in\mathbb{N}}$ contained in a fixed ball and accumulating only at a point outside Ω. The functions

$$u_j := \lambda_n(B_j)^{-\frac{1}{p}} \mathbf{1}_{B_j}$$

are constant on each component of Ω, so all their positive-order weak derivatives vanish and $\|u_j\|_{W^{m,p}(\Omega)} = 1$. For $j \neq \ell$, we have $\|u_j - u_\ell\|_{L^p(\Omega)} = 2^{1/p}$. Thus, even the loss of one derivative at the same exponent does not yield compactness on this $W^{m,p}(\Omega)$.

The condition $m > k$ is also essential: on a ball of finite measure, the index inequality holds with $m = k = 0$ and $q < p$, but the inclusion $L^p \hookrightarrow L^q$ is not compact. On a cube, functions alternating between 1 and -1 on dyadic intervals of the first coordinate are bounded in L^p, and any two differ on a set whose measure is half that of the cube. Their distances in L^q are therefore bounded away from zero.

3.4.1 Global embeddings and localization in $\mathbb{R}^n$

Proposition 3.4.7. (Global continuous embeddings with finite exponents) Let $m > k \geq 0$ be integers and $1 \leq p \leq q < \infty$ satisfy $m - \frac{n}{p} \geq k - \frac{n}{q}$. Then

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow W^{k,q}(\mathbb{R}^n)$$

continuously. The same conclusion holds on an extension domain for $W^{m,p}$ by restricting its extension operator.

Proof. Let $Q_a := a + (0,1)^n$, with $a \in \mathbb{Z}^n$, and $\chi_a(x) := \chi(x-a)$, where $\chi \in C_c^\infty((-1,2)^n)$ equals one on $[0,1]^n$. Proposition 3.3.13 and the Leibniz rule give $\chi_a u \in W_0^{m,p}(a + (-1,2)^n)$. Applying Theorem 3.3.16 on these translated cubes yields

$$\|u\|_{W^{k,q}(Q_a)} \leq C\|u\|_{W^{m,p}(a+(-1,2)^n)},$$

with the same constant for every a. The cubes Q_a cover $\mathbb{R}^n$ except for a null set, and the enlarged cubes have multiplicity at most 3^n. Since $p \leq q$, for every nonnegative family (b_a) we have $(\sum_{a\in\mathbb{Z}^n} b_a^q)^{1/q} \leq (\sum_{a\in\mathbb{Z}^n} b_a^p)^{1/p}$. Consequently,

$$\begin{aligned}\|u\|_{W^{k,q}(\mathbb{R}^n)} &\leq C\left(\sum_{a\in\mathbb{Z}^n} \|u\|^p_{W^{m,p}(a+(-1,2)^n)}\right)^{1/p} \\ &\leq C3^{n/p}\|u\|_{W^{m,p}(\mathbb{R}^n)}.\end{aligned}$$

Composition with an extension operator followed by restriction proves the final assertion. □

Corollary 3.4.8. (Compactness with common support and local compactness) Let $m > k \geq 0$, $1 \leq p, q < \infty$, and $m - \frac{n}{p} > k - \frac{n}{q}$. For each compact set $K \subseteq \mathbb{R}^n$, the inclusion into $W^{k,q}(\mathbb{R}^n)$ of the subspace

$$W^{m,p}_K(\mathbb{R}^n) := \{u \in W^{m,p}(\mathbb{R}^n) \mid \operatorname{supp} u \subseteq K\}$$

is compact. For any open subset Ω, every bounded sequence in $W^{m,p}(\Omega)$ has a subsequence converging in $W^{k,q}(U)$ for each open subset $U \Subset \Omega$.

Proof. If $K \Subset B$ for a ball B, Proposition 3.3.13 places the functions in $W^{m,p}_0(B)$. Theorem 3.4.2 provides a subsequence converging in $W^{k,q}_0(B)$, and extension by zero preserves this convergence in $\mathbb{R}^n$.
For the local assertion, take an exhaustion of Ω and cutoffs $\chi_j \in C^\infty_c(\Omega)$ equal to one on its successive members. The Leibniz rule and the first part apply to each localized sequence. Extract successive subsequences and then the diagonal subsequence. The limits agree on intersections by uniqueness of the local limit in measure. Every $U \Subset \Omega$ is contained in a member of the exhaustion, proving the stated convergence. □

Remark 3.4.9. (Translations toward infinity) The equality $W^{m,p}_0(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$ for $p < \infty$ does not make the global continuous embeddings compact. If $0 \neq \varphi \in C^\infty_c(\mathbb{R}^n)$ and we choose points x_j so that the supports of $u_j(x) := \varphi(x - x_j)$ are disjoint, the $W^{m,p}$ norms remain constant and

$$\|u_j - u_\ell\|_{L^q(\mathbb{R}^n)} = 2^{1/q}\|\varphi\|_{L^q(\mathbb{R}^n)} \qquad (j \neq \ell, 1 \leq q < \infty).$$

For $q = \infty$, the distance is $\|\varphi\|_{L^\infty(\mathbb{R}^n)} > 0$. Global compactness fails even with a strict gain in indices. Localization or a common compact support prevents this escape.
The condition $q \geq p$ in the global continuous embeddings must also be retained. If $1 \leq q < p < \infty$, choose a with $n/p < a \leq n/q$. The function $u(x) = (1 + |x|^2)^{-a/2}$ and all its derivatives of every order belong to $L^p(\mathbb{R}^n)$, whereas $u \notin L^q$. Inclusions based on finite measure apply to bounded domains or fixed supports, not to the whole Euclidean space.

3.5 Sobolev embeddings into Hölder spaces

The Sobolev embeddings studied so far improve the integrability of a weakly differentiable function. In the supercritical regime, the gain is stronger: when the number of derivatives and the integrability exponent satisfy $mp > n$, functions in $W^{m,p}$ admit representatives with Hölder continuous classical derivatives. This property is fundamental in regularity theory. The embedding does not itself produce new derivatives: one must first obtain a priori estimates placing the weak solution in a sufficiently high space $W^{m,p}$; Morrey's embedding then converts this Sobolev regularity into quantitative classical regularity. The development in this section follows Morrey's strategy as presented in [97, Section 12.3].

Definition 3.5.1. (Functions of class C^k and $C^{k,\alpha}$ on the closure of an open subset) Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. We begin with the usual convention

$$C^0(\overline{\Omega}) := \left\{v : \overline{\Omega} \longrightarrow \mathbb{R} \,\middle|\, v \text{ is continuous}\right\}.$$

For $k \in \mathbb{N} \cup \{\infty\}$, write

$$C^k(\overline{\Omega}) := \left\{ v \in C^0(\overline{\Omega}) \;\middle|\; \begin{array}{l} \text{there are an open subset } V \subseteq \mathbb{R}^n, \text{ with } \overline{\Omega} \subseteq V, \text{ and a function} \\ \widetilde{v} \in C^k(V) \text{ such that } \widetilde{v} \restriction_{\overline{\Omega}} = v \end{array} \right\}.$$

Continuity shows that an extension initially defined on Ω determines, whenever it exists, a unique function on $\overline{\Omega}$. In applications, we shall therefore identify a function on Ω with its continuous representative on the closure. Moreover, for $k \in \mathbb{N}_0 \cup \{\infty\}$, define

$$C_c^k(\overline{\Omega}) := \left\{ v \in C^k(\overline{\Omega}) \middle| \operatorname{supp}_{\overline{\Omega}}(v) \text{ is compact} \right\},$$

where the support is taken in the topological space $\overline{\Omega}$. Thus, for $k \geq 1$, the notation $C^k(\overline{\Omega})$ does not merely mean that the derivatives have limits at the boundary. It requires the function to be the restriction of a function of class C^k defined on an open neighborhood of $\overline{\Omega}$. This is also the convention used in trace theory.

Now let $K \subseteq \mathbb{R}^n$ be nonempty and let $w \colon K \longrightarrow \mathbb{R}$ be a bounded function. For $0 < \alpha < 1$, set

$$[w]_{C^{0,\alpha}(K)} := \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|w(x) - w(y)|}{|x - y|^\alpha}, \qquad \|w\|_{C^{0,\alpha}(K)} := \|w\|_{C^0(K)} + [w]_{C^{0,\alpha}(K)}, \qquad \|w\|_{C^0(K)} := \sup_{x \in K} |w(x)|,$$

with the convention that the supremum is zero when K consists of a single point. For orders $k \geq 1$, we also assume $K = \overline{\operatorname{Int}(K)}$. This condition ensures that derivatives on K are determined by the restriction of the function: two extensions agreeing on K have the same derivatives on its interior and, by continuity, on K. If $k \in \mathbb{N}$ and w is of class C^k on a neighborhood of K, define the intrinsic norm

$$\|w\|_{C^{k,\alpha}_{\mathrm{int}}(K)} := \sum_{|\beta| \leq k} \|D^\beta w\|_{C^0(K)} + \sum_{|\beta| = k} [D^\beta w]_{C^{0,\alpha}(K)}. \tag{3.3}$$

For $k = 0$, write $\|w\|_{C^{0,\alpha}_{\mathrm{int}}(K)} := \|w\|_{C^{0,\alpha}(K)}$. Denote by $C^{k,\alpha}_{\mathrm{int}}(K)$ the space of restrictions to K of functions of class C^k on a neighborhood of K for which this norm is finite. The subscript int recalls that the norm compares only the values of the function and its derivatives on K; it does not require an extension with global control.

Let $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. For $v \in C^k(\mathbb{R}^n)$, define

$$[v]_{C^{k,\alpha}(\mathbb{R}^n)} := \sum_{|\beta| = k} [D^\beta v]_{C^{0,\alpha}(\mathbb{R}^n)}$$

and

$$\|v\|_{C^{k,\alpha}(\mathbb{R}^n)} := \sum_{|\beta| \leq k} \|D^\beta v\|_{L^\infty(\mathbb{R}^n)} + [v]_{C^{k,\alpha}(\mathbb{R}^n)}.$$

Denote by $C^{k,\alpha}(\mathbb{R}^n)$ the space of functions for which this norm is finite.

Finally, suppose that Ω is bounded. For $v \in C^k(\overline{\Omega})$, let

$$\mathcal{E}^{k,\alpha}_\Omega(v) := \left\{ \widetilde{v} \in C^{k,\alpha}(\mathbb{R}^n) \middle| \widetilde{v} \restriction_{\overline{\Omega}} = v \right\}.$$

Define

$$C^{k,\alpha}(\overline{\Omega}) := \left\{ v \in C^k(\overline{\Omega}) \,\middle|\, \mathcal{E}^{k,\alpha}_{\Omega}(v) \neq \varnothing \right\}$$

and equip this space with the extension norm

$$\|v\|_{C^{k,\alpha}(\overline{\Omega})} := \inf_{\widetilde{v} \in \mathcal{E}^{k,\alpha}_{\Omega}(v)} \|\widetilde{v}\|_{C^{k,\alpha}(\mathbb{R}^n)}. \tag{3.4}$$

In particular, $C^{k,\alpha}(\overline{\Omega}) \subseteq C^k(\overline{\Omega})$.

Remark 3.5.2. (Local and global formulations by extensions) The preceding definition admits a local formulation. Indeed, a function v belongs to $C^{k,\alpha}(\overline{\Omega})$ if and only if there are open sets

$$\overline{\Omega} \subseteq V_0 \Subset V_1 \Subset V$$

and an extension $\widehat{v} \in C^k(V)$ of v, that is, $\widehat{v}\,|_{\overline{\Omega}} = v$, such that

$$\|\widehat{v}\|_{C^{k,\alpha}_{\mathrm{int}}(\overline{V_1})} < \infty.$$

A global extension satisfies this condition immediately. Conversely, choose a cutoff function $\chi \in C^\infty_c(V_1)$ equal to one on a neighborhood of $\overline{V_0}$. The extension by zero of $\chi\widehat{v}$ outside V_1, denoted by $\widetilde{\chi\widehat{v}}$, belongs to $C^{k,\alpha}(\mathbb{R}^n)$ and agrees with v on $\overline{\Omega}$. The intermediate open set V_1 ensures that all derivatives involved in the Leibniz rule are controlled on a compact subset of the domain of $\widehat{v}$. For $k = \infty$, the part of this formulation concerning the existence of a smooth extension recovers exactly the convention used for $C^\infty(\overline{\Omega})$ in trace theory.

When $k = 0$, the class thus defined agrees with the usual class of Hölder continuous functions on the compact set $\overline{\Omega}$. Indeed, if $H = [v]_{C^{0,\alpha}(\overline{\Omega})}$, McShane's formula

$$\widehat{v}(x) := \inf_{y \in \overline{\Omega}} \bigl(v(y) + H|x-y|^\alpha \bigr)$$

extends v to a Hölder continuous function of exponent α on $\mathbb{R}^n$, since $(x,y) \mapsto |x-y|^\alpha$ is a metric for $0 < \alpha \leq 1$. Multiplication by a cutoff function equal to one near $\overline{\Omega}$ then produces a bounded extension. Therefore, when $k = 0$, the extension norm in (3.4) is equivalent to the usual intrinsic norm on $\overline{\Omega}$.

For $k \geq 1$, the extension condition is part of the convention. In the applications in this section, the functions come from $W^{m,p}_0(\Omega)$, and their extensions by zero provide the required extensions. In the geometric applications, the coordinate domains are balls and the localized functions vanish near the coordinate boundary.

Remark 3.5.3. (Changing the Hölder exponent) If $0 < \alpha \leq \beta < 1$ and Ω is bounded, then

$$C^{k,\beta}(\overline{\Omega}) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$$

continuously. To see this, fix a cutoff function $\chi \in C^\infty_c(\mathbb{R}^n)$ equal to one on a neighborhood of $\overline{\Omega}$. If $\widetilde{v}$ is an extension of v, then $\chi\widetilde{v}$ is still an extension, and the Leibniz rule gives

$$\|\chi\widetilde{v}\|_{C^{k,\beta}(\mathbb{R}^n)} \leq C_\chi \|\widetilde{v}\|_{C^{k,\beta}(\mathbb{R}^n)}.$$

Since $\chi\widetilde{v}$ has support in a fixed compact set, each of its derivatives of order k satisfies

$$[w]_{C^{0,\alpha}(\mathbb{R}^n)} \le C_{\chi,\alpha,\beta}\big(\|w\|_{L^\infty(\mathbb{R}^n)} + [w]_{C^{0,\beta}(\mathbb{R}^n)}\big).$$

Taking the infimum over extensions $\widetilde{v}$ proves continuity of the embedding. We shall also use the fact that a function of class C^1 with bounded derivative is Lipschitz, and hence Hölder continuous with every exponent $0 < \alpha < 1$ on sets of bounded diameter.

The quantity defined in (3.4) is a norm. Homogeneity and the triangle inequality follow by taking extensions and adding them. Moreover,

$$\|v\|_{C^0(\overline{\Omega})} \le \|v\|_{C^{k,\alpha}(\overline{\Omega})},$$

since the same inequality holds for every extension. The norm therefore vanishes only when $v = 0$.

Proposition 3.5.4. (Completeness of Hölder spaces) For $k \in \mathbb{N}_0$ and $0 < \alpha < 1$, the space $C^{k,\alpha}(\mathbb{R}^n)$ is a Banach space. If $\Omega \subseteq \mathbb{R}^n$ is open and bounded, then $C^{k,\alpha}(\overline{\Omega})$, with the norm (3.4), is also a Banach space.

Proof. Let $(v_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $C^{k,\alpha}(\mathbb{R}^n)$. For each multi-index β with $|\beta| \le k$, the sequence $(D^\beta v_j)$ converges uniformly to a bounded continuous function v_β. If $|\beta| < k$, the fundamental theorem of calculus along segments allows passage to the limit in

$$D^\beta v_j(x + te_i) - D^\beta v_j(x) = \int_0^t D^{\beta+e_i} v_j(x + se_i)\, ds.$$

It follows that $v_0 \in C^k(\mathbb{R}^n)$ and $D^\beta v_0 = v_\beta$ for $|\beta| \le k$. If $|\beta| = k$, passing to the limit in the Hölder inequalities gives $v_\beta \in C^{0,\alpha}(\mathbb{R}^n)$. Finally, letting one index tend to infinity in the Cauchy condition yields $v_j \to v_0$ in $C^{k,\alpha}(\mathbb{R}^n)$.

Now let $(u_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $C^{k,\alpha}(\overline{\Omega})$. Extract a subsequence $(u_{j_\nu})_{\nu\in\mathbb{N}}$ such that

$$\|u_{j_{\nu+1}} - u_{j_\nu}\|_{C^{k,\alpha}(\overline{\Omega})} < 2^{-\nu-1}.$$

Choose $U_1 \in \mathcal{E}^{k,\alpha}_\Omega(u_{j_1})$. For each $\nu \in \mathbb{N}$, the definition of the infimum gives $G_\nu \in \mathcal{E}^{k,\alpha}_\Omega(u_{j_{\nu+1}} - u_{j_\nu})$ such that

$$\|G_\nu\|_{C^{k,\alpha}(\mathbb{R}^n)} < 2^{-\nu}.$$

The series

$$U_1 + \sum_{\nu=1}^{\infty} G_\nu$$

converges in $C^{k,\alpha}(\mathbb{R}^n)$ to a function U. Upon restriction to $\overline{\Omega}$, the partial sum of order N is $u_{j_{N+1}}$. Therefore, $u := U\restriction_{\overline{\Omega}}$ belongs to $C^{k,\alpha}(\overline{\Omega})$, and $u_{j_\nu} \to u$ in the extension norm. Since the original sequence is Cauchy, the whole sequence converges to u. □

The central analytic ingredient is the following local estimate. Its advantage is that the power of r is determined by the scaling of the problem.

Lemma 3.5.5. (Morrey's estimate on cubes) Let $n < p < \infty$ and let $Q \subseteq \mathbb{R}^n$ be a cube with sides parallel to the coordinate axes and side length $r > 0$. There is a constant $C = C(n,p) > 0$ such that

$$|v(x) - v(y)| \le C r^{1-\frac{n}{p}} \|\nabla v\|_{L^p(Q)} \tag{3.5}$$

for every $v \in C^1(\overline{Q})$ and all $x, y \in Q$.

Proof. Denote by

$$v_Q := \frac{1}{\lambda_n(Q)} \int_Q v(z)\, dz$$

the average of v on Q. Fix $x \in Q$. Since Q is convex, for every $y \in Q$ the segment joining x to y is contained in Q and

$$v(y) - v(x) = \int_0^1 \langle \nabla v(x + t(y-x)), y - x \rangle \, dt.$$

Averaging with respect to y and making, for each $t \in (0,1)$, the change of variables $z = x + t(y-x)$, we obtain

$$|v(x) - v_Q| \le \frac{1}{r^n} \int_0^1 \int_{x + t(Q-x)} |x - z|\, t^{-n-1} |\nabla v(z)|\, dz\, dt.$$

If $z \in x + t(Q-x)$, then $|z - x| \le \sqrt{n}\, r t$ and therefore $t \ge \frac{|z-x|}{\sqrt{n}\, r}$. Tonelli's theorem allows us to change the order of integration and gives

$$|v(x) - v_Q| \le C_n \int_Q \frac{|\nabla v(z)|}{|x - z|^{n-1}}\, dz.$$

Let $p' = \frac{p}{p-1}$. Hölder's inequality and the inclusion $Q \subseteq B_{\mathrm{euc}}(x, \sqrt{n}\, r)$ imply

$$|v(x) - v_Q| \le C_n \left(\int_{B_{\mathrm{euc}}(x, \sqrt{n} r)} |x - z|^{(1-n)p'}\, dz \right)^{\frac{1}{p'}} \|\nabla v\|_{L^p(Q)}.$$

The radial integral is finite precisely because $p > n$, and its power of r is $1 - \frac{n}{p}$. Consequently,

$$|v(x) - v_Q| \le C r^{1-\frac{n}{p}} \|\nabla v\|_{L^p(Q)}.$$

Applying the same inequality to y and using $|v(x) - v(y)| \le |v(x) - v_Q| + |v(y) - v_Q|$ gives (3.5). □

Theorem 3.5.6. (First-order Morrey embedding) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset and let $n < p < \infty$. If

$$\mu := 1 - \frac{n}{p},$$

then every element of $W_0^{1,p}(\Omega)$ has a unique representative in $C^{0,\mu}(\overline{\Omega})$, and there is a constant $C = C(n,p,\Omega) > 0$ such that

$$\|u\|_{C^{0,\mu}(\overline{\Omega})} \le C \|u\|_{W^{1,p}(\Omega)}. \tag{3.6}$$

Proof. First let $\varphi \in C_c^\infty(\Omega)$, and denote by $\widetilde{\varphi}$ its extension by zero to $\mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, choose a cube of side length $2|x-y|$ containing both points. Lemma 3.5.5, applied to $\widetilde{\varphi}$, gives

$$|\widetilde{\varphi}(x) - \widetilde{\varphi}(y)| \le C|x-y|^\mu \|\nabla \widetilde{\varphi}\|_{L^p(\mathbb{R}^n)}.$$

Therefore,

$$[\widetilde{\varphi}]_{C^{0,\mu}(\mathbb{R}^n)} \le C\|\nabla \varphi\|_{L^p(\Omega)}.$$

Take a fixed cube Q_0 containing $\overline{\Omega}$ in its interior and a point $y_0 \in Q_0 \setminus \overline{\Omega}$. Since $\widetilde{\varphi}(y_0) = 0$, the preceding estimate also gives

$$\|\widetilde{\varphi}\|_{L^\infty(\mathbb{R}^n)} \le C_\Omega \|\nabla \varphi\|_{L^p(\Omega)}.$$

Thus, (3.6) holds for every $\varphi \in C_c^\infty(\Omega)$.

Now let $u \in W_0^{1,p}(\Omega)$ and choose $(\varphi_j)_{j\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ such that $\varphi_j \to u$ in $W^{1,p}(\Omega)$. The global inequality already proved shows that the sequence of extensions by zero $(\widetilde{\varphi}_j)_{j\in\mathbb{N}}$ is Cauchy in $C^{0,\mu}(\mathbb{R}^n)$. By Proposition 3.5.4, it converges to a function $\widetilde{u} \in C^{0,\mu}(\mathbb{R}^n)$. At the same time, $\widetilde{\varphi}_j$ converges in $L^p(\mathbb{R}^n)$ to the extension by zero of u, so $\widetilde{u} = u$ almost everywhere on Ω. The restriction of $\widetilde{u}$ to $\overline{\Omega}$ is the desired representative and, by the definition of the extension norm, satisfies (3.6).

If two continuous representatives of the same class in $L^p(\Omega)$ agree almost everywhere, they agree throughout Ω, since the set on which they differ would otherwise be open and have positive measure. Continuity up to the boundary completes uniqueness on $\overline{\Omega}$. □

The higher-order result admits a uniform formulation in terms of the Sobolev regularity number

$$\sigma := m - \frac{n}{p}.$$

The restriction $0 < \alpha < 1$ avoids the Lipschitz endpoint, which requires separate treatment when σ is an integer and does not in general follow from the same argument.

Theorem 3.5.7. (Higher-order Sobolev–Morrey embedding) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset, let $m \in \mathbb{N}$ and $1 \le p < \infty$, and suppose that

$$\sigma := m - \frac{n}{p} > 0.$$

If $k \in \mathbb{N}_0$ and $0 < \alpha < 1$ satisfy

$$k + \alpha \le \sigma, \tag{3.7}$$

then every element of $W_0^{m,p}(\Omega)$ has a unique representative in $C^{k,\alpha}(\overline{\Omega})$, and there is a constant $C = C(n, m, p, k, \alpha, \Omega) > 0$ such that

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \le C\|u\|_{W^{m,p}(\Omega)}. \tag{3.8}$$

More precisely, the following assertions hold.

(a) If $\sigma \notin \mathbb{N}$, $\ell := \lfloor \sigma \rfloor$, and $\theta := \sigma - \ell$, then

$$W_0^{m,p}(\Omega) \hookrightarrow C^{\ell,\theta}(\overline{\Omega})$$

continuously.

(b) If $\sigma = s \in \mathbb{N}$, then for every $0 < \theta < 1$,

$$W_0^{m,p}(\Omega) \hookrightarrow C^{s-1,\theta}(\overline{\Omega})$$

continuously.

Proof. We begin with functions in $C_c^\infty(\Omega)$ and identify them with their extensions by zero to $\mathbb{R}^n$.
First suppose that $\sigma \notin \mathbb{N}$. Set

$$\ell := \lfloor \sigma \rfloor, \qquad \theta := \sigma - \ell \in (0,1), \qquad d := m - \ell.$$

Then

$$(d-1)p < n < dp.$$

If $d = 1$, we have $p > n$. For every multi-index β with $|\beta| \le \ell$, the function $D^\beta u$ belongs to $W_0^{1,p}(\Omega)$, so Theorem 3.5.6 gives

$$\|D^\beta u\|_{C^{0,\theta}(\overline{\Omega})} \le C\|u\|_{W^{m,p}(\Omega)}, \qquad \theta = 1 - \frac{n}{p}.$$

Now suppose that $d \ge 2$ and define

$$q := p_{d-1}^* = \frac{np}{n-(d-1)p}.$$

Corollary 3.3.17, applied with $j = \ell + 1$ and a gain of $d-1$ derivatives, gives the continuous embedding

$$W_0^{m,p}(\Omega) = W_0^{\ell+1+(d-1),p}(\Omega) \hookrightarrow W_0^{\ell+1,q}(\Omega).$$

The inequality $dp > n$ implies $q > n$, and a direct computation shows that

$$1 - \frac{n}{q} = d - \frac{n}{p} = m - \ell - \frac{n}{p} = \theta.$$

For each $|\beta| \le \ell$, we have $D^\beta u \in W_0^{1,q}(\Omega)$. Theorem 3.5.6 yields

$$\|D^\beta u\|_{C^{0,\theta}(\overline{\Omega})} \le C\|u\|_{W^{m,p}(\Omega)}.$$

Summing over the finite family of multi-indices proves part (a) for smooth functions. Density of $C_c^\infty(\Omega)$ in $W_0^{m,p}(\Omega)$ and completeness of $C^{\ell,\theta}(\overline{\Omega})$ extend the map to the whole space. Simultaneous convergence in $W^{m,p}$ and in $C^{\ell,\theta}$ identifies the classical derivatives of the representative with the weak derivatives of the original class.
Now suppose that $\sigma = s \in \mathbb{N}$. Then

$$d := m - s = \frac{n}{p} \in \mathbb{N}.$$

If $d = 1$, we already have $W_0^{m,p}(\Omega) = W_0^{s+1,n}(\Omega)$. If $d \ge 2$, Corollary 3.3.17, applied with $j = s + 1$ and a gain of $d - 1$ derivatives, gives

$$W_0^{m,p}(\Omega) = W_0^{s+1+(d-1),p}(\Omega) \hookrightarrow W_0^{s+1,n}(\Omega),$$

since $p^*_{d-1} = n$. Let $0 < \theta < 1$ and choose $q > n$ sufficiently large that

$$1 - \frac{n}{q} > \theta.$$

For every $|\gamma| \le s$, the derivative $D^\gamma u$ belongs to $W^{1,n}_0(\Omega)$. The part of Theorem 3.3.7 corresponding to the critical case $p = n$ shows that $D^\gamma u \in L^q(\Omega)$ and

$$\|D^\gamma u\|_{L^q(\Omega)} \le C_q \|u\|_{W^{m,p}(\Omega)}.$$

In particular, for $|\beta| \le s-1$, we have $D^\beta u \in W^{1,q}_0(\Omega)$. To retain the zero subscript, apply the same estimates to differences of approximants of u in $C^\infty_c(\Omega)$; all their derivatives of order at most s converge in L^q, so the derivatives of order at most $s-1$ converge in $W^{1,q}$ to elements of its closed subspace $W^{1,q}_0(\Omega)$. Theorem 3.5.6 gives a representative of $D^\beta u$ that is Hölder continuous with exponent $1 - \frac{n}{q}$. Since Ω is bounded and $1 - \frac{n}{q} > \theta$, we obtain

$$\|D^\beta u\|_{C^{0,\theta}(\overline{\Omega})} \le C \|u\|_{W^{m,p}(\Omega)}.$$

This proves part (b), first for smooth functions and then for all of $W^{m,p}_0(\Omega)$ by density.
Finally, let (k, α) be a pair satisfying (3.7). If σ is not an integer, part (a) and the elementary inclusions between Hölder spaces give (3.8). If $\sigma = s$ is an integer, choose $\theta \in (\alpha, 1)$ and use part (b). When $k < s-1$, the derivatives of order k of the extensions by zero in $\mathbb{R}^n$ have bounded first derivatives, and are therefore Lipschitz; when $k = s-1$, it suffices to use the inclusion $C^{s-1,\theta} \hookrightarrow C^{s-1,\alpha}$. Uniqueness of representatives is proved as in Theorem 3.5.6: two functions of class C^k agreeing almost everywhere on Ω agree on Ω and, by continuity, on its closure. □

To prove compactness, we shall need an elementary inequality between Hölder seminorms.

Lemma 3.5.8. (Interpolation of Hölder seminorms) Let $K \subseteq \mathbb{R}^n$, $0 < \alpha < \beta < 1$, and $v \in C^{0,\beta}(K)$. Then

$$[v]_{C^{0,\alpha}(K)} \le 2^{1-\frac{\alpha}{\beta}} \|v\|^{1-\frac{\alpha}{\beta}}_{C^0(K)} [v]^{\frac{\alpha}{\beta}}_{C^{0,\beta}(K)}. \tag{3.9}$$

Proof. Set $M := \|v\|_{C^0(K)}$ and $H := [v]_{C^{0,\beta}(K)}$. For $x \neq y$, we have simultaneously

$$|v(x) - v(y)| \le 2M, \qquad |v(x) - v(y)| \le H|x-y|^\beta.$$

If $H = 0$, the assertion is immediate. If $H > 0$, separate the cases $|x-y| \le \left(\frac{2M}{H}\right)^{\frac{1}{\beta}}$ and $|x-y| > \left(\frac{2M}{H}\right)^{\frac{1}{\beta}}$. Both give

$$\frac{|v(x) - v(y)|}{|x-y|^\alpha} \le (2M)^{1-\frac{\alpha}{\beta}} H^{\frac{\alpha}{\beta}}.$$

Taking the supremum completes the proof. □

Lemma 3.5.9. (Arzelà–Ascoli on compact Euclidean subsets) Let $K \subseteq \mathbb{R}^n$ be compact and let $(v_j)_{j\in\mathbb{N}}$ be a sequence of continuous functions on K. Suppose that there is $C > 0$ such that

$$\|v_j\|_{C^0(K)} \le C \qquad \text{for every } j \in \mathbb{N},$$

and that the family is equicontinuous. Then there is a subsequence converging uniformly on K.

Proof. For each $m \in \mathbb{N}$, compactness of K provides a finite set $D_m \subseteq K$ such that the balls of radius $\frac{1}{m}$ centered at D_m cover K. The set $D := \bigcup_{m\in\mathbb{N}} D_m$ is countable and dense. Enumerate it as $D = \{x_1, x_2, \dots\}$. The numerical sequence $(v_j(x_1))_{j\in\mathbb{N}}$ is bounded. By Bolzano–Weierstrass, it has a convergent subsequence. Extract from this a subsequence converging at x_2 and continue inductively. The diagonal subsequence, again denoted by (v_j), converges at each point of D.
We show that it is uniformly Cauchy. Let $\varepsilon > 0$. By equicontinuity, there is $\delta > 0$ such that

$$|v_j(x) - v_j(y)| < \frac{\varepsilon}{3}$$

for every j and any $x, y \in K$ with $|x - y| < \delta$. Compactness of K allows us to choose $x_{i_1}, \dots, x_{i_N} \in D$ so that the balls of radius δ centered at these points cover K. Since the subsequence converges at each of these points, there is j_0 such that

$$|v_j(x_{i_\nu}) - v_\ell(x_{i_\nu})| < \frac{\varepsilon}{3}$$

for $j, \ell \geq j_0$ and $\nu \in \{1, \dots, N\}$. Given $x \in K$, choose ν with $|x - x_{i_\nu}| < \delta$. Then

$$|v_j(x) - v_\ell(x)| \leq |v_j(x) - v_j(x_{i_\nu})| + |v_j(x_{i_\nu}) - v_\ell(x_{i_\nu})| + |v_\ell(x_{i_\nu}) - v_\ell(x)| < \varepsilon.$$

Thus, the subsequence is Cauchy in $C^0(K)$ and converges uniformly, since this space is Banach. □

Theorem 3.5.10. (Rellich–Kondrashov into Hölder spaces) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset, let $m \in \mathbb{N}$ and $1 \leq p < \infty$, and suppose that

$$\sigma := m - \frac{n}{p} > 0.$$

If $k \in \mathbb{N}_0$ and $0 < \alpha < 1$ satisfy

$$k + \alpha < \sigma, \tag{3.10}$$

then the embedding

$$W_0^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$$

is compact.

Proof. Choose β such that

$$\alpha < \beta < 1, \qquad k + \beta \leq \sigma.$$

This choice is possible by (3.10). Theorem 3.5.7 shows that every bounded sequence $(u_j)_{j\in\mathbb{N}}$ in $W_0^{m,p}(\Omega)$, after choosing the representatives obtained by extension by zero, is bounded in $C^{k,\beta}(\mathbb{R}^n)$. These extensions vanish outside $\overline{\Omega}$. Fix a compact cube Q containing $\overline{\Omega}$ in its interior.
For each $|\gamma| = k$, the family $(D^\gamma u_j)$ is uniformly bounded and uniformly Hölder continuous on Q. If $|\gamma| < k$, the family is uniformly Lipschitz, since its first derivatives are uniformly bounded. Lemma 3.5.9 and a diagonal argument over the finite family of multi-indices allow us to extract a subsequence, which we do not relabel, such that

$$D^\gamma u_j \longrightarrow v_\gamma \qquad \text{uniformly on } Q$$

for every $|\gamma| \le k$. The identities obtained from the fundamental theorem of calculus pass to the limit, so there is $v \in C^k(Q)$ with $D^\gamma v = v_\gamma$.
For $|\gamma| = k$, apply Lemma 3.5.8 to $D^\gamma(u_j - u_\ell)$. The seminorms of order β are uniformly bounded, while the uniform norms tend to zero. Consequently,

$$[D^\gamma(u_j - u_\ell)]_{C^{0,\alpha}(Q)} \longrightarrow 0 \qquad \text{as } j, \ell \to \infty.$$

Thus, the subsequence is Cauchy in the $\|\cdot\|_{C^{k,\alpha}_{\mathrm{int}}(Q)}$ norm and converges to v in that norm. All extensions are supported in the fixed compact set $\overline{\Omega} \Subset \mathrm{Int}(Q)$. Hence convergence in the $C^{k,\alpha}_{\mathrm{int}}(Q)$ norm implies convergence of these same extensions in $C^{k,\alpha}(\mathbb{R}^n)$: outside Q the functions vanish, and pairs with one point outside Q are controlled using the positive distance between $\overline{\Omega}$ and ∂Q. Restricting to $\overline{\Omega}$ and using the extension norm, we obtain a subsequence converging in $C^{k,\alpha}(\overline{\Omega})$. □

Remark 3.5.11. (Optimal form of the exponents) Let $\sigma = m - \frac{n}{p} > 0$.

(a) If $\sigma = \ell + \theta$ with $\ell \in \mathbb{N}_0$ and $0 < \theta < 1$, the embedding

$$W^{m,p}_0(\Omega) \hookrightarrow C^{\ell,\theta}(\overline{\Omega})$$

is continuous, and becomes compact upon replacing θ by any $0 < \alpha < \theta$.

(b) If $\sigma = s \in \mathbb{N}$, then for every $0 < \alpha < 1$ the embedding

$$W^{m,p}_0(\Omega) \hookrightarrow C^{s-1,\alpha}(\overline{\Omega})$$

is compact and, in particular, continuous.

Corollary 3.5.12. (Infinite exponents) Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $m > k \ge 0$ and $1 \le p < \infty$. If $m - \frac{n}{p} > k$, the embedding

$$W^{m,p}_0(\Omega) \hookrightarrow W^{k,\infty}(\Omega)$$

is compact. The same conclusion holds for $W^{m,p}(\Omega)$ if Ω is an extension domain for $W^{m,p}$. At the endpoint $p = 1$ and $m - k = n$, both embeddings into $W^{k,\infty}$ remain continuous, although they are not compact when Ω is nonempty.
If $p = \infty$, $m > k \ge 0$, and Ω is an extension domain for $W^{m,\infty}$, the embeddings $W^{m,\infty}(\Omega) \hookrightarrow W^{k,q}(\Omega)$ are compact for $1 \le q \le \infty$. They are also compact on $W^{m,\infty}_0(\Omega) := \overline{C^\infty_c(\Omega)}^{W^{m,\infty}(\Omega)}$ for any bounded open subset. On $\mathbb{R}^n$, for $1 \le p < \infty$ and $m - \frac{n}{p} > k$, and also for $p = 1$ and $m - k = n$, the embedding $W^{m,p}(\mathbb{R}^n) \hookrightarrow W^{k,\infty}(\mathbb{R}^n)$ is continuous. Restriction through an extension operator gives the same continuity on the half-space.

Proof. For $p < \infty$, choose $0 < \alpha < 1$ with $k + \alpha < m - \frac{n}{p}$. Theorem 3.5.10 gives compactness in $C^{k,\alpha}(\overline{\Omega})$, whose norm controls all derivatives of order at most k in L^∞. On an extension domain, apply the same argument to $Tu = \chi \mathcal{E} u$ on a ball, as in Theorem 3.4.2.
At the endpoint $p = 1$, for $\varphi \in C^\infty_c(\mathbb{R}^n)$, successive applications of the fundamental theorem of calculus in

the n coordinates give

$$\varphi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} D_1 \cdots D_n \varphi(t_1, \ldots, t_n)\, dt_n \cdots dt_1, \qquad \|\varphi\|_\infty \le \|D_1 \cdots D_n \varphi\|_{L^1(\mathbb{R}^n)}.$$

Apply this bound to each $D^\alpha \varphi$, $|\alpha| \le k$. All derivatives on the right-hand side have order at most $k + n = m$. Density shows that the extensions by zero of the approximants converge uniformly together with their derivatives of order at most k. The fundamental theorem of calculus along segments identifies the limit as a function of class C^k and gives continuity into $W^{k,\infty}$. Applying a cutoff to an extension treats $W^{m,1}(\Omega)$. To see that compactness fails, take $u_\varepsilon(x) = \varepsilon^k \varphi((x - x_0)/\varepsilon)$ with support in an interior ball: their $W^{m,1}$ norms are bounded, and some derivative of order k has a constant nonzero uniform norm, although it converges to zero almost everywhere.

For $p = \infty$, choose $1 \le r < \infty$ sufficiently large that $m - \frac{n}{r} > k$. The functions Tu have support in a fixed compact set and are bounded in $W^{m,r}$ by Hölder's inequality on that set. The first part gives compactness in $W^{k,\infty}$, which implies compactness in $W^{k,q}(\Omega)$ by finiteness of the measure. For $W_0^{m,\infty}$, the closure argument in Proposition 3.3.13 remains valid in the $W^{m,\infty}$ norm and provides extension by zero. Their test function approximants also converge in $W^{m,r}$ on a fixed ball. The first part applies again.

For global continuity, repeat the localization on the cubes Q_a from Proposition 3.4.7. The Sobolev–Morrey embedding, applied to $\chi_a u$, controls $\|u\|_{W^{k,\infty}(Q_a)}$ by $C\|u\|_{W^{m,p}(a+(-1,2)^n)}$, with C independent of a. Taking the supremum over $a \in \mathbb{Z}^n$ gives the global estimate. □

Equality of indices with an infinite exponent requires its own analysis. For example, in dimension one, $W_0^{1,1}(a,b) \hookrightarrow C^0([a,b])$ continuously: the estimate $\|\varphi\|_\infty \le \|\varphi'\|_{L^1((a,b))}$ and density give the continuous representative. This embedding is not compact. If $c \in (a,b)$ and $0 \le \varphi \in C_c^\infty((-1,1))$ satisfies $\varphi(0) = 1$, the functions $u_j(x) = \varphi(j(x-c))$, for sufficiently large j, are bounded in $W_0^{1,1}(a,b)$, have uniform norm one, and converge to zero at every point other than c. They have no uniformly convergent subsequence. The preceding assertions do not infer compactness at any equality endpoint solely from a comparison of indices.

3.6 Localized embeddings on the half-space

The upper half-space is the Euclidean model for boundary charts. Since it is unbounded, compactness cannot be formulated globally without a condition preventing functions from moving toward infinity. In applications to compact manifolds, each function localized by a partition of unity has support contained in a fixed compact set. This is exactly the situation recorded below.

Definition 3.6.1. (Uniformly Lipschitz continuous boundary) Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. We say that its boundary $\partial\Omega$ is **uniformly Lipschitz continuous** if there exist $\varepsilon, L > 0$, $M \in \mathbb{N}$, and a countable locally finite open cover $(\Omega_j)_{j\in\mathbb{N}}$ of $\partial\Omega$ such that:

(a) for each $x \in \partial\Omega$, there is $j \in \mathbb{N}$ such that $B_{\mathrm{euc}}(x,\varepsilon) \subseteq \Omega_j$;

(b) no point of $\mathbb{R}^n$ belongs to more than M sets in the family $(\Omega_j)_{j\in\mathbb{N}}$;

(c) for each $j \in \mathbb{N}$, there exist a coordinate system $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and a Lipschitz function

$f_j\colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, both depending on j, such that $\mathrm{Lip}(f_j) \le L$ and

$$\Omega_j \cap \Omega = \Omega_j \cap V_j, \qquad V_j := \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y_n > f_j(y')\}.$$

The numbers ε, L, and M are called the **uniformity parameters** of the boundary.

Theorem 3.6.2. (Extension on open subsets with uniformly Lipschitz boundary) Let $\Omega \subseteq \mathbb{R}^n$ be an open subset whose boundary is uniformly Lipschitz continuous with parameters $\varepsilon, L > 0$ and $M \in \mathbb{N}$. Let $1 \le p \le \infty$ and $m \in \mathbb{N}$. Then there is a continuous linear operator

$$\mathcal{E}\colon W^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^n)$$

such that, for every $u \in W^{m,p}(\Omega)$, we have $\mathcal{E}(u)(x) = u(x)$ for almost every $x \in \Omega$. Moreover, there is a constant $c = c(n, m, p) > 0$ for which

$$\|\mathcal{E}(u)\|_{L^p(\mathbb{R}^n)} \le c(1 + M)\|u\|_{L^p(\Omega)} \tag{3.11}$$

and, for each $k \in \{1, \dots, m\}$,

$$\|\nabla^k \mathcal{E}(u)\|_{L^p(T^{(0,k)}(\mathbb{R}^n))} \le c\big(1 + M(1 + L^k)\big) \sum_{i=0}^{k} \left(\frac{M}{\varepsilon}\right)^{k-i} \|\nabla^i u\|_{L^p(T^{(0,i)}(\Omega))}, \tag{3.12}$$

where $\nabla^0 u := u$ and the covariant derivatives are those induced by the Euclidean metric. In particular, there is a constant $C = C(n, m, p, \varepsilon, M, L) > 0$ such that

$$\|\mathcal{E}(u)\|_{W^{m,p}(\mathbb{R}^n)} \le C\|u\|_{W^{m,p}(\Omega)}.$$

The construction of the operator and the estimates (3.11) and (3.12) can be found in [97, Theorem 13.17].

Corollary 3.6.3. (Embeddings on a ball without boundary conditions) Let $B \subset \mathbb{R}^n$ be a bounded open ball. Let $\ell > k \ge 0$ be integers and $1 \le p, q < \infty$. If

$$\ell - \frac{n}{p} \ge k - \frac{n}{q},$$

then $W^{\ell,p}(B) \hookrightarrow W^{k,q}(B)$ continuously; if the inequality is strict, the embedding is compact.

Proof. The boundary of a ball is smooth. Near each point, after a translation and a rotation, it can be represented as the graph of a smooth function on a ball in $\mathbb{R}^{n-1}$. Shrinking this ball makes the derivative of the function bounded, and the mean value theorem then shows that the graph is Lipschitz. Multiplying the function by a cutoff equal to one on a smaller ball gives a Lipschitz function defined on all of $\mathbb{R}^{n-1}$ without changing this local representation. Compactness of ∂B allows us to choose finitely many representations on open sets V_i and smaller open sets U_i that still cover ∂B, with $\overline{U_i} \Subset V_i$. The minimum of the positive distances $\mathrm{dist}(\overline{U_i}, \mathbb{R}^n \setminus V_i)$, the maximum of the Lipschitz constants, and the number of charts give the parameters in Definition 3.6.1. For $n = 1$, the boundary consists of the two endpoints of an interval, and the same description reduces to half-lines.

By Theorem 3.6.2, B is an extension domain for $W^{\ell,p}$. Apply the second part of Theorem 3.3.16 and, under the strict inequality, of Theorem 3.4.2. Both results apply to the full space $W^{\ell,p}(B)$, without requiring interior support or membership in $W_0^{\ell,p}(B)$. □

Corollary 3.6.4. (Half-space extension operator) Let $k \in \mathbb{N}_0$ and $1 \le p \le \infty$. There is a continuous linear operator

$$\mathcal{E}_k \colon W^{k,p}(\mathbb{R}^n_+) \longrightarrow W^{k,p}(\mathbb{R}^n)$$

such that $(\mathcal{E}_k u) \restriction_{\mathbb{R}^n_+} = u$ at almost every point of $\mathbb{R}^n_+$. In particular, there is $C_{k,p,n} > 0$ such that

$$\|\mathcal{E}_k u\|_{W^{k,p}(\mathbb{R}^n)} \le C_{k,p,n} \|u\|_{W^{k,p}(\mathbb{R}^n_+)}$$

for every $u \in W^{k,p}(\mathbb{R}^n_+)$.

Proof. If $k = 0$, extension by zero suffices. If $k \ge 1$, the half-space $\mathbb{R}^n_+ := \{(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid t > 0\}$ has uniformly Lipschitz boundary, and Theorem 3.6.2 gives the operator for every $1 \le p \le \infty$.
For finite exponents, one can see explicitly how higher-order derivatives are preserved. Choose $a_1, \dots, a_k \in \mathbb{R}$ solving

$$\sum_{j=1}^{k} a_j (-j)^r = 1, \qquad r \in \{0, \dots, k-1\}.$$

The matrix is a Vandermonde matrix at the distinct points $-1, \dots, -k$, and is therefore invertible. For a function smooth up to the boundary, define

$$(\mathcal{E}_k u)(x',t) := \begin{cases} u(x',t), & t > 0, \\ \displaystyle\sum_{j=1}^{k} a_j u(x',-jt), & t < 0. \end{cases}$$

The tangential and normal derivatives of total order at most $k-1$ agree on both sides of $t = 0$ by the preceding equations. Integration by parts on both half-spaces cancels the boundary terms. Applying this argument successively to each derivative of order at most $k-1$ shows that the weak derivatives of order at most k are the ordinary derivatives on each side. For $\alpha = (\alpha', r)$, $|\alpha| \le k$, the change of variable $s = -jt$ and Minkowski's inequality give

$$\|D^\alpha \mathcal{E}_k u\|_{L^p(\{t<0\})} \le \sum_{j=1}^{k} |a_j| j^{r-1/p} \|D^\alpha u\|_{L^p(\mathbb{R}^n_+)}.$$

The extension operator already obtained and Proposition 3.3.12 show that restrictions of $C_c^\infty(\mathbb{R}^n)$ are dense in $W^{k,p}(\mathbb{R}^n_+)$ for $p < \infty$. The estimate allows this formula to be extended by continuity and verifies all its derivatives. For $k = 1$, we obtain even reflection; for $k \ge 2$, the finite combination also ensures agreement of the necessary normal derivatives. □

Remark 3.6.5. (Density and boundary conditions on the half-space) Restrictions of $C_c^\infty(\mathbb{R}^n)$ are dense in $W^{m,p}(\mathbb{R}^n_+)$ for $m \ge 1$ and $1 \le p < \infty$. These functions need not vanish on the boundary. In contrast, $W_0^{m,p}(\mathbb{R}^n_+)$ is defined using approximants compactly supported in the open half-space. Theorem 3.7.1 will characterize

this latter space by the vanishing of traces of order less than m.

For a compact set $K \subseteq \overline{\mathbb{R}^n_+}$, we shall write

$$W^{k,p}_K(\mathbb{R}^n_+) := \left\{ u \in W^{k,p}(\mathbb{R}^n_+) \middle| \operatorname{supp} u \subseteq K \right\},$$

where support is understood in the essential sense within $\overline{\mathbb{R}^n_+}$.

Theorem 3.6.6. (Localized Sobolev and Rellich–Kondrashov embeddings on the half-space) Let $K \subseteq \overline{\mathbb{R}^n_+}$ be compact, let $m > k \geq 0$ be integers, and let $1 \leq p, q < \infty$. If

$$m - \frac{n}{p} \geq k - \frac{n}{q},$$

the inclusion $W^{m,p}_K(\mathbb{R}^n_+) \hookrightarrow W^{k,q}(\mathbb{R}^n_+)$ is continuous. If the inequality is strict, this inclusion is compact. In particular, $W^{m,p}_K(\mathbb{R}^n_+) \hookrightarrow W^{m-1,p}(\mathbb{R}^n_+)$ is compact for every $m \geq 1$ and $1 \leq p < \infty$.
If $q = \infty$ and $p < \infty$, the inclusion is compact when $m - \frac{n}{p} > k$; for $p = 1$ and $m - k = n$, it is continuous. If $p = \infty$, it is compact for every $1 \leq q \leq \infty$ and $m > k$. These assertions also hold for the subspace $W^{m,p}_K(\mathbb{R}^n_+) \cap W^{m,p}_0(\mathbb{R}^n_+)$; for finite exponents, its limit belongs to $W^{k,q}_0(\mathbb{R}^n_+)$.

Proof. Take $R > 0$ such that $K \subseteq B_{\text{euc}}(0, R) \cap \overline{\mathbb{R}^n_+}$, and $\chi \in C^\infty_c(B_{\text{euc}}(0, 2R))$ equal to one on a neighborhood of K. Define

$$Tu := \chi \mathcal{E}_m u.$$

The Leibniz rule and Corollary 3.6.4 give

$$\|Tu\|_{W^{m,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{m,p}(\mathbb{R}^n_+)}.$$

Moreover, Tu has compact support contained in $B := B_{\text{euc}}(0, 2R)$, and $(Tu) \restriction_{\mathbb{R}^n_+} = u$, since u vanishes outside K. For $p < \infty$, Proposition 3.3.13 gives $Tu \in W^{m,p}_0(B)$. Continuity for finite exponents follows from Theorem 3.3.16 on B, and compactness from Theorem 3.4.2. In the compactness assertion, the resulting sequence converges to an element of $W^{k,q}_0(B)$, since this is the target space in Theorem 3.4.2. Extension by zero is continuous by Proposition 3.3.13. Hence this convergence is preserved in $W^{k,q}(\mathbb{R}^n)$ and, upon restriction, in $W^{k,q}(\mathbb{R}^n_+)$. The same cutoff and the same ball are used for the entire sequence.
For $p < \infty$ and $q = \infty$, use Corollary 3.5.12 on B. For $p = \infty$, the functions Tu are bounded in $W^{m,r}(\mathbb{R}^n)$ for each $r < \infty$ because their support is fixed. Choose $r > n/(m-k)$ and apply that corollary again; the finite exponents follow from the finite measure of the support.
If each u_j also belongs to $W^{m,p}_0(\mathbb{R}^n_+)$ and the exponents are finite, we may use $Z_{\mathbb{R}^n_+} u_j$ in place of $\mathcal{E}_m u_j$. To justify the zero condition in the target space, approximate each u_j in $W^{m,p}$ by test functions and multiply them by χ. Multiplication preserves the approximation and places all supports in B. The continuous embedding on B shows that the restrictions converge in $W^{k,q}$ to u_j, so $u_j \in W^{k,q}_0(\mathbb{R}^n_+)$. Closedness of this subspace preserves the condition in the limit supplied by compactness. □

Remark 3.6.7. (Support up to the boundary and local compactness) The compact set K in the theorem may meet $\{t = 0\}$. In this case, a function in $W^{m,p}_K(\mathbb{R}^n_+)$ may have nonzero trace, and its extension by zero

may fail to belong to $W^{m,p}(\mathbb{R}^n)$; this is why $\mathcal{E}_m$ is used. If $K \Subset \mathbb{R}^n_+$ is contained in the interior, choose a cutoff in $C_c^\infty(\mathbb{R}^n_+)$ equal to one near K. The weak Leibniz rule and interior mollification then show that $W_K^{m,p}(\mathbb{R}^n_+) \subseteq W_0^{m,p}(\mathbb{R}^n_+)$ for $p < \infty$.
For a bounded sequence in $W^{m,p}(\mathbb{R}^n_+)$, cutoffs on balls and a diagonal extraction give local compactness up to the boundary: convergence holds in $W^{k,q}(B_{\mathrm{euc}}(0,R) \cap \mathbb{R}^n_+)$ for every $R > 0$ in the strict finite range. Without a condition on the tails, the whole half-space lacks global compactness. In dimension $n \geq 2$, tangential translations of a nonzero test function have constant norms and may have disjoint supports. In dimension $n = 1$, translate that function toward $+\infty$. Both examples reproduce the separation of norms in Remark 3.4.9, even within $W_0^{m,p}(\mathbb{R}^n_+)$.

To apply the half-space embeddings to functions defined on a half-ball, we must first extend them across its spherical portion. Separation of the support from that portion prevents jumps; no vanishing on the flat boundary is required.

Lemma 3.6.8. (Extension by zero from a half-ball) Let $m \in \mathbb{N}_0$, $1 \leq p < \infty$, and $R > 0$. Let $B^+ = B_{\mathrm{euc}}(0,R) \cap \mathbb{R}^n_+$ and $B' = B_{\mathrm{euc}}(0,R) \cap \partial\mathbb{R}^n_+$. Suppose that $v \in W^{m,p}(B^+)$ and that there is a compact set $K \subset B_{\mathrm{euc}}(0,R) \cap \{x_n \geq 0\}$, lying at a positive distance from the spherical portion $\partial B_{\mathrm{euc}}(0,R) \cap \{x_n \geq 0\}$, such that $v = 0$ almost everywhere on $B^+ \setminus K$.
Define $\widetilde{v} \colon \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ by

$$\widetilde{v}(x) = \begin{cases} v(x), & x \in B^+, \\ 0, & x \in \mathbb{R}^n_+ \setminus B^+. \end{cases}$$

Then $\widetilde{v} \in W^{m,p}(\mathbb{R}^n_+)$, and for every multi-index β with $|\beta| \leq m$, we have $D^\beta \widetilde{v} = \widetilde{D^\beta v}$ in the weak sense, where $\widetilde{D^\beta v}$ denotes the extension by zero of $D^\beta v$ to $\mathbb{R}^n_+$. In particular,

$$\|\widetilde{v}\|_{W^{m,p}(\mathbb{R}^n_+)} = \|v\|_{W^{m,p}(B^+)}.$$

Proof. If $m = 0$, the definition of extension by zero gives $\|\widetilde{v}\|_{L^p(\mathbb{R}^n_+)} = \|v\|_{L^p(B^+)}$, and the identity corresponding to $\beta = 0$ is immediate. Since $W^{0,p} = L^p$, this proves the result in that case. Henceforth assume $m \geq 1$.
Denote the spherical portion of the half-ball by $\Gamma_R := \partial B_{\mathrm{euc}}(0,R) \cap \{x_n \geq 0\}$. If $K = \varnothing$, then $v = 0$ almost everywhere on B^+, and the result is immediate. Thus assume $K \neq \varnothing$.
Since K lies at a positive distance from Γ_R, we may take $\delta := \operatorname{dist}(K, \Gamma_R) > 0$. First consider what this implies. For each $x \in K$, we have $\operatorname{dist}(x, \Gamma_R) = R - \|x\|$. If $x \neq 0$, the point $\dfrac{R}{\|x\|}x$ belongs to Γ_R, and hence $\operatorname{dist}(x, \Gamma_R) \leq \left\| x - \dfrac{R}{\|x\|}x \right\| = R - \|x\|$. The reverse inequality follows from $\|x - \mathbf{y}\| \geq \big|\|\mathbf{y}\| - \|x\|\big| = R - \|x\|$ for every $\mathbf{y} \in \Gamma_R$. If $x = 0$, the equality is also clear. Therefore, $R - \|x\| \geq \delta$ for every $x \in K$; that is, $\|x\| \leq R - \delta$ for every $x \in K$. In particular, $K \subset B_{\mathrm{euc}}(0, R - \dfrac{\delta}{2}) \cap \{x_n \geq 0\}$.
Define $U := B^+ \cap \left\{ x \in \mathbb{R}^n \mid \|x\| > R - \dfrac{\delta}{2} \right\}$. Then U is an interior strip adjacent to Γ_R, and moreover $U \cap K = \varnothing$. Since $v = 0$ almost everywhere on $B^+ \setminus K$, it follows that $v = 0$ almost everywhere on U.
By locality of weak derivatives, $D^\beta v = 0$ almost everywhere on U for every $|\beta| \leq m$. If $w \in W^{k,p}(B^+)$ vanishes almost everywhere on a relatively open subset $V \subset B^+$, then for every $\psi \in C_c^\infty(V)$, viewed as a test function

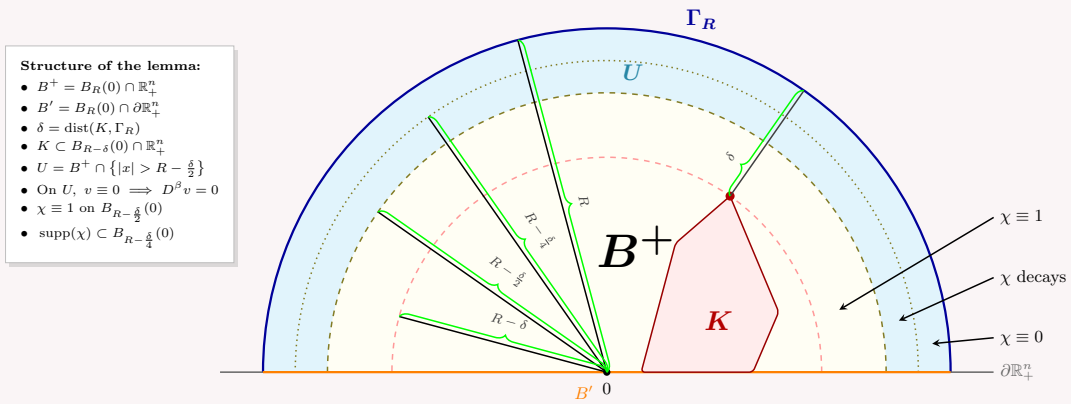


Figure 3.1. Outline of the proof of extension by zero. The essential support of v is contained in a compact set K separated from the spherical portion Γ_R by a positive distance δ. This separation allows us to construct a neighborhood U of Γ_R on which v and its weak derivatives vanish, as well as a cutoff function χ equal to 1 near K and vanishing before it reaches Γ_R. This prevents boundary terms from appearing on the spherical portion and gives $D^\beta \widetilde{v} = \widetilde{D^\beta v}$.

on B^+, we have

$$\int_V D^\beta w\, \psi\, dx = \int_{B^+} D^\beta w\, \psi\, dx = (-1)^{|\beta|} \int_{B^+} w\, D^\beta \psi\, dx = (-1)^{|\beta|} \int_V w\, D^\beta \psi\, dx = 0.$$

For each $|\beta| \le k$, since $D^\beta w \in L^p(V)$, it follows that $D^\beta w = 0$ almost everywhere on V.
We now prove that $D^\beta \widetilde{v} = \widetilde{D^\beta v}$ in the weak sense for every $|\beta| \le m$. Fix such an β and take $\varphi \in C_c^\infty(\mathbb{R}^n_+)$. Choose a function $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ on $B_{\mathrm{euc}}(0, R - \frac{\delta}{2})$ and $\mathrm{supp}\,\chi \subset B_{\mathrm{euc}}(0, R - \frac{\delta}{4})$. This choice is possible because $R - \frac{\delta}{2} < R - \frac{\delta}{4} < R$. In particular, $\chi \equiv 1$ on a neighborhood of K, and $(\chi|_{\mathbb{R}^n_+})\varphi \restriction_{B^+} \in C_c^\infty(B^+)$, since χ vanishes near the spherical portion and φ has compact support in $\mathbb{R}^n_+$.
Since $\widetilde{v} = 0$ on $\mathbb{R}^n_+ \setminus B^+$, we have

$$\int_{\mathbb{R}^n_+} \widetilde{v}\, D^\beta \varphi\, dx = \int_{B^+} v\, D^\beta \varphi\, dx.$$

Moreover, since $v = 0$ almost everywhere on $B^+ \setminus K$, it suffices to compare the integrands on a neighborhood of K. On that neighborhood, $\chi \equiv 1$, and therefore $D^\beta(\chi\varphi) = D^\beta \varphi$, where $\chi\varphi$ denotes $(\chi|_{\mathbb{R}^n_+})\varphi$. It follows that $v D^\beta \varphi = v D^\beta(\chi\varphi)$ almost everywhere on B^+. Thus,

$$\int_{B^+} v\, D^\beta \varphi\, dx = \int_{B^+} v\, D^\beta(\chi\varphi)\, dx.$$

Since $(\chi|_{\mathbb{R}^n_+})\varphi \restriction_{B^+} \in C_c^\infty(B^+)$, the definition of weak derivative gives

$$\int_{B^+} v\, D^\beta(\chi\varphi)\, dx = (-1)^{|\beta|} \int_{B^+} D^\beta v\, \chi\varphi\, dx.$$

Now observe that within B^+, the set where $\chi \neq 1$ is contained in U, except possibly on the sphere

$\left\{\|x\| = R - \frac{\delta}{2}\right\}$, which has measure zero. Since we already showed that $D^\beta v = 0$ almost everywhere on U, we obtain

$$\int_{B^+} D^\beta v\, \chi\varphi\, dx = \int_{B^+} D^\beta v\, \varphi\, dx = \int_{\mathbb{R}^n_+} \widetilde{D^\beta v}\, \varphi\, dx.$$

Combining the preceding equalities gives

$$\int_{\mathbb{R}^n_+} \widetilde{v}\, D^\beta \varphi\, dx = (-1)^{|\beta|} \int_{\mathbb{R}^n_+} \widetilde{D^\beta v}\, \varphi\, dx.$$

This proves that $D^\beta \widetilde{v} = \widetilde{D^\beta v}$ in the weak sense for every $|\beta| \le m$.
Since $\widetilde{D^\beta v} \in L^p(\mathbb{R}^n_+)$ for every $|\beta| \le m$, we conclude that $\widetilde{v} \in W^{m,p}(\mathbb{R}^n_+)$. Moreover,

$$\|\widetilde{v}\|^p_{W^{m,p}(\mathbb{R}^n_+)} = \sum_{|\beta|\le m} \int_{\mathbb{R}^n_+} |D^\beta \widetilde{v}|^p\, dx = \sum_{|\beta|\le m} \int_{\mathbb{R}^n_+} |\widetilde{D^\beta v}|^p\, dx = \sum_{|\beta|\le m} \int_{B^+} |D^\beta v|^p\, dx = \|v\|^p_{W^{m,p}(B^+)}.$$

Therefore, $\|\widetilde{v}\|_{W^{m,p}(\mathbb{R}^n_+)} = \|v\|_{W^{m,p}(B^+)}$. □

Corollary 3.6.9. (Approximation and localized embeddings on a half-ball) Let $B^+ = B_{\mathrm{euc}}(0,R) \cap \mathbb{R}^n_+$, $B' = B_{\mathrm{euc}}(0,R) \cap \{x_n = 0\}$, and let $K \subset B_{\mathrm{euc}}(0,R) \cap \{x_n \ge 0\}$ be compact.

(a) If $s \in \mathbb{N}_0$, $1 \le p < \infty$, and $v \in W^{s,p}(B^+)$ vanishes almost everywhere on $B^+ \setminus K$, there exist $f_\nu \in C^\infty_c(B_{\mathrm{euc}}(0,R))$ whose restrictions converge to v in $W^{s,p}(B^+)$. Their supports are contained in a common compact subset of $B_{\mathrm{euc}}(0,R)$.

(b) If $\ell > k \ge 0$ are integers, $1 \le p,q < \infty$, and $\ell - n/p \ge k - n/q$, there is $C > 0$ such that every $v \in W^{\ell,p}(B^+)$ with that support satisfies

$$\|v\|_{W^{k,q}(B^+)} \le C\|v\|_{W^{\ell,p}(B^+)}.$$

If the index inequality is strict, every bounded sequence of these functions, with the same K, has a subsequence converging in $W^{k,q}(B^+)$.

The restrictions in the first part are smooth up to B', but need not vanish there.

Proof. Extend v by zero within the half-space using Lemma 3.6.8. The function $\widetilde{v}$ has the same Sobolev norm as v and vanishes outside K. For the first part, restrictions of $C^\infty_c(\mathbb{R}^n)$ are dense in $W^{s,p}(\mathbb{R}^n_+)$: for $s \ge 1$, apply the operator from Corollary 3.6.4 and Proposition 3.3.12; for $s = 0$, approximate the extension by zero in $L^p(\mathbb{R}^n)$. Thus, there exist $h_\nu \in C^\infty_c(\mathbb{R}^n)$ with $h_\nu \restriction_{\mathbb{R}^n_+} \to \widetilde{v}$ in $W^{s,p}$.
Choose a fixed cutoff $\eta \in C^\infty_c(B_{\mathrm{euc}}(0,R))$ equal to one near K and set $f_\nu := \eta h_\nu$. Since $\eta\widetilde{v} = \widetilde{v}$, the Leibniz rule and restriction give

$$\begin{aligned}\|f_\nu \restriction_{B^+} - v\|_{W^{s,p}(B^+)} &\le \|\eta(h_\nu \restriction_{\mathbb{R}^n_+} - \widetilde{v})\|_{W^{s,p}(\mathbb{R}^n_+)} \\ &\le C_{s,p,\eta}\|h_\nu \restriction_{\mathbb{R}^n_+} - \widetilde{v}\|_{W^{s,p}(\mathbb{R}^n_+)} \longrightarrow 0.\end{aligned}$$

The support of each f_ν is contained in $\operatorname{supp}\eta \Subset B_{\mathrm{euc}}(0,R)$.
For the second part, let $B := B_{\mathrm{euc}}(0,2R)$ and define

$$z := (\mathcal{E}_\ell \widetilde{v}) \restriction_B .$$

Half-space extension and Lemma 3.3.11 imply $z\restriction_{B^+} = v$ and

$$\|z\|_{W^{\ell,p}(B)} \le \|\mathcal{E}_\ell \widetilde{v}\|_{W^{\ell,p}(\mathbb{R}^n)} \le C_{\mathcal{E}} \|v\|_{W^{\ell,p}(B^+)}.$$

By Corollary 3.6.3,

$$\|v\|_{W^{k,q}(B^+)} \le \|z\|_{W^{k,q}(B)} \le C_B C_{\mathcal{E}} \|v\|_{W^{\ell,p}(B^+)}.$$

In this application we do not need to apply a cutoff to $\mathcal{E}_\ell \widetilde{v}$: the embedding is applied to its restriction in the full space on the ball. The cutoff in the first part serves a different purpose, keeping the supports of the approximants within the chart.

If $(v_\nu)_{\nu\in\mathbb{N}}$ is bounded and the inequality is strict, use the same operator and the same ball to define z_ν. Then

$$\sup_{\nu\in\mathbb{N}} \|z_\nu\|_{W^{\ell,p}(B)} \le C_{\mathcal{E}} \sup_{\nu\in\mathbb{N}} \|v_\nu\|_{W^{\ell,p}(B^+)} < \infty.$$

The same corollary provides a subsequence $z_{\nu_h} \to z_\infty$ in $W^{k,q}(B)$. Finally,

$$\|v_{\nu_h} - z_\infty\restriction_{B^+}\|_{W^{k,q}(B^+)} \le \|z_{\nu_h} - z_\infty\|_{W^{k,q}(B)} \longrightarrow 0.$$

This proves compactness without imposing a zero boundary condition. □

3.6.1 Summary of domains and embeddings

In the table, $m > k \ge 0$ are integers, p, q are finite unless otherwise stated, and $\delta := (m - \frac{n}{p}) - (k - \frac{n}{q})$. The extension condition concerns the source space. On every open subset, $W_0^{m,p}$ is defined by closure; interior smooth density and the equality $W_0 = W$ on $\mathbb{R}^n$ are given in Proposition 3.3.12.

Domain and space	Assumptions and range	Conclusion	Reference
Ω bounded, $W_0^{m,p}$	$1 \le p, q < \infty$	Continuous if $\delta \ge 0$; compact if $\delta > 0$, into $W_0^{k,q}$	3.3.16, 3.4.2
Ω bounded, $W^{m,p}$	Extension domain; $1 \le p, q < \infty$	Continuous if $\delta \ge 0$; compact if $\delta > 0$	3.3.16, 3.4.2
$\mathbb{R}^n$, $W^{m,p} = W_0^{m,p}$	$1 \le p \le q < \infty$	Globally continuous if $\delta \ge 0$	3.4.7
$\mathbb{R}^n$, support in compact K	$1 \le p, q < \infty$; $\delta > 0$	Compact with common support; locally compact	3.4.8
$\mathbb{R}^n_+$, $W_K^{m,p}$; also $W_K^{m,p} \cap W_0^{m,p}$	$K \subseteq \overline{\mathbb{R}^n_+}$ compact; $1 \le p, q < \infty$	Continuous if $\delta \ge 0$; compact if $\delta > 0$	3.6.6
Ω bounded; $W_0^{m,p}$ or $W^{m,p}$ with extension	$p < \infty$, $q = \infty$, $m - n/p > k$; or $p = \infty$	Compact into $W^{k,q}$ in the indicated ranges	3.5.12

3.7 The trace theorem on the half-space

The trace assigns boundary values to a Sobolev function through smooth approximation. On the half-space, an estimate in the normal direction allows its construction for every $1 \le p < \infty$; applying it to derivatives will also give the characterization of $W_0^{m,p}(\mathbb{R}^n_+)$.

We shall use the convention in Definition 3.5.1: a function belongs to $C^\infty(\overline{\Omega})$ if it is the restriction to $\overline{\Omega}$ of a

smooth function defined on an open neighborhood of $\overline{\Omega}$. Moreover,

$$C_c^\infty(\overline{\Omega}) = \left\{ f \in C^\infty(\overline{\Omega}) \middle| \operatorname{supp}_{\overline{\Omega}}(f) \text{ is compact} \right\}.$$

In particular, on the half-space we shall work with $C_c^\infty(\overline{\mathbb{R}^n_+})$. For each $m \in \mathbb{N}$ and $1 \le p < \infty$, this space is dense in $W^{m,p}(\mathbb{R}^n_+)$, and ordinary restriction to the flat boundary initially defines an operator on these functions. The trace theorem states that this operator extends uniquely and continuously to all of $W^{m,p}(\mathbb{R}^n_+)$.

Theorem 3.7.1. (Trace theorem on the half-space) Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$, let $\partial\mathbb{R}^n_+ = \{x_n = 0\} \cong \mathbb{R}^{n-1}$, and let $m \ge 1$ and $1 \le p < \infty$. Then, for each multi-index β with $|\beta| \le m-1$, the operator

$$f \longmapsto (D^\beta f) \restriction_{\partial\mathbb{R}^n_+}, \qquad f \in C_c^\infty(\overline{\mathbb{R}^n_+}),$$

admits a unique continuous linear extension

$$\gamma_0 D^\beta \colon W^{m,p}(\mathbb{R}^n_+) \longrightarrow W^{m-1-|\beta|,p}(\mathbb{R}^{n-1}).$$

Moreover,

$$W_0^{m,p}(\mathbb{R}^n_+) = \left\{ f \in W^{m,p}(\mathbb{R}^n_+) \middle| \gamma_0 D^\beta f = 0 \text{ for every } |\beta| \le m-1 \right\},$$

where $W_0^{m,p}(\mathbb{R}^n_+)$ denotes the closure of $C_c^\infty(\mathbb{R}^n_+)$ in $W^{m,p}(\mathbb{R}^n_+)$.

Proof. Write $x = (x', t) \in \mathbb{R}^{n-1} \times (0, \infty)$. When $n = 1$, interpret $\mathbb{R}^0$ as a single point equipped with counting measure; the tangential derivatives appearing below then reduce to the derivative of order zero. By Corollary 3.6.4 and density of $C_c^\infty(\mathbb{R}^n)$ in $W^{m,p}(\mathbb{R}^n)$, restriction of smooth approximations of an extension of f proves density of $C_c^\infty(\overline{\mathbb{R}^n_+})$ in $W^{m,p}(\mathbb{R}^n_+)$.
Begin with $m = 1$. If $f \in C_c^\infty(\overline{\mathbb{R}^n_+})$, the fundamental theorem of calculus gives, for $p = 1$,

$$|f(x', 0)| \le \int_0^\infty |\partial_n f(x', t)| \, dt.$$

Integrating with respect to x' gives

$$\| f \restriction_{\partial\mathbb{R}^n_+} \|_{L^1(\mathbb{R}^{n-1})} \le \| \partial_n f \|_{L^1(\mathbb{R}^n_+)}.$$

For $1 < p < \infty$, apply the same argument to $|f|^p$ and integrate. Hölder's and Young's inequalities yield

$$\begin{aligned} \| f \restriction_{\partial\mathbb{R}^n_+} \|^p_{L^p(\mathbb{R}^{n-1})} &\le p \int_{\mathbb{R}^n_+} |f|^{p-1} |\partial_n f| \, dx \\ &\le p \| f \|^{p-1}_{L^p(\mathbb{R}^n_+)} \| \partial_n f \|_{L^p(\mathbb{R}^n_+)} \\ &\le (p-1) \| f \|^p_{L^p(\mathbb{R}^n_+)} + \| \partial_n f \|^p_{L^p(\mathbb{R}^n_+)}. \end{aligned}$$

By density and completeness of $L^p(\mathbb{R}^{n-1})$, restriction has a unique continuous linear extension $\gamma_0 \colon W^{1,p}(\mathbb{R}^n_+) \to L^p(\mathbb{R}^{n-1})$ for every $1 \le p < \infty$.
Now let $m \ge 1$, $|\beta| \le m-1$, and $f \in W^{m,p}(\mathbb{R}^n_+)$. Each $D^\beta f$ belongs to $W^{1,p}(\mathbb{R}^n_+)$, so its trace is defined. Take $f_\nu \in C_c^\infty(\overline{\mathbb{R}^n_+})$ with $f_\nu \to f$ in $W^{m,p}$. For every tangential multi-index $\alpha \in \mathbb{N}_0^{n-1}$ with $|\alpha| \le m-1-|\beta|$, the

first-order estimate implies

$$\gamma_0 D^{\beta+(\alpha,0)} f_\nu \longrightarrow \gamma_0 D^{\beta+(\alpha,0)} f \quad \text{on } L^p(\mathbb{R}^{n-1}).$$

For smooth functions,

$$D^\alpha_{x'}(\gamma_0 D^\beta f_\nu) = \gamma_0 D^{\beta+(\alpha,0)} f_\nu.$$

Integrate this identity against a test function $\varphi \in C_c^\infty(\mathbb{R}^{n-1})$ and pass to the limit on both sides. Hölder's inequality justifies this passage also for $p = 1$, using $\varphi, D^\alpha\varphi \in L^\infty$. Thus,

$$D^\alpha_{x'}(\gamma_0 D^\beta f) = \gamma_0 D^{\beta+(\alpha,0)} f$$

in the weak sense. Summing the first-order estimates for these finitely many multi-indices gives

$$\|\gamma_0 D^\beta f\|_{W^{m-1-|\beta|,p}(\mathbb{R}^{n-1})} \le C(n,m,p)\|f\|_{W^{m,p}(\mathbb{R}^n_+)}.$$

The density already established also proves uniqueness of this extension.

To characterize the kernel, if $f \in W_0^{m,p}(\mathbb{R}^n_+)$, approximate by functions in $C_c^\infty(\mathbb{R}^n_+)$. All their derivatives restrict to zero on the boundary; continuity of the traces implies $\gamma_0 D^\beta f = 0$ for every $|\beta| \le m-1$.

Conversely, suppose that these traces vanish. First justify the integration-by-parts identity that allows extension by zero. If $v \in W^{1,p}(\mathbb{R}^n_+)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, smooth approximation up to the boundary and continuity of γ_0 give

$$\int_{\mathbb{R}^n_+} v\,\partial_i\varphi\,dx = -\int_{\mathbb{R}^n_+} (D_i v)\varphi\,dx - \delta_{in}\int_{\mathbb{R}^{n-1}} (\gamma_0 v)(x')\varphi(x',0)\,dx'.$$

For smooth functions, the formula follows from the fundamental theorem of calculus and tangential integration by parts; all integrals pass to the limit by Hölder's inequality on the fixed support of φ. If $\gamma_0 v = 0$ and $E_0 v$ denotes extension by zero to $\mathbb{R}^n$, the preceding identity is precisely

$$D_i(E_0 v) = E_0(D_i v) \quad \text{in the sense of distributions on } \mathbb{R}^n.$$

Apply this identity successively to $v = D^\alpha f$. At order zero, $E_0 f \in L^p(\mathbb{R}^n)$. If we already know $D^\alpha(E_0 f) = E_0(D^\alpha f)$ for some $|\alpha| \le m-1$, then $D^\alpha f \in W^{1,p}(\mathbb{R}^n_+)$ and $\gamma_0 D^\alpha f = 0$. For each $i \in \{1,\dots,n\}$, the first-order step gives

$$D^{\alpha+e_i}(E_0 f) = D_i E_0(D^\alpha f) = E_0(D^{\alpha+e_i} f).$$

Induction proves, for all multi-indices of order at most m,

$$D^\alpha(E_0 f) = E_0(D^\alpha f), \qquad \|E_0 f\|_{W^{m,p}(\mathbb{R}^n)} = \|f\|_{W^{m,p}(\mathbb{R}^n_+)}.$$

Set $F := E_0 f$. For $t > 0$, the function $F_t(x) := F(x - te_n)$ belongs to $W^{m,p}(\mathbb{R}^n)$, vanishes when $x_n < t$, and converges to F in $W^{m,p}(\mathbb{R}^n)$ as $t \to 0^+$. Indeed, $D^\alpha F_t = (D^\alpha F)(\cdot - te_n)$, and continuity of translations in L^p applies to each of the finitely many derivatives $|\alpha| \le m$.

For fixed t, choose $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi = 1$ on a neighborhood of $\overline{B_{\mathrm{euc}}(0,1)}$ and set $\chi_R(x) = \chi(x/R)$. The Leibniz rule and $\|D^\alpha \chi_R\|_\infty = R^{-|\alpha|}\|D^\alpha\chi\|_\infty$ show that $\chi_R F_t \to F_t$ in $W^{m,p}(\mathbb{R}^n)$ as $R \to \infty$: terms with no derivatives on χ_R converge by domination, while the others contain the factor $R^{-|\alpha|}$, $|\alpha| \ge 1$. The product

still vanishes on $\{x_n < t\}$. If ρ_ε is a mollifier supported in $\overline{B_{\mathrm{euc}}(0,\varepsilon)}$ and $0 < \varepsilon < t/2$, then

$$\rho_\varepsilon * (\chi_R F_t) \in C_c^\infty(\mathbb{R}^n_+),$$

since its support is contained in $\{x_n \geq t - \varepsilon\}$. For fixed t, R, these convolutions converge to $\chi_R F_t$ in $W^{m,p}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. For each $\nu \in \mathbb{N}$, choose successively $t_\nu > 0$, R_ν, and $0 < \varepsilon_\nu < t_\nu/2$ so that each of the three errors is less than $1/(3\nu)$. The restriction of $\rho_{\varepsilon_\nu} * (\chi_{R_\nu} F_{t_\nu})$ converges to f in $W^{m,p}(\mathbb{R}^n_+)$. By definition, $f \in W_0^{m,p}(\mathbb{R}^n_+)$, completing the characterization of the kernel. □

We shall use the following convention from now on. If $v \in W^{m,p}(B^+)$ satisfies the support assumption in Lemma 3.6.8 and $\widetilde{v}$ denotes its extension by zero to $\mathbb{R}^n_+$, then for $|\beta| \leq m-1$ we shall write $\gamma_0 D^\beta v = 0$ on B' to mean that

$$(\gamma_0 D^\beta \widetilde{v}) \restriction_{B'} = 0 \qquad \text{on } W^{m-1-|\beta|,p}(B').$$

This notation does not introduce an independent trace on the half-ball; it merely abbreviates the half-space trace after extension by zero.

Lemma 3.7.2. (Locality of the half-space trace) Let $1 \leq p < \infty$, let $O \subset \partial\mathbb{R}^n_+$ be a relatively open subset, and let $F \in W^{1,p}(\mathbb{R}^n_+)$. Suppose that there is an open subset $V \subset \mathbb{R}^n$, with $O \subset V \cap \partial\mathbb{R}^n_+$, such that $F = 0$ almost everywhere on $V \cap \mathbb{R}^n_+$. Then $\gamma_0 F = 0$ in O.

Proof. Fix $x_0 \in O$. Since O is relatively open in $\partial\mathbb{R}^n_+$ and V is open in $\mathbb{R}^n$, we may choose $r > 0$ such that $B_{\mathrm{euc}}(x_0, 2r) \subset V$ and $B_{\mathrm{euc}}(x_0, 2r) \cap \partial\mathbb{R}^n_+ \subset O$. Take a function $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\chi \subset B_{\mathrm{euc}}(x_0, 2r)$ and $\chi \equiv 1$ on $B_{\mathrm{euc}}(x_0, r)$. Upon restricting χ to $\mathbb{R}^n_+$, the product χF is well defined as an element of $W^{1,p}(\mathbb{R}^n_+)$. Since $\operatorname{supp}\chi \cap \mathbb{R}^n_+ \subset V \cap \mathbb{R}^n_+$ and $F = 0$ almost everywhere on $V \cap \mathbb{R}^n_+$, we have $\chi F = 0$ in $W^{1,p}(\mathbb{R}^n_+)$. Therefore, $\gamma_0(\chi F) = 0$.

We now prove that $\gamma_0(\chi G) = \chi \restriction_{\partial\mathbb{R}^n_+} \gamma_0 G$ for every $G \in W^{1,p}(\mathbb{R}^n_+)$. If $G \in C_c^\infty(\overline{\mathbb{R}^n_+})$, the identity is immediate, since both sides are the restriction of χG to $\partial\mathbb{R}^n_+$. For $G \in W^{1,p}(\mathbb{R}^n_+)$, take a sequence $G_j \in C_c^\infty(\overline{\mathbb{R}^n_+})$ such that $G_j \to G$ in $W^{1,p}(\mathbb{R}^n_+)$. Since multiplication by the fixed function $\chi|_{\mathbb{R}^n_+}$ is continuous on $W^{1,p}(\mathbb{R}^n_+)$, we have $\chi G_j \to \chi G$ in $W^{1,p}(\mathbb{R}^n_+)$. Moreover, continuity of the trace gives

$$\gamma_0(\chi G_j) \longrightarrow \gamma_0(\chi G) \qquad \text{on } L^p(\partial\mathbb{R}^n_+),$$

and, since multiplication by $\chi \restriction_{\partial\mathbb{R}^n_+}$ is continuous on $L^p(\partial\mathbb{R}^n_+)$,

$$\chi \restriction_{\partial\mathbb{R}^n_+} \gamma_0 G_j \longrightarrow \chi \restriction_{\partial\mathbb{R}^n_+} \gamma_0 G \qquad \text{on } L^p(\partial\mathbb{R}^n_+).$$

Passing to the limit in the smooth identity yields

$$\gamma_0(\chi G) = \chi \restriction_{\partial\mathbb{R}^n_+} \gamma_0 G.$$

Applying this to $G = F$ gives

$$0 = \gamma_0(\chi F) = \chi \restriction_{\partial\mathbb{R}^n_+} \gamma_0 F.$$

Since $\chi \equiv 1$ on $B_{\mathrm{euc}}(x_0, r)$, it follows that $\gamma_0 F = 0$ in $B_{\mathrm{euc}}(x_0, r) \cap \partial\mathbb{R}^n_+$. Second countability of O allows us to cover it by a countable family of these neighborhoods. The union of their exceptional sets has measure

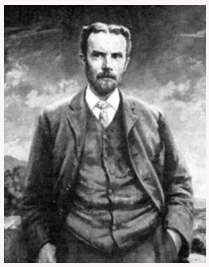
zero, so $\gamma_0 F = 0$ almost everywhere on O. □

Corollary 3.7.3. (Local characterization on a half-ball) Let $B^+ = B_{\text{euc}}(0,R) \cap \mathbb{R}^n_+$, let $B' = B_{\text{euc}}(0,R) \cap \partial\mathbb{R}^n_+$, and let $v \in W^{m,p}(B^+)$. Suppose that there is a compact set $K \subset B_{\text{euc}}(0,R) \cap \{x_n \geq 0\}$, lying at a positive distance from the spherical portion $\partial B_{\text{euc}}(0,R) \cap \{x_n \geq 0\}$, such that $v = 0$ almost everywhere on $B^+ \setminus K$.
If $\gamma_0 D^\beta v = 0$ on B', in the sense of the preceding convention, for every $|\beta| \leq m-1$, then v belongs to the closure of $C_c^\infty(B^+)$ in $W^{m,p}(B^+)$.

Proof. If $K = \varnothing$, then $v = 0$ almost everywhere on B^+, and the result is immediate. Thus assume $K \neq \varnothing$.
Let $\widetilde{v}$ be the extension by zero of v to $\mathbb{R}^n_+$. By Lemma 3.6.8, we have $\widetilde{v} \in W^{m,p}(\mathbb{R}^n_+)$ and $D^\beta \widetilde{v} = \widetilde{D^\beta v}$ for every $|\beta| \leq m$. By hypothesis, $(\gamma_0 D^\beta \widetilde{v}) \restriction_{B'} = 0$ for every $|\beta| \leq m-1$. We show that in fact $\gamma_0 D^\beta \widetilde{v} = 0$ on all of $\partial\mathbb{R}^n_+$.
Fix $|\beta| \leq m-1$. We already know that the trace vanishes upon restriction to B'. It remains to examine what happens on $\partial\mathbb{R}^n_+ \setminus B'$. Again denote the spherical portion of the half-ball by $\Gamma_R := \partial B_{\text{euc}}(0,R) \cap \{x_n \geq 0\}$, take $\delta := \operatorname{dist}(K, \Gamma_R) > 0$, and set $U := B^+ \cap \{x \in \mathbb{R}^n \mid \|x\| > R - \frac{\delta}{2}\}$. As in the proof of the lemma, $D^\alpha v = 0$ almost everywhere on U for every $|\alpha| \leq m$.
Let $x_0 = (x_0', 0) \in \partial\mathbb{R}^n_+ \setminus B'$. Then $\|x_0'\| \geq R$. If $\|x_0'\| > R$, take $\rho := \frac{\|x_0'\| - R}{2} > 0$. Then $B_{\text{euc}}(x_0, \rho) \cap \mathbb{R}^n_+ \subset \mathbb{R}^n_+ \setminus B^+$, and the definition of extension by zero gives $D^\beta \widetilde{v} = 0$ almost everywhere on $B_{\text{euc}}(x_0, \rho) \cap \mathbb{R}^n_+$.
If $\|x_0'\| = R$, take $\rho := \frac{\delta}{4}$. If $x \in B_{\text{euc}}(x_0, \rho) \cap \mathbb{R}^n_+$, then either $x \notin B^+$, where $D^\beta \widetilde{v} = 0$, or $x \in B^+$. In the latter case, $\|x\| \geq R - \rho = R - \frac{\delta}{4} > R - \frac{\delta}{2}$, so $x \in U$, where $D^\beta v = 0$ almost everywhere. Thus in this case too, $D^\beta \widetilde{v} = 0$ almost everywhere on $B_{\text{euc}}(x_0, \rho) \cap \mathbb{R}^n_+$.
Thus, $D^\beta \widetilde{v}$ vanishes almost everywhere on a relative neighborhood of each point of $\partial\mathbb{R}^n_+ \setminus B'$. Since $|\beta| \leq m-1$, we have $D^\beta \widetilde{v} \in W^{1,p}(\mathbb{R}^n_+)$, and Lemma 3.7.2, applied to $F = D^\beta \widetilde{v}$, yields $\gamma_0 D^\beta \widetilde{v} = 0$ in $\partial\mathbb{R}^n_+$. Since this holds for every $|\beta| \leq m-1$, Theorem 3.7.1 implies $\widetilde{v} \in W_0^{m,p}(\mathbb{R}^n_+)$. Hence there is a sequence $(\eta_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n_+)$ such that $\eta_k \to \widetilde{v}$ in $W^{m,p}(\mathbb{R}^n_+)$.
Take a function $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\chi \subset B_{\text{euc}}(0,R)$ and $\chi \equiv 1$ on a neighborhood of K. Since multiplication by $\chi|_{\mathbb{R}^n_+}$ is continuous on $W^{m,p}(\mathbb{R}^n_+)$, we have $\chi\eta_k \to \chi\widetilde{v} = \widetilde{v}$ in $W^{m,p}(\mathbb{R}^n_+)$. Moreover, $(\chi\eta_k) \restriction_{B^+} \in C_c^\infty(B^+)$, because η_k has compact support in $\mathbb{R}^n_+$ and $\operatorname{supp}\chi \subset B_{\text{euc}}(0,R)$. Restricting to B^+ gives a sequence in $C_c^\infty(B^+)$ converging to v in $W^{m,p}(B^+)$. Therefore, v belongs to the closure of $C_c^\infty(B^+)$ in $W^{m,p}(B^+)$. □

4

Distributions and Fourier analysis



Oliver Heaviside (1850–1925)

Figure 4.1. Oliver Heaviside systematically used the step function and operational calculus methods to study electrical problems, anticipating ideas that later received a rigorous formulation in distribution theory.

Distribution theory extends the notion of a function and makes it possible to differentiate objects that have neither a classical derivative nor a weak derivative representable by a locally integrable function. Its modern formulation, due to Laurent Schwartz, rigorously organizes ideas that had already appeared in operational calculus and in the study of discontinuous signals.

Sobolev spaces showed that integration by parts makes it possible to assign derivatives to functions of limited regularity. The distributional framework takes this idea one step further: a distribution is defined by its action on test functions, and its derivatives are obtained by formally transferring differentiation to those functions. This provides a unified treatment of the Heaviside function, the Dirac delta, and many other singularities arising in differential equations, mathematical physics, and Fourier analysis.

A weak solution may therefore have derivatives only in the distributional sense. For differential operators with constant coefficients, the Fourier transform turns those derivatives into multiplication: with the unitary normalization to be fixed below,

$$\widehat{D^{\alpha}u}(\xi) = (i\xi)^{\alpha}\,\widehat{u}(\xi).$$

This identity reduces linear equations to algebraic relations in the frequency variable. It also allows us to construct fundamental solutions and regularizations, and to distinguish the behavior of low frequencies from that of high

frequencies. Tempered distributions will provide the common domain on which these operations remain well defined.

The step function introduced by Oliver Heaviside provides an immediate motivation. Among other phenomena, it models the instantaneous switching on of a signal and has a jump at the origin; its derivative cannot be represented by a locally integrable function, but it can be represented by a distribution.

Example 4.0.1. (The Heaviside function) We define the Heaviside function $H\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Although H is locally integrable, it is not weakly differentiable in the sense of having its derivative represented by a function in $L^1_{\mathrm{loc}}(\mathbb{R})$. This difficulty motivates the distributional extension of the derivative. Suppose, seeking a contradiction, that there exists $v \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} H(x)\varphi'(x)dx = -\int_{\mathbb{R}} v(x)\varphi(x)dx \qquad \forall \varphi \in C_c^\infty(\mathbb{R}).$$

Since $H = 0$ on $(-\infty, 0)$ and $H = 1$ on $[0, \infty)$, we obtain

$$\int_{\mathbb{R}} H(x)\varphi'(x)dx = \int_0^\infty \varphi'(x)dx.$$

Since φ has compact support, there exists $R > 0$ such that $\varphi(x) = 0$ for $x \geq R$, so

$$\int_0^\infty \varphi'(x)dx = \int_0^R \varphi'(x)dx.$$

This integral is defined over a finite interval, and since φ is smooth, the Lebesgue integral agrees with the Riemann integral. Theorem B.6.5 gives

$$\int_0^R \varphi'(x)dx = \varphi(R) - \varphi(0) = -\varphi(0).$$

Consequently, the preceding equality would imply

$$\int_{\mathbb{R}} v(x)\varphi(x)dx = \varphi(0), \qquad \forall \varphi \in C_c^\infty(\mathbb{R}).$$

Now, if we take a family of functions $\varphi_\varepsilon \in C_c^\infty(\mathbb{R})$ with $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon(0) = 1$, and $\mathrm{supp}(\varphi_\varepsilon) \subset (-\varepsilon, \varepsilon)$, it follows that

$$\int_{\mathbb{R}} v(x)\varphi_\varepsilon(x)dx = 1.$$

But since $v \in L^1_{\mathrm{loc}}(\mathbb{R})$,

$$\left| \int_{\mathbb{R}} v(x)\varphi_\varepsilon(x)dx \right| \leq \int_{-\varepsilon}^{\varepsilon} |v(x)|dx \xrightarrow[\varepsilon \to 0]{} 0,$$

where the limit is justified by applying the *Lebesgue dominated convergence theorem* (Theorem B.6.14), since $v \in L^1_{\mathrm{loc}}(\mathbb{R})$ ensures integrability on the intervals $(-\varepsilon, \varepsilon)$. This contradicts the fact that the integral always

equals 1. Thus no such function v exists, and we conclude that H has no weak derivative.

This calculation makes clear the need to extend the notion of a weak derivative: we must allow new objects that, although not ordinary functions, act consistently on test functions. This new object is precisely the *Dirac delta*.

Before formalizing the definition, it is helpful to recall the physical motivation: in his study of quantum mechanics, Paul Dirac (1902–1984) formally introduced the symbol δ_0 to model *point sources of energy*. In that setting, the delta was an indispensable tool for describing systems in which all the mass or charge is concentrated at a single point, and engineers and physicists used it long before a rigorous mathematical foundation became available through the distribution theory developed by Laurent Schwartz. We now formalize the Dirac delta.

Example 4.0.2. (The Dirac delta) We define the ***Dirac delta*** to be the linear functional $\delta_0 \colon C_c^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ given by

$$\delta_0(\varphi) = \varphi(0).$$

The preceding identity suggests that δ_0 should be interpreted as the derivative of the Heaviside function. It is not an ordinary function, but it can act linearly on test functions. For this action to have analytical stability, linearity alone is insufficient: continuity is needed. Before defining a distribution, we must therefore specify the topology of the space $C_c^\infty(\mathbb{R}^n)$.

4.1 The space $\mathcal{D}(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. The space $C_c^\infty(\Omega)$ of smooth functions with compact support will be the basic space of test functions. To use it in functional analysis, we need to equip it with a topology that simultaneously records the behavior of a function and all its derivatives, without reducing this information to a single norm.

We first carry out the construction by fixing a compact set $K \subseteq \Omega$ and considering the functions whose support is contained in K. We will use the general results on topologies induced by families of seminorms, locally convex spaces, metrizability, and Fréchet spaces collected in Appendix B.10.

In particular, we will use Definition B.10.18, Theorem B.10.19, Propositions B.10.21, B.10.23, and B.10.24, as well as Theorem B.10.38.

We first consider the space of all smooth functions with support contained in a fixed compact subset $K \subseteq \Omega$, which we denote by $\mathcal{D}_K(\Omega)$:

$$\mathcal{D}_K(\Omega) := \{\varphi \in C_c^\infty(\Omega) \mid \operatorname{supp}(\varphi) \subseteq K\}.$$

The topology on this subspace must be consistent with the natural notion of uniform convergence of functions and their partial derivatives of every order on the compact set K. For each $j \in \mathbb{N}_0$, we define

$$\|\varphi\|_{C^j(K)} := \max_{|\alpha| \leq j} \sup_{x \in K} |D^\alpha \varphi(x)|.$$

Since the case $\alpha = 0$ is included and $\operatorname{supp}(\varphi) \subseteq K$, each $\|\cdot\|_{C^j(K)}$ is a norm on $\mathcal{D}_K(\Omega)$. If $\|\varphi\|_{C^j(K)} = 0$, then in particular $\|\varphi\|_{C^0(K)} = 0$, so $\varphi = 0$ on K. On the other hand, since $\operatorname{supp}(\varphi) \subseteq K$, if $x \in \Omega \setminus K$, then $\varphi(x) = 0$. Therefore, $\varphi = 0$ throughout Ω.

By Theorem B.10.19, the family of norms $\{\|\cdot\|_{C^j(K)}\}_{j \in \mathbb{N}_0}$ induces a topology τ_K on $\mathcal{D}_K(\Omega)$ under which this space is a locally convex topological vector space. A neighborhood base at the origin for this topology consists of finite

intersections of sets of the form

$$\bigcap_{q=1}^{k} \left\{ \varphi \in \mathcal{D}_K(\Omega) \,\middle|\, \|\varphi\|_{C^{j_q}(K)} < r_q \right\},$$

where $j_1, \dots, j_k \in \mathbb{N}_0$, $r_1, \dots, r_k > 0$, and $k \in \mathbb{N}$. The neighborhoods of an arbitrary element $\varphi \in \mathcal{D}_K(\Omega)$ have the form $\varphi + U$, where U is a neighborhood of the origin.

Lemma 4.1.1. The family

$$V_{K,j,\ell} := \left\{ \varphi \in \mathcal{D}_K(\Omega) \,\middle|\, \|\varphi\|_{C^j(K)} < \frac{1}{\ell} \right\}, \qquad j \in \mathbb{N}_0, \quad \ell \in \mathbb{N},$$

is a countable neighborhood base at the origin for τ_K. In particular, $(\mathcal{D}_K(\Omega), \tau_K)$ has a countable local base at each point.

Proof. First, observe that each $V_{K,j,\ell}$ is a neighborhood of the origin, since it is one of the neighborhoods determined by the norm $\|\cdot\|_{C^j(K)}$.
Now let U be a neighborhood of the origin in τ_K. By the description of the local base, there exists a basic neighborhood

$$W := \bigcap_{q=1}^{k} \left\{ \varphi \in \mathcal{D}_K(\Omega) \,\middle|\, \|\varphi\|_{C^{j_q}(K)} < r_q \right\}$$

such that $W \subseteq U$. We define $j := \max\{j_1, \dots, j_k\}$ and choose $\ell \in \mathbb{N}$ such that $\frac{1}{\ell} < \min\{r_1, \dots, r_k\}$.
Since the norms $\|\cdot\|_{C^j(K)}$ are increasing in j, if $\psi \in V_{K,j,\ell}$, then, for each $q \in \{1, \dots, k\}$,

$$\|\psi\|_{C^{j_q}(K)} \le \|\psi\|_{C^j(K)} < \frac{1}{\ell} < r_q.$$

Therefore, $\psi \in W$. Thus, $V_{K,j,\ell} \subseteq W \subseteq U$.
This proves that $\{V_{K,j,\ell} \mid j \in \mathbb{N}_0,\ \ell \in \mathbb{N}\}$ is a local base at the origin. Moreover, this family is countable, since $\mathbb{N}_0 \times \mathbb{N}$ is countable.
Finally, since translations are homeomorphisms in the topology induced by seminorms, for each $\varphi \in \mathcal{D}_K(\Omega)$ the family

$$\{\varphi + V_{K,j,\ell} \mid j \in \mathbb{N}_0,\ \ell \in \mathbb{N}\}$$

is a countable neighborhood base at φ. □

Since the family $\{\|\cdot\|_{C^j(K)}\}_{j \in \mathbb{N}_0}$ separates points, Proposition B.10.21 implies that the space $(\mathcal{D}_K(\Omega), \tau_K)$ is Hausdorff. If $\varphi \neq 0$, then, since $\operatorname{supp}(\varphi) \subseteq K$, it is impossible for φ to vanish throughout K; therefore, $\|\varphi\|_{C^0(K)} > 0$.

In particular, singletons in $(\mathcal{D}_K(\Omega), \tau_K)$ are closed. Moreover, by Lemma 4.1.1, this space has a countable local base at the origin. By Theorem B.10.9, we conclude that $(\mathcal{D}_K(\Omega), \tau_K)$ is metrizable.

Furthermore, in this case we can write down a compatible metric explicitly. By Proposition B.10.24, the topology τ_K is induced by the translation-invariant metric

$$d_K(\varphi, \psi) := \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{\|\varphi - \psi\|_{C^j(K)}}{1 + \|\varphi - \psi\|_{C^j(K)}}, \qquad \varphi, \psi \in \mathcal{D}_K(\Omega).$$

By Proposition B.10.23, a sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in $\mathcal{D}_K(\Omega)$ if and only if

$$\|\varphi_n - \varphi\|_{C^j(K)} \to 0 \qquad \text{for every } j \in \mathbb{N}_0.$$

This means that $\varphi_n \to \varphi$ if and only if, for every multi-index α, the sequence $(D^\alpha \varphi_n)_{n\in\mathbb{N}}$ converges uniformly on K to $D^\alpha \varphi$. If $\|\varphi_n - \varphi\|_{C^j(K)} \to 0$ for all j, then, given a multi-index α, it suffices to take $j \geq |\alpha|$ to obtain

$$\|D^\alpha \varphi_n - D^\alpha \varphi\|_{C^0(K)} \leq \|\varphi_n - \varphi\|_{C^j(K)} \to 0.$$

Conversely, if for every multi-index α the sequence $(D^\alpha \varphi_n)_{n\in\mathbb{N}}$ converges uniformly on K to $D^\alpha \varphi$, then, for each $j \in \mathbb{N}_0$, the set of multi-indices α with $|\alpha| \leq j$ is finite. Therefore,

$$\|\varphi_n - \varphi\|_{C^j(K)} = \max_{|\alpha|\leq j} \|D^\alpha \varphi_n - D^\alpha \varphi\|_{C^0(K)} \to 0.$$

Likewise, a sequence $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{D}_K(\Omega)$ if and only if

$$\|\varphi_n - \varphi_m\|_{C^j(K)} \to 0 \qquad \text{as } n, m \to \infty,$$

for every $j \in \mathbb{N}_0$. Equivalently, for every multi-index α, the sequence $(D^\alpha \varphi_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on K.

4.2 Projective limits

A natural question is whether the space $\mathcal{D}_K(\Omega)$ is complete under the topology τ_K we have just constructed. It is indeed complete. Moreover, $\mathcal{D}_K(\Omega)$ is not only a locally convex space generated by a countable family of norms, but can also be described as a *projective limit of Banach spaces*.

The same description by projective limits will be used again for spaces of test sections of smooth vector bundles and for the space of rapidly decreasing functions, usually called the *Schwartz class*. The latter will be the test function space for tempered distributions.

From the topological point of view, the projective topology we will use is not a new construction: it is precisely an initial topology in the sense of Definition B.10.39. We call it *projective* because it is associated with a family of linear maps into known topological spaces.

Definition 4.2.1. (Projective topology) Let X be a vector space, let $\{X_i\}_{i\in I}$ be a family of topological vector spaces, and, for each $i \in I$, let $T_i \colon X \longrightarrow X_i$ be a linear map. The *projective topology* on X associated with the family $(T_i)_{i\in I}$ is the initial topology determined by these maps.
Equivalently, it is the coarsest topology on X for which all the maps $T_i \colon X \longrightarrow X_i$ are continuous. A neighborhood base at the origin for this topology consists of the sets

$$\bigcap_{q=1}^{r} T_{i_q}^{-1}(U_{i_q}),$$

where $r \in \mathbb{N}$, $i_1, \ldots, i_r \in I$, and each U_{i_q} is a neighborhood of the origin in X_{i_q}.

Proposition 4.2.2. Let X be a vector space, let $\{X_i\}_{i\in I}$ be a family of topological vector spaces, and, for each $i \in I$, let $T_i \colon X \longrightarrow X_i$ be a linear map. Then the projective topology associated with $(T_i)_{i\in I}$ makes X a topological vector space.
Moreover:

(a) if each X_i is locally convex, then X is locally convex;

(b) if each X_i is Hausdorff and the family $(T_i)_{i\in I}$ separates points of X, then X is Hausdorff.

Proof. Denote the projective topology on X by τ_{proj}.
We first prove that addition is continuous. Since it suffices to check continuity on basic neighborhoods, let $x, y \in X$ and let W be a basic neighborhood of $x+y$ in τ_{proj}. We can write

$$W = \bigcap_{q=1}^{r} T_{i_q}^{-1}(U_q),$$

where, for each $q \in \{1,\dots,r\}$, the set U_q is a neighborhood of $T_{i_q}(x+y)$ in X_{i_q}.
Since addition is continuous in X_{i_q}, there exist neighborhoods V_q of $T_{i_q}(x)$ and W_q of $T_{i_q}(y)$ such that $V_q + W_q \subseteq U_q$. We define

$$V := \bigcap_{q=1}^{r} T_{i_q}^{-1}(V_q), \qquad W' := \bigcap_{q=1}^{r} T_{i_q}^{-1}(W_q).$$

Then V is a neighborhood of x and W' is a neighborhood of y. If $u \in V$ and $v \in W'$, then, for each $q \in \{1,\dots,r\}$, we have $T_{i_q}(u) \in V_q$ and $T_{i_q}(v) \in W_q$. Since T_{i_q} is linear,

$$T_{i_q}(u+v) = T_{i_q}(u) + T_{i_q}(v) \in V_q + W_q \subseteq U_q.$$

Therefore, $u+v \in W$. Thus, $V + W' \subseteq W$, and addition is continuous.
We now show that scalar multiplication is continuous. Since it suffices to check continuity on basic neighborhoods, let $\lambda \in \mathbb{K}$, $x \in X$, and let W be a basic neighborhood of λx. We write

$$W = \bigcap_{q=1}^{r} T_{i_q}^{-1}(U_q),$$

where U_q is a neighborhood of $\lambda T_{i_q}(x)$ in X_{i_q}. Since scalar multiplication is continuous in X_{i_q}, there exist a neighborhood A_q of λ in $\mathbb{K}$ and a neighborhood V_q of $T_{i_q}(x)$ in X_{i_q} such that $A_q V_q \subseteq U_q$. We define

$$A := \bigcap_{q=1}^{r} A_q, \qquad V := \bigcap_{q=1}^{r} T_{i_q}^{-1}(V_q).$$

Then A is a neighborhood of λ in $\mathbb{K}$ and V is a neighborhood of x in X. If $\mu \in A$ and $u \in V$, then, for each $q \in \{1,\dots,r\}$, we have $\mu \in A_q$ and $T_{i_q}(u) \in V_q$. Therefore,

$$T_{i_q}(\mu u) = \mu T_{i_q}(u) \in A_q V_q \subseteq U_q.$$

Thus, $\mu u \in W$. We conclude that scalar multiplication is continuous.

This proves that (X, τ_{proj}) is a topological vector space.
We prove (a). Suppose that each X_i is locally convex. For each $i \in I$, take a neighborhood base at the origin in X_i consisting of convex balanced sets. Then the basic neighborhoods of the origin in X may be taken to have the form

$$\bigcap_{q=1}^{r} T_{i_q}^{-1}(U_{i_q}),$$

where each U_{i_q} is convex and balanced. Since the inverse image of a convex balanced set under a linear map retains both properties, and since finite intersections of convex balanced sets are also convex and balanced, we conclude that X is locally convex.
We prove (b). Suppose that each X_i is Hausdorff and that the family $(T_i)_{i\in I}$ separates points of X. Then

$$\bigcap_{i\in I} \ker(T_i) = \{0\}.$$

Since X_i is Hausdorff, the set $\{0\}$ is closed in X_i. Since T_i is continuous, $\ker(T_i) = T_i^{-1}(\{0\})$ is closed in X. Therefore, $\{0\}$ is closed in X, being an intersection of closed sets. By Proposition B.10.20, we conclude that X is Hausdorff. □

The general formulation of projective limits uses systems of spaces and transition maps. In the applications in this book, we will mainly encounter the concrete case in which a space is realized as an intersection of Banach spaces contained in a common topological vector space.

Definition 4.2.3. (Projective limit realized as an intersection) Let Y be a Hausdorff topological vector space and let $\{X_n\}_{n\in\mathbb{N}}$ be a family of Banach spaces such that each X_n is a vector subspace of Y and the inclusion $X_n \hookrightarrow Y$ is continuous. We define

$$X := \bigcap_{n\in\mathbb{N}} X_n.$$

For each $n \in \mathbb{N}$, denote the natural inclusion by $\iota_n \colon X \longrightarrow X_n$ and equip X with the projective topology associated with the family $(\iota_n)_{n\in\mathbb{N}}$.
In this setting we write

$$X = \varprojlim_{n\in\mathbb{N}} X_n$$

and say that X is the *projective limit* of the family $\{X_n\}_{n\in\mathbb{N}}$. Some texts also use the term *inverse limit*.

Proposition 4.2.4. (Sequential completeness of a projective intersection) Let Y be a Hausdorff topological vector space and let $\{X_n\}_{n\in\mathbb{N}}$ be a family of Banach spaces such that each X_n is a vector subspace of Y and the inclusion $X_n \hookrightarrow Y$ is continuous. Consider

$$X = \bigcap_{n\in\mathbb{N}} X_n$$

equipped with the projective topology associated with the inclusions $\iota_n \colon X \longrightarrow X_n$. Then X is sequentially complete: every Cauchy sequence in X converges in X.

Proof. Let $(x^{(q)})_{q\in\mathbb{N}}$ be a Cauchy sequence in X with respect to the projective topology. Fix $n \in \mathbb{N}$. We will prove that $(x^{(q)})_{q\in\mathbb{N}}$ is a Cauchy sequence in X_n.
Let U_n be a neighborhood of the origin in X_n. Since $\iota_n \colon X \longrightarrow X_n$ is continuous by definition of the projective topology, the set $\iota_n^{-1}(U_n)$ is a neighborhood of the origin in X. Since $(x^{(q)})_{q\in\mathbb{N}}$ is Cauchy in X, there exists $q_0 \in \mathbb{N}$ such that

$$x^{(q)} - x^{(p)} \in \iota_n^{-1}(U_n) \qquad \forall q, p \geq q_0.$$

Therefore,

$$x^{(q)} - x^{(p)} \in U_n \qquad \forall q, p \geq q_0.$$

This proves that $(x^{(q)})_{q\in\mathbb{N}}$ is Cauchy in X_n.
Since X_n is Banach, there exists $x_n \in X_n$ such that $x^{(q)} \to x_n$ in X_n as $q \to \infty$.
We show that the limits x_n do not depend on n when regarded as elements of Y. Let $n, m \in \mathbb{N}$. Since the inclusions $X_n \hookrightarrow Y$ and $X_m \hookrightarrow Y$ are continuous, $x^{(q)} \to x_n$ in X_n implies $x^{(q)} \to x_n$ in Y, and $x^{(q)} \to x_m$ in X_m implies $x^{(q)} \to x_m$ in Y. Since Y is Hausdorff, the limit of a convergent sequence is unique. Consequently, $x_n = x_m$ in Y.
There is therefore a unique element $x \in Y$ such that $x = x_n$ for every $n \in \mathbb{N}$. Since $x_n \in X_n$ for each n, we conclude that $x \in \bigcap_{n\in\mathbb{N}} X_n = X$.
It remains to prove that $x^{(q)} \to x$ in the projective topology of X. Let V be a neighborhood of the origin in X. By definition of this topology, there is a basic neighborhood

$$V_0 = \bigcap_{s=1}^{r} \iota_{n_s}^{-1}(U_{n_s})$$

such that $V_0 \subseteq V$, where $r \in \mathbb{N}$, $n_1, \ldots, n_r \in \mathbb{N}$, and each U_{n_s} is a neighborhood of the origin in X_{n_s}.
For each $s \in \{1, \ldots, r\}$, we know that $x^{(q)} \to x$ in X_{n_s}. Therefore, there exists $q_s \in \mathbb{N}$ such that $x^{(q)} - x \in U_{n_s}$ for every $q \geq q_s$. Taking $q_* := \max\{q_1, \ldots, q_r\}$, we obtain, for every $q \geq q_*$,

$$x^{(q)} - x \in \bigcap_{s=1}^{r} \iota_{n_s}^{-1}(U_{n_s}) = V_0 \subseteq V.$$

Consequently, $x^{(q)} \to x$ in X. We conclude that X is sequentially complete. □

Remark 4.2.5. If completeness of a topological vector space is formulated in terms of Cauchy nets or filters, the same argument shows that $\varprojlim_{n\in\mathbb{N}} X_n$ is complete in that sense. In the immediate applications, the sequential formulation will suffice, since the spaces we will study are metrizable.

We now return to the space $\mathcal{D}_K(\Omega)$. For each $j \in \mathbb{N}_0$, we define

$$C_K^j(\Omega) := \left\{ \varphi \in C^j(\Omega) \;\middle|\; \operatorname{supp}(\varphi) \subseteq K \right\},$$

and equip it with the norm

$$\|\varphi\|_{C^j(K)} := \max_{|\alpha|\leq j} \sup_{x\in K} |D^\alpha \varphi(x)|.$$

Lemma 4.2.6. For each $j \in \mathbb{N}_0$, the space $\left(C_K^j(\Omega), \|\cdot\|_{C^j(K)}\right)$ is a Banach space.

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C_K^j(\Omega)$. Fix a multi-index α with $|\alpha| \le j$.
If $x \in \Omega \setminus K$, then, since $\operatorname{supp}(\varphi_n) \subseteq K$, we have $x \notin \operatorname{supp}(\varphi_n)$. Therefore, there is an open neighborhood of x on which φ_n vanishes. Consequently, $D^\alpha \varphi_n(x) = 0$. Thus, $D^\alpha \varphi_n$ vanishes on $\Omega \setminus K$ for every $n \in \mathbb{N}$.
Moreover, for any $n, m \in \mathbb{N}$,

$$\|D^\alpha \varphi_n - D^\alpha \varphi_m\|_{L^\infty(\Omega)} = \|D^\alpha \varphi_n - D^\alpha \varphi_m\|_{C^0(K)} \le \|\varphi_n - \varphi_m\|_{C^j(K)}.$$

Since $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy in $C_K^j(\Omega)$, it follows that $(D^\alpha \varphi_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on Ω.
We now define the corresponding limit. For each $x \in \Omega$, the real sequence $(D^\alpha \varphi_n(x))_{n\in\mathbb{N}}$ is Cauchy and therefore converges in $\mathbb{R}$. We define

$$g_\alpha(x) := \lim_{n\to\infty} D^\alpha \varphi_n(x), \qquad x \in \Omega.$$

We show that the convergence is uniform. Let $\varepsilon > 0$. Since $(D^\alpha \varphi_n)_{n\in\mathbb{N}}$ is uniformly Cauchy, there exists $N \in \mathbb{N}$ such that, if $n, m \ge N$, then

$$\|D^\alpha \varphi_n - D^\alpha \varphi_m\|_{L^\infty(\Omega)} < \varepsilon.$$

Fix $n \ge N$ and let $m \to \infty$. For each $x \in \Omega$,

$$|D^\alpha \varphi_n(x) - g_\alpha(x)| \le \varepsilon.$$

Taking the supremum over $x \in \Omega$, we obtain

$$\|D^\alpha \varphi_n - g_\alpha\|_{L^\infty(\Omega)} \le \varepsilon.$$

Therefore, $D^\alpha \varphi_n \to g_\alpha$ uniformly on Ω.
Since each $D^\alpha \varphi_n$ is continuous and the convergence is uniform, g_α is continuous on Ω. Moreover, since $D^\alpha \varphi_n = 0$ on $\Omega \setminus K$ for every n, passing to the limit gives

$$g_\alpha = 0 \qquad \text{in } \Omega \setminus K.$$

We define $\varphi := g_0$. We will prove that $\varphi \in C_K^j(\Omega)$ and that $D^\alpha \varphi = g_\alpha$ for every multi-index α with $|\alpha| \le j$.
We begin by showing compatibility between the limits. We claim that, if β is a multi-index with $|\beta| \le j-1$ and $i \in \{1, \dots, n\}$, then

$$\partial_i g_\beta = g_{\beta+e_i} \qquad \text{in } \Omega.$$

Fix $x \in \Omega$. Since Ω is open, there exists $\delta > 0$ such that $x + te_i \in \Omega$ for every $t \in [-\delta, \delta]$. If $h \in (-\delta, \delta)$, then, for each $n \in \mathbb{N}$, Theorem B.6.5 applied to the function $t \mapsto D^\beta \varphi_n(x + te_i)$ gives

$$D^\beta \varphi_n(x + he_i) - D^\beta \varphi_n(x) = \int_0^h D^{\beta+e_i} \varphi_n(x + te_i)\, dt.$$

Let $n \to \infty$. By the uniform convergence $D^\beta \varphi_n \to g_\beta$, the left-hand side converges to

$$g_\beta(x + he_i) - g_\beta(x).$$

On the other hand,

$$\left| \int_0^h \left(D^{\beta+e_i}\varphi_n(x+te_i) - g_{\beta+e_i}(x+te_i) \right) dt \right| \le |h| \| D^{\beta+e_i}\varphi_n - g_{\beta+e_i} \|_{L^\infty(\Omega)}.$$

Since $D^{\beta+e_i}\varphi_n \to g_{\beta+e_i}$ uniformly on Ω, the right-hand side converges to 0. Therefore,

$$g_\beta(x + he_i) - g_\beta(x) = \int_0^h g_{\beta+e_i}(x+te_i)\, dt.$$

If $h \neq 0$, we divide by h:

$$\frac{g_\beta(x + he_i) - g_\beta(x)}{h} = \frac{1}{h} \int_0^h g_{\beta+e_i}(x+te_i)\, dt.$$

Since $g_{\beta+e_i}$ is continuous at x, we have

$$\left| \frac{1}{h} \int_0^h g_{\beta+e_i}(x+te_i)\, dt - g_{\beta+e_i}(x) \right| \le \sup_{t \in [-|h|,|h|]} |g_{\beta+e_i}(x+te_i) - g_{\beta+e_i}(x)| \longrightarrow 0$$

as $h \to 0$. Consequently,

$$\partial_i g_\beta(x) = g_{\beta+e_i}(x).$$

Since $x \in \Omega$ was arbitrary, we conclude that $\partial_i g_\beta = g_{\beta+e_i}$ on Ω.
Using this relation, we will prove by induction on $r \in \{0, \dots, j\}$ that:

$$\varphi \in C^r(\Omega) \qquad \text{and} \qquad D^\alpha \varphi = g_\alpha \quad \text{for every } |\alpha| \le r.$$

For $r = 0$, this is immediate, since $\varphi = g_0$ and g_0 is continuous.
Now suppose that the assertion holds for some $r \in \{0, \dots, j-1\}$. We will prove that it holds for $r+1$. Let α be a multi-index with $|\alpha| = r+1$. Choose $i \in \{1, \dots, n\}$ such that $\alpha_i \ge 1$ and write $\alpha = \beta + e_i$, where $|\beta| = r$.
By the induction hypothesis, $D^\beta \varphi = g_\beta$. Since $|\beta| = r \le j-1$, the identity proved above gives

$$\partial_i g_\beta = g_{\beta+e_i} = g_\alpha.$$

Therefore,

$$D^\alpha \varphi = \partial_i (D^\beta \varphi) = \partial_i g_\beta = g_\alpha.$$

Moreover, g_α is continuous. Since this holds for every multi-index α with $|\alpha| = r+1$, we conclude that $\varphi \in C^{r+1}(\Omega)$ and that $D^\alpha \varphi = g_\alpha$ for every $|\alpha| \le r+1$.
By induction, $\varphi \in C^j(\Omega)$ and $D^\alpha \varphi = g_\alpha$ for every multi-index α with $|\alpha| \le j$.
Since $g_0 = 0$ on $\Omega \setminus K$, we have $\varphi = 0$ on $\Omega \setminus K$. Since K is closed in Ω, we conclude that $\operatorname{supp}(\varphi) \subseteq K$. Therefore, $\varphi \in C_K^j(\Omega)$.
Finally, for every multi-index α with $|\alpha| \le j$, we already know that $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on Ω, and in

particular uniformly on K. Since the set of multi-indices α with $|\alpha| \le j$ is finite,

$$\|\varphi_n - \varphi\|_{C^j(K)} = \max_{|\alpha|\le j} \|D^\alpha \varphi_n - D^\alpha \varphi\|_{C^0(K)} \longrightarrow 0.$$

Thus every Cauchy sequence in $C_K^j(\Omega)$ converges in $C_K^j(\Omega)$. We conclude that $\left(C_K^j(\Omega), \|\cdot\|_{C^j(K)}\right)$ is a Banach space. □

Now observe that

$$\mathcal{D}_K(\Omega) = \bigcap_{j\in\mathbb{N}_0} C_K^j(\Omega).$$

If $\varphi \in \mathcal{D}_K(\Omega)$, then $\varphi \in C_K^j(\Omega)$ for every $j \in \mathbb{N}_0$. Conversely, if $\varphi \in \bigcap_{j\in\mathbb{N}_0} C_K^j(\Omega)$, then φ is of class C^j for every $j \in \mathbb{N}_0$, so $\varphi \in C^\infty(\Omega)$. Moreover, $\operatorname{supp}(\varphi) \subseteq K$. Therefore, $\varphi \in \mathcal{D}_K(\Omega)$.

For each $j \in \mathbb{N}_0$, consider the natural inclusion $\iota_j \colon \mathcal{D}_K(\Omega) \longrightarrow C_K^j(\Omega)$. Equip $\mathcal{D}_K(\Omega)$ with the projective topology associated with the family $(\iota_j)_{j\in\mathbb{N}_0}$, and denote it by τ_K^{proj}.

A neighborhood base at the origin for τ_K^{proj} consists of finite intersections of sets of the form

$$\iota_j^{-1}\left(\left\{\varphi \in C_K^j(\Omega) \;\middle|\; \|\varphi\|_{C^j(K)} < r\right\}\right), \qquad j \in \mathbb{N}_0, \quad r > 0.$$

Since ι_j is the natural inclusion, these sets agree with

$$\left\{\varphi \in \mathcal{D}_K(\Omega) \;\middle|\; \|\varphi\|_{C^j(K)} < r\right\}.$$

Therefore, a neighborhood base at the origin for τ_K^{proj} consists of the sets

$$\bigcap_{q=1}^{k} \left\{\varphi \in \mathcal{D}_K(\Omega) \;\middle|\; \|\varphi\|_{C^{j_q}(K)} < r_q\right\},$$

where $k \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}_0$, and $r_1, \dots, r_k > 0$.

This family is the neighborhood base at the origin defining the topology τ_K constructed earlier from the family of norms $\{\|\cdot\|_{C^j(K)}\}_{j\in\mathbb{N}_0}$. Consequently, $\tau_K^{\text{proj}} = \tau_K$.

In other words,

$$\left(\mathcal{D}_K(\Omega), \tau_K\right) = \varprojlim_{j\in\mathbb{N}_0} C_K^j(\Omega).$$

Since each $C_K^j(\Omega)$ is a Banach space and the family is indexed by $\mathbb{N}_0$, we can reindex it using $n \mapsto n-1$ to write it as a family indexed by $\mathbb{N}$. Proposition 4.2.4 then implies that $\mathcal{D}_K(\Omega)$ is sequentially complete with respect to τ_K.

It remains to translate this sequential completeness into completeness of the metric d_K already constructed. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to d_K. Fix $j \in \mathbb{N}_0$ and $\varepsilon > 0$. Since $\dfrac{1}{2^j}\dfrac{\varepsilon}{1+\varepsilon} > 0$, there exists $N \in \mathbb{N}$ such that, if $n, m \ge N$, then

$$d_K(\varphi_n, \varphi_m) < \frac{1}{2^j}\frac{\varepsilon}{1+\varepsilon}.$$

By definition of d_K,

$$\frac{1}{2^j}\frac{\|\varphi_n-\varphi_m\|_{C^j(K)}}{1+\|\varphi_n-\varphi_m\|_{C^j(K)}} \le d_K(\varphi_n,\varphi_m).$$

Therefore,

$$\frac{\|\varphi_n-\varphi_m\|_{C^j(K)}}{1+\|\varphi_n-\varphi_m\|_{C^j(K)}} < \frac{\varepsilon}{1+\varepsilon}.$$

Since the function $t \mapsto \dfrac{t}{1+t}$ is strictly increasing on $[0,\infty)$, we conclude that $\|\varphi_n-\varphi_m\|_{C^j(K)} < \varepsilon$ for any $n,m \ge N$.

We have proved that, for each $j \in \mathbb{N}_0$, $\|\varphi_n-\varphi_m\|_{C^j(K)} \to 0$ as $n,m \to \infty$. By Proposition B.10.23, $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy with respect to τ_K. Since $\mathcal{D}_K(\Omega)$ is sequentially complete for this topology, there exists $\varphi \in \mathcal{D}_K(\Omega)$ such that $\varphi_n \to \varphi$ in τ_K.

Finally, since the metric d_K induces the topology τ_K, it follows that $d_K(\varphi_n,\varphi) \to 0$. Thus every Cauchy sequence with respect to d_K converges in $\mathcal{D}_K(\Omega)$, and therefore d_K is complete.

We conclude that $(\mathcal{D}_K(\Omega),\tau_K)$ is a locally convex, Hausdorff, metrizable, and complete space. By Theorem B.10.38, $\mathcal{D}_K(\Omega)$ is a Fréchet space.

4.3 The topology of $\mathcal{D}(\Omega)$

The construction in this section follows the treatment in [97, Section 10.1]. To construct a topology on $C_c^\infty(\Omega)$, let $\mathcal{B}_0$ be the collection of subsets $U \subseteq C_c^\infty(\Omega)$ such that:

(a) $0 \in U$,

(b) U is balanced[1] and convex,

(c) for every compact set $K \subseteq \Omega$, the set $U \cap \mathcal{D}_K(\Omega)$ is a neighborhood of the origin in $(\mathcal{D}_K(\Omega),\tau_K)$.

The collection $\mathcal{B}_0$ is designed to serve as a neighborhood base at the origin in $C_c^\infty(\Omega)$. From it, we define

$$\mathcal{B} := \{\varphi + V \mid \varphi \in C_c^\infty(\Omega),\ V \in \mathcal{B}_0\}.$$

These conditions determine the topology through its neighborhoods.

Theorem 4.3.1. Let $\Omega \subseteq \mathbb{R}^n$ be open. There is a unique topology τ_{LF} on $C_c^\infty(\Omega)$ for which $\mathcal{B}_0$ is a neighborhood base at the origin and the vector space operations are continuous. This topology is Hausdorff and locally convex. For each $\varphi \in C_c^\infty(\Omega)$, the family $\varphi+\mathcal{B}_0$ is a neighborhood base at φ; a set O is open if and only if for each $\varphi \in O$ there exists $V \in \mathcal{B}_0$ with $\varphi + V \subseteq O$. We denote the topological vector space $(C_c^\infty(\Omega),\tau_{LF})$ by $\mathcal{D}(\Omega)$ and call its elements *test functions*.

Proof. The family $\mathcal{B}_0$ is nonempty, since it contains the entire space. Each $U \in \mathcal{B}_0$ is absorbing: if $\varphi \in C_c^\infty(\Omega)$, take a compact set K containing its support. The neighborhood $U \cap \mathcal{D}_K(\Omega)$ absorbs φ in that space, and hence U also absorbs it in $C_c^\infty(\Omega)$. Moreover, $\mathcal{B}_0$ is stable under finite intersections and multiplication by positive scalars. Indeed, these operations preserve convexity and balancedness, and the intersections with each $\mathcal{D}_K(\Omega)$ remain neighborhoods of the origin. Proposition B.10.26 therefore yields a unique locally convex topology with neighborhood base $\mathcal{B}_0$.

[1] A subset U of a vector space is said to be balanced if for every scalar λ with $|\lambda| \le 1$ we have $\lambda U \subseteq U$.

To check that it is Hausdorff, consider

$$V_\varepsilon := \left\{ \varphi \in C_c^\infty(\Omega) \;\middle|\; \|\varphi\|_{L^\infty(\Omega)} < \varepsilon \right\}, \qquad \varepsilon > 0.$$

This set belongs to $\mathcal{B}_0$, because its intersection with $\mathcal{D}_K(\Omega)$ is the open ball of radius ε for the uniform norm. If $\varphi \neq \psi$, choose $3\varepsilon < \|\varphi - \psi\|_{L^\infty(\Omega)}$. The neighborhoods $\varphi + V_\varepsilon$ and $\psi + V_\varepsilon$ are disjoint: otherwise, the triangle inequality would give a uniform distance less than 2ε between φ and ψ. Their interiors are disjoint open neighborhoods. The description of the open sets and uniqueness follow from Proposition B.10.2. □

The norms introduced earlier depend on the compact set K containing the support of the functions. A natural question is whether it is possible to avoid this dependence and define a topology directly on $C_c^\infty(\Omega)$ using global norms.

With this in mind, we could consider, for each $j \in \mathbb{N}_0$, the function

$$\|\varphi\|_j := \sup_{\substack{x \in \Omega \\ |\alpha| \leq j}} |D^\alpha \varphi(x)|. \tag{4.1}$$

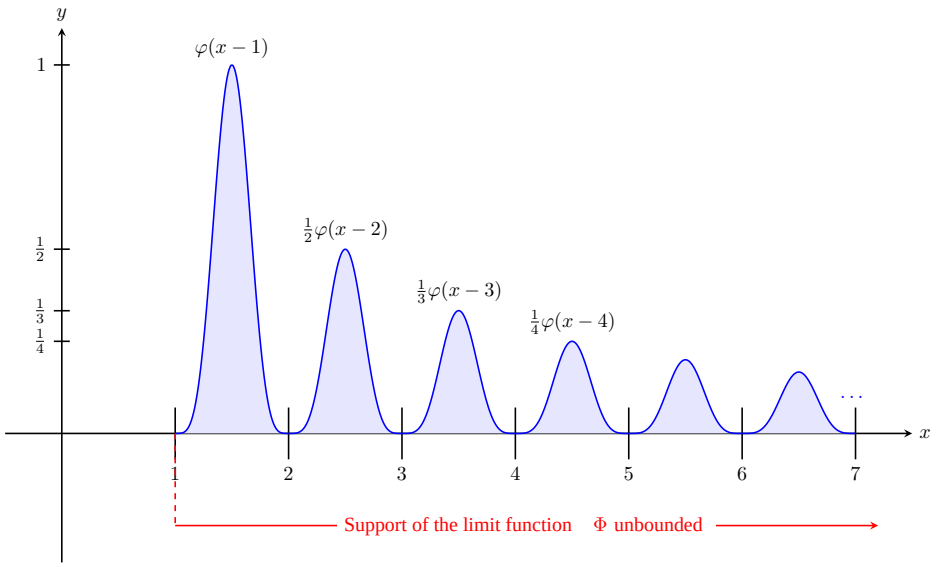
Observe that this supremum is finite, since every function $\varphi \in C_c^\infty(\Omega)$ has compact support. Moreover, each $\|\cdot\|_j$ defines a norm on $C_c^\infty(\Omega)$.

We could then try to define a topology on $C_c^\infty(\Omega)$ as the topology induced by the family of norms $\{\|\cdot\|_j\}_{j \in \mathbb{N}_0}$. However, this construction is unsuitable. Intuitively, these norms do not uniformly control the behavior of the supports of the functions, which leads to a topology that is too weak.

In particular, the resulting space is not complete, as the following example shows.

Example 4.3.2. Consider $\Omega = \mathbb{R}$, and let $\varphi \in C_c^\infty(\mathbb{R})$ be a function such that $\operatorname{supp}(\varphi) \subseteq [0,1]$ and $\varphi(x) > 0$ for every $x \in (0,1)$. Define the sequence of functions $(\varphi_k)_{k \in \mathbb{N}}$ by

$$\varphi_k(x) := \sum_{m=1}^{k} \frac{1}{m} \varphi(x-m) = \varphi(x-1) + \frac{1}{2}\varphi(x-2) + \cdots + \frac{1}{k}\varphi(x-k).$$



We will show that this is a Cauchy sequence with respect to the topology generated by the family of seminorms

$$\|\cdot\|_j = \sup_{\substack{x\in\mathbb{R}\\ i\le j}} |D^i(\cdot)(x)|,$$

but its limit does not have compact support, concluding that $C_c^\infty(\mathbb{R})$ is not complete under this topology.
Let $l,k\in\mathbb{N}$ with $k>l$. The difference between the terms of the sequence is

$$\varphi_k(x)-\varphi_l(x)=\sum_{m=l+1}^{k}\frac{1}{m}\varphi(x-m).$$

Since $\operatorname{supp}(\varphi)\subseteq[0,1]$, the support of each translated term $\varphi(x-m)$ is contained in the interval $[m,m+1]$. The interiors of these intervals are disjoint and all derivatives of φ vanish at their endpoints. Therefore, at each point there is at most one nonzero term, even after differentiation.
We evaluate the jth seminorm of the difference. For any derivative of order $i\le j$, we have

$$|D^i(\varphi_k-\varphi_l)(x)|=\left|\sum_{m=l+1}^{k}\frac{1}{m}D^i\varphi(x-m)\right|.$$

Since at each point there is at most one nonzero summand, the supremum over $\mathbb{R}$ is obtained by maximizing over each interval $[m,m+1]$, giving

$$\|D^i(\varphi_k-\varphi_l)\|_{L^\infty(\mathbb{R})}=\max_{l+1\le m\le k}\left(\frac{1}{m}\|D^i\varphi\|_{L^\infty(\mathbb{R})}\right).$$

Since $\varphi\in C_c^\infty(\mathbb{R})$, its derivatives are bounded, so

$$C_j:=\max_{i\le j}\|D^i\varphi\|_{L^\infty(\mathbb{R})}<\infty.$$

Then

$$\|\varphi_k-\varphi_l\|_j\le\frac{1}{l+1}C_j.$$

As $l,k\to\infty$, the quantity $\dfrac{1}{l+1}C_j$ tends to zero, which shows that $(\varphi_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to each seminorm in the family.
In particular, for $j=0$ we obtain

$$\|\varphi_k-\varphi_l\|_{L^\infty(\mathbb{R})}\le\frac{C_0}{l+1}\to 0,\qquad \text{if } l,k\to\infty,$$

so (φ_k) is uniformly Cauchy on $\mathbb{R}$, and consequently there exists a bounded function Φ such that

$$\|\varphi_k-\Phi\|_{L^\infty(\mathbb{R})}\longrightarrow 0,\qquad k\to\infty,$$

that is, $\varphi_k\to\Phi$ uniformly on $\mathbb{R}$.
The same argument applied to each derivative shows that for every $i\in\mathbb{N}_0$ the sequence $(D^i\varphi_k)$ is uniformly

Cauchy on $\mathbb{R}$, and therefore there exists a continuous bounded function ψ_i such that

$$\|D^i\varphi_k - \psi_i\|_{L^\infty(\mathbb{R})} \longrightarrow 0, \qquad k \to \infty.$$

Since also $\varphi_k(0) = 0$ for every k, the sequence converges pointwise on 0. The theorem on uniform convergence of derivatives implies that $\Phi \in C^\infty(\mathbb{R})$ and that

$$D^i\Phi(x) = \psi_i(x) \qquad \text{for every } x \in \mathbb{R},\ i \in \mathbb{N}_0.$$

Moreover, since φ_k are the partial sums of the series, we have

$$\Phi(x) = \sum_{m=1}^{\infty} \frac{1}{m}\varphi(x-m).$$

To prove convergence in each seminorm, fix $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$. For every $l > k$, the triangle inequality gives

$$\|\varphi_k - \Phi\|_j \le \|\varphi_k - \varphi_l\|_j + \|\varphi_l - \Phi\|_j.$$

By the estimate already obtained, for every $l > k$ we have

$$\|\varphi_k - \varphi_l\|_j \le \frac{C_j}{k+1}.$$

On the other hand, since $D^i\varphi_l \to D^i\Phi$ uniformly on $\mathbb{R}$ for every $i \le j$, we have

$$\|\varphi_l - \Phi\|_j = \max_{i\le j} \|D^i\varphi_l - D^i\Phi\|_{L^\infty(\mathbb{R})} \longrightarrow 0, \qquad l \to \infty.$$

Letting $l \to \infty$ in the preceding triangle inequality gives

$$\|\varphi_k - \Phi\|_j \le \frac{C_j}{k+1} \to 0, \qquad \text{if } k \to \infty.$$

Since j was arbitrary, we conclude that for each $j \in \mathbb{N}_0$ we have $\|\varphi_k - \Phi\|_j \to 0$ as $k \to \infty$. This shows that φ_k converges to Φ with respect to each seminorm in the family, and therefore φ_k converges to Φ with respect to the topology generated by this family of seminorms.

To analyze the support of Φ, note that by hypothesis $\varphi > 0$ on $(0,1)$. For each $m \in \mathbb{N}$, choose the point $x_m = m + \frac{1}{2}$. Evaluating the limit function at this point gives

$$\Phi\left(m + \frac{1}{2}\right) = \frac{1}{m}\varphi\left(\frac{1}{2}\right) > 0.$$

This implies that Φ takes strictly positive values at arbitrarily large points, so its support is unbounded. Consequently, $\operatorname{supp}(\Phi)$ is not compact and $\Phi \notin C_c^\infty(\mathbb{R})$.

Although the family of seminorms $\{\|\cdot\|_j\}_{j\in\mathbb{N}_0}$ does not generate the topology of $\mathcal{D}(\Omega)$—since these seminorms do not control the supports of the functions—it is possible to describe the topology τ_{LF} using suitable families of seminorms that simultaneously incorporate the behavior of the derivatives and of the supports. The following exercise presents one such construction.

Exercise 4.3.3. Let $\Omega \subseteq \mathbb{R}^n$ be open. Consider a sequence $\{K_l\}_{l\in\mathbb{N}_0}$ of compact sets in Ω, with $K_0 := \varnothing$, such that for $l \geq 1$ the sequence is increasing and nested, $K_l \subseteq \operatorname{int}(K_{l+1})$, and $\Omega = \bigcup_{l=1}^{\infty} K_l$.

Let $\mathbf{m} := \{m_l\}_{l\in\mathbb{N}_0}$ be a sequence in $\mathbb{N}_0$ such that $m_l \to \infty$, and let $\mathbf{a} := \{a_l\}_{l\in\mathbb{N}_0}$ be a sequence in $(0,\infty)$ such that $a_l \to 0$. For each $\varphi \in \mathcal{D}(\Omega)$ we define

$$p_{\mathbf{m},\mathbf{a}}(\varphi) := \sup_{l\in\mathbb{N}_0} \sup_{x\in\Omega\setminus K_l} \frac{1}{a_l} \sum_{|\alpha|\leq m_l} |D^\alpha \varphi(x)|.$$

(a) Prove that $p_{\mathbf{m},\mathbf{a}}$ is a seminorm on $\mathcal{D}(\Omega)$.

(b) Prove that the family of seminorms $\{p_{\mathbf{m},\mathbf{a}}\}_{\mathbf{m},\mathbf{a}}$ generates the topology τ_{LF} of Theorem 4.3.1.

Observe that the preceding family of seminorms is, in general, uncountable. This reflects the fact that $\mathcal{D}(\Omega)$ will turn out not to be metrizable (as the reader will justify later) and, in particular, is not a Fréchet space.

Fixing the support recovers exactly the topology we already defined on $\mathcal{D}_K(\Omega)$.

Theorem 4.3.4. Let $\Omega \subseteq \mathbb{R}^n$ be open. For any compact set $K \subseteq \Omega$, the topology τ_K agrees with the subspace topology on $\mathcal{D}_K(\Omega)$ induced by τ_{LF} as a subset of $\mathcal{D}(\Omega)$.

Proof. The inclusion $\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}(\Omega)$ is continuous: the inverse image of each $U \in \mathcal{B}_0$ is, by definition, a neighborhood of the origin in $\mathcal{D}_K(\Omega)$. Thus the induced topology is no finer than τ_K.
Conversely, for $j \in \mathbb{N}_0$ and $\varepsilon > 0$, the ball

$$V_{j,\varepsilon} := \{\varphi \in C_c^\infty(\Omega) \mid \|\varphi\|_j < \varepsilon\}$$

belongs to $\mathcal{B}_0$: on any compact support, the global norm in (4.1) agrees with the norm corresponding to that compact set. Hence

$$V_{j,\varepsilon} \cap \mathcal{D}_K(\Omega) = \{\varphi \in \mathcal{D}_K(\Omega) \mid \|\varphi\|_{C^j(K)} < \varepsilon\}$$

is a neighborhood for the induced topology. These balls form a neighborhood base for τ_K, proving the reverse inclusion. □

We next study the topologically bounded sets[2] in $\mathcal{D}(\Omega)$, for which we will need the following result:

Lemma 4.3.5. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $K \subseteq \Omega$ be compact, and let $x_0 \in \Omega$. Then, for every $r > 0$, the set $U := \{\varphi \in \mathcal{D}_K(\Omega) \mid |\varphi(x_0)| < r\}$ is open in $(\mathcal{D}_K(\Omega), \tau_K)$.

Proof. First observe that if $x_0 \notin K$, then every function $\varphi \in \mathcal{D}_K(\Omega)$ satisfies $\varphi(x_0) = 0$, so $U = \mathcal{D}_K(\Omega)$, which is open.
Now suppose that $x_0 \in K$. Let $\varphi \in U$, that is, $|\varphi(x_0)| < r$. Define $\varepsilon := r - |\varphi(x_0)| > 0$. Consider the neighborhood of φ given by $V := \{\psi \in \mathcal{D}_K(\Omega) \mid \|\psi - \varphi\|_{K,0} < \varepsilon\}$. Recall that $\|\psi-\varphi\|_{C^0(K)} = \|\psi-\varphi\|_{C^0(K)}$. Then, for every $\psi \in V$, we have in particular $|\psi(x_0) - \varphi(x_0)| \leq \|\psi - \varphi\|_{C^0(K)} < \varepsilon$.
Therefore, $|\psi(x_0)| \leq |\varphi(x_0)| + |\psi(x_0) - \varphi(x_0)| < |\varphi(x_0)| + \varepsilon = r$. This shows that $\psi \in U$, and consequently $V \subseteq U$.
We have proved that every $\varphi \in U$ has a neighborhood contained in U, so U is open in τ_K. □

[2] Let X be a topological vector space. A subset $B \subseteq X$ is said to be topologically bounded if for every neighborhood U of the origin there exists $\lambda > 0$ such that $B \subseteq \lambda U$.

Theorem 4.3.6. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $W \subseteq \mathcal{D}(\Omega)$ be a topologically bounded set. Then there exists a compact set $K \subseteq \Omega$ such that $W \subseteq \mathcal{D}_K(\Omega)$. Moreover, for each $j \in \mathbb{N}_0$, there exists $M_j > 0$ such that $\|\varphi\|_j \leq M_j$ for every $\varphi \in W$, where $\|\cdot\|_j$ is given by (4.1).

Proof. We argue by contradiction. Suppose that $W \nsubseteq \mathcal{D}_K(\Omega)$ for every compact set $K \subseteq \Omega$. Let $\{K_l\}_{l\in\mathbb{N}}$ be an increasing sequence of compact sets such that $K_l \subseteq \operatorname{int}(K_{l+1})$ and $\Omega = \bigcup_{l=1}^{\infty} K_l$.

Then, for each $l \in \mathbb{N}$, there exists $\varphi_l \in W$ such that $\operatorname{supp}(\varphi_l) \nsubseteq K_l$. Therefore, we can choose a point $y_l \in \operatorname{supp}(\varphi_l) \setminus K_l$.

Since K_l is closed in Ω, the set $\Omega \setminus K_l$ is an open neighborhood of y_l. Moreover, by definition of support, $y_l \in \operatorname{supp}(\varphi_l)$ implies that every neighborhood of y_l intersects the set $\{x \in \Omega \mid \varphi_l(x) \neq 0\}$. In particular, there exists a point $x_l \in \Omega \setminus K_l$ such that $\varphi_l(x_l) \neq 0$.

We define

$$U := \left\{ \varphi \in \mathcal{D}(\Omega) \;\middle|\; |\varphi(x_l)| < \frac{|\varphi_l(x_l)|}{l},\ \forall l \in \mathbb{N} \right\}.$$

We claim that $U \in \mathcal{B}_0$. First, $0 \in U$, since $|0(x_l)| = 0 < \dfrac{|\varphi_l(x_l)|}{l}$ for every $l \in \mathbb{N}$.

Moreover, U is balanced. If $\varphi \in U$ and $|\lambda| \leq 1$, then

$$|(\lambda\varphi)(x_l)| = |\lambda||\varphi(x_l)| \leq |\varphi(x_l)| < \frac{|\varphi_l(x_l)|}{l}$$

for every $l \in \mathbb{N}$. Therefore, $\lambda\varphi \in U$.

Also, U is convex. If $\varphi, \psi \in U$ and $s \in [0,1]$, then, for every $l \in \mathbb{N}$,

$$|(s\varphi + (1-s)\psi)(x_l)| \leq s|\varphi(x_l)| + (1-s)|\psi(x_l)| < s\frac{|\varphi_l(x_l)|}{l} + (1-s)\frac{|\varphi_l(x_l)|}{l} = \frac{|\varphi_l(x_l)|}{l}.$$

Thus, $s\varphi + (1-s)\psi \in U$.

It remains to check that, for every compact set $K \subseteq \Omega$, the set $U \cap \mathcal{D}_K(\Omega)$ is a neighborhood of the origin in $(\mathcal{D}_K(\Omega), \tau_K)$. Let $K \subseteq \Omega$ be compact. Since $\Omega = \bigcup_{l=1}^{\infty} K_l$ and the sequence (K_l) is increasing, there exists $L \in \mathbb{N}$ such that $K \subseteq K_L$. Then, if $l \geq L$, we have $K \subseteq K_l$ and, since $x_l \notin K_l$, we conclude that $x_l \notin K$.

We define

$$I_K := \{l \in \mathbb{N} \mid x_l \in K\}.$$

By the preceding argument, $I_K \subseteq \{1, \ldots, L-1\}$, so I_K is finite.

If $\varphi \in \mathcal{D}_K(\Omega)$ and $l \notin I_K$, then $x_l \notin K$ and, since $\operatorname{supp}(\varphi) \subseteq K$, we have $\varphi(x_l) = 0$. Consequently, the conditions corresponding to these indices are automatically satisfied. Therefore,

$$U \cap \mathcal{D}_K(\Omega) = \left\{ \varphi \in \mathcal{D}_K(\Omega) \;\middle|\; |\varphi(x_l)| < \frac{|\varphi_l(x_l)|}{l},\ \forall l \in I_K \right\}.$$

Since I_K is finite, this set is a finite intersection of sets of the form

$$\{\varphi \in \mathcal{D}_K(\Omega) \mid |\varphi(x_l)| < c_l\},$$

which are open in $(\mathcal{D}_K(\Omega), \tau_K)$ by Lemma 4.3.5. Moreover, they contain the origin. Consequently, $U \cap \mathcal{D}_K(\Omega)$ is a neighborhood of the origin in $(\mathcal{D}_K(\Omega), \tau_K)$.
Since this holds for every compact set $K \subseteq \Omega$, we conclude that $U \in \mathcal{B}_0$. In particular, $U = 0 + U \in \mathcal{B}$, so U is a neighborhood of the origin in $(\mathcal{D}(\Omega), \tau_{LF})$.
Since W is topologically bounded, there exists $t > 0$ such that $W \subseteq tU$. Let $l \in \mathbb{N}$ satisfy $l \geq t$. Then

$$\frac{|\varphi_l(x_l)|}{t} \geq \frac{|\varphi_l(x_l)|}{l},$$

and therefore

$$\left|t^{-1}\varphi_l(x_l)\right| = \frac{|\varphi_l(x_l)|}{t} \geq \frac{|\varphi_l(x_l)|}{l}.$$

Thus, $t^{-1}\varphi_l \notin U$, which is equivalent to $\varphi_l \notin tU$. This contradicts the fact that $\varphi_l \in W \subseteq tU$.
Therefore, there exists a compact set $K \subseteq \Omega$ such that $W \subseteq \mathcal{D}_K(\Omega)$.
Finally, by Theorem 4.3.4, the set $W = W \cap \mathcal{D}_K(\Omega)$ is bounded with respect to the topology τ_K. By Corollary B.10.22, for each $j \in \mathbb{N}_0$ there exists $M_j > 0$ such that

$$\|\varphi\|_{C^j(K)} \leq M_j \qquad \text{for every } \varphi \in W.$$

Since $\operatorname{supp}(\varphi) \subseteq K$ for every $\varphi \in W$, we have $\|\varphi\|_j = \|\varphi\|_{C^j(K)}$ on W. Therefore,

$$\|\varphi\|_j \leq M_j \qquad \text{for every } \varphi \in W.$$

This completes the proof. □

The space $\mathcal{D}(\Omega)$ with the topology τ_{LF} of Theorem 4.3.1 will turn out to be a (sequentially) complete topological vector space, and we will be able to characterize convergence of sequences quite conveniently:

Theorem 4.3.7. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then the space $(\mathcal{D}(\Omega), \tau_{LF})$ is sequentially complete. Moreover, a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq \mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ with respect to τ_{LF} if and only if:

(a) there exists a compact set $K \subseteq \Omega$ such that $\operatorname{supp}(\varphi_k) \subseteq K$ for every $k \in \mathbb{N}$,

(b) $\lim_{k\to\infty} D^\alpha \varphi_k = D^\alpha \varphi$ uniformly on Ω for every multi-index $\alpha \in \mathbb{N}_0^n$.

Proof. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}(\Omega)$. By Proposition B.10.13, the set $\{\varphi_k \mid k \in \mathbb{N}\}$ is topologically bounded in $\mathcal{D}(\Omega)$. By Theorem 4.3.6, there exists a compact set $K \subseteq \Omega$ such that

$$\{\varphi_k \mid k \in \mathbb{N}\} \subseteq \mathcal{D}_K(\Omega).$$

By Theorem 4.3.4, the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{D}_K(\Omega)$ with respect to τ_K. Since $\mathcal{D}_K(\Omega)$ is complete, there exists $\varphi \in \mathcal{D}_K(\Omega)$ such that

$$\varphi_k \longrightarrow \varphi$$

in $\mathcal{D}_K(\Omega)$ with respect to τ_K.
Applying Theorem 4.3.4 again, we conclude that $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$ with respect to τ_{LF}. Therefore, $\mathcal{D}(\Omega)$ is sequentially complete.

We now prove the equivalence. Let $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq \mathcal{D}(\Omega)$ and let $\varphi \in \mathcal{D}(\Omega)$.
($\Rightarrow$) Suppose that $\varphi_k \to \varphi$ with respect to τ_{LF}. Then the sequence is Cauchy and, by Proposition B.10.13, $\{\varphi_k \mid k \in \mathbb{N}\}$ is topologically bounded in $\mathcal{D}(\Omega)$. By Theorem 4.3.6, there exists a compact set $K \subseteq \Omega$ such that

$$\varphi_k \in \mathcal{D}_K(\Omega)$$

for every $k \in \mathbb{N}$. That is, $\operatorname{supp}(\varphi_k) \subseteq K$ for every $k \in \mathbb{N}$. Since also $\varphi \in \mathcal{D}(\Omega)$, we have $\varphi \in \mathcal{D}_{K'}(\Omega)$ for some compact set $K' \subseteq \Omega$. Replacing K by $K \cup K'$, we may assume that $\varphi \in \mathcal{D}_K(\Omega)$.
By Theorem 4.3.4, the topology induced by τ_{LF} on $\mathcal{D}_K(\Omega)$ agrees with τ_K. In particular, $\varphi_k \to \varphi$ in $\mathcal{D}_K(\Omega)$ with respect to τ_K. By definition of τ_K, this is equivalent to saying that, for every $j \in \mathbb{N}_0$,

$$\|\varphi_k - \varphi\|_{C^j(K)} \to 0.$$

Consequently, for every multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$D^\alpha \varphi_k \to D^\alpha \varphi$$

uniformly on K. Since $\varphi_k, \varphi \in \mathcal{D}_K(\Omega)$, their derivatives vanish on $\Omega \setminus K$. Consequently,

$$\|D^\alpha \varphi_k - D^\alpha \varphi\|_{L^\infty(\Omega)} = \|D^\alpha \varphi_k - D^\alpha \varphi\|_{C^0(K)} \longrightarrow 0.$$

The convergence is therefore uniform on Ω. This proves both conditions.
($\Leftarrow$) Now suppose that there exists a compact set $K \subseteq \Omega$ such that $\operatorname{supp}(\varphi_k) \subseteq K$ for every $k \in \mathbb{N}$ and that

$$D^\alpha \varphi_k \to D^\alpha \varphi$$

uniformly on Ω for every multi-index $\alpha \in \mathbb{N}_0^n$. The case $\alpha = 0$ implies pointwise convergence throughout Ω. If $x \in \Omega \setminus K$, then $\varphi_k(x) = 0$ for every k, so $\varphi(x) = \lim_{k\to\infty} \varphi_k(x) = 0$. Since K is closed in Ω, it follows that $\operatorname{supp}(\varphi) \subseteq K$. Then $\varphi_k, \varphi \in \mathcal{D}_K(\Omega)$ and, for every $j \in \mathbb{N}_0$,

$$\|\varphi_k - \varphi\|_{C^j(K)} = \max_{|\alpha|\le j} \|D^\alpha \varphi_k - D^\alpha \varphi\|_{C^0(K)} \to 0.$$

Therefore, $\varphi_k \to \varphi$ in $\mathcal{D}_K(\Omega)$ with respect to τ_K. Since τ_K agrees with the subspace topology induced by τ_{LF} on $\mathcal{D}_K(\Omega)$, we conclude that $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$ with respect to τ_{LF}. □

When $\Omega \subseteq \mathbb{R}^n$ is open and nonempty, with $n \ge 1$, the space $\mathcal{D}(\Omega)$ is not metrizable:

Exercise 4.3.8. Let $\Omega \subseteq \mathbb{R}^n$ be open and nonempty, with $n \ge 1$.

(a) Prove that for every compact set $K \subseteq \Omega$, $\mathcal{D}_K(\Omega)$ is closed and has empty interior in $\mathcal{D}(\Omega)$.

(b) Prove that $\mathcal{D}(\Omega)$ is not metrizable.

Hint: To prove that $\mathcal{D}_K(\Omega)$ has empty interior in the first part, construct a sequence of functions in $\mathcal{D}(\Omega) \setminus \mathcal{D}_K(\Omega)$ converging to the constant 0 in the topology of $\mathcal{D}(\Omega)$. For the second part, consider an exhaustion by compact sets $(K_j)_{j\in\mathbb{N}}$ such that $\Omega = \bigcup_{j=1}^{\infty} K_j$. Use the preceding part and the Baire category theorem

(Theorem B.1.14), recalling the relation between complete metric spaces and Baire spaces.

4.4 LF spaces

The spaces $\mathcal{D}(\Omega)$ can be obtained as a particular case of a general construction in functional analysis known as a *strict inductive limit of Fréchet spaces*. Spaces obtained in this way are called *LF spaces*.

To formalize this notion, recall that, given a family of maps $g_i\colon X_i \longrightarrow X$, there is a final topology on X, characterized as the finest topology making all the g_i continuous. In the setting of topological vector spaces, however, this purely topological final topology need not be compatible with the linear structure of X, nor need it be locally convex.

Thus, when the spaces X_i are locally convex and the maps g_i are linear, we are interested in the finest among the locally convex topologies on X making all the g_i continuous. In the case of inclusions of subspaces, this construction takes the following form.

Theorem 4.4.1. (Locally convex final topology) Let X be a vector space and let $(X_i)_{i\in I}$ be a family of subspaces of X such that

$$X = \bigcup_{i\in I} X_i.$$

Suppose that each X_i is equipped with a locally convex topology τ_i. Then there is a unique locally convex topology τ_{LC} on X that is the finest among all locally convex topologies making the canonical inclusions

$$\iota_i\colon X_i \hookrightarrow X, \qquad i \in I.$$

continuous.
The topology τ_{LC} is called the *locally convex final topology* associated with the family $\big((X_i,\tau_i),\iota_i\big)_{i\in I}$.

Proof. Let $\mathcal{U}$ be the collection of all absolutely convex subsets[3] $U \subseteq X$ such that, for each $i \in I$, $U \cap X_i$ is a neighborhood of the origin in (X_i,τ_i).
First observe that every element of $\mathcal{U}$ is absorbing. Let $U \in \mathcal{U}$ and let $x \in X$. Since

$$X = \bigcup_{i\in I} X_i,$$

there exists $i \in I$ such that $x \in X_i$. By hypothesis, $U \cap X_i$ is a neighborhood of the origin in the topological vector space X_i and is therefore absorbing in X_i. Consequently, there exists $\lambda > 0$ such that

$$x \in t(U \cap X_i) \qquad \text{for every } t \geq \lambda.$$

Since $t(U \cap X_i) \subseteq tU$ for every $t > 0$, it follows that

$$x \in tU \qquad \text{for every } t \geq \lambda.$$

Since $x \in X$ was arbitrary, we conclude that U is absorbing.
We now show that $\mathcal{U}$ satisfies the hypotheses of Proposition B.10.26.

(a) By definition, each $U \in \mathcal{U}$ is absolutely convex, and we have just proved that it is also absorbing.

(b) Let $U, V \in \mathcal{U}$. Then $U \cap V$ is absolutely convex. Moreover, for each $i \in I$,

$$(U \cap V) \cap X_i = (U \cap X_i) \cap (V \cap X_i).$$

Since $U \cap X_i$ and $V \cap X_i$ are neighborhoods of the origin in (X_i, τ_i), their intersection is as well. Therefore, $U \cap V \in \mathcal{U}$.

(c) Let $U \in \mathcal{U}$ and let $\lambda > 0$. Define $W := \frac{\lambda}{2} U$. Since U is absolutely convex, so is W. Moreover, for each $i \in I$, since X_i is a subspace of X, we have

$$W \cap X_i = \left(\frac{\lambda}{2} U\right) \cap X_i = \frac{\lambda}{2}(U \cap X_i).$$

Since multiplication by the nonzero scalar $\frac{\lambda}{2}$ is a homeomorphism of (X_i, τ_i) onto itself, the set $\frac{\lambda}{2}(U \cap X_i)$ is a neighborhood of the origin in X_i. Thus, W is absolutely convex and $W \cap X_i$ is a neighborhood of the origin in X_i for every $i \in I$. By definition of $\mathcal{U}$, we conclude that $W \in \mathcal{U}$.

Finally, since U is balanced and $\frac{\lambda}{2} \leq \lambda$, we have

$$W = \frac{\lambda}{2} U \subseteq \lambda U.$$

By Proposition B.10.26, the family $\mathcal{U}$ is a local base at the origin for a unique locally convex topology τ_{LC} on X.

We show that each canonical inclusion

$$\iota_i \colon X_i \hookrightarrow X$$

is continuous with respect to τ_{LC}. Since ι_i is linear, it suffices to check continuity at the origin. Let A be a neighborhood of the origin in (X, τ_{LC}). Since $\mathcal{U}$ is a local base at the origin, there exists $U \in \mathcal{U}$ such that $U \subseteq A$. Then $\iota_i^{-1}(U) = U \cap X_i$ is a neighborhood of the origin in (X_i, τ_i), and $\iota_i^{-1}(U) \subseteq \iota_i^{-1}(A)$. Therefore, $\iota_i^{-1}(A)$ is a neighborhood of the origin in X_i, proving that ι_i is continuous.

Finally, we show that τ_{LC} is the finest among all locally convex topologies with this property. Let τ' be another locally convex topology on X such that all the inclusions ι_i are continuous. Let V be an absolutely convex neighborhood of the origin in (X, τ'). Then, for each $i \in I$,

$$V \cap X_i = \iota_i^{-1}(V)$$

is a neighborhood of the origin in (X_i, τ_i), by continuity of ι_i. Therefore, $V \in \mathcal{U}$.

Now, by Proposition B.10.15, the absolutely convex neighborhoods of the origin form a local base in (X, τ'). Thus, if A is a neighborhood of the origin in (X, τ'), there is an absolutely convex neighborhood V of the origin such that $V \subseteq A$. By the preceding argument, $V \in \mathcal{U}$. Since $\mathcal{U}$ is a local base at the origin for τ_{LC}, the set V is a neighborhood of the origin with respect to τ_{LC}; since it is contained in A, we conclude that A is also a neighborhood of the origin with respect to τ_{LC}. We have proved that every neighborhood of the origin for τ' is also a neighborhood of the origin for τ_{LC}. Consequently, $\tau' \subseteq \tau_{LC}$.

Thus, τ_{LC} is the finest among all locally convex topologies making the inclusions ι_i continuous.

Uniqueness follows immediately from this maximality property. □

With this construction, we can formally define LF spaces.

Definition 4.4.2. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Suppose there is a sequence of subspaces $(X_n)_{n\in\mathbb{N}}$ such that:

(a) $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ and $X = \bigcup_{n=1}^{\infty} X_n$.

(b) For each n, the space X_n, equipped with a topology τ_n, is a Fréchet space.

(c) For each n, the topology τ_n agrees with the subspace topology induced on X_n by (X_{n+1}, τ_{n+1}).

Let τ_{LF} be the locally convex final topology on X associated with the inclusions $\iota_n \colon (X_n, \tau_n) \hookrightarrow X$, guaranteed by Theorem 4.4.1. Then (X, τ_{LF}) is called the *strict inductive limit* of the sequence (X_n, τ_n), and we write

$$(X, \tau_{LF}) = \varinjlim_{n\in\mathbb{N}} (X_n, \tau_n).$$

In this case, (X, τ_{LF}) is said to be an *LF space*. The adjective *strict* refers to compatibility of the topologies of the stages, not to the inclusions being proper. Stationary sequences are allowed; in particular, every Fréchet space is obtained by taking $X_n = X$ with the same topology for every n.

Remark 4.4.3. If in the preceding definition each space X_n is Banach, then X is called an LB space. In particular, every LB space is an LF space.

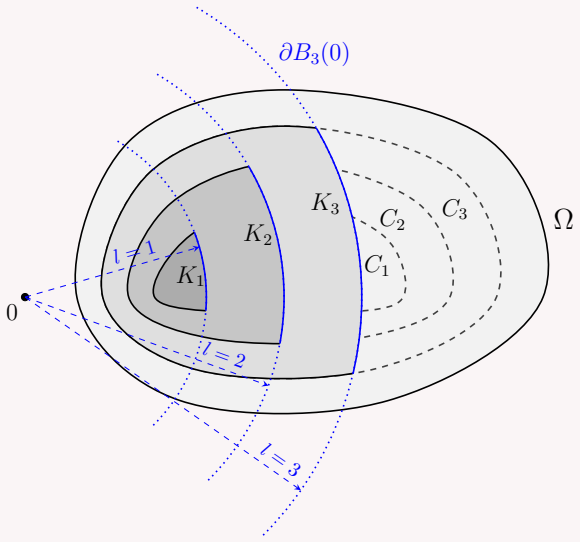
Remark 4.4.4. The notion of an LF space arose naturally in the development of functional analysis and distribution theory during the first half of the twentieth century. At that time, it was observed that many fundamental spaces in analysis could not be adequately described within the category of Banach spaces or even Fréchet spaces, although they could be constructed as increasing unions of the latter.
One of the most important examples appeared in Laurent Schwartz's distribution theory. The space of smooth functions with compact support, $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$, has a topological structure essential to the study of distributions and partial differential equations, but is not metrizable under its natural topology. However, it can be written as a countable increasing union of the spaces $\mathcal{D}_K(\Omega) = \{f \in C^\infty(\Omega) \mid \operatorname{supp}(f) \subseteq K\}$, where $K \Subset \Omega$ ranges over the compact subsets of Ω. Each of these spaces turns out to be a Fréchet space, which motivated the systematic study of countable inductive limits of Fréchet spaces.
The theory was subsequently developed within the general framework of locally convex spaces by authors such as Jean Dieudonné and Alexander Grothendieck, becoming a fundamental tool in functional analysis, distribution theory, differential geometry, and global analysis.

To show that $\mathcal{D}(\Omega)$ is an LF space, we first construct an exhaustion of Ω by compact sets.
For each $l \in \mathbb{N}$, we define $C_l := \left\{ x \in \Omega \;\middle|\; d\left(x, \mathbb{R}^n \setminus \Omega\right) \geq \frac{1}{l} \right\}$, and then $K_l := C_l \cap \overline{B}_{\text{euc}}(0, l)$.

The geometric idea behind this construction is twofold: on the one hand, the sets C_l force points to remain at a positive distance from the boundary $\partial\Omega$; on the other, intersection with the closed ball $\overline{B}_{\text{euc}}(0, l)$ prevents points from escaping to infinity.

It is immediate to check that each K_l is compact, that $K_l \subseteq \operatorname{int}(K_{l+1})$, and that $\Omega = \bigcup_{l=1}^{\infty} K_l$.

[3] A subset U of a vector space is said to be *absolutely convex* (or a *disk*) if it is both convex and balanced.



Geometric interpretation

The set $K_l = C_l \cap \overline{B}_{\mathrm{euc}}(0,l)$ is obtained by combining two complementary restrictions:

(a) C_l removes points too close to $\partial\Omega$;

(b) $\overline{B}_{\mathrm{euc}}(0,l)$ controls behavior at infinity.

Consequently, the boundary of K_l consists of:

$$\partial K_l \subseteq \left\{ x \in \Omega \mid d(x,\partial\Omega) = \frac{1}{l} \right\} \cup \partial B_{\mathrm{euc}}(0,l).$$

In this way, the sequence $(K_l)_{l\in\mathbb{N}}$ provides a natural exhaustion of Ω by compact sets.

Figure 4.2. Construction of a compact exhaustion of Ω.

Now taking $X_l := \mathcal{D}_{K_l}(\Omega)$, conditions (a)–(c) of Definition 4.4.2 are satisfied. The family is increasing and exhaustive, each space X_l is Fréchet, and the topologies are compatible by Theorem 4.3.4.

It remains to check that the topology τ_{LF} agrees with the locally convex inductive topology associated with the inclusions $\mathcal{D}_{K_l}(\Omega) \hookrightarrow \mathcal{D}(\Omega)$.

Proposition 4.4.5. The topology τ_{LF} on $\mathcal{D}(\Omega)$ agrees with the locally convex inductive topology associated with the family of inclusions

$$\iota_l \colon \mathcal{D}_{K_l}(\Omega) \hookrightarrow \mathcal{D}(\Omega), \qquad l \in \mathbb{N}.$$

Proof. Denote by τ_{LC} the locally convex inductive topology on $\mathcal{D}(\Omega)$ associated with the family

$$\big((\mathcal{D}_{K_l}(\Omega), \tau_{K_l}), \iota_l\big)_{l\in\mathbb{N}}.$$

By Theorem 4.4.1, a local base at the origin for τ_{LC} is given by the family $\mathcal{U}$ of all absolutely convex subsets $U \subseteq \mathcal{D}(\Omega)$ such that $U \cap \mathcal{D}_{K_l}(\Omega)$ is a neighborhood of the origin in $(\mathcal{D}_{K_l}(\Omega), \tau_{K_l})$ for every $l \in \mathbb{N}$.
We compare this family with the local base $\mathcal{B}_0$ defining the topology τ_{LF}. Recall that, by definition, $V \in \mathcal{B}_0$ if and only if $0 \in V$, V is absolutely convex, and $V \cap \mathcal{D}_K(\Omega)$ is a neighborhood of the origin in $(\mathcal{D}_K(\Omega), \tau_K)$ for every compact set $K \subseteq \Omega$.
Thus, the only apparent difference between the families $\mathcal{U}$ and $\mathcal{B}_0$ is that, in the definition of $\mathcal{U}$, the neighborhood condition is required only for the compact sets in the exhaustion $(K_l)_{l\in\mathbb{N}}$, whereas in the definition of $\mathcal{B}_0$ it is required for every compact set $K \subseteq \Omega$. We show that the two conditions are equivalent.
First we prove that $\mathcal{B}_0 \subseteq \mathcal{U}$. Let $V \in \mathcal{B}_0$. Since each K_l is compact, the definition of $\mathcal{B}_0$ implies that $V \cap \mathcal{D}_{K_l}(\Omega)$ is a neighborhood of the origin in $(\mathcal{D}_{K_l}(\Omega), \tau_{K_l})$ for every $l \in \mathbb{N}$. Since V is also absolutely convex, we conclude that $V \in \mathcal{U}$.

We now prove the reverse inclusion. Let $U \in \mathcal{U}$ and let $K \subseteq \Omega$ be an arbitrary compact set. Since $(K_l)_{l\in\mathbb{N}}$ is an exhaustion of Ω by compact sets, there exists $l \in \mathbb{N}$ such that $K \subseteq K_l$.
Then $\mathcal{D}_K(\Omega) \subseteq \mathcal{D}_{K_l}(\Omega)$.
Moreover, the topology τ_K agrees with the topology induced by τ_{K_l} on the subspace $\mathcal{D}_K(\Omega)$. Since $U \cap \mathcal{D}_{K_l}(\Omega)$ is a neighborhood of the origin in $(\mathcal{D}_{K_l}(\Omega), \tau_{K_l})$, it follows that

$$\big(U \cap \mathcal{D}_{K_l}(\Omega)\big) \cap \mathcal{D}_K(\Omega) = U \cap \mathcal{D}_K(\Omega)$$

is a neighborhood of the origin in $(\mathcal{D}_K(\Omega), \tau_K)$.
Since $K \subseteq \Omega$ was an arbitrary compact set, U satisfies the neighborhood condition required in the definition of $\mathcal{B}_0$. Moreover, since $U \cap \mathcal{D}_{K_l}(\Omega)$ is a neighborhood of the origin for any $l \in \mathbb{N}$, we have $0 \in U$. Therefore, $U \in \mathcal{B}_0$.
We have proved that $\mathcal{B}_0 = \mathcal{U}$.
Consequently, τ_{LF} and τ_{LC} have the same local base at the origin. By Proposition B.10.2, the two topologies agree. That is, $\tau_{LF} = \tau_{LC}$. □

Consequently, $(\mathcal{D}(\Omega), \tau_{LF})$ is the strict inductive limit

$$\mathcal{D}(\Omega) = \varinjlim_{l\in\mathbb{N}} \mathcal{D}_{K_l}(\Omega),$$

and is therefore an LF space.

The advantage of this description is that many fundamental properties of $\mathcal{D}(\Omega)$ follow directly from its LF structure in a purely topological framework. These properties will allow us to construct spaces of distributions on vector bundles without repeating the constructions and proofs.

Theorem 4.4.6. Let (X, τ) be an *LF* space with defining sequence $(X_n)_{n\in\mathbb{N}}$. Then:

(a) X is Hausdorff.

(b) A set $B \subseteq X$ is topologically bounded if and only if there exists $m \in \mathbb{N}$ such that $B \subseteq X_m$ and B is topologically bounded in X_m.

(c) X is complete; in particular, it is sequentially complete.

(d) A sequence $(x_k)_{k\in\mathbb{N}}$ converges to $x \in X$ if and only if there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_k \mid k \in \mathbb{N}\} \subseteq X_m$ and $x_k \to x$ in the topology of X_m.

The argument for strict inductive limits comes from [149, Chapter 13].

Proof. We will prove completeness using filters, since an *LF* space need not be metrizable.
We will repeatedly use the following elementary observation. If A is an absolutely convex set, $u, v \in A$ and $\alpha, \beta \in \mathbb{K}$ satisfy $|\alpha| + |\beta| \le 1$, then $\alpha u + \beta v \in A$.
If $\alpha = 0$ or $\beta = 0$, the assertion follows from the balancedness of A. Now suppose that $\alpha \neq 0$ and $\beta \neq 0$, and define $s := |\alpha| + |\beta|$. Since $0 < s \le 1$, we have

$$\alpha u + \beta v = s\left(\frac{|\alpha|}{s}\frac{\alpha}{|\alpha|}u + \frac{|\beta|}{s}\frac{\beta}{|\beta|}v\right).$$

Since A is balanced, $\frac{\alpha}{|\alpha|}u, \frac{\beta}{|\beta|}v \in A$; since $\frac{|\alpha|}{s} + \frac{|\beta|}{s} = 1$, the term in parentheses belongs to A by convexity. Finally, since $s \leq 1$ and A is balanced, we conclude that $\alpha u + \beta v \in A$.

(a) Let $x \in X \backslash \{0\}$. Since $X = \bigcup_{n=1}^{\infty} X_n$, there exists $m \in \mathbb{N}$ such that $x \in X_m$. Since X_m is Fréchet, it is Hausdorff; therefore, there is a neighborhood of the origin in X_m that does not contain x. By Proposition B.10.15, we can choose an absolutely convex neighborhood U_m of the origin in X_m such that $x \notin U_m$.

For $n < m$, define $U_n := U_m \cap X_n$. Since the topology of X_n agrees with the topology induced by X_m, each U_n is an absolutely convex neighborhood of the origin in X_n. Moreover, for $2 \leq n \leq m$ we have $U_n \cap X_{n-1} = U_{n-1}$.

We now construct U_n inductively for $n > m$. Suppose that U_{n-1} has been defined as an absolutely convex neighborhood of the origin in X_{n-1}. Since the topology of X_{n-1} agrees with the topology induced by X_n, there exists an absolutely convex neighborhood V_n of the origin in X_n such that $V_n \cap X_{n-1} \subseteq U_{n-1}$. Define

$$U_n := \operatorname{cobal}(V_n \cup U_{n-1})^4 = \left\{ \alpha v + \beta u \mid v \in V_n,\ u \in U_{n-1},\ |\alpha| + |\beta| \leq 1 \right\}.$$

By construction, U_n is absolutely convex and contains V_n; in particular, it is a neighborhood of the origin in X_n.

We claim that $U_n \cap X_{n-1} = U_{n-1}$. The inclusion $U_{n-1} \subseteq U_n \cap X_{n-1}$ is immediate, since $U_{n-1} \subseteq U_n$ and $U_{n-1} \subseteq X_{n-1}$. To prove the other inclusion, take $z \in U_n \cap X_{n-1}$. Then there exist $v \in V_n$, $u \in U_{n-1}$, and scalars $\alpha, \beta \in \mathbb{K}$ such that $|\alpha| + |\beta| \leq 1$ and $z = \alpha v + \beta u$. If $\alpha = 0$, then $z = \beta u \in U_{n-1}$, since U_{n-1} is balanced. If $\alpha \neq 0$, the preceding equality gives

$$v = \alpha^{-1}(z - \beta u) \in X_{n-1},$$

since $z, u \in X_{n-1}$. Therefore, $v \in V_n \cap X_{n-1} \subseteq U_{n-1}$. Since also $u \in U_{n-1}$ and $|\alpha| + |\beta| \leq 1$, the preliminary observation about absolutely convex sets gives $z = \alpha v + \beta u \in U_{n-1}$. The equality is proved.

Now define

$$U := \bigcup_{n=1}^{\infty} U_n.$$

The sequence $(U_n)_{n \in \mathbb{N}}$ is increasing, since $U_{n-1} = U_n \cap X_{n-1} \subseteq U_n$. It follows that U is absolutely convex. If $y, z \in U$, there exist indices $r, s \in \mathbb{N}$ such that $y \in U_r$ and $z \in U_s$. Taking $N := \max\{r, s\}$, we obtain $y, z \in U_N$; therefore, if $\alpha, \beta \in \mathbb{K}$ satisfy $|\alpha| + |\beta| \leq 1$, then $\alpha y + \beta z \in U_N \subseteq U$, since U_N is absolutely convex.

We show that $U \cap X_n = U_n$ for every $n \in \mathbb{N}$. If $r \geq n$, the equalities $U_j \cap X_{j-1} = U_{j-1}$ allow us to descend successively from r to n:

$$U_r \cap X_n = (U_r \cap X_{r-1}) \cap X_n = U_{r-1} \cap X_n = \cdots = U_n.$$

If, instead, $r < n$, then $U_r \subseteq U_n \subseteq X_n$, so $U_r \cap X_n = U_r \subseteq U_n$. Consequently,

$$U \cap X_n = \left(\bigcup_{r=1}^{\infty} U_r \right) \cap X_n = \bigcup_{r=1}^{\infty} (U_r \cap X_n) = U_n.$$

Thus, $U \cap X_n$ is a neighborhood of the origin in X_n for every $n \in \mathbb{N}$. By definition of the topology LF, U is a neighborhood of the origin in X. Applying the preceding equality to the index m fixed at the outset, we obtain $U \cap X_m = U_m$. Since $x \in X_m$ and $x \notin U_m$, we conclude that $x \notin U$.

For each $x \in X \setminus \{0\}$, we have found a neighborhood of the origin that does not contain x. Therefore, X is Hausdorff.

(b) First suppose that $B \subseteq X_m$ is topologically bounded in X_m. Let U be a neighborhood of the origin in X. Since the inclusion $\iota_m \colon X_m \hookrightarrow X$ is continuous, $U \cap X_m$ is a neighborhood of the origin in X_m. By the topological boundedness of B in X_m, there exists $\lambda > 0$ such that

$$B \subseteq \lambda (U \cap X_m) \subseteq \lambda U.$$

Thus, B is topologically bounded in X.

Conversely, suppose that B is topologically bounded in X and is not contained in any X_m. We recursively choose points $x_k \in B$ and indices $m_k \in \mathbb{N}$ as follows. Take $x_1 \in B \setminus X_1$ and let m_1 be the least index such that $x_1 \in X_{m_1}$. Once m_{k-1} has been chosen, take $x_k \in B \setminus X_{m_{k-1}}$ and let m_k be the least index such that $x_k \in X_{m_k}$. Then $(m_k)_{k \in \mathbb{N}}$ is strictly increasing and, for each k, $x_k \in X_{m_k} \setminus X_{m_k - 1}$.

Since the topology of $X_{m_k - 1}$ agrees with the topology induced by X_{m_k} and $X_{m_k - 1}$ is complete, being Fréchet, it follows that $X_{m_k - 1}$ is a closed subspace of X_{m_k}. Since $x_k \notin X_{m_k - 1}$, Corollary B.10.35 guarantees the existence of a continuous linear functional $f_k \in X'_{m_k}$ such that $f_k|_{X_{m_k - 1}} = 0$ and $f_k(x_k) = k$.

We construct a neighborhood U of the origin in X from an increasing sequence of absolutely convex neighborhoods $U_n \subseteq X_n$.

For $n = 1$, fix an absolutely convex neighborhood U_1 of the origin in X_1. Suppose that U_{n-1} has been constructed as an absolutely convex neighborhood of the origin in X_{n-1}. Since the topology of X_{n-1} agrees with the topology induced by X_n, there exists an absolutely convex neighborhood V_n of the origin in X_n such that $V_n \cap X_{n-1} \subseteq U_{n-1}$. Define

$$W_n := \operatorname{cobal}(V_n \cup U_{n-1}) = \left\{ \alpha v + \beta u \mid v \in V_n,\ u \in U_{n-1},\ |\alpha| + |\beta| \leq 1 \right\}.$$

Since W_n contains V_n, it is a neighborhood of the origin in X_n; moreover, by construction, it is absolutely convex. The same argument used in the preceding part shows that $W_n \cap X_{n-1} = U_{n-1}$.

Now define

$$U_n := \begin{cases} W_n, & \text{if } n \neq m_k \text{ for every } k, \\ W_n \cap \{x \in X_n \mid |f_k(x)| \leq 1\}, & \text{if } n = m_k. \end{cases}$$

Each U_n is absolutely convex. Moreover, if $n = m_k$, the set $\{x \in X_n \mid |f_k(x)| \leq 1\}$ is a neighborhood of the origin in X_n, since it contains the open set $\{x \in X_n \mid |f_k(x)| < 1\}$. Therefore, each U_n is an absolutely convex neighborhood of the origin in X_n.

The equality $U_n \cap X_{n-1} = U_{n-1}$ is preserved at every step. If $n \neq m_k$ for every k, we simply have $U_n \cap X_{n-1} = W_n \cap X_{n-1} = U_{n-1}$. If $n = m_k$, then

$$\begin{aligned} U_{m_k} \cap X_{m_k-1} &= W_{m_k} \cap \{x \in X_{m_k} \mid |f_k(x)| \leq 1\} \cap X_{m_k-1} \\ &= W_{m_k} \cap X_{m_k-1} = U_{m_k-1}, \end{aligned}$$

since f_k vanishes on X_{m_k-1}.

Since $U_{n-1} = U_n \cap X_{n-1} \subseteq U_n$, the sequence $(U_n)_{n\in\mathbb{N}}$ is increasing. Define $U := \bigcup_{n=1}^{\infty} U_n$. As the union of an increasing sequence of absolutely convex sets, U is absolutely convex; moreover, the preceding equalities give $U \cap X_n = U_n$ for every $n \in \mathbb{N}$. By definition of the topology LF, U is a neighborhood of the origin in X.

Since B is topologically bounded in X, there exists $\lambda > 0$ such that $B \subseteq \lambda U$. In particular, $x_k \in \lambda U$ for every k. Since $x_k \in X_{m_k}$, we have

$$x_k \in \lambda U \cap X_{m_k} = \lambda(U \cap X_{m_k}) = \lambda U_{m_k}.$$

By definition of U_{m_k}, every element $y \in U_{m_k}$ satisfies $|f_k(y)| \leq 1$. Since $\dfrac{x_k}{\lambda} \in U_{m_k}$,

$$\left|f_k\left(\frac{x_k}{\lambda}\right)\right| \leq 1,$$

and therefore $|f_k(x_k)| \leq \lambda$. However, by construction, $f_k(x_k) = k$. Thus, $k \leq \lambda$ for every $k \in \mathbb{N}$, which is impossible.

This contradiction shows that B must be contained in some X_m. It remains to show that it is topologically bounded in X_m. Let V be a neighborhood of the origin in X_m. Since the topology of X_m agrees with the topology induced by X, there exists a neighborhood W of the origin in X such that $V = W \cap X_m$. Since B is topologically bounded in X, there exists $\lambda > 0$ such that $B \subseteq \lambda W$. Since also $B \subseteq X_m$,

$$B \subseteq \lambda W \cap X_m = \lambda(W \cap X_m) = \lambda V.$$

Therefore, B is topologically bounded in X_m.

(c) We prove completeness using filters, since X is not necessarily metrizable. Recall that a filter $\mathscr{F}$ on X is a family of nonempty subsets closed under finite intersections and passage to supersets. We say that $\mathscr{F}$ is Cauchy if, for every neighborhood U of the origin, there exists $M \in \mathscr{F}$ such that $M - M \subseteq U$; the space is complete if every Cauchy filter converges.

Let $\mathscr{F}$ be a Cauchy filter on X. The sets

$$M + V, \qquad M \in \mathscr{F}, \quad V \text{ a neighborhood of the origin in } X,$$

form a filter base: the intersection of two of them contains $(M_1 \cap M_2) + (V_1 \cap V_2)$. Denote the filter they generate by $\mathscr{G}$. Each basic set $M + V$ belongs to $\mathscr{F}$, since it contains M; therefore, $\mathscr{F}$ is finer than $\mathscr{G}$. Moreover, $\mathscr{G}$ is Cauchy. Indeed, given a neighborhood U of the origin, choose an absolutely convex

neighborhood V such that $V + V - V \subseteq U$ and a set $M \in \mathscr{F}$ such that $M - M \subseteq V$. Then

$$(M + V) - (M + V) \subseteq V + V - V \subseteq U.$$

We claim that there exists $p \in \mathbb{N}$ such that

$$A \cap X_p \neq \varnothing \qquad \text{for every } A \in \mathscr{G}. \tag{$*$}$$

Suppose otherwise. For each n we could choose a basic set $A_n = M_n + V_n \in \mathscr{G}$ such that $A_n \cap X_n = \varnothing$, with $M_n \in \mathscr{F}$. By successively shrinking the neighborhoods, we may assume that each V_n is absolutely convex and that $V_{n+1} \subseteq V_n$. Define

$$W_n := \operatorname{conv}\left(V_n \cup \bigcup_{k<n}(V_k \cap X_k)\right).$$

We have $(M_n + W_n) \cap X_n = \varnothing$. For $n = 1$ this follows from $W_1 = V_1$. If $n \geq 2$, every element of W_n can be written as $ty + (1-t)z$, with $0 \leq t \leq 1$, $y \in V_n$, and $z \in X_{n-1}$, after replacing z by an element of the convex hull of $\bigcup_{k<n}(V_k \cap X_k)$. If $m + ty + (1-t)z \in X_n$, with $m \in M_n$, then $m + ty \in (M_n + V_n) \cap X_n$, because V_n is balanced, contradicting the choice of A_n.

Now set

$$W := \operatorname{conv}\left(\bigcup_{k=1}^{\infty}(V_k \cap X_k)\right).$$

The set W is absolutely convex and $W \cap X_k$ contains the neighborhood $V_k \cap X_k$ of the origin in X_k; by definition of the topology LF, W is a neighborhood of the origin in X. The monotonicity of (V_k) also shows that $W \subseteq W_n$ for every n: the terms with index less than n appear in the definition of W_n, whereas those with index greater than or equal to n are contained in V_n.

Since $\mathscr{G}$ is Cauchy, there exists $B \in \mathscr{G}$ such that $B - B \subseteq W$. The set B also belongs to $\mathscr{F}$, and therefore $B \cap M_n \neq \varnothing$ for every n. If $b_n \in B \cap M_n$, then

$$B \subseteq b_n + (B - B) \subseteq M_n + W_n.$$

Consequently, $B \cap X_n = \varnothing$ for every n, which is impossible because $X = \bigcup_n X_n$ and a filter does not contain the empty set. This proves assertion $(*)$.

Fix p as in $(*)$. The sets $A \cap X_p$, with $A \in \mathscr{G}$, form a base for a filter $\mathscr{G}_p$ on X_p. This filter is Cauchy: if U_p is a neighborhood of the origin in X_p, equality of the original topology of X_p and the one induced by X allows us to choose a neighborhood U of the origin in X such that $U \cap X_p \subseteq U_p$, and it suffices to apply the Cauchy property of $\mathscr{G}$. Since X_p is Fréchet, there exists $x \in X_p$ such that $\mathscr{G}_p$ converges to x.

This implies that x is a cluster point of $\mathscr{G}$. A Cauchy filter with a cluster point converges to it: given a neighborhood U of the origin, choose a balanced neighborhood V with $V + V \subseteq U$ and $A \in \mathscr{G}$ with $A - A \subseteq V$; since $A \cap (x + V) \neq \varnothing$, if a belongs to this intersection, then $A \subseteq x + V + V \subseteq x + U$. Thus, $\mathscr{G}$ converges to x. Since $\mathscr{F}$ is finer than $\mathscr{G}$, $\mathscr{F}$ also converges to x. This proves that X is complete.

(d) First suppose that $x_k \to x$ in X. Then $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, and by Proposition B.10.13, the

set $\{x_k \mid k \in \mathbb{N}\}$ is topologically bounded in X. Since the singleton $\{x\}$ is also topologically bounded, the union $B := \{x\} \cup \{x_k \mid k \in \mathbb{N}\}$ is topologically bounded in X. By part (b), there exists $m \in \mathbb{N}$ such that $B \subseteq X_m$. Since the topology of X_m agrees with the topology induced by X, it follows that $x_k \to x$ in X_m.

Conversely, if there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_k \mid k \in \mathbb{N}\} \subseteq X_m$ and $x_k \to x$ in X_m, then $x_k \to x$ in X by continuity of the canonical inclusion $X_m \hookrightarrow X$.

□

4.5 Distributions: the dual of $\mathcal{D}(\Omega)$

The concept of a distribution will be obtained from the dual of $\mathcal{D}(\Omega)$. To this end, we first study some properties of continuous linear functionals defined on $\mathcal{D}(\Omega)$.

Theorem 4.5.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ be linear. Then the following properties are equivalent:

(a) T is continuous.

(b) T is bounded.

(c) If $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ converges to $\varphi \in \mathcal{D}(\Omega)$ with respect to τ, then

$$\lim_{\ell\to\infty} T(\varphi_\ell) = T(\varphi).$$

(d) $T \restriction_{\mathcal{D}_K(\Omega)}$ is continuous for every compact set $K \subseteq \Omega$.

(e) For every compact set $K \subseteq \Omega$, there exist $j \in \mathbb{N}_0$ and $c_{K,j} > 0$ such that

$$|T(\varphi)| \le c_{K,j}\|\varphi\|_{C^j(K)} \qquad \forall \varphi \in \mathcal{D}_K(\Omega). \tag{4.2}$$

Proof. We show that (a) implies (b). If T is continuous, then T is bounded by Theorem B.10.29.
For (b)⇒(c), suppose that T is bounded and let $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$ such that $\varphi_\ell \to \varphi$ with respect to τ. By part (d) of Theorem 4.4.6, there exists a compact set $K \subseteq \Omega$ such that

$$\{\varphi\} \cup \{\varphi_\ell \mid \ell \in \mathbb{N}\} \subseteq \mathcal{D}_K(\Omega)$$

and $\varphi_\ell \to \varphi$ in the Fréchet topology of $\mathcal{D}_K(\Omega)$.
The restriction

$$T \restriction_{\mathcal{D}_K(\Omega)} : \mathcal{D}_K(\Omega) \longrightarrow \mathbb{R}$$

is bounded. Indeed, the inclusion $\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}(\Omega)$ is continuous and therefore maps topologically bounded

[4] If $A \subseteq X$ is a subset of a vector space, its absolutely convex hull is defined by

$$\operatorname{cobal}(A) = \left\{ \sum_{j=1}^{r} \lambda_j x_j \;\middle|\; x_j \in A,\ \sum_{j=1}^{r} |\lambda_j| \le 1 \right\}.$$

It is the smallest absolutely convex subset of X containing A.

sets to topologically bounded sets; one then applies the boundedness of T. Since $\mathcal{D}_K(\Omega)$ is metrizable, Theorem B.10.29 shows that this restriction is continuous. Consequently,

$$T(\varphi_\ell) \longrightarrow T(\varphi).$$

(c)⇒(d). Let $K \subseteq \Omega$ be compact. Suppose that $\{\varphi_\ell\}_{\ell\in\mathbb{N}} \subseteq \mathcal{D}_K(\Omega)$ and that

$$d_K(\varphi_\ell, \varphi) \to 0.$$

By Proposition B.10.23, this means that $\varphi_\ell \to \varphi$ in $(\mathcal{D}_K(\Omega), \tau_K)$. Since the inclusion

$$\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}(\Omega)$$

is continuous, it follows that $\varphi_\ell \to \varphi$ with respect to τ. By (c),

$$T(\varphi_\ell) \to T(\varphi).$$

Thus, $T \restriction_{\mathcal{D}_K(\Omega)}$ is sequentially continuous. Since $\mathcal{D}_K(\Omega)$ is metrizable, we conclude that $T \restriction_{\mathcal{D}_K(\Omega)}$ is continuous.
(d)⇒(a). Let $V \subseteq \mathbb{R}$ be an absolutely convex neighborhood of the origin and set

$$U := T^{-1}(V).$$

The set U is absolutely convex. For every compact set $K \subseteq \Omega$, continuity of $T \restriction_{\mathcal{D}_K(\Omega)}$ implies that

$$U \cap \mathcal{D}_K(\Omega) = \left(T \restriction_{\mathcal{D}_K(\Omega)}\right)^{-1}(V)$$

is a neighborhood of the origin in $\mathcal{D}_K(\Omega)$. By the description of neighborhoods of the origin for the locally convex final topology in Theorem 4.4.1, U is a neighborhood of the origin in $\mathcal{D}(\Omega)$. Thus, T is continuous at the origin and, by linearity, at every point.
(d)⇔(e). Suppose that (d) holds and let $K \subseteq \Omega$ be compact. Since $T \restriction_{\mathcal{D}_K(\Omega)}$ is continuous at the origin, given $\varepsilon = 1$, there exist $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that

$$V_{K,j,n} \subseteq T^{-1}((-1,1)).$$

This means that

$$|T(\varphi)| \leq 1 \qquad \forall \varphi \in \mathcal{D}_K(\Omega) \text{ with } \|\varphi\|_{C^j(K)} < \frac{1}{n}.$$

If $\varphi \in \mathcal{D}_K(\Omega)$ and $\varphi \neq 0$, then

$$\left\| \frac{1}{2n} \frac{\varphi}{\|\varphi\|_{C^j(K)}} \right\|_{C^j(K)} < \frac{1}{n}.$$

Therefore,

$$\left| T\left(\frac{1}{2n} \frac{\varphi}{\|\varphi\|_{C^j(K)}} \right) \right| \leq 1.$$

By linearity of T, it follows that

$$|T(\varphi)| \leq 2n\|\varphi\|_{C^j(K)}.$$

This proves (e). Conversely, suppose that (e) holds and let $K \subseteq \Omega$ be compact. We wish to prove that

$$T \restriction_{\mathcal{D}_K(\Omega)} : \mathcal{D}_K(\Omega) \longrightarrow \mathbb{R}$$

is continuous.
By hypothesis, there exist $j \in \mathbb{N}_0$ and $c_K > 0$ such that

$$|T(\varphi)| \leq c_{K,j}\|\varphi\|_{C^j(K)} \qquad \forall \varphi \in \mathcal{D}_K(\Omega).$$

It suffices to show that $T \restriction_{\mathcal{D}_K(\Omega)}$ is continuous at 0. Since T is linear, continuity at the origin implies continuity at every point.
Let $\varepsilon > 0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{c_{K,j}}{\varepsilon}$, and therefore

$$\frac{c_{K,j}}{n} < \varepsilon.$$

Now consider $\varphi \in V_{K,j,n}$. By definition of $V_{K,j,n}$, this means that

$$\|\varphi\|_{C^j(K)} < \frac{1}{n}.$$

Applying the bound given by (e), we obtain

$$|T(\varphi)| \leq c_{K,j}\|\varphi\|_{C^j(K)} < \frac{c_{K,j}}{n} < \varepsilon.$$

Therefore,

$$\varphi \in V_{K,j,n} \quad \Longrightarrow \quad T(\varphi) \in (-\varepsilon, \varepsilon),$$

and consequently

$$V_{K,j,n} \subseteq \left(T \restriction_{\mathcal{D}_K(\Omega)}\right)^{-1}((-\varepsilon, \varepsilon)).$$

Since $V_{K,j,n}$ is a neighborhood of 0 in $\mathcal{D}_K(\Omega)$, we conclude that

$$\left(T \restriction_{\mathcal{D}_K(\Omega)}\right)^{-1}((-\varepsilon, \varepsilon))$$

is a neighborhood of 0. This proves that $T \restriction_{\mathcal{D}_K(\Omega)}$ is continuous at 0 and, by linearity, is continuous throughout $\mathcal{D}_K(\Omega)$. □

Definition 4.5.2. Let $\Omega \subseteq \mathbb{R}^n$ be open. The dual of $(\mathcal{D}(\Omega), \tau)$, denoted by $\mathcal{D}'(\Omega)$, is called ***the space of distributions defined on*** Ω, and its elements are called ***distributions***. We will frequently use the duality notation $\langle T, \varphi \rangle$ to denote $T(\varphi)$. The space $\mathcal{D}'(\Omega)$ is equipped with the weak-$*$ topology. Thus, a sequence $\{T_k\}_{k\in\mathbb{N}} \subseteq \mathcal{D}'(\Omega)$ converges to $T \in \mathcal{D}'(\Omega)$ if

$$T_k(\varphi) \to T(\varphi), \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

In this case, we say that $\{T_k\}_{k\in\mathbb{N}}$ converges to T *in the sense of distributions* or *in the distributional sense*.

Example 4.5.3. If $\psi \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, then the functional $\psi T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$(\psi T)(\varphi) := T(\psi\varphi), \qquad \varphi \in \mathcal{D}(\Omega),$$

is a distribution.
First observe that ψT is linear. If $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$ and $a, b \in \mathbb{R}$, then

$$(\psi T)(a\varphi_1 + b\varphi_2) = T\big(\psi(a\varphi_1 + b\varphi_2)\big) = T(a\psi\varphi_1 + b\psi\varphi_2) = a(\psi T)(\varphi_1) + b(\psi T)(\varphi_2).$$

We now show that ψT is continuous. Let $K \subseteq \Omega$ be compact. Since T is a distribution, by part (e) of Theorem 4.5.1, there exist $j \in \mathbb{N}_0$ and $c_K > 0$ such that

$$|T(\eta)| \leq c_K \|\eta\|_{C^j(K)}, \qquad \forall \eta \in \mathcal{D}_K(\Omega).$$

Now let $\varphi \in \mathcal{D}_K(\Omega)$. Then $\psi\varphi \in \mathcal{D}_K(\Omega)$ and therefore

$$|(\psi T)(\varphi)| = |T(\psi\varphi)| \leq c_K \|\psi\varphi\|_{C^j(K)}.$$

To estimate this seminorm, we use the Leibniz rule for partial derivatives with multi-indices (Proposition 1.1.17). If $|\alpha| \leq j$, then

$$\partial^\alpha(\psi\varphi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta \psi)(\partial^{\alpha-\beta}\varphi).$$

Taking the supremum over K gives

$$\|\partial^\alpha(\psi\varphi)\|_{C^0(K)} \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|\partial^\beta \psi\|_{C^0(K)} \|\partial^{\alpha-\beta}\varphi\|_{C^0(K)}.$$

Since $\psi \in C^\infty(\Omega)$ and K is compact, each derivative $\partial^\beta\psi$ is bounded on K. Define

$$M_\beta := \|\partial^\beta \psi\|_{C^0(K)}.$$

Then

$$\|\partial^\alpha(\psi\varphi)\|_{C^0(K)} \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} M_\beta \|\partial^{\alpha-\beta}\varphi\|_{C^0(K)}.$$

Summing over $|\alpha| \leq j$ shows that there exists a constant $C_K > 0$ such that

$$\|\psi\varphi\|_{C^j(K)} \leq C_K \|\varphi\|_{C^j(K)}.$$

Substituting into the preceding estimate yields

$$|(\psi T)(\varphi)| \leq c_K C_K \|\varphi\|_{C^j(K)}.$$

Applying part (e) of Theorem 4.5.1 again, we conclude that ψT is a distribution, that is,

$$\psi T \in \mathcal{D}'(\Omega).$$

Example 4.5.4. Let $f \in L^1_{\text{loc}}(\Omega)$. Define the functional

$$T_f : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}, \qquad T_f(\varphi) := \int_\Omega f(x)\varphi(x)\,dx.$$

Then T_f is a distribution. Linearity follows from linearity of the integral. To prove continuity, let $K \subseteq \Omega$ be compact and let $\varphi \in \mathcal{D}_K(\Omega)$. Then

$$|T_f(\varphi)| = \left| \int_K f(x)\varphi(x)\,dx \right| \leq \int_K |f(x)||\varphi(x)|\,dx.$$

Since $|\varphi(x)| \leq \|\varphi\|_{K,0}$ for every $x \in K$, we obtain

$$|T_f(\varphi)| \leq \|\varphi\|_{K,0} \int_K |f(x)|\,dx.$$

Defining

$$c_K := \int_K |f(x)|\,dx,$$

we have

$$|T_f(\varphi)| \leq c_K \|\varphi\|_{K,0}.$$

By part (e) of Theorem 4.5.1, we conclude that T_f is a distribution.

Example 4.5.5. Let $x_0 \in \Omega$. The functional

$$\delta_{x_0} : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}, \qquad \delta_{x_0}(\varphi) := \varphi(x_0),$$

is called the **Dirac delta** centered at x_0. Linearity is immediate. We show that it is continuous. Let $K \subseteq \Omega$ be compact with $x_0 \in K$ and let $\varphi \in \mathcal{D}_K(\Omega)$. Then

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \|\varphi\|_{C^0(K)} = \|\varphi\|_{K,0}.$$

Therefore,

$$|\delta_{x_0}(\varphi)| \leq \|\varphi\|_{K,0}.$$

Applying part (e) of Theorem 4.5.1, we conclude that δ_{x_0} is a distribution.

An important operation on distributions is the *restriction* of a distribution:

Definition 4.5.6. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $v \in \mathcal{D}'(\mathbb{R}^n)$. We define the restriction of v to Ω to be the distribution $v \restriction_\Omega \in \mathcal{D}'(\Omega)$ given by

$$\langle v \restriction_\Omega, \varphi \rangle := \langle v, \tilde{\varphi} \rangle, \qquad \forall \varphi \in \mathcal{D}(\Omega),$$

where $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^n)$ is the extension of φ by zero outside Ω.

4.6 Order of a distribution

In this section we define the order of a distribution. The preceding examples show that the integer $j \in \mathbb{N}_0$ in (4.2) may change with the choice of distribution and compact set. If the same integer works for every compact set $K \subseteq \Omega$, then the smallest integer for which (4.2) holds is known as the *order of the distribution* T.

Definition 4.6.1. Let $T \in \mathcal{D}'(\Omega)$. We say that T is a distribution of *order less than or equal to* m if there exists $m \in \mathbb{N}_0$ such that for every compact set $K \subseteq \Omega$ there exists a constant $c_K > 0$ with the property that

$$|T(\varphi)| \leq c_K \|\varphi\|_{K,m} \qquad \forall \varphi \in \mathcal{D}_K(\Omega).$$

If such an integer m exists, the smallest one is called the *order of the distribution* T and is denoted by $\operatorname{ord}(T)$. If no such integer exists, we say that T is a distribution of *infinite order*.

Example 4.6.2. The preceding examples allow us to identify the order of several important distributions.

(a) If $f \in L^1_{\text{loc}}(\Omega)$, then the distribution

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x)\,dx$$

has order 0.

If $K \subseteq \Omega$ is compact and $\varphi \in \mathcal{D}_K(\Omega)$, then

$$|T_f(\varphi)| = \left| \int_K f(x)\varphi(x)\,dx \right| \leq \int_K |f(x)||\varphi(x)|\,dx \leq \|\varphi\|_{K,0} \int_K |f(x)|\,dx.$$

Defining

$$c_K := \int_K |f(x)|\,dx,$$

we obtain

$$|T_f(\varphi)| \leq c_K \|\varphi\|_{K,0}.$$

Therefore, the integer $m = 0$ works in the preceding definition, and we conclude that

$$\operatorname{ord}(T_f) = 0.$$

(b) For $x_0 \in \Omega$, the Dirac delta

$$\delta_{x_0}(\varphi) = \varphi(x_0)$$

is also a distribution of order 0.

If $K \subseteq \Omega$ is compact and $x_0 \in K$, then for $\varphi \in \mathcal{D}_K(\Omega)$

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \|\varphi\|_{C^0(K)} = \|\varphi\|_{K,0}.$$

Therefore,

$$|\delta_{x_0}(\varphi)| \leq \|\varphi\|_{K,0},$$

and again the integer $m = 0$ satisfies the condition in the definition, implying

$$\operatorname{ord}(\delta_{x_0}) = 0.$$

(c) If $\psi \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$ has order m, then ψT also has order less than or equal to m.

The estimate in the preceding example provides a constant $C_K > 0$ such that

$$\|\psi\varphi\|_{K,m} \le C_K \|\varphi\|_{K,m}.$$

Since T has order m, there exists $c_K > 0$ such that

$$|T(\eta)| \le c_K \|\eta\|_{K,m}.$$

Taking $\eta = \psi\varphi$, we obtain

$$|(\psi T)(\varphi)| = |T(\psi\varphi)| \le c_K \|\psi\varphi\|_{K,m} \le c_K C_K \|\varphi\|_{K,m}.$$

Therefore, ψT satisfies the definition with the same integer m, and thus

$$\operatorname{ord}(\psi T) \le m.$$

(d) Now let μ be a signed Radon measure[1] on Ω. Define

$$T_\mu(\varphi) = \int_\Omega \varphi \, d\mu, \qquad \varphi \in \mathcal{D}(\Omega).$$

First observe that T_μ is linear by linearity of the integral.

We show that it is continuous. Let $K \subseteq \Omega$ be compact and let $\varphi \in \mathcal{D}_K(\Omega)$. Since φ vanishes outside K, we have

$$T_\mu(\varphi) = \int_K \varphi \, d\mu.$$

Let $|\mu|$ be the total variation[2] of μ. Then

$$|T_\mu(\varphi)| = \left| \int_K \varphi \, d\mu \right| \le \int_K |\varphi| \, d|\mu|.$$

Since $|\varphi(x)| \le \|\varphi\|_{K,0}$ for every $x \in K$, we obtain

$$|T_\mu(\varphi)| \le \|\varphi\|_{K,0} \int_K d|\mu| = |\mu|(K) \, \|\varphi\|_{K,0}.$$

Defining

$$c_K := |\mu|(K),$$

we obtain

$$|T_\mu(\varphi)| \le c_K \|\varphi\|_{K,0}.$$

By the definition of the order of a distribution, we conclude that

$$\operatorname{ord}(T_\mu) = 0.$$

It turns out that all distributions of order 0 can be identified with measures.

Theorem 4.6.3. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $T \in \mathcal{D}'(\Omega)$.

(a) If T is positive, that is, if

$$T(\varphi) \geq 0, \qquad \forall \varphi \in \mathcal{D}(\Omega) \text{ with } \varphi \geq 0,$$

then there exists a unique positive Radon measure $\mu\colon \mathcal{B}(\Omega) \longrightarrow [0,\infty]$ such that

$$T(\varphi) = \int_\Omega \varphi \, d\mu, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

(b) If T has order 0, then there exist two positive Radon measures

$$\mu_1, \mu_2\colon \mathcal{B}(\Omega) \longrightarrow [0,\infty]$$

such that

$$T(\varphi) = \int_\Omega \varphi \, d\mu_1 - \int_\Omega \varphi \, d\mu_2, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Proof. Recall that $C_c(\Omega)$ is equipped with the strict inductive limit topology

$$C_c(\Omega) = \varinjlim_{i\in\mathbb{N}} C_{K_i}(\Omega),$$

where $\{K_i\}_{i\in\mathbb{N}}$ is an increasing exhaustion of Ω by compact sets and

$$C_{K_i}(\Omega) := \{u \in C_c(\Omega) \mid \operatorname{supp}(u) \subseteq K_i\}$$

is equipped with the uniform norm

$$\|u\|_{K_i,\infty} := \|u\|_{C^0(K_i)}.$$

In particular, a linear functional $L\colon C_c(\Omega) \longrightarrow \mathbb{R}$ is continuous if and only if for every compact set $K \subset \Omega$ there exists a constant $C_K > 0$ such that

$$|L(u)| \leq C_K \|u\|_{K,\infty}, \qquad \forall u \in C_c(\Omega) \text{ with } \operatorname{supp}(u) \subset K.$$

[1] A *signed measure* on Ω is a function $\mu\colon \mathcal{B}(\Omega) \longrightarrow [-\infty,\infty]$ defined on the Borel σ-algebra $\mathcal{B}(\Omega)$ such that $\mu(\varnothing) = 0$, taking at most one of the values ∞ and $-\infty$, and satisfying $\mu\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu(E_k)$ for every countable family $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{B}(\Omega)$ of disjoint sets. This is the global notion of Definition B.5.19. To be Radon, its positive and negative variations must additionally be Radon; see Definition B.5.22.

[2] Let μ be a signed measure. Its *total variation* is the positive measure $|\mu|\colon \mathcal{B}(\Omega) \longrightarrow [0,\infty]$ defined by

$$|\mu|(E) = \sup\left\{ \sum_{k=1}^\infty |\mu(E_k)| \;\middle|\; E = \bigcup_{k=1}^\infty E_k,\ E_k \in \mathcal{B}(\Omega),\ E_i \cap E_j = \varnothing,\ i \neq j \right\}.$$

Equivalently, if $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ, then $|\mu| = \mu^+ + \mu^-$.

(a) First suppose that T is positive. We will show that T has order 0.

Let $K \subset \Omega$ be compact. Take a bounded open set $U \subset \Omega$ such that

$$K \subset U \subset \overline{U} \subset \Omega.$$

Consider a cutoff function $\psi \in C^\infty(\Omega)$ such that

$$\psi \equiv 1 \text{ in } K, \qquad \operatorname{supp}(\psi) \subset U, \qquad 0 \le \psi \le 1.$$

In particular, $\psi \in \mathcal{D}(\Omega)$.

Now let $\varphi \in \mathcal{D}_K(\Omega)$. Since $\psi \equiv 1$ on K and $\operatorname{supp}(\varphi) \subset K$, we have

$$|\varphi(x)| \le \|\varphi\|_{K,0}\psi(x), \qquad \forall x \in \Omega.$$

It follows that

$$\|\varphi\|_{K,0}\psi - \varphi \ge 0, \qquad \|\varphi\|_{K,0}\psi + \varphi \ge 0.$$

Since T is positive,

$$T(\|\varphi\|_{K,0}\psi - \varphi) \ge 0, \qquad T(\|\varphi\|_{K,0}\psi + \varphi) \ge 0.$$

By linearity,

$$-\|\varphi\|_{K,0}T(\psi) \le T(\varphi) \le \|\varphi\|_{K,0}T(\psi),$$

and therefore

$$|T(\varphi)| \le T(\psi)\|\varphi\|_{K,0}, \qquad \forall \varphi \in \mathcal{D}_K(\Omega).$$

This shows that T has order 0.

Now take a sequence of bounded open sets $\{\Omega_i\}_{i\in\mathbb{N}}$ such that

$$\Omega_i \Subset \Omega_{i+1}, \qquad \bigcup_{i=1}^{\infty} \Omega_i = \Omega,$$

and define $K_i := \overline{\Omega_i}$. Then each K_i is compact and $K_i \subset \Omega$.

Since T has order 0, for each i there exists $c_i > 0$ such that

$$|T(\varphi)| \le c_i\|\varphi\|_{K_i,0}, \qquad \forall \varphi \in \mathcal{D}_{K_i}(\Omega). \tag{4.3}$$

Let $u \in C_c(\Omega_i)$. Since $\operatorname{supp}(u)$ is compact and contained in Ω_i, there exists a sequence

$$u_k \in \mathcal{D}_{K_i}(\Omega)$$

such that

$$u_k \to u$$

uniformly on K_i.

By (4.3),

$$|T(u_k - u_\ell)| \le c_i \|u_k - u_\ell\|_{K_i,0} \to 0 \qquad (k,\ell \to \infty).$$

Consequently, $\{T(u_k)\}$ is Cauchy in $\mathbb{R}$, and we define

$$T_i(u) := \lim_{k\to\infty} T(u_k).$$

We show that this definition does not depend on the approximating sequence. Suppose that

$$v_k \in \mathcal{D}_{K_i}(\Omega), \qquad v_k \to u$$

uniformly on K_i. Then

$$|T(u_k - v_k)| \le c_i \|u_k - v_k\|_{K_i,0} \le c_i \left(\|u_k - u\|_{K_i,0} + \|u - v_k\|_{K_i,0}\right) \to 0,$$

and it follows that

$$\lim_{k\to\infty} T(u_k) = \lim_{k\to\infty} T(v_k).$$

Therefore, $T_i(u)$ is well defined.

Now let $u, v \in C_c(\Omega_i)$ and $\alpha, \beta \in \mathbb{R}$. Take sequences

$$u_k \to u, \qquad v_k \to v$$

uniformly on K_i, with

$$u_k, v_k \in \mathcal{D}_{K_i}(\Omega).$$

Then

$$\alpha u_k + \beta v_k \to \alpha u + \beta v$$

uniformly on K_i, and by linearity of T we obtain

$$T_i(\alpha u + \beta v) = \lim_{k\to\infty} T(\alpha u_k + \beta v_k) = \alpha T_i(u) + \beta T_i(v).$$

Thus, T_i is linear.

Moreover, if $u \ge 0$, we can choose an approximating sequence $u_k \ge 0$. Then

$$T(u_k) \ge 0,$$

and passing to the limit gives

$$T_i(u) \ge 0.$$

Therefore, T_i is positive.

Passing to the limit in (4.3) gives

$$|T_i(u)| \le c_i \|u\|_{K_i,0}, \qquad \forall u \in C_c(\Omega_i),$$

and consequently T_i is continuous with respect to the uniform norm.

If $u \in C_c(\Omega_i)$, any approximating sequence valid for T_i is also valid for T_{i+1}, so

$$T_{i+1}(u) = T_i(u).$$

Thus the functionals T_i are compatible and define a linear functional

$$L\colon C_c(\Omega) \longrightarrow \mathbb{R}$$

given by

$$L(u) = T_i(u) \qquad (u \in C_c(\Omega_i)).$$

Now let $K \subset \Omega$ be compact. Choose i such that $K \subset \Omega_i$. If $\operatorname{supp}(u) \subset K$, then

$$|L(u)| = |T_i(u)| \le c_i \|u\|_{K_i,0} = c_i \|u\|_{K,0}.$$

This shows that L is locally bounded and therefore continuous on $C_c(\Omega)$.

Applying the Riesz representation theorem (Theorem B.5.23), we obtain a unique positive Radon measure μ such that

$$L(u) = \int_\Omega u \, d\mu, \qquad \forall u \in C_c(\Omega).$$

Since L extends T, we conclude that

$$T(\varphi) = \int_\Omega \varphi \, d\mu, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

(b) Now suppose that T has order 0. Then, for every compact set $K \subset \Omega$ there exists $C_K > 0$ such that

$$|T(\varphi)| \le C_K \|\varphi\|_{K,0}, \qquad \forall \varphi \in \mathcal{D}_K(\Omega).$$

Repeating the preceding procedure, we obtain mutually compatible continuous linear functionals

$$T_i\colon C_c(\Omega_i) \longrightarrow \mathbb{R}$$

inducing a linear functional

$$L\colon C_c(\Omega) \longrightarrow \mathbb{R}$$

that extends T.

Moreover, the same argument shows that L is locally bounded. Since Ω is locally compact, Hausdorff, and σ-compact, part (b) of Theorem B.5.23 directly provides two positive Radon measures μ_1, μ_2 such that

$$L(u) = \int_\Omega u \, d\mu_1 - \int_\Omega u \, d\mu_2, \qquad \forall u \in C_c(\Omega).$$

Each integral is finite because u has compact support. The difference of the measures need not be defined on every Borel subset of Ω; the representation is understood in the local sense of Definition B.5.22. Therefore,

$$T(\varphi) = \int_\Omega \varphi \, d\mu_1 - \int_\Omega \varphi \, d\mu_2, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

□

4.7 Distributional derivatives

The notion of a weak derivative motivated the concept of a distribution because, as we saw earlier, some functions have no weak derivatives but are *sufficiently well behaved* in the sense that they should admit some kind of *generalized* derivative. The concept of a distribution will allow us to extend the notion of a derivative as follows:

Definition 4.7.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $T \in \mathcal{D}'(\Omega)$. Given a multi-index $\alpha \in \mathbb{N}_0^n \setminus \{0\}$, we define the αth *distributional* derivative of T by

$$D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \varphi), \qquad \varphi \in \mathcal{D}(\Omega).$$

For $j \in \mathbb{N}$, the symbol $\nabla^j T$ represents the collection of all αth distributional derivatives of T with $|\alpha| = j$.

Remark 4.7.2. Naturally, the αth distributional derivative of a distribution T must again be a distribution. To prove this, note that by Theorem 4.5.1, if $T \in \mathcal{D}'(\Omega)$ and $K \subseteq \Omega$ is compact, then there exist $j \in \mathbb{N}_0$ and a constant $c_K > 0$ such that

$$|T(\varphi)| \leq c_K \|\varphi\|_{C^j(K)}, \qquad \forall \varphi \in \mathcal{D}_K(\Omega).$$

Then

$$|D^\alpha T(\varphi)| = |T(D^\alpha \varphi)| \leq c_K \|D^\alpha \varphi\|_{C^j(K)} \leq c_K \|\varphi\|_{C^{j+|\alpha|}(K)}, \qquad \forall \varphi \in \mathcal{D}_K(\Omega).$$

Again by Theorem 4.5.1, we obtain $D^\alpha T \in \mathcal{D}'(\Omega)$.
Moreover, this estimate allows us to control the order of the distributional derivative. If T has finite order $m = \operatorname{ord}(T)$, taking $j = m$ in the preceding inequality gives

$$\operatorname{ord}(D^\alpha T) \leq m + |\alpha|.$$

If T does not have finite order, then $\operatorname{ord}(T) = \infty$ and the preceding inequality remains valid in $\mathbb{N}_0 \cup \{\infty\}$, since

$$\operatorname{ord}(D^\alpha T) \leq \infty = \infty + |\alpha|.$$

In particular, if $u \in L^1_{\mathrm{loc}}(\Omega)$, T_u is the distribution induced by u, and α is a multi-index, then the αth weak derivative of u, $D^\alpha u \in L^1_{\mathrm{loc}}(\Omega)$ (if it exists), has the property that its induced distribution $T_{D^\alpha u} \in \mathcal{D}'(\Omega)$ is the αth distributional derivative of T_u, that is,

$$D^\alpha T_u = T_{D^\alpha u}.$$

If $\varphi \in \mathcal{D}(\Omega)$, then by definition of distributional derivative

$$D^\alpha T_u(\varphi) = (-1)^{|\alpha|} T_u(D^\alpha \varphi) = (-1)^{|\alpha|} \int_\Omega u\, D^\alpha \varphi.$$

Since $D^\alpha u$ is the weak derivative of u, we have

$$\int_\Omega u\, D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega D^\alpha u\, \varphi,$$

and therefore

$$D^\alpha T_u(\varphi) = \int_\Omega D^\alpha u\, \varphi = T_{D^\alpha u}(\varphi).$$

Since this holds for every $\varphi \in \mathcal{D}(\Omega)$, we conclude that

$$D^\alpha T_u = T_{D^\alpha u}.$$

Directional derivatives extend similarly:

Definition 4.7.3. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $T \in \mathcal{D}'(\Omega)$. Given a vector $\nu \in \mathbb{R}^n \setminus \{0\}$, we define the kth directional derivative of T in the direction of ν by

$$\frac{\partial^k T}{\partial \nu^k}(\varphi) := (-1)^k T\left(\frac{\partial^k \varphi}{\partial \nu^k}\right), \qquad \varphi \in \mathcal{D}(\Omega).$$

In particular, if $u \in L^1_{\mathrm{loc}}(\Omega)$, $\nu \in \mathbb{R}^n \setminus \{0\}$, and $k \in \mathbb{N}$, then the kth weak or distributional directional derivative of u in the direction ν is the distribution

$$\frac{\partial^k T_u}{\partial \nu^k} \in \mathcal{D}'(\Omega).$$

Other differential operators, such as the Laplacian, admit extensions to the distributional sense.

Example 4.7.4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u \in L^1_{\mathrm{loc}}(\Omega)$. The Laplacian of u in the distributional sense is defined as the distribution

$$T_{\Delta u} := \sum_{i=1}^n \partial_i^2 T_u.$$

In other words, for each $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} T_{\Delta u}(\varphi) &= \sum_{i=1}^n \partial_i^2 T_u(\varphi) = \sum_{i=1}^n T_u(\partial_i^2 \varphi) \\ &= \sum_{i=1}^n \int_\Omega u\, \partial_i^2 \varphi = \int_\Omega u\, \Delta \varphi. \end{aligned}$$

We say that u is *subharmonic* on Ω if its distributional Laplacian is positive, that is, if

$$T_{\Delta u}(\varphi) \geq 0, \qquad \forall \varphi \in \mathcal{D}(\Omega) \text{ with } \varphi \geq 0.$$

In this case, $T_{\Delta u}$ is a positive distribution. By Theorem 4.6.3, every positive distribution is a positive Radon measure. Therefore, there exists a unique positive Radon measure μ on Ω such that

$$T_{\Delta u}(\varphi) = \int_\Omega \varphi\, d\mu, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Consequently, the distributional Laplacian of a subharmonic function can be identified with a positive Radon measure.

Example 4.7.5. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $F = (F_1, \dots, F_n) \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$. The divergence of F in the distributional sense is defined as the distribution

$$T_{\operatorname{div} F} := \sum_{i=1}^{n} \partial_i T_{F_i}.$$

In other words, for each $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} T_{\operatorname{div} F}(\varphi) &= \sum_{i=1}^{n} \partial_i T_{F_i}(\varphi) = -\sum_{i=1}^{n} T_{F_i}(\partial_i \varphi) \\ &= -\sum_{i=1}^{n} \int_\Omega F_i \, \partial_i \varphi = -\int_\Omega F \cdot \nabla \varphi. \end{aligned}$$

Exercise 4.7.6. Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$u(x) := \begin{cases} \cos(x), & \text{if } -\pi \le x \le 0, \\ 1 - \dfrac{x}{\pi}, & \text{if } 0 < x \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute the derivative of u in the distributional sense and find its order.

(b) Let $v := u \restriction_{(-\pi,\pi)}$. Compute the first and second derivatives of v in the distributional sense on $(-\pi, \pi)$ and find their orders.

(c) Let $w : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable on $\mathbb{R} \setminus \{a\}$ and suppose that

$$\lim_{x \to a^-} w(x), \qquad \lim_{x \to a^+} w(x).$$

exist. Compute the derivative of w in the distributional sense. What can you say about its order? Under what conditions can one conclude that it has order 0?

Solution. (a) Let $\varphi \in \mathcal{D}(\mathbb{R})$. Since $u \in L^1_{\text{loc}}(\mathbb{R})$, the function u induces a distribution

$$T_u : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}, \qquad T_u(\varphi) := \int_{\mathbb{R}} u(x)\varphi(x)\, dx.$$

By definition of distributional derivative,

$$T_u'(\varphi) := -T_u(\varphi') = -\int_{\mathbb{R}} u(x)\varphi'(x)\, dx.$$

Since u is given by different expressions on $[-\pi, 0]$ and on $(0, \pi]$, we have

$$T_u'(\varphi) = -\int_{-\pi}^{0} \cos(x)\varphi'(x)\,dx - \int_{0}^{\pi} \left(1 - \frac{x}{\pi}\right)\varphi'(x)\,dx.$$

We integrate each term by parts. For the first,

$$\begin{aligned} -\int_{-\pi}^{0} \cos(x)\varphi'(x)\,dx &= -\Big[\cos(x)\varphi(x)\Big]_{-\pi}^{0} + \int_{-\pi}^{0} (-\sin x)\varphi(x)\,dx \\ &= -\varphi(0) - \varphi(-\pi) + \int_{-\pi}^{0} (-\sin x)\varphi(x)\,dx, \end{aligned}$$

since $\cos(0) = 1$ and $\cos(-\pi) = -1$.

For the second term,

$$-\int_{0}^{\pi} \left(1 - \frac{x}{\pi}\right)\varphi'(x)\,dx = -\Big[\left(1 - \frac{x}{\pi}\right)\varphi(x)\Big]_{0}^{\pi} - \frac{1}{\pi}\int_{0}^{\pi} \varphi(x)\,dx$$

$$= \varphi(0) - \frac{1}{\pi}\int_{0}^{\pi} \varphi(x)\,dx.$$

Adding the two expressions gives

$$T_u'(\varphi) = -\varphi(-\pi) + \int_{-\pi}^{0} (-\sin x)\varphi(x)\,dx - \frac{1}{\pi}\int_{0}^{\pi} \varphi(x)\,dx.$$

This can be written as

$$T_u'(\varphi) = T_f(\varphi) - \delta_{-\pi}(\varphi),$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$f(x) := \begin{cases} -\sin x, & \text{if } -\pi < x < 0, \\ -\dfrac{1}{\pi}, & \text{if } 0 < x < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$T_u' = T_f - \delta_{-\pi}.$$

In other words,

$$u' = f - \delta_{-\pi}$$

in the distributional sense.

We now determine its order. Since $f \in L^1_{\text{loc}}(\mathbb{R})$, the distribution T_f has order 0. Moreover, $\delta_{-\pi}$ also has order 0. It follows that u', being a sum of distributions of order 0, is a distribution of order 0.

(b) Now consider $v := u \restriction_{(-\pi,\pi)}$. Then

$$v(x) = \begin{cases} \cos(x), & \text{if } -\pi < x \le 0, \\ 1 - \dfrac{x}{\pi}, & \text{if } 0 < x < \pi. \end{cases}$$

Moreover,

$$\lim_{x\to 0^-} v(x) = \cos(0) = 1 \qquad \text{and} \qquad \lim_{x\to 0^+} v(x) = 1 - \frac{0}{\pi} = 1,$$

so v is continuous at 0. In particular, $v \in L^1_{\text{loc}}((-\pi,\pi))$, and therefore it induces a distribution

$$T_v \colon \mathcal{D}((-\pi,\pi)) \longrightarrow \mathbb{R}, \qquad T_v(\varphi) := \int_{-\pi}^{\pi} v(x)\varphi(x)\,dx.$$

We compute its first distributional derivative. Let $\varphi \in \mathcal{D}((-\pi,\pi))$. By definition,

$$T_v'(\varphi) := -T_v(\varphi') = -\int_{-\pi}^{\pi} v(x)\varphi'(x)\,dx.$$

Splitting the integral according to the definition of v gives

$$T_v'(\varphi) = -\int_{-\pi}^{0} \cos(x)\varphi'(x)\,dx - \int_{0}^{\pi} \left(1 - \frac{x}{\pi}\right)\varphi'(x)\,dx.$$

We integrate by parts on each interval. For the first,

$$\begin{aligned} -\int_{-\pi}^{0} \cos(x)\varphi'(x)\,dx &= -\Big[\cos(x)\varphi(x)\Big]_{-\pi}^{0} + \int_{-\pi}^{0} (-\sin x)\varphi(x)\,dx \\ &= -\varphi(0) - \varphi(-\pi) + \int_{-\pi}^{0} (-\sin x)\varphi(x)\,dx. \end{aligned}$$

Now, since $\varphi \in \mathcal{D}((-\pi,\pi))$, we have $\varphi(-\pi) = 0$. Consequently,

$$-\int_{-\pi}^{0} \cos(x)\varphi'(x)\,dx = -\varphi(0) + \int_{-\pi}^{0} (-\sin x)\varphi(x)\,dx.$$

For the second term,

$$\begin{aligned} -\int_{0}^{\pi} \left(1 - \frac{x}{\pi}\right)\varphi'(x)\,dx &= -\left[\left(1 - \frac{x}{\pi}\right)\varphi(x)\right]_{0}^{\pi} - \frac{1}{\pi}\int_{0}^{\pi} \varphi(x)\,dx \\ &= \varphi(0) - \frac{1}{\pi}\int_{0}^{\pi} \varphi(x)\,dx, \end{aligned}$$

since $\left(1 - \frac{\pi}{\pi}\right)\varphi(\pi) = 0$.

Adding the two expressions, the terms at $\varphi(0)$ cancel and we obtain

$$T_v'(\varphi) = \int_{-\pi}^{0} (-\sin x)\varphi(x)\,dx - \frac{1}{\pi}\int_{0}^{\pi} \varphi(x)\,dx.$$

This can be written as

$$T_v'(\varphi) = \int_{-\pi}^{\pi} g(x)\varphi(x)\,dx,$$

where

$$g(x) := \begin{cases} -\sin x, & \text{if } -\pi < x < 0, \\ -\dfrac{1}{\pi}, & \text{if } 0 < x < \pi. \end{cases}$$

Therefore,

$$T_v' = T_g.$$

Consequently,

$$v' = \begin{cases} -\sin x, & \text{if } -\pi < x < 0, \\ -\dfrac{1}{\pi}, & \text{if } 0 < x < \pi, \end{cases} \qquad \text{in the distributional sense.}$$

Since $g \in L^1_{\mathrm{loc}}((-\pi,\pi))$, the distribution T_g has order 0. Consequently, v' is a distribution of order 0.

We now compute the second distributional derivative of v. Since $T_v' = T_g$, we have

$$T_v'' = (T_v')' = (T_g)'.$$

Let $\varphi \in \mathcal{D}((-\pi,\pi))$. Then

$$T_v''(\varphi) = T_g'(\varphi) = -T_g(\varphi') = -\int_{-\pi}^{0} (-\sin x)\varphi'(x)\,dx + \frac{1}{\pi}\int_0^{\pi} \varphi'(x)\,dx.$$

We integrate each term by parts. For the first,

$$-\int_{-\pi}^{0} (-\sin x)\varphi'(x)\,dx = -\Big[(-\sin x)\varphi(x)\Big]_{-\pi}^{0} + \int_{-\pi}^{0} (-\cos x)\varphi(x)\,dx.$$

Since $\sin(0) = 0$ and $\sin(-\pi) = 0$, the boundary term vanishes, so

$$-\int_{-\pi}^{0} (-\sin x)\varphi'(x)\,dx = \int_{-\pi}^{0} (-\cos x)\varphi(x)\,dx.$$

For the second term,

$$\frac{1}{\pi}\int_0^{\pi} \varphi'(x)\,dx = \frac{1}{\pi}\big[\varphi(x)\big]_0^{\pi} = -\frac{1}{\pi}\varphi(0),$$

since $\varphi(\pi) = 0$.

Substituting into the expression for $T_v''(\varphi)$ gives

$$T_v''(\varphi) = \int_{-\pi}^{0} (-\cos x)\varphi(x)\,dx - \frac{1}{\pi}\varphi(0).$$

This means that

$$T_v'' = T_h - \frac{1}{\pi}\delta_0,$$

where

$$h(x) := \begin{cases} -\cos x, & \text{if } -\pi < x < 0, \\ 0, & \text{if } 0 < x < \pi. \end{cases}$$

In other words,

$$v'' = \begin{cases} -\cos x, & \text{if } -\pi < x < 0, \\ 0, & \text{if } 0 < x < \pi, \end{cases} \; - \frac{1}{\pi}\delta_0$$

in the distributional sense.

We now examine its order. Since $h \in L^1_{\mathrm{loc}}((-\pi,\pi))$, the distribution T_h has order 0. Moreover, δ_0 has order 0, so v'' is a distribution of order 0.

(c) Now let $w\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function differentiable on $\mathbb{R}\setminus\{a\}$, and suppose that the one-sided limits

$$\lim_{x\to a^-} w(x) \qquad \text{and} \qquad \lim_{x\to a^+} w(x).$$

exist. We denote these limits by

$$w(a^-) := \lim_{x\to a^-} w(x), \qquad w(a^+) := \lim_{x\to a^+} w(x).$$

By hypothesis, both real numbers are well defined.

Since w is differentiable on $\mathbb{R}\setminus\{a\}$, it is in particular continuous on $(-\infty,a)$ and on (a,∞). Moreover, since the finite one-sided limits at a exist, the function w is locally bounded near a. It follows that

$$w \in L^1_{\mathrm{loc}}(\mathbb{R}),$$

and therefore w induces a distribution

$$T_w\colon \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}, \qquad T_w(\varphi) := \int_{\mathbb{R}} w(x)\varphi(x)\,dx.$$

We compute its distributional derivative. Let $\varphi \in \mathcal{D}(\mathbb{R})$. By definition,

$$T_w'(\varphi) := -T_w(\varphi') = -\int_{\mathbb{R}} w(x)\varphi'(x)\,dx.$$

We split the integral over the intervals $(-\infty,a)$ and (a,∞):

$$T_w'(\varphi) = -\int_{-\infty}^{a} w(x)\varphi'(x)\,dx - \int_{a}^{\infty} w(x)\varphi'(x)\,dx.$$

Suppose also that the classical derivative of w on $\mathbb{R}\setminus\{a\}$, denoted by w'_{cl}, belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ after being extended arbitrarily at the point a. We can then integrate by parts on each of the two preceding intervals. We obtain

$$\begin{aligned} -\int_{-\infty}^{a} w(x)\varphi'(x)\,dx &= -\Big[w(x)\varphi(x)\Big]_{-\infty}^{a} + \int_{-\infty}^{a} w'_{\mathrm{cl}}(x)\varphi(x)\,dx \\ &= -w(a^-)\varphi(a) + \int_{-\infty}^{a} w'_{\mathrm{cl}}(x)\varphi(x)\,dx, \end{aligned}$$

since φ has compact support.

Similarly,

$$\begin{aligned} -\int_{a}^{\infty} w(x)\varphi'(x)\,dx &= -\Big[w(x)\varphi(x)\Big]_{a}^{\infty} + \int_{a}^{\infty} w'_{\mathrm{cl}}(x)\varphi(x)\,dx \\ &= w(a^+)\varphi(a) + \int_{a}^{\infty} w'_{\mathrm{cl}}(x)\varphi(x)\,dx. \end{aligned}$$

Adding the two equalities gives

$$T_w'(\varphi) = \int_{-\infty}^{a} w_{\mathrm{cl}}'(x)\varphi(x)\,dx + \int_{a}^{\infty} w_{\mathrm{cl}}'(x)\varphi(x)\,dx + \big(w(a^+) - w(a^-)\big)\varphi(a).$$

Since a point has measure zero, we can write

$$\int_{-\infty}^{a} w_{\mathrm{cl}}'(x)\varphi(x)\,dx + \int_{a}^{\infty} w_{\mathrm{cl}}'(x)\varphi(x)\,dx = \int_{\mathbb{R}} w_{\mathrm{cl}}'(x)\varphi(x)\,dx.$$

Consequently,

$$T_w'(\varphi) = \int_{\mathbb{R}} w_{\mathrm{cl}}'(x)\varphi(x)\,dx + \big(w(a^+) - w(a^-)\big)\varphi(a),$$

that is,

$$T_w' = T_{w_{\mathrm{cl}}'} + \big(w(a^+) - w(a^-)\big)\delta_a.$$

In other words,

$$w' = w_{\mathrm{cl}}' + \big(w(a^+) - w(a^-)\big)\delta_a$$

in the distributional sense.

We now consider the order of T_w'. Since T_w is a distribution of order 0, Remark 4.7.2 implies that T_w' is a distribution of order at most 1.

If also $w_{\mathrm{cl}}' \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $T_{w_{\mathrm{cl}}'}$ has order 0, and since δ_a also has order 0, we conclude that

$$T_w' = T_{w_{\mathrm{cl}}'} + \big(w(a^+) - w(a^-)\big)\delta_a$$

is a distribution of order 0.

In particular, this occurs if the two branches of w, extended at a with the values $w(a^-)$ and $w(a^+)$, are absolutely continuous on $[b,a]$ and $[a,c]$ for any $b < a < c$. The fundamental theorem of calculus for absolutely continuous functions gives integrability of $|w_{\mathrm{cl}}'|$ on (b,a) and on (a,c). Every compact subset of $\mathbb{R}$ is contained in some interval $[b,c]$ of this type, so $w_{\mathrm{cl}}' \in L^1_{\mathrm{loc}}(\mathbb{R})$. This condition controls the derivative as one approaches a; absolute continuity on compact intervals separated from a is insufficient for this.

□

After studying several examples of distributions obtained as derivatives, it is natural to ask to what extent every distribution admits such a representation. The following theorem gives a local answer:

Theorem 4.7.7. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $T \in \mathcal{D}'(\Omega)$. Then for every compact set $K \subseteq \Omega$ there exist a continuous function $u\colon \Omega \longrightarrow \mathbb{R}$ and a multi-index α (both depending on K) such that

$$T(\varphi) = (-1)^{|\alpha|} \int_{\Omega} u D^{\alpha}\varphi, \qquad \forall \varphi \in \mathcal{D}_K(\Omega).$$

Proof. Let $K \subseteq \Omega$ be a fixed compact set. Since K is compact and Ω is open, there exists an open set U_0 such that $K \subseteq U_0 \Subset \Omega$. By Proposition 1.3.6, there exists $\chi \in C^{\infty}(\Omega)$ such that $\chi = 1$ on K and $\operatorname{supp}(\chi) \subseteq U_0$. In

particular, $\chi \in \mathcal{D}(\Omega)$. Define $S \in \mathcal{D}'(\mathbb{R}^n)$ by

$$S(\psi) := T\big((\chi\psi)\restriction_\Omega\big), \qquad \psi \in \mathcal{D}(\mathbb{R}^n).$$

Multiplication by χ and restriction to Ω are continuous on the corresponding test function spaces; therefore, S is a distribution. Moreover, $\operatorname{supp}(S) \subseteq \operatorname{supp}(\chi)$.
Choose a closed cube

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

such that $\operatorname{supp}(\chi) \subseteq \operatorname{Int}(Q)$. This choice does not require Q to be contained in Ω. If $\varphi \in \mathcal{D}_K(\Omega)$ and $\widetilde{\varphi}$ denotes its extension by zero to $\mathbb{R}^n$, then $\widetilde{\varphi} \in \mathcal{D}_Q(\mathbb{R}^n)$ and

$$S(\widetilde{\varphi}) = T(\chi\varphi) = T(\varphi). \tag{4.4}$$

Denote by $\beta := (1, \dots, 1)$ the multi-index whose components are all equal to 1.
Now let $\psi \in \mathcal{D}_Q(\mathbb{R}^n)$. For $x \in Q$ define the subcube

$$Q(x) := \{y \in Q \mid a_i \le y_i \le x_i,\ i = 1, \dots, n\}.$$

Since $\psi \in \mathcal{D}_Q(\mathbb{R}^n)$, its support is contained in the closed cube Q. Since ψ is of class C^∞ and vanishes identically on $\mathbb{R}^n \setminus Q$, both ψ and all its partial derivatives must vanish on the boundary ∂Q. This ensures that when Theorem B.6.5 is applied in each variable, integrating from the lower endpoint a_i, the boundary terms vanish.
Theorem B.6.5, applied in the first variable, gives

$$\psi(x) = \psi(a_1, x_2, \dots, x_n) + \int_{a_1}^{x_1} \partial_1\psi(y_1, x_2, \dots, x_n)\, dy_1 = \int_{a_1}^{x_1} \partial_1\psi(y_1, x_2, \dots, x_n)\, dy_1.$$

Since $\partial_1\psi$ vanishes on ∂Q, it is in particular zero when the second coordinate is exactly a_2. This allows us to apply Theorem B.6.5 to the integrand with respect to the second variable without boundary terms appearing. Repeating this argument successively in each of the n variables—and using at each step the fact that the intermediate derivatives also vanish on the boundary of Q—gives the integral identity

$$\psi(x) = \int_{Q(x)} D^\beta\psi(y)\, dy, \qquad \forall x \in Q. \tag{4.5}$$

Since $S \in \mathcal{D}'(\mathbb{R}^n)$, the restriction of S to $\mathcal{D}_Q(\mathbb{R}^n)$ is continuous by Theorem 4.5.1. Therefore, there exist an integer $j \ge 0$ and a constant $C'_Q > 0$ such that

$$|S(\psi)| \le C'_Q \|\psi\|_{C^j(Q)}, \qquad \forall \psi \in \mathcal{D}_Q(\mathbb{R}^n),$$

where

$$\|\psi\|_{C^j(Q)} = \max_{|\alpha| \le j} \|D^\alpha\psi\|_{C^0(Q)}.$$

Now let α be a multi-index with $|\alpha| \le j$ and set $\gamma := (j+1)\beta - \alpha$. Since $\alpha_i \le j$, we have $\gamma_i \ge 1$ for every i. Applying the fundamental theorem of calculus γ_i times in the variable i and using the fact that all

intermediate derivatives vanish when $x_i = a_i$ gives

$$D^{\alpha}\psi(x) = \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \left(\prod_{i=1}^{n} \frac{(x_i - y_i)^{\gamma_i - 1}}{(\gamma_i - 1)!} \right) D^{(j+1)\beta}\psi(y)\, dy_n \cdots dy_1.$$

Indeed, for a single variable this is the repeated integration formula, proved by induction on γ_i; the displayed formula results from applying it successively in the n variables and using Fubini. Since $0 \le x_i - y_i \le b_i - a_i$ on the integration region, taking absolute values and enlarging that region to all of Q gives

$$|D^{\alpha}\psi(x)| \le C_{Q,\alpha} \int_Q |D^{(j+1)\beta}\psi(y)|\, dy,$$

where the constant

$$C_{Q,\alpha} := \prod_{i=1}^{n} \frac{(b_i - a_i)^{\gamma_i - 1}}{(\gamma_i - 1)!}$$

depends only on the side lengths of the cube Q and the multi-index $\gamma = (j+1)\beta - \alpha$. Taking the supremum over $x \in Q$, the maximum over $|\alpha| \le j$, and defining

$$C_Q := \max_{|\alpha| \le j} C_{Q,\alpha},$$

we deduce that

$$\|\psi\|_{C^j(Q)} \le C_Q \int_Q |D^{(j+1)\beta}\psi(y)|\, dy.$$

Consequently, absorbing C_Q into a new global constant $C := C'_Q C_Q$, we obtain the bound

$$|S(\psi)| \le C \int_Q |D^{(j+1)\beta}\psi(y)|\, dy, \qquad \forall \psi \in \mathcal{D}_Q(\mathbb{R}^n). \tag{4.6}$$

Now define the linear operator

$$L\colon \mathcal{D}_Q(\mathbb{R}^n) \longrightarrow L^1(Q), \qquad L(\psi) = D^{(j+1)\beta}\psi.$$

The operator L is injective. Indeed, suppose that $D^{(j+1)\beta}\psi = 0$. We will prove by descending induction that $D^{q\beta}\psi = 0$ for $q = j, j-1, \ldots, 0$. If $q = j$, identity (4.5), applied to $D^{j\beta}\psi$, gives $D^{j\beta}\psi = 0$ on Q. If $0 \le q < j$ and we already know that $D^{(q+1)\beta}\psi = 0$, the same identity, now applied to $D^{q\beta}\psi$, gives $D^{q\beta}\psi = 0$. Upon reaching $q = 0$ we obtain $\psi = 0$ on Q; the argument also directly covers the case $j = 0$. All applications are legitimate because the derivatives of ψ belong to $\mathcal{D}_Q(\mathbb{R}^n)$ and hence vanish on ∂Q. Outside Q the function ψ is already zero, so $\psi = 0$ on $\mathbb{R}^n$.

Let $Y := L(\mathcal{D}_Q(\mathbb{R}^n))$ be the image of L. Define a linear functional $T_1 : Y \longrightarrow \mathbb{R}$ by $T_1(L(\psi)) := S(\psi)$. Injectivity of L ensures that T_1 is well defined. Moreover, by (4.6),

$$|T_1(w)| \le C\|w\|_{L^1(Q)}, \qquad \forall w \in Y.$$

By the analytic form of the Hahn–Banach theorem (Theorem B.2.27), we can extend T_1 to a continuous linear functional on all of $L^1(Q)$. Since the dual space $(L^1(Q))^*$ is isometrically identified with $L^\infty(Q)$ with

respect to Lebesgue measure, there exists a function $v \in L^\infty(Q)$ such that

$$T_1(\psi) = \int_Q v\psi, \qquad \forall \psi \in L^1(Q).$$

In particular, for $\psi \in \mathcal{D}_Q(\mathbb{R}^n)$ we obtain

$$S(\psi) = \int_Q v D^{(j+1)\beta}\psi.$$

Extend v by zero outside Q and define

$$u_0(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} v(y_1, \ldots, y_n)\, dy_n \cdots dy_1.$$

Since v is bounded and has compact support, the function u_0 is well defined. Moreover, if $x, x' \in \mathbb{R}^n$, the symmetric difference between the regions $\{y \in Q : y \le x\}$ and $\{y \in Q : y \le x'\}$ is contained in the union of n strips whose measures tend to zero as $x' \to x$. Consequently,

$$|u_0(x') - u_0(x)| \le \|v\|_{L^\infty(Q)} \lambda_n\big(\{y \in Q : y \le x'\} \triangle \{y \in Q : y \le x\}\big) \longrightarrow 0,$$

and u_0 is continuous on $\mathbb{R}^n$. We check that it satisfies the distributional equation $D^\beta u_0 = v$. For any test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we evaluate the dual action:

$$\begin{aligned}\langle D^\beta u_0, \varphi\rangle &= (-1)^{|\beta|} \int_{\mathbb{R}^n} u_0(x)\, D^\beta\varphi(x)\, dx \\ &= (-1)^n \int_{\mathbb{R}^n} \left(\int_{\{y \le x\}} v(y_1, \ldots, y_n)\, dy_1 \cdots dy_n \right) D^\beta\varphi(x)\, dx,\end{aligned}$$

where the notation $y \le x$ means $y_i \le x_i$ for every i. By Theorem B.6.18, we interchange the order of integration; for fixed y, the variable x ranges over the domain $\{x \ge y\}$:

$$\langle D^\beta u_0, \varphi\rangle = (-1)^n \int_{\mathbb{R}^n} v(y_1, \ldots, y_n) \left(\int_{\{x \ge y\}} D^\beta\varphi(x)\, dx \right) dy_1 \cdots dy_n.$$

Integrating $D^\beta\varphi = \partial_1 \cdots \partial_n \varphi$ in each variable x_i from y_i to ∞ gives exactly $(-1)^n \varphi(y_1, \ldots, y_n)$ because φ has compact support. Therefore,

$$\langle D^\beta u_0, \varphi\rangle = (-1)^n \int_{\mathbb{R}^n} v(y_1, \ldots, y_n)(-1)^n\, \varphi(y_1, \ldots, y_n)\, dy_1 \cdots dy_n = \langle v, \varphi\rangle.$$

This proves that $D^\beta u_0 = v$ in the sense of distributions.
Substituting this fact into the preceding expression for $S(\psi)$ and using the definition of distributional derivative, we obtain, for every $\psi \in \mathcal{D}_Q(\mathbb{R}^n)$,

$$S(\psi) = \langle D^\beta u_0, D^{(j+1)\beta}\psi\rangle = (-1)^{|\beta|} \int_{\mathbb{R}^n} u_0\, D^{(j+2)\beta}\psi = (-1)^n \int_{\mathbb{R}^n} u_0\, D^{(j+2)\beta}\psi.$$

Now define the multi-index $\alpha := (j+2)\beta$, so that $|\alpha| = (j+2)n$. To adjust the sign, define the continuous

function

$$\widehat{u} := (-1)^{n-|\alpha|} u_0 = (-1)^{(j+1)n} u_0,$$

where in the second equality we used $n - |\alpha| = -(j+1)n$ and $(-1)^{-(j+1)n} = (-1)^{(j+1)n}$ (since an integer and its negative have the same parity). Let $u := \widehat{u}\,|_\Omega$. With this choice, we have $(-1)^{|\alpha|}\widehat{u} = (-1)^n u_0$. Finally, if $\varphi \in \mathcal{D}_K(\Omega)$, then, by (4.4) and because $\operatorname{supp}(D^\alpha \widetilde{\varphi}) \subseteq K \subseteq \Omega$,

$$T(\varphi) = S(\widetilde{\varphi}) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \widehat{u}\, D^\alpha \widetilde{\varphi} = (-1)^{|\alpha|} \int_\Omega u\, D^\alpha \varphi.$$

This completes the proof. □

4.8 Rapidly decreasing functions and the Schwartz class

Compactly supported test functions are suitable for studying local phenomena, but compact support is not preserved by the Fourier transform. To analyze a function and its frequencies simultaneously, it is helpful to replace this condition with one controlling behavior at infinity. The Schwartz class consists of smooth functions whose derivatives decrease faster than any power. Its topology records all these controls at once and will be the natural test function space for tempered distributions. The development in this section follows [97, Chapter 10, §§10.4–10.8], with the unitary Fourier normalization to be fixed below.

We work with complex-valued functions. Real-valued functions are viewed through their canonical inclusion into the complex-valued ones; whenever a real result is needed, it will suffice to restrict the resulting operators.

Definition 4.8.1. (Schwartz class) The *Schwartz class* on $\mathbb{R}^n$ is

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(\mathbb{R}^n, \mathbb{C}) \;\middle|\; p_{\alpha,\beta}(\varphi) < \infty \text{ for any } \alpha, \beta \in \mathbb{N}_0^n \right\},$$

where $p_{\alpha,\beta}(\varphi) := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)|$. For $N \in \mathbb{N}_0$ we also write

$$q_N(\varphi) := \max_{|\alpha|+|\beta| \le N} p_{\alpha,\beta}(\varphi).$$

We equip $\mathcal{S}(\mathbb{R}^n)$ with the locally convex topology generated by the countable family $(q_N)_{N \in \mathbb{N}_0}$.

The preceding condition is equivalent to saying that, for each $m \in \mathbb{N}_0$ and each multi-index β, the function $(1 + \|x\|)^m D^\beta \varphi(x)$ is bounded. In particular, $D^\beta \varphi(x) \to 0$ faster than $\|x\|^{-m}$ for any m. The function $x \mapsto e^{-\|x\|^2}$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and does not have compact support, whereas $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$.

Proposition 4.8.2. (Equivalent seminorms) For $N \in \mathbb{N}_0$ define

$$r_N(\varphi) := \max_{|\beta| \le N} \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |D^\beta \varphi(x)|.$$

The families (q_N) and (r_N) generate the same topology on $\mathcal{S}(\mathbb{R}^n)$. More precisely, for each N there exist $M \in \mathbb{N}_0$ and positive constants c_N, C_N such that $q_N \le c_N r_N$ and $r_N \le C_N q_M$.

Proof. If $|\alpha|+|\beta| \le N$, then $|x^\alpha D^\beta \varphi(x)| \le (1+\|x\|)^N |D^\beta \varphi(x)|$, giving $q_N \le r_N$. For the reverse inequality, we use the fact that there exists a constant $C_N > 0$ such that

$$(1+\|x\|)^N \le C_N \sum_{|\alpha|\le N} |x^\alpha|, \qquad x \in \mathbb{R}^n.$$

This estimate is obtained by expanding $(1+|x_1|+\cdots+|x_n|)^N$ and observing that $\|x\| \le |x_1|+\cdots+|x_n|$. Therefore, $r_N(\varphi) \le C_N q_{2N}(\varphi)$. □

Theorem 4.8.3. (Fréchet structure of the Schwartz class) The space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space. A sequence (φ_j) converges to φ in $\mathcal{S}(\mathbb{R}^n)$ if and only if $q_N(\varphi_j - \varphi) \to 0$ for every $N \in \mathbb{N}_0$.

Proof. By Proposition B.10.24, the countable family (q_N) induces an invariant metric compatible with the topology. It remains to check completeness. Let (φ_j) be a Cauchy sequence. For each pair of multi-indices α, β, the sequence $(x^\alpha D^\beta \varphi_j)_j$ is Cauchy in the Banach space $C_b(\mathbb{R}^n)$ and converges uniformly to a continuous function $g_{\alpha,\beta}$.
Set $\varphi := g_{0,0}$. For $i \in \{1,\dots,n\}$, $x \in \mathbb{R}^n$, and $t \in \mathbb{R}$, Theorem B.6.5 gives

$$\varphi_j(x+te_i) - \varphi_j(x) = \int_0^t D_i\varphi_j(x+se_i)\,ds.$$

Uniform convergence allows us to pass to the limit inside the integral and obtain

$$\varphi(x+te_i) - \varphi(x) = \int_0^t g_{0,e_i}(x+se_i)\,ds.$$

Thus, $D_i\varphi = g_{0,e_i}$. Repeating the argument, we conclude that $\varphi \in C^\infty(\mathbb{R}^n)$ and $D^\beta\varphi = g_{0,\beta}$ for every β. Since multiplication by x^α commutes with the uniform limit, $x^\alpha D^\beta \varphi = g_{\alpha,\beta}$ and therefore $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Finally, $p_{\alpha,\beta}(\varphi_j - \varphi) \to 0$ for every α, β, proving convergence in $\mathcal{S}(\mathbb{R}^n)$. □

Proposition 4.8.4. (Continuous operations on $\mathcal{S}$) Let $\alpha \in \mathbb{N}_0^n$, $a \in \mathbb{R}^n$, and $A \in \mathrm{GL}(n,\mathbb{R})$. The maps

$$\varphi \longmapsto D^\alpha\varphi, \qquad \varphi \longmapsto x^\alpha\varphi, \qquad \varphi \longmapsto \varphi(\cdot - a), \qquad \varphi \longmapsto \varphi \circ A$$

are continuous linear endomorphisms of $\mathcal{S}(\mathbb{R}^n)$. Multiplication $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is bilinear and continuous.

Proof. The first two assertions follow directly from the definition. For translations, we use $\|x\| \le \|x-a\|+\|a\|$ and the binomial formula. If $A \in \mathrm{GL}(n,\mathbb{R})$, then

$$\|A^{-1}\|^{-1}\|x\| \le \|Ax\| \le \|A\|\|x\|, \qquad x \in \mathbb{R}^n.$$

The chain rule and these inequalities show that, for each pair of multi-indices α, β, there exists a constant

$C_{\alpha,\beta,A} > 0$, depending only on α, β, A, and the fixed Euclidean norm, such that

$$p_{\alpha,\beta}(\varphi \circ A) \le C_{\alpha,\beta,A} \max_{\substack{|\delta|=|\alpha| \\ |\gamma|=|\beta|}} p_{\delta,\gamma}(\varphi).$$

Indeed, $\|x\|^{|\alpha|} \le \|A^{-1}\|^{|\alpha|}\|Ax\|^{|\alpha|}$, the power $\|Ax\|^{|\alpha|}$ is bounded by a finite sum of monomials of degree $|\alpha|$, and each derivative of $\varphi \circ A$ is a finite linear combination of derivatives of φ of order $|\beta|$, with coefficients depending only on A. Finally, the Leibniz rule gives

$$x^\alpha D^\beta(\varphi\psi) = \sum_{\gamma \le \beta} \binom{\beta}{\gamma} (x^\alpha D^\gamma \varphi) D^{\beta-\gamma}\psi,$$

and each summand is controlled by a seminorm of φ and a seminorm of ψ. □

Proposition 4.8.5. (Density) The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$. Moreover, for every $1 \le p < \infty$, the inclusions

$$\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$$

have dense image and the inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous.

Proof. Take $\eta \in C_c^\infty(\mathbb{R}^n)$ with $0 \le \eta \le 1$ and $\eta = 1$ on $\overline{B}_{\mathrm{euc}}(0,1)$, and define $\eta_R(x) := \eta(\frac{x}{R})$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the function $\eta_R\varphi$ belongs to $\mathcal{D}(\mathbb{R}^n)$. The Leibniz rule shows that $D^\beta((1-\eta_R)\varphi)$ is a sum of terms of the form $R^{-|\gamma|}(D^\gamma\eta)(\frac{x}{R})D^{\beta-\gamma}\varphi(x)$. When $\gamma = 0$, the term vanishes on $\|x\| \le R$ and its seminorm tends to zero by the rapid decrease of φ; when $|\gamma| > 0$, it is supported in an annulus with radii comparable to R, and the same decrease absorbs any power of R. Consequently, $\eta_R\varphi \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$.
If $m > \frac{n}{p}$, then

$$\|\varphi\|_{L^p(\mathbb{R}^n)}^p \le r_m(\varphi)^p \int_{\mathbb{R}^n} (1+\|x\|)^{-mp}\, dx,$$

so the inclusion into $L^p(\mathbb{R}^n)$ is continuous. Density follows from $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ and Proposition B.7.18. □

4.9 Tempered distributions

The topology of $\mathcal{S}$ makes it possible to impose controlled growth on a distribution. Unlike a general distribution, a tempered distribution can be Fourier transformed because the transform preserves the Schwartz class.

Definition 4.9.1. (Tempered distribution) A *tempered distribution* is a continuous linear functional $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$. The continuous dual of $\mathcal{S}(\mathbb{R}^n)$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$ and is equipped, unless otherwise stated, with the weak-$*$ topology $\sigma(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$.

Proposition 4.9.2. (Continuity criterion) A linear functional $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ if and only if there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$|\langle T, \varphi\rangle| \le C q_N(\varphi) \qquad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proof. Sufficiency is immediate. Suppose that T is continuous at the origin. There exists a basic neighborhood

$$U = \{\varphi \in \mathcal{S}(\mathbb{R}^n) \mid q_{N_1}(\varphi) < \varepsilon_1, \dots, q_{N_k}(\varphi) < \varepsilon_k\}$$

such that $|T(\varphi)| < 1$ for $\varphi \in U$. Since the seminorms q_N are increasing, if $N := \max_j N_j$ and $\varepsilon := \min_j \varepsilon_j$, then $q_N(\varphi) < \varepsilon$ implies $\varphi \in U$. For $q_N(\varphi) > 0$, we apply the preceding argument to $\frac{\varepsilon\varphi}{2q_N(\varphi)}$ and obtain $|T(\varphi)| \le 2\varepsilon^{-1} q_N(\varphi)$. If $q_N(\varphi) = 0$, then $\varphi = 0$. □

Proposition 4.9.3. (Natural inclusions) For every $1 \le p \le \infty$ there is a continuous injective linear embedding

$$L^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \qquad f \longmapsto T_f, \qquad \langle T_f, \varphi\rangle := \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx.$$

Every locally integrable function of at most polynomial growth also defines a tempered distribution. Every distribution with compact support is tempered.

Proof. Let p' be the conjugate exponent. By Hölder's inequality in Proposition B.7.9 and continuity of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^{p'}(\mathbb{R}^n)$,

$$|\langle T_f, \varphi\rangle| \le \|f\|_{L^p(\mathbb{R}^n)} \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} q_N(\varphi)$$

for some N. Injectivity follows from Proposition B.6.20.
If $|f(x)| \le C(1 + \|x\|)^m$ almost everywhere, choose $N > m + n$ and estimate

$$\int_{\mathbb{R}^n} |f\varphi| \le C r_N(\varphi) \int_{\mathbb{R}^n} (1 + \|x\|)^{m-N}\,dx.$$

For the last assertion, let $T \in \mathcal{D}'(\mathbb{R}^n)$ have compact support K. Choose $\chi \in \mathcal{D}(\mathbb{R}^n)$ equal to one on a neighborhood of K and define, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \widetilde{T}, \varphi\rangle := \langle T, \chi\varphi\rangle.$$

The definition does not depend on the cutoff: if χ_1 and χ_2 equal one near K, then $(\chi_1 - \chi_2)\varphi$ vanishes on a neighborhood of $\operatorname{supp}(T)$ and, by definition of the support of a distribution, $\langle T, (\chi_1 - \chi_2)\varphi\rangle = 0$. By the local finite-order estimate for T on $\operatorname{supp}(\chi)$, there exist $C > 0$ and $N \in \mathbb{N}_0$ such that

$$|\langle \widetilde{T}, \varphi\rangle| \le C \max_{|\alpha| \le N} \|D^\alpha(\chi\varphi)\|_{C^0(\operatorname{supp}(\chi))} \le C_\chi q_N(\varphi).$$

Thus, $\widetilde{T} \in \mathcal{S}'(\mathbb{R}^n)$ and its restriction to $\mathcal{D}(\mathbb{R}^n)$ agrees with T. □

Example 4.9.4. Every polynomial defines a tempered distribution. More generally, if μ is a Radon measure for which there exist $C, m > 0$ with $|\mu|(B_{\mathrm{euc}}(0,R)) \le C(1+R)^m$ for $R \ge 1$, then $\langle T_\mu, \varphi\rangle := \int \varphi\,d\mu$ defines an element of $\mathcal{S}'(\mathbb{R}^n)$. To see this, decompose $\mathbb{R}^n$ into the unit ball and the annuli $k \le \|x\| < k+1$; the growth of μ is absorbed by a seminorm $r_N(\varphi)$ with $N > m + 1$.

Definition 4.9.5. (Differentiation and multiplication) Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Define

$$\langle D^\alpha T, \varphi\rangle := (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle, \qquad \langle x^\alpha T, \varphi\rangle := \langle T, x^\alpha\varphi\rangle.$$

Both functionals belong to $\mathcal{S}'(\mathbb{R}^n)$.

The final assertion follows from continuity of the operations in Proposition 4.8.4. The same argument allows us to multiply T by any function $a \in C^\infty(\mathbb{R}^n)$ whose derivatives have at most polynomial growth: the Leibniz rule shows that $\varphi \mapsto a\varphi$ is continuous on $\mathcal{S}(\mathbb{R}^n)$.

4.10 Convolution and regularization

Convolution connects regularization by mollifiers with multiplication in frequency. We first recall the inequality ensuring its existence in the spaces $L^p(\mathbb{R}^n)$.

Theorem 4.10.1. (Young's inequality for convolutions) Let $1 \le p, q, r \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g$ is defined almost everywhere, belongs to $L^r(\mathbb{R}^n)$, and

$$\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

In particular, $L^1(\mathbb{R}^n) * L^p(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ continuously.

Proof. The case $r = \infty$ is Proposition B.7.9 applied to $y \mapsto f(x-y)g(y)$. The case $p = 1$ follows from Theorem B.8.2 and translation invariance:

$$\|f * g\|_{L^q(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} |f(y)| \, \|g(\cdot - y)\|_{L^q(\mathbb{R}^n)} \, dy = \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

The case $q = 1$ follows by commutativity. In the other cases $1 < p, q \le r < \infty$; if $p = r$ or $q = r$, the factor whose conjugate exponent is infinite is omitted in the following application of Hölder. In the remaining case, we write

$$|f(x-y)g(y)| = \left(|f(x-y)|^p |g(y)|^q\right)^{\frac{1}{r}} |f(x-y)|^{1-\frac{p}{r}} |g(y)|^{1-\frac{q}{r}}.$$

Proposition B.7.9 with exponents r, $\frac{pr}{r-p}$, and $\frac{qr}{r-q}$, followed by Theorem B.6.17 and the case $L^1(\mathbb{R}^n) * L^1(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$, gives

$$\|f * g\|_{L^r(\mathbb{R}^n)}^r \le \|f\|_{L^p(\mathbb{R}^n)}^p \|g\|_{L^q(\mathbb{R}^n)}^q \|f\|_{L^p(\mathbb{R}^n)}^{r-p} \|g\|_{L^q(\mathbb{R}^n)}^{r-q},$$

which is the asserted inequality. □

Proposition 4.10.2. (Convolution in the Schwartz class) If $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$ and

$$D^\alpha(\varphi * \psi) = (D^\alpha \varphi) * \psi = \varphi * (D^\alpha \psi).$$

The map $(\varphi, \psi) \mapsto \varphi * \psi$ is bilinear and continuous from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Proof. Differentiation under the integral sign is justified by rapid decrease, which provides an integrable majorant for the difference quotients; we apply the dominated convergence theorem B.6.14. To control the variable x, we use $x^\alpha = (x - y + y)^\alpha$ and the multinomial formula. Each term of $x^\alpha D^\beta(\varphi * \psi)$ is the convolution of a function $y^\gamma D^\delta \psi(y)$ with a function $(x-y)^{\alpha-\gamma} D^{\beta-\delta}\varphi(x-y)$. Young's inequality $L^1(\mathbb{R}^n) * L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ in Theorem 4.10.1 controls that term by Schwartz seminorms. The same estimate proves

bilinear continuity. □

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we write $\check{\varphi}(x) := \varphi(-x)$.

Definition 4.10.3. (Convolution of a tempered distribution with a Schwartz function) Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Define

$$(T * \varphi)(x) := \langle T, \varphi(x - \cdot)\rangle, \qquad x \in \mathbb{R}^n.$$

Proposition 4.10.4. (Tempered regularization) The function $T * \varphi$ is smooth and

$$D^\alpha(T * \varphi) = T * (D^\alpha \varphi) = (D^\alpha T) * \varphi.$$

For each α there exist $C_\alpha > 0$ and $N_\alpha \in \mathbb{N}_0$ such that $|D^\alpha(T * \varphi)(x)| \le C_\alpha(1 + \|x\|)^{N_\alpha}$. Moreover, if $\rho \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\displaystyle\int_{\mathbb{R}^n} \rho = 1$ and $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(\tfrac{x}{\varepsilon})$, then $T * \rho_\varepsilon \to T$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \to 0^+$.

Proof. The difference quotient of $x \mapsto \varphi(x-\cdot)$ converges in $\mathcal{S}(\mathbb{R}^n)$ to the corresponding derivative; continuity of T allows passage to the limit and proves smoothness. The identity with $D^\alpha T$ is obtained by transferring the derivative to the test function.
By Proposition 4.9.2, $|\langle T, \psi\rangle| \le C q_N(\psi)$. The inequality $\|y\| \le \|x - y\| + \|x\|$ and the binomial formula show that $q_N(\varphi(x - \cdot)) \le C_\varphi(1 + \|x\|)^N$, giving polynomial growth.
Finally, for $\psi \in \mathcal{S}(\mathbb{R}^n)$, Fubini's theorem B.6.18 and the definition of convolution give

$$\langle T * \rho_\varepsilon, \psi\rangle = \langle T, \check{\rho}_\varepsilon * \psi\rangle.$$

Since $\displaystyle\int_{\mathbb{R}^n} \check{\rho} = 1$, the change of variable $y = \varepsilon z$ gives

$$(\check{\rho}_\varepsilon * \psi)(x) - \psi(x) = \int_{\mathbb{R}^n} \check{\rho}(z)\big(\psi(x - \varepsilon z) - \psi(x)\big)\, dz.$$

The fundamental theorem of calculus for Banach-space-valued functions B.6.5, applied after differentiating along the corresponding segment, and the inequality

$$1 + \|x\| \le (1 + \|x - \varepsilon z\|)(1 + \varepsilon\|z\|)$$

imply, for each $N \in \mathbb{N}_0$ and $0 < \varepsilon \le 1$,

$$q_N(\check{\rho}_\varepsilon * \psi - \psi) \le C_N \varepsilon\, q_{N+1}(\psi) \int_{\mathbb{R}^n} |\rho(z)|\,\|z\|(1 + \|z\|)^N\, dz.$$

The integral is finite because $\rho \in \mathcal{S}(\mathbb{R}^n)$. Therefore, $\check{\rho}_\varepsilon * \psi \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$, and continuity of T completes the proof. □

4.11 The Fourier transform

The Fourier transform turns derivatives into multiplication and convolutions into products. This correspondence makes it possible to measure regularity through the behavior of a function in frequency. We choose the unitary normalization because it makes the transform an isometry of $L^2(\mathbb{R}^n)$ without additional constants. For the classical properties, the extension to distributions, and compatibility among the different definitions, we follow the development in [29, Sections 10.3–10.5], with our normalization in $\mathbb{R}^n$; the proofs are included to fix all constants and to avoid confusing the integral formula with the extensions by density or duality.

Definition 4.11.1. (Fourier transform on L^1) For $f \in L^1(\mathbb{R}^n)$ define

$$\widehat{f}(\xi) := \mathcal{F}f(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx, \qquad \xi \in \mathbb{R}^n.$$

For $g \in L^1(\mathbb{R}^n)$, the formal inverse transform is defined by

$$\mathcal{F}^{-1}g(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

Proposition 4.11.2. (Properties on L^1) The map $\mathcal{F}\colon L^1(\mathbb{R}^n) \longrightarrow C_b(\mathbb{R}^n)$ is linear and

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover, $\widehat{f}$ is uniformly continuous.

Proof. The estimate follows by taking the absolute value inside the integral. If $h \to 0$, then

$$|\widehat{f}(\xi+h) - \widehat{f}(\xi)| \le (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |e^{-ix\cdot h} - 1|\,|f(x)|\,dx.$$

The right-hand side does not depend on ξ, converges to zero pointwise in the variable x, and is dominated by $2(2\pi)^{-\frac{n}{2}}|f| \in L^1(\mathbb{R}^n)$. Theorem B.6.14 proves uniform continuity. □

Proposition 4.11.3. (Fourier calculus on $\mathcal{S}$) If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, then

$$\mathcal{F}(D^\alpha \varphi)(\xi) = (i\xi)^\alpha \widehat{\varphi}(\xi), \qquad \mathcal{F}(x^\alpha \varphi)(\xi) = i^{|\alpha|} D^\alpha \widehat{\varphi}(\xi).$$

In particular, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear operators from $\mathcal{S}(\mathbb{R}^n)$ to itself.

Proof. The first identity follows by integration by parts; the boundary terms disappear by rapid decrease. The second follows by differentiation under the integral sign. Combining both identities with

$$\|h\|_{L^1(\mathbb{R}^n)} \le C_N \sup_{x\in\mathbb{R}^n} (1+\|x\|)^N |h(x)|, \qquad N > n,$$

the Leibniz rule applied to $D^\alpha(x^\beta \varphi)$ and the preceding bound with the weight $(1+\|x\|)^{n+1}$ give explicitly

$$p_{\alpha,\beta}(\widehat{\varphi}) \le C_{\alpha,\beta}\, q_{n+1+|\alpha|+|\beta|}(\varphi).$$

Indeed, each summand contains a derivative of order at most $|\alpha|$ and a monomial whose degree, after incorporating the weight, does not exceed $|\beta|+n+1$; the factor $(1+\|x\|)^{-n-1}$ is integrable on $\mathbb{R}^n$. This proves that $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and establishes continuity. The same calculation, with the sign of the phase reversed, proves the assertion for $\mathcal{F}^{-1}$. □

Lemma 4.11.4. (Transform of the Gaussian) For $a > 0$, the function $g_a(x) := e^{-\frac{a\|x\|^2}{2}}$ satisfies

$$\widehat{g_a}(\xi) = a^{-\frac{n}{2}} e^{-\frac{\|\xi\|^2}{2a}}.$$

In particular, g_1 is fixed by the Fourier transform.

Proof. By Fubini's theorem B.6.18, it suffices to treat $n = 1$. Let $G(\xi) := \widehat{g_a}(\xi)$. Proposition 4.11.3 and the identity $g_a' = -axg_a$ give $i\xi G = -aiG'$, that is, $G'(\xi) = -\dfrac{\xi G(\xi)}{a}$. Therefore, $G(\xi) = G(0)e^{-\frac{\xi^2}{2a}}$. To calculate the constant, set $I_a := \displaystyle\int_{\mathbb{R}} e^{-\frac{ax^2}{2}}\,dx$. Tonelli's theorem B.6.17 and the change to polar coordinates, justified by the change of variables theorem B.6.15, give

$$I_a^2 = \int_{\mathbb{R}^2} e^{-\frac{a\|x\|^2}{2}}\,dx = 2\pi \int_0^\infty e^{-\frac{ar^2}{2}}\, r\,dr = \frac{2\pi}{a}.$$

Thus, $G(0) = (2\pi)^{-\frac{1}{2}} I_a = a^{-\frac{1}{2}}$. The product of the one-dimensional formulas completes the proof. □

Lemma 4.11.5. (Gaussian approximation to the identity) For $\varepsilon > 0$, let

$$k_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{2\varepsilon}}.$$

Then $k_\varepsilon \geq 0$, $\displaystyle\int_{\mathbb{R}^n} k_\varepsilon = 1$, and, for every $\delta > 0$,

$$\int_{\{\|x\|\geq\delta\}} k_\varepsilon(x)\,dx \longrightarrow 0.$$

If $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then $k_\varepsilon * f \to f$ in $L^p(\mathbb{R}^n)$; convergence is uniform when f is bounded and uniformly continuous.

Proof. The mass is computed using Lemma 4.11.4. The change of variable $x = \sqrt{\varepsilon}z$ and integrability of the Gaussian prove concentration. Moreover, Theorem B.8.2 gives

$$\|k_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} k_\varepsilon(y)\|\tau_y f - f\|_{L^p(\mathbb{R}^n)}\,dy.$$

Split the integral over $\{\|y\| < \delta\}$ and its complement; on the first part use Proposition B.8.17, and on the second use the bound $2\|f\|_{L^p(\mathbb{R}^n)}$ and concentration. The same argument with the uniform norm proves the final assertion. □

Theorem 4.11.6. (Fourier inversion in the Schwartz class) For every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\mathcal{F}^{-1}(\mathcal{F}\varphi) = \varphi = \mathcal{F}(\mathcal{F}^{-1}\varphi).$$

Consequently, $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a topological isomorphism with inverse $\mathcal{F}^{-1}$.

Proof. Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Fubini's theorem B.6.18 and Lemma 4.11.4 give

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{\varepsilon\|\xi\|^2}{2}} \widehat{\varphi}(\xi)\, d\xi = \int_{\mathbb{R}^n} \varphi(y) k_\varepsilon(x-y)\, dy,$$

where k_ε is the function in Lemma 4.11.5. That lemma shows that the right-hand side converges to $\varphi(x)$. On the left-hand side, $e^{-\frac{\varepsilon\|\xi\|^2}{2}} \to 1$ and the integrand is dominated by $|\widehat{\varphi}| \in L^1(\mathbb{R}^n)$. Theorem B.6.14 gives $\mathcal{F}^{-1}\widehat{\varphi}(x) = \varphi(x)$. The second identity follows by reversing the sign of the phase. □

Corollary 4.11.7. (Inversion in L^1) If $f \in L^1(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n)$, then f agrees almost everywhere with a continuous function vanishing at infinity and

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi$$

for almost every $x \in \mathbb{R}^n$. In particular, the Fourier transform is injective on $L^1(\mathbb{R}^n)$.

Proof. Let k_ε be the Gaussian in the preceding proof. By Fubini's theorem B.6.18,

$$(f * k_\varepsilon)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{\varepsilon\|\xi\|^2}{2}} \widehat{f}(\xi)\, d\xi.$$

The right-hand side converges uniformly to $\mathcal{F}^{-1}\widehat{f}$ by Theorem B.6.14. The left-hand side converges to f in $L^1(\mathbb{R}^n)$ by Lemma 4.11.5. A subsequence converges pointwise almost everywhere, so $f = \mathcal{F}^{-1}\widehat{f}$ almost everywhere. Set $h := \mathcal{F}^{-1}\widehat{f}$. By Proposition 4.11.2, applied with the opposite sign in the phase, h is uniformly continuous. Moreover, $h = f$ almost everywhere and $f \in L^1(\mathbb{R}^n)$, so $h \in L^1(\mathbb{R}^n)$.

These two properties imply that h vanishes at infinity. Indeed, otherwise there would exist $\varepsilon > 0$ and a sequence (x_j) with $\|x_j\| \to \infty$ and $|h(x_j)| \geq \varepsilon$. Uniform continuity provides a radius $r > 0$, independent of j, such that $|h(x)| \geq \varepsilon/2$ if $\|x - x_j\| < r$. Passing to a subsequence, we may assume that the balls $B(x_j, r)$ are pairwise disjoint: after finitely many centers have been chosen, take the next one at a distance greater than $2r$ from all of them. Then

$$\int_{\mathbb{R}^n} |h(x)|\, dx \geq \sum_{j=1}^{\infty} \int_{B(x_j,r)} |h(x)|\, dx \geq \sum_{j=1}^{\infty} \frac{\varepsilon}{2} \lambda_n(B(0,r)) = \infty,$$

which contradicts integrability of h. □

Theorem 4.11.8. (Riemann–Lebesgue lemma) The Fourier transform induces a continuous operator

$$\mathcal{F}\colon L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n).$$

In particular, $\widehat{f}(\xi) \to 0$ as $\|\xi\| \to \infty$.

Proof. By Proposition 4.8.5, there exists $(\varphi_j) \subseteq \mathcal{S}(\mathbb{R}^n)$ with $\varphi_j \to f$ in $L^1(\mathbb{R}^n)$. Proposition 4.11.2 implies that $\widehat{\varphi}_j \to \widehat{f}$ uniformly. Each $\widehat{\varphi}_j$ belongs to $\mathcal{S}(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$, and $C_0(\mathbb{R}^n)$ is closed in the uniform norm. □

Proposition 4.11.9. (Convolution, product, and symmetries) Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\mathcal{F}(f * g) = (2\pi)^{\frac{n}{2}} \widehat{f}\,\widehat{g}, \qquad \mathcal{F}(fg) = (2\pi)^{-\frac{n}{2}} \widehat{f} * \widehat{g}.$$

If $a \in \mathbb{R}^n$, $\lambda \neq 0$, and $A \in \mathrm{GL}(n, \mathbb{R})$, then

$$\mathcal{F}(f(\cdot - a))(\xi) = e^{-ia\cdot\xi} \widehat{f}(\xi), \qquad \mathcal{F}(e^{ia\cdot(\cdot)} f)(\xi) = \widehat{f}(\xi - a),$$

and

$$\mathcal{F}(f \circ A)(\xi) = |\det A|^{-1} \widehat{f}(A^{-T}\xi).$$

Proof. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$, the function $(x, y) \mapsto f(y)g(x-y)$ is integrable on $\mathbb{R}^{2n}$ and Fubini's theorem B.6.18 applies. With $z = x - y$,

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(y) g(x-y)\, dy\, dx$$

$$= (2\pi)^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\, dy \right) \left(\int_{\mathbb{R}^n} e^{-iz\cdot\xi} g(z)\, dz \right) = (2\pi)^{\frac{n}{2}} \widehat{f}(\xi)\widehat{g}(\xi).$$

Applying this identity to $\mathcal{F}^{-1}\widehat{f}$ and $\mathcal{F}^{-1}\widehat{g}$, and using Theorem 4.11.6, gives $\mathcal{F}(fg) = (2\pi)^{-\frac{n}{2}} \widehat{f} * \widehat{g}$.
For translation, substitute $y = x - a$; for modulation, combine the phase as $e^{-ix\cdot(\xi - a)}$. Finally, with $y = Ax$,

$$\mathcal{F}(f \circ A)(\xi) = (2\pi)^{-\frac{n}{2}} |\det A|^{-1} \int e^{-iy\cdot A^{-T}\xi} f(y)\, dy,$$

which is the last formula. Each interchange of order or change of variable is justified by absolute integrability of the Schwartz functions. □

4.11.1 The transform on L^2 and on the L^p scale

The initial definition of Fourier on integrable functions is insufficient for energy arguments. Plancherel's theorem extends the transform to $L^2(\mathbb{R}^n)$ and provides the starting point for studying which estimates survive away from the Hilbert space exponent.

Theorem 4.11.10. (Parseval and Plancherel) For $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi.$$

In particular, $\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$. The Fourier transform extends uniquely to a unitary operator $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, whose inverse is the extension of $\mathcal{F}^{-1}$.

Proof. Since $f, h \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$, the corresponding double integral is absolutely convergent. Fubini's theorem B.6.18 gives the bilinear identity

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) h(\xi)\, d\xi = \int_{\mathbb{R}^n} f(x) \widehat{h}(x)\, dx$$

in which both pairings are bilinear. Take $h = \overline{\widehat{g}}$. By inversion,

$$\widehat{h}(x) = \mathcal{F}(\overline{\widehat{g}})(x) = \overline{\mathcal{F}^{-1}(\widehat{g})(x)} = \overline{g(x)}.$$

Substitution of this equality proves Parseval's identity. With $g = f$ we obtain $\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.
Now let $f \in L^2(\mathbb{R}^n)$ and choose $f_m \in \mathcal{S}(\mathbb{R}^n)$ with $f_m \to f$ in $L^2(\mathbb{R}^n)$. The isometry shows that $(\widehat{f_m})$ is Cauchy; define $\mathcal{F}f := \lim_{m\to\infty} \widehat{f_m}$ in $L^2(\mathbb{R}^n)$. If $(\widetilde{f_m})$ is another approximation, then

$$\|\widehat{f_m} - \widehat{\widetilde{f_m}}\|_{L^2(\mathbb{R}^n)} = \|f_m - \widetilde{f_m}\|_{L^2(\mathbb{R}^n)} \longrightarrow 0,$$

so the definition is independent of the sequence and preserves the norm. The same procedure extends $\mathcal{F}^{-1}$; the inversion identities on $\mathcal{S}(\mathbb{R}^n)$ and passage to the limit show that the two extended operators are inverses. In particular, $\mathcal{F}$ is unitary and surjective. By approximation, this also proves Parseval's identity for every pair of elements of $L^2(\mathbb{R}^n)$. □

When $f \in L^2(\mathbb{R}^n)$, the integral expression defining $\widehat{f}$ may fail to converge. The transform is then understood as the limit in $L^2(\mathbb{R}^n)$ constructed in the proof. If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, take truncations and regularizations $f_m \in \mathcal{S}(\mathbb{R}^n)$ converging simultaneously in $L^1(\mathbb{R}^n)$ and in $L^2(\mathbb{R}^n)$. Their transforms converge uniformly to the integral transform by Proposition 4.11.2, and converge in $L^2(\mathbb{R}^n)$ to the Plancherel transform. A subsequence of the second limit converges almost everywhere; by uniqueness of the pointwise limit, the two transforms agree almost everywhere.

Proposition 4.11.11. (L^p functions as tempered distributions) For every $1 \le p \le \infty$, the map

$$L^p(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n), \qquad f \longmapsto T_f, \qquad \langle T_f, \varphi \rangle := \int f(x)\varphi(x)\, dx,$$

is linear, continuous, and injective. Therefore, $\widehat{f} := \widehat{T_f}$ is always defined as a tempered distribution. For $p = 2$ it agrees with the Plancherel transform, and for $f \in L^1(\mathbb{R}^n)$ it agrees with the integral formula.

Compatibility with regular distributions is recorded in [29, Section 10.5].

Proof. If $p < \infty$, Hölder's inequality in Proposition B.7.9 gives

$$|\langle T_f, \varphi \rangle| \le \|f\|_{L^p(\mathbb{R}^n)} \|\varphi\|_{L^{p'}(\mathbb{R}^n)}.$$

Choosing $N > \frac{n}{p'}$, we have

$$\|\varphi\|_{L^{p'}(\mathbb{R}^n)} \le \|(1 + \|x\|)^{-N}\|_{L^{p'}(\mathbb{R}^n)} \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |\varphi(x)|.$$

For $p = \infty$, use the same estimate with $\|\varphi\|_{L^1(\mathbb{R}^n)}$ and any $N > n$. This proves continuity in the topology of

$\mathcal{S}(\mathbb{R}^n)$ and therefore $T_f \in \mathcal{S}'(\mathbb{R}^n)$. If $T_f = 0$, then the integral of f against every function in $C_c^\infty(\mathbb{R}^n)$ is zero; Proposition B.6.20 implies that $f = 0$ almost everywhere, proving injectivity.
Compatibility with Plancherel is first checked on $\mathcal{S}(\mathbb{R}^n)$ and extended to $L^2(\mathbb{R}^n)$ by density and continuity. For the integral definition on $L^1(\mathbb{R}^n)$, the identity follows directly by pairing with a Schwartz function and applying Fubini's theorem B.6.18. □

4.11.2 Fourier on tempered distributions

Weak solutions and symbols of polynomial growth require us to go beyond the framework of functions. Since the transform preserves the Schwartz space, duality provides a canonical extension to $\mathcal{S}'(\mathbb{R}^n)$.

Definition 4.11.12. (Transform of a tempered distribution) For $T \in \mathcal{S}'(\mathbb{R}^n)$ define $\widehat{T} \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}^{-1}T \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \widehat{T}, \varphi \rangle := \langle T, \widehat{\varphi} \rangle, \qquad \langle \mathcal{F}^{-1}T, \varphi \rangle := \langle T, \mathcal{F}^{-1}\varphi \rangle.$$

The definition uses the topological transpose of the transform on $\mathcal{S}(\mathbb{R}^n)$. No conjugation is introduced because distributions are viewed as linear functionals.

Theorem 4.11.13. (Fourier on $\mathcal{S}'$) The Fourier transform is a linear, weak-$*$ continuous isomorphism from $\mathcal{S}'(\mathbb{R}^n)$ to itself, with inverse $\mathcal{F}^{-1}$. For every $T \in \mathcal{S}'(\mathbb{R}^n)$ and every multi-index α,

$$\mathcal{F}(D^\alpha T) = (i\xi)^\alpha \widehat{T}, \qquad \mathcal{F}(x^\alpha T) = i^{|\alpha|} D^\alpha \widehat{T}.$$

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\mathcal{F}(T * \varphi) = (2\pi)^{\frac{n}{2}} \widehat{T}\, \widehat{\varphi}.$$

Proof. Continuity and the inversion formula follow by transposing Theorem 4.11.6. For the derivative,

$$\langle \mathcal{F}(D^\alpha T), \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \widehat{\varphi} \rangle = \langle T, \mathcal{F}((i\,\cdot)^\alpha \varphi) \rangle,$$

which is the action of $(i\xi)^\alpha \widehat{T}$. The identity involving $x^\alpha T$ is analogous. The convolution formula is first checked by pairing both sides with a Schwartz function and applying Fubini's theorem B.6.18; continuity then gives the conclusion for every $T \in \mathcal{S}'(\mathbb{R}^n)$. □

4.12 Littlewood–Paley dyadic decomposition

The Fourier transform makes it possible to separate a distribution into frequency bands. Instead of studying the entire spectrum simultaneously, we first isolate the low frequencies and then annuli whose characteristic scale is 2^j, with $j \in \mathbb{N}$. This inhomogeneous decomposition preserves the information in the entire distribution and will allow us to measure regularity by observing how each block decreases as frequency increases. It will provide the common foundation for Bessel, Besov, and Triebel–Lizorkin spaces.

Proposition 4.12.1. (Dyadic resolution of unity) There exists a radial function $\chi \in C_c^\infty(\mathbb{R}^n)$ such that

$0 \le \chi \le 1$, $\chi = 1$ on $\overline{B}_{\mathrm{euc}}(0,1)$, and $\operatorname{supp}(\chi) \subseteq B_{\mathrm{euc}}(0,2)$. If

$$\varphi_0(\xi) := \chi(\xi), \qquad \varphi_j(\xi) := \chi(2^{-j}\xi) - \chi(2^{-j+1}\xi), \quad j \in \mathbb{N},$$

then $\varphi_j \in C_c^\infty(\mathbb{R}^n)$, φ_j is supported in an annulus $c2^j \le \|\xi\| \le C2^j$ for $j \in \mathbb{N}$, and

$$\sum_{j \in \mathbb{N}_0} \varphi_j(\xi) = 1, \qquad \xi \in \mathbb{R}^n.$$

At each point, only a uniformly bounded number of enlarged annuli intersect.

Proof. The existence of χ follows from a radial cutoff function. The partial sum is telescoping:

$$\sum_{j=0}^{N} \varphi_j(\xi) = \chi(2^{-N}\xi),$$

and converges to one. The assertions about supports follow from where $\chi(2^{-j}\xi)$ and $\chi(2^{-j+1}\xi)$ can differ. □

Definition 4.12.2. (Dyadic blocks) Fixing a resolution as above, for $T \in \mathcal{S}'(\mathbb{R}^n)$ define

$$\Delta_j T := \mathcal{F}^{-1}(\varphi_j \widehat{T}), \qquad j \in \mathbb{N}_0.$$

The distributions $\Delta_j T$ are called the *dyadic blocks* of T.

Theorem 4.12.3. (Littlewood–Paley decomposition) For every $T \in \mathcal{S}'(\mathbb{R}^n)$,

$$T = \sum_{j \in \mathbb{N}_0} \Delta_j T$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. If $T \in \mathcal{S}(\mathbb{R}^n)$, convergence takes place in $\mathcal{S}(\mathbb{R}^n)$.

Proof. The partial sum is $\mathcal{F}^{-1}(\chi(2^{-N}\cdot)\widehat{T})$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it suffices to prove that $\chi(2^{-N}\cdot)\varphi \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$. The difference vanishes on a ball whose radius tends to infinity; differentiating it produces terms $2^{-N|\alpha|}(D^\alpha \chi)(2^{-N}\xi) D^\beta \varphi(\xi)$. The rapid decrease of φ makes each seminorm tend to zero. Continuity of T proves convergence in $\mathcal{S}'(\mathbb{R}^n)$. If $T \in \mathcal{S}(\mathbb{R}^n)$, the same calculation applies directly to $\widehat{T}$. □

Theorem 4.12.4. (Bernstein inequalities) Let $1 \le p \le q \le \infty$, $m \in \mathbb{N}_0$, and $R > 0$.

(a) If $f \in L^p(\mathbb{R}^n)$ and $\operatorname{supp}(\widehat{f}) \subseteq \overline{B}_{\mathrm{euc}}(0,R)$, then

$$\|D^\alpha f\|_{L^q(\mathbb{R}^n)} \le C R^{|\alpha| + n(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)}, \qquad |\alpha| \le m.$$

(b) If $f \in L^p(\mathbb{R}^n)$ and $\operatorname{supp}(\widehat{f}) \subseteq \{\xi \in \mathbb{R}^n \mid cR \le \|\xi\| \le CR\}$, then

$$C_1 R^m \|f\|_{L^p(\mathbb{R}^n)} \le \sum_{|\alpha| = m} \|D^\alpha f\|_{L^p(\mathbb{R}^n)} \le C_2 R^m \|f\|_{L^p(\mathbb{R}^n)}.$$

The constants depend only on the indicated parameters and on c, C, not on R or f.

Proof. By scaling, it suffices to consider $R = 1$. Choose $\eta \in C_c^\infty(\mathbb{R}^n)$ equal to one on a neighborhood of the support of $\widehat{f}$. Then $f = (2\pi)^{-\frac{n}{2}}(\mathcal{F}^{-1}\eta) * f$ and

$$D^\alpha f = (2\pi)^{-\frac{n}{2}}(D^\alpha \mathcal{F}^{-1}\eta) * f.$$

Theorem 4.10.1, with $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, proves the first estimate. For the lower bound in the second, choose η supported away from the origin and equal to one on the annulus. If $m = 0$, the assertion is immediate. If $m \geq 1$, set

$$D_m(\xi) := \sum_{\ell=1}^n \xi_\ell^{2m}, \qquad m_{me_\ell}(\xi) := i^{-m}\eta(\xi)\frac{\xi_\ell^m}{D_m(\xi)},$$

where $(e_\ell)_{\ell=1}^n$ is the standard basis. The denominator is positive away from the origin; therefore, $m_{me_\ell} \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ and

$$\eta(\xi) = \sum_{\ell=1}^n m_{me_\ell}(\xi)(i\xi_\ell)^m.$$

Since $\eta\widehat{f} = \widehat{f}$, taking the inverse Fourier transform gives

$$f = (2\pi)^{-\frac{n}{2}} \sum_{\ell=1}^n (\mathcal{F}^{-1} m_{me_\ell}) * D_\ell^m f.$$

Each $\mathcal{F}^{-1} m_{me_\ell}$ belongs to $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$; Theorem 4.10.1 then gives

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C \sum_{\ell=1}^n \|D_\ell^m f\|_{L^p(\mathbb{R}^n)}.$$

The upper bound is the first part. □

The finite overlap of the dyadic annuli will be the mechanism allowing us later to compare norms constructed from different resolutions.

5

Interpolation spaces

Intermediate spaces often arise between two spaces measuring different levels of size or regularity. Interpolation makes this idea precise and allows continuity of an operator at intermediate levels to be deduced from its bounds at two endpoints. It also yields compactness results under assumptions to be specified. In this book, the theory will serve three purposes: deriving fractional inequalities from integer-order results, distinguishing the real and complex Sobolev scales, and transferring Euclidean definitions to charts and trivializations. We shall use the functional analysis, integration, and Hardy inequality results from the analysis appendix, as well as the Fourier transform from Chapter 4. The results of this chapter will be applied to the scales in Chapter 6 and, later, to spaces of functions and sections on manifolds.

We shall develop the Lions–Peetre real method and the Calderón complex method. Our general references will be [97, Chapters 16–17] and [20]; applications to function spaces are compared with [146, §1.6] and [130, §2.3].

Interpolating instead of proving anew

Interpolation theory matured with the recognition that many endpoint estimates contain information about all intermediate levels. The Lions–Peetre methods quantify the decomposition of an element between two spaces, while Calderón's complex method uses holomorphic families. An estimate for an operator on two spaces can thus yield estimates on an entire family of intermediate spaces. In the study of regularity, this allows passage from integer to fractional orders.

5.1 Compatible pairs of Banach spaces

Interpolation compares two levels of regularity within a common ambient space. Compatibility ensures that the sum and intersection of the endpoint spaces are well defined and suitable for measuring decompositions.

Definition 5.1.1. (Compatible pair) A pair (X_0, X_1) of normed spaces is called *compatible* if there are continuous linear embeddings of X_0 and X_1 into a common Hausdorff topological vector space X. We shall identify both spaces with their images in X.

The ambient space serves only to give meaning to equality between elements of X_0 and X_1. Define

$$X_0 \cap X_1 \quad \text{with} \quad \|x\|_{X_0\cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\},$$

and

$$X_0 + X_1 := \{x_0 + x_1 \mid x_0 \in X_0,\ x_1 \in X_1\}, \qquad \|x\|_{X_0+X_1} := \inf_{\substack{x_0\in X_0,\ x_1\in X_1\\ x=x_0+x_1}} \big(\|x_0\|_{X_0} + \|x_1\|_{X_1}\big).$$

Proposition 5.1.2. If (X_0, X_1) is a compatible pair of Banach spaces, then $X_0 \cap X_1$ and $X_0 + X_1$ are Banach spaces. The inclusions $X_0 \cap X_1 \hookrightarrow X_j \hookrightarrow X_0 + X_1$, $j = 0, 1$, are continuous.

Proof. A Cauchy sequence in $X_0 \cap X_1$ converges in each X_j. The two limits agree in X because the embeddings are continuous and X is Hausdorff; their common limit belongs to the intersection.
For the sum, consider $X_0 \times X_1$ with the norm $\|(x_0, x_1)\| := \|x_0\|_{X_0} + \|x_1\|_{X_1}$ and the subspace

$$N := \{(x, -x) \mid x \in X_0 \cap X_1\}.$$

The subspace N is closed: if $(x_k, -x_k) \to (a, b)$ in $X_0 \times X_1$, then $x_k \to a$ and $x_k \to -b$ in X, whence $a = -b \in X_0 \cap X_1$. The map $(x_0, x_1) \mapsto x_0 + x_1$ induces an isometry from the quotient $(X_0 \times X_1)/N$ onto $X_0 + X_1$. Completeness of quotients of Banach spaces by closed subspaces completes the proof. □

Unless otherwise stated, all compatible pairs used from now on will consist of Banach spaces.

5.2 The real method: the K-functional

To compare x with the two endpoint spaces, we decompose it as $x = x_0 + x_1$. The same decomposition may be suitable at one scale and inefficient at another. The following functional chooses the smallest possible cost when the norm of the second component is multiplied by t.

Definition 5.2.1. (K-functional) Let (X_0, X_1) be a compatible pair. For $x \in X_0 + X_1$ and $t > 0$, define

$$K(t, x; X_0, X_1) := \inf_{\substack{x_0\in X_0,\ x_1\in X_1\\ x=x_0+x_1}} \big(\|x_0\|_{X_0} + t\|x_1\|_{X_1}\big).$$

When the pair is clear, we shall simply write $K(t, x)$.

For each $t > 0$, $K(t, \cdot)$ is a norm on $X_0 + X_1$ and satisfies

$$\min\{1, t\}\,\|x\|_{X_0+X_1} \le K(t, x) \le \max\{1, t\}\,\|x\|_{X_0+X_1}, \qquad x \in X_0 + X_1.$$

The parameter t weights the component x_1: when t is small, its norm contributes less to the cost. If X_1 represents greater regularity than X_0, the functional quantifies the cost of separating a more regular part from x.

Proposition 5.2.2. (Elementary properties of the K-functional) Let $x, y \in X_0 + X_1$ and $s, t > 0$. Then:

(a) $K(t, x + y) \le K(t, x) + K(t, y)$ and $K(t, \lambda x) = |\lambda| K(t, x)$;

(b) $t \mapsto K(t, x)$ is increasing and concave;

(c) $t \mapsto t^{-1}K(t,x)$ is decreasing;

(d) $K(t,x;X_0,X_1) = tK(t^{-1},x;X_1,X_0)$;

(e) $K(t,x) \le \min\{\|x\|_{X_0}, t\|x\|_{X_1}\}$ when $x \in X_0 \cap X_1$;

(f) $\min\{1,t\}\|x\|_{X_0+X_1} \le K(t,x) \le \max\{1,t\}\|x\|_{X_0+X_1}$.

Proof. The algebraic properties follow by adding decompositions and taking the infimum. Each function $t \mapsto \|x_0\|_{X_0} + t\|x_1\|_{X_1}$ is affine and increasing; the infimum of this family is increasing and concave. Moreover, $t^{-1}K(t,x)$ is the infimum of the decreasing functions $t \mapsto t^{-1}\|x_0\|_{X_0} + \|x_1\|_{X_1}$, taken over decompositions of x, and is therefore decreasing. Symmetry follows by interchanging x_0 and x_1. The last two estimates follow from the decompositions $x = x + 0$, $x = 0 + x$ and from comparing the coefficients $1, t$ with their minimum and maximum. □

Intermediate regularity requires simultaneous control at all scales $t \in (0,\infty)$. The factor $t^{-\theta}$ fixes the intermediate level; the exponent q determines how the costs are combined. The measure dt/t assigns the same weight to intervals related by dilation.

Definition 5.2.3. (Real interpolation space) Let $0 < \theta < 1$ and $1 \le q \le \infty$. Define

$$(X_0,X_1)_{\theta,q} := \{x \in X_0 + X_1 \mid \|x\|_{\theta,q;K} < \infty\},$$

where, if $q < \infty$,

$$\|x\|_{\theta,q;K} := \left(\int_0^\infty \left(t^{-\theta}K(t,x)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}},$$

and $\|x\|_{\theta,\infty;K} := \sup_{t\in(0,\infty)} t^{-\theta}K(t,x)$.

Theorem 5.2.4. (Basic properties of the K-method) If (X_0,X_1) is a compatible pair of Banach spaces, then $(X_0,X_1)_{\theta,q}$ is a Banach space and

$$X_0 \cap X_1 \hookrightarrow (X_0,X_1)_{\theta,q} \hookrightarrow X_0 + X_1$$

continuously. If $X_1 \hookrightarrow X_0$, let C_E be the norm of the embedding, let $T_E := \max\{1, C_E\}$, and set, for $q < \infty$,

$$N_{\theta,q}(x) := \|x\|_{X_0} + \left(\int_0^1 \left(t^{-\theta}K(t,x)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

and, for $q = \infty$,

$$N_{\theta,\infty}(x) := \|x\|_{X_0} + \sup_{t\in(0,1]} t^{-\theta}K(t,x).$$

Then, if $q < \infty$,

$$\frac{\|x\|_{\theta,q;K}}{\max\{1,(\theta q)^{-\frac{1}{q}}\}} \le N_{\theta,q}(x) \le \left(1 + (\theta q)^{\frac{1}{q}} T_E^\theta\right)\|x\|_{\theta,q;K},$$

whereas

$$\|x\|_{\theta,\infty;K} \le N_{\theta,\infty}(x) \le (1 + T_E^{\theta})\|x\|_{\theta,\infty;K}.$$

Proof. If $x \in X_0 \cap X_1$, Proposition 5.2.2 gives $K(t,x) \le \min\{1,t\}\|x\|_{X_0\cap X_1}$. Since

$$\int_0^1 t^{(1-\theta)q}\,\frac{dt}{t} + \int_1^\infty t^{-\theta q}\,\frac{dt}{t} < \infty,$$

we obtain the first embedding when $q < \infty$. For $q = \infty$, use $\sup_{t\in(0,\infty)} t^{-\theta}\min\{1,t\} = 1$ directly. For the second embedding, Proposition 5.2.2 implies $\|x\|_{X_0+X_1} \le K(t,x)$ for $1 \le t \le 2$. Thus, if $q < \infty$,

$$\|x\|_{X_0+X_1} \le (\log 2)^{-\frac{1}{q}}\, 2^{\theta}\|x\|_{\theta,q;K};$$

if $q = \infty$, evaluation at $t = 1$ suffices.
Now let (x_j) be Cauchy in $(X_0,X_1)_{\theta,q}$. The second embedding implies $x_j \to x$ in $X_0 + X_1$ for some x. For $q < \infty$, continuity of the norm $K(t,\cdot)$ on $X_0 + X_1$ and Fatou's lemma B.6.12 give

$$\|x_j - x\|_{\theta,q;K}^q \le \liminf_{k\to\infty} \|x_j - x_k\|_{\theta,q;K}^q.$$

The case $q = \infty$ follows by taking the supremum after passing to the limit for each t. Thus $x_j \to x$ in the interpolation space.
If $X_1 \hookrightarrow X_0$, let C_E be the norm of the embedding and let $T_E := \max\{1, C_E\}$. Then $X_0 + X_1 = X_0$ and, for every decomposition $x = x_0 + x_1$,

$$\|x\|_{X_0} \le \|x_0\|_{X_0} + C_E\|x_1\|_{X_1}.$$

Consequently, if $t \ge T_E$, we have $K(t,x) \ge \|x\|_{X_0}$; the decomposition $x = x + 0$ gives the reverse inequality. Hence

$$\int_{T_E}^\infty \left(t^{-\theta}K(t,x)\right)^q \frac{dt}{t} = \frac{T_E^{-\theta q}}{\theta q}\|x\|_{X_0}^q.$$

Moreover, $K(t,x) \le \|x\|_{X_0}$ for every $t > 0$. If $I_0 := \displaystyle\int_0^1 (t^{-\theta}K(t,x))^q\,dt/t$, it follows that

$$\|x\|_{\theta,q;K} \le \left(I_0 + (\theta q)^{-1}\|x\|_{X_0}^q\right)^{\frac{1}{q}} \le \max\{1, (\theta q)^{-\frac{1}{q}}\}N_{\theta,q}(x).$$

The tail identity starting at T_E gives, in the opposite direction, $\|x\|_{X_0} \le (\theta q)^{\frac{1}{q}} T_E^{\theta}\|x\|_{\theta,q;K}$, while $I_0^{\frac{1}{q}} \le \|x\|_{\theta,q;K}$. This proves both bounds for $q < \infty$. For $q = \infty$, the inequality $K(t,x) \le \|x\|_{X_0}$ shows that $\|x\|_{\theta,\infty;K} \le N_{\theta,\infty}(x)$, and the equality $K(T_E,x) = \|x\|_{X_0}$ gives $\|x\|_{X_0} \le T_E^{\theta}\|x\|_{\theta,\infty;K}$; this yields the remaining bound. □

Lemma 5.2.5. (Dyadic discretization) For $0 < \theta < 1$ and $1 \le q \le \infty$, there are constants $c_{\theta,q}, C_{\theta,q} > 0$ such that

$$c_{\theta,q}\left\|\left(2^{-k\theta}K(2^k,x)\right)_{k\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})} \le \|x\|_{\theta,q;K} \le C_{\theta,q}\left\|\left(2^{-k\theta}K(2^k,x)\right)_{k\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})}.$$

The constants depend only on θ and q.

Proof. If $2^k \le t \le 2^{k+1}$, monotonicity of K and $\frac{K(t,x)}{t}$ gives

$$2^{-\theta}2^{-k\theta}K(2^k,x) \le t^{-\theta}K(t,x) \le 2^{1-\theta}2^{-k\theta}K(2^k,x).$$

Integrate over each interval $[2^k, 2^{k+1}]$ with respect to dt/t and sum. The argument with suprema covers $q = \infty$. □

Corollary 5.2.6. (Monotonicity in the second index) If $1 \le q_0 \le q_1 \le \infty$, then

$$(X_0, X_1)_{\theta,q_0} \hookrightarrow (X_0, X_1)_{\theta,q_1}.$$

Moreover, $(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}$ and, for every x,

$$\|x\|_{\theta,q;K,(X_0,X_1)} = \|x\|_{1-\theta,q;K,(X_1,X_0)}.$$

Proof. The first assertion follows from Lemma 5.2.5 and $\ell^{q_0} \hookrightarrow \ell^{q_1}$. For the second, use $K(t,x;X_0,X_1) = tK(t^{-1},x;X_1,X_0)$ and the change of variable $s = t^{-1}$. □

Proposition 5.2.7. (Multiplicative inequality) There is $C_{\theta,q} > 0$ such that for every $x \in X_0 \cap X_1$,

$$\|x\|_{(X_0,X_1)_{\theta,q}} \le C_{\theta,q}\|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}.$$

Proof. If $x = 0$, the assertion is immediate. Otherwise, set $a := \|x\|_{X_0} > 0$, $b := \|x\|_{X_1} > 0$, and $t_0 := a/b$. If $q < \infty$, Proposition 5.2.2 gives

$$\begin{aligned}\|x\|_{\theta,q;K}^q &\le b^q \int_0^{t_0} t^{(1-\theta)q}\,\frac{dt}{t} + a^q \int_{t_0}^{\infty} t^{-\theta q}\,\frac{dt}{t} \\ &= \left(\frac{1}{(1-\theta)q} + \frac{1}{\theta q}\right) a^{(1-\theta)q} b^{\theta q}.\end{aligned}$$

For $q = \infty$, the function $t^{-\theta}\min\{a, tb\}$ attains its maximum at $t = t_0$, where its value is $a^{1-\theta}b^{\theta}$. □

5.3 Interpolation of operators

The analytic value of the K-method lies in transferring known endpoint bounds to intermediate spaces. The following result makes this transfer precise and retains control of the operator norm.

Theorem 5.3.1. (Interpolation theorem for the K-method) Let (X_0, X_1) and (Y_0, Y_1) be compatible pairs of Banach spaces. Let $T\colon X_0 + X_1 \longrightarrow Y_0 + Y_1$ be linear, and suppose that its restrictions $T\colon X_j \longrightarrow Y_j$ are continuous, with norms M_j, $j = 0, 1$. Then

$$T\colon (X_0, X_1)_{\theta,q} \longrightarrow (Y_0, Y_1)_{\theta,q}$$

is continuous and

$$\|T\| \le M_0^{1-\theta} M_1^{\theta}.$$

Proof. If $x = x_0 + x_1$, then

$$K(t, Tx; Y_0, Y_1) \le M_0 \|x_0\|_{X_0} + tM_1 \|x_1\|_{X_1}.$$

Taking the infimum,

$$K(t, Tx; Y_0, Y_1) \le M_0 K\left(t\frac{M_1}{M_0}, x; X_0, X_1\right).$$

The conclusion follows from the change of variable $s = \frac{tM_1}{M_0}$. If one of the norms M_j is zero, approximate it by $M_j + \varepsilon$ and let $\varepsilon \to 0^+$. □

Corollary 5.3.2. (Interpolation of embeddings) If $X_j \hookrightarrow Y_j$ for $j \in \{0,1\}$, then $(X_0, X_1)_{\theta,q} \hookrightarrow (Y_0, Y_1)_{\theta,q}$. If there are linear operators $R\colon X_j \longrightarrow Y_j$ and $S\colon Y_j \longrightarrow X_j$, continuous at both endpoints and satisfying $SR = I$ on $X_0 + X_1$, then S induces a retraction and R a coretraction between the interpolated spaces.

Proof. The inclusion $I\colon X_0 + X_1 \to Y_0 + Y_1$ restricted to each endpoint is a continuous operator $I\colon X_j \to Y_j$. The operator interpolation theorem, applied to I, provides a continuous operator

$$I\colon (X_0, X_1)_{\theta,q} \longrightarrow (Y_0, Y_1)_{\theta,q},$$

which is precisely the embedding in the first assertion.
In the second situation, applying the same theorem separately to R and S gives continuous operators

$$R\colon (X_0, X_1)_{\theta,q} \longrightarrow (Y_0, Y_1)_{\theta,q}, \qquad S\colon (Y_0, Y_1)_{\theta,q} \longrightarrow (X_0, X_1)_{\theta,q}.$$

For $x \in (X_0, X_1)_{\theta,q} \subseteq X_0 + X_1$, the assumed identity on the sum space implies

$$S(Rx) = x.$$

Thus $S \circ R = I$ on the interpolated space: S is a retraction, and R is its corresponding coretraction. □

Theorem 5.3.3. (Compactness into a common target space) Let Y be a Banach space and let $T\colon X_0 + X_1 \longrightarrow Y$ be linear. If $T\colon X_0 \longrightarrow Y$ is compact and $T\colon X_1 \longrightarrow Y$ is continuous, then

$$T\colon (X_0, X_1)_{\theta,q} \longrightarrow Y$$

is compact for $0 < \theta < 1$ and $1 \le q \le \infty$.

Proof. Let B be a bounded subset of $(X_0, X_1)_{\theta,q}$. By Lemma 5.2.5, including when $q = \infty$, there is $C > 0$ such that $K(t,x) \le Ct^\theta$ for every $t > 0$ and $x \in B$. For each $x \in B$, choose $x = a_{x,t} + b_{x,t}$ with

$$\|a_{x,t}\|_{X_0} + t\|b_{x,t}\|_{X_1} \le 2Ct^\theta.$$

Given a tolerance $\varepsilon > 0$, choose t sufficiently large that $2C\|T\|_{X_1 \to Y} t^{\theta-1} < \varepsilon$. The set $\{a_{x,t} \mid x \in B\}$ is bounded in X_0 and, by compactness of $T\colon X_0 \longrightarrow Y$, its image can be covered by finitely many balls of radius ε. Since

$$\|Tb_{x,t}\|_Y \le 2C\|T\|_{X_1 \to Y} t^{\theta-1} < \varepsilon,$$

$T(B)$ admits a finite cover by balls of radius 2ε. Thus $T(B)$ is totally bounded; since Y is complete, its closure is compact. This proves the assertion. □

5.4 The J-method

The K-functional describes an element through a single scale-dependent decomposition. The J-method instead uses a continuous superposition of elements belonging to both endpoint spaces simultaneously.

Definition 5.4.1. (J-functional) For $a \in X_0 \cap X_1$ and $t > 0$, define

$$J(t, a; X_0, X_1) := \max\{\|a\|_{X_0}, t\|a\|_{X_1}\}.$$

A function $u : (0,\infty) \longrightarrow X_0 \cap X_1$ is admissible for $x \in X_0 + X_1$ if it is strongly measurable, the Bochner integral $\int_0^\infty u(t)\,dt/t$ converges in $X_0 + X_1$, and

$$x = \int_0^\infty u(t)\,\frac{dt}{t}.$$

Definition 5.4.2. (Space defined by the J-method) Let $0 < \theta < 1$ and $1 \le q \le \infty$. Define $(X_0, X_1)_{\theta,q;J}$ as the set of $x \in X_0 + X_1$ admitting a representation as above for which

$$t \longmapsto t^{-\theta} J(t, u(t))$$

belongs to $L^q((0,\infty), dt/t)$. Its norm is the infimum of the $L^q((0,\infty), dt/t)$ norm over all admissible representations.

Lemma 5.4.3. (From a J-representation to a K-estimate) If $x = \int_0^\infty u(t)\,dt/t$ is an admissible representation with $t^{-\theta} J(t, u(t)) \in L^q((0,\infty), dt/t)$, then for $r > 0$,

$$K(r, x) \le \int_0^r \|u(t)\|_{X_0}\,\frac{dt}{t} + r\int_r^\infty \|u(t)\|_{X_1}\,\frac{dt}{t}.$$

Proof. Split the integral over $(0, r)$ and (r,∞). If $g(t) := t^{-\theta} J(t, u(t)) \in L^q((0,\infty), dt/t)$, then

$$\int_0^r \|u(t)\|_{X_0}\,\frac{dt}{t} \le \int_0^r t^\theta g(t)\,\frac{dt}{t} < \infty$$

and

$$\int_r^\infty \|u(t)\|_{X_1}\,\frac{dt}{t} \le \int_r^\infty t^{\theta-1} g(t)\,\frac{dt}{t} < \infty.$$

Finiteness follows from Proposition B.7.9, or directly if $q = 1, \infty$. By the Bochner integrability criterion in Proposition B.6.23, the first part converges in X_0 and the second in X_1. This splitting is an admissible decomposition for $K(r, x)$, and the triangle inequality for the Bochner integral gives the stated estimate. □

Theorem 5.4.4. (Equivalence of the K- and J-methods) For every compatible pair of Banach spaces, $0 < \theta < 1$ and $1 \le q \le \infty$,

$$(X_0, X_1)_{\theta,q;K} = (X_0, X_1)_{\theta,q;J}$$

and there are constants $c_{\theta,q}, C_{\theta,q} > 0$, independent of the pair and the element, such that

$$c_{\theta,q}\|x\|_{\theta,q;K} \le \|x\|_{\theta,q;J} \le C_{\theta,q}\|x\|_{\theta,q;K}$$

for every x in the common space.

Proof. First let x be given by a J-representation. Lemma 5.4.3, followed by the two integral Hardy inequalities from Theorem B.8.6,

$$\left\| r^{-\theta} \int_0^r h(t)\,\frac{dt}{t} \right\|_{L^q((0,\infty),dr/r)} \le \frac{1}{\theta}\|t^{-\theta}h(t)\|_{L^q((0,\infty),dt/t)}$$

and

$$\left\| r^{1-\theta} \int_r^\infty h(t)\,\frac{dt}{t} \right\|_{L^q((0,\infty),dr/r)} \le \frac{1}{1-\theta}\|t^{1-\theta}h(t)\|_{L^q((0,\infty),dt/t)},$$

gives $\|x\|_{\theta,q;K} \le C_{\theta,q}\|x\|_{\theta,q;J}$. In the first inequality, we use $h(t) = \|u(t)\|_{X_0}$, and in the second, $h(t) = \|u(t)\|_{X_1}$. For the reverse inclusion, use a dyadic construction. Choose decompositions $x = a_k + b_k$ such that

$$\|a_k\|_{X_0} + 2^k\|b_k\|_{X_1} \le 2K(2^k, x).$$

Lemma 5.2.5 gives $K(2^k, x) \le C2^{k\theta}\|x\|_{\theta,q;K}$. Set $v_k := a_{k+1} - a_k = b_k - b_{k+1}$. Then

$$\|a_k\|_{X_0} \longrightarrow 0 \quad (k \to -\infty), \qquad \|b_k\|_{X_1} \longrightarrow 0 \quad (k \to \infty).$$

Truncating the sum gives $\displaystyle\sum_{k=-N}^{N} v_k = a_{N+1} - a_{-N} \to x$ in $X_0 + X_1$. Therefore, $x = \displaystyle\sum_{k\in\mathbb{Z}} v_k$ in $X_0 + X_1$. Moreover,

$$J(2^k, v_k) \le C\big(K(2^k, x) + K(2^{k+1}, x)\big).$$

Define $u(t) := (\log 2)^{-1} v_k$ for $2^k \le t < 2^{k+1}$. For $k \le 0$, use $\|v_k\|_{X_0} \le C2^{k\theta}\|x\|_{\theta,q;K}$; for $k > 0$, use $\|v_k\|_{X_1} \le C2^{-k(1-\theta)}\|x\|_{\theta,q;K}$. Both geometric series converge. Thus u is integrable in $X_0 + X_1$ and $x = \displaystyle\int_0^\infty u(t)\,dt/t$. Lemma 5.2.5 and the preceding estimate give $\|x\|_{\theta,q;J} \le C\|x\|_{\theta,q;K}$. □

Lemma 5.4.5. (Discrete characterization of the J-method) Let $0 < \theta < 1$ and $1 \le q \le \infty$. An element $x \in X_0 + X_1$ belongs to $(X_0, X_1)_{\theta,q}$ if and only if it admits a representation

$$x = \sum_{k\in\mathbb{Z}} u_k \quad \text{in } X_0 + X_1, \qquad u_k \in X_0 \cap X_1,$$

for which

$$\big(2^{-k\theta} J(2^k, u_k; X_0, X_1)\big)_{k\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

Moreover, the interpolation norm is equivalent to the infimum of the norm of this sequence over all such representations. If $q < \infty$, $X_0 \cap X_1$ is dense in $(X_0, X_1)_{\theta,q}$.

Proof. First suppose that a discrete representation is given, and write $c_k := 2^{-k\theta} J(2^k, u_k)$. For $k \le 0$,

$$\|u_k\|_{X_0} \le 2^{k\theta} c_k,$$

and, for $k > 0$,

$$\|u_k\|_{X_1} \le 2^{-k(1-\theta)} c_k.$$

Proposition B.7.9 for sequences and summation of the geometric series show that the two tails converge in X_0 and X_1, respectively. Defining $u(t) := (\log 2)^{-1} u_k$ on $[2^k, 2^{k+1})$ yields an admissible representation and, for every $t \in [2^k, 2^{k+1})$,

$$\frac{2^{-\theta}}{\log 2} 2^{-k\theta} J(2^k, u_k) \le t^{-\theta} J(t, u(t)) \le \frac{2}{\log 2} 2^{-k\theta} J(2^k, u_k).$$

This proves one of the norm estimates.
Conversely, given an admissible representation $u(t)$, set

$$u_k := \int_{2^k}^{2^{k+1}} u(t) \frac{dt}{t}.$$

Proposition B.6.27 allows these integrals to be summed in $X_0 + X_1$ and gives $x = \sum_{k\in\mathbb{Z}} u_k$. The triangle inequality for the Bochner integral and $2^k \le t$ imply

$$J(2^k, u_k) \le \int_{2^k}^{2^{k+1}} J(t, u(t)) \frac{dt}{t}.$$

Proposition B.7.9 on each dyadic interval, followed by Theorem B.6.17, yields

$$\left\| \left(2^{-k\theta} J(2^k, u_k)\right)_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \le C_{\theta,q} \| t^{-\theta} J(t, u(t)) \|_{L^q((0,\infty), dt/t)}.$$

The case $q = \infty$ follows by taking suprema. Taking the infimum over representations and using Theorem 5.4.4 completes the equivalence of norms. If $q < \infty$, the finite sums $\sum_{\substack{k\in\mathbb{Z} \\ |k|\le N}} u_k \in X_0 \cap X_1$ converge to x in the interpolation norm, since tails of a sequence in $\ell^q(\mathbb{Z})$ tend to zero. This proves the final assertion. □

5.5 Reiteration

Interpolation can be applied more than once. The following result states that interpolating between two already interpolated levels produces no new scales.

Lemma 5.5.1. (Holmstedt's formula) Let (X_0, X_1) be a compatible pair of Banach spaces, and let $0 < \theta_0 < \theta_1 < 1$ and $1 \le q_0, q_1 \le \infty$. Set $X_{\theta_j} := (X_0, X_1)_{\theta_j, q_j}$. There are constants $c, C > 0$ such that for $x \in X_0 + X_1$ and $r > 0$,

$$c\big(A_0(r,x) + r^{\theta_1-\theta_0} A_1(r,x)\big) \le K(r^{\theta_1-\theta_0}, x; X_{\theta_0}, X_{\theta_1}) \le C\big(A_0(r,x) + r^{\theta_1-\theta_0} A_1(r,x)\big),$$

where

$$A_0(r,x) := \left\| t^{-\theta_0} K(t,x;X_0,X_1) \right\|_{L^{q_0}((0,r),dt/t)},$$

and

$$A_1(r,x) := \left\| t^{-\theta_1} K(t,x;X_0,X_1) \right\|_{L^{q_1}((r,\infty),dt/t)},$$

with suprema when some $q_j = \infty$. If $x \notin X_{\theta_0} + X_{\theta_1}$, the functional in the middle is interpreted as $+\infty$. The constants c and C depend only on $\theta_0, \theta_1, q_0, q_1$.

Proof. Set $\delta := \theta_1 - \theta_0$ and choose $m \in \mathbb{Z}$ so that $2^m \le r < 2^{m+1}$. First suppose that $A_0(r,x) + r^\delta A_1(r,x) < \infty$. For each $k \in \mathbb{Z}$, choose a decomposition $x = a_k + b_k$ such that

$$\|a_k\|_{X_0} + 2^k \|b_k\|_{X_1} \le 2K(2^k, x; X_0, X_1)$$

and set $u_k := a_{k+1} - a_k = b_k - b_{k+1}$. Comparison of K on dyadic intervals gives $K(2^k,x) \le C 2^{k\theta_0} A_0(r,x)$ for $2^{k+1} \le r$ and $K(2^k,x) \le C 2^{k\theta_1} A_1(r,x)$ for $2^{k-1} \ge r$. Thus $a_k \to 0$ in X_0 as $k \to -\infty$, and $b_k \to 0$ in X_1 as $k \to \infty$. Therefore, $x = \sum_{k\in\mathbb{Z}} u_k$ in $X_0 + X_1$, and

$$\|u_k\|_{X_0} + 2^k \|u_k\|_{X_1} \le C\big(K(2^k,x) + K(2^{k+1},x)\big).$$

Define

$$x_0 := \sum_{\substack{k\in\mathbb{Z}\\ k\le m}} u_k, \qquad x_1 := \sum_{\substack{k\in\mathbb{Z}\\ k>m}} u_k.$$

The discrete characterization of the J-method in Lemma 5.4.5, Lemma 5.2.5, and comparison of r with 2^m give

$$\|x_0\|_{X_{\theta_0}} \le C A_0(r,x), \qquad \|x_1\|_{X_{\theta_1}} \le C A_1(r,x).$$

For the terms near the splitting point, use $K(2^{m+1},x) \le 4K(2^{m-1},x)$ and the interval $[2^{m-1}, 2^m] \subseteq (0,r)$; for the other tail, use $[2^{m+1}, 2^{m+2}] \subseteq (r,\infty)$. Thus the constants also control the terms adjacent to r. Each series converges in $X_0 + X_1$ and defines a J-representation of the indicated space; this proves membership even when $q_j = \infty$, without requiring convergence of partial sums in the norm of that space. Since $x = x_0 + x_1$, this decomposition proves

$$K(r^\delta, x; X_{\theta_0}, X_{\theta_1}) \le C\big(A_0(r,x) + r^\delta A_1(r,x)\big).$$

If either tail is infinite, this inequality holds automatically.
For the reverse bound, let $x = y_0 + y_1$ with $y_j \in X_{\theta_j}$. Lemma 5.2.5 implies the pointwise estimate

$$K(t, y_j; X_0, X_1) \le C t^{\theta_j} \|y_j\|_{X_{\theta_j}}, \qquad t > 0.$$

The triangle inequality for $K(t,\cdot)$ gives

$$A_0(r,x) \le C\big(\|y_0\|_{X_{\theta_0}} + r^\delta \|y_1\|_{X_{\theta_1}}\big)$$

and

$$r^\delta A_1(r,x) \le C\big(\|y_0\|_{X_{\theta_0}} + r^\delta \|y_1\|_{X_{\theta_1}}\big).$$

Here we integrate only the powers $t^{\delta q_0}$ on $(0,r)$ and $t^{-\delta q_1}$ on (r,∞); if an exponent is infinite, take suprema. Taking the infimum over decompositions $x = y_0 + y_1$ yields the remaining inequality. □

Theorem 5.5.2. (Real reiteration theorem) Let $0 \le \theta_0 < \theta_1 \le 1$, $0 < \eta < 1$, and $1 \le q, q_0, q_1 \le \infty$. If $X_{\theta_j} = (X_0, X_1)_{\theta_j, q_j}$ when $0 < \theta_j < 1$, while $X_{\theta_0} = X_0$ if $\theta_0 = 0$ and $X_{\theta_1} = X_1$ if $\theta_1 = 1$, then

$$(X_{\theta_0}, X_{\theta_1})_{\eta,q} = (X_0, X_1)_{\theta,q}, \qquad \theta := (1-\eta)\theta_0 + \eta\theta_1,$$

and there are constants $c, C > 0$, depending only on $\theta_0, \theta_1, \eta, q, q_0, q_1$, such that

$$c\|x\|_{(X_0,X_1)_{\theta,q}} \le \|x\|_{(X_{\theta_0},X_{\theta_1})_{\eta,q}} \le C\|x\|_{(X_0,X_1)_{\theta,q}}$$

for every x in the common space.

Proof. First assume $0 < \theta_0 < \theta_1 < 1$ and set $\delta := \theta_1 - \theta_0$, so that $\theta = \theta_0 + \eta\delta$. Write

$$K_*(\tau, x) := K(\tau, x; X_{\theta_0}, X_{\theta_1}), \qquad c_k := 2^{-k\theta} K(2^k, x; X_0, X_1).$$

The proof of Lemma 5.2.5, with geometric base 2^δ in place of 2, provides constants $d_*, D_* > 0$ such that

$$d_* \left\| (2^{-k\eta\delta} K_*(2^{k\delta}, x))_k \right\|_{\ell^q} \le \|x\|_{(X_{\theta_0},X_{\theta_1})_{\eta,q}} \le D_* \left\| (2^{-k\eta\delta} K_*(2^{k\delta}, x))_k \right\|_{\ell^q}.$$

By Lemma 5.5.1, the sequence between the norm signs is bounded above and below, term by term, by multiples of

$$B_{0,k} := 2^{-k\eta\delta} A_0(2^k, x), \qquad B_{1,k} := 2^{k(1-\eta)\delta} A_1(2^k, x).$$

Splitting the integrals defining A_0 and A_1 into the intervals $[2^j, 2^{j+1}]$ and using monotonicity of K and $\frac{K(t,x)}{t}$, as in the proof of the discretization lemma, reduces the upper bound to the two sequence operators

$$(\mathsf{H}^-_{a,r} c)_k := \left(\sum_{\substack{j \in \mathbb{Z} \\ j \le k}} (2^{-a(k-j)} |c_j|)^r \right)^{\frac{1}{r}},$$

$$(\mathsf{H}^+_{a,r} c)_k := \left(\sum_{\substack{j \in \mathbb{Z} \\ j \ge k}} (2^{-a(j-k)} |c_j|)^r \right)^{\frac{1}{r}},$$

with suprema if $r = \infty$, taking $(a, r) = (\eta\delta, q_0)$ and $(a, r) = ((1-\eta)\delta, q_1)$, respectively. For any $a > 0$ and $1 \le q, r \le \infty$, there is $C_{a,q,r} > 0$ such that

$$\|\mathsf{H}^-_{a,r} c\|_{\ell^q} + \|\mathsf{H}^+_{a,r} c\|_{\ell^q} \le C_{a,q,r} \|c\|_{\ell^q}.$$

Indeed, if $q \ge r < \infty$, raise to the power r and apply Young's inequality in $\ell^{\frac{q}{r}}$ to the geometric kernel

$(2^{-arm}\mathbf{1}_{\{m\ge 0\}})_m$. If $q<r$, use

$$\left(\sum_{j\in\mathbb{Z}} d_j^r\right)^{\frac{q}{r}} \le \sum_{j\in\mathbb{Z}} d_j^q$$

and sum first over the outer index; the geometric series $\sum_{m\in\mathbb{N}_0} 2^{-aqm}$ appears. For $r=\infty$, the supremum is bounded by the sum of powers q, and for $q=\infty$ the assertion follows directly. The tail operator is treated with the reflected kernel. We deduce

$$\|x\|_{(X_{\theta_0},X_{\theta_1})_{\eta,q}} \le C\|(c_k)_k\|_{\ell^q} \le C'\|x\|_{(X_0,X_1)_{\theta,q}}.$$

For the reverse bound, the interval $[2^k,2^{k+1}]$ in the definition of $A_0(2^{k+1},x)$ and monotonicity of K give a constant $d_0=d_0(\theta_0,q_0)>0$ such that

$$2^{-(k+1)\eta\delta}A_0(2^{k+1},x) \ge d_0 c_k.$$

This also holds for $q_0=\infty$, interpreting the norm as a supremum. Holmstedt's lower bound and the two discretizations yield

$$\|x\|_{(X_0,X_1)_{\theta,q}} \le C\|x\|_{(X_{\theta_0},X_{\theta_1})_{\eta,q}}.$$

Now consider $\theta_0=0<\theta_1<1$. We first prove

$$K(r^{\theta_1},x;X_0,X_{\theta_1}) \asymp K(r,x)+r^{\theta_1}A_1(r,x), \tag{5.1}$$

where $\asymp$ means that each side is bounded by a positive constant times the other, uniformly in x and r.
If the right-hand side is finite and $x\neq 0$, choose $x=a+b$ with

$$\|a\|_{X_0}+r\|b\|_{X_1} \le 2K(r,x).$$

For $t<r$, use $K(t,b)\le t\|b\|_{X_1}$; for $t\ge r$, the triangle inequality and monotonicity give

$$K(t,b) \le K(t,x)+\|a\|_{X_0} \le 3K(t,x).$$

Integrating on both intervals, or taking suprema, yields

$$\|b\|_{X_{\theta_1}} \le C\big(r^{-\theta_1}K(r,x)+A_1(r,x)\big).$$

This decomposition proves one bound in (5.1). For the other, write $x=y_0+y_1$, where $y_0\in X_0$ and $y_1\in X_{\theta_1}$. The pointwise estimate for K and the integral of $t^{-\theta_1 q_1}$ over (r,∞) give

$$K(r,x) \le \|y_0\|_{X_0}+Cr^{\theta_1}\|y_1\|_{X_{\theta_1}},$$
$$r^{\theta_1}A_1(r,x) \le C\|y_0\|_{X_0}+r^{\theta_1}\|y_1\|_{X_{\theta_1}}.$$

Take the infimum over y_0,y_1.
The case $0<\theta_0<\theta_1=1$ follows by applying the preceding formula to the pair (X_1,X_0) with parameter $1-\theta_0$ and scale r^{-1}. Indeed, symmetry of the K-functional and the substitution $t\mapsto t^{-1}$ transform that

formula into

$$K(r^{1-\theta_0}, x; X_{\theta_0}, X_1) \asymp A_0(r,x) + r^{-\theta_0} K(r,x). \tag{5.2}$$

Multiply (5.1) by $r^{-\eta\theta_1}$, or (5.2) by $r^{-\eta(1-\theta_0)}$. In both cases, the term containing $K(r,x)$ is $r^{-\theta}K(r,x)$. The other term is controlled by $\mathsf{H}^+_{(1-\eta)\theta_1,q_1}$ or $\mathsf{H}^-_{\eta(1-\theta_0),q_0}$, respectively. The estimates already proved for these operators give the upper bound, and the term $r^{-\theta}K(r,x)$ gives the lower bound. If $\theta_0 = 0$ and $\theta_1 = 1$, the assertion is the definition. □

Definition 5.5.3. (ℓ^q-sums of a family of Banach spaces) Let I be a set and let $(Z_i)_{i\in I}$ be a family of Banach spaces. For $1 \le q < \infty$, define

$$\ell^q\big(I;(Z_i)_{i\in I}\big) := \left\{ z = (z_i)_{i\in I} \;\middle|\; z_i \in Z_i \text{ for every } i \in I, \quad \sum_{i\in I} \|z_i\|_{Z_i}^q < \infty \right\},$$

with the norm

$$\|z\|_{\ell^q(I;(Z_i)_{i\in I})} := \left(\sum_{i\in I} \|z_i\|_{Z_i}^q \right)^{\frac{1}{q}}.$$

A sum over an arbitrary set is understood as the supremum of sums over finite subsets. For $q = \infty$, set

$$\ell^\infty\big(I;(Z_i)_{i\in I}\big) := \left\{ z = (z_i)_{i\in I} \;\middle|\; z_i \in Z_i \text{ for every } i \in I, \quad \sup_{i\in I} \|z_i\|_{Z_i} < \infty \right\},$$

with the supremum norm. If all the spaces equal a single space Z, write simply $\ell^q(I;Z)$. When $I = \mathbb{Z}$ and the family is indicated in the argument, we shall also use the abbreviation $\ell^q((Z_k)_{k\in\mathbb{Z}})$.

Lemma 5.5.4. (Duality of ℓ^q-sums) Let I be a set, let $(Z_i)_{i\in I}$ be a family of Banach spaces, and let $1 \le q < \infty$. Then

$$\big(\ell^q\big(I;(Z_i)_{i\in I}\big)\big)' = \ell^{q'}\big(I;(Z_i')_{i\in I}\big)$$

isometrically, through the pairing

$$\langle z, \lambda\rangle := \sum_{i\in I} \lambda_i(z_i).$$

Proof. Hölder's inequality for summable families shows that each $\lambda = (\lambda_i)_{i\in I}$ in the space on the right defines a continuous functional whose norm does not exceed $\|\lambda\|_{\ell^{q'}(I;(Z_i')_{i\in I})}$.
Conversely, let Λ be a continuous functional on the space on the left. For each $i \in I$, denote by

$$\iota_i \colon Z_i \longrightarrow \ell^q\big(I;(Z_j)_{j\in I}\big)$$

the inclusion into coordinate i and set $\lambda_i := \Lambda \circ \iota_i$. If $F \subseteq I$ is finite, choose $z_i \in Z_i$ of norm one that realizes $\|\lambda_i\|_{Z_i'}$ with arbitrarily small error. Duality of $\ell^q(F)$ gives

$$\left(\sum_{i\in F} \|\lambda_i\|_{Z_i'}^{q'} \right)^{\frac{1}{q'}} \le \|\Lambda\|,$$

with the usual interpretation as a maximum when $q' = \infty$. Taking the supremum over finite subsets F, we

obtain $\lambda \in \ell^{q'}(I;(Z_i')_{i\in I})$ and $\|\lambda\| \le \|\Lambda\|$.
Finitely supported families are dense in $\ell^q(I;(Z_i)_{i\in I})$ for $q<\infty$. Thus the equality

$$\Lambda(z) = \sum_{i\in I} \lambda_i(z_i)$$

is first proved for finite support and then extended by continuity. The first estimate gives equality of norms. The argument includes $q=1$, in which case $q'=\infty$. □

A Banach pair (X_0,X_1) is called *regular* if $X_0\cap X_1$ is dense in each X_j. In that case, the duals X_0' and X_1' are viewed as subspaces of $(X_0\cap X_1)'$ by restriction and form a compatible pair.

Lemma 5.5.5. (Dual of the intersection of a regular pair) If (X_0,X_1) is a regular pair, restriction gives the isometric identification

$$(X_0\cap X_1)' = X_0' + X_1',$$

where the intersection carries the maximum norm and the sum of the duals carries the quotient norm. More generally, for $t>0$,

$$\|y\|_{(X_0\cap X_1, J(t,\cdot))'} = K(t^{-1}, y; X_0', X_1').$$

Proof. Consider the diagonal inclusion

$$D\colon X_0\cap X_1 \longrightarrow X_0\times X_1, \qquad Dx=(x,x),$$

where the product carries the norm $\max\{\|x_0\|_{X_0}, t\|x_1\|_{X_1}\}$. Every functional y on the diagonal extends without increasing its norm by the Hahn–Banach corollary B.2.29. A functional on the product has the form $(x_0,x_1)\mapsto y_0(x_0)+y_1(x_1)$ and norm $\|y_0\|_{X_0'}+t^{-1}\|y_1\|_{X_1'}$. Restricting it to the diagonal and taking the infimum over all extensions gives the formula. Regularity ensures that restriction $X_j'\to(X_0\cap X_1)'$ is injective, so the sum belongs to a compatible pair and is not merely an abstract quotient. □

5.6 Duality for real interpolation

Theorem 5.6.1. (Duality of the real method) Let (X_0,X_1) be a regular pair, let $0<\theta<1$, and let $1\le q<\infty$. Then the pairing between X_0+X_1 and $X_0'\cap X_1'$ extends to a topological isomorphism

$$\big((X_0,X_1)_{\theta,q}\big)' \cong (X_0',X_1')_{\theta,q'},$$

where q' is the conjugate exponent. For $q=\infty$, the continuous embedding of the space on the right into the dual on the left is retained, but surjectivity may fail. More precisely, if $1\le q<\infty$ and $\Lambda_y(x):=\langle x,y\rangle$, there are constants $c_{\theta,q}, C_{\theta,q}>0$ such that

$$c_{\theta,q}\|y\|_{(X_0',X_1')_{\theta,q'}} \le \|\Lambda_y\|_{((X_0,X_1)_{\theta,q})'} \le C_{\theta,q}\|y\|_{(X_0',X_1')_{\theta,q'}}.$$

Proof. Set $\Delta := X_0 \cap X_1$, $\Sigma := X_0 + X_1$, and $Y := (X_0', X_1')_{\theta,q'}$. Regularity allows all elements of Y to be viewed as functionals on Δ.
If $a \in \Delta$ and $v \in X_0' \cap X_1'$, every decomposition $a = a_0 + a_1$ satisfies

$$|\langle a, v\rangle| \le \bigl(\|a_0\|_{X_0} + t^{-1}\|a_1\|_{X_1}\bigr) J(t, v; X_0', X_1').$$

Thus,

$$|\langle a, v\rangle| \le K(t^{-1}, a; X_0, X_1) J(t, v; X_0', X_1'). \tag{5.3}$$

Represent $y \in Y$ by the J-method for the dual pair:

$$y = \int_0^\infty v(t)\,\frac{dt}{t} \quad \text{in } X_0' + X_1' = \Delta'.$$

The identification of the sum space with Δ' comes from Lemma 5.5.5. Evaluation at $a \in \Delta$ is continuous on this space, so (5.3), Hölder's inequality, and the substitution $r = t^{-1}$ give

$$\begin{aligned} |\langle a, y\rangle| &\le \int_0^\infty \bigl(t^\theta K(t^{-1}, a)\bigr)\bigl(t^{-\theta} J(t, v(t))\bigr)\,\frac{dt}{t} \\ &\le \|a\|_{\theta,q;K} \bigl\|t^{-\theta} J(t, v(t))\bigr\|_{L^{q'}((0,\infty), dt/t)}. \end{aligned}$$

Take the infimum over representations and apply Theorem 5.4.4. If $q < \infty$, density of Δ from Lemma 5.4.5 allows y to extend uniquely to a continuous functional Λ_y on $(X_0, X_1)_{\theta,q}$, with the asserted bound.
Conversely, let $\Lambda \in ((X_0, X_1)_{\theta,q})'$ and assume $q < \infty$. Define

$$Z_k := (\Delta, 2^{-k\theta} J(2^k, \cdot)), \qquad \mathcal{J}_{\theta,q} := \ell^q\bigl(\mathbb{Z}; (Z_k)_k\bigr).$$

Each Z_k is Banach by Proposition 5.1.2. Lemma 5.4.5 shows that $S((a_k)_{k\in\mathbb{Z}}) := \sum_{k\in\mathbb{Z}} a_k$ is continuous and surjective from $\mathcal{J}_{\theta,q}$ onto the interpolated space. Apply Lemma 5.5.4 directly to the functional $\Lambda \circ S$. We obtain functionals $\lambda_k \in \Delta'$ such that

$$\Bigl\|\bigl(2^{k\theta}\|\lambda_k\|_{(\Delta, J(2^k,\cdot))'}\bigr)_k\Bigr\|_{\ell^{q'}} \le C_{\theta,q}\|\Lambda\|.$$

Evaluation on a sequence whose only nonzero coordinate is $a \in \Delta$ gives $\lambda_k(a) = \Lambda(a)$. Thus all λ_k agree with $y := \Lambda|_\Delta$. Lemma 5.5.5 transforms the estimate into

$$\Bigl\|\bigl(2^{k\theta} K(2^{-k}, y; X_0', X_1')\bigr)_k\Bigr\|_{\ell^{q'}} \le C_{\theta,q}\|\Lambda\|.$$

Lemma 5.2.5, with the change of index $k \mapsto -k$, implies $y \in Y$ and gives the other norm bound. Since Δ is dense, $\Lambda = \Lambda_y$ on the whole space.
For $q = \infty$, the pairing outside Δ requires clarification. Let $x \in (X_0, X_1)_{\theta,\infty}$ and take the partial sums x_N of a nearly optimal discrete J-representation. Then $x_N \in \Delta$, $x_N \to x$ in Σ, and

$$\sup_{N\in\mathbb{N}} \|x_N\|_{\theta,\infty;K} \le C_\theta \|x\|_{\theta,\infty;K}.$$

If $y \in X_0' \cap X_1'$, its action is continuous on Σ. The estimate already obtained on Δ, followed by the limit in

N, gives

$$|\langle x, y\rangle| \le C_\theta \|x\|_{\theta,\infty;K} \|y\|_{(X_0', X_1')_{\theta,1}}.$$

Lemma 5.4.5, applied to the dual pair with second index 1, extends this pairing to all of $y \in (X_0', X_1')_{\theta,1}$. It is bilinear, continuous, and injective in y, since on Δ it agrees with the original action. The preceding proof of surjectivity requires duality of ℓ^q with $q < \infty$ and does not apply at the endpoint $q = \infty$. □

5.7 The complex method

In this section, the Banach spaces are complex. A function $F\colon U \subseteq \mathbb{C} \longrightarrow X$ is called holomorphic if the limit

$$F'(z) := \lim_{h\to 0} \frac{F(z+h) - F(z)}{h}$$

exists in the norm of X for each $z \in U$.

To apply the method to a real space X, use its complexification $X_{\mathbb{C}} := X + iX$ with the norm

$$\|x + iy\|_{X_{\mathbb{C}}} := \sup_{t\in\mathbb{R}} \|x\cos t - y \sin t\|_X.$$

Multiplication by a complex scalar rotates the pair (x, y) and multiplies it by the modulus of the scalar, so this expression is a complex norm. Moreover,

$$\max\{\|x\|_X, \|y\|_X\} \le \|x + iy\|_{X_{\mathbb{C}}} \le \|x\|_X + \|y\|_X,$$

which proves completeness. The inclusions of a real pair are complexified within a common ambient space. Define its real interpolated space as the fixed-point space of conjugation on $[X_{0,\mathbb{C}}, X_{1,\mathbb{C}}]_\theta$. Conjugation preserves the Calderón norm upon replacing $F(z)$ by $\overline{F(\bar z)}$. If T is real and continuous, its complex extension has the same norm: the inequality $\|T_{\mathbb{C}}(x + iy)\| \le \|T\|\|x + iy\|$ follows for each term in the supremum, and the reverse bound follows on X. Thus the operator theorems also apply to real spaces. The isometric equalities below are formulated for complex spaces with the indicated norms. Passage to their real parts retains the identifications with equivalent norms.

We shall denote by

$$\mathcal{T} := \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}, \qquad \overline{\mathcal{T}} = \{z \in \mathbb{C} \mid 0 \le \operatorname{Re} z \le 1\}$$

the open strip and its closure, respectively.

Before defining the Calderón space, we establish the complex analysis tools to be used in both the scalar and vector-valued cases.

Proposition 5.7.1. (Cauchy's integral formula with values in a Banach space) Let $U \subseteq \mathbb{C}$ be open, let X be a complex Banach space, and let $F\colon U \longrightarrow X$ be holomorphic. If $\overline{B}_{\mathrm{euc}}(z_0, r) \subseteq U$, then for $z \in B_{\mathrm{euc}}(z_0, r)$ and $m \in \mathbb{N}_0$,

$$F^{(m)}(z) = \frac{m!}{2\pi i} \int_{|\zeta - z_0| = r} \frac{F(\zeta)}{(\zeta - z)^{m+1}}\, d\zeta, \tag{5.4}$$

where the circle is traversed with positive orientation and the integral is a Bochner integral. In particular, F admits a Taylor series around each point, convergent in the norm of X. If $X = \mathbb{C}$ and two holomorphic functions agree on a set with an interior accumulation point, they agree on every connected component of

U containing that point.

Proof. Begin with the scalar case and Goursat's argument. If the integral of a holomorphic function over the boundary of a triangle were nonzero, subdividing the triangle into four congruent triangles would allow us to choose one whose integral has modulus at least one quarter of the original. Iteration produces nested triangles with diameters tending to zero and a common point a. For each $\varepsilon > 0$, differentiability of f at a provides $\delta > 0$ such that

$$|f(\zeta) - f(a) - f'(a)(\zeta - a)| \le \varepsilon |\zeta - a| \qquad \text{if } |\zeta - a| < \delta.$$

The integrals of the two affine terms over a triangular boundary vanish. If $d_k := \operatorname{diam}(T_k) < \delta$, every point of T_k is at distance at most d_k from a, and the length of ∂T_k does not exceed $3d_k$; consequently,

$$\left| \int_{\partial T_k} f(\zeta)\, d\zeta \right| \le 3\varepsilon d_k^2.$$

On the other hand, the triangles are congruent after rescaling by 2^{-k}, so $d_k = 2^{-k} d_0$, whereas the inductive choice implies

$$\left| \int_{\partial T_k} f(\zeta)\, d\zeta \right| \ge 4^{-k} \left| \int_{\partial T_0} f(\zeta)\, d\zeta \right|.$$

If the last integral were nonzero, division by d_k^2 and the choice $3\varepsilon < d_0^{-2} |\int_{\partial T_0} f(\zeta)\, d\zeta|$ would give a contradiction. Thus the integral over every triangular boundary is zero. Summing over a triangulation cancels the integrals along interior edges. This gives the result for polygonal regions, including those with holes. For the circles we shall use, approximate by inscribed polygons; uniform continuity of the integrand on a compact neighborhood and convergence of these parametrizations allow passage to the limit in the integrals.

Apply this fact to $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ on the region between $|\zeta - z_0| = r$ and a small circle centered at z. Shrinking the latter circle yields

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Differentiation of the kernel under the integral gives the formula for $f^{(m)}$.

Return to the vector-valued case. Continuity of F on the circle ensures existence of the Bochner integral. For every $\lambda \in X'$, the function $\lambda \circ F$ is holomorphic, and compatibility of the integral with continuous linear operators, Proposition B.6.27, allows application of the scalar formula to $m = 0$. The Hahn–Banach theorem B.2.27 shows that X' separates points and gives the vector-valued identity for $F(z)$. On each concentric disk of radius less than r, the difference quotients of $(\zeta - z)^{-1}$ converge uniformly on the circle to $(\zeta - z)^{-2}$. The Bochner integral inequality allows differentiation of the identity in the norm of X. Repetition gives all derivatives and the formula for each m.

If $|z - z_0| < \rho < r$, the geometric series for the kernel converges uniformly for $|z - z_0| \le \rho$; we may integrate term by term and obtain the Taylor series of F, converging uniformly in norm on that disk. In the scalar case, the first nonzero coefficient in this series shows that the zeros of a holomorphic function that is not identically zero are isolated. This proves the final assertion. □

Lemma 5.7.2. (Maximum principle and holomorphic limits) The following assertions hold.

(a) If $Q \subseteq \mathbb{C}$ is a compact rectangle and $f\colon Q \longrightarrow \mathbb{C}$ is continuous and holomorphic in the interior, then $\max_{z\in Q}|f(z)| = \max_{z\in \partial Q}|f(z)|$. Moreover, if $|f|$ attains its maximum at an interior point, then f is constant.

(b) A uniform limit on compact subsets of scalar holomorphic functions is holomorphic.

(c) If X is a Banach space, a uniform limit on compact subsets of holomorphic functions with values in X is holomorphic.

Proof. Proposition 5.7.1, with $m = 0$, gives the mean value property. If $|f|$ attains an interior maximum, equality in the triangle inequality for that formula forces f to be constant on a neighborhood; the final assertion of the same proposition propagates constancy throughout the connected interior of the rectangle. This proves (a).
For (b) and (c), let (F_j) be a sequence converging uniformly on compact subsets to F, with values in $\mathbb{C}$ or X, respectively. Each F_j satisfies Cauchy's formula from Proposition 5.7.1. Uniform convergence on the circle allows passage to the limit in the Bochner integral and gives

$$F(z) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = r} \frac{F(\zeta)}{\zeta - z}\, d\zeta$$

on every disk whose closure is contained in the common domain. On a smaller concentric disk, the kernel and its derivative with respect to z are uniformly bounded on the circle. The fundamental theorem of calculus B.6.5 then allows differentiation under the integral and proves holomorphicity of F. □

Lemma 5.7.3. (Hadamard's three-lines lemma) Let $F\colon \overline{\mathcal{T}} \longrightarrow \mathbb{C}$ be continuous and bounded, holomorphic on $\mathcal{T}$, and suppose that $|F(j+it)| \le M_j$ for $t \in \mathbb{R}$ and $j = 0, 1$. Then

$$|F(\theta)| \le M_0^{1-\theta} M_1^{\theta}, \qquad 0 < \theta < 1.$$

Proof. We may assume $M_0 M_1 > 0$. For $\varepsilon > 0$, consider

$$G_\varepsilon(z) := F(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z-\theta)^2}.$$

On the vertical sides of a rectangle $0 \le \operatorname{Re} z \le 1$, $|\operatorname{Im} z| \le R$, its modulus is bounded by $e^{\varepsilon}\max\{\theta^2, (1-\theta)^2\}$; on the horizontal sides, it tends to zero as $R \to \infty$. The maximum principle from Lemma 5.7.2, applied to the rectangle and followed by the limit $R \to \infty$, gives $|G_\varepsilon(\theta)| \le e^{\varepsilon}\max\{\theta^2, (1-\theta)^2\}$. Finally, let $\varepsilon \to 0^+$. If $M_j = 0$, apply the preceding argument with $M_j + \delta$ and let $\delta \to 0^+$. □

The two lines bounding $\mathcal{T}$ represent the endpoint spaces: on it we measure in X_0, and on $1 + it$ in X_1, for $t \in \mathbb{R}$. A holomorphic function joins these boundary values. Evaluation at $z = \theta$ produces the intermediate element, and the three-lines lemma controls its size.

Definition 5.7.4. (Calderón space) Let (X_0, X_1) be a compatible pair of complex Banach spaces. Denote by $\mathscr{F}(X_0, X_1)$ the space of functions $F\colon \overline{\mathcal{T}} \longrightarrow X_0 + X_1$ that are continuous and bounded, holomorphic on $\mathcal{T}$, satisfy $F(j+it) \in X_j$, have restrictions $t \mapsto F(j+it)$ continuous in the norm of X_j with $F(j+it) \to 0$ in X_j as

$|t| \to \infty$, and have norm

$$\|F\|_{\mathscr{F}} := \max_{j\in\{0,1\}} \sup_{t\in\mathbb{R}} \|F(j+it)\|_{X_j} < \infty.$$

For $0 < \theta < 1$, define

$$[X_0, X_1]_\theta := \{F(\theta) \mid F \in \mathscr{F}(X_0, X_1)\}, \qquad \|x\|_{[X_0,X_1]_\theta} := \inf_{\substack{F\in\mathscr{F}(X_0,X_1)\\ F(\theta)=x}} \|F\|_{\mathscr{F}}.$$

Theorem 5.7.5. (Complex interpolation of operators) The space $[X_0, X_1]_\theta$ is Banach. If $T\colon X_0 + X_1 \longrightarrow Y_0 + Y_1$ is linear and $T\colon X_j \longrightarrow Y_j$ has norm M_j, then

$$T\colon [X_0, X_1]_\theta \longrightarrow [Y_0, Y_1]_\theta$$

is continuous and $\|T\| \le M_0^{1-\theta} M_1^\theta$.

Proof. Write $\Sigma := X_0 + X_1$. For $\lambda \in \Sigma'$ of norm at most one, the restrictions of λ to X_j have norm at most one. Lemma 5.7.3, applied to $z \mapsto \lambda(F(z+it))$, and Hahn–Banach give

$$\sup_{z\in\overline{T}} \|F(z)\|_\Sigma \le \|F\|_{\mathscr{F}}. \tag{5.5}$$

Thus a Cauchy sequence in $\mathscr{F}$ converges uniformly in Σ on the closed strip and uniformly in X_j on each boundary line. The limit is continuous in the indicated norms, its boundary values tend to zero, and Lemma 5.7.2 preserves holomorphicity. Hence $\mathscr{F}$ is Banach. Evaluation $F \mapsto F(\theta)$ is continuous by (5.5), so its kernel is closed. The space $[X_0, X_1]_\theta$, with its defining norm, is the quotient by that kernel and is Banach. In particular,

$$\|x\|_\Sigma \le \|x\|_{[X_0,X_1]_\theta}.$$

If $x \in X_0 \cap X_1$, the representative $F(z) = e^{\varepsilon(z-\theta)^2} x$ also gives the continuous embedding $X_0 \cap X_1 \hookrightarrow [X_0, X_1]_\theta$. The operator T is continuous between the sum spaces: apply its endpoint bounds to a decomposition and take the infimum. If $M_0 M_1 > 0$ and $F \in \mathscr{F}(X_0, X_1)$, set

$$G(z) := M_0^{z-1} M_1^{-z} T(F(z)).$$

This function belongs to $\mathscr{F}(Y_0, Y_1)$ and satisfies $\|G\|_{\mathscr{F}} \le \|F\|_{\mathscr{F}}$. Since $G(\theta) = M_0^{\theta-1} M_1^{-\theta} T(F(\theta))$, the infimum over F gives the required bound. If some M_j is zero, use the bounds $M_j + \varepsilon$ and let $\varepsilon \to 0^+$. □

Lemma 5.7.6. (Dual criterion for the Lebesgue norm) Let (A, μ) be a σ-finite measure space and let h be a measurable function finite almost everywhere. For $1 \le s \le \infty$,

$$\|h\|_{L^s(A)} = \sup\left\{ \left| \int_A hg\, d\mu \right| \;\middle|\; \begin{array}{l} g \text{ simple, supported on a set of finite measure,} \\ hg \in L^1(A), \quad \|g\|_{L^{s'}(A)} \le 1 \end{array} \right\},$$

allowing both sides to be infinite.

Proof. Hölder's inequality proves one bound if $h \in L^s$; if the norm of h is infinite, this bound holds automatically. For the reverse bound, take an increasing sequence of sets of finite measure whose union is A and intersect them with $\{|h| \le N\}$. On each resulting set, when $1 < s < \infty$, use the function $\overline{\operatorname{sgn} h}|h|^{s-1}$ normalized in $L^{s'}$. It is bounded and can be approximated uniformly by simple functions; if necessary, divide each approximant by the maximum of one and its norm. The supremum is at least the L^s norm of h restricted to that set. For $s = 1$, use $\overline{\operatorname{sgn} h}$, of norm at most one in L^∞. Monotone convergence gives the conclusion in both cases. If $s = \infty$, for each $0 < a < \|h\|_\infty$ choose a set of positive finite measure on which $a < |h| \le N$, and use its normalized indicator function multiplied by $\overline{\operatorname{sgn} h}$. The same simple approximations show that the supremum is at least a. □

Theorem 5.7.7. (Riesz–Thorin) Let (A, μ) and (B, ν) be σ-finite measure spaces, let $1 \le p_0, p_1, q_0, q_1 \le \infty$, and let

$$T : L^{p_0}(A) + L^{p_1}(A) \longrightarrow L^{q_0}(B) + L^{q_1}(B)$$

be a linear operator. Suppose that

$$\|Tf\|_{L^{q_j}(B)} \le M_j \|f\|_{L^{p_j}(A)}, \qquad j = 0, 1.$$

If $0 < \theta < 1$ and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then the restriction of T to $L^{p_\theta}(A)$ takes values in $L^{q_\theta}(B)$ and

$$\|Tf\|_{L^{q_\theta}(B)} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^{p_\theta}(A)}.$$

Proof. First assume $p_\theta < \infty$ and $q_\theta > 1$. By homogeneity, it suffices to consider simple functions

$$f = \sum_{k=1}^{N} a_k \mathbf{1}_{E_k}, \qquad g = \sum_{\ell=1}^{L} b_\ell \mathbf{1}_{F_\ell},$$

with disjoint sets of finite measure and $\|f\|_{L^{p_\theta}(A)} = \|g\|_{L^{q'_\theta}(B)} = 1$. For $a \neq 0$, write $\omega(a) := \frac{a}{|a|}$ and set

$$\frac{1}{p(z)} := \frac{1-z}{p_0} + \frac{z}{p_1}, \qquad \frac{1}{q'(z)} := \frac{1-z}{q'_0} + \frac{z}{q'_1}.$$

With the convention that zero coefficients are omitted, define

$$f_z := \sum_{k=1}^{N} |a_k|^{\frac{p_\theta}{p(z)}} \omega(a_k) \mathbf{1}_{E_k}, \qquad g_z := \sum_{\ell=1}^{L} |b_\ell|^{\frac{q'_\theta}{q'(z)}} \omega(b_\ell) \mathbf{1}_{F_\ell}.$$

Then $f_\theta = f$, $g_\theta = g$, and, for $j \in \{0, 1\}$ and $t \in \mathbb{R}$,

$$\|f_{j+it}\|_{L^{p_j}(A)} = 1, \qquad \|g_{j+it}\|_{L^{q'_j}(B)} = 1.$$

If $p_j = \infty$ or $q'_j = \infty$, the equality is interpreted with $\frac{1}{\infty} = 0$ and verified by taking the essential maximum.

The function

$$\Phi(z) := \int_B (Tf_z)(y) g_z(y)\, d\nu(y)$$

is holomorphic and bounded on the closed strip; expanding the two sums reduces it to a finite sum of exponential functions of z. The boundary hypothesis and Hölder's inequality from Proposition B.7.9 give $|\Phi(j+it)| \le M_j$. Lemma 5.7.3 yields

$$\left| \int_B (Tf) g\, d\nu \right| \le M_0^{1-\theta} M_1^{\theta}.$$

Undoing the normalization and using the criterion in Lemma 5.7.6 gives the estimate for $1 < q_\theta < \infty$. If $q_\theta = 1$, necessarily $q_0 = q_1 = 1$; take $g_z = g \in L^\infty(B)$, and the same argument works. If $q_\theta = \infty$, necessarily $q_0 = q_1 = \infty$; obtain the bound by applying the argument to functions $g \in L^1(B)$ and using duality again. If $p_\theta = \infty$, then $p_0 = p_1 = \infty$, and take $f_z = f$.

Finally, Proposition B.7.5 allows passage from simple functions to every function in $L^{p_\theta}(A)$ when $p_\theta < \infty$. This limit preserves the original operator: $L^{p_\theta}(A)$ embeds continuously into $L^{p_0}(A) + L^{p_1}(A)$. To see this, if $p_0 < p_1$ and $f \neq 0$, split f over the sets $\{|f| > \|f\|_{p_\theta}\}$ and $\{|f| \le \|f\|_{p_\theta}\}$; the two parts have norms in L^{p_0} and L^{p_1} at most $\|f\|_{p_\theta}$. If $p_1 < p_0$, interchange the roles, and if $p_0 = p_1$ the embedding is immediate. The operator T is continuous between the sum spaces by its endpoint bounds, and both limits agree locally in measure. When $p_\theta = \infty$, the argument with $f_z = f$ applies to each $f \in L^\infty(A)$ without approximation in that norm. This completes the proof. □

Theorem 5.7.8. (Hausdorff–Young inequality) Let $1 \le p \le 2$ and let p' be its conjugate exponent. The Fourier transform extends uniquely to a continuous operator

$$\mathcal{F} : L^p(\mathbb{R}^n) \longrightarrow L^{p'}(\mathbb{R}^n)$$

and satisfies

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n)} \le (2\pi)^{-n(\frac{1}{p}-\frac{1}{2})} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. The norm of the operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ is at most $(2\pi)^{-\frac{n}{2}}$ by Proposition 4.11.2; on $L^2(\mathbb{R}^n)$ it is one by Plancherel's theorem 4.11.10. The Riesz–Thorin theorem 5.7.7, applied initially to simple functions of finite support, gives the result for $1 < p < 2$ with the indicated constant. Density of that subspace in $L^p(\mathbb{R}^n)$, given by Proposition B.7.5, provides the unique extension. □

The interval $1 \le p \le 2$ in Hausdorff–Young is essential. For $p > 2$, Proposition 4.11.11 still defines the transform in $\mathcal{S}'(\mathbb{R}^n)$, but in general there is no continuous operator $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$. For $p = \infty$, the simplest instance already appears: the constant function one satisfies

$$\mathcal{F}1 = (2\pi)^{\frac{n}{2}} \delta_0$$

in $\mathcal{S}'(\mathbb{R}^n)$, which is not a function. For $2 < p < \infty$, failure of a bound $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ can be seen through sums of modulations. If $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\psi}$ is supported in a sufficiently small ball, and $L > 0$ separates its translates, consider

$$f_N(x) := \psi(x) \sum_{k=1}^{N} \varepsilon_k e^{ikLx_1}, \qquad \varepsilon_k \in \{-1, 1\}.$$

Theorem B.8.8, integrated in x by Theorem B.6.17, allows the signs to be chosen so that $\|f_N\|_{L^p(\mathbb{R}^n)} \le C_p N^{\frac{1}{2}} \|\psi\|_{L^p(\mathbb{R}^n)}$. Disjointness of the supports of the frequency blocks gives $\|\widehat{f_N}\|_{L^{p'}(\mathbb{R}^n)} = N^{\frac{1}{p'}} \|\widehat{\psi}\|_{L^{p'}(\mathbb{R}^n)}$. Since $\frac{1}{p'} > \frac{1}{2}$, a uniform bound would be impossible.

Corollary 5.7.9. (Complex interpolation of Lebesgue spaces) Let (A, μ) be a σ-finite measure space, let $1 \le p_0, p_1 \le \infty$, and let $0 < \theta < 1$. If

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

then

$$[L^{p_0}(A), L^{p_1}(A)]_\theta = L^p(A)$$

with equality of norms under the usual normalization of the complex method.

Proof. Let $F \in \mathscr{F}(L^{p_0}(A), L^{p_1}(A))$. For a simple function g supported on a set of finite measure, with $\|g\|_{L^{p'}(A)} = 1$, construct as in the proof of Theorem 5.7.7 an analytic family g_z such that $g_\theta = g$ and $\|g_{j+it}\|_{L^{p_j'}(A)} = 1$. The three-lines lemma 5.7.3, applied to

$$z \longmapsto \int_A F(z,x) g_z(x)\, d\mu(x)$$

gives $\left| \int_A F(\theta, x) g(x)\, d\mu(x) \right| \le \|F\|_{\mathscr{F}}$. The dual criterion in Lemma 5.7.6 yields

$$\|F(\theta)\|_{L^p(A)} \le \|F\|_{\mathscr{F}},$$

and taking the infimum gives $[L^{p_0}(A), L^{p_1}(A)]_\theta \hookrightarrow L^p(A)$.

For the reverse inclusion, if $p < \infty$, begin with a nonzero simple function f supported on a set of finite measure and consider

$$F(z,x) := e^{\varepsilon(z-\theta)^2} \|f\|_{L^p(A)}^{1-\frac{p}{p(z)}} |f(x)|^{\frac{p}{p(z)}} \operatorname{sgn} f(x), \qquad \frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}.$$

Then $F(\theta) = f$ and, on the boundary $j + it$, the modulus of the power of f is $|f|^{\frac{p}{p_j}}$; hence

$$\|F(j+it)\|_{L^{p_j}(A)} \le e^{\varepsilon(j-\theta)^2} \|f\|_{L^p(A)}.$$

The Gaussian factor also ensures decay as $|t| \to \infty$. Letting $\varepsilon \to 0^+$ gives

$$\|f\|_{[L^{p_0}(A), L^{p_1}(A)]_\theta} \le \|f\|_{L^p(A)}.$$

Proposition B.7.5 and completeness from Theorem 5.7.5 allow passage to every $f \in L^p(A)$ when $p < \infty$. If an endpoint exponent is infinite, use the convention $\frac{1}{\infty} = 0$ in the same formula; if $p = \infty$, necessarily $p_0 = p_1 = \infty$: the function $z \mapsto e^{\varepsilon(z-\theta)^2} f$ gives one bound, and the three-lines lemma gives the other. □

5.7.1 Translations, regularization, and boundary estimates

To interpolate families of spaces, we shall need more than a bound by the maximum of the boundary norms. We first justify the approximations to be used, then obtain an estimate averaging those norms.

Lemma 5.7.10. (Translations and Gaussian regularization) Let $F \in \mathscr{F}(X_0, X_1)$ and $0 < \alpha < 1$. Then

$$t \longmapsto F(\alpha + it)$$

is continuous with values in $[X_0, X_1]_\alpha$, tends to zero as $|t| \to \infty$, and satisfies

$$\sup_{t\in\mathbb{R}} \|F(\alpha + it)\|_{[X_0,X_1]_\alpha} \le \|F\|_{\mathscr{F}}.$$

Moreover, there are functions $F_\varepsilon \in \mathscr{F}(X_0, X_1)$, entire with values in $X_0 \cap X_1$, such that $F_\varepsilon \to F$ in $\mathscr{F}$. In particular, $X_0 \cap X_1$ is dense in $[X_0, X_1]_\alpha$.

Proof. The function $z \mapsto F(z + it)$ belongs to $\mathscr{F}$ with the same norm as F, giving the bound. The boundary functions are continuous and tend to zero, so they are uniformly continuous. Hence

$$\|F(\cdot + ih) - F\|_{\mathscr{F}} \longrightarrow 0 \qquad (h \to 0),$$

and evaluation at $\alpha + it$ proves the asserted continuity. To justify the limit at infinity, fix $d > 0$ and use the representative $z \mapsto e^{d(z-\alpha)^2} F(z + it)$. We obtain

$$\|F(\alpha + it)\|_{[X_0,X_1]_\alpha} \le e^{d \max\{\alpha^2,(1-\alpha)^2\}} \max_{j\in\{0,1\}} \sup_{s\in\mathbb{R}} e^{-ds^2} \|F(j + i(t + s))\|_{X_j}.$$

On $|s| > R$, the Gaussian factor makes the right-hand side small uniformly in t; on $|s| \le R$, the boundary values tend uniformly to zero as $|t| \to \infty$.
For $\varepsilon > 0$, define, initially on the strip,

$$F_\varepsilon(z) := \frac{1}{\sqrt{\pi\varepsilon}} \int_{\mathbb{R}} e^{-t^2/\varepsilon} F(z + it)\, dt.$$

The integrals are Bochner integrals. Cauchy's theorem used in Proposition 5.7.1, applied to the integrand over rectangles, allows the integration line to be shifted and gives

$$F_\varepsilon(z) = \frac{1}{\sqrt{\pi\varepsilon}} \int_{\mathbb{R}} e^{(z-j-it)^2/\varepsilon} F(j + it)\, dt, \qquad j = 0, 1. \tag{5.6}$$

Indeed, on the horizontal sides, the modulus of the kernel is bounded by a constant times $e^{-(R-|\operatorname{Im} z|)^2/\varepsilon}$; their integrals tend to zero. First shift interior lines, then pass to the boundary by continuity and dominated convergence.
The right-hand side of (5.6) defines an entire function with values in X_j: on every compact subset of z, all derivatives of the kernel admit a bound integrable in t. The two expressions agree on the strip and, by the identity principle, throughout $\mathbb{C}$ with values in $X_0 + X_1$. They thus define an entire function in $X_0 \cap X_1$. On

the line $j + i\mathbb{R}$, they are Gaussian convolutions of the boundary function in X_j, whence

$$\|F_\varepsilon\|_{\mathscr{F}} \le \|F\|_{\mathscr{F}}, \qquad \|F_\varepsilon - F\|_{\mathscr{F}} \longrightarrow 0.$$

For the second assertion, split the integral into $|t| < \delta$, where uniform continuity is used, and $|t| \ge \delta$, where the Gaussian mass tends to zero. Evaluation at α proves density. □

Lemma 5.7.11. (Integral estimate for the interpolated norm) For $0 < \alpha < 1$, set

$$P_0(\alpha, t) := \frac{\sin(\pi\alpha)}{2(\cosh(\pi t) - \cos(\pi\alpha))}, \qquad P_1(\alpha, t) := \frac{\sin(\pi\alpha)}{2(\cosh(\pi t) + \cos(\pi\alpha))}.$$

These kernels are positive and integrable, with masses $m_0 := 1 - \alpha$ and $m_1 := \alpha$, respectively. If $F \in \mathscr{F}(X_0, X_1)$, then

$$\|F(\alpha)\|_{[X_0,X_1]_\alpha} \le \exp\left(\sum_{j=0}^{1} \int_{\mathbb{R}} P_j(\alpha, t) \log \|F(j + it)\|_{X_j}\, dt\right). \tag{5.7}$$

We use $\log 0 = -\infty$. In particular, if $1 \le r_0, r_1 \le \infty$,

$$\|F(\alpha)\|_{[X_0,X_1]_\alpha} \le B_0^{1-\alpha} B_1^{\alpha}, \tag{5.8}$$

where

$$B_j := \left(\frac{1}{m_j} \int_{\mathbb{R}} P_j(\alpha, t) \|F(j + it)\|_{X_j}^{r_j}\, dt\right)^{1/r_j}$$

if $r_j < \infty$, and $B_j := \sup_{t\in\mathbb{R}} \|F(j + it)\|_{X_j}$ if $r_j = \infty$. We also have

$$\|F(\alpha)\|_{[X_0,X_1]_\alpha} \le \sum_{j=0}^{1} \int_{\mathbb{R}} P_j(\alpha, t) \|F(j + it)\|_{X_j}\, dt. \tag{5.9}$$

The same estimates hold for a continuous bounded function on the strip, holomorphic in its interior, whose boundary restrictions are continuous and bounded in X_j, even if they do not tend to zero.

Proof. We first describe the harmonic function used in the proof. For smooth real functions φ_0, φ_1 constant outside a compact interval, set

$$u(x, y) := \sum_{j=0}^{1} \int_{\mathbb{R}} P_j(x, y - t)\varphi_j(t)\, dt, \qquad 0 < x < 1.$$

The kernels are obtained through the substitution $w = e^{\pi i z}$ in the Poisson kernel for the upper half-plane

$$\frac{\operatorname{Im} w}{\pi |s - w|^2}\, ds.$$

For $s > 0$, write $s = e^{-\pi t}$, and for $s < 0$, $s = -e^{-\pi t}$; these substitutions give exactly P_0 and P_1. The half-plane kernel is the imaginary part of $\pi^{-1}(s - w)^{-1}$ and is therefore harmonic. Its integral in s is one, and its mass outside a neighborhood of the boundary point tends to zero. This proves harmonicity of u and its boundary

values $u(j,y)=\varphi_j(y)$. Integrating the kernel over the positive and negative half-lines, using an arctangent antiderivative, gives the masses $1-x$ and x. In particular, u lies between the minimum and maximum of the boundary data. For smooth data as chosen, u and its first derivatives extend continuously to the boundary lines. This can be checked in the half-plane kernel: for the tangential derivative, integrate by parts and apply the kernel to the differentiated datum. For the normal derivative, subtract the first-order Taylor polynomial of the datum near the boundary point. The linear term has zero integral over symmetric intervals; the remainder is of second order and provides an integrable bound for the differentiated kernel. Outside that neighborhood there is no singularity. Dominated convergence justifies the limits and local continuity of both derivatives.

The function u has a harmonic conjugate on the strip. With the normalization $v(\alpha,0)=0$, we can construct it by

$$v(x,y):=-\int_\alpha^x u_y(s,0)\,ds+\int_0^y u_x(x,t)\,dt.$$

The equality $u_{xx}+u_{yy}=0$ gives $v_y=u_x$ and $v_x=-u_y$. Thus $h:=u+iv$ is holomorphic, $h(\alpha)=u(\alpha,0)$ is real, and $\operatorname{Re} h(j+it)=\varphi_j(t)$.

Fix $\varepsilon,\delta>0$. The functions $t\mapsto\log(\|F(j+it)\|_{X_j}+\varepsilon)$ are continuous and tend to $\log\varepsilon$. Using a cutoff outside a large interval and convolution with a smooth function, choose φ_j as above so that

$$\log(\|F(j+it)\|_{X_j}+\varepsilon)\le\varphi_j(t)\le\log(\|F(j+it)\|_{X_j}+\varepsilon)+\delta.$$

Construct h with these data and define

$$G(z):=e^{h(\alpha)-h(z)}F(z).$$

Boundedness of $\operatorname{Re} h$ proves that $G\in\mathscr{F}$; moreover, $G(\alpha)=F(\alpha)$ and

$$\|G(j+it)\|_{X_j}\le e^{u(\alpha,0)}\frac{\|F(j+it)\|_{X_j}}{\|F(j+it)\|_{X_j}+\varepsilon}\le e^{u(\alpha,0)}.$$

By the definition of the interpolated norm,

$$\log\|F(\alpha)\|_{[X_0,X_1]_\alpha}\le\delta+\sum_{j=0}^{1}\int_{\mathbb{R}}P_j(\alpha,t)\log(\|F(j+it)\|_{X_j}+\varepsilon)\,dt.$$

Let first $\delta\to0$ and then $\varepsilon\to0^+$. Monotone convergence, applied after subtracting a constant upper bound, justifies the limit even when the integral equals $-\infty$.

To obtain (5.8), apply concavity of the logarithm with respect to the probability measures $m_j^{-1}P_j(\alpha,t)\,dt$ to the powers r_j. This integral inequality follows from $\log s\le s-1$, first dividing by the average. If $r_j=\infty$, use the supremum. The same inequality with respect to the measure of total mass one on both lines gives (5.9).

Finally, for a function with bounded boundary values that do not vanish at infinity, apply the preceding argument to $e^{d(z-\alpha)^2}F(z)$. This function belongs to $\mathscr{F}$ and has the same value at α. Let $d\to0^+$ in the estimates, using dominated convergence; the kernels decay exponentially and the boundary values are bounded. □

5.7.2 Sums of spaces and Banach-valued functions

Applications to Fourier blocks and families of charts require interpolation of several components at once. We first study a finite family, where each component admits a Calderón representative. We then pass to infinite families by truncation and to measurable functions by approximation with simple functions. The restrictions on the exponents express exactly when these approximations converge in norm.

For a Banach space X, write $L^r(A;X)$ for the classes of strongly measurable functions with finite norm

$$\|f\|_{L^r(A;X)} := \Big(\int_A \|f(a)\|_X^r \, d\mu(a)\Big)^{1/r}$$

or with the essential supremum norm if $r = \infty$. These spaces are complete. Indeed, from a Cauchy sequence, extract a subsequence whose differences have summable norms. Minkowski's inequality and monotone convergence show that the pointwise sum of the norms of these differences belongs to L^r if $r < \infty$. Thus the series converges almost everywhere in X, and its tails tend to zero in L^r. If $r = \infty$, the same conclusion holds outside the countable union of the null sets for the essential bounds. The limit is strongly measurable by Lemma B.6.24.

Proposition 5.7.12. (Interpolation of ℓ^p-sums and Bochner spaces) Let I be a set, let $1 \le r_0, r_1 < \infty$ and $0 < \theta < 1$, and set

$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

For a family of compatible pairs of complex Banach spaces $(X_{0,i}, X_{1,i})_{i\in I}$, we have isometrically

$$\big[\ell^{r_0}\big(I;(X_{0,i})_{i\in I}\big), \ell^{r_1}\big(I;(X_{1,i})_{i\in I}\big)\big]_\theta = \ell^r\big(I;([X_{0,i},X_{1,i}]_\theta)_{i\in I}\big). \tag{5.10}$$

If $w_{0,i}, w_{1,i} > 0$, the same identity with weighted norms is

$$\big[\ell^{r_0}\big(I;(w_{0,i}X_{0,i})_{i\in I}\big), \ell^{r_1}\big(I;(w_{1,i}X_{1,i})_{i\in I}\big)\big]_\theta = \ell^r\big(I;(w_{0,i}^{1-\theta}w_{1,i}^{\theta}[X_{0,i},X_{1,i}]_\theta)_{i\in I}\big), \tag{5.11}$$

where wX denotes X with norm $w\|\cdot\|_X$. For a finite family, these identities also hold if some r_j is infinite, with $1/\infty = 0$.

If (A,μ) is σ-finite and (X_0, X_1) is a compatible pair, then for $r_0, r_1 < \infty$ we also have isometrically

$$[L^{r_0}(A;X_0), L^{r_1}(A;X_1)]_\theta = L^r(A;[X_0,X_1]_\theta). \tag{5.12}$$

Proof. Begin with a finite family. Let F be a Calderón function for the endpoint sums, and write F_i for its i coordinate. The coordinate projections are continuous at the endpoints, so $F_i \in \mathscr{F}(X_{0,i}, X_{1,i})$. Apply (5.8) to each F_i and set

$$b_{j,i} := \left(\frac{1}{m_j}\int_{\mathbb{R}} P_j(\theta,t)\|F_i(j+it)\|_{X_{j,i}}^{r_j}\, dt\right)^{1/r_j}, \quad m_0 = 1-\theta, \quad m_1 = \theta,$$

with suprema if $r_j = \infty$. We obtain

$$\|F_i(\theta)\|_{[X_{0,i},X_{1,i}]_\theta} \le b_{0,i}^{1-\theta} b_{1,i}^{\theta}.$$

If $r < \infty$, Hölder's inequality for sums, with $r(1-\theta)/r_0 + r\theta/r_1 = 1$, gives

$$\|(F_i(\theta))_i\|_{\ell^r([X_{0,i},X_{1,i}]_\theta)} \le \|(b_{0,i})_i\|_{\ell^{r_0}}^{1-\theta}\|(b_{1,i})_i\|_{\ell^{r_1}}^{\theta} \le \|F\|_{\mathscr{F}}.$$

The last bound follows by interchanging sum and integral using Tonelli's theorem. If some r_j is infinite, use the maximum of the $b_{j,i}$; if $r = \infty$, both endpoints are infinite and take the maximum in the coordinate estimate. This proves one inclusion.

For the other inclusion, let $x = (x_i)_i$ and set $a_i := \|x_i\|_{[X_{0,i},X_{1,i}]_\theta}$ and $A := \|(a_i)_i\|_{\ell^r}$. If $A = 0$, there is nothing to prove. Omit zero coordinates and choose representatives $F_i(\theta) = x_i$ with $\|F_i\|_{\mathscr{F}} \le (1+\varepsilon)a_i$. When $r < \infty$, define

$$\frac{1}{r(z)} := \frac{1-z}{r_0} + \frac{z}{r_1}, \qquad F(z) := \left(A^{1-r/r(z)} a_i^{r/r(z)-1} F_i(z)\right)_i.$$

This is a Calderón function for the sums, and $F(\theta) = x$. On each boundary,

$$\|F(j+it)\|_{\ell^{r_j}(X_{j,i})} \le (1+\varepsilon)A^{1-r/r_j} \left(\sum_{i\in I} a_i^r\right)^{1/r_j} = (1+\varepsilon)A,$$

with the maximum interpretation if $r_j = \infty$. If both endpoints are infinite, simply take $F = (F_i)_i$; the family is finite. Letting $\varepsilon \to 0^+$ proves equality of norms.

For arbitrary I and $r_0, r_1 < \infty$, restriction to finite subsets gives the first inclusion and its bound, since the norm of $\ell^r(I)$ is the supremum of the norms of its finite restrictions. Conversely, every element of $\ell^r(I)$ has at most countably many nonzero coordinates: for each integer $k \ge 1$, only finitely many coordinates have norm greater than $1/k$. Its truncations converge in ℓ^r, and the bound already proved makes them Cauchy in the interpolated space. Completeness and coordinate projections identify the limit. No density in an infinite ℓ^∞-sum has been used.

The weighted formula is obtained first in each coordinate. Indeed, multiplying a representative by $w_0^{z-\theta} w_1^{\theta-z}$ preserves its value at θ and changes its weighted boundary norms by the common factor $w_0^{1-\theta} w_1^{\theta}$. Multiplication by the inverse gives the reverse inequality. Then apply the sum formula just proved.

Turn now to Bochner functions. For disjoint measurable sets $E_1, \dots, E_N$ of positive finite measure, define

$$Pf := \sum_{\nu=1}^{N} \mathbf{1}_{E_\nu} \frac{1}{\mu(E_\nu)} \int_{E_\nu} f \, d\mu,$$

and set $Pf = 0$ outside their union. The Bochner integral inequality and Hölder's inequality show, for every Banach space X and $1 \le s < \infty$,

$$\|Pf\|_{L^s(A;X)} \le \|f\|_{L^s(A;X)}.$$

Its image is identified with a finite sum of copies of X whose norm has weights $\mu(E_\nu)^{1/s}$. The weighted formula gives exactly the weight $\mu(E_\nu)^{1/r}$ in the interpolated space.

Let $F \in \mathscr{F}(L^{r_0}(A;X_0), L^{r_1}(A;X_1))$. For each j, the image of the boundary line together with zero is compact in $L^{r_j}(A;X_j)$. Choose a finite net in that image and approximate its elements by simple functions supported on sets of finite measure. Such approximations exist by strong measurability and integrability of the norm: first truncate the domain and the size of the function, then use the simple functions in its definition. A common partition of the supports of these approximants produces an operator P fixing them. Its contractivity implies, for each function f in the net and its approximation g,

$$\|Pf - f\| \le \|P(f-g)\| + \|g - f\| \le 2\|f - g\|.$$

Taking sufficiently fine nets and small errors gives operators P_k such that $P_k F \to F$ in $\mathscr{F}$.

The equality for finite sums implies

$$\|P_k F(\theta) - P_l F(\theta)\|_{L^r(A;[X_0,X_1]_\theta)} \le \|(P_k - P_l)F\|_{\mathscr{F}}.$$

To use it, simply refine both partitions to a common partition. Thus $P_k F(\theta)$ converges in the Bochner space on the right-hand side of (5.12). That limit agrees with $F(\theta)$: both convergences imply local convergence in measure with values in $X_0 + X_1$, by Hölder's inequality on sets of finite measure. This proves the inclusion from the left-hand side. The bound is one, since $\|P_k F\|_{\mathscr{F}} \le \|F\|_{\mathscr{F}}$.
Finally, simple functions supported on sets of finite measure with values in $[X_0, X_1]_\theta$ are dense in their Bochner space. For each such function, the construction using coordinate representatives and the preceding weights gives the reverse inclusion with bound one. Differences of two simple functions are treated on a common partition; completeness allows passage to the limit. This completes the proof. □

5.7.3 Complex reiteration

To prove reiteration while retaining the general assumptions, we shall use a description through Fourier coefficients of periodic functions. The idea is to express an element in two ways: as the sum of a sequence and as the constant value of a sequence decomposed between the endpoints. The relationship between them is obtained by taking differences and partial sums. This procedure is related to the approach of [100, §§4 and 6].

Definition 5.7.13. (Fourier coefficient spaces) Let X be a complex Banach space. Denote by $\mathcal{C}(X)$ the space of sequences $u = (u_n)_{n\in\mathbb{Z}}$ of the form

$$u_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt, \qquad f \in L^1((-\pi,\pi);X),$$

with the endpoints of the interval identified. Its norm is

$$\|u\|_{\mathcal{C}(X)} := \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(t)\|_X\,dt.$$

For $b > 1$ and $\rho \in \mathbb{R}$, write

$$\mathcal{C}_b(\rho;X) := \{u \mid (b^{\rho n}u_n)_n \in \mathcal{C}(X)\}, \qquad \|u\|_{\mathcal{C}_b(\rho;X)} := \|(b^{\rho n}u_n)_n\|_{\mathcal{C}(X)}.$$

These periodic coefficients use the measure $dt/(2\pi)$; they are not the Fourier transform on $\mathbb{R}$ defined in the preceding chapter.

We justify that the norm is well defined. The Fejér kernel

$$\Phi_N(t) := \frac{1}{N+1}\left|\sum_{k=0}^{N} e^{ikt}\right|^2$$

is nonnegative, has integral one with respect to $dt/(2\pi)$, and satisfies $\Phi_N(t) \le ((N+1)\sin^2(t/2))^{-1}$ outside $t = 0$. Its convolutions therefore approximate continuous periodic functions uniformly: use uniform continuity near zero and the preceding bound on the complement. Approximating a Bochner function in L^1 by simple functions and their scalar indicator functions by continuous periodic functions gives the same result in L^1, since the convolution

operators have norm at most one. Their coefficients are

$$(Q_N u)_n := \left(1 - \frac{|n|}{N+1}\right)_+ u_n.$$

Consequently, if all coefficients of f vanish, then $\Phi_N * f = 0$ for every N and $f = 0$. This proves uniqueness of f and completeness of $\mathcal{C}(X)$ through its isometric identification with L^1 with normalized measure. It also proves

$$\|Q_N u\|_{\mathcal{C}_b(\rho;X)} \le \|u\|_{\mathcal{C}_b(\rho;X)}, \qquad Q_N u \longrightarrow u \quad \text{in } \mathcal{C}_b(\rho;X). \tag{5.13}$$

Each coordinate satisfies $\|b^{\rho n} u_n\|_X \le \|u\|_{\mathcal{C}_b(\rho;X)}$. Thus the coefficient spaces form compatible pairs in the space of sequences with values in $X_0 + X_1$, equipped with the topology of coordinatewise convergence.

Lemma 5.7.14. (Discrete representations of the complex method) Let $b > 1$ and $0 < \theta < 1$. For a compatible pair (X_0, X_1), set

$$\mathcal{U}_0 := \mathcal{C}_b(-\theta; X_0), \qquad \mathcal{U}_1 := \mathcal{C}_b(1-\theta; X_1).$$

Then $x \in [X_0, X_1]_\theta$ if and only if there is $u \in \mathcal{U}_0 \cap \mathcal{U}_1$ such that

$$x = \sum_{n\in\mathbb{Z}} u_n \quad \text{in } X_0 + X_1.$$

Its norm is equivalent to the infimum of $\max\{\|u\|_{\mathcal{U}_0}, \|u\|_{\mathcal{U}_1}\}$ over these representations. It is also equivalent to

$$\|\underline{x}\|_{\mathcal{U}_0+\mathcal{U}_1}, \qquad \underline{x} := (x)_{n\in\mathbb{Z}}.$$

In particular, the last quantity is finite exactly for elements of $[X_0, X_1]_\theta$. The constants depend only on b and θ.

Proof. If $u \in \mathcal{U}_0 \cap \mathcal{U}_1$, the coordinate bounds give

$$\|u_n\|_{X_0} \le b^{\theta n}\|u\|_{\mathcal{U}_0}, \qquad \|u_n\|_{X_1} \le b^{-(1-\theta)n}\|u\|_{\mathcal{U}_1}.$$

Use the first for $n < 0$ and the second for $n \ge 0$. The geometric series prove absolute convergence in $X_0 + X_1$. The exponential polynomials

$$H_N(z) := \sum_{n\in\mathbb{Z}} b^{(z-\theta)n} (Q_N u)_n$$

take values in $X_0 \cap X_1$ and have imaginary period $P := 2\pi/\log b$. On the boundary $j + it$, the corresponding periodic function is the Fejér sum of the function with coefficients $b^{(j-\theta)n} u_n$.
Lemma 5.7.11, in its version for bounded functions, implies

$$\|H_N(\theta) - H_M(\theta)\|_{[X_0,X_1]_\theta} \le C_{b,\theta} \max_{j\in\{0,1\}} \|(Q_N - Q_M)u\|_{\mathcal{U}_j}. \tag{5.14}$$

To check the constant, split each Poisson integral into intervals of length P. The periodized kernel $\sum_{k\in\mathbb{Z}} P_j(\theta, t+kP)$ is uniformly bounded on $[0,P]$, since P_j is continuous and decays exponentially. This bounds the integral by a constant times the average $P^{-1}\int_0^P \|H_N(j+it) - H_M(j+it)\|_{X_j}\, dt$. By (5.13), the values $H_N(\theta)$

converge in the interpolated space; in the sum space they converge to $\sum_{n\in\mathbb{Z}} u_n$. This gives the first norm bound.

We prove the converse direction. If H is holomorphic on the strip, continuous and bounded up to the boundary in the corresponding norms, and has imaginary period P, define

$$u_n := b^{-(j-\theta)n}\frac{1}{P}\int_0^P H(j+it)e^{-in(\log b)t}\,dt, \qquad j=0,1. \tag{5.15}$$

The two expressions agree: apply Cauchy's theorem to $b^{-n(z-\theta)}H(z)$ on a rectangle of height P; the horizontal integrals cancel by periodicity. Continuity allows passage to the boundary lines. Thus $u_n \in X_0\cap X_1$, $u\in \mathcal{U}_0\cap\mathcal{U}_1$, and

$$\max_{j\in\{0,1\}}\|u\|_{\mathcal{U}_j} \le \max_{j\in\{0,1\}}\sup_{t\in\mathbb{R}}\|H(j+it)\|_{X_j}.$$

The same integrals on the line $\operatorname{Re} z=\theta$ show that $(u_n)_n$ are the coefficients of $H(\theta+it)$. Fejér convergence on that line and the absolute convergence already proved give $H(\theta)=\sum_{n\in\mathbb{Z}} u_n$.

Now let $x\in[X_0,X_1]_\theta$, $x\neq 0$, and choose $F(\theta)=x$ with $\|F\|_{\mathscr{F}}\le 2\|x\|_{[X_0,X_1]_\theta}$. Fix $d>0$ sufficiently large that

$$2\sum_{k\in\mathbb{Z}\setminus\{0\}} e^{-dk^2P^2}\le\frac{1}{2}$$

and periodize:

$$H(z):=\sum_{k\in\mathbb{Z}} e^{d(z+ikP-\theta)^2}F(z+ikP).$$

The series converges uniformly on compact subsets and on one period of each boundary, with values in X_j on the boundary j. Moreover,

$$\max_{j\in\{0,1\}}\sup_{t\in\mathbb{R}}\|H(j+it)\|_{X_j}\le C_{b,\theta}\|F\|_{\mathscr{F}},$$

since $\sup_{t\in\mathbb{R}}\sum_{k\in\mathbb{Z}} e^{-d(t+kP)^2}<\infty$. If $y:=H(\theta)$, Lemma 5.7.10 and the choice of d give

$$\|x-y\|_{[X_0,X_1]_\theta}\le\sum_{k\in\mathbb{Z}\setminus\{0\}} e^{-dk^2P^2}\|F(\theta+ikP)\|_{[X_0,X_1]_\theta}\le\frac{1}{2}\|x\|_{[X_0,X_1]_\theta}.$$

Formula (5.15) represents y with norm in $\mathcal{U}_0\cap\mathcal{U}_1$ at most $C_{b,\theta}\|x\|_{[X_0,X_1]_\theta}$. Repeat the construction for $x-y$ and successive remainders. Their norms are at most $2^{-l}\|x\|$. The resulting coefficient sequences form an absolutely convergent series in the Banach space $\mathcal{U}_0\cap\mathcal{U}_1$. Its sum u satisfies $\sum_{n\in\mathbb{Z}} u_n = x$ and has the required bound. Interchange of the two sums is justified by continuity of the operator $u\mapsto\sum_{n\in\mathbb{Z}} u_n$ into the sum space, proved using geometric series at the start.

It remains to establish the characterization through $\underline{x}$. Given a representation $x=\sum_{n\in\mathbb{Z}} u_n$, set

$$a_n:=\sum_{\substack{k\in\mathbb{Z}\\ k<n}} u_k\in X_0, \qquad c_n:=\sum_{\substack{k\in\mathbb{Z}\\ k\ge n}} u_k\in X_1.$$

Then $a_n + c_n = x$. For the weighted coefficients,

$$b^{-\theta n} a_n = \sum_{l \in \mathbb{N}} b^{-\theta l} b^{-\theta(n-l)} u_{n-l},$$
$$b^{(1-\theta)n} c_n = \sum_{l \in \mathbb{N}_0} b^{-(1-\theta)l} b^{(1-\theta)(n+l)} u_{n+l}.$$

Shifting the indices of a sequence in $\mathcal{C}(X)$ corresponds to multiplying its function by e^{ilt}, an isometry on L^1. The preceding geometric series give

$$\|a\|_{\mathcal{U}_0} + \|c\|_{\mathcal{U}_1} \le C_{b,\theta} \max_{j \in \{0,1\}} \|u\|_{\mathcal{U}_j}.$$

Conversely, if $x = a_n + c_n$ with $a \in \mathcal{U}_0$ and $c \in \mathcal{U}_1$, define

$$u_n := a_{n+1} - a_n = c_n - c_{n+1} \in X_0 \cap X_1.$$

The same shift invariance implies

$$\|u\|_{\mathcal{U}_0} \le (1 + b^{\theta}) \|a\|_{\mathcal{U}_0}, \qquad \|u\|_{\mathcal{U}_1} \le (1 + b^{-(1-\theta)}) \|c\|_{\mathcal{U}_1}.$$

The coordinate bounds show that $a_n \to 0$ in X_0 as $n \to -\infty$, and $c_n \to 0$ in X_1 as $n \to \infty$. Thus $\sum_{n=-N}^{N} u_n = a_{N+1} - a_{-N} \to x$ in the sum space. Taking infima completes the proof. □

Lemma 5.7.15. (Decomposition of coefficients at an intermediate level) Let $b > 1$ and $0 \le \alpha \le 1$, and set $E_\alpha := [X_0, X_1]_\alpha$ if $0 < \alpha < 1$, $E_0 := X_0$, and $E_1 := X_1$. Then

$$\mathcal{C}(E_\alpha) \hookrightarrow \mathcal{C}_b(-\alpha; X_0) + \mathcal{C}_b(1-\alpha; X_1) \tag{5.16}$$

with norm at most one.

Proof. If $\alpha = 0$ or 1, the space on the left equals one of the summands, and the assertion is immediate. Assume $0 < \alpha < 1$ and denote the sum space on the right by V. Begin with a trigonometric polynomial

$$f(t) = \sum_{\substack{n \in \mathbb{Z} \\ |n| \le N}} u_n e^{int}, \qquad u_n \in X_0 \cap X_1.$$

Thus its coefficient sequence belongs to V. Let $\Lambda \in V'$ have norm at most one. By (5.13), the functionals $\Lambda \circ Q_M$ have norm at most one and converge to Λ on each element of V. Each depends on only finitely many coordinates, so it has the form

$$(\Lambda \circ Q_M)(v) = \sum_{n \in \mathbb{Z}} y_n(v_n), \qquad y_n \in (X_0 + X_1)',$$

with only finitely many nonzero y_n. For $j \in \{0,1\}$, set

$$H(z,t) := \sum_{n\in\mathbb{Z}} b^{(\alpha-z)n} e^{-int} y_n.$$

The norm of $\Lambda \circ Q_M$ restricted to $\mathcal{C}_b(j-\alpha; X_j)$ is

$$\sup_{t\in[-\pi,\pi]} \|H(j,t)\|_{X_j'} \le 1. \tag{5.17}$$

To verify the equality giving this norm, if $(b^{(j-\alpha)n} v_n)_n$ are the coefficients of $g \in L^1((-\pi,\pi); X_j)$, then

$$(\Lambda \circ Q_M)(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j,t)(g(t))\, dt.$$

This proves one inequality. For the other, choose a point t_0 and a unit vector in X_j realizing the value of $\|H(j,t_0)\|_{X_j'}$ with arbitrarily small error, and multiply it by the indicator of a small interval around t_0, normalized in L^1. Continuity of $H(j,\cdot)$ gives the required limit. No identification of the full dual of a Bochner space is needed.

If $F \in \mathscr{F}(X_0, X_1)$ and t is fixed, the scalar function $z \mapsto H(z,t)(F(z))$ is holomorphic and bounded on the strip. On the boundary $j + is$,

$$H(j+is,t) = H(j, t + s\log b),$$

so (5.17) and the three-lines lemma imply

$$|H(\alpha,t)(F(\alpha))| \le \|F\|_{\mathscr{F}}.$$

Taking the infimum over F shows that $H(\alpha,t)$ has norm at most one on E_α. Consequently,

$$|(\Lambda \circ Q_M)(u)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\alpha,t)(f(t))\, dt \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(t)\|_{E_\alpha}\, dt.$$

Let $M \to \infty$ and take the supremum over Λ. Hahn–Banach gives $\|u\|_V \le \|u\|_{\mathcal{C}(E_\alpha)}$.

Lemma 5.7.10 ensures that $X_0 \cap X_1$ is dense in E_α. Hence trigonometric polynomials with coefficients in that intersection are dense in $L^1((-\pi,\pi); E_\alpha)$: first apply Fejér approximation, then approximate its finitely many coefficients. The bound obtained and completeness of V extend the embedding to all of $\mathcal{C}(E_\alpha)$. Coordinate projections show that this extension preserves each coefficient, so it is the inclusion in (5.16). □

Theorem 5.7.16. (Complex reiteration) Let $0 \le \theta_0 < \theta_1 \le 1$, $0 < \eta < 1$, and $\theta := (1-\eta)\theta_0 + \eta\theta_1$. If $X_{\theta_j} := [X_0, X_1]_{\theta_j}$ at interior points, with X_0, X_1 retained at the endpoints, then

$$[X_{\theta_0}, X_{\theta_1}]_\eta = [X_0, X_1]_\theta$$

with equivalent norms. The constants depend only on θ_0, θ_1, η.

Proof. Set $Y_j := X_{\theta_j}$ and $\delta := \theta_1 - \theta_0$. For the first inclusion, let $F \in \mathscr{F}(X_0, X_1)$ and consider its regularizations F_ε from Lemma 5.7.10. The functions

$$G_\varepsilon(z) := F_\varepsilon(\theta_0 + \delta z)$$

are holomorphic on $X_0 \cap X_1$ and belong to $\mathscr{F}(Y_0, Y_1)$. The bound on interior lines in the same lemma gives

$$\|G_\varepsilon\|_{\mathscr{F}(Y_0,Y_1)} \le \|F_\varepsilon\|_{\mathscr{F}(X_0,X_1)} \le \|F\|_{\mathscr{F}(X_0,X_1)}.$$

On lines coinciding with an endpoint, use the boundary norm directly. To check boundedness in the sum space and the required holomorphicity, use the fact that F_ε is entire in the intersection and formula (5.6): its norms at both endpoints are bounded on every vertical strip of finite width. The same estimate applied to $F_\varepsilon - F_{\varepsilon'}$ shows that (G_ε) is Cauchy in $\mathscr{F}(Y_0, Y_1)$. Its limit takes the value $F(\theta)$ at η by continuity of the inclusions into $X_0 + X_1$. Thus,

$$\|F(\theta)\|_{[Y_0,Y_1]_\eta} \le \|F\|_{\mathscr{F}(X_0,X_1)}.$$

For the reverse inclusion, fix $b = e$ and $c := b^\delta > 1$. Let $x \in [Y_0, Y_1]_\eta$. The constant-sequence characterization in Lemma 5.7.14, applied to the pair (Y_0, Y_1) with base c, allows us to write

$$\underline{x} = v_0 + v_1, \qquad v_j \in \mathcal{C}_c(j - \eta; Y_j),$$

with sum of norms at most $C\|x\|_{[Y_0,Y_1]_\eta}$, up to arbitrarily small error. Since $\delta(j - \eta) = \theta_j - \theta$, the sequences

$$w_{j,n} := b^{(\theta_j - \theta)n} v_{j,n}$$

belong to $\mathcal{C}(Y_j)$ with the same norm as v_j in its weighted space. Lemma 5.7.15 decomposes them as

$$w_j = a_{j,0} + a_{j,1}, \qquad a_{j,0} \in \mathcal{C}_b(-\theta_j; X_0), \qquad a_{j,1} \in \mathcal{C}_b(1 - \theta_j; X_1),$$

with sum of norms at most $\|w_j\|_{\mathcal{C}(Y_j)}$, up to arbitrarily small error. Multiplying each coordinate by $b^{(\theta - \theta_j)n}$ and summing over $j \in \{0, 1\}$ gives

$$\underline{x} = a_0 + a_1, \qquad a_0 \in \mathcal{C}_b(-\theta; X_0), \qquad a_1 \in \mathcal{C}_b(1 - \theta; X_1).$$

The weights cancel exactly: for example, $b^{-\theta n} b^{(\theta - \theta_j)n} = b^{-\theta_j n}$. Therefore,

$$\|\underline{x}\|_{\mathcal{C}_b(-\theta;X_0)+\mathcal{C}_b(1-\theta;X_1)} \le C\|x\|_{[Y_0,Y_1]_\eta}.$$

Another application of Lemma 5.7.14 implies $x \in [X_0, X_1]_\theta$ and gives the reverse bound. The constants depend only on the bases b, c and the indicated parameters; the decomposition lemma has norm one. This includes $\theta_0 = 0$ or $\theta_1 = 1$. □

Lemma 5.7.17. (Calderón's gluing lemma) Let $0 \le \alpha_0 < \alpha_1 \le 1$, let $0 < \eta < 1$, and set $\alpha := (1 - \eta)\alpha_0 + \eta\alpha_1$. If $G \in \mathscr{F}(X_{\alpha_0}, X_{\alpha_1})$, then for each $\varepsilon > 0$ there is $F_\varepsilon \in \mathscr{F}(X_0, X_1)$ such that

$$F_\varepsilon(\alpha) = G(\eta), \qquad \|F_\varepsilon\|_{\mathscr{F}(X_0,X_1)} \le C_{\alpha_0,\alpha_1,\eta}\big(\|G\|_{\mathscr{F}(X_{\alpha_0},X_{\alpha_1})} + \varepsilon\big).$$

Also, for $F \in \mathscr{F}(X_0, X_1)$ and $0 < \alpha < 1$,

$$\sup_{t\in\mathbb{R}} \|F(\alpha + it)\|_{[X_0,X_1]_\alpha} \le \|F\|_{\mathscr{F}}.$$

Proof. Theorem 5.7.16 gives $G(\eta) \in [X_0, X_1]_\alpha$ with norm at most $C\|G\|_{\mathscr{F}}$. Choose a representative whose value at α is $G(\eta)$ and whose norm differs by less than ε from the infimum defining that norm. The final assertion is Lemma 5.7.10. □

Proposition 5.7.18. (Real interpolation of ℓ^p-sums) Let $1 \le p < \infty$. For every compatible pair (X_0, X_1) and $0 < \theta < 1$,

$$(\ell^p(X_0), \ell^p(X_1))_{\theta,p} = \ell^p((X_0, X_1)_{\theta,p})$$

as sets, and for every sequence $x = (x_j)$,

$$\|x\|_{\ell^p((X_0,X_1)_{\theta,p})} \le \|x\|_{(\ell^p(X_0),\ell^p(X_1))_{\theta,p}} \le 2^{1-\frac{1}{p}} \|x\|_{\ell^p((X_0,X_1)_{\theta,p})}.$$

Proof. For a sequence $x = (x_j)_{j\in\mathbb{N}}$, set $J := \mathbb{N}$ and

$$A(t,x) := \left(\sum_{j\in J} K(t, x_j; X_0, X_1)^p \right)^{\frac{1}{p}}.$$

If $x = a + b$, Minkowski's inequality from Theorem B.7.3, applied to the discrete measure space J, gives

$$A(t,x) \le \|a\|_{\ell^p(X_0)} + t\|b\|_{\ell^p(X_1)}.$$

Taking the infimum yields $A(t,x) \le K(t,x;\ell^p(X_0),\ell^p(X_1))$. Conversely, for each coordinate choose $x_j = a_j + b_j$ with $\|a_j\|_{X_0} + t\|b_j\|_{X_1} \le K(t,x_j) + \varepsilon_j$, where $(\varepsilon_j) \in \ell^p(J)$ is arbitrarily small. Then

$$\|a\|_{\ell^p(X_0)} + t\|b\|_{\ell^p(X_1)} \le 2^{1-\frac{1}{p}} \left(\sum_{j\in J} (\|a_j\|_{X_0} + t\|b_j\|_{X_1})^p \right)^{\frac{1}{p}}.$$

Letting (ε_j) tend to zero gives

$$A(t,x) \le K(t,x;\ell^p(X_0),\ell^p(X_1)) \le 2^{1-\frac{1}{p}} A(t,x).$$

Raise to the power p, integrate with respect to $t^{-\theta p}dt/t$, and apply Tonelli's theorem, Theorem B.6.17. The estimates also hold if either side is infinite. If the right-hand side is finite, the estimate at $t = 1$ and the chosen decompositions show that x belongs to the sum space; hence both inclusions have been justified. □

5.8 Lorentz spaces as interpolation spaces

Fractional embeddings obtained by the real method naturally take values in Lorentz spaces. We recall only what will be used later.

Definition 5.8.1. (Decreasing rearrangement and Lorentz spaces) Let (A,μ) be a σ-finite nonatomic measure space, and let $f : A \longrightarrow \mathbb{C}$ be measurable. Define

$$\mu_f(\lambda) := \mu(\{x \in A \mid |f(x)| > \lambda\}), \qquad f^*(t) := \inf\{\lambda \in (0,\infty) \mid \mu_f(\lambda) \le t\}, \qquad t > 0,$$

with $\inf \varnothing = \infty$. For $1 < p < \infty$ and $1 \le q \le \infty$, the space $L^{p,q}(A)$ consists of the classes of functions for which

$$\|f\|_{L^{p,q}(A)} := \|t^{1/p} f^*(t)\|_{L^q((0,\infty),dt/t)} < \infty,$$

and, if $q = \infty$, define explicitly

$$\|f\|_{L^{p,\infty}(A)} = \sup_{t\in(0,\infty)} t^{1/p} f^*(t).$$

We shall also use

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad \|f\|^{\#}_{L^{p,q}(A)} := \|t^{1/p} f^{**}(t)\|_{L^q((0,\infty),dt/t)}.$$

The first expression is a quasinorm; the second will be an equivalent norm, as we shall prove below.

Lemma 5.8.2. (Rearrangement identities) Under the assumptions of the preceding definition, for $t > 0$ we have

$$\int_0^t f^*(s)\,ds = \sup_{\substack{E\subseteq A \text{ measurable}\\ \mu(E)\le t}} \int_E |f|\,d\mu.$$

Consequently, if $f = g + h$,

$$\int_0^t f^*(s)\,ds \le \int_0^t g^*(s)\,ds + \int_0^t h^*(s)\,ds.$$

Moreover, for $0 < p < \infty$,

$$\int_A |f|^p\,d\mu = \int_0^\infty f^*(t)^p\,dt.$$

Proof. The definition of the generalized inverse and right continuity of μ_f give

$$|\{s > 0 \mid f^*(s) > \lambda\}| = \mu_f(\lambda), \qquad \lambda > 0.$$

Indeed, $s \ge \mu_f(\lambda)$ implies $f^*(s) \le \lambda$. If $s < \mu_f(\lambda)$, right continuity provides $\delta > 0$ with $\mu_f(\lambda+\delta) > s$, and then $f^*(s) \ge \lambda + \delta$. This argument includes the case $\mu_f(\lambda) = \infty$, by continuity from below of measures as the level decreases to λ. Cavalieri's formula from Proposition B.8.1 therefore gives

$$\int_0^t f^*(s)\,ds = \int_0^\infty \min\{t, \mu_f(\lambda)\}\,d\lambda. \tag{5.18}$$

For $\mu(E) \le t$, Cavalieri's formula also implies

$$\int_E |f|\,d\mu = \int_0^\infty \mu(E \cap \{|f| > \lambda\})\,d\lambda \le \int_0^\infty \min\{t, \mu_f(\lambda)\}\,d\lambda.$$

For the reverse inequality, we shall use the fact that in a nonatomic space, a measurable set of finite measure contains subsets of every measure between zero and its own measure. Here is a justification. Every set of positive measure contains a subset of positive measure less than any $\varepsilon > 0$: split it into two parts of positive measure and repeat with the smaller part. To obtain a prescribed measure a, choose disjoint parts without exceeding a and, at each step, with measure greater than half the supremum of still admissible measures. If the sum of their measures tended to a value less than a, the remainder would contain a fixed part of positive measure smaller than the difference; it would remain admissible at every step, contradicting convergence

of the sum. The union of the parts has measure a.
If $|f|$ is simple and supported on a set of finite measure, choose E by taking its highest level sets first and, when necessary, a portion of the last level to reach measure t. This attains the quantity in (5.18). For a general function, approximate $|f|$ by an increasing sequence of nonnegative simple functions supported on sets of finite measure. Their distribution functions increase to μ_f; monotone convergence in (5.18) allows passage to the limit and proves equality of the suprema.
Applying that equality to $|g+h| \le |g|+|h|$ gives subadditivity. Finally, $|f|$ and f^* have the same distribution function; Cavalieri's formula applied to their pth powers gives the final identity. □

The preceding subadditivity and Minkowski's inequality show that $\|\cdot\|^{\#}_{L^{p,q}(A)}$ is a norm. Since $f^* \le f^{**}$, one comparison is immediate. For the other, set $g(t) := t^{1/p} f^*(t)$ and observe that

$$t^{1/p} f^{**}(t) = \int_0^t \left(\frac{s}{t}\right)^{1-1/p} g(s)\,\frac{ds}{s}.$$

Theorem B.8.6, with $1 - 1/p = 1/p' > 0$, gives

$$\|f\|_{L^{p,q}(A)} \le \|f\|^{\#}_{L^{p,q}(A)} \le p' \|f\|_{L^{p,q}(A)}. \tag{5.19}$$

This also proves the quasi-triangle inequality for the first expression. The next theorem will provide completeness for the equivalent norm.

Theorem 5.8.3. (Interpolation of L^p spaces) Let (A,μ) be a σ-finite nonatomic measure space. Let $1 \le p_0 < p_1 \le \infty$, $0 < \theta < 1$, $1 \le q \le \infty$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$(L^{p_0}(A), L^{p_1}(A))_{\theta,q} = L^{p,q}(A)$$

as sets. More precisely, there are $c_{p_0,p_1,\theta,q}, C_{p_0,p_1,\theta,q} > 0$, independent of f and (A,μ), such that

$$c_{p_0,p_1,\theta,q} \|f\|_{L^{p,q}(A)} \le \|f\|_{(L^{p_0}(A),L^{p_1}(A))_{\theta,q}} \le C_{p_0,p_1,\theta,q} \|f\|_{L^{p,q}(A)}.$$

Moreover, $L^{p,p}(A) = L^p(A)$ with equality of norms, and if $q_0 \le q_1$, then $L^{p,q_0}(A) \hookrightarrow L^{p,q_1}(A)$.

Proof. We first prove the identity

$$K(t, f; L^1(A), L^\infty(A)) = \int_0^t f^*(s)\,ds. \tag{5.20}$$

If $f = g + h$, Lemma 5.8.2 gives

$$\int_0^t f^*(s)\,ds \le \|g\|_{L^1(A)} + t\|h\|_{L^\infty(A)}.$$

Taking the infimum proves one inequality. For the other, suppose that the integral is finite and set $\lambda := f^*(t) < \infty$. With $\operatorname{sgn} f = f/|f|$ where $f \ne 0$, and zero at its zeros, define

$$g := \operatorname{sgn} f\,(|f| - \lambda)_+, \qquad h := f - g.$$

Monotonicity of μ_f and the definition of λ imply $\mu_f(s) > t$ if $0 < s < \lambda$ and $\mu_f(s) \le t$ if $s > \lambda$. By Cavalieri's formula and (5.18),

$$\|g\|_{L^1(A)} + t\|h\|_{L^\infty(A)} \le \int_\lambda^\infty \mu_f(s)\,ds + t\lambda = \int_0^t f^*(s)\,ds.$$

In particular, this construction proves $f \in L^1 + L^\infty$ whenever the integral is finite. If it is infinite, the first inequality already precludes a decomposition of finite cost. This gives (5.20) with the extended interpretation.

For $1 < p < \infty$, set $\alpha := 1 - 1/p$. The preceding identity gives

$$\|f\|_{(L^1(A),L^\infty(A))_{\alpha,q}} = \|t^{1/p} f^{**}(t)\|_{L^q((0,\infty),dt/t)} = \|f\|^{\#}_{L^{p,q}(A)}.$$

The comparisons (5.19) prove $(L^1, L^\infty)_{1-1/p,q} = L^{p,q}$, as well as completeness with the norm $\|\cdot\|^{\#}$, by Theorem 5.2.4.

For general exponents, write $\alpha_j := 1 - 1/p_j$. If $1 < p_j < \infty$, the result just proved with second index p_j, together with the final identity in Lemma 5.8.2, identifies

$$L^{p_j}(A) = L^{p_j,p_j}(A) = (L^1(A), L^\infty(A))_{\alpha_j,p_j}$$

with equivalent norms. If $p_0 = 1$ or $p_1 = \infty$, retain the corresponding endpoint. Apply Theorem 5.5.2; its final parameter is

$$(1-\theta)\alpha_0 + \theta\alpha_1 = 1 - \left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right) = 1 - \frac{1}{p}.$$

We obtain the asserted identity. Equivalence of norms at the endpoints is transported by Theorem 5.3.1 applied to the identity in both directions; this gives the indicated dependence of the constants.

The equality $L^{p,p} = L^p$ is isometric for the quasinorm in Definition 5.8.1, since

$$\|f\|^p_{L^{p,p}(A)} = \int_0^\infty f^*(t)^p\,dt = \|f\|^p_{L^p(A)}.$$

Finally, monotonicity of f^* allows comparison, on each interval $[2^k, 2^{k+1}]$, of the norm defining $L^{p,q}$ with that of the sequence $(2^{k/p} f^*(2^k))_k$ in $\ell^q(\mathbb{Z})$. The lower bound uses the right endpoint and the upper bound the left; shifting the index only introduces the factor $2^{1/p}$. The embedding $\ell^{q_0} \hookrightarrow \ell^{q_1}$ when $q_0 \le q_1$ proves the final assertion. □

6

Intermediate regularity

Regularity does not increase only in integer steps. Between having m and $m+1$ integrable derivatives lies a continuous range of behaviors arising in trace problems, interpolation, nonlocal equations, and spectral theory. To measure these intermediate orders, we shall use differences of values, Fourier multipliers, and frequency decompositions.

Several scales for the same question

Bessel potentials connected regularity with the Fourier transform; Slobodeckij and Gagliardo developed difference-based seminorms; Besov and Triebel organized scales capable of separating the order of smoothness from the way frequencies are summed. Each construction reflects different properties of functions. Their agreements and differences allow us to choose the appropriate scale for localization in charts, taking traces, and interpolating operators.

The spaces $W^{m,p}(\mathbb{R}^n)$ measure regularity through integer-order derivatives. Nonlocal problems show that intermediate orders also need to be quantified. For $0 < s < 1$, the formal equation

$$(-\Delta)^s u = f$$

is associated with the energy

$$\mathcal{E}_s(u) = \frac{c_{n,s}}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx\, dy - \int_{\mathbb{R}^n} f\, u \, dx.$$

The first term compares values of u at pairs of points instead of using a local derivative, leading to the Gagliardo–Slobodeckij seminorm. For $1 < p < \infty$, the analogous expression

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy$$

motivates the spaces $W^{s,p}(\mathbb{R}^n)$ and appears in energies associated with the fractional p-Laplacian.

There is another route to fractional regularity. The operator $(I - \Delta)^{s/2}$ is defined through Fourier analysis and

leads to Bessel potentials and the spaces $H^{s,p}(\mathbb{R}^n)$. Both constructions recover $W^{m,p}(\mathbb{R}^n)$ when $s = m \in \mathbb{N}_0$, but for $s \notin \mathbb{N}_0$ they agree in general only in the Hilbert case $p = 2$. Traces and elliptic regularity also require negative orders and scales well suited to localization.

This is why Sobolev–Slobodeckij, Bessel, Besov, and Triebel–Lizorkin spaces appear together. Each scale organizes integrability, smoothness order, and frequency distribution differently. This chapter constructs these scales, relates their parameters, and proves the interpolation, localization, and embedding properties to be used on manifolds. We follow [98, Chapters 6–8 and 11–12], [146, Chapter 1], and [130, §§2.2–2.3].

6.1 Bessel potentials and the spaces $H^{s,p}(\mathbb{R}^n)$

Write

$$\langle \xi \rangle := (1 + \|\xi\|^2)^{\frac{1}{2}}, \qquad \xi \in \mathbb{R}^n.$$

Lemma 6.1.1. (Peetre's inequality) For every $\mu \in \mathbb{R}$ and any $\xi, \eta \in \mathbb{R}^n$,

$$\langle \xi + \eta \rangle^{\mu} \le 2^{\frac{|\mu|}{2}} \langle \xi \rangle^{\mu} \langle \eta \rangle^{|\mu|}.$$

Proof. The triangle inequality and $2ab \le a^2 + b^2$ give

$$\begin{aligned} \langle \xi + \eta \rangle^2 &= 1 + \|\xi + \eta\|^2 \\ &\le 1 + 2\|\xi\|^2 + 2\|\eta\|^2 \\ &\le 2(1 + \|\xi\|^2)(1 + \|\eta\|^2) = 2\langle \xi \rangle^2 \langle \eta \rangle^2. \end{aligned}$$

If $\mu \ge 0$, raise this inequality to the power $\frac{\mu}{2}$. If $\mu < 0$, apply the preceding inequality to $\xi = (\xi + \eta) - \eta$ to obtain

$$\langle \xi \rangle \le \sqrt{2} \langle \xi + \eta \rangle \langle \eta \rangle.$$

Dividing by $\langle \xi \rangle \langle \xi + \eta \rangle$ and raising to the power $|\mu|$ gives

$$\langle \xi + \eta \rangle^{\mu} \le 2^{\frac{|\mu|}{2}} \langle \xi \rangle^{\mu} \langle \eta \rangle^{|\mu|},$$

as required. □

For each $s \in \mathbb{R}$ and each multi-index α, there is $C_{s,\alpha} > 0$ such that

$$|D^{\alpha} \langle \xi \rangle^{s}| \le C_{s,\alpha} \langle \xi \rangle^{s - |\alpha|}, \qquad \xi \in \mathbb{R}^n.$$

In particular, all these derivatives have at most polynomial growth. By the remark following Definition 4.9.5, multiplication $T \mapsto \langle \cdot \rangle^s T$ is well defined and continuous on $\mathcal{S}'(\mathbb{R}^n)$.

Definition 6.1.2. (Bessel operator and space) Let $s \in \mathbb{R}$ and $1 < p < \infty$. Define

$$J^s u := \mathcal{F}^{-1}\big(\langle \cdot \rangle^s \widehat{u}\big), \qquad u \in \mathcal{S}'(\mathbb{R}^n),$$

and

$$H^{s,p}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) \mid J^s u \in L^p(\mathbb{R}^n)\}, \qquad \|u\|_{H^{s,p}(\mathbb{R}^n)} := \|J^s u\|_{L^p(\mathbb{R}^n)}.$$

The operator J^{-s} is called the *Bessel potential* of order s.

Proposition 6.1.3. (Structure of the Bessel scale) Let $s, t \in \mathbb{R}$ and $1 < p < \infty$.

(a) $H^{s,p}(\mathbb{R}^n)$ is a Banach space, and $J^s \colon H^{s,p}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is an isometric isomorphism with inverse J^{-s}.

(b) $J^t \colon H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-t,p}(\mathbb{R}^n)$ is an isometric isomorphism.

(c) $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s,p}(\mathbb{R}^n)$.

(d) With respect to the distributional pairing extended from $\mathcal{S}(\mathbb{R}^n)$, the dual of $H^{s,p}(\mathbb{R}^n)$ is identified isometrically with $H^{-s,p'}(\mathbb{R}^n)$: if $\Lambda_v(u) := \langle v, u\rangle$, then

$$\|\Lambda_v\|_{(H^{s,p}(\mathbb{R}^n))'} = \|v\|_{H^{-s,p'}(\mathbb{R}^n)}.$$

Proof. The identities $J^s J^{-s} = I = J^{-s} J^s$ hold in $\mathcal{S}'(\mathbb{R}^n)$ because the symbols multiply. Thus J^s identifies $H^{s,p}(\mathbb{R}^n)$ with $L^p(\mathbb{R}^n)$, which is complete by Theorem B.7.3; this proves (a) and (b). If $f \in L^p(\mathbb{R}^n)$, density established in Proposition 4.8.5 allows us to choose $f_j \in \mathcal{S}(\mathbb{R}^n)$ with $f_j \to f$ in $L^p(\mathbb{R}^n)$. Set $u_j := J^{-s} f_j \in \mathcal{S}(\mathbb{R}^n)$; then $u_j \to J^{-s} f$ in $H^{s,p}(\mathbb{R}^n)$.
For duality, if $v \in H^{-s,p'}(\mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$, Parseval's identity from Theorem 4.11.10, applied first to Schwartz functions and then by density, allows us to write

$$\langle v, u\rangle = \int_{\mathbb{R}^n} (J^{-s} v)(x)(J^s u)(x)\, dx.$$

Hölder's inequality from Proposition B.7.9 gives

$$|\Lambda_v(u)| \le \|J^{-s} v\|_{L^{p'}(\mathbb{R}^n)} \|J^s u\|_{L^p(\mathbb{R}^n)} = \|v\|_{H^{-s,p'}(\mathbb{R}^n)} \|u\|_{H^{s,p}(\mathbb{R}^n)}.$$

Conversely, a continuous functional on $H^{s,p}(\mathbb{R}^n)$ is transported by $J^{-s} \colon L^p(\mathbb{R}^n) \longrightarrow H^{s,p}(\mathbb{R}^n)$ to a functional on $L^p(\mathbb{R}^n)$. Theorem B.7.7 gives $g \in L^{p'}(\mathbb{R}^n)$, and $v := J^s g \in H^{-s,p'}(\mathbb{R}^n)$ represents the original functional and satisfies

$$\|v\|_{H^{-s,p'}(\mathbb{R}^n)} = \|g\|_{L^{p'}(\mathbb{R}^n)} = \|\Lambda_v\|_{(H^{s,p}(\mathbb{R}^n))'}.$$

□

For $s > 0$, the potential J^{-s} admits a convolution representation. This description links the Fourier construction to the later construction through the heat semigroup.

Proposition 6.1.4. (Euclidean Bessel kernel) Let $s > 0$. There is a positive radial function $G_s \in L^1(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} G_s = 1$, such that

$$J^{-s} f = G_s * f$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and, by continuity, for $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. Moreover,

$$G_s(x) = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty e^{-t} (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}} t^{\frac{s}{2}-1}\, dt, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

The value at the origin may be chosen arbitrarily without changing the element of L^1; the integral there may diverge when $s \le n$.

Proof. The gamma identity

$$\langle \xi \rangle^{-s} = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty e^{-t(1+\|\xi\|^2)} t^{\frac{s}{2}-1}\, dt$$

and the Gaussian transform formula from Lemma 4.11.4 show that the Fourier transform of G_s, with the normalization in Definition 4.11.1, is $(2\pi)^{-\frac{n}{2}} \langle \xi \rangle^{-s}$. The convolution formula from Proposition 4.11.9 gives $J^{-s} f = G_s * f$ for $f \in \mathcal{S}(\mathbb{R}^n)$. Tonelli's theorem B.6.17 proves that $G_s \ge 0$ and

$$\int_{\mathbb{R}^n} G_s(x)\, dx = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty e^{-t} t^{\frac{s}{2}-1}\, dt = 1.$$

Young's inequality from Theorem 4.10.1, together with density of $\mathcal{S}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ when $p < \infty$, extends the identity to these exponents. For $p = \infty$, the same formula directly defines $G_s * f \in L^\infty(\mathbb{R}^n)$, and the identity is understood in $\mathcal{S}'(\mathbb{R}^n)$, where it is verified by duality against Schwartz functions. □

Definition 6.1.5. (Bessel spaces on an open subset) Let $\Omega \subseteq \mathbb{R}^n$ be open. Define $H^{s,p}(\Omega)$ as the space of restrictions to Ω of elements of $H^{s,p}(\mathbb{R}^n)$, with the quotient norm

$$\|u\|_{H^{s,p}(\Omega)} := \inf \left\{ \|U\|_{H^{s,p}(\mathbb{R}^n)} \;\middle|\; U \restriction_\Omega = u \right\}.$$

This definition is extrinsic: it does not assert that a seminorm computed only within Ω is equivalent to the quotient norm. Such an equivalence requires an extension property of the open subset and will be proved for Lipschitz domains. This distinction will be essential when comparing $H^{s,p}(\Omega)$ with intrinsic Slobodeckij spaces.

6.1.1 The Euclidean Hilbert scale and localization

For elliptic regularity, we shall need only the case $p = 2$. Write $H^s(\mathbb{R}^n) := H^{s,2}(\mathbb{R}^n)$ and $\langle \xi \rangle := (1 + \|\xi\|^2)^{\frac{1}{2}}$. If $r \in \mathbb{N}$, define the space $H^s(\mathbb{R}^n, \mathbb{K}^r)$ componentwise and equip it with the norm

$$\|u\|^2_{H^s(\mathbb{R}^n,\mathbb{K}^r)} := \sum_{a=1}^{r} \|u^a\|^2_{H^s(\mathbb{R}^n)}.$$

Although the preceding definition is available for every $s \in \mathbb{R}$, the proof of local elliptic regularity will use only integer indices, including negative ones. Noninteger orders will be revisited when studying the scales associated with the heat semigroup.

Proposition 6.1.6. (Elementary properties of the $H^s(\mathbb{R}^n)$ scale) Let $s, t \in \mathbb{R}$ and $t \ge s$, and let α be a multi-index. Then the following assertions hold.

(a) $H^s(\mathbb{R}^n, \mathbb{K}^r)$ is a Hilbert space, and $\mathcal{S}(\mathbb{R}^n, \mathbb{K}^r)$ is dense in it.

(b) The inclusion $H^t(\mathbb{R}^n,\mathbb{K}^r) \hookrightarrow H^s(\mathbb{R}^n,\mathbb{K}^r)$ is continuous.

(c) The distributional derivative induces a continuous linear operator

$$D^\alpha : H^s(\mathbb{R}^n,\mathbb{K}^r) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^n,\mathbb{K}^r).$$

(d) Translation $\tau_h u(x) := u(x-h)$ is an isometry of $H^s(\mathbb{R}^n,\mathbb{K}^r)$, and

$$\|\tau_h u - u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} \longrightarrow 0 \qquad \text{when } h \to 0.$$

(e) In the real case, the dual of $H^s(\mathbb{R}^n,\mathbb{R}^r)$ is identified isometrically with $H^{-s}(\mathbb{R}^n,\mathbb{R}^r)$. In the complex case, the correct complex-linear Hermitian identification is

$$H^{-s}(\mathbb{R}^n,\overline{\mathbb{C}^r}) \xrightarrow{\cong} \big(H^s(\mathbb{R}^n,\mathbb{C}^r)\big)'. \tag{6.1}$$

Moreover, the standard basis of $\mathbb{C}^r$ defines the complex-bilinear distributional pairing

$$H^{-s}(\mathbb{R}^n,\mathbb{C}^r) \times H^s(\mathbb{R}^n,\mathbb{C}^r) \longrightarrow \mathbb{C}, \qquad (v,u) \longmapsto \langle v,u\rangle_{\mathcal{D}',\mathcal{D}}, \tag{6.2}$$

which also induces an isometric complex-linear isomorphism from the first factor onto the dual of the second. This second identification uses the standard trivialization; the first is the intrinsic Hermitian form.

Proof. By Plancherel's theorem 4.11.10, the map

$$\Lambda^s : H^s(\mathbb{R}^n,\mathbb{K}^r) \longrightarrow L^2(\mathbb{R}^n,\mathbb{K}^r), \qquad \widehat{\Lambda^s u}(\xi) := \langle\xi\rangle^s \widehat{u}(\xi),$$

is a surjective isometry with inverse Λ^{-s}. In the real case, complexification is used when taking Fourier transforms. The multipliers $\langle\xi\rangle^{\pm s}$, being real and even, commute with conjugation in physical space and preserve real distributions. Thus the isometry and its inverse are well defined also for $\mathbb{K} = \mathbb{R}$. Completeness of $L^2(\mathbb{R}^n,\mathbb{K}^r)$ proves that $H^s(\mathbb{R}^n,\mathbb{K}^r)$ is a Hilbert space with the inner product transported by Λ^s.
For density, let $u \in H^s(\mathbb{R}^n,\mathbb{K}^r)$. Density in L^2 allows us to choose $(f_j) \subseteq C_c^\infty(\mathbb{R}^n,\mathbb{K}^r)$ such that $f_j \to \Lambda^s u$ in $L^2(\mathbb{R}^n,\mathbb{K}^r)$. Set $u_j := \Lambda^{-s} f_j$. Derivatives of the symbol $\langle\xi\rangle^{-s}$ have at most polynomial growth, so multiplication by it preserves the Schwartz class. Since the Fourier transform and its inverse also preserve that class, $u_j \in \mathcal{S}(\mathbb{R}^n,\mathbb{C}^r)$; if $\mathbb{K} = \mathbb{R}$, compatibility with conjugation also gives $u_j = \overline{u_j}$. Thus, in both cases, $u_j \in \mathcal{S}(\mathbb{R}^n,\mathbb{K}^r)$ and

$$\|u_j - u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} = \|f_j - \Lambda^s u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} \longrightarrow 0.$$

If $t \geq s$, then $\langle\xi\rangle^s \leq \langle\xi\rangle^t$, whence

$$\|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} = \|\langle\xi\rangle^s \widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{C}^r)} \leq \|\langle\xi\rangle^t \widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{C}^r)} = \|u\|_{H^t(\mathbb{R}^n,\mathbb{K}^r)}.$$

Likewise, since $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi)$ and $|\xi^\alpha| \leq \langle\xi\rangle^{|\alpha|}$,

$$\|D^\alpha u\|_{H^{s-|\alpha|}(\mathbb{R}^n,\mathbb{K}^r)} \leq \|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}.$$

The Fourier transform of $\tau_h u$ is $e^{-ih\cdot\xi}\widehat{u}(\xi)$, so $\|\tau_h u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} = \|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}$. Moreover,

$$\|\tau_h u - u\|^2_{H^s(\mathbb{R}^n,\mathbb{K}^r)} = \int_{\mathbb{R}^n} \langle\xi\rangle^{2s}|e^{-ih\cdot\xi}-1|^2|\widehat{u}(\xi)|^2\,d\xi.$$

The integrand converges pointwise to zero and is dominated by $4\langle\xi\rangle^{2s}\|\widehat{u}(\xi)\|^2_{\mathbb{C}^r} \in L^1(\mathbb{R}^n)$. Lebesgue's dominated convergence theorem B.6.14 gives continuity of translations.
Finally, first let $\mathbb{K} = \mathbb{C}$. By the complex-linear form of Riesz's theorem, Corollary B.3.13, every functional F on H^s has a unique representative $\overline{g} \in \overline{L^2(\mathbb{R}^n,\mathbb{C}^r)}$ such that

$$F(u) = \langle \Lambda^s u, g\rangle_{L^2}.$$

If $w := \Lambda^s g \in H^{-s}(\mathbb{R}^n,\mathbb{C}^r)$, this equality reads

$$F(u) = \langle \Lambda^s u, \Lambda^{-s} w\rangle_{L^2}.$$

Pointwise conjugation identifies $\overline{H^{-s}(\mathbb{R}^n,\mathbb{C}^r)}$ complex-linearly with $H^{-s}(\mathbb{R}^n,\overline{\mathbb{C}^r})$; this gives (6.1), including equality of norms.
For the bilinear pairing, if u, v are Schwartz functions, Plancherel's theorem and the Fourier convention give

$$\langle v,u\rangle_{\mathcal{D}',\mathcal{D}} = \int_{\mathbb{R}^n}\sum_{a=1}^{r} \widehat{v^{,a}}(-\xi)\widehat{u^{,a}}(\xi)\,d\xi.$$

Inserting the weights $\langle\xi\rangle^{-s}$ and $\langle\xi\rangle^s$, Cauchy–Schwarz and the substitution $\xi \mapsto -\xi$ prove that its norm is at most $\|v\|_{H^{-s}(\mathbb{R}^n)}\|u\|_{H^s(\mathbb{R}^n)}$. Equality of norms and surjectivity follow from the preceding Hermitian identification through conjugation and reflection $\xi \mapsto -\xi$. Density of $\mathcal{S}$ extends the pairing and proves (6.2). In the real case, omit conjugation, and the same argument gives the stated assertion. □

Definition 6.1.7. (Local Sobolev spaces of real order) Let $\Omega \subseteq \mathbb{R}^n$ be open and let $s \in \mathbb{R}$. Define

$$H^s_{\mathrm{loc}(\mathbb{R}^n)}(\Omega,\mathbb{K}^r) := \left\{ u \in \mathcal{D}'(\Omega,\mathbb{K}^r) \;\middle|\; \chi u \in H^s(\mathbb{R}^n,\mathbb{K}^r) \text{ for every } \chi \in C^\infty_c(\Omega)\right\},$$

where χu is extended by zero outside Ω.

6.1.2 Multiplication, regularization, and commutators

Ordinary continuity of multiplication by a smooth function does not suffice to freeze coefficients at negative orders. We shall need to separate the principal term, controlled by the uniform norm of the coefficient, from a remainder losing one derivative.

Lemma 6.1.8. (Refined multiplication estimate) Let $s \in \mathbb{R}$ and $a \in C^\infty_c(\mathbb{R}^n)$. There is a constant $C_{a,s} > 0$ such that

$$\|av\|_{H^s(\mathbb{R}^n)} \le \|a\|_{L^\infty(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + C_{a,s}\|v\|_{H^{s-1}(\mathbb{R}^n)}$$

for every $v \in H^s(\mathbb{R}^n)$. For a matrix-valued function $A \in C^\infty_c(\mathbb{R}^n, M_{q\times r}(\mathbb{K}))$, there is a constant $C_{A,s} > 0$ such

that

$$\|Av\|_{H^s(\mathbb{R}^n,\mathbb{K}^q)} \le C_{q,r}\|A\|_{L^\infty(\mathbb{R}^n,M_{q\times r}(\mathbb{K}))}\|v\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} + C_{A,s}\|v\|_{H^{s-1}(\mathbb{R}^n,\mathbb{K}^r)}.$$

Proof. By density, it suffices to consider $v \in \mathcal{S}(\mathbb{R}^n)$. Write Λ^s for the Fourier multiplier with symbol $\langle\xi\rangle^s$. Then

$$\Lambda^s(av) = a\Lambda^s v + R_s(a,v),$$

where, up to the fixed constant determined by the normalization of the Fourier transform,

$$\widehat{R_s(a,v)}(\xi) = \int_{\mathbb{R}^n} \big(\langle\xi\rangle^s - \langle\xi-\eta\rangle^s\big)\widehat{a}(\eta)\widehat{v}(\xi-\eta)\,d\eta.$$

The fundamental theorem of calculus for Banach-valued functions B.6.5, applied to $t \mapsto \langle\xi - t\eta\rangle^s$, together with Peetre's inequality from Lemma 6.1.1, provides a constant $C_s > 0$ for which

$$\big|\langle\xi\rangle^s - \langle\xi-\eta\rangle^s\big| \le C_s\|\eta\|\langle\eta\rangle^{|s|+1}\langle\xi-\eta\rangle^{s-1}.$$

Since $a \in C_c^\infty$, its Fourier transform is rapidly decreasing, and the function

$$K_s(\eta) := C_s\|\eta\|\langle\eta\rangle^{|s|+1}|\widehat{a}(\eta)|$$

belongs to $L^1(\mathbb{R}^n)$. Young's convolution inequality from Theorem 4.10.1 gives

$$\|R_s(a,v)\|_{L^2(\mathbb{R}^n)} \le \|K_s\|_{L^1(\mathbb{R}^n)}\|v\|_{H^{s-1}(\mathbb{R}^n)}.$$

On the other hand,

$$\|a\Lambda^s v\|_{L^2(\mathbb{R}^n)} \le \|a\|_{L^\infty(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}.$$

Adding the two estimates proves the scalar inequality. The matrix assertion follows by applying the result to each entry and summing finitely many components. □

Lemma 6.1.9. (Localization criterion) Let $u \in \mathcal{D}'(\Omega,\mathbb{K}^r)$ and $s \in \mathbb{R}$. The following conditions are equivalent.

(a) $u \in H^s_{\mathrm{loc}(\mathbb{R}^n)}(\Omega,\mathbb{K}^r)$.

(b) For every compact set $K \subseteq \Omega$, there is $\chi \in C_c^\infty(\Omega)$, with $\chi = 1$ on a neighborhood of K, such that $\chi u \in H^s(\mathbb{R}^n,\mathbb{K}^r)$.

(c) Every point $x \in \Omega$ has an open neighborhood $V \Subset \Omega$ and a function $\chi \in C_c^\infty(\Omega)$, with $\chi = 1$ on a neighborhood of $\overline{V}$, for which $\chi u \in H^s(\mathbb{R}^n,\mathbb{K}^r)$.

Proof. The implication (a)⇒(b) is immediate after choosing, by Proposition 1.3.6, a bump function equal to one on a neighborhood of K; (b)⇒(c) follows by taking a compact set containing the point in its interior. Assume (c) and let $\zeta \in C_c^\infty(\Omega)$. The compact set $\mathrm{supp}(\zeta)$ is covered by finitely many open sets $V_1,\dots,V_N$ as in (c). Choose a partition of unity $\eta_1,\dots,\eta_N$ on a neighborhood of $\mathrm{supp}(\zeta)$ subordinate to this cover, and

functions $\chi_j \in C_c^\infty(\Omega)$ such that $\chi_j = 1$ on a neighborhood of $\overline{V_j}$ and $\chi_j u \in H^s(\mathbb{R}^n)$. Then

$$\zeta u = \sum_{j=1}^{N} (\zeta \eta_j)(\chi_j u).$$

The refined multiplication estimate in Lemma 6.1.8 shows that each summand belongs to $H^s(\mathbb{R}^n, \mathbb{K}^r)$. The sum is finite, so $\zeta u \in H^s(\mathbb{R}^n)$. Therefore, $u \in H^s_{\mathrm{loc}(\mathbb{R}^n)}$. □

Proposition 6.1.10. (Local Sobolev order of a distribution) Let $\Omega \subseteq \mathbb{R}^n$ be open and let $T \in \mathcal{D}'(\Omega, \mathbb{K}^r)$. For every $\chi \in C_c^\infty(\Omega)$, there is $N \in \mathbb{N}$ such that

$$\chi T \in H^{-N}(\mathbb{R}^n, \mathbb{K}^r).$$

In particular, every distribution belongs locally to some Sobolev space of negative integer order, although the integer may depend on the compact set under consideration.

Proof. The argument is componentwise, so it suffices to treat the scalar case. Let $K \Subset \Omega$ be a compact set containing the support of χ in its interior. Continuity of T on $\mathcal{D}_K(\Omega)$ provides an integer $q \in \mathbb{N}_0$ and a constant $C > 0$ such that

$$|\langle T, \psi \rangle| \le C \max_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le q}} \sup_{x \in K} |D^\alpha \psi(x)| \qquad \text{for every } \psi \in \mathcal{D}_K(\Omega).$$

Applying this inequality to $\psi = \chi\varphi$, with $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and using the Leibniz rule gives $C_1 > 0$ such that

$$|\langle \chi T, \varphi \rangle| \le C_1 \max_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta| \le q}} \|D^\beta \varphi\|_{L^\infty(\mathbb{R}^n)}.$$

Choose an integer $N > q + \dfrac{n}{2}$. By the Fourier inversion theorem 4.11.6 and the Cauchy–Schwarz inequality from Theorem B.3.2, for $|\beta| \le q$ we have

$$|D^\beta \varphi(x)| \le C_2 \int_{\mathbb{R}^n} \|\xi\|^{|\beta|} |\widehat{\varphi}(\xi)| \, d\xi \le C_3 \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{-2(N-|\beta|)} \, d\xi \right)^{\frac{1}{2}} \|\varphi\|_{H^N(\mathbb{R}^n)}.$$

The integral is finite because $N - |\beta| > \dfrac{n}{2}$. Thus,

$$|\langle \chi T, \varphi \rangle| \le C_4 \|\varphi\|_{H^N(\mathbb{R}^n)}.$$

Density of $\mathcal{S}(\mathbb{R}^n)$ in $H^N(\mathbb{R}^n)$, proved in Proposition 6.1.6, extends χT to a continuous functional on $H^N(\mathbb{R}^n)$. The dual identification in Proposition 6.1.6 yields $\chi T \in H^{-N}(\mathbb{R}^n)$. □

Let ρ be the radial mollifier from Theorem B.9.1, and set $J_\varepsilon u := \rho_\varepsilon * u$.

Proposition 6.1.11. (Mollifiers on the $H^s(\mathbb{R}^n)$ scale) Let $s \in \mathbb{R}$ and $0 < \varepsilon \le 1$. Then J_ε induces a continuous

linear operator on $H^s(\mathbb{R}^n,\mathbb{K}^r)$, and

$$\|J_\varepsilon u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} \le \|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}.$$

Moreover,

$$J_\varepsilon u \longrightarrow u \quad \text{in } H^s(\mathbb{R}^n,\mathbb{K}^r) \quad \text{when } \varepsilon \to 0^+.$$

If u has compact support, then $J_\varepsilon u \in C^\infty(\mathbb{R}^n,\mathbb{K}^r)$ and

$$\operatorname{supp}(J_\varepsilon u) \subseteq \{x \in \mathbb{R}^n \mid \operatorname{dist}_{\text{euc}}(x,\operatorname{supp}(u)) \le \varepsilon\}.$$

Finally, since ρ is radial, J_ε is self-adjoint with respect to the $L^2(\mathbb{R}^n,\mathbb{K}^r)$ pairing and the $H^{-s}(\mathbb{R}^n,\mathbb{K}^r)$–$H^s(\mathbb{R}^n,\mathbb{K}^r)$ duality.

Proof. The change of variable $y = \varepsilon z$ allows us to write

$$J_\varepsilon u = \int_{\mathbb{R}^n} \rho(z)\tau_{\varepsilon z} u \, dz,$$

where the integral is taken in the Hilbert space $H^s(\mathbb{R}^n,\mathbb{K}^r)$. Since translations are isometries,

$$\|J_\varepsilon u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} \le \int_{\mathbb{R}^n} \rho(z)\|\tau_{\varepsilon z} u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}\, dz = \|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}.$$

Likewise,

$$\|J_\varepsilon u - u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} \le \int_{\mathbb{R}^n} \rho(z)\|\tau_{\varepsilon z} u - u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}\, dz.$$

For each fixed z, the integrand converges to zero by Proposition 6.1.6 and is dominated by $2\rho(z)\|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}$. The Bochner dominated convergence theorem B.6.25 proves convergence in $H^s(\mathbb{R}^n,\mathbb{K}^r)$.
If u has compact support, convolution of a compactly supported distribution with $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ is smooth. The support inclusion follows directly from the fact that ρ_ε vanishes outside $\overline{B}_{\text{euc}}(0,\varepsilon)$. Finally, radiality implies $\rho_\varepsilon(x) = \rho_\varepsilon(-x)$; Fubini's theorem B.6.18, applied first to smooth functions, and density give $\langle J_\varepsilon u, v\rangle = \langle u, J_\varepsilon v\rangle$ and then the same identity by duality. □

Lemma 6.1.12. (Friedrichs commutator) Let $j \in \mathbb{Z}$ and $a \in C_c^\infty(\mathbb{R}^n)$. There is a constant $C_{a,j} > 0$, independent of $0 < \varepsilon \le 1$, such that

$$\|[a, J_\varepsilon]v\|_{H^{j+1}(\mathbb{R}^n)} \le C_{a,j}\|v\|_{H^j(\mathbb{R}^n)}$$

for every $v \in H^j(\mathbb{R}^n)$, where $[a, J_\varepsilon] := aJ_\varepsilon - J_\varepsilon a$. The same assertion holds componentwise for matrix coefficients.

Proof. Begin with $j = 0$ and fix $\ell \in \{1,\dots,n\}$. For $v \in C_c^\infty(\mathbb{R}^n)$,

$$[a, J_\varepsilon]v(x) = \int_{\mathbb{R}^n} \big(a(x) - a(x-y)\big)\rho_\varepsilon(y)v(x-y)\, dy.$$

The fundamental theorem of calculus for Banach-valued functions B.6.5, applied to $t \mapsto a(x - ty)$, gives

$$|a(x) - a(x-y)| \le \|Da\|_{L^\infty(\mathbb{R}^n)}\|y\|.$$

By Young's inequality from Theorem 4.10.1,

$$\|[a, J_\varepsilon]v\|_{L^2(\mathbb{R}^n)} \le \|Da\|_{L^\infty(\mathbb{R}^n)} \big\| \|y| \rho_\varepsilon(y) \big\|_{L^1(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} \le C\varepsilon \|Da\|_{L^\infty(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}.$$

Differentiating the identity $[a, J_\varepsilon]v = aJ_\varepsilon v - J_\varepsilon(av)$ yields

$$D_\ell[a, J_\varepsilon]v = (D_\ell a)J_\varepsilon v + \int_{\mathbb{R}^n} \big(a(x) - a(x-y)\big)(D_\ell \rho_\varepsilon)(y) v(x-y)\, dy.$$

The first term is bounded in $L^2(\mathbb{R}^n)$ by $\|D_\ell a\|_{L^\infty(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}$. For the second, use Theorem B.6.5 again and observe that

$$\int_{\mathbb{R}^n} \|y\| |D_\ell \rho_\varepsilon(y)|\, dy = \int_{\mathbb{R}^n} \|z\| |D_\ell \rho(z)|\, dz,$$

which is independent of ε. This gives

$$\|[a, J_\varepsilon]v\|_{H^1(\mathbb{R}^n)} \le C_a \|v\|_{L^2(\mathbb{R}^n)}.$$

Density extends the inequality to all of $v \in L^2(\mathbb{R}^n)$.
Proceed by induction. Suppose that for some $q \in \mathbb{N}_0$, every coefficient $b \in C_c^\infty(\mathbb{R}^n)$ satisfies

$$\|[b, J_\varepsilon]w\|_{H^{q+1}(\mathbb{R}^n)} \le C_{b,q} \|w\|_{H^q(\mathbb{R}^n)}.$$

If $v \in H^{q+1}(\mathbb{R}^n)$, the distributional identity

$$D_\ell[a, J_\varepsilon]v = [D_\ell a, J_\varepsilon]v + [a, J_\varepsilon]D_\ell v$$

and the induction hypothesis, applied to $D_\ell a$ and a, give

$$\|D_\ell[a, J_\varepsilon]v\|_{H^{q+1}(\mathbb{R}^n)} \le C \|v\|_{H^{q+1}(\mathbb{R}^n)}.$$

Together with the bound in $L^2(\mathbb{R}^n)$, this yields

$$\|[a, J_\varepsilon]v\|_{H^{q+2}(\mathbb{R}^n)} \le C \|v\|_{H^{q+1}(\mathbb{R}^n)}.$$

By induction, the assertion holds for every $j \in \mathbb{N}_0$.
Now let $j = -q$, with $q \in \mathbb{N}$. Since J_ε is self-adjoint,

$$[a, J_\varepsilon]^* = -[\overline{a}, J_\varepsilon].$$

The part already proved, with index $q-1$, states that $[\overline{a}, J_\varepsilon] \colon H^{q-1}(\mathbb{R}^n) \longrightarrow H^q(\mathbb{R}^n)$ is uniformly continuous. By duality, $[a, J_\varepsilon] \colon H^{-q}(\mathbb{R}^n) \longrightarrow H^{-q+1}(\mathbb{R}^n)$ is uniformly continuous. This completes the proof for every $j \in \mathbb{Z}$. □

Corollary 6.1.13. (Commutator of a differential operator with a mollifier) Let

$$Q = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le m}} A_\alpha(x) D^\alpha$$

be a differential system on $\mathbb{R}^n$ whose matrix coefficients belong to C_c^∞. For every $j \in \mathbb{Z}$, there is a constant $C_{Q,j} > 0$, independent of $0 < \varepsilon \le 1$, such that

$$\|[Q, J_\varepsilon]v\|_{H^{j-m+1}(\mathbb{R}^n)} \le C_{Q,j} \|v\|_{H^j(\mathbb{R}^n)}$$

for every $v \in H^j(\mathbb{R}^n, \mathbb{K}^r)$.

Proof. Constant-coefficient derivatives commute with convolution, so

$$[Q, J_\varepsilon]v = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le m}} [A_\alpha, J_\varepsilon] D^\alpha v.$$

By Proposition 6.1.6, $D^\alpha v \in H^{j-|\alpha|}(\mathbb{R}^n)$. Lemma 6.1.12 gives

$$[A_\alpha, J_\varepsilon] D^\alpha v \in H^{j-|\alpha|+1}(\mathbb{R}^n).$$

Since $|\alpha| \le m$, the inclusion $H^{j-|\alpha|+1}(\mathbb{R}^n) \hookrightarrow H^{j-m+1}(\mathbb{R}^n)$ is continuous. Summing the finitely many terms gives the stated estimate. □

6.2 Slobodeckij spaces

Bessel potentials measure regularity through a Fourier multiplier. A second approach, intrinsic to the spatial variable, compares the values of a function at two points. The power in the denominator is chosen to make the expression invariant under Euclidean scaling. This construction leads to Sobolev–Slobodeckij spaces and remains meaningful on an open subset without first extending the function outside it.

Definition 6.2.1. (Gagliardo seminorm and Slobodeckij space) Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $0 < \sigma < 1$ and $1 \le p < \infty$. For $u \in L^p(\Omega)$, define

$$[u]_{W^{\sigma,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{\|x - y\|^{n+\sigma p}} \, dx \, dy \right)^{\frac{1}{p}}.$$

The space $W^{\sigma,p}(\Omega)$ consists of functions for which this seminorm is finite and is equipped with

$$\|u\|_{W^{\sigma,p}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{W^{\sigma,p}(\Omega)}.$$

If $s = m + \sigma$, where $m \in \mathbb{N}_0$ and $0 < \sigma < 1$, set

$$W^{s,p}(\Omega) := \{ u \in W^{m,p}(\Omega) \mid D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for } |\alpha| = m \},$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{W^{m,p}(\Omega)} + \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=m}} [D^\alpha u]_{W^{\sigma,p}(\Omega)}.$$

For $s = m \in \mathbb{N}_0$, retain the usual definition of $W^{m,p}(\Omega)$.

The definition at integers is not obtained by formally setting $\sigma = 0$ or $\sigma = 1$: the corresponding integrals do not define the integer-order norm. We shall see later that this apparent discontinuity reflects a real difference between the fine indices of the Besov and Triebel–Lizorkin scales.

Proposition 6.2.2. (Increment formulation) If $u \in L^p(\mathbb{R}^n)$, $0 < \sigma < 1$, and $1 \le p < \infty$, then

$$[u]^p_{W^{\sigma,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{\|\tau_h u - u\|^p_{L^p(\mathbb{R}^n)}}{\|h\|^{n+\sigma p}}\, dh, \qquad \tau_h u(x) := u(x+h).$$

In particular, the functional on the right is translation invariant.

Proof. The integrand is nonnegative. Theorem B.6.17 and the change of variables $y = x + h$ give

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x+h)-u(x)|^p}{\|h\|^{n+\sigma p}}\, dx\, dh = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{\|y-x\|^{n+\sigma p}}\, dy\, dx.$$

Neither integral needs to be assumed finite beforehand. □

Theorem 6.2.3. (Functional structure and density) Let $s > 0$, $s \notin \mathbb{N}$, and $1 \le p < \infty$.

(a) $W^{s,p}(\Omega)$ is a Banach space for every open subset $\Omega \subseteq \mathbb{R}^n$.

(b) If $1 < p < \infty$, then $W^{s,p}(\Omega)$ is reflexive; for every $1 \le p < \infty$, it is separable.

(c) $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.

Proof. First treat $0 < s < 1$. Let

$$d\mu_s(x,y) := \|x-y\|^{-n-sp}\, dx\, dy$$

on $\Omega\times\Omega \setminus \{x = y\}$. The map

$$\mathcal{J}\colon u \longmapsto \left(u, \frac{u(x)-u(y)}{\|x-y\|^{\frac{n}{p}+s}}\right)$$

identifies $W^{s,p}(\Omega)$ isometrically with a subspace of

$$L^p(\Omega) \times L^p(\Omega\times\Omega, dx\, dy)$$

when the product carries the sum norm. If instead the product carries the norm ℓ^p, and we set $a := \|u\|_{L^p(\Omega)}$ and $b := [u]_{W^{s,p}(\Omega)}$, then

$$(a^p + b^p)^{\frac{1}{p}} \le a + b \le 2^{1-\frac{1}{p}}(a^p + b^p)^{\frac{1}{p}}.$$

This subspace is closed. Indeed, if $u_j \to u$ in $L^p(\Omega)$ and the quotients converge to v in $L^p(\Omega\times\Omega)$, a subse-

quence converges pointwise almost everywhere in both spaces. Outside a null set, we obtain

$$v(x,y) = \frac{u(x)-u(y)}{\|x-y\|^{\frac{n}{p}+s}}.$$

Completeness of the product proves (a). The same argument, together with reflexivity and separability of the L^p spaces established in Theorem B.7.7, proves (b). For $s = m+\sigma$, apply the argument to weak derivatives of order m and use the fact that the operator $u \mapsto (D^\alpha u)_{|\alpha|\le m}$ has closed graph in the product of $L^p(\Omega)$ spaces.

We prove (c). Choose $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\eta = 1$ on $B_{\mathrm{euc}}(0,1)$ and set $\eta_R(x) = \eta(\frac{x}{R})$. For $0 < s < 1$,

$$\begin{aligned}&|(1-\eta_R(x))u(x) - (1-\eta_R(y))u(y)|\\ &\qquad \le |1-\eta_R(x)|\,|u(x)-u(y)| + |u(y)|\,|\eta_R(x)-\eta_R(y)|.\end{aligned}$$

The first term converges to zero in the weighted product measure by Theorem B.6.14. For the second, use

$$|\eta_R(x)-\eta_R(y)| \le C\min\left\{1, \frac{\|x-y\|}{R}\right\}.$$

After the substitution $h = x-y$,

$$\int_{\mathbb{R}^n} \frac{\min\left\{1, \frac{\|h\|}{R}\right\}^p}{\|h\|^{n+sp}}\,dh = C_{n,s,p}R^{-sp};$$

so the second term tends to zero. We conclude that $\eta_R u \to u$ in $W^{s,p}(\mathbb{R}^n)$.

Now let ρ be a standard mollifier and $u_\varepsilon = \rho_\varepsilon * u$. Theorem B.8.2 and translation invariance give

$$\|u_\varepsilon - u\|_{W^{s,p}(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} \rho(z)\|\tau_{\varepsilon z}u - u\|_{W^{s,p}(\mathbb{R}^n)}\,dz.$$

Translations are continuous in $W^{s,p}(\mathbb{R}^n)$: this follows by applying Proposition B.8.17 both to $u \in L^p(\mathbb{R}^n)$ and to the two-variable function representing its difference quotient in $L^p(\mathbb{R}^{2n})$. Theorem B.6.14 implies $u_\varepsilon \to u$ in $W^{s,p}(\mathbb{R}^n)$. Apply the cutoff first, then regularization; the result belongs to $C_c^\infty(\mathbb{R}^n)$. For $s = m+\sigma$, $0 < \sigma < 1$, also expand the terms produced by the cutoff. If $|\alpha| = m$, the Leibniz rule gives

$$D^\alpha(\eta_R u - u) = (\eta_R - 1)D^\alpha u + \sum_{\substack{\beta\in\mathbb{N}_0^n\\ \beta\le\alpha,\ \beta\ne\alpha}} \binom{\alpha}{\beta} D^{\alpha-\beta}\eta_R\, D^\beta u.$$

The first term converges to zero in $W^{\sigma,p}$ by what has already been proved. For the remaining terms, $D^\beta u \in W^{1,p}$. The increment lemma, whose proof uses only integer-order Sobolev properties, gives

$$\|w\|_{W^{\sigma,p}(\mathbb{R}^n)} \le C\|w\|_{W^{1,p}(\mathbb{R}^n)}, \qquad w \in W^{1,p}(\mathbb{R}^n):$$

in the increment integral, use $\|\tau_h w - w\|_{L^p(\mathbb{R}^n)} \le \|h\|\|\nabla w\|_{L^p(\mathbb{R}^n)}$ for $\|h\| < 1$, and $2\|w\|_{L^p(\mathbb{R}^n)}$ on the complement. If $\gamma = \alpha - \beta \ne 0$, both $\|D^\gamma\eta_R\|_{L^\infty(\mathbb{R}^n)}$ and $\|D(D^\gamma\eta_R)\|_{L^\infty(\mathbb{R}^n)}$ tend to zero. The formula for the difference of a product and these two bounds imply

$$\|D^\gamma\eta_R\, D^\beta u\|_{W^{\sigma,p}(\mathbb{R}^n)} \le C\left(\|D^\gamma\eta_R\|_{L^\infty(\mathbb{R}^n)} + \|D(D^\gamma\eta_R)\|_{L^\infty(\mathbb{R}^n)}\right)\|D^\beta u\|_{W^{\sigma,p}(\mathbb{R}^n)} \longrightarrow 0.$$

Lower-order derivatives are controlled by the same Leibniz rule in L^p. This justifies the cutoff in the full norm. Regularization commutes with each weak derivative, and the continuity of translations already proved applies to the quotients of derivatives of order m. Thus regularization after the cutoff establishes density also at higher orders. □

6.2.1 The Slobodeckij interpolation identity

The Gagliardo seminorm is not an isolated choice: it is exactly the quantity measuring the position of u between $L^p(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$. This interpretation explains both the exponent $n+sp$ and many subsequent properties.

Lemma 6.2.4. (Increments of a Sobolev function) If $w \in W^{1,p}(\mathbb{R}^n)$ and $1 \le p < \infty$, then

$$\|\tau_h w - w\|_{L^p(\mathbb{R}^n)} \le |h| \, \|\nabla w\|_{L^p(\mathbb{R}^n)} \qquad \text{for every } h \in \mathbb{R}^n.$$

Proof. For $w \in C_c^\infty(\mathbb{R}^n)$, Theorem B.6.5 gives

$$w(x+h) - w(x) = \int_0^1 \nabla w(x+th) \cdot h \, dt.$$

Theorem B.8.2 and translation invariance of the $L^p(\mathbb{R}^n)$ norm yield the inequality. Density of $C_c^\infty(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$, Proposition 3.2.10, allows passage to every $w \in W^{1,p}(\mathbb{R}^n)$. □

Theorem 6.2.5. (Slobodeckij as a real interpolation space) Let $0 < s < 1$ and $1 \le p < \infty$. Then

$$\big(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n)\big)_{s,p} = W^{s,p}(\mathbb{R}^n)$$

and there are constants $c_{n,s,p}, C_{n,s,p} > 0$ such that

$$c_{n,s,p} \|u\|_{W^{s,p}(\mathbb{R}^n)} \le \|u\|_{(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{s,p}} \le C_{n,s,p} \|u\|_{W^{s,p}(\mathbb{R}^n)}$$

for every $u \in W^{s,p}(\mathbb{R}^n)$.

Proof. Write $K(t,u)$ for the functional of the pair $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))$. If $u = v + w$, with $v \in L^p(\mathbb{R}^n)$ and $w \in W^{1,p}(\mathbb{R}^n)$, Lemma 6.2.4 and the elementary inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ give

$$\|\tau_h u - u\|_{L^p(\mathbb{R}^n)}^p \le C_p \big(\|v\|_{L^p(\mathbb{R}^n)} + |h| \|w\|_{W^{1,p}(\mathbb{R}^n)} \big)^p.$$

Taking the infimum and using polar coordinates,

$$[u]_{W^{s,p}(\mathbb{R}^n)}^p \le C \int_{\mathbb{R}^n} \frac{K(|h|,u)^p}{|h|^{n+sp}} \, dh = C |\mathbb{S}^{n-1}| \int_0^\infty \big(t^{-s} K(t,u) \big)^p \, \frac{dt}{t}.$$

The embedding of the interpolated space into $L^p(\mathbb{R}^n) + W^{1,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, proved in Theorem 5.2.4, also controls the $L^p(\mathbb{R}^n)$ norm. This proves one inclusion.

For the converse, take a nonnegative mollifier ρ supported in the unit ball and set $u_t = \rho_t * u$. The decom-

position

$$u = (u - u_t) + u_t$$

is admissible for $K(t,u)$. Proposition B.7.9, with respect to the probability measure $\rho_t(h)\,dh$, gives

$$\|u - u_t\|^p_{L^p(\mathbb{R}^n)} \le Ct^{-n} \int_{\{h\in\mathbb{R}^n \,|\, \|h\|<t\}} \|\tau_h u - u\|^p_{L^p(\mathbb{R}^n)}\,dh.$$

Since $\displaystyle\int_{\mathbb{R}^n} D_i\rho_t(h)\,dh = 0$,

$$D_i u_t(x) = \int_{\mathbb{R}^n} D_i\rho_t(h)\big(u(x-h) - u(x)\big)\,dh,$$

and another application of Proposition B.7.9 gives

$$\|\nabla u_t\|^p_{L^p(\mathbb{R}^n)} \le Ct^{-n-p} \int_{\{h\in\mathbb{R}^n \,|\, \|h\|<t\}} \|\tau_h u - u\|^p_{L^p(\mathbb{R}^n)}\,dh.$$

Moreover, Theorem 4.10.1 gives $\|u_t\|_{L^p(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)}$. Consequently,

$$K(t,u)^p \le C\left[t^{-n}\int_{\{\|h\|<t\}} \|\tau_h u - u\|^p_{L^p(\mathbb{R}^n)}\,dh + t^p\|u\|^p_{L^p(\mathbb{R}^n)}\right]$$

for $0 < t < 1$. Multiply by t^{-sp}, integrate with respect to dt/t, and apply Theorem B.6.17. The first term gives

$$\int_{\{\|h\|<1\}} \|\tau_h u - u\|^p_{L^p(\mathbb{R}^n)} \left(\int_{\|h\|}^1 t^{-n-sp-1}\,dt\right) dh \le C[u]^p_{W^{s,p}(\mathbb{R}^n)}.$$

The second term is $C\|u\|^p_{L^p(\mathbb{R}^n)} \displaystyle\int_0^1 t^{(1-s)p}\,dt/t$. Apply the two quantified inequalities for the truncated norm in Theorem 5.2.4 and absorb their constants, which depend only on s and p; this gives

$$\|u\|_{(L^p(\mathbb{R}^n),W^{1,p}(\mathbb{R}^n))_{s,p}} \le C\|u\|_{W^{s,p}(\mathbb{R}^n)}.$$

□

Corollary 6.2.6. (Interpolation between consecutive integer orders) Let $m \in \mathbb{N}_0$, $0 < s < 1$, and $1 < p < \infty$. Then

$$\big(W^{m,p}(\mathbb{R}^n), W^{m+1,p}(\mathbb{R}^n)\big)_{s,p} = W^{m+s,p}(\mathbb{R}^n)$$

as sets, and there are constants $c_{n,m,s,p}, C_{n,m,s,p} > 0$ such that

$$c_{n,m,s,p}\|u\|_{W^{m+s,p}(\mathbb{R}^n)} \le \|u\|_{(W^{m,p}(\mathbb{R}^n),W^{m+1,p}(\mathbb{R}^n))_{s,p}} \le C_{n,m,s,p}\|u\|_{W^{m+s,p}(\mathbb{R}^n)}$$

for every $u \in W^{m+s,p}(\mathbb{R}^n)$.

Proof. For $m = 0$, this is the preceding theorem. Assume $m \ge 1$. If $u = v + w$, with $v \in W^{m,p}$ and $w \in W^{m+1,p}$, Lemma 6.2.4 gives, for $\alpha \in \mathbb{N}_0^n$, $|\alpha| = m$,

$$\|\tau_h D^\alpha u - D^\alpha u\|_{L^p(\mathbb{R}^n)} \le 2\|v\|_{W^{m,p}(\mathbb{R}^n)} + C\|h\|\|w\|_{W^{m+1,p}(\mathbb{R}^n)}.$$

Take the infimum over these decompositions and integrate with weight $\|h\|^{-n-sp}$ exactly as in the proof for $m=0$. This controls all seminorms of order $m+s$. Moreover, $W^{m+1,p}\hookrightarrow W^{m,p}$ implies

$$K(1,u;W^{m,p},W^{m+1,p})\ge c\|u\|_{W^{m,p}(\mathbb{R}^n)}.$$

Monotonicity of K and the portion of the interpolation integral corresponding to $1\le t\le 2$ also control this norm. We obtain the first continuous inclusion.

For the converse, fix the mollifier ρ from the preceding proof, supported in the unit ball, and write $u=(u-u_t)+u_t$, where $u_t=\rho_t*u$. If $|\alpha|<m$, the increment lemma gives

$$\|D^\alpha(u-u_t)\|_{L^p(\mathbb{R}^n)}\le Ct\|u\|_{W^{m,p}(\mathbb{R}^n)},\qquad 0<t<1.$$

For $|\alpha|=m$, Minkowski's and Hölder's inequalities give

$$\|D^\alpha(u-u_t)\|_{L^p(\mathbb{R}^n)}^p\le Ct^{-n}\int_{\|h\|<t}\|\tau_hD^\alpha u-D^\alpha u\|_{L^p(\mathbb{R}^n)}^p\,dh.$$

Since $\int_{\mathbb{R}^n}D_\ell\rho_t=0$, also

$$t^p\|D_\ell D^\alpha u_t\|_{L^p(\mathbb{R}^n)}^p\le Ct^{-n}\int_{\|h\|<t}\|\tau_hD^\alpha u-D^\alpha u\|_{L^p(\mathbb{R}^n)}^p\,dh,\qquad \ell\in\{1,\dots,n\}.$$

Derivatives of u_t of order at most m are controlled by Young's inequality, so

$$K(t,u;W^{m,p},W^{m+1,p})^p\le Ct^p\|u\|_{W^{m,p}(\mathbb{R}^n)}^p+Ct^{-n}\int_{\|h\|<t}\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|=m}}\|\tau_hD^\alpha u-D^\alpha u\|_{L^p(\mathbb{R}^n)}^p\,dh.$$

Multiply by t^{-sp} and integrate with dt/t over $0<t<1$. The first term is integrable because $s<1$. For the second, Tonelli's theorem and

$$\int_{\|h\|}^1 t^{-n-sp-1}\,dt\le\frac{\|h\|^{-n-sp}}{n+sp},\qquad 0<\|h\|<1,$$

yield the Gagliardo seminorms of derivatives of order m. For $t\ge1$, use the decomposition $u=u+0$; the remaining integral is bounded by $\|u\|_{W^{m,p}(\mathbb{R}^n)}^p/(sp)$. This proves the second inclusion.

The integer-order norms used are the ℓ^p norms of this book. If a sum of norms appears in an estimate, the comparison is explicit: for $r\in\{m,m+1\}$, with $N_r=\#\{\alpha\in\mathbb{N}_0^n\mid|\alpha|\le r\}$,

$$\left(\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|\le r}}a_\alpha^p\right)^{1/p}\le\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|\le r}}a_\alpha\le N_r^{1-1/p}\left(\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|\le r}}a_\alpha^p\right)^{1/p}.$$

The constants therefore depend only on n,m,s,p and the fixed mollifier. The same argument holds for $p=1$. □

6.3 Besov and Triebel–Lizorkin spaces

The Littlewood–Paley decomposition from Chapter 4 allows a distribution to be studied by separating its content into frequency scales. Each block $\Delta_j u$ measures the part of u lying roughly at frequencies of size 2^j, while the block $\Delta_0 u$ collects low frequencies. Once this separation is made, there are two natural ways to combine the information. In the first, one takes the spatial norm of each block and then sums over scales; in the second, one first combines all blocks pointwise and only then takes the spatial norm. The first procedure leads to Besov spaces, the second to Triebel–Lizorkin spaces.

The difference between these two orders of combination is not merely notational. When the indices agree, Tonelli's theorem permits interchange of sum and integral; otherwise, the position of the sequence norm generally changes the resulting space. This distinction will later explain why Sobolev–Slobodeckij spaces are identified with $B^s_{p,p}(\mathbb{R}^n)$, while Bessel potentials are identified with $F^s_{p,2}(\mathbb{R}^n)$. This organization of the scales is the classical Littlewood–Paley description; see, for example, [146, Chapters 1–2] and [126, Chapter 2].

We use the resolution $(\varphi_j)_{j\in\mathbb{N}_0}$ and the blocks Δ_j from Proposition 4.12.1 and Definition 4.12.2. Also choose real radial functions $\widetilde{\varphi}_j \in C_c^\infty(\mathbb{R}^n)$, $j \in \mathbb{N}_0$, equal to one on a neighborhood of $\operatorname{supp}(\varphi_j)$ and having uniform dyadic supports. Write

$$\widetilde{\Delta}_j := \mathcal{F}^{-1}\widetilde{\varphi}_j\mathcal{F}, \qquad j \in \mathbb{N}_0.$$

Then $\widetilde{\Delta}_j\Delta_j = \Delta_j$ for every $j \in \mathbb{N}_0$.

6.3.1 Multiplier tools

The dyadic construction uses two estimates: continuity of certain Fourier multipliers and control of a function by its frequency blocks. Their constants will need to be uniform in the block index.

Theorem 6.3.1. (Mikhlin multipliers) Let $1 < p < \infty$ and let $N \geq n+2$ be an integer. Suppose that $m \in C^N(\mathbb{R}^n \setminus \{0\})$ satisfies

$$M_N(m) := \max_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\leq N}} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \|\xi\|^{|\alpha|}|D^\alpha m(\xi)| < \infty.$$

Then the operator initially defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$T_m f := \mathcal{F}^{-1}(m\widehat{f})$$

extends uniquely to a continuous operator on $L^p(\mathbb{R}^n)$, and

$$\|T_m f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p,N} M_N(m) \|f\|_{L^p(\mathbb{R}^n)}.$$

The same bound holds for a family of symbols with uniform Mikhlin constants.

Proof. If $M_N(m) = 0$, the symbol vanishes. Otherwise, first divide m by $M_N(m)$ and restore this factor at the end. We may thus assume $M_N(m) = 1$. Let $\vartheta \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ satisfy $\sum_{j\in\mathbb{Z}} \vartheta(2^{-j}\xi) = 1$ for $\xi \neq 0$, and set

$m_j(\xi) = m(\xi)\vartheta(2^{-j}\xi)$. The substitution $\xi = 2^j\zeta$ and the hypotheses show that

$$\max_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|\le N}} \|D^\alpha[m(2^j\cdot)\vartheta]\|_{L^\infty(\mathbb{R}^n)} \le CM_N(m)$$

uniformly in j. Integration by parts in the inverse transform gives, for the kernel $K_j = \mathcal{F}^{-1}m_j$,

$$|K_j(x)| \le CM_N(m)2^{jn}(1+2^j\|x\|)^{-N}$$

and applying the same computation to a derivative gives

$$|\nabla K_j(x)| \le CM_N(m)2^{j(n+1)}(1+2^j\|x\|)^{-N}.$$

The sum $K = \sum_{j\in\mathbb{Z}} K_j$ converges locally uniformly on $\mathbb{R}^n\setminus\{0\}$. Separate indices according to whether $2^j\|x\| \le 1$ or $2^j\|x\| > 1$, and sum the two geometric series; this gives

$$|K(x)| \le CM_N(m)\|x\|^{-n}, \qquad |K(x-h)-K(x)| \le CM_N(m)\frac{\|h\|}{\|x\|^{n+1}}$$

when $\|h\| \le \frac{1}{2}\|x\|$. The operator is bounded on $L^2(\mathbb{R}^n)$ by Theorem 4.11.10, since m is bounded. With the unitary Fourier normalization, the convolution kernel is $(2\pi)^{-n/2}K$. This representation is first verified outside the support of a compactly supported function, where the kernel series converges absolutely; the partial sums of the symbols also converge in L^2 by Plancherel's theorem.

Let $f \in L^1\cap L^2$, let $\lambda > 0$, and apply the Calderón–Zygmund decomposition from Theorem B.8.15:

$$f = g + \sum_{Q\in\mathcal{Q}} b_Q, \qquad \|g\|_{L^2(\mathbb{R}^n)}^2 \le C\lambda\|f\|_{L^1(\mathbb{R}^n)}, \qquad \sum_{Q\in\mathcal{Q}} \|b_Q\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{L^1(\mathbb{R}^n)}.$$

Each b_Q has integral zero and support in Q, and $\sum_{Q\in\mathcal{Q}} |Q| \le C\|f\|_{L^1(\mathbb{R}^n)}/\lambda$. Write $Q^* = 4\sqrt{n}\,Q$, with the same center c_Q. If $x \notin Q^*$ and $y \in Q$, then $\|y-c_Q\| \le \|x-c_Q\|/2$. By cancellation,

$$T_m b_Q(x) = (2\pi)^{-n/2}\int_Q [K(x-y)-K(x-c_Q)]b_Q(y)\,dy,$$

and the kernel estimate implies

$$|T_m b_Q(x)| \le C\frac{\ell(Q)}{\|x-c_Q\|^{n+1}}\|b_Q\|_{L^1(\mathbb{R}^n)}.$$

The integral of $\ell(Q)\|x-c_Q\|^{-n-1}$ outside Q^* is bounded by a dimensional constant. Thus, if $E = \bigcup_{Q\in\mathcal{Q}} Q^*$,

$$|E| \le C\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}, \qquad \int_{\mathbb{R}^n\setminus E}\sum_{Q\in\mathcal{Q}} |T_m b_Q| \le C\|f\|_{L^1(\mathbb{R}^n)}.$$

Convergence of the decomposition in L^2 identifies this sum outside E with $T_m(f-g)$. Chebyshev's inequality

and the L^2 bound for the good part give

$$|\{x \in \mathbb{R}^n \mid |T_m f(x)| > \lambda\}| \le |E| + \frac{4}{\lambda^2}\|T_m g\|_{L^2(\mathbb{R}^n)}^2 + \frac{2}{\lambda}\int_{\mathbb{R}^n \setminus E} |T_m(f-g)|$$
$$\le C\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}.$$

Theorem B.8.9 interpolates this weak $(1,1)$ bound with the strong $(2,2)$ bound, yielding continuity on L^p for $1 < p \le 2$. If $p > 2$, apply the preceding argument to the symbol $\overline{m}$ on $L^{p'}$ and use the adjoint identity and duality of L^p. Density of $\mathcal{S}$ in L^p gives a unique continuous extension. Finally, multiplying again by $M_N(m)$ provides the linear bound in the statement and its uniformity. □

Theorem 6.3.2. (Littlewood–Paley square function) Let $1 < p < \infty$. There are constants $c_p, C_p > 0$ such that for every $f \in L^p(\mathbb{R}^n)$,

$$c_p\|f\|_{L^p(\mathbb{R}^n)} \le \left\| \left(\sum_{j \in \mathbb{N}_0} |\Delta_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We first bound the square function by the norm of f. Let $J \subseteq \mathbb{N}_0$ be finite. For each choice of signs $(\varepsilon_j)_{j \in J} \in \{-1,1\}^J$, the symbol

$$m_\varepsilon(\xi) := \sum_{j \in J} \varepsilon_j \varphi_j(\xi)$$

satisfies the hypotheses of Theorem 6.3.1 with constants independent of J and the signs: finite interaction of the supports ensures that at each $\xi \neq 0$ only a uniformly bounded number of terms occur, and after scaling, the derivatives of φ_j have the required bounds. Therefore,

$$\left\| \sum_{j \in J} \varepsilon_j \Delta_j f \right\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}.$$

Take $\varepsilon_j = r_j(t)$, where $(r_j)_{j \in \mathbb{N}_0}$ are the Rademacher functions, integrate in $t \in [0,1]$, and use Khintchine's theorem B.8.8 pointwise in x. Theorem B.6.17 allows the integrals to be interchanged and gives

$$\left\| \left(\sum_{j \in J} |\Delta_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}.$$

Letting J increase to $\mathbb{N}_0$ and applying monotone convergence yields the full upper bound.
For the reverse inequality, first take $f, g \in \mathcal{S}(\mathbb{R}^n)$. Since $\widetilde{\Delta}_j \Delta_j = \Delta_j$ and the dyadic resolution reconstructs the identity in $\mathcal{S}'(\mathbb{R}^n)$,

$$\langle f, g \rangle = \sum_{j \in \mathbb{N}_0} \langle \Delta_j f, \widetilde{\Delta}_j g \rangle.$$

Finite interaction of the supports allows the enlarged blocks to be chosen so that the series converges abso-

lutely for Schwartz functions. Cauchy–Schwarz in $\ell^2(\mathbb{N}_0)$ gives

$$|\langle f, g\rangle| \le \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{N}_0} |\Delta_j f(x)|^2\right)^{\frac{1}{2}} \left(\sum_{j\in\mathbb{N}_0} |\widetilde{\Delta}_j g(x)|^2\right)^{\frac{1}{2}} dx.$$

Apply Hölder's inequality with exponents p and p' and the square-function bound already proved, now for the enlarged resolution and $g \in L^{p'}(\mathbb{R}^n)$. Thus,

$$|\langle f, g\rangle| \le C \left\| \left(\sum_{j\in\mathbb{N}_0} |\Delta_j f|^2\right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Duality from Theorem B.7.7 gives the reverse inequality for $f \in \mathcal{S}(\mathbb{R}^n)$. Finally, density of $\mathcal{S}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, Proposition 4.8.5, and continuity of the square function extend the equivalence to all of $L^p(\mathbb{R}^n)$. □

For $a > 0$, we shall use the abbreviation

$$\mathcal{M}_a f := \left[\mathcal{M}(|f|^a)\right]^{\frac{1}{a}}.$$

Lemma 6.3.3. (Dyadic multipliers on sequences) Let $0 < p < \infty$, $0 < q \le \infty$, and $0 < a < \min\{p, q\}$, with the convention that q is omitted from this minimum if $q = \infty$. Suppose that $(f_j)_{j\in\mathbb{N}_0} \subseteq \mathcal{S}'(\mathbb{R}^n)$ is a family whose elements are represented by functions and satisfy, for every $j \in \mathbb{N}_0$,

$$\operatorname{supp}(\widehat{f_j}) \subseteq \{\xi \in \mathbb{R}^n \mid \|\xi\| \le A2^j\}.$$

If $T_j f_j = K_j * f_j$, where $K_j(x) = 2^{jn} K^{(j)}(2^j x)$ and

$$\sup_{j\in\mathbb{N}_0} \sup_{x\in\mathbb{R}^n} (1 + \|x\|)^{N+n+1} |K^{(j)}(x)| < \infty$$

for some $N > \frac{n}{a}$, then

$$\left\| \left(\sum_{j\in\mathbb{N}_0} |T_j f_j|^q\right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \le C \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^q\right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)},$$

with the usual suprema when $q = \infty$.

Proof. Lemma B.8.16 gives $|T_j f_j| \le C\mathcal{M}_a f_j$ pointwise. If $q < \infty$, apply Theorem B.8.14 to the sequence $(|f_j|^a)_{j\in\mathbb{N}_0}$ with exponents $\frac{p}{a}$ and $\frac{q}{a}$, both greater than one, and raise the result to the power $\frac{1}{a}$. If $q = \infty$, use

$$\sup_{j\in\mathbb{N}_0} \mathcal{M}(|f_j|^a) \le \mathcal{M}\left(\sup_{j\in\mathbb{N}_0} |f_j|^a\right)$$

and continuity of $\mathcal{M}$ on $L^{\frac{p}{a}}(\mathbb{R}^n)$ from Theorem B.8.11. □

Definition 6.3.4. (Besov and Triebel–Lizorkin spaces) Let $s \in \mathbb{R}$ and $1 \le q \le \infty$. In this section, we work in the Banach range of these scales; classical theory extends the definitions to more general quasi-Banach

indices, but these endpoints will not be needed for later results.

(a) If $1 \le p \le \infty$, define

$$B^s_{p,q}(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \|u\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} := \left\| \left(2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^n)} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^q(\mathbb{N}_0)}.$$

(b) If $1 \le p < \infty$, define

$$F^s_{p,q}(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \|u\|_{F^s_{p,q}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|u\|_{F^s_{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{j \in \mathbb{N}_0} 2^{jsq} |\Delta_j u|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)},$$

and, if $q = \infty$,

$$\|u\|_{F^s_{p,\infty}(\mathbb{R}^n)} := \left\| \sup_{j \in \mathbb{N}_0} 2^{js} |\Delta_j u| \right\|_{L^p(\mathbb{R}^n)}.$$

Definition 6.3.5. (Retraction and coretraction) Let X and Y be Banach spaces. A continuous linear operator $R\colon X \longrightarrow Y$ is a *retraction* if there is a continuous linear operator $S\colon Y \longrightarrow X$ such that

$$RS = I_Y.$$

In that case, S is called a *coretraction* of R or a continuous right inverse. In particular, every retraction is surjective and every coretraction is injective.

We also adopt the convention $F^s_{\infty,\infty}(\mathbb{R}^n) := B^s_{\infty,\infty}(\mathbb{R}^n)$. The usual scale $F^s_{\infty,q}$, with $q < \infty$, is defined through local averages and will not be used in this chapter; see [146, Section 1.5.2].

If $p = q < \infty$, Theorem B.6.17 directly shows that

$$B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n)$$

with the same norm for a fixed resolution. For distinct indices, interchanging $L^p(\mathbb{R}^n)$ and $\ell^q(\mathbb{N}_0)$ generally changes the space.

Reconstruction of a distribution requires retaining information about the frequencies of its summands. The next lemma expresses this condition and also allows treatment of the endpoint $p = 1$.

Lemma 6.3.6. (Synthesis of blocks with controlled spectral support) Let $s \in \mathbb{R}$, $1 \le q \le \infty$, and $0 < c < C$. Suppose that $v_j \in L^p(\mathbb{R}^n)$, $j \in \mathbb{N}_0$, and

$$\operatorname{supp} \widehat{v}_0 \subseteq \{\xi \in \mathbb{R}^n \mid \|\xi\| \le C\}, \qquad \operatorname{supp} \widehat{v}_j \subseteq \{\xi \in \mathbb{R}^n \mid c2^j \le \|\xi\| \le C2^j\} \quad (j \in \mathbb{N}).$$

If $1 \le p \le \infty$, then

$$\left\| \sum_{j \in \mathbb{N}_0} v_j \right\|_{B^s_{p,q}(\mathbb{R}^n)} \le C_1 \left\| \left(2^{js} \|v_j\|_{L^p(\mathbb{R}^n)}\right)_{j \in \mathbb{N}_0} \right\|_{\ell^q}.$$

If $1 \le p < \infty$, also

$$\left\| \sum_{j \in \mathbb{N}_0} v_j \right\|_{F^s_{p,q}(\mathbb{R}^n)} \le C_2 \left\| \left(2^{js} v_j\right)_{j \in \mathbb{N}_0} \right\|_{L^p(\mathbb{R}^n;\ell^q)}.$$

In each assertion, finiteness of the right-hand side ensures convergence of the series in $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Choose multipliers P_j equal to one on the indicated supports, supported on slightly larger annuli when $j \ge 1$. For $\zeta \in \mathcal{S}(\mathbb{R}^n)$ and any $N > 0$,

$$\|P_j \zeta\|_{L^{p'}(\mathbb{R}^n)} \le C_N 2^{-jN} \mathfrak{p}_N(\zeta), \qquad j \in \mathbb{N}_0,$$

where $\mathfrak{p}_N$ is a finite sum of Schwartz seminorms. To obtain this bound, differentiate the product of the symbol of P_j and $\widehat{\zeta}$, use decay of all derivatives of $\widehat{\zeta}$ on the annulus, and integrate by parts in the inversion formula. The number of integrations is chosen greater than n; this simultaneously controls the uniform norm and the integral norm of the result, and the remaining norms follow from Hölder's inequality.
Each sequence norm in the statement dominates $2^{js}\|v_j\|_{L^p(\mathbb{R}^n)}$. Since $\langle v_j, \zeta \rangle = \langle v_j, P_j \zeta \rangle$, Hölder's inequality gives

$$|\langle v_j, \zeta \rangle| \le C_N 2^{-j(N+s)} \mathfrak{p}_N(\zeta) \mathcal{N},$$

where $\mathcal{N}$ is the corresponding right-hand side. Choosing $N > |s|$ proves absolute convergence of the pairing and its continuity on ζ.
The spectral supports imply $\Delta_k v_j = 0$ if $|k - j| > N_0$, for a fixed integer N_0. Young's inequality gives

$$2^{ks} \left\| \Delta_k \sum_{j \in \mathbb{N}_0} v_j \right\|_{L^p(\mathbb{R}^n)} \le C \sum_{\substack{j \in \mathbb{N}_0 \\ |j-k| \le N_0}} 2^{js} \|v_j\|_{L^p(\mathbb{R}^n)}.$$

Proposition B.8.4 proves the Besov bound. For Triebel–Lizorkin, choose $0 < a < 1$, so that $a < p$ and, if $q < \infty$, also $a < q$. Lemma 6.3.3, applied to the blocks v_j, allows the $L^p(\ell^q)$ norm to be taken in the same finite sum of shifts. This gives the second bound. Apply the estimates first to finite sums; distributional convergence identifies each block of the limit, and Fatou's lemma allows passage to the series. For $q = \infty$, use the supremum over $j \in \mathbb{N}_0$. □

Theorem 6.3.7. (Independence of the dyadic resolution) Let $\|\cdot\|_{A,\Delta}$ and $\|\cdot\|_{A,\nabla}$ be the norms constructed from two dyadic resolutions, where $A = B^s_{p,q}$ or $A = F^s_{p,q}$, with the indices in Definition 6.3.4. There are $c_A, C_A > 0$, independent of u, such that

$$c_A \|u\|_{A,\Delta} \le \|u\|_{A,\nabla} \le C_A \|u\|_{A,\Delta}.$$

These spaces are Banach, and their inclusions into $\mathcal{S}'(\mathbb{R}^n)$ are continuous.

Proof. Inserting the enlarged blocks gives

$$\nabla_k u = \sum_{\substack{j\in\mathbb{N}_0\\ |j-k|\le N_0}} \nabla_k \widetilde{\Delta}_j \Delta_j u, \qquad k \in \mathbb{N}_0.$$

The kernels have uniformly bounded integral norms. Young's inequalities for convolutions and sequences prove the Besov bound. For the Triebel–Lizorkin norm, Lemma 6.3.3 applies to $f_j = \Delta_j u$, whose spectral supports are controlled. The sum contains only $2N_0+1$ index shifts and therefore gives the required bound. Interchanging the resolutions proves equivalence.

For completeness, take the sequence spaces

$$b^s_{p,q} := \left\{ (v_j)_{j\in\mathbb{N}_0} \mid \left\| \left(2^{js}\|v_j\|_{L^p(\mathbb{R}^n)}\right)_{j\in\mathbb{N}_0} \right\|_{\ell^q} < \infty \right\},$$

$$f^s_{p,q} := \left\{ (v_j)_{j\in\mathbb{N}_0} \mid \left\| \left(2^{js} v_j\right)_{j\in\mathbb{N}_0} \right\|_{L^p(\mathbb{R}^n;\ell^q(\mathbb{N}_0))} < \infty \right\}.$$

In $L^p(\ell^\infty)$, we mean sequences of measurable functions, identified coordinatewise almost everywhere, with norm $\| \sup_{j\in\mathbb{N}_0} |v_j| \|_{L^p(\mathbb{R}^n)}$. This is a mixed norm; the map taking values in ℓ^∞ need not be strongly measurable in the Bochner sense. Completeness of both sequence spaces follows directly: from a Cauchy sequence, extract a subsequence whose differences have summable norms. The sum of the absolute values of these differences is finite almost everywhere in the corresponding sequence norm. This gives a coordinatewise measurable limit, and the summable-tail inequality proves convergence in norm. For the remaining indices, this agrees with the usual completeness of Lebesgue spaces and Banach sums. In each space, retain only the sequences whose Fourier supports lie in the annuli of the preceding lemma, chosen to contain the supports of φ_j. This subspace is closed: convergence in either norm implies convergence of each coordinate in L^p and hence in $\mathcal{S}'$; a distribution vanishing outside a fixed closed set retains this property in the limit.

On this subspace, define

$$R(v_j)_{j\in\mathbb{N}_0} := \sum_{j\in\mathbb{N}_0} v_j, \qquad Su := (\Delta_j u)_{j\in\mathbb{N}_0}.$$

The lemma proves continuity of R; the definitions of the norms prove continuity of S. Dyadic reconstruction gives $RS = I$. Thus SR is a continuous projection from the sequence subspace onto $S(A)$. Its image is closed and Banach, and S identifies A isometrically with it. The pairing estimate obtained in the lemma, applied to $v_j = \Delta_j u$, also proves continuity of $A \hookrightarrow \mathcal{S}'$. □

For interpolation of Bessel spaces, it will be useful to have synthesis defined on sequences without spectral restrictions.

Lemma 6.3.8. (Synthesis on the full sequence space) The operator

$$\widetilde{R}(v_j)_{j\in\mathbb{N}_0} := \sum_{j\in\mathbb{N}_0} \widetilde{\Delta}_j v_j$$

is continuous from $b^s_{p,q}$ to $B^s_{p,q}$ for $1 \le p,q \le \infty$. It is also continuous from $f^s_{p,q}$ to $F^s_{p,q}$ if $1 < p < \infty$ and $1 \le q \le \infty$. In both cases, $\widetilde{R}S = I$.

Proof. The enlarged blocks have kernels dominated by

$$Ch_j(x), \qquad h_j(x) := 2^{jn}(1 + 2^j\|x\|)^{-n-1}.$$

The Besov bound follows from Young's inequality and the spectral synthesis lemma. For $1 < p < \infty$, $1 < q < \infty$, domination by the maximal operator and Fefferman–Stein control $\|(\widetilde{\Delta}_j v_j)_{j\in\mathbb{N}_0}\|_{L^p(\mathbb{R}^n;\ell^q(\mathbb{N}_0))}$. If $q = \infty$, use

$$\sup_{j\in\mathbb{N}_0} |\widetilde{\Delta}_j v_j| \le C\mathcal{M}\left(\sup_{j\in\mathbb{N}_0} |v_j|\right).$$

For $q = 1$, take a finite family and a function $g \in L^{p'}(\mathbb{R}^n)$, $g \ge 0$. Tonelli's theorem and symmetry of h_j give

$$\begin{aligned}\int_{\mathbb{R}^n} g \sum_{j=0}^{N} |\widetilde{\Delta}_j v_j| &\le C \int_{\mathbb{R}^n} \sum_{j=0}^{N} |v_j|(h_j * g) \\ &\le C \int_{\mathbb{R}^n} \left(\sum_{j=0}^{N} |v_j|\right) \mathcal{M}g \\ &\le C_p \left\| \sum_{j=0}^{N} |v_j| \right\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

Duality of L^p and monotone convergence yield the bound for the full family. Incorporate the weights 2^{js} into v_j. Then apply the spectral synthesis lemma to $\widetilde{\Delta}_j v_j$. □

The requirement $p > 1$ in the second assertion matters: the proof of completeness for $F^s_{1,q}$ uses the subspace with controlled spectral supports from the preceding theorem.

Corollary 6.3.9. (Approximation and loss of regularity) For $p, q < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{p,q}$ and in $F^s_{p,q}$. For any admissible fine index, $1 \le p < \infty$, and $\varepsilon > 0$,

$$A^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s-\varepsilon}_{p,p}(\mathbb{R}^n), \qquad A \in \{B, F\}.$$

The dyadic partial sums converge in this latter space.

Proof. If both indices are finite, the tails of the block sequence tend to zero in the corresponding norm by dominated convergence. The spectral synthesis lemma gives the same conclusion for tails of the reconstruction. Each of the finitely many remaining blocks is approximated in L^p by Schwartz functions; applying a multiplier equal to one on its spectral support preserves convergence and keeps the support in a slightly larger annulus. Finite synthesis gives approximation in the norm of the space.
In every case,

$$2^{js}\|\Delta_j u\|_{L^p(\mathbb{R}^n)} \le \|u\|_{A^s_{p,q}}, \qquad j \in \mathbb{N}_0.$$

Multiplying by $2^{-j\varepsilon}$ and summing the pth powers proves the second assertion and convergence of the tails. For $q = \infty$, the asserted convergence takes place in the space of order $s - \varepsilon$. □

6.3.2 Fundamental identifications and interpolation

In this subsection, we use the abstract interpolation results developed in Chapter 5; in particular, we apply the real and complex methods to the Euclidean scales constructed here.

The newly defined scales contain several spaces introduced earlier. Two mechanisms should be distinguished. Identification of Bessel potentials with $F^s_{p,2}(\mathbb{R}^n)$ is a Littlewood–Paley statement and therefore depends on the quadratic combination of frequency blocks. In contrast, identification of noninteger Sobolev–Slobodeckij spaces with $B^s_{p,p}(\mathbb{R}^n)$ comes from real interpolation. The two proofs explain how different extensions of the integer Sobolev scale arise.

Theorem 6.3.10. (Bessel potentials and integer-order Sobolev spaces) Let $1 < p < \infty$ and $s \in \mathbb{R}$.

(a) For every $s \in \mathbb{R}$,

$$H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$$

and there exist $c_{s,p}, C_{s,p} > 0$ such that

$$c_{s,p}\|u\|_{H^{s,p}(\mathbb{R}^n)} \le \|u\|_{F^s_{p,2}(\mathbb{R}^n)} \le C_{s,p}\|u\|_{H^{s,p}(\mathbb{R}^n)}.$$

(b) If $m \in \mathbb{N}_0$, then

$$H^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$$

and there exist $c_{m,p}, C_{m,p} > 0$ such that

$$c_{m,p}\|u\|_{W^{m,p}(\mathbb{R}^n)} \le \|u\|_{H^{m,p}(\mathbb{R}^n)} \le C_{m,p}\|u\|_{W^{m,p}(\mathbb{R}^n)}.$$

The classical identification $F^s_{p,2} = H^{s,p}$ can be found in [146, Chapter 1].

Proof. First extend the Littlewood–Paley criterion to distributions. If $v \in \mathcal{S}'(\mathbb{R}^n)$ and

$$G_v := \left(\sum_{j \in \mathbb{N}_0} |\Delta_j v|^2 \right)^{1/2} \in L^p(\mathbb{R}^n),$$

dyadic reproduction and Cauchy–Schwarz give, for $\zeta \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} |\langle v, \zeta \rangle| &= \left| \sum_{j \in \mathbb{N}_0} \langle \Delta_j v, \widetilde{\Delta}_j \zeta \rangle \right| \\ &\le \|G_v\|_{L^p(\mathbb{R}^n)} \left\| \left(\sum_{j \in \mathbb{N}_0} |\widetilde{\Delta}_j \zeta|^2 \right)^{1/2} \right\|_{L^{p'}(\mathbb{R}^n)} \le C\|G_v\|_{L^p(\mathbb{R}^n)} \|\zeta\|_{L^{p'}(\mathbb{R}^n)}. \end{aligned}$$

The sum of pairings converges absolutely by the same inequality. The argument with signs used in Theorem 6.3.2 also applies to the enlarged blocks, since their symbols have the same uniform bounds and finite overlap. Density of $\mathcal{S}$ in $L^{p'}$ and duality of $L^{p'}$ show that v is represented by a function in L^p, with $\|v\|_{L^p(\mathbb{R}^n)} \le C\|G_v\|_{L^p(\mathbb{R}^n)}$. Together with the direct Littlewood–Paley bound, this proves the criterion in both directions for $v \in \mathcal{S}'$.

On the support of φ_j, the weight $\langle\xi\rangle^s$ is comparable to 2^{js} uniformly in $j \in \mathbb{N}_0$. To pass from this comparison of symbols to one of norms, consider

$$m_j(\xi) = 2^{-js}\langle\xi\rangle^s \widetilde{\varphi}_j(\xi), \qquad \widetilde{m}_j(\xi) = 2^{js}\langle\xi\rangle^{-s}\widetilde{\varphi}_j(\xi).$$

After $\xi = 2^j\zeta$, all their derivatives are uniformly bounded on a fixed compact set. Integration by parts shows that their kernels satisfy the hypotheses of Lemma 6.3.3. Applying it with fine index 2, first to the blocks of u and then to those of $J^s u$, yields

$$c\left\|\left(\sum_{j\in\mathbb{N}_0} 2^{2js}|\Delta_j u|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \le \left\|\left(\sum_{j\in\mathbb{N}_0} |\Delta_j J^s u|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \le C\left\|\left(\sum_{j\in\mathbb{N}_0} 2^{2js}|\Delta_j u|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)}.$$

The preceding distributional criterion states that these quantities are finite exactly when $J^s u \in L^p$, and are then equivalent to $\|J^s u\|_{L^p(\mathbb{R}^n)}$. This proves (a).

For (b), if $\alpha \in \mathbb{N}_0^n$ and $|\alpha| \le m$, the symbol

$$b_\alpha(\xi) = (i\xi)^\alpha \langle\xi\rangle^{-m}$$

satisfies the Mikhlin conditions: each differentiation lowers its order by one, and its initial order is $|\alpha| - m \le 0$. Thus $D^\alpha u = T_{b_\alpha} J^m u$ and

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le m}} \|D^\alpha u\|_{L^p(\mathbb{R}^n)} \le C\|J^m u\|_{L^p(\mathbb{R}^n)}.$$

To recover $J^m u$ from the derivatives, introduce the positive polynomial

$$P_m(\xi) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le m}} \xi^{2\alpha}.$$

We have $c_m\langle\xi\rangle^{2m} \le P_m(\xi) \le C_m\langle\xi\rangle^{2m}$. The upper bound follows by estimating each monomial. For the lower bound, if $\|\xi\| \ge 1$, some coordinate satisfies $|\xi_\ell| \ge \|\xi\|/\sqrt{n}$, and the summand ξ_ℓ^{2m} gives the bound; if $\|\xi\| \le 1$, use the constant term.

For the same multi-indices, define

$$a_\alpha(\xi) = \frac{\langle\xi\rangle^m (-i\xi)^\alpha}{P_m(\xi)}.$$

Differentiating $P_m P_m^{-1} = 1$ and inducting on $|\beta|$ gives

$$|D^\beta(P_m^{-1})(\xi)| \le C_\beta\langle\xi\rangle^{-2m-|\beta|}, \qquad \beta \in \mathbb{N}_0^n.$$

The Leibniz rule then implies $|D^\beta a_\alpha(\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{|\alpha|-m-|\beta|}$; hence the a_α are also Mikhlin symbols. The algebraic identity

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le m}} a_\alpha(\xi)(i\xi)^\alpha = \langle\xi\rangle^m$$

gives, in $\mathcal{S}'$,

$$J^m u = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le m}} T_{a_\alpha} D^\alpha u.$$

Continuity of each multiplier on L^p proves the reverse inequality. The sum norms and the ℓ^p norms for this finite number of derivatives are equivalent, giving exactly the norm of $W^{m,p}$ fixed in this book. □

The preceding theorem allows interpolation methods to be transferred to dyadic scales. For the real method, regularity is encoded in the weight 2^{js} of each block; for the complex method, these weights interpolate geometrically. This difference explains why the real method produces the Besov scale while the complex method preserves the Bessel scale.

Theorem 6.3.11. (Interpolation of the scales) Let $s_0 < s_1$, $0 < \theta < 1$, and $s = (1-\theta)s_0 + \theta s_1$.

(a) If $1 < p < \infty$ and $1 \le q \le \infty$, then

$$(H^{s_0,p}(\mathbb{R}^n), H^{s_1,p}(\mathbb{R}^n))_{\theta,q} = B^s_{p,q}(\mathbb{R}^n)$$

and there exist $c, C > 0$, depending on $n, s_0, s_1, p, q, \theta$ and the fixed resolution, such that

$$c\|u\|_{B^s_{p,q}(\mathbb{R}^n)} \le \|u\|_{(H^{s_0,p}(\mathbb{R}^n), H^{s_1,p}(\mathbb{R}^n))_{\theta,q}} \le C\|u\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

(b) If $1 < p_0, p_1 < \infty$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

then

$$[H^{s_0,p_0}(\mathbb{R}^n), H^{s_1,p_1}(\mathbb{R}^n)]_\theta = H^{s,p}(\mathbb{R}^n)$$

and there exist $c, C > 0$, depending on $n, s_0, s_1, p_0, p_1, \theta$ and the fixed resolution, such that

$$c\|u\|_{H^{s,p}(\mathbb{R}^n)} \le \|u\|_{[H^{s_0,p_0}, H^{s_1,p_1}]_\theta} \le C\|u\|_{H^{s,p}(\mathbb{R}^n)}.$$

(c) In particular, if $m \in \mathbb{N}$ and $0 < \theta < 1$, then

$$(L^p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n))_{\theta,q} = B^{\theta m}_{p,q}(\mathbb{R}^n),$$

and

$$[L^p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)]_\theta = H^{\theta m,p}(\mathbb{R}^n).$$

Remark 6.3.12. (Why p is fixed in the real method) In part (a), p cannot simply be replaced by two distinct exponents p_0, p_1. If $p^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$, already in the low-frequency block one would obtain

$$(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta,q} = L^{p,q}(\mathbb{R}^n),$$

a Lorentz space, rather than $L^p(\mathbb{R}^n)$ except in special cases. Formulas with $p_0 \neq p_1$ lead to Besov–Lorentz scales. The complex method in part (b), in contrast, interpolates smoothness and the spatial exponent

simultaneously within the Bessel scale.

Proof. Define the analysis operator

$$Su := (\Delta_j u)_{j\in\mathbb{N}_0}$$

and the synthesis operator

$$R(v_j)_{j\in\mathbb{N}_0} := \sum_{j\in\mathbb{N}_0} \widetilde{\Delta}_j v_j.$$

Lemma 6.3.8, with $1 < p < \infty$ and fine index 2, proves that both operators are continuous on the scales below and satisfy $RS = I$. Thus the function scales are identified as retracts of the weighted sequence spaces associated with their dyadic blocks.
We prove (a). Write

$$\delta := s_1 - s_0 > 0, \qquad t_k := 2^{-k\delta}, \qquad a_j := \|\Delta_j u\|_{L^p(\mathbb{R}^n)}, \quad j,k \in \mathbb{N}_0.$$

For $k \in \mathbb{N}_0$, separate high and low frequencies by

$$u = u_0^{(k)} + u_1^{(k)},$$

with

$$u_0^{(k)} := \sum_{\substack{j\in\mathbb{N}_0\\ j>k}} \widetilde{\Delta}_j \Delta_j u, \qquad u_1^{(k)} := \sum_{j=0}^{k} \widetilde{\Delta}_j \Delta_j u.$$

Finite interaction of the blocks, the triangle inequality, and the identification $H^{s_i,p}(\mathbb{R}^n) = F^{s_i}_{p,2}(\mathbb{R}^n)$ from Theorem 6.3.10 give

$$K(t_k, u; H^{s_0,p}(\mathbb{R}^n), H^{s_1,p}(\mathbb{R}^n)) \le C \left(\sum_{\substack{j\in\mathbb{N}_0\\ j>k}} 2^{js_0} a_j + t_k \sum_{j=0}^{k} 2^{js_1} a_j \right). \tag{6.3}$$

In the reverse direction, let $u = v_0 + v_1$ with $v_i \in H^{s_i,p}(\mathbb{R}^n)$. Uniform boundedness of the block Δ_j and the description $H^{s_i,p} = F^{s_i}_{p,2}$ imply

$$2^{js_0} a_j \le C \left(\|v_0\|_{H^{s_0,p}(\mathbb{R}^n)} + t_j \|v_1\|_{H^{s_1,p}(\mathbb{R}^n)} \right).$$

Taking the infimum over all decompositions of u gives

$$2^{js} a_j \le C t_j^{-\theta} K(t_j, u; H^{s_0,p}(\mathbb{R}^n), H^{s_1,p}(\mathbb{R}^n)). \tag{6.4}$$

Set $b_j := 2^{js} a_j$. Multiplying (6.3) by $t_k^{-\theta}$ transforms its right-hand side into

$$C \left(\sum_{\substack{j\in\mathbb{N}_0\\ j>k}} 2^{-\theta\delta(j-k)} b_j + \sum_{j=0}^{k} 2^{-(1-\theta)\delta(k-j)} b_j \right).$$

Both coefficient sequences are geometrically summable. Lemma 5.2.5 and the Hardy inequalities from Theorem B.8.5, together with (6.4), give constants $c, C > 0$ such that

$$c\left\|(2^{js}\|\Delta_j u\|_{L^p(\mathbb{R}^n)})_{j\in\mathbb{N}_0}\right\|_{\ell^q(\mathbb{N}_0)} \le \|u\|_{(H^{s_0,p}(\mathbb{R}^n),H^{s_1,p}(\mathbb{R}^n))_{\theta,q}} \le C\left\|(2^{js}\|\Delta_j u\|_{L^p(\mathbb{R}^n)})_{j\in\mathbb{N}_0}\right\|_{\ell^q(\mathbb{N}_0)}.$$

This discretization describes the portion $0 < t \le 1$. For $t \ge 1$, the embedding $H^{s_1,p}(\mathbb{R}^n) \hookrightarrow H^{s_0,p}(\mathbb{R}^n)$ implies that there is $c_0 > 0$, independent of t and u, such that

$$c_0\|u\|_{H^{s_0,p}(\mathbb{R}^n)} \le K(t,u;H^{s_0,p}(\mathbb{R}^n),H^{s_1,p}(\mathbb{R}^n)) \le \|u\|_{H^{s_0,p}(\mathbb{R}^n)}$$

for every $t \ge 1$. This norm is also controlled by the Besov norm. To see this, dyadic reconstruction and the triangle inequality give

$$\|u\|_{H^{s_0,p}(\mathbb{R}^n)} \le C\sum_{j\in\mathbb{N}_0} 2^{js_0}a_j = C\sum_{j\in\mathbb{N}_0} 2^{-j(s-s_0)}b_j.$$

The sequence $(2^{-j(s-s_0)})_{j\in\mathbb{N}_0}$ belongs to $\ell^{q'}(\mathbb{N}_0)$ when $1 < q < \infty$; for $q = 1$ use its uniform bound, and for $q = \infty$ its summability. By Hölder's inequality for sequences, with these endpoint interpretations,

$$\|u\|_{H^{s_0,p}(\mathbb{R}^n)} \le C\|(b_j)_{j\in\mathbb{N}_0}\|_{\ell^q(\mathbb{N}_0)}.$$

Thus the interpolation integral over $[1,\infty)$ is controlled by the same norm, since $t^{-\theta} \in L^q((1,\infty),dt/t)$ for $q < \infty$ and is bounded for $q = \infty$. This proves (a), including the endpoints $q = 1$ and $q = \infty$.
For (b), define, for $i \in \{0,1\}$,

$$\ell^2_{s_i}(\mathbb{N}_0) := \left\{(z_j)_{j\in\mathbb{N}_0} \;\middle|\; \sum_{j\in\mathbb{N}_0} 2^{2js_i}|z_j|^2 < \infty\right\},$$

with its natural norm, and set

$$Y_i := L^{p_i}(\mathbb{R}^n;\ell^2_{s_i}(\mathbb{N}_0)).$$

By Theorem 6.3.10, S and R identify $H^{s_i,p_i}(\mathbb{R}^n)$ with a retract of Y_i. The weights interpolate geometrically,

$$(2^{js_0})^{1-\theta}(2^{js_1})^{\theta} = 2^{js},$$

and the spatial exponent satisfies $p^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$. Proposition 5.7.12, applied to weighted sequences on $\mathbb{N}_0$ (or equivalently, after extending them by zero to $\mathbb{Z}$), and then to the Bochner spaces, gives

$$[Y_0, Y_1]_\theta = L^p(\mathbb{R}^n;\ell^2_s(\mathbb{N}_0)).$$

Theorem 5.7.5, applied to S and R, preserves the retraction. Finally, the identification

$$H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$$

from Theorem 6.3.10 identifies the resulting retract with $H^{s,p}(\mathbb{R}^n)$ and proves (b).
For (c), use $H^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $H^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$, both from Theorem 6.3.10, and apply (a) and (b) with $s_0 = 0$ and $s_1 = m$. □

Corollary 6.3.13. (Real interpolation with different fine indices) Let $s_0 < s_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $1 < p < \infty$, and $1 \le q, q_0, q_1 \le \infty$. If $A_0, A_1 \in \{B, F\}$, then

$$\big((A_0)^{s_0}_{p,q_0}(\mathbb{R}^n), (A_1)^{s_1}_{p,q_1}(\mathbb{R}^n)\big)_{\theta,q} = B^s_{p,q}(\mathbb{R}^n)$$

with equivalent norms. Here $(A_i)^{s_i}_{p,q_i}$ means $B^{s_i}_{p,q_i}$ or $F^{s_i}_{p,q_i}$ according to the choice of A_i, for $i \in \{0,1\}$.

Proof. The proof of part (a) of the preceding theorem uses two properties of the endpoints $Y_i = H^{s_i,p}$:

$$2^{js_i}\|\Delta_j v\|_{L^p(\mathbb{R}^n)} \le C\|v\|_{Y_i}, \qquad \left\|\sum_{j\in J} v_j\right\|_{Y_i} \le C\sum_{j\in J} 2^{js_i}\|v_j\|_{L^p(\mathbb{R}^n)},$$

where $J \subseteq \mathbb{N}_0$ and the v_j have spectral support in the corresponding annuli. The second assertion includes series when the right-hand side is finite. Both properties also hold for $Y_i = (A_i)^{s_i}_{p,q_i}$: the first follows from the definition, and the second from Lemma 6.3.6 and the triangle inequality.
Thus, with $t_k = 2^{-k(s_1-s_0)}$ and $a_j = \|\Delta_j u\|_{L^p(\mathbb{R}^n)}$, the same frequency splitting yields

$$K(t_k, u; Y_0, Y_1) \le C\left(\sum_{\substack{j\in\mathbb{N}_0\\ j>k}} 2^{js_0}a_j + t_k\sum_{j=0}^{k} 2^{js_1}a_j\right), \qquad 2^{ks}a_k \le Ct_k^{-\theta}K(t_k, u; Y_0, Y_1).$$

Multiplying the first inequality by $t_k^{-\theta}$ produces exactly the two geometric sequences from the preceding proof. Young's inequality and discretization of K compare the Besov norm with the portion of the interpolation norm corresponding to $0 < t \le 1$. If $u \in B^s_{p,q}$, the second property, with $i = 0$, also gives

$$\|u\|_{Y_0} \le C\sum_{j\in\mathbb{N}_0} 2^{-j(s-s_0)}\big(2^{js}a_j\big) \le C'\|u\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

For $t \ge 1$, use $K(t, u; Y_0, Y_1) \le \|u\|_{Y_0}$; integrability of $t^{-\theta}$ completes the bound on that interval. This proves both inclusions. □

The real interpolation formula now identifies the spaces defined through differences with the Besov scale.

Corollary 6.3.14. (Sobolev–Slobodeckij, Besov, and Triebel–Lizorkin) Let $1 < p < \infty$ and $s > 0$, and assume $s \notin \mathbb{N}$. Then

$$W^{s,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n)$$

and there are constants $c_{n,s,p}, C_{n,s,p} > 0$ such that

$$c_{n,s,p}\|u\|_{W^{s,p}(\mathbb{R}^n)} \le \|u\|_{B^s_{p,p}(\mathbb{R}^n)} = \|u\|_{F^s_{p,p}(\mathbb{R}^n)} \le C_{n,s,p}\|u\|_{W^{s,p}(\mathbb{R}^n)}.$$

The identification through real interpolation is classical; see [146, Chapters 1–2].

Proof. Write $s = m + \sigma$, with $m \in \mathbb{N}_0$ and $0 < \sigma < 1$. Corollary 6.2.6 gives

$$W^{s,p}(\mathbb{R}^n) = (W^{m,p}(\mathbb{R}^n), W^{m+1,p}(\mathbb{R}^n))_{\sigma,p}.$$

By Theorem 6.3.10, the two integer-order spaces in the pair may be replaced by $H^{m,p}(\mathbb{R}^n)$ and $H^{m+1,p}(\mathbb{R}^n)$. Part (a) of Theorem 6.3.11 then gives

$$W^{s,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n).$$

On the other hand, for a fixed dyadic resolution, Theorem B.6.17 allows sum and integral to be interchanged:

$$\begin{aligned} \|u\|^p_{F^s_{p,p}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{N}_0} 2^{jsp} |\Delta_j u(x)|^p \, dx \\ &= \sum_{j\in\mathbb{N}_0} 2^{jsp} \|\Delta_j u\|^p_{L^p(\mathbb{R}^n)} = \|u\|^p_{B^s_{p,p}(\mathbb{R}^n)}. \end{aligned}$$

This yields the identification and equivalence of norms. □

Corollary 6.3.15. (When $H^{s,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ coincide) Let $s > 0$ and $s \notin \mathbb{N}$, and let $1 < p < \infty$.

(a) If $1 < p \le 2$, then $W^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n)$.

(b) If $2 \le p < \infty$, then $H^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n)$.

(c) Equality holds if $p = 2$. If $p \ne 2$, the corresponding embedding is strict.

For $s = m \in \mathbb{N}_0$, however,

$$H^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$$

for every $1 < p < \infty$.

Proof. By Corollary 6.3.14 and Theorem 6.3.10, it suffices to compare $F^s_{p,p}$ with $F^s_{p,2}$. The inclusion $\ell^{q_0}(\mathbb{N}_0) \hookrightarrow \ell^{q_1}(\mathbb{N}_0)$, for $q_0 \le q_1$, applied at each point to $(2^{js}|\Delta_j u(x)|)_{j\in\mathbb{N}_0}$, proves the embeddings and equality for $p = 2$.

We construct examples establishing strictness inside any open ball $B \subseteq \mathbb{R}^n$. This will also allow their use on domains. By independence of the resolution, we may choose a radial cutoff χ equal to one when $\|\xi\| \le 1$, zero when $\|\xi\| \ge 3/2$, and nonincreasing in the radius. For $\varphi_j(\xi) = \chi(2^{-j}\xi) - \chi(2^{-j+1}\xi)$, $j \ge 1$, the block equals one when $3/4 < 2^{-j}\|\xi\| < 1$. Take $\xi_* = (7/8, 0, \dots, 0)$, a number $0 < \varepsilon < 1/16$, and

$$j_m = J + 4m, \qquad \lambda_m = 2^{j_m}\xi_*, \qquad m \in \mathbb{N},$$

with $J \in \mathbb{N}$. The balls of center λ_m and radius $\varepsilon 2^{j_m}$ lie in regions where $\varphi_{j_m} = 1$; the other blocks vanish there.

Fix $0 \ne \zeta \in C^\infty_c(B)$ and a cutoff $\omega \in C^\infty_c(B_{\mathrm{euc}}(0,\varepsilon))$ equal to one near the origin. Define

$$\zeta_m = \mathcal{F}^{-1}\big(\omega(2^{-j_m}\cdot)\widehat{\zeta}\big).$$

For any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and any $N > 0$, rapid decay of $\widehat{\zeta}$ gives

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta (\zeta_m - \zeta)(x)| \le C_{\alpha,\beta,N} 2^{-j_m N}. \tag{6.5}$$

Indeed, the symbol $1 - \omega(2^{-j_m}\xi)$ vanishes on a ball of radius comparable to 2^{j_m}. Differentiating in ξ to produce x^α, multiplying by ξ^β, and estimating the inverse transform by its L^1 norm reduces the bound to tail integrals of a Schwartz function. These tails decay with every power of the radius.
Let $(a_m)_{m\in\mathbb{N}}$ be a bounded sequence. Since $s > 0$, the series

$$w = \sum_{m\in\mathbb{N}} a_m 2^{-j_m s} e^{i\langle \lambda_m, x\rangle} \zeta_m(x), \qquad u = \zeta(x) \sum_{m\in\mathbb{N}} a_m 2^{-j_m s} e^{i\langle \lambda_m, x\rangle}$$

converge uniformly and in $\mathcal{S}'$. The second function has support in B. By the choice of frequencies,

$$2^{j_m s} \Delta_{j_m} w = a_m e^{i\langle \lambda_m, x\rangle} \zeta_m(x), \qquad \Delta_j w = 0 \quad \text{if } j \notin \{j_m \mid m \in \mathbb{N}\}.$$

The sequence $(a_m(\zeta_m - \zeta))_{m\in\mathbb{N}}$ belongs to $L^p(\mathbb{R}^n; \ell^q(\mathbb{N}))$ for every $1 \le q \le \infty$: by (6.5), the sum of its L^p norms is finite. On the other hand,

$$\big\| (a_m \zeta)_{m\in\mathbb{N}} \big\|_{L^p(\mathbb{R}^n; \ell^q(\mathbb{N}_0))} = \|\zeta\|_{L^p(\mathbb{R}^n)} \|(a_m)_{m\in\mathbb{N}}\|_{\ell^q}.$$

The triangle inequality, also applied to the difference, shows that

$$w \in F^s_{p,q} \quad \Longleftrightarrow \quad (a_m)_{m\in\mathbb{N}} \in \ell^q(\mathbb{N}).$$

Moreover, $u - w \in \mathcal{S}$. To verify this, differentiate each summand of the difference. A derivative of the exponential produces at most a power of 2^{j_m}, and any polynomial weight in x is absorbed into the seminorm in (6.5). Choosing N larger than these powers makes the series summable in every Schwartz seminorm. Therefore,

$$u \in F^s_{p,q} \quad \Longleftrightarrow \quad (a_m)_{m\in\mathbb{N}} \in \ell^q(\mathbb{N}).$$

If $p < 2$, choose $a_m = m^{-\gamma}$ with $1/2 < \gamma \le 1/p$; then $(a_m) \in \ell^2 \setminus \ell^p$ and $u \in H^{s,p} \setminus W^{s,p}$. If $p > 2$, choose $1/p < \gamma \le 1/2$ to obtain $u \in W^{s,p} \setminus H^{s,p}$. These choices prove strictness, even with compact support in B. The integer-order case follows from Theorem 6.3.10. □

Remark 6.3.16. (Two different extensions of the integer scale) The corollary clarifies a common source of terminological ambiguity. Some texts call both $H^{s,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ "fractional Sobolev spaces." We shall retain different letters. The scale

$$H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$$

is stable under complex interpolation and suited to multipliers and elliptic operators. The scale

$$W^{s,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n), \qquad s \notin \mathbb{N},$$

arises from real interpolation and is suited to differences, traces, and nonlocal energies. For $p = 2$, both perspectives come together.

6.4 Multipliers, changes of coordinates, and localization

Euclidean theory can be transferred to a manifold only if two operations are stable: multiplication by a function in a partition of unity and change of coordinates. The next result treats both operations. Uniformity of the constants will be decisive in the chapter on bounded geometry.

Definition 6.4.1. (The spaces BC^j and BUC^j) Let $U \subseteq \mathbb{R}^m$ be an open subset, let F be a Banach space, and let $j \in \mathbb{N}_0$. Adopt the conventions $D^0 f = f$ and $\mathcal{L}^0(\mathbb{R}^m;F) = F$. For $0 \le \ell \le j$, the space $\mathcal{L}^\ell(\mathbb{R}^m;F)$ carries the norm

$$\|A\|_{\mathcal{L}^\ell} := \sup\bigl\{\|A(v_1,\dots,v_\ell)\|_F \mid |v_1| \le 1,\dots,|v_\ell| \le 1\bigr\}.$$

Define

$$BC^j(U,F) := \left\{ f \in C^j(U,F) \,\middle|\, \sum_{\ell=0}^{j} \sup_{x\in U} \|D^\ell f(x)\|_{\mathcal{L}^\ell} < +\infty \right\},$$

with the norm

$$\|f\|_{BC^j(U,F)} := \sum_{\ell=0}^{j} \sup_{x\in U} \|D^\ell f(x)\|_{\mathcal{L}^\ell}.$$

The space $BUC^j(U,F)$ is the subspace of $BC^j(U,F)$ consisting of functions for which

$$D^\ell f \colon U \longrightarrow \mathcal{L}^\ell(\mathbb{R}^m;F)$$

is uniformly continuous for each $0 \le \ell \le j$; it carries the norm induced from $BC^j(U,F)$. In particular, $BC^0(U,F)$ is the space of continuous bounded functions, while $BUC^0(U,F)$ additionally requires uniform continuity.

Now let $Q^m_+ = [0,1)\times(-1,1)^{m-1}$ and let $V \subseteq Q^m_+$ be a relatively open subset. Set

$$V^\circ := V \cap \bigl((0,1)\times(-1,1)^{m-1}\bigr).$$

By $C^j(V,F)$, we mean the space of functions $f\colon V \longrightarrow F$ for which $f\restriction_{V^\circ}$ is of class C^j and each derivative $D^\ell(f\restriction_{V^\circ})$, $0 \le \ell \le j$, admits a continuous extension to V, requiring the order-zero extension to agree with f. The extension is unique because V° is dense in V and is also denoted by $D^\ell f$. Define $BC^j(V,F)$ and its norm by the preceding formulas with V in place of U, and let $BUC^j(V,F)$ be the subspace for which each extended function

$$D^\ell f \colon V \longrightarrow \mathcal{L}^\ell(\mathbb{R}^m;F)$$

is uniformly continuous with respect to the induced Euclidean distance. In particular, if $V = Q^m_+$, each derivative of a function in $BUC^j(Q^m_+,F)$ has a unique continuous extension to the Euclidean closure of Q^m_+. The convention also applies to overlap domains relatively open in a half-box and agrees with the usual notion of regularity up to the face $\{0\}\times(-1,1)^{m-1}$ on a manifold with boundary.

Finally, if Ω is any of the preceding domains,

$$BC^\infty(\Omega,F) := \bigcap_{j\in\mathbb{N}_0} BC^j(\Omega,F), \qquad BUC^\infty(\Omega,F) := \bigcap_{j\in\mathbb{N}_0} BUC^j(\Omega,F),$$

with the families of seminorms $\|\cdot\|_{BC^j}$; no single uniform bound in j is required. For matrix-valued func-

tions and tensor coefficients, use the corresponding finite-dimensional space as codomain, with its operator norm or the tensor norm induced by the Euclidean norm. Thus the bounds hold simultaneously for all components.

This is the sum convention used in Amann's uniform regularity. If one introduces the auxiliary norm $\|f\|_{\max,j} := \max_{\ell\in\{0,\dots,j\}} \sup_{x\in U} \|D^\ell f(x)\|_{\mathcal{L}^\ell}$, then explicitly,

$$\|f\|_{\max,j} \le \|f\|_{BC^j} \le (j+1)\|f\|_{\max,j}.$$

The class of spaces is also independent of the chosen Euclidean norm. Indeed, if $a|v|_1 \le |v|_2 \le b|v|_1$ for every $v \in \mathbb{R}^m$, with $a, b > 0$, then for each ℓ,

$$\|A\|_{\mathcal{L}^\ell,2} \le a^{-\ell}\|A\|_{\mathcal{L}^\ell,1}, \qquad \|A\|_{\mathcal{L}^\ell,1} \le b^\ell \|A\|_{\mathcal{L}^\ell,2}.$$

Consequently,

$$\|f\|_{BC^j,2} \le \max\{1, a^{-j}\}\|f\|_{BC^j,1}, \qquad \|f\|_{BC^j,1} \le \max\{1, b^j\}\|f\|_{BC^j,2}.$$

The two domain metrics are bi-Lipschitz equivalent and therefore determine the same uniform continuity. If F is finite dimensional and the codomain norms satisfy

$$c_F\|w\|_{F,1} \le \|w\|_{F,2} \le C_F\|w\|_{F,1} \qquad (w \in F),$$

then the preceding inequalities become

$$\|f\|_{BC^j,2} \le C_F \max\{1, a^{-j}\}\|f\|_{BC^j,1}, \qquad \|f\|_{BC^j,1} \le c_F^{-1} \max\{1, b^j\}\|f\|_{BC^j,2}.$$

Thus the classes BC^j and BUC^j are independent of these choices.

Estimates for changes of coordinates require control of both frequency separation and spatial decay of the kernel. Write

$$h_{r,D}(x) := 2^{rn}(1 + 2^r\|x\|)^{-D}, \qquad r \in \mathbb{N}_0, \quad D > n,$$

and, for a band-limited function v,

$$P_{r,\lambda}v(x) := \sup_{z\in\mathbb{R}^n} \frac{|v(z)|}{(1 + 2^r\|x - z\|)^\lambda}.$$

Lemma B.8.16 ensures that $P_{r,\lambda}v \le C\mathcal{M}_a v$ when $\lambda > n/a$ and the support of $\widehat{v}$ lies in a ball of radius comparable to 2^r.

Lemma 6.4.2. (Kernel estimate for a change of coordinates) Let $M \in \mathbb{N}$, let $g \in BC^M(\mathbb{R}^n)$, and let ψ be a global diffeomorphism of class C^{M+1}. Suppose that ψ and ψ^{-1} are Lipschitz and that their derivatives of orders $1, \dots, M+1$ are bounded. The kernel $L_{j,r}(x,z)$ of

$$\Delta_j\big[g\,((\widetilde{\Delta}_r v) \circ \psi)\big](x) = \int_{\mathbb{R}^n} L_{j,r}(x,z)v(z)\,dz$$

satisfies, for any $D > n$,

$$|L_{j,r}(x,z)| \le C_{M,D}\,2^{-M|j-r|}h_{\min\{j,r\},D}(\psi(x) - z), \qquad j, r \in \mathbb{N}_0. \tag{6.6}$$

Proof. Denote by K_j and $\widetilde{K}_r$ the real convolution kernels of the blocks, including their Fourier constants. Then

$$L_{j,r}(x,z) = \int_{\mathbb{R}^n} K_j(x-y)g(y)\widetilde{K}_r(\psi(y)-z)\,dy.$$

Positive-index kernels have all moments zero. Moreover, for any multi-index α and any $D_1 > 0$,

$$|D^\alpha K_j(y)| + |D^\alpha \widetilde{K}_j(y)| \le C_{\alpha,D_1} 2^{j(n+|\alpha|)}(1+2^j\|y\|)^{-D_1}.$$

If $j \ge r$ and $j \ge 1$, subtract from $y \mapsto g(y)\widetilde{K}_r(\psi(y)-z)$ its Taylor polynomial at x of degree $M-1$. The polynomial terms vanish upon integration against $K_j(x-y)$. The Leibniz and Faà di Bruno formulas, Theorem 1.1.19, show that derivatives of order M of the remainder are bounded by

$$C2^{r(n+M)}(1+2^r\|\psi(x)-z\|)^{-D}(1+2^r\|x-y\|)^{D}$$

on the segment between x and y. Here we use

$$1+2^r\|\psi(x)-z\| \le C(1+2^r\|\psi(x+t(y-x))-z\|)(1+2^r\|x-y\|), \qquad 0 \le t \le 1.$$

The integral remainder also contains $\|x-y\|^M$. The substitution $w = 2^j(x-y)$, with $D_1 > M+D+n$, gives

$$\int_{\mathbb{R}^n} |K_j(x-y)|\,\|x-y\|^M(1+2^r\|x-y\|)^D\,dy \le C2^{-jM}.$$

This yields (6.6) in this case.
If $r > j$, first make the substitution $w = \psi(y)$. The factor against which $\widetilde{K}_r(w-z)$ acts is

$$K_j(x-\psi^{-1}(w))\,g(\psi^{-1}(w))\,|\det D\psi^{-1}(w)|.$$

Subtract its Taylor polynomial at z of degree $M-1$. Its derivatives of order M require at most $M+1$ derivatives of ψ^{-1} because of the determinant. The same computation, now with $2^{jM}\|w-z\|^M$, gives $2^{-M(r-j)}$. The bi-Lipschitz condition compares $\|x-\psi^{-1}(z)\|$ with $\|\psi(x)-z\|$. When $j = r = 0$, the bound follows directly from the integral of two rapidly decreasing kernels. This also covers equal positive indices.

□

Lemma 6.4.3. (Multipliers and diffeomorphisms on the Sobolev scale) Let $s \in \mathbb{R}$ and $k \in \mathbb{N}$, with $k > |s|+n+1$. For $H^{s,p}$, assume $1 < p < \infty$; for $B^s_{p,q}$, assume $1 \le p,q \le \infty$; and for $F^s_{p,q}$, assume $1 \le p < \infty$, $1 \le q \le \infty$.

(a) If $g \in BC^k(\mathbb{R}^n)$, then

$$\|gu\|_{A^s_{p,q}} \le C\|g\|_{BC^k}\|u\|_{A^s_{p,q}}, \qquad A \in \{B,F\},$$

and the same assertion holds in $H^{s,p}$.

(b) Let $\psi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a diffeomorphism of class C^{k+1}, with

$$|\det D\psi(x)| \ge c > 0, \qquad \|D^\alpha\psi\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_\alpha \quad (1 \le |\alpha| \le k+1).$$

Then $u \mapsto u \circ \psi$ is continuous on the same scales. The constants depend only on the parameters and the indicated bounds.

(c) Let $\psi\colon U \longrightarrow V$ be a diffeomorphism and let $\eta \in C_c^{k+1}(U)$. Suppose that the derivative bounds and bi-Lipschitz conditions of the preceding lemma hold on a neighborhood of the support of η. The operator

$$T_{\eta,\psi}u(x) := \begin{cases} \eta(x)u(\psi(x)), & x \in U, \\ 0, & x \notin U \end{cases}$$

is continuous on the same scales. The bound is uniform for families with the same derivative bounds, bi-Lipschitz constants, an upper bound for the diameters of the supports, and a positive lower bound for their distance from the boundary of these neighborhoods.

Proof. The inverse matrix formula and the lower bound for the determinant control $D\psi^{-1}$. Differentiating $\psi^{-1} \circ \psi = I$ and applying Faà di Bruno, the term containing $D^m\psi^{-1}$ is

$$(D^m\psi^{-1}) \circ \psi\,[D\psi, \dots, D\psi].$$

The other terms contain derivatives of ψ^{-1} of order less than m. Induction on $m = 2, \dots, k+1$, followed by multiplication by $D\psi^{-1}$ in each argument, controls all derivatives of the inverse. Integrating the first derivative along segments proves that both functions are globally Lipschitz.
Choose $0 < a < 1$, $\lambda > n/a$, and an integer $M < k$ so that $M - \lambda > |s|$. This is possible because $k > |s| + n + 1$. Apply (6.6) to $v_r = \Delta_r u$, with $D > n + \lambda$. Peetre's definition gives

$$|v_r(z)| \le P_{r,\lambda}v_r(\psi(x))(1 + 2^r\|\psi(x) - z\|)^{\lambda}.$$

The integral of the second factor against $h_{\min\{j,r\},D}$ is bounded by $C2^{\lambda(r-j)_+}$. Thus,

$$|\Delta_j[g\,((\widetilde{\Delta}_r\Delta_r u) \circ \psi)](x)| \le C2^{-(M-\lambda)|j-r|}\mathcal{M}_a(\Delta_r u)(\psi(x)). \tag{6.7}$$

Multiplication corresponds to $\psi = I$, and composition to $g = 1$.
Multiply by 2^{js} and sum in $r \in \mathbb{N}_0$. The coefficients

$$\left(2^{-(M-\lambda-|s|)|j-r|}\right)_{j-r\in\mathbb{Z}}$$

form a summable sequence. For Besov spaces, scalar boundedness of the maximal operator on $L^{p/a}$, followed by Young's inequality for sequences, gives the bound; if $p = \infty$, use its bound on L^∞ directly. For Triebel–Lizorkin spaces, apply Fefferman–Stein with exponents $p/a > 1$ and $q/a > 1$. If $q = \infty$, use the maximal operator of the supremum, as in Lemma 6.3.3. Change of variables introduces only the bound for $|\det D\psi^{-1}|$. The identity $H^{s,p} = F^s_{p,2}$ gives the assertion for Bessel spaces.
For part (c), reduce the local change to finitely many global changes. Fix x_0 near the support of η and write

$$A_0 = D\psi(x_0), \qquad a_0(x) = \psi(x_0) + A_0(x - x_0).$$

If $B(x_0, 2r)$ lies in the neighborhood under consideration, choose a cutoff χ equal to one on $B(x_0, r)$ and supported in $B(x_0, 2r)$, with $\|D\chi\|_{L^\infty(\mathbb{R}^n)} \le C/r$. The fundamental theorem of calculus, applied to $\psi - a_0$ and

its derivative, gives

$$\|\psi - a_0\|_{L^\infty(B(x_0,2r))} \le Cr^2 \|D^2\psi\|_{L^\infty(B_{\mathrm{euc}}(x_0,2r))}, \qquad \|D(\psi - a_0)\|_{L^\infty(B(x_0,2r))} \le Cr \|D^2\psi\|_{L^\infty(B_{\mathrm{euc}}(x_0,2r))}.$$

Extend $F_0 = \chi(\psi - a_0)$ by zero. By the Leibniz rule, $\|DF_0\|_{L^\infty(B_{\mathrm{euc}}(x_0,2r))} \le Cr\|D^2\psi\|_{L^\infty(B_{\mathrm{euc}}(x_0,2r))}$. Choose $r > 0$, uniformly under the assumptions of this part, so that $\|A_0^{-1}\|\|DF_0\|_{L^\infty(\mathbb{R}^n)} < 1/2$. The function

$$\Psi_0(x) = a_0(x) + F_0(x), \qquad x \in \mathbb{R}^n,$$

agrees with ψ on $B(x_0, r)$ and is bi-Lipschitz. To check surjectivity as well, given $y \in \mathbb{R}^n$, apply the fixed-point theorem to

$$x \longmapsto x_0 + A_0^{-1}\big(y - \psi(x_0) - F_0(x)\big),$$

whose contraction constant is less than $1/2$. Its fixed point is the unique solution of $\Psi_0(x) = y$. The inverse function theorem and the derivative bounds already established give a global diffeomorphism with the bounds required in (b).

Cover the support of η by finitely many of these balls and choose a subordinate smooth partition $(\beta_\ell)_{\ell=1}^N$. Then

$$T_{\eta,\psi}u = \sum_{\ell=1}^{N} \eta\beta_\ell\,(u \circ \Psi_\ell).$$

Applying (a) and (b) to each summand proves (c). The ball radius, the number N, and the cutoff derivatives are controlled by the bounds in the statement, giving the asserted uniformity as well.

We specify the distributional interpretation when $s < 0$ and the coefficients have finite regularity. The pairing estimate in the proof of Lemma 6.3.6, for a test function supported on a compact set K, gives

$$|\langle u, \zeta\rangle| \le C_K \|u\|_{A^s_{p,q}} \max_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le m}} \|D^\alpha \zeta\|_{L^\infty(\mathbb{R}^n)},$$

if m is an integer with $|s| + n < m < k$. Thus u also acts continuously on C_c^m functions. Define the operators by

$$\langle gu, \zeta\rangle = \langle u, g\zeta\rangle, \qquad \langle u \circ \psi, \zeta\rangle = \left\langle u, (\zeta \circ \psi^{-1})|\det D\psi^{-1}|\right\rangle.$$

The available derivatives make both pairings legitimate. The dyadic partial sums converge in this sense: approximate the test function C_c^k in C_c^m by smooth functions and use the preceding uniform bound. Thus estimates for finite sums pass to the distributional operator by Fatou's lemma. This argument includes $q = \infty$ without requiring density in the norm of order s. □

To formulate the localization principle, call a family $(\eta_\nu)_{\nu\in I} \subseteq C_c^\infty(\mathbb{R}^n)$ a *uniform localization system* if it satisfies:

(i) $0 \le \eta_\nu \le 1$ and $\displaystyle\sum_{\nu\in I} \eta_\nu = 1$;

(ii) each point belongs to at most N supports;

(iii) support diameters and all $\|\eta_\nu\|_{BC^k(\mathbb{R}^n)}$ norms are uniformly bounded;

(iv) there exist $\chi_\nu \in C_c^\infty(\mathbb{R}^n)$, with $\chi_\nu = 1$ near $\operatorname{supp}(\eta_\nu)$, satisfying the same bounds and the same type of overlap.

Theorem 6.4.4. (Localization principle) Let (η_ν) be a uniform localization system.

(a) If $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$, there are constants $c, C > 0$, independent of u, such that

$$c\|u\|_{F^s_{p,q}(\mathbb{R}^n)} \le \left(\sum_{\nu \in I} \|\eta_\nu u\|^p_{F^s_{p,q}(\mathbb{R}^n)} \right)^{\frac{1}{p}} \le C\|u\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

In particular, the formula holds for $H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$ and, if $s > 0$ is not an integer, for $W^{s,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n)$.

(b) For $B^s_{p,q}(\mathbb{R}^n)$ with $q \neq p$, there is in general no analogous equivalence with an outer ℓ^p sum. Such an equivalence does hold when $q = p$, because $B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n)$.

Proof. Omit indices for which $\eta_\nu = 0$ and choose $x_\nu \in \operatorname{supp} \eta_\nu$, $\nu \in I$. The diameter bounds place the supports of η_ν and χ_ν in balls of fixed radius around x_ν. Moreover, there is $r_0 > 0$, independent of ν, such that $\chi_\nu > 1/2$ on $B(x_\nu, r_0)$: simply take

$$r_0 < \frac{1}{2\left(1 + \sup_{\nu \in I} \|D\chi_\nu\|_{L^\infty(\mathbb{R}^n)}\right)}.$$

Since the supports of χ_ν have bounded overlap, comparing the volumes of these balls gives

$$\#\{\nu \in I \mid \|x - x_\nu\| \le R\} \le C(1+R)^n, \qquad x \in \mathbb{R}^n, \quad R > 0.$$

In particular, I is countable and, for $D > n$,

$$\sup_{x \in \mathbb{R}^n} \sum_{\nu \in I} (1 + \|x - x_\nu\|)^{-D} < \infty. \tag{6.8}$$

To verify the last bound, group the centers in $\|x - x_\nu\| \le 1$ and in the annuli $2^m < \|x - x_\nu\| \le 2^{m+1}$, $m \in \mathbb{N}_0$. The resulting series is dominated by $\sum_{m \in \mathbb{N}_0} 2^{m(n-D)}$.

We shall use the preceding kernel estimate with the spatial decay supplied by the cutoffs. For $a_\nu = \eta_\nu$ or $a_\nu = \chi_\nu$, let $L_{\nu,j,r}$ be the kernels of $\Delta_j a_\nu \widetilde{\Delta}_r$. For any $M, D, E > 0$, with $E > n$,

$$|L_{\nu,j,r}(x,z)| \le C_{M,D,E}\, 2^{-M|j-r|}(1 + \|x - x_\nu\|)^{-D} h_{\min\{j,r\},E}(x - z). \tag{6.9}$$

The constant is uniform in ν, j, r.

We explain the spatial factor. In the integral

$$L_{\nu,j,r}(x,z) = \int_{\mathbb{R}^n} K_j(x-y) a_\nu(y) \widetilde{K}_r(y-z)\, dy,$$

y belongs to a ball of fixed radius around x_ν. If x lies in a slightly larger ball, that factor is comparable to one, and Lemma 6.4.2 applies. On the complement, $1 + \|x - x_\nu\| \le C(1 + \|x - y\|)$. For $j \ge r$, decay of $K_j(x-y)$ allows extraction of $(1+\|x-x_\nu\|)^{-D} 2^{-M(j-r)}$; the remaining powers are integrated with $\widetilde{K}_r$. For $r > j$, subtract Taylor polynomials against $\widetilde{K}_r$. Each derivative of $K_j(x-y)a_\nu(y)$ retains the same decay outside the ball and

contributes at most a factor 2^j. Integration of the remainder contributes 2^{-rM}. In both cases, the inequality

$$1 + 2^{\min\{j,r\}}\|x-z\| \le (1 + 2^{\min\{j,r\}}\|x-y\|)(1 + 2^{\min\{j,r\}}\|y-z\|)$$

allows the factor $h_{\min\{j,r\},E}(x-z)$ to be retained. Initially choose decay powers larger than the sum of the extracted powers plus $n+1$, so all remaining integrals are uniformly finite. Only finitely many derivative bounds of the cutoffs are involved.

Fix $0 < a < 1$, $\lambda > n/a$, $D > n$, and $M > |s| + \lambda$. Integrating (6.9) against $\Delta_r u$, as in (6.7), gives

$$2^{js}|\Delta_j(a_\nu u)(x)| \le C w_\nu(x) \sum_{r\in\mathbb{N}_0} 2^{-(M-\lambda-|s|)|j-r|} 2^{rs} P_{r,\lambda}(\Delta_r u)(x),$$

where $w_\nu(x) := (1 + \|x - x_\nu\|)^{-D}$. Take the ℓ^q norm in j, use Young's inequality for sequences, and set

$$G_u(x) := \left\| \left(2^{rs} P_{r,\lambda}(\Delta_r u)(x)\right)_{r\in\mathbb{N}_0} \right\|_{\ell^q}.$$

Peetre control and Fefferman–Stein, with the supremum argument if $q = \infty$, imply

$$\|G_u\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

Thus (6.8) and Tonelli's theorem give

$$\sum_{\nu\in I} \|\eta_\nu u\|^p_{F^s_{p,q}(\mathbb{R}^n)} \le C \int_{\mathbb{R}^n} \sum_{\nu\in I} w_\nu(x)^p G_u(x)^p \, dx \le C\|u\|^p_{F^s_{p,q}(\mathbb{R}^n)}.$$

For the reverse inequality, first consider a finite family $(v_\nu)_{\nu\in I}$ and the operator

$$R(v_\nu)_{\nu\in I} := \sum_{\nu\in I} \chi_\nu v_\nu.$$

The same estimate, followed by the triangle inequality in ℓ^q, gives pointwise

$$\left\| \left(2^{js}\Delta_j R(v_\nu)(x)\right)_{j\in\mathbb{N}_0} \right\|_{\ell^q} \le C \sum_{\nu\in I} w_\nu(x) G_{v_\nu}(x).$$

Hölder's inequality for the weighted sum over I implies

$$\left(\sum_{\nu\in I} w_\nu G_{v_\nu}\right)^p \le \left(\sum_{\nu\in I} w_\nu\right)^{p-1} \sum_{\nu\in I} w_\nu G^p_{v_\nu}.$$

Integration yields

$$\|R(v_\nu)\|^p_{F^s_{p,q}(\mathbb{R}^n)} \le C \sum_{\nu\in I} \|v_\nu\|^p_{F^s_{p,q}(\mathbb{R}^n)}.$$

Finite families are dense in the outer ℓ^p sum, so R extends continuously to it. Apply this bound to $v_\nu = \eta_\nu u$. The sum is locally finite in the distributional sense, and $\chi_\nu \eta_\nu = \eta_\nu$, so $R(\eta_\nu u) = u$. This completes (a).

For (b), construct functions with compact spatial support. Choose a smooth partition obtained by lattice translation that equals one on small balls around its centers, and $\zeta \in C^\infty_c(\mathbb{R}^n)$, $\zeta \neq 0$, supported in one of these balls. Choose distinct lattice centers x_m sufficiently far apart that each translate of ζ lies in a region

where only one cutoff equals one.

We may use a dyadic resolution whose annular symbol equals one near some point $\xi_* \neq 0$. Indeed, choose the initial cutoff equal to one on $\|\xi\| \le 1$ and zero for $\|\xi\| \ge 3/2$; its dyadic difference equals one on $3/4 < \|\xi\| < 1$. Fix ξ_* in that annulus. For integers $j_m = J + Lm$, $m \in \mathbb{N}$, set

$$v_m(x) := 2^{-j_m s}\zeta(x - x_m)e^{i2^{j_m}\xi_*\cdot x}, \qquad u_N := \sum_{m=1}^{N} v_m.$$

The integers J, L will be fixed once, independently of N. For any $M > |s|$, regularity of ζ gives

$$2^{ks}\|\Delta_k v_m\|_{L^p(\mathbb{R}^n)} \le C_M 2^{-(M-|s|)|k-j_m|}, \qquad k \in \mathbb{N}_0, \quad m \in \mathbb{N}. \tag{6.10}$$

For $k > j_m$, use M derivatives of $\zeta e^{i2^{j_m}\xi_*\cdot x}$ and the moments of K_k. For $k < j_m$ separated by more than a fixed number of indices, integrate by parts in the direction ξ_* in the convolution formula: each integration contributes 2^{-j_m}, while differentiating the kernel contributes at most 2^k. Derivatives of ζ have fixed L^p norms. The remaining indices are controlled by Young's inequality. These operations prove (6.10) also for $p = 1$ and $p = \infty$.

Moreover,

$$2^{j_m s}e^{-i2^{j_m}\xi_*\cdot x}\Delta_{j_m} v_m(x) \longrightarrow \zeta(x - x_m) \quad \text{in } L^p$$

uniformly in the translations as $j_m \to \infty$. To see this, the modulated kernel has mass one and scale 2^{-j_m}; write the difference using $\zeta(\cdot - y) - \zeta$ and use continuity of translations. For $p = \infty$, use uniform continuity of ζ. Choose J so that the norm of these principal blocks is at least $c_0 > 0$. Then choose L sufficiently large that

$$C_M \sum_{\ell \in \mathbb{Z}\setminus\{0\}} 2^{-(M-|s|)L|\ell|} < c_0/2.$$

The reverse triangle inequality in each block j_m gives $2^{j_m s}\|\Delta_{j_m} u_N\|_{L^p(\mathbb{R}^n)} \ge c_0/2$. The upper bound for all blocks follows by summing (6.10) and applying Young's inequality. Thus,

$$cN^{1/q} \le \|u_N\|_{B^s_{p,q}(\mathbb{R}^n)} \le CN^{1/q}.$$

By the choice of supports, the nonzero localizations of u_N are exactly $v_1, \dots, v_N$, and each has Besov norm between two positive constants. Consequently,

$$cN^{1/p} \le \left\| \left(\|\eta_\nu u_N\|_{B^s_{p,q}(\mathbb{R}^n)} \right)_{\nu \in I} \right\|_{\ell^p} \le CN^{1/p}.$$

We interpret $N^{1/\infty} = 1$. Equivalence of the two norms with constants independent of N would require $p = q$. For $p = q < \infty$, the identity $B^s_{p,p} = F^s_{p,p}$ and (a) give the asserted equivalence. □

Remark 6.4.5. (Besov localization on manifolds) Part (b) does not prevent localization of a Besov function with finitely many charts, but it does prevent using an unchanged ℓ^p sum over an infinite atlas. On a noncompact manifold without boundary and of bounded geometry, one first defines the scale F, which does localize, and then

$$B^s_{p,q}(M) := (F^{s_0}_{p,p}(M), F^{s_1}_{p,p}(M))_{\theta,q}, \qquad s = (1-\theta)s_0 + \theta s_1.$$

In the Euclidean case, Corollary 6.3.13 justifies this formula for any $s_0 < s < s_1$, with $\theta = (s - s_0)/(s_1 - s_0)$. In the construction on the manifold, independence of auxiliary parameters follows by combining localization formulas with reiteration. Uniform bounds on the charts and their transitions are the assumptions controlling the constants over an infinite atlas. The procedure will be revisited in Chapter 13 and agrees with the Große–Schneider construction [72] and Amann's localization principle [6].

6.5 Open subsets, extension operators, and comparison of scales

For $A \in \{B, F\}$, define the restriction space

$$A^s_{p,q}(\Omega) := \{U\restriction_\Omega \mid U \in A^s_{p,q}(\mathbb{R}^n)\}, \qquad \|u\|_{A^s_{p,q}(\Omega)} := \inf_{\substack{U \in A^s_{p,q}(\mathbb{R}^n) \\ U\restriction_\Omega = u}} \|U\|_{A^s_{p,q}(\mathbb{R}^n)}.$$

This convention agrees with Definition 6.1.5 when $A^s_{p,q} = F^s_{p,2}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$.

Extension must use only values of the distribution inside the domain. To achieve this, we construct kernels supported in a cone that remains within the domain when subtracted from a point of the domain. Cancellation of all moments will allow the same kernels to be used at every regularity order.

Lemma 6.5.1. (Conical reproducing kernels) Let $\omega\colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ be a Lipschitz function with constant L, and set

$$\Omega_\omega := \{(x', x_n) \in \mathbb{R}^n \mid x_n > \omega(x')\}, \qquad \mathcal{K}_L := \{(y', y_n) \in \mathbb{R}^n \mid y_n < -(L+1)\|y'\|\}.$$

There exist $\varphi_0, \varphi, \psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that, upon defining

$$\varphi_j(x) := 2^{jn}\varphi(2^j x), \qquad \psi_j(x) := 2^{jn}\psi(2^j x), \qquad j \in \mathbb{N},$$

the following properties hold.

(a) The supports of φ_j, ψ_j are contained in $\mathcal{K}_L \cap \{x_n \le -c2^{-j}\}$, with $c > 0$. Moreover,

$$\int_{\mathbb{R}^n} \varphi_0 = \int_{\mathbb{R}^n} \psi_0 = 1, \qquad \int_{\mathbb{R}^n} x^\alpha \varphi(x)\,dx = \int_{\mathbb{R}^n} x^\alpha \psi(x)\,dx = 0 \quad (\alpha \in \mathbb{N}_0^n).$$

(b) Denote by

$$\mathcal{S}'(\Omega_\omega) := \{U\restriction_{\Omega_\omega} \mid U \in \mathcal{S}'(\mathbb{R}^n)\}$$

the space of restrictions of tempered distributions. For u in this space, $\varphi_j * u$ is a well-defined smooth function on Ω_ω, independent of the extension, and

$$u = \sum_{j \in \mathbb{N}_0} \psi_j * (\varphi_j * u) \quad \text{in } \mathcal{D}'(\Omega_\omega). \tag{6.11}$$

(c) For $U \in \mathcal{S}'(\mathbb{R}^n)$, write

$$Q_{j,\lambda}U(x) := \sup_{y \in \mathbb{R}^n} \frac{|(\varphi_j * U)(y)|}{(1 + 2^j\|x - y\|)^\lambda}.$$

If $\lambda > n/a$, where $0 < a < 1$, then

$$\left\| \left(2^{js} \|Q_{j,\lambda} U\|_{L^p(\mathbb{R}^n)}\right)_{j\in\mathbb{N}_0} \right\|_{\ell^q} \le C\|U\|_{B^s_{p,q}(\mathbb{R}^n)}, \tag{6.12}$$

$$\left\| \left(2^{js} Q_{j,\lambda} U\right)_{j\in\mathbb{N}_0} \right\|_{L^p(\mathbb{R}^n;\ell^q(\mathbb{N}_0))} \le C\|U\|_{F^s_{p,q}(\mathbb{R}^n)} \tag{6.13}$$

for the Banach indices defined in this chapter. The first bound also allows $p = \infty$. If $f_j = (\varphi_j * U)\restriction_{\Omega_\omega}$, then for every $N > 0$,

$$|\Delta_k[\psi_j * (\mathbf{1}_{\Omega_\omega} f_j)](x)| \le C_N 2^{-N|j-k|} Q_{j,\lambda} U(x), \qquad j,k \in \mathbb{N}_0. \tag{6.14}$$

Proof. First construct a function $g \in \mathcal{S}(\mathbb{R})$ supported in $[1,\infty)$, with integral one and all positive-order moments zero. Choose $\rho \in C^\infty_c((1,2))$, with $\int_{\mathbb{R}} \rho = 1$. For $0 \le \ell \le N$, the polynomial interpolation formula at the points $2^0,\dots,2^N$, evaluated at zero, gives the coefficients

$$a_{\ell,N} := \prod_{\substack{r\in\{0,\dots,N\}\\ r\ne\ell}} \frac{2^r}{2^r - 2^\ell}.$$

Indeed, the polynomials

$$\prod_{\substack{r\in\{0,\dots,N\}\\ r\ne\ell}} \frac{t - 2^r}{2^\ell - 2^r}$$

equal one at $t = 2^\ell$ and zero at the other points. Their combination with the values of a polynomial of degree at most N agrees with that polynomial, since their difference has $N+1$ roots. In particular,

$$\sum_{\ell=0}^{N} a_{\ell,N} 2^{\ell m} = \begin{cases} 1, & m = 0, \\ 0, & 1 \le m \le N. \end{cases}$$

Separating factors with $r < \ell$ and $r > \ell$ shows that

$$|a_{\ell,N}| \le C 2^{-\ell(\ell+1)/2}.$$

The constant is uniform because $\prod_{r\in\mathbb{N}} (1 - 2^{-r}) > 0$. This last assertion follows by taking logarithms and using $-\log(1-t) \le 2t$ for $0 \le t \le 1/2$. For each ℓ, the products have a limit a_ℓ as $N \to \infty$. The preceding bound allows passage to the limit in every moment sum, even after multiplying by $2^{\ell m}$. Thus,

$$\sum_{\ell\in\mathbb{N}_0} a_\ell 2^{\ell m} = \delta_{m0}, \qquad m \in \mathbb{N}_0.$$

Define

$$g(t) := \sum_{\ell\in\mathbb{N}_0} a_\ell 2^{-\ell} \rho(2^{-\ell} t).$$

The seminorm $\sup_{t\in\mathbb{R}} (1+|t|)^A |D^B g(t)|$, with $A, B \in \mathbb{N}_0$, is dominated by a constant times $\sum_{\ell\in\mathbb{N}_0} |a_\ell| 2^{\ell(A-B-1)}$. All

these series converge. Thus g is Schwartz, has the required support, and its moments can be computed term by term.

Choose an invertible matrix A whose columns lie in a narrow cone around the positive vertical half-axis and whose positive cone lies in $\{y_n > (L+1)\|y'\|\}$. This choice is obtained by taking n linearly independent vectors sufficiently close to the vertical vector. Set $B := -A$ and

$$\vartheta(x) := |\det B|^{-1} \prod_{i=1}^{n} g((B^{-1}x)_i), \qquad \vartheta_j(x) := 2^{jn}\vartheta(2^j x), \quad j \in \mathbb{Z}.$$

The change of variables $x = By$ proves that ϑ has integral one. Every positive-degree monomial in By expands as a sum of monomials in y, each containing a positive-order moment of some factor g. Thus all positive-order moments of ϑ vanish. Its support lies in $\mathcal{K}_L$, separated from the vertex by a plane $x_n = -c$.

Now define

$$\varphi_0 := \vartheta, \qquad \psi_0 := 3\vartheta - 2\vartheta * \vartheta,$$

and, for $j \in \mathbb{N}$,

$$\varphi_j := \vartheta_j - \vartheta_{j-1},$$

$$\psi_j := 3(\vartheta_j + \vartheta_{j-1}) - 2(\vartheta_j * \vartheta_j + \vartheta_j * \vartheta_{j-1} + \vartheta_{j-1} * \vartheta_{j-1}).$$

These are dilations of two fixed functions. Stability of the cone under addition preserves the supports. The convolution moment formula, obtained by expanding $(x+y)^\alpha$, shows that φ_j and ψ_j have all moments zero. In particular, the integral of ψ_j is $6-6=0$.

The polynomial identity

$$(a-b)\big(3(a+b) - 2(a^2+ab+b^2)\big) = (3a^2 - 2a^3) - (3b^2 - 2b^3)$$

yields, in the convolution algebra, the telescoping sum

$$\sum_{j=0}^{N} \psi_j * \varphi_j = 3\vartheta_N * \vartheta_N - 2\vartheta_N * \vartheta_N * \vartheta_N.$$

Since $\int_{\mathbb{R}^n} \vartheta = 1$, $\vartheta_N * \zeta \to \zeta$ in $\mathcal{S}$ for $\zeta \in \mathcal{S}$. This is checked by writing the difference as

$$\int_{\mathbb{R}^n} \vartheta(y)\big(\zeta(\cdot - 2^{-N}y) - \zeta\big)\, dy$$

and using continuity of translations in each Schwartz seminorm, dominated by a power of $1 + \|y\|$. By duality, the telescoping sum converges to the identity in $\mathcal{S}'$. With the unitary Fourier normalization, this is equivalent to

$$\sum_{j \in \mathbb{N}_0} \widehat{\psi}_j(\xi)\widehat{\varphi}_j(\xi) = (2\pi)^{-n}, \qquad \xi \in \mathbb{R}^n. \tag{6.15}$$

If $x \in \Omega_\omega$ and $y \in \mathcal{K}_L$, then

$$x_n - y_n > \omega(x') + (L+1)\|y'\| \geq \omega(x'-y') + \|y'\|.$$

Thus $x - y \in \Omega_\omega$. For a tempered extension U of u, define $(\varphi_j * u)(x) := \langle U, \varphi_j(x - \cdot)\rangle$. The Schwartz function

in the pairing has support inside Ω_ω at a positive distance from its boundary. Approximate it in $\mathcal{S}$ by compact cutoffs whose supports remain in that domain. Two extensions agreeing on Ω_ω act identically on each cutoff and, after passage to the limit, on the whole function. This proves independence. Derivatives with respect to x are taken in the kernel, so the convolution is smooth. The same argument applies to ψ_j, and restriction of the global identity already proved gives (6.11).

We prove the bounds. If κ_k is the kernel of Δ_k, cancellation and Schwartz decay imply, for any $N, \lambda > 0$,

$$\int_{\mathbb{R}^n} |(\kappa_k * \psi_j)(y)|(1 + 2^j\|y\|)^\lambda \, dy \le C_{N,\lambda} 2^{-N|j-k|}. \tag{6.16}$$

If $j > k$, subtract from the kernel at scale 2^{-k} its Taylor polynomial of degree $M-1$ and use the moments of ψ_j. The remainder contributes $2^{-M(j-k)}$; the weight in the integral costs at most $2^{\lambda(j-k)}$. If $k > j$, the moments of κ_k contribute $2^{-M(k-j)}$. In both computations, first integrate the factor $\|y\|^M$ at the scale of the canceling kernel, then the decay of the other kernel. Choosing $M > N + \lambda$ and decay powers larger than $M + \lambda + n + 1$ proves the bound. Zero indices are treated in the same way using cancellation at the positive index; for $j = k = 0$, weighted integrability of two Schwartz functions suffices.

The same estimate holds for $\varphi_j * \widetilde{\kappa}_r$. Inserting the reconstruction of U and comparing weights at scales j and r, we obtain, for any $N > 0$,

$$Q_{j,\lambda} U(x) \le C_N \sum_{r \in \mathbb{N}_0} 2^{-N|j-r|} P_{r,\lambda}(\Delta_r U)(x). \tag{6.17}$$

In detail, bound $|\Delta_r U(z)|$ by $P_{r,\lambda}(\Delta_r U)(x)(1 + 2^r\|x - z\|)^\lambda$. After dividing by the weight at scale j, the triangle inequality for the weights introduces at most $2^{\lambda|j-r|}$. Absorb this factor by taking a higher cancellation order in (6.16). The pairings and sums first converge in $\mathcal{S}'$; the estimate holds with possibly infinite sides and, when the norm of U is finite, controls their pointwise convergence.

Take $N > |s|$. Young's inequality for sequences and control of Peetre's maximal function by $\mathcal{M}_a$ give (6.12) and (6.13). For Besov spaces, use the scalar maximal operator on each block; for Triebel–Lizorkin spaces, use the exponents p/a and q/a, both greater than one. The case $q = \infty$ follows using the maximal operator of the supremum.

Finally,

$$|\varphi_j * U(y)| \le Q_{j,\lambda} U(x)(1 + 2^j\|x - y\|)^\lambda.$$

Inserting this bound into convolution with $\kappa_k * \psi_j$ and using (6.16) proves (6.14). The indicator of the domain only decreases the integral of absolute values. □

This construction realizes Rychkov's conical reproduction [124, Theorem 4.1]. The distinction between $\mathcal{S}'(\Omega)$ and $\mathcal{D}'(\Omega)$, as well as the double-convolution formulation, can also be found in [133, Section 1.1].

The preceding computation gives a property to be used when differentiating the extension. Let η_j, θ_j and $j \in \mathbb{N}$ be dyadic dilations of Schwartz functions with all moments zero. For a measurable set $D \subseteq \mathbb{R}^n$ and $m \ge 0$, define

$$T_m f := \sum_{j \in \mathbb{N}} 2^{-jm} \eta_j * \big(\mathbf{1}_D (\theta_j * f)\big).$$

Estimates (6.16) and (6.17), applied to these two families, prove

$$\|T_m f\|_{A^{s+m}_{p,q}} \le C\|f\|_{A^s_{p,q}}, \qquad A \in \{B, F\}. \tag{6.18}$$

Indeed, after multiplying the output block by $2^{k(s+m)}$, the factor 2^{-jm} leaves

$$2^{k(s+m)}2^{-jm}2^{-N|j-k|} \le 2^{js}2^{-(N-|s+m|)|j-k|}.$$

Choosing $N > |s+m|$ allows the geometric summation in the proof to be repeated. The series converges in $\mathcal{S}'$: a tempered distribution gives $|\theta_j * f(x)| \le C2^{jM_0}(1+\|x\|)^{M_0}$ for some M_0, whereas pairing with $\eta_j(-\cdot) * \zeta$ decays faster than every power of 2^{-j}. This last bound is exactly the preceding Taylor computation applied in Schwartz seminorms. The same justification applies to the series defining the extension operator.

Theorem 6.5.2. (Extension on Lipschitz domains) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. There is a single linear operator E, independent of the parameters, with $Eu\restriction_\Omega = u$, inducing continuous operators

$$E\colon A^s_{p,q}(\Omega) \longrightarrow A^s_{p,q}(\mathbb{R}^n), \qquad A \in \{B, F\}, \quad s \in \mathbb{R},$$

for all indices defined in this chapter. Moreover,

$$E\colon W^{s,p}(\Omega) \longrightarrow W^{s,p}(\mathbb{R}^n) \quad \text{is continuous if } s \ge 0, \quad 1 < p < \infty.$$

If $s > 0$ is not an integer, then

$$W^{s,p}(\Omega) = B^s_{p,p}(\Omega) = F^s_{p,p}(\Omega)$$

with equivalent norms. The constants may depend on the domain and the parameters, but the operator is the same.

Proof. Begin on Ω_ω. For $u \in \mathcal{S}'(\Omega_\omega)$, define

$$E_\omega u := \sum_{j\in\mathbb{N}_0} \psi_j * \big(\mathbf{1}_{\Omega_\omega}(\varphi_j * u)\big).$$

The preceding lemma shows that this formula depends only on u. For a tempered extension U, the functions $\varphi_j * U$ are smooth and of polynomial growth, so their product with the indicator defines a tempered distribution. The justification following (6.18) proves convergence of the series. If the evaluation point lies in Ω_ω, the support of ψ_j allows the indicator to be removed. Conical reproduction then gives $E_\omega u = u$ on that domain.

Multiply (6.14) by 2^{ks}, take $N > |s|$, and apply Young's inequality and the local-means bounds. For any extension U in the scale under consideration, we obtain

$$\|E_\omega u\|_{A^s_{p,q}(\mathbb{R}^n)} \le C\|U\|_{A^s_{p,q}(\mathbb{R}^n)}.$$

Taking the infimum over $U\restriction_{\Omega_\omega} = u$ proves the bound by the quotient norm.

We now relate this norm to the intrinsic norms. First let $0 < \sigma < 1$ and $u \in W^{\sigma,p}(\Omega_\omega)$. Its extension by zero

belongs to $L^p(\mathbb{R}^n)$, so $u \in \mathcal{S}'(\Omega_\omega)$. For $j \geq 1$, cancellation of φ_j gives

$$\varphi_j * u(x) = \int_{K_L} \varphi_j(y)\big(u(x-y) - u(x)\big)\, dy.$$

Both arguments of u remain in the domain. Hölder's inequality with respect to the measure $|\varphi_j(y)|\,dy$ and Tonelli's theorem imply

$$\begin{aligned}\sum_{j\in\mathbb{N}} 2^{j\sigma p}\|\varphi_j * u\|^p_{L^p(\Omega_\omega)} &\leq C \int_{K_L} \left(\sum_{j\in\mathbb{N}} 2^{j\sigma p}|\varphi_j(y)|\right) \int_{\Omega_\omega} |u(x-y)-u(x)|^p\, dx\, dy \\ &\leq C[u]^p_{W^{\sigma,p}(\Omega_\omega)}.\end{aligned}$$

The second inequality uses

$$\sum_{j\in\mathbb{N}} 2^{j\sigma p}|\varphi_j(y)| \leq C\|y\|^{-n-\sigma p}.$$

This bound follows by separating $2^j\|y\| \leq 1$ and $2^j\|y\| > 1$ with a decay exponent greater than $n + \sigma p$. The zero block is controlled by Young's inequality. For Besov synthesis, the unweighted version of (6.16) and Young's inequality suffice, so

$$\|E_\omega u\|_{B^\sigma_{p,p}(\mathbb{R}^n)} \leq C\left(\|u\|_{L^p(\Omega_\omega)} + [u]_{W^{\sigma,p}(\Omega_\omega)}\right).$$

The global identification with $W^{\sigma,p}$ proves extension in this first interval.
To treat higher-order derivatives, construct primitives preserving the cone. With the function g used above, set

$$g_-(t) := \tfrac{1}{2}g(t/2), \qquad d(t) := g(t) - g_-(t).$$

All moments of d vanish. For $m \in \mathbb{N}$,

$$h_m(t) := \frac{1}{(m-1)!}\int_{-\infty}^{t} (t-r)^{m-1} d(r)\, dr$$

satisfies $D^m h_m = d$, is supported in $[1,\infty)$, and is Schwartz. Indeed, the vanishing moments allow the integral to be written as the negative of the integral from t to ∞, proving decay of all its derivatives. Integrating by parts m times gives

$$\int_{\mathbb{R}} t^\ell h_m(t)\, dt = (-1)^m \frac{\ell!}{(\ell+m)!}\int_{\mathbb{R}} t^{\ell+m} d(t)\, dt = 0, \qquad \ell \in \mathbb{N}_0.$$

The telescoping identity for a product gives

$$\prod_{r=1}^{n} g(y_r) - \prod_{r=1}^{n} g_-(y_r) = \sum_{i=1}^{n} \left(\prod_{r=1}^{i-1} g(y_r)\right) d(y_i) \left(\prod_{r=i+1}^{n} g_-(y_r)\right),$$

with the empty product equal to one. Replace d by $D^m h_m$ and make the linear change $x = By$. This gives

functions $\tau_{i,m} \in \mathcal{S}(\mathbb{R}^n)$ supported in the same cone, with all moments zero, such that

$$\varphi = \sum_{i=1}^{n} D_{v_i}^m \tau_{i,m}, \qquad v_i := Be_i.$$

Here D_{v_i} is differentiation in the constant direction v_i. If $\tau_{i,m,j}(x) := 2^{jn}\tau_{i,m}(2^j x)$, then

$$\varphi_j * u = 2^{-jm} \sum_{i=1}^{n} \tau_{i,m,j} * D_{v_i}^m u \quad \text{in } \Omega_\omega.$$

Integration by parts is distributional; the kernel supports remain inside the domain, so no boundary terms appear.

If $u \in W^{m,p}(\Omega_\omega)$, extend each $D_{v_i}^m u$ by zero to a function $f_i \in L^p(\mathbb{R}^n)$. For $|\alpha| \le m$, differentiating the series for $E_\omega u$ gives the zero block and the sums

$$\sum_{j\in\mathbb{N}} 2^{j(|\alpha|-m)} \psi_j^\alpha * \left(\mathbf{1}_{\Omega_\omega}(\tau_{i,m,j} * f_i)\right), \qquad \psi_j^\alpha := 2^{-j|\alpha|} D^\alpha \psi_j.$$

Bound (6.18), with scale $F_{p,2}^0 = L^p$, controls these sums in $F_{p,2}^{m-|\alpha|} \hookrightarrow L^p$. The zero block is controlled by Young's inequality with $D^\alpha \psi_0$. Since directional derivatives are finite combinations of partial derivatives of order m, we conclude

$$\|E_\omega u\|_{W^{m,p}(\mathbb{R}^n)} \le C\|u\|_{W^{m,p}(\Omega_\omega)}.$$

The case $m = 0$ follows directly from the bound in $F_{p,2}^0$, using the extension by zero of u.

If $s = m + \sigma$, $0 < \sigma < 1$, each $D_{v_i}^m u$ belongs to $W^{\sigma,p}(\Omega_\omega)$. The first case of the proof gives an extension f_i in $B_{p,p}^\sigma(\mathbb{R}^n)$ with controlled norm. The same representation, now without differentiating the output kernel, and (6.18) give

$$\|E_\omega u\|_{B_{p,p}^{m+\sigma}(\mathbb{R}^n)} \le C\left(\|u\|_{L^p(\Omega_\omega)} + \sum_{i=1}^{n} \|D_{v_i}^m u\|_{W^{\sigma,p}(\Omega_\omega)}\right) \le C\|u\|_{W^{m+\sigma,p}(\Omega_\omega)}.$$

The global identification $B_{p,p}^{m+\sigma} = W^{m+\sigma,p}$ completes the intrinsic bound at all noninteger orders.

For a bounded Lipschitz domain, choose a finite cover by boundary cylinders and an interior region. After rotations and translations, each boundary portion is the graph of a Lipschitz function. Extend that function to all of $\mathbb{R}^{n-1}$ by $\inf_{y\in D}(\omega(y) + L\|x' - y\|)$, where D is the graph domain; the Lipschitz inequality proves agreement with ω on D. This gives special domains Ω_i agreeing with Ω in smaller cylinders. Choose a smooth partition $\eta_0, \dots, \eta_N$ on a neighborhood of $\overline{\Omega}$, with η_0 supported in Ω, and cutoffs χ_i equal to one near $\operatorname{supp}\eta_i$ and supported in these smaller cylinders. Define

$$E_\Omega u := (\eta_0 u)^0 + \sum_{i=1}^{N} \chi_i E_i(\eta_i u).$$

On each special domain, set $\eta_i u$ equal to zero outside the cylinder; the cutoff vanishes near this artificial boundary. The superscript 0 indicates extension by zero of the interior part. On Ω, the formula equals

$\sum_{i=0}^{N} \eta_i u = u$. Smooth multipliers and finiteness of the cover give the bounds in restriction spaces.

For intrinsic norms, the Leibniz rule controls derivatives of the products. In the Gagliardo quotients, terms crossing an artificial boundary are estimated using the positive distance d from the cutoff support to that boundary and

$$\int_{\{h\in\mathbb{R}^n : \|h\|\geq d\}} \|h\|^{-n-\sigma p}\, dh = C_{n,\sigma,p} d^{-\sigma p}.$$

Interior terms are controlled by the seminorm of u and by $|\eta_i(x)-\eta_i(y)| \leq C\min\{1,\|x-y\|\}$. For higher-order products, the Leibniz rule also produces derivatives of u of order less than m. Gluing at integer orders, which uses only the Leibniz rule, already gives a continuous operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^n)$. Each of these lower derivatives belongs to $W^{1,p}(\Omega)$; extend it with this operator and apply the increment lemma to control its $W^{\sigma,p}$ norm. This justifies all fractional terms in the products and preserves the intrinsic bounds under gluing.

Finally, restricting a global function never increases either its derivative norms or its Gagliardo integral. This gives $\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{B^s_{p,p}(\Omega)}$. The constructed extension gives the reverse inequality. The identity $B^s_{p,p} = F^s_{p,p}$ passes to the quotients and completes the proof. □

Corollary 6.5.3. (Comparison of H/W on a domain) Let Ω be a bounded Lipschitz domain, let $s > 0$ be noninteger, and let $1 < p < \infty$. Then

$$\begin{cases} W^{s,p}(\Omega) \hookrightarrow H^{s,p}(\Omega), & 1 < p \leq 2, \\ H^{s,p}(\Omega) \hookrightarrow W^{s,p}(\Omega), & 2 \leq p < \infty. \end{cases}$$

Equality holds for $p = 2$ and, if Ω has nonempty interior, the embedding is strict for $p \neq 2$. On an arbitrary open subset, only the embedding of the restriction space $B^s_{p,p}(\Omega)$ into the intrinsic Gagliardo space is always guaranteed; equality, even for $p = 2$, requires an extension property.

Proof. Apply the extension operator from Theorem 6.5.2, the global embedding from Corollary 6.3.15, and restriction to the domain. The identity for $p = 2$ follows in the same way.

For strictness, fix a ball $B \Subset \Omega$. The construction in Corollary 6.3.15 gives a function u supported in B that belongs to the larger scale but not the smaller one on $\mathbb{R}^n$. Its restriction belongs to the larger scale on Ω. If it also belonged to the smaller scale, the extension operator would give an extension V in that global scale. Take $\chi \in C^\infty_c(\Omega)$ equal to one near $\overline{B}$. Since $V = u$ on Ω, we have $\chi V = u$ on $\mathcal{S}'(\mathbb{R}^n)$. Continuity of multiplication by χ would place u in the smaller global scale, a contradiction.

On an arbitrary open subset, restricting a global extension decreases the integrals defining the intrinsic norm. This gives $B^s_{p,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$. Our proof of the reverse inclusion requires the continuous extension constructed for Lipschitz domains. □

6.6 Sobolev embeddings of real order

The quantity

$$\operatorname{ind}(s,p) := s - \frac{n}{p}$$

is called the differential index. Bernstein's inequalities from Theorem 4.12.4 show that passage from p_0 to a larger exponent p_1 requires losing $n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)$ derivatives. Interpolation explains the same rule starting from integer-order embeddings.

Theorem 6.6.1. (Embeddings between scales) Let $1 < p_0 \le p_1 < \infty$ and $s_0, s_1 \in \mathbb{R}$.

(a) If $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$, then

$$B^{s_0}_{p_0,q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1,q}(\mathbb{R}^n)$$

for $1 \le q \le \infty$, and also

$$F^{s_0}_{p_0,q}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^n)$$

for the same fine indices.

(b) If $s_0 > s_1$ and p is fixed, then

$$A^{s_0}_{p,q_0}(\mathbb{R}^n) \hookrightarrow A^{s_1}_{p,q_1}(\mathbb{R}^n), \qquad A \in \{B, F\},$$

for any admissible fine indices q_0, q_1.

(c) If $s_0 = s_1$, the embedding within a single scale holds when $q_0 \le q_1$.

Proof. For Besov spaces, Bernstein's inequality gives, for each $j \in \mathbb{N}_0$,

$$2^{js_1}\|\Delta_j u\|_{L^{p_1}(\mathbb{R}^n)} \le C 2^{j(s_1 + n(1/p_0 - 1/p_1))}\|\Delta_j u\|_{L^{p_0}(\mathbb{R}^n)} = C 2^{js_0}\|\Delta_j u\|_{L^{p_0}(\mathbb{R}^n)}.$$

Taking the $\ell^q(\mathbb{N}_0)$ norm proves the first embedding.
For Triebel–Lizorkin spaces, the blocks must be estimated together before integration. If $p_0 = p_1$, then also $s_0 = s_1$, and the assertion is immediate. Assume $p_0 < p_1$ and set

$$d = s_0 - s_1 = n(1/p_0 - 1/p_1), \qquad f_j = 2^{js_0}\Delta_j u, \qquad G(x) = \|(f_j(x))_{j\in\mathbb{N}_0}\|_{\ell^q}.$$

We have $0 < d < n/p_0$. A band-limited reproducing kernel gives $f_j = \widetilde{K}_j * f_j$, where

$$|\widetilde{K}_j(y)| \le C_N 2^{jn}(1 + 2^j\|y\|)^{-N}.$$

By Minkowski's inequality in ℓ^q, with the supremum if $q = \infty$,

$$\begin{aligned}\|(2^{-jd}f_j(x))_{j\in\mathbb{N}_0}\|_{\ell^q} &\le C_N \int_{\mathbb{R}^n} \sup_{j\in\mathbb{N}_0}\left[2^{j(n-d)}(1 + 2^j\|y\|)^{-N}\right] G(x-y)\,dy \\ &\le C\int_{\mathbb{R}^n} \|y\|^{d-n} G(x-y)\,dy =: C I_d G(x).\end{aligned}$$

The second inequality follows because $t^{n-d}(1+t)^{-N}$ is bounded for $t > 0$ if $N > n - d$.
We prove the scalar bound needed for I_d. For $R > 0$, split the integral into $\|y\| < R$ and $\|y\| \ge R$. Decomposing the first region into annuli $2^{-k-1}R \le \|y\| < 2^{-k}R$, $k \in \mathbb{N}_0$, and using the Hardy–Littlewood maximal operator

gives

$$\int_{\|y\|<R} \|y\|^{d-n} G(x-y)\,dy \le CR^d \mathcal{M}G(x) \sum_{k\in\mathbb{N}_0} 2^{-kd} \le C' R^d \mathcal{M}G(x).$$

Hölder's inequality on the second region yields

$$\int_{\|y\|\ge R} \|y\|^{d-n} G(x-y)\,dy \le CR^{d-n/p_0} \|G\|_{L^{p_0}(\mathbb{R}^n)},$$

since $(n-d)p_0' > n$. When $\mathcal{M}G(x) > 0$, choose $R = (\|G\|_{L^{p_0}(\mathbb{R}^n)}/\mathcal{M}G(x))^{p_0/n}$; if the maximal function vanishes, both integrals vanish. This gives

$$I_d G(x) \le C\|G\|_{L^{p_0}(\mathbb{R}^n)}^{dp_0/n} \big(\mathcal{M}G(x)\big)^{1-dp_0/n}.$$

Since $p_1(1-dp_0/n) = p_0$, boundedness of the maximal operator on L^{p_0} implies $\|I_d G\|_{L^{p_1}(\mathbb{R}^n)} \le C\|G\|_{L^{p_0}(\mathbb{R}^n)}$. This inequality and the preceding vector-valued estimate prove $F^{s_0}_{p_0,q} \hookrightarrow F^{s_1}_{p_1,q}$, including $q = \infty$.
For (b), write $\delta = s_0 - s_1 > 0$. If $(b_j)_{j\in\mathbb{N}_0}$ is a nonnegative sequence,

$$\|(2^{-j\delta} b_j)_{j\in\mathbb{N}_0}\|_{\ell^{q_1}} \le \|(2^{-j\delta})_{j\in\mathbb{N}_0}\|_{\ell^{q_1}} \sup_{j\in\mathbb{N}_0} b_j \le C_{\delta,q_1} \|(b_j)_{j\in\mathbb{N}_0}\|_{\ell^{q_0}}.$$

Apply this inequality to $b_j = 2^{js_0}\|\Delta_j u\|_{L^p(\mathbb{R}^n)}$ for Besov spaces and pointwise to $b_j(x) = 2^{js_0}|\Delta_j u(x)|$ for Triebel–Lizorkin spaces. Finally, (c) is the inclusion $\ell^{q_0} \hookrightarrow \ell^{q_1}$ when $q_0 \le q_1$, applied in these same two positions. □

Lemma 6.6.2. (Fractional Morrey estimate for the Gagliardo seminorm) Let $0 < s < 1$, $1 < p < \infty$, and $sp > n$. If $\alpha = s - \frac{n}{p}$, every $u \in W^{s,p}(\mathbb{R}^n)$ has a representative in $C_b^{0,\alpha}(\mathbb{R}^n)$, and

$$\|u\|_{C_b^{0,\alpha}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\mathbb{R}^n)}.$$

Proof. Let $B = B(x,r)$. By Proposition B.7.9 and Theorem B.6.17,

$$\begin{aligned}\int_B |u-u_B|^p &\le \frac{1}{|B|}\int_B\int_B |u(y)-u(z)|^p\,dy\,dz \\ &\le Cr^{-n}(2r)^{n+sp}[u]^p_{W^{s,p}(\mathbb{R}^n)} = Cr^{n+p\alpha}[u]^p_{W^{s,p}(\mathbb{R}^n)}.\end{aligned}$$

We explain the passage from this integral estimate to pointwise continuity. If $B_k = B(x, 2^{-k}r)$, Proposition B.7.9 gives

$$|u_{B_{k+1}} - u_{B_k}| \le C(2^{-k}r)^\alpha [u]_{W^{s,p}(\mathbb{R}^n)}.$$

The geometric series shows that $(u_{B_k})_{k\in\mathbb{N}_0}$ converges. By Theorem B.8.12, its limit agrees with $u(x)$ for almost every x and defines a representative at every point. If $r = \|x-y\|$, compare the averages on $B(x,r)$ and $B(y,r)$ with the average on a ball containing both; the same estimate and the two dyadic telescoping sums give

$$|u(x)-u(y)| \le C\|x-y\|^\alpha [u]_{W^{s,p}(\mathbb{R}^n)}.$$

Finally, Proposition B.7.9 bounds the average on the unit ball by $\|u\|_{L^p(\mathbb{R}^n)}$, which also controls the uniform norm. □

Theorem 6.6.3. (Sobolev, Morrey, and the critical case) Let $s > 0$ and $1 < p < \infty$.

(a) If $sp < n$ and $p_s^* = \dfrac{np}{n-sp}$, then

$$H^{s,p}(\mathbb{R}^n) \hookrightarrow L^{p_s^*}(\mathbb{R}^n).$$

If $s \notin \mathbb{N}$, also

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow L^{p_s^*,p}(\mathbb{R}^n) \hookrightarrow L^{p_s^*}(\mathbb{R}^n).$$

(b) If $sp = n$, both $H^{s,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ embed into $L^q(\mathbb{R}^n)$ for every $p \le q < \infty$, but not into $L^\infty(\mathbb{R}^n)$ in general. The limiting embedding

$$B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$$

is continuous.

(c) If $sp > n$, the elements have continuous representatives. More precisely, if $k \in \mathbb{N}_0$ and $0 < \alpha < 1$ satisfy $k + \alpha < s - \frac{n}{p}$, then

$$H^{s,p}(\mathbb{R}^n),\ W^{s,p}(\mathbb{R}^n) \hookrightarrow C_b^{k,\alpha}(\mathbb{R}^n).$$

When $0 < s < 1$ and $sp > n$, the Slobodeckij scale reaches the exponent $\alpha = s - \frac{n}{p}$.

The same results hold on a bounded Lipschitz domain after restriction; there, in the critical case, one obtains $L^q(\Omega)$ for every $1 \le q < \infty$.

Proof. We first explain the interpolation mechanism. By Theorem 6.3.10, the embedding of the F scale at constant differential index gives

$$H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n) \hookrightarrow F^0_{p_s^*,2}(\mathbb{R}^n) = L^{p_s^*}(\mathbb{R}^n).$$

To obtain the Lorentz refinement without varying the spatial exponent in the real interpolation formula, choose

$$0 \le s_0 < s < s_1 < \frac{n}{p}, \qquad s = (1-\theta)s_0 + \theta s_1,$$

and define q_i by $q_i^{-1} = p^{-1} - \frac{s_i}{n}$. The embedding just proved gives $H^{s_i,p}(\mathbb{R}^n) \hookrightarrow L^{q_i}(\mathbb{R}^n)$. Interpolate the identity operator using Theorem 5.3.1 and use Theorems 6.3.11 and 5.8.3:

$$W^{s,p}(\mathbb{R}^n) = (H^{s_0,p}(\mathbb{R}^n), H^{s_1,p}(\mathbb{R}^n))_{\theta,p} \hookrightarrow (L^{q_0}(\mathbb{R}^n), L^{q_1}(\mathbb{R}^n))_{\theta,p} = L^{p_s^*,p}(\mathbb{R}^n).$$

Indeed, $(p_s^*)^{-1} = (1-\theta)q_0^{-1} + \theta q_1^{-1}$. Since $p < p_s^*$, monotonicity of the second index in Lorentz spaces, included in Theorem 5.8.3, implies $L^{p_s^*,p}(\mathbb{R}^n) \hookrightarrow L^{p_s^*}(\mathbb{R}^n)$. This proves (a).
If $sp = n$, apply (a) with any $s - \varepsilon$ and choose ε to obtain the desired finite exponent. On $\mathbb{R}^n$, this covers $q \ge p$; on a domain of finite measure, Proposition B.7.9 also covers $q < p$.
We show explicitly that the critical embedding into $L^\infty(\mathbb{R}^n)$ fails. Choose $\eta \in \mathcal{S}(\mathbb{R}^n)$ with $\eta(0) > 0$ and

$$\operatorname{supp}(\widehat{\eta}) \subseteq \left\{ \xi \in \mathbb{R}^n \,\middle|\, \frac{3}{4} \le \|\xi\| \le \frac{4}{3} \right\},$$

and define

$$u_N(x) := \sum_{j=1}^{N} \frac{1}{j} \eta(2^j x).$$

Finite interaction of the blocks and the change of variable $y = 2^j x$ give

$$\sup_{N\in\mathbb{N}} \|u_N\|_{B^{\frac{n}{p}}_{p,p}(\mathbb{R}^n)} \le C \left(\sum_{j=1}^{\infty} \frac{1}{j^p} \right)^{\frac{1}{p}} < \infty.$$

The critical Triebel–Lizorkin norm is also uniformly bounded. Indeed, on the annulus $2^{-k-1} < \|x\| \le 2^{-k}$, geometric growth for $j \le k$ and Schwartz decay for $j > k$ give

$$\left(\sum_{j=1}^{N} \frac{2^{\frac{2jn}{p}}}{j^2} |\eta(2^j x)|^2 \right)^{\frac{1}{2}} \le C \frac{2^{\frac{kn}{p}}}{1+k}.$$

Use this bound for $1 \le k < N$. Raising to the power p and integrating over these annuli gives $\sum_{k=1}^{N-1} (1+k)^{-p}$. On the ball $\|x\| \le 2^{-N}$, the geometric sum gives a bound $C2^{Nn/p}/N$, whose integral to the power p is at most CN^{-p}. On $\|x\| > 1$, choose $M > n/p$ and use $|\eta(2^j x)| \le C_M 2^{-jM} \|x\|^{-M}$; the sum is bounded by $C_M \|x\|^{-M}$, whose pth power is integrable on that region. The region $1/2 < \|x\| \le 1$ likewise has a uniform bound. Lemma 6.3.6 converts these estimates for the sequence of summands into the norm of u_N. Thus,

$$\sup_{N\in\mathbb{N}} \|u_N\|_{F^{\frac{n}{p}}_{p,2}(\mathbb{R}^n)} < \infty.$$

However, $u_N(0) = \eta(0) \sum_{j=1}^{N} \frac{1}{j} \to \infty$. Theorem 6.3.10 always identifies $H^{\frac{n}{p},p}(\mathbb{R}^n)$ with $F^{\frac{n}{p}}_{p,2}(\mathbb{R}^n)$. If $\frac{n}{p} \notin \mathbb{N}$, Corollary 6.3.14 identifies $W^{\frac{n}{p},p}(\mathbb{R}^n)$ with $B^{\frac{n}{p}}_{p,p}(\mathbb{R}^n)$; if $\frac{n}{p} \in \mathbb{N}$, the same Theorem 6.3.10 gives $W^{\frac{n}{p},p}(\mathbb{R}^n) = H^{\frac{n}{p},p}(\mathbb{R}^n)$. Therefore, in every case, neither $W^{\frac{n}{p},p}(\mathbb{R}^n)$ nor $H^{\frac{n}{p},p}(\mathbb{R}^n)$ embeds into $L^\infty(\mathbb{R}^n)$.
On the other hand, Theorem 4.12.4 and absolute summation give

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le \sum_{j\in\mathbb{N}_0} \|\Delta_j u\|_{L^\infty(\mathbb{R}^n)} \le C \sum_{j\in\mathbb{N}_0} 2^{\frac{jn}{p}} \|\Delta_j u\|_{L^p(\mathbb{R}^n)} = C\|u\|_{B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)}.$$

If $sp > n$, Theorem 4.12.4, applied block by block, maps the source space into $B^{s-\frac{n}{p}}_{\infty,\infty}(\mathbb{R}^n)$, the Hölder–Zygmund scale. If $k + \alpha < s - \frac{n}{p}$, the uniform block sum and the estimate

$$\|D^k \Delta_j u(\cdot + h) - D^k \Delta_j u\|_{L^\infty(\mathbb{R}^n)} \le C \min\{1, 2^j\|h\|\}\, 2^{jk} \|\Delta_j u\|_{L^\infty(\mathbb{R}^n)}$$

are split into $2^j\|h\| \le 1$ and $2^j\|h\| > 1$; the two geometric series give the modulus $\|h\|^\alpha$. For $W^{s,p}(\mathbb{R}^n)$ with $0 < s < 1$, Lemma 6.6.2 directly provides the limiting exponent $s - \frac{n}{p}$. The extension from Theorem 6.5.2 transfers all embeddings to the domain. □

Theorem 6.6.4. (Rellich–Kondrashov of real order) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, and let $1 < p < \infty$ and $s > t \geq 0$. Then

$$H^{s,p}(\Omega) \hookrightarrow H^{t,p}(\Omega)$$

is compact. If s, t are not integers, the same result holds for $W^{s,p}(\Omega) \hookrightarrow W^{t,p}(\Omega)$; with the usual integer-order interpretations, it holds for all nonnegative orders. Moreover, if $sp < n$ and $1 \leq q < p_s^*$, the embeddings into $L^q(\Omega)$ are compact.

Proof. Let $(u_\nu)_{\nu\in\mathbb{N}}$ be a bounded sequence in one of the spaces of order s. Extend it by Theorem 6.5.2 and multiply by a cutoff $\zeta \in C_c^\infty(\mathbb{R}^n)$ equal to one on a neighborhood of $\overline{\Omega}$. This gives a sequence $(v_\nu)_{\nu\in\mathbb{N}}$ supported in a fixed compact set $K \subset \mathbb{R}^n$ and bounded in the corresponding global space. If

$$P_J := \sum_{j=0}^{J} \Delta_j,$$

the block definition and Proposition B.8.4 give the uniform tail estimate

$$\|(I - P_J)v_\nu\|_{A^t_{p,r}(\mathbb{R}^n)} \leq C 2^{-J(s-t)} \|v_\nu\|_{A^s_{p,r}(\mathbb{R}^n)}, \qquad A \in \{B, F\}.$$

Here take $A = F$, $r = 2$ for the Bessel scale and $A = B$, $r = p$ for the noninteger Slobodeckij scale. We first prove compactness within each of these two scales, then treat integer or mixed orders of $W^{s,p}$.
Fix J. The operator P_J is convolution with a kernel $K_J \in \mathcal{S}(\mathbb{R}^n)$. The sequence $(v_\nu)_{\nu\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n)$ and vanishes outside K. By Theorem 4.10.1 and Proposition B.8.17,

$$\|P_J v_\nu(\cdot + h) - P_J v_\nu\|_{L^p(\mathbb{R}^n)} \leq \|K_J(\cdot + h) - K_J\|_{L^1(\mathbb{R}^n)} \|v_\nu\|_{L^p(\mathbb{R}^n)} \longrightarrow 0$$

uniformly in ν as $\|h\| \to 0$. Moreover, Minkowski's integral inequality from Theorem B.8.2 gives

$$\|P_J v_\nu\|_{L^p(\{x\in\mathbb{R}^n \mid \|x\| > R\})} \leq \|v_\nu\|_{L^1(K)} \sup_{y\in K} \|K_J(\cdot - y)\|_{L^p(\{x\in\mathbb{R}^n \mid \|x\| > R\})} \longrightarrow 0$$

uniformly in ν as $R \to \infty$; we also used $\|v_\nu\|_{L^1(K)} \leq |K|^{\frac{1}{p'}} \|v_\nu\|_{L^p(\mathbb{R}^n)}$. The Fréchet–Kolmogorov criterion from Proposition B.7.19 shows that $(P_J v_\nu)_{\nu\in\mathbb{N}}$ is relatively compact in $L^p(\mathbb{R}^n)$.
Since $P_J v_\nu$ has frequencies in a fixed ball, Theorem 4.12.4 implies

$$\|P_J(v_\nu - v_\mu)\|_{A^t_{p,r}(\mathbb{R}^n)} \leq C_J \|P_J(v_\nu - v_\mu)\|_{L^p(\mathbb{R}^n)}.$$

Write $M_s := \sup_{\nu\in\mathbb{N}} \|v_\nu\|_{A^s_{p,r}(\mathbb{R}^n)} < \infty$. For each $k \in \mathbb{N}$, choose J_k so that

$$2C 2^{-J_k(s-t)} M_s < 2^{-k}.$$

Starting from the original sequence, recursively extract nested subsequences $(v_\nu^{(k)})_\nu$: once the subsequence at level $k-1$ is chosen, compactness of $(P_{J_k} v_\nu^{(k-1)})_\nu$ in $L^p(\mathbb{R}^n)$ and Bernstein's inequality allow us to choose $(v_\nu^{(k)})_\nu$ so that $(P_{J_k} v_\nu^{(k)})_\nu$ converges in $A^t_{p,r}(\mathbb{R}^n)$. Take the diagonal subsequence $w_k := v_k^{(k)}$. For each fixed k_0, the tail $(w_k)_{k\geq k_0}$ is a subsequence of $(v_\nu^{(k_0)})_\nu$; thus $(P_{J_{k_0}} w_k)_{k\geq k_0}$ is Cauchy in the space of order t. Given $\varepsilon > 0$,

choose k_0 with $2^{-k_0} < \frac{\varepsilon}{2}$ and then $N \geq k_0$ so that for $k, \ell \geq N$,

$$\|P_{J_{k_0}}(w_k - w_\ell)\|_{A^t_{p,r}(\mathbb{R}^n)} < \frac{\varepsilon}{2}.$$

The uniform estimate for the two tails then gives

$$\|w_k - w_\ell\|_{A^t_{p,r}(\mathbb{R}^n)} < \varepsilon.$$

Thus (w_k) converges in the space of order t. Restriction to Ω is continuous, proving compactness between orders within each scale.

For integer or mixed orders of $W^{s,p}$, choose two nonintegers a, b with $t < b < a < s$. The strict loss in order allows the fine block index to change. Indeed, if $A_j := 2^{js}\|\Delta_j v\|_{L^p(\mathbb{R}^n)}$, the source norm dominates $\sup_{j\in\mathbb{N}_0} A_j$, both for $B^s_{p,p}$ and for $F^s_{p,2}$. By convergence of the geometric series,

$$\|v\|_{B^a_{p,p}(\mathbb{R}^n)} \leq C\left(\sum_{j=0}^{\infty} 2^{-jp(s-a)} A_j^p\right)^{1/p} \leq C_{s-a}\|v\|_{W^{s,p}(\mathbb{R}^n)}.$$

When s is an integer, use $W^{s,p} = H^{s,p} = F^s_{p,2}$ here, with equivalence of norms. At the target endpoint, the triangle inequality in $L^p(\ell^2)$ and Hölder's inequality for sequences give

$$\|v\|_{F^t_{p,2}(\mathbb{R}^n)} \leq C\sum_{j=0}^{\infty} 2^{jt}\|\Delta_j v\|_{L^p(\mathbb{R}^n)} \leq C_{b-t}\|v\|_{B^b_{p,p}(\mathbb{R}^n)}.$$

If t is not an integer, the inclusion $B^b_{p,p} \hookrightarrow B^t_{p,p}$ follows directly from the weights; if it is an integer, the last estimate and $F^t_{p,2} = W^{t,p}$ give the required inclusion. Applying these estimates to extensions with common support and using the compactness from $B^a_{p,p}$ to $B^b_{p,p}$ just proved gives a subsequence converging in $W^{t,p}(\Omega)$. This includes all nonnegative orders without identifying $B^j_{p,p}$ with $W^{j,p}$ at an integer order.

For the embedding into $L^q(\Omega)$, choose $0 < t < s$ such that $q < p^*_t < p^*_s$ when $q > p$; if $q \leq p$, any $0 < t < s$ and Proposition B.7.9 on the domain of finite measure suffice. In both cases, use

$$A^s_{p,r}(\Omega) \hookrightarrow\hookrightarrow A^t_{p,r}(\Omega) \hookrightarrow L^q(\Omega).$$

The second embedding is Theorem 6.6.3. This completes the proof. □

6.7 Fractional-order traces

Restriction to the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ lowers dimension by one. A function concentrated in a strip of thickness 2^{-j} has a spatial norm containing the factor $2^{-j/p}$; evaluation at the center of the strip loses that factor. This explains the loss of $1/p$ derivatives. In Triebel–Lizorkin spaces, integration in the normal direction also determines the fine index p of the target norm.

Assume $n \geq 2$. When $n = 1$, omit tangential variables and interpret the space on a point as $\mathbb{K}$ with its absolute value; the same arguments give evaluation at the endpoints of an interval.

For a compact smooth hypersurface Σ, define $B^\sigma_{p,q}(\Sigma)$ using a finite atlas, a partition of unity, and the Euclidean

norms of the local expressions. Lemma 6.4.3, applied in dimension $n-1$, compares any two such norms: insert one partition into the other and apply the coordinate-change operators on each overlap. Since there are finitely many terms, both inequalities have constants independent of the distribution. The resulting space is independent of the atlas.

Lemma 6.7.1. (Dyadic trace estimate) Let $1 < p < \infty$ and $j \in \mathbb{N}_0$, and let $v \in L^p(\mathbb{R}^n)$ satisfy

$$\operatorname{supp} \widehat{v} \subseteq \{\xi \in \mathbb{R}^n \mid \|\xi\| \le A2^j\}.$$

Then

$$\|v(\cdot,0)\|_{L^p(\mathbb{R}^{n-1})} \le C_{A,p}\, 2^{j/p} \|v\|_{L^p(\mathbb{R}^n)}. \tag{6.19}$$

Let $s > 1/p$, $1 \le q \le \infty$, $u_j = \Delta_j u$, $j \in \mathbb{N}_0$, and $\sigma = s - 1/p > 0$. If the corresponding right-hand norm is finite, then

$$\left\| \left(2^{\ell\sigma} \left\| \Delta'_\ell \sum_{j\in\mathbb{N}_0} u_j(\cdot,0) \right\|_{L^p(\mathbb{R}^{n-1})} \right)_{\ell\in\mathbb{N}_0} \right\|_{\ell^q} \le C \left\| \left(2^{js}\|u_j\|_{L^p(\mathbb{R}^n)}\right)_{j\in\mathbb{N}_0} \right\|_{\ell^q}, \tag{6.20}$$

and

$$\left(\sum_{\ell\in\mathbb{N}_0} 2^{\ell\sigma p} \left\| \Delta'_\ell \sum_{j\in\mathbb{N}_0} u_j(\cdot,0) \right\|^p_{L^p(\mathbb{R}^{n-1})} \right)^{1/p} \le C \left\| \left(2^{js} u_j\right)_{j\in\mathbb{N}_0} \right\|_{L^p(\mathbb{R}^n;\ell^q(\mathbb{N}_0))}. \tag{6.21}$$

Here Δ'_ℓ are blocks on $\mathbb{R}^{n-1}$. Suprema replace the ℓ^∞ norms. The constant in (6.21) can be chosen independently of $q \in [1,\infty]$.

Proof. Choose a kernel $\eta \in \mathcal{S}(\mathbb{R})$ whose convolution multiplier $(2\pi)^{1/2}\widehat{\eta}$ equals one on $[-A,A]$. Reproduction in the normal variable gives

$$v(x',0) = \int_{\mathbb{R}} 2^j \eta(-2^j t) v(x',t)\, dt.$$

Hölder's inequality in t implies

$$|v(x',0)|^p \le C 2^j \int_{\mathbb{R}} |v(x',t)|^p\, dt.$$

Integration in x' proves (6.19). The formula is first obtained for Schwartz functions, then for v through reproduction by a kernel of enlarged frequency support. This kernel provides the smooth representative of v and allows passage to the limit in L^p.

Set $g_j = u_j(\cdot,0)$. Its tangential Fourier support lies in a ball of radius $C2^j$, although it may contain low tangential frequencies. Thus,

$$\Delta'_\ell \sum_{j\in\mathbb{N}_0} g_j = \sum_{\substack{j\in\mathbb{N}_0 \\ j\ge \ell - N_0}} \Delta'_\ell g_j. \tag{6.22}$$

Young's inequality and (6.19) give

$$2^{\ell\sigma}\|\Delta'_\ell g_j\|_{L^p(\mathbb{R}^{n-1})} \le C 2^{-(j-\ell)\sigma} 2^{js}\|u_j\|_{L^p(\mathbb{R}^n)}, \qquad j \ge \ell - N_0. \tag{6.23}$$

The sequence $\left(2^{-m\sigma}\mathbf{1}_{\{m\ge -N_0\}}\right)_{m\in\mathbb{Z}}$ is summable. Young's inequality for sequences proves (6.20).

For the second estimate, integrate each block over a different strip. Define

$$I_j := (2^{-j}, 2^{1-j}], \qquad j \in \mathbb{N}_0.$$

These intervals are disjoint and have length 2^{-j}. If $t \in I_j$, Peetre's definition implies

$$|u_j(x', 0)| \le 3^{\lambda} P_{j,\lambda} u_j(x', t).$$

Multiply by 2^{jsp} and integrate in I_j and then in x'. Summing in j, disjointness of the intervals gives

$$\begin{aligned}\sum_{j\in\mathbb{N}_0} 2^{j(sp-1)} \|g_j\|_{L^p(\mathbb{R}^{n-1})}^p &\le C \sum_{j\in\mathbb{N}_0} \int_{\mathbb{R}^{n-1}\times I_j} \left(2^{js} P_{j,\lambda} u_j(x)\right)^p dx \\ &\le C \int_{\mathbb{R}^n} \sup_{j\in\mathbb{N}_0} \left(2^{js} P_{j,\lambda} u_j(x)\right)^p dx.\end{aligned}$$

Choose $0 < a < 1$ and $\lambda > n/a$. Peetre control and monotonicity of the maximal operator give

$$\sup_{j\in\mathbb{N}_0} 2^{js} P_{j,\lambda} u_j \le C \left[\mathcal{M}\left(\sup_{j\in\mathbb{N}_0} |2^{js} u_j|^a \right) \right]^{1/a}.$$

Since $p/a > 1$, boundedness of the maximal operator implies

$$\left(\sum_{j\in\mathbb{N}_0} 2^{j\sigma p} \|g_j\|_{L^p(\mathbb{R}^{n-1})}^p \right)^{1/p} \le C \left\| \sup_{j\in\mathbb{N}_0} |2^{js} u_j| \right\|_{L^p(\mathbb{R}^n)} \le C \left\| \left(2^{js} u_j\right)_{j\in\mathbb{N}_0} \right\|_{L^p(\mathbb{R}^n;\ell^q(\mathbb{N}_0))}.$$

Finally, (6.22), Young's convolution inequality, and the same geometric sequence from (6.23) transfer this norm of the g_j to the $B^{\sigma}_{p,p}$ norm of their sum. This proves (6.21) with a constant independent of q.
The trace series converge in $L^p(\mathbb{R}^{n-1})$: in both situations, the sequence $(2^{j\sigma}\|g_j\|_{L^p(\mathbb{R}^{n-1})})_{j\in\mathbb{N}_0}$ is bounded and $\sum_{j\in\mathbb{N}_0} 2^{-j\sigma} < \infty$. Thus all steps for finite families pass to the limit. □

Theorem 6.7.2. (Hyperplane trace and coretraction) Let $1 < p < \infty$, $1 \le q \le \infty$, and $s > 1/p$. Restriction to the hyperplane induces continuous surjective operators

$$\operatorname{tr}\colon B^s_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-1/p}_{p,q}(\mathbb{R}^{n-1}), \qquad \operatorname{tr}\colon F^s_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}). \tag{6.24}$$

Both have a continuous linear coretraction. In particular,

$$\operatorname{tr}\colon H^{s,p}(\mathbb{R}^n) \longrightarrow B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \tag{6.25}$$

is a retraction.

Proof. For each element of the spaces in the statement, define

$$\operatorname{tr} u := \sum_{j\in\mathbb{N}_0} (\Delta_j u)(\cdot, 0).$$

The lemma proves convergence and continuity. This definition agrees with canonical evaluation in the normal direction. Indeed, (6.19) holds on every hyperplane of height t with the same constant, and

$$\sup_{t\in\mathbb{R}} \|(\Delta_j u)(\cdot,t)\|_{L^p(\mathbb{R}^{n-1})} \le C 2^{-j(s-1/p)} \|u\|_{A^s_{p,q}}.$$

Each block is a continuous function of t with values in $L^p(\mathbb{R}^{n-1})$. This follows from one-dimensional reproduction and continuity of translations of its kernel. The series converges uniformly in this space of continuous functions. Its sum represents u as a distribution by dyadic reconstruction, and its value at zero is the defined trace. Thus the trace is independent of the resolution. If u vanishes on the half-space $t > 0$, its continuous representative in t vanishes for $t > 0$ and also at $t = 0$. This observation will give independence from extensions.

Choose $\rho \in \mathcal{S}(\mathbb{R})$ with $\rho(0) = 1$ and compact Fourier support. For data g in the target space, set

$$(\operatorname{ex} g)(x',t) := \sum_{j\in\mathbb{N}_0} \rho(2^j t)\Delta'_j g(x'). \tag{6.26}$$

The summands have spectral support in annuli of $\mathbb{R}^n$ comparable to 2^j, except for the zero block, whose support lies in a ball. Moreover,

$$\|\rho(2^j\cdot)\Delta'_j g\|_{L^p(\mathbb{R}^n)} = 2^{-j/p}\|\rho\|_{L^p(\mathbb{R})}\|\Delta'_j g\|_{L^p(\mathbb{R}^{n-1})}. \tag{6.27}$$

Lemma 6.3.6 gives the Besov bound and distributional convergence.

For Triebel–Lizorkin spaces, we prove the stronger bound with fine index one; the inclusion $\ell^1 \hookrightarrow \ell^q$ will give the other indices. Write

$$b_j(x') := 2^{j(s-1/p)}|\Delta'_j g(x')|, \qquad j \in \mathbb{N}_0.$$

Spectral synthesis and decay of ρ give

$$\|\operatorname{ex} g\|_{F^s_{p,1}(\mathbb{R}^n)} \le C \left\| \sum_{j\in\mathbb{N}_0} 2^{j/p}(1+2^j|t|)^{-M} b_j(x') \right\|_{L^p(\mathbb{R}^{n-1}\times\mathbb{R},\,dx'\,dt)}.$$

Take $M > 1/p$ and split $\mathbb{R}\setminus\{0\}$ into

$$A_m := \{t \in \mathbb{R} \mid 2^{-m-1} < |t| \le 2^{-m}\}, \qquad m \in \mathbb{Z}.$$

Their measure is 2^{-m}. For $t \in A_m$, multiplying the sum by $2^{-m/p}$ bounds it by

$$C \sum_{j\in\mathbb{N}_0} c_{j-m} b_j(x'), \qquad c_\ell := \begin{cases} 2^{\ell/p}, & \ell \le 0, \\ 2^{-\ell(M-1/p)}, & \ell > 0. \end{cases}$$

The sequence $(c_\ell)_{\ell\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z})$. Young's inequality in the indices j, m, followed by Tonelli's theorem in x', yields

$$\|\operatorname{ex} g\|^p_{F^s_{p,q}(\mathbb{R}^n)} \le C \sum_{j\in\mathbb{N}_0} 2^{jp(s-1/p)} \|\Delta'_j g\|^p_{L^p(\mathbb{R}^{n-1})} = C\|g\|^p_{B^{s-1/p}_{p,p}(\mathbb{R}^{n-1})}. \tag{6.28}$$

The constants are independent of the number of summands.

Finally, $s - 1/p > 0$ implies $\sum_{j\in\mathbb{N}_0} \|\Delta'_j g\|_{L^p(\mathbb{R}^{n-1})} < \infty$. The series in (6.26) converges uniformly in t with values in L^p and equals $\sum_{j\in\mathbb{N}_0} \Delta'_j g = g$ at $t = 0$. The canonical interpretation of the trace proves $\operatorname{tr}\operatorname{ex} = I$. □

Theorem 6.7.3. (Fractional trace on smooth domains) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with boundary C^∞, and let $1 < p < \infty$, $1 \le q \le \infty$, and $s > 1/p$. There are continuous surjective operators

$$\operatorname{Tr}\colon B^s_{p,q}(\Omega) \longrightarrow B^{s-1/p}_{p,q}(\partial\Omega), \qquad \operatorname{Tr}\colon F^s_{p,q}(\Omega) \longrightarrow B^{s-1/p}_{p,p}(\partial\Omega).$$

Each has a continuous linear right inverse. In particular,

$$\operatorname{Tr}\colon H^{s,p}(\Omega) \longrightarrow B^{s-1/p}_{p,p}(\partial\Omega)$$

is a retraction; the same holds with $W^{s,p}(\Omega)$ as source space when s is not an integer.

Proof. Choose an extension U of u to $\mathbb{R}^n$, boundary charts $\Phi_i\colon Q \longrightarrow U_i$, $i \in \{1,\dots,N\}$, and subordinate smooth cutoffs η_i whose sum equals one near $\partial\Omega$. The charts satisfy

$$\Phi_i(Q \cap \{t > 0\}) = U_i \cap \Omega, \qquad \Phi_i(Q \cap \{t = 0\}) = U_i \cap \partial\Omega.$$

Choose the cutoffs compactly supported inside each chart. Define

$$\operatorname{Tr}_\Omega u := \sum_{i=1}^{N} \big[\operatorname{tr}\big((\eta_i U) \circ \Phi_i\big)\big] \circ (\Phi_i \restriction_{\{t=0\}})^{-1}, \tag{6.29}$$

with extension by zero of each localized expression. The multiplication and composition lemmas and the hyperplane theorem bound the sum by $C\|U\|_{A^s_{p,q}}$.

If U_1, U_2 extend the same u, the localized expression of $U_1 - U_2$ vanishes on $t > 0$. The final observation about the canonical trace in the preceding theorem shows that its trace is zero. Thus the formula does not depend on the extension. Taking the infimum over all extensions gives the bound by the quotient norm of u.

To compare two systems of charts, fix $0 < \varepsilon < s - 1/p$. Corollary 6.3.9 allows approximation of U, after localization, in $B^{s-\varepsilon}_{p,p}$ by smooth functions. The two trace constructions agree for these functions. Multiplication, composition, and trace are continuous at this lower order, which still exceeds $1/p$. Passing to the limit, the resulting traces agree as distributions on the boundary. Since both already belong to the target space of order $s - 1/p$, this identity is also equality of their elements in that space. The argument covers $q = \infty$.

We now construct a right inverse. Choose cutoffs $\zeta_i \in C^\infty(\partial\Omega)$ supported inside the charts, with $\sum_{i=1}^{N} \zeta_i^2 = 1$ To obtain them, start with cutoffs whose sum is positive and divide each by the square root of the sum of their squares. Given g, express

$$g_i := (\zeta_i g) \circ (\Phi_i \restriction_{\{t=0\}})$$

on $\mathbb{R}^{n-1}$, extending it by zero. Apply ex from (6.26). Choose on Q a cutoff χ_i whose restriction to $t = 0$ is the expression of ζ_i, and extend it smoothly with compact support inside Q. The sum of the functions $\chi_i \operatorname{ex} g_i$

transported by Φ_i^{-1} and restricted to Ω defines a continuous operator. Its trace is

$$\sum_{i=1}^{N} \zeta_i(\zeta_i g) = g.$$

The Bessel and Slobodeckij identifications give the final assertions. □

Remark 6.7.4. (Lipschitz boundaries) On a Lipschitz boundary, coordinate changes do not have the derivatives used in the preceding theorem. The trace formulation at this boundary regularity requires a separate development; in the interval

$$1/p < s < 1 + 1/p$$

the target space has order between zero and one and is described through differences in boundary charts. The corresponding classical theorem can be found in [146, Section 4.4.2]. Boundary charts of the smooth manifolds in this book will be treated using the preceding theorem.

Part III

Nonlinear analysis on Riemannian manifolds

Introduction to Part III

We now return to manifolds with the tools developed in the Euclidean setting. This is one of the central steps in the book. It is not enough to replace partial derivatives with covariant derivatives and repeat the same formulas: working on a manifold requires the Riemannian measure, bundle metrics, connections, supports of sections, and transformations between charts and trivializations. Each of these structures determines how an analytic notion is correctly formulated.

The first chapters construct Sobolev spaces of functions and of sections of vector bundles, and develop the integration-by-parts formulas used to define weak covariant derivatives and formal adjoints. Differential operators on bundles and vector-valued distributions are then introduced. The principal symbol and ellipticity allow regularity to be recovered from the equation. Later, the connection Laplacian and the heat semigroup provide intrinsic scales of fractional regularity.

The final stage of this development requires us to leave behind the convenience of compactness. On a noncompact manifold, a local estimate can be globalized only if its constants remain controlled as the chart moves. Bounded geometry provides precisely this uniform control. Normal and Fermi coordinates, partitions of unity, and uniform trivializations then allow the results to be extended to submanifolds and manifolds with boundary. Throughout these chapters, we will continually alternate between two operations: expressing a geometric object in coordinates in order to calculate, and reconstructing a global, intrinsic statement from local estimates.

Aubin and analysis on manifolds

The development of geometric equations in the twentieth century made it necessary to adapt the tools of functional analysis to spaces whose geometry could not be ignored. Thierry Aubin played a central role in systematizing Sobolev inequalities and variational problems on manifolds, particularly in the study of conformal equations. Subsequently, Hebey, Strichartz, Eichhorn, Grigor'yan, Amann, and many others extended these ideas to complete and noncompact manifolds, manifolds of bounded geometry, manifolds with boundary, and spaces with more general structures. In this part we follow the same course: beginning with the local theory and identifying the geometric hypotheses that turn it into a global theory.

7

Sobolev spaces on Riemannian manifolds

The Dirichlet energy provides an intrinsic starting point for transferring Sobolev spaces to a Riemannian manifold $(M, \mathbf{g})$. For scalar data f, we formally consider

$$\mathcal{E}_{\mathbf{g}}(u) = \frac{1}{2}\int_M |du|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} - \int_M f u \, d\lambda_{\mathbf{g}}.$$

For smooth u and φ, with interior support or boundary conditions that eliminate the boundary term, the first variation gives

$$\int_M \langle du, d\varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M f \varphi \, d\lambda_{\mathbf{g}}.$$

This identity is the weak formulation of $-\Delta_{\mathbf{g}} u = f$ and remains meaningful when u has only one square-integrable weak derivative.

The metric enters both the norm of du and the measure $\lambda_{\mathbf{g}}$, but the definition must be independent of coordinates. Charts allow a function to be compared locally with its Euclidean representative; partitions of unity assemble these comparisons, and covariant derivatives provide a global description of higher orders. On a compact manifold, finiteness of the cover makes the norms obtained from these constructions equivalent.

This chapter will define the spaces $W^{m,p}(M)$, prove their completeness, and establish independence from the relevant auxiliary data. It will then transfer the density, embedding, and Rellich–Kondrashov theorems to the manifold. The following example first shows how the same geometric operator takes different expressions in different charts while still representing the Laplace–Beltrami operator.

Example 7.0.1. (The Laplace–Beltrami operator on the sphere in different coordinates) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and let $\iota\colon S^2 \hookrightarrow \mathbb{R}^3$ be the canonical inclusion. Denote the Euclidean metric on $\mathbb{R}^3$ by $\overline{\mathbf{g}}$. The Riemannian metric on S^2 is defined as the pullback of $\overline{\mathbf{g}}$ by the inclusion:

$$\mathbf{g} = \iota^* \overline{\mathbf{g}}, \qquad \mathbf{g}_p(X_p, Y_p) = \overline{\mathbf{g}}_{\iota(p)}(d\iota_p(X_p), d\iota_p(Y_p)), \quad X_p, Y_p \in T_p S^2.$$

If (x^1, x^2) are local coordinates induced by a chart $\varphi\colon U \subseteq S^2 \longrightarrow V \subseteq \mathbb{R}^2$, denote by $\Phi := \iota \circ \varphi^{-1}\colon V = \varphi(U) \longrightarrow \mathbb{R}^3$ the composition of the inverse chart with the inclusion. By the definition of the induced

metric,

$$\mathbf{g} = \iota^*\overline{\mathbf{g}}, \qquad \mathbf{g}_p(X_p, Y_p) = \overline{\mathbf{g}}_{\iota(p)}\big(d\iota_p(X_p), d\iota_p(Y_p)\big), \quad X_p, Y_p \in T_pS^2.$$

In particular, for the coordinate vectors

$$\partial_i\big|_p = (d\varphi^{-1})_{\varphi(p)}\left(\partial_i\Big|_{\varphi(p)}\right),$$

we have

$$d\iota_p\left(\partial_i\big|_p\right) = d\iota_p \circ \left((d\varphi^{-1})_{\varphi(p)}\left(\partial_i\Big|_{\varphi(p)}\right)\right) = d(\iota\circ\varphi^{-1})_{\varphi(p)}\left(\partial_i\Big|_{\varphi(p)}\right) = d\Phi_{\varphi(p)}\left(\partial_i\Big|_{\varphi(p)}\right).$$

Thus, evaluating the definition of $\mathbf{g}$ at the point p,

$$g_{ij}(p) = \mathbf{g}_p\left(\partial_i\big|_p, \partial_j\big|_p\right) = \overline{\mathbf{g}}_{\iota(p)}\left(d\Phi_{\varphi(p)}\left(\partial_i\Big|_{\varphi(p)}\right), d\Phi_{\varphi(p)}\left(\partial_j\Big|_{\varphi(p)}\right)\right).$$

Finally, since $d\Phi_{\varphi(p)}\left(\partial_i\Big|_{\varphi(p)}\right)$ agrees with the partial derivative $\dfrac{\partial\Phi}{\partial x_i}(\varphi(p))$ at $\mathbb{R}^3$, we obtain

$$g_{ij}(p) = \left\langle \frac{\partial\Phi}{\partial x^i}(\varphi(p)), \frac{\partial\Phi}{\partial x^j}(\varphi(p)) \right\rangle_{\mathbb{R}^3},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ is the Euclidean inner product. This identity shows that the local coefficients of the induced metric on S^2 are computed, for each $p \in U$, as the inner products of the partial derivatives of the parametrization $\Phi = \iota\circ\varphi^{-1}$ evaluated at $\varphi(p) \in \mathbb{R}^2$.

We shall use spherical coordinates (θ, φ) with the following convention: θ is the radial angle in the xy plane, $0 < \theta < 2\pi$, and φ is the azimuthal angle measured from the positive z axis, $0 < \varphi < \pi$. The inverse of these coordinates defines a chart on an open subset of S^2. The corresponding map is

$$\Phi(\theta, \varphi) = \big(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi\big).$$

Its partial derivatives are

$$\partial_\theta\Phi = \big(-\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0\big), \qquad \partial_\varphi\Phi = \big(\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi\big).$$

With $\mathbf{g} = \iota^*\overline{\mathbf{g}}$, we obtain pointwise

$$g_{\theta\theta} = \langle\partial_\theta\Phi, \partial_\theta\Phi\rangle_{\mathbb{R}^3} = \sin^2\varphi, \qquad g_{\theta\varphi} = \big\langle\partial_\theta\Phi, \partial_\varphi\Phi\big\rangle_{\mathbb{R}^3} = 0, \qquad g_{\varphi\varphi} = \big\langle\partial_\varphi\Phi, \partial_\varphi\Phi\big\rangle_{\mathbb{R}^3} = 1.$$

Therefore,

$$(g_{ij}) = \begin{pmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} \csc^2\varphi & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sqrt{\det(\mathbf{g})} = \sin\varphi,$$

and the local Laplace–Beltrami formula is

$$\Delta u = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_i\big(\sqrt{\det(\mathbf{g})}g^{ij}\partial_j u\big) = \frac{1}{\sin\varphi}\partial_\varphi\big(\sin\varphi\partial_\varphi u\big) + \frac{1}{\sin^2\varphi}\partial_{\theta\theta}u.$$

Now let $N = (0,0,1)$ be the north pole and consider stereographic projection

$$\sigma\colon S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2, \qquad \sigma(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Its inverse

$$\sigma^{-1}(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right)$$

composed with the inclusion $\iota\colon S^2 \hookrightarrow \mathbb{R}^3$ gives

$$\widetilde{\Phi} = \iota\circ\sigma^{-1}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad \widetilde{\Phi}(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right).$$

Differentiating and taking Euclidean inner products gives

$$g_{xx} = \left\langle \partial_x\widetilde{\Phi}, \partial_x\widetilde{\Phi}\right\rangle_{\mathbb{R}^3} = \frac{4}{\big(1+x^2+y^2\big)^2}, \qquad g_{yy} = \left\langle \partial_y\widetilde{\Phi}, \partial_y\widetilde{\Phi}\right\rangle_{\mathbb{R}^3} = \frac{4}{\big(1+x^2+y^2\big)^2},$$

$$g_{xy} = g_{yx} = \left\langle \partial_x\widetilde{\Phi}, \partial_y\widetilde{\Phi}\right\rangle_{\mathbb{R}^3} = 0,$$

whence

$$(g_{ij}) = \frac{4}{\big(1+x^2+y^2\big)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad (g^{ij}) = \frac{\big(1+x^2+y^2\big)^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \sqrt{\det(\mathbf{g})} = \frac{4}{\big(1+x^2+y^2\big)^2}.$$

In these coordinates,

$$\Delta u = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_i\big(\sqrt{\det(\mathbf{g})}g^{ij}\partial_j u\big) = \frac{\big(1+x^2+y^2\big)^2}{4}\big(\partial_{xx}u + \partial_{yy}u\big).$$

PDEs on Riemannian manifolds arise inevitably in several fields:

- In **differential geometry**, problems such as the Yamabe problem (existence of metrics of constant scalar curvature in a conformal class) and the study of geometric flows such as Ricci flow (used by Perelman in resolving the Poincaré conjecture) depend essentially on Sobolev inequalities and estimates on manifolds.
- In **mathematical physics**, the equations of general relativity are formulated on Lorentzian manifolds, and analysis of their associated operators requires suitable Sobolev spaces. The positive mass theorem is a paradigmatic example in which geometry, physics, and Sobolev inequalities interact.
- In **global analysis and topology**, the *Hodge theorem* ensures that each differential cohomology class has a harmonic representative, while the *Atiyah–Singer index theorem* relates the index of elliptic operators to topological invariants of the manifold. These results show how analysis of differential operators on manifolds becomes a central tool for understanding their global structure.

Thierry Aubin and Sobolev spaces on manifolds

In 1976, Thierry Aubin published a systematic treatment of Sobolev spaces on Riemannian manifolds and of the inequalities used to study geometric equations [15]. His work helped consolidate an intrinsic approach based on the metric and connection, particularly fruitful in conformal problems and nonlinear equations. Seeley developed spectral tools, and Strichartz extended several constructions to complete manifolds. Contemporary theory combines these ideas with uniform localization, heat kernels, and bounded geometry.

Sobolev spaces on manifolds are more than a generalization of Euclidean tools. They provide the language in which energy, regularity, and geometry can be studied simultaneously.

7.1 Definition and basic properties

We shall adopt an intrinsic definition using covariant derivatives. In a chart, these derivatives are expressed in terms of partial derivatives and Christoffel symbols; globally, the metric allows them to be measured and integrated without privileging coordinates.

For a function $u \in C^\infty(M)$, the information of order m is encoded by $\nabla^m u$. The following results describe its local expression and its relationship with the Riemannian metric.

Proposition 7.1.1. Let M be a smooth manifold with or without boundary. Let ∇ be a connection on TM, let (U,φ) be a chart on M, and let $\{\Gamma_{ij}^k\}$ be the Christoffel symbols of the connection with respect to (U,φ). Given $u \in C^\infty(M)$, $\nabla^m u$ has the following local expression:

$$(\nabla^m u)_{i_1\dots i_m} = \frac{\partial (\nabla^{m-1}u)_{i_2\dots i_m}}{\partial x_{i_1}} - \sum_{l=2}^{m} \Gamma_{i_1 i_l}^k (\nabla^{m-1}u)_{i_2\dots i_{l-1} k i_{l+1}\dots i_m}.$$

That is, according to the convention fixed in Proposition 2.4.29, the new covariant index occupies the first slot.

Proof. Using Proposition 2.4.13 and the convention for the position of the new index, we have

$$\begin{aligned}
(\nabla^m u)_{i_1\dots i_m} &= (\nabla_{\partial_{i_1}}(\nabla^{m-1}u))(\partial_{i_2},\dots,\partial_{i_m}) \\
&= \partial_{i_1}\left((\nabla^{m-1}u)(\partial_{i_2},\dots,\partial_{i_m})\right) \\
&\quad - \sum_{l=2}^{m} (\nabla^{m-1}u)\left(\partial_{i_2},\dots,\partial_{i_{l-1}},\nabla_{\partial_{i_1}}\partial_{i_l},\partial_{i_{l+1}},\dots,\partial_{i_m}\right) \\
&= \frac{\partial (\nabla^{m-1}u)_{i_2\dots i_m}}{\partial x_{i_1}} - \sum_{l=2}^{m} \Gamma_{i_1 i_l}^k (\nabla^{m-1}u)_{i_2\dots i_{l-1} k i_{l+1}\dots i_m}.
\end{aligned}$$

□

Proposition 7.1.2. Let M be a smooth manifold with or without boundary of dimension n, and let ∇ be a

connection on TM. If $u \in C^\infty(M)$ and (U, φ) is a smooth chart on M, then

$$\nabla^m u = \sum_{i_1,\dots,i_m=1}^{n} \sum_{0\le|\alpha|\le m} D^\alpha u P_\alpha^{i_1\dots i_m} \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_m}$$

on U, where $P_\alpha^{i_1\dots i_m}$ is a polynomial in the Christoffel symbols of ∇ and their derivatives up to order $m-1$.

Proof. We argue by induction on m. For $m = 1$, $\nabla u = \dfrac{\partial u}{\partial x^i}\mathbf{d}x^i$; for $m = 2$,

$$\nabla^2 u = \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k} \right) \mathbf{d}x^i \otimes \mathbf{d}x^j.$$

Suppose the assertion holds for some $m \ge 2$. By Proposition 7.1.1,

$$(\nabla^{m+1} u)_{i_1\dots i_{m+1}} = \frac{\partial (\nabla^m u)_{i_2\dots i_{m+1}}}{\partial x_{i_1}} - \sum_{l=2}^{m+1} \Gamma_{i_1 i_l}^k (\nabla^m u)_{i_2\dots i_{l-1} k i_{l+1}\dots i_{m+1}}.$$

The induction hypothesis tells us that

$$(\nabla^m u)_{i_2\dots i_{m+1}} = \sum_{0\le|\alpha|\le m} D^\alpha u P_\alpha^{i_2\dots i_{m+1}}.$$

Then

$$\begin{aligned}(\nabla^{m+1} u)_{i_1\dots i_{m+1}} = &\sum_{0\le|\alpha|\le m} \frac{\partial (D^\alpha u)}{\partial x_{i_1}} P_\alpha^{i_2\dots i_{m+1}} \\ &+ \sum_{0\le|\alpha|\le m} D^\alpha u \frac{\partial P_\alpha^{i_2\dots i_{m+1}}}{\partial x_{i_1}} \\ &- \sum_{l=2}^{m+1} \sum_{k=1}^{n} \Gamma_{i_1 i_l}^k \sum_{0\le|\alpha|\le m} D^\alpha u \, P_\alpha^{i_2\dots i_{l-1} k i_{l+1}\dots i_{m+1}}.\end{aligned}$$

The first line contains derivatives of u of order at most $m+1$. In the second, the coefficients are polynomials in the Christoffel symbols and their derivatives up to order m, because we differentiate once a polynomial that, by hypothesis, contains derivatives only up to order $m-1$. In the third line, that polynomial is multiplied by a Christoffel symbol without increasing the maximum order of derivatives of its coefficients. Grouping the terms multiplying each $D^\beta u$, $|\beta| \le m+1$, gives the required polynomials $P_\beta^{i_1\dots i_{m+1}}$. This completes the induction. □

Since $(M, \mathcal{L}_M, \lambda_{\mathbf{g}})$ is a measure space and $\lambda_{\mathbf{g}}$ is a Radon measure, the results of measure theory apply directly and allow us to consider the Lebesgue spaces $L^p(M)$.

If $u \in C^\infty(M)$, Proposition 2.3.3 gives the following local coordinate expression for $|\nabla^k u|_{\mathbf{g}} := \sqrt{\mathbf{g}(\nabla^k u, \nabla^k u)}\colon M \longrightarrow [0,\infty)$:

$$|\nabla^k u|_{\mathbf{g}} = \sqrt{g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1\dots i_k} (\nabla^k u)_{j_1\dots j_k}}.$$

Moreover, this function is at least continuous, and hence measurable. We may consider

$$\int_M |\nabla^k u|_{\mathbf{g}}^p d\lambda_{\mathbf{g}}$$

for each $p \in [1,\infty)$. We can define

$$\|u\|_{W^{m,p}(M)} := \left(\sum_{j=0}^{m} \int_M |\nabla^j u|_{\mathbf{g}}^p d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}}$$

and ask which $u \in C^\infty(M)$ satisfy $\|u\|_{W^{m,p}(M)} < \infty$. This motivates the following definition.

Definition 7.1.3. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. For an integer $m \geq 0$ and $p \in [1,\infty)$, define

(a) $C^{m,p}(M) := \{u \in C^\infty(M) \mid \|u\|_{W^{m,p}(M)} < \infty\}$

(b) $W^{m,p}(M)$ as the completion of $C^{m,p}(M)$ under the norm $\|\cdot\|_{W^{m,p}(M)}$.

Remark 7.1.4. We have

$$C^{m,p}(M) = \{u \in C^\infty(M) \mid |\nabla^j u|_{\mathbf{g}} \in L^p(M) \text{ for every } j \in \{0,\dots,m\}\}.$$

To see that this definition is consistent, we shall prove that $\left(C^{m,p}(M), \|\cdot\|_{W^{m,p}(M)}\right)$ is a normed space.

Proposition 7.1.5. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $m \in \mathbb{N}_0$ and $1 \leq p < \infty$. Then $\left(C^{m,p}(M), \|\cdot\|_{W^{m,p}(M)}\right)$ is a normed space.

Proof. Let $u \in C^\infty(M)$ satisfy $\|u\|_{W^{m,p}(M)} = 0$. Since $0 \leq \|u\|_{L^p(M)} \leq \|u\|_{W^{m,p}(M)} = 0$, we have $\|u\|_{L^p(M)} = 0$, so $u = 0$ almost everywhere. Moreover, since u is continuous, we conclude that $u \equiv 0$ on M.

Now let $\lambda \in \mathbb{R}$. Then

$$\|\lambda u\|_{W^{m,p}(M)} := \left(\sum_{j=0}^{m} \int_M |\nabla^j(\lambda u)|_{\mathbf{g}}^p d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}} = \left(\sum_{j=0}^{m} \int_M |\lambda \nabla^j(u)|_{\mathbf{g}}^p d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}} = \left(\sum_{j=0}^{m} \int_M |\lambda|^p |\nabla^j u|_{\mathbf{g}}^p d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}}$$

$$= |\lambda| \left(\sum_{j=0}^{m} \int_M |\nabla^j u|_{\mathbf{g}}^p d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}} = |\lambda| \|u\|_{W^{m,p}(M)},$$

where we used $|\lambda \nabla^j u|_{\mathbf{g}} = \sqrt{\mathbf{g}(\lambda \nabla^j u, \lambda \nabla^j u)} = \sqrt{\lambda^2 \mathbf{g}(\nabla^j u, \nabla^j u)} = |\lambda| \sqrt{\mathbf{g}(\nabla^j u, \nabla^j u)} = |\lambda| |\nabla^j u|_{\mathbf{g}}$ for each $j \in \{0,\dots,m\}$.

Let $u, v \in C^{m,p}(M)$. For $w \in C^{m,p}(M)$, define on the measure space

$$X_m := M \times \{0,\dots,m\}, \qquad \mu_m := \lambda_{\mathbf{g}} \otimes \#,$$

the function $A_w(x,j) := |\nabla^j w(x)|_{\mathbf{g}}$, where # is counting measure. Then

$$\|A_w\|_{L^p(X_m,\mu_m)} = \left(\sum_{j=0}^{m} \int_M |\nabla^j w|_{\mathbf{g}}^p \, d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}} = \|w\|_{W^{m,p}(M)}.$$

Since $|\nabla^j(u+v)|_{\mathbf{g}} \le |\nabla^j u|_{\mathbf{g}} + |\nabla^j v|_{\mathbf{g}}$, we have $A_{u+v} \le A_u + A_v$. Monotonicity of the norm and Minkowski's inequality in $L^p(X_m,\mu_m)$, given in Theorem B.7.3, yield

$$\|u+v\|_{W^{m,p}(M)} \le \|A_u + A_v\|_{L^p(X_m,\mu_m)} \le \|u\|_{W^{m,p}(M)} + \|v\|_{W^{m,p}(M)}.$$

In particular, $C^{m,p}(M)$ is closed under addition and multiplication by scalars in $C^\infty(M)$; thus it is a vector space, and $\|\cdot\|_{W^{m,p}(M)}$ is a norm. □

We shall also define the analogue of $W_0^{m,p}(\Omega)$.

Definition 7.1.6. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $m \in \mathbb{N}_0$ and $1 \le p < \infty$. Define

$$W_0^{m,p}(M) := \overline{C_c^\infty(\operatorname{Int} M)}^{W^{m,p}(M)}.$$

Functions in $C_c^\infty(\operatorname{Int} M)$ are viewed as smooth functions on M by extension by zero; this is legitimate because their supports are compactly contained in the interior.

The sum norm defined below is often used in the literature; see, for example, [79] and [1]. The following proposition quantifies its comparison with the norm adopted in this book.

Proposition 7.1.7. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $m \in \mathbb{N}_0$ and $1 \le p < \infty$. For each $u \in C^{m,p}(M)$, define

$$\|u\|_{\widetilde{W}^{m,p}(M)} := \sum_{j=0}^{m} \left(\int_M |\nabla^j u|_{\mathbf{g}}^p \, d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}}.$$

Then, for every $u \in C^{m,p}(M)$,

$$\|u\|_{W^{m,p}(M)} \le \|u\|_{\widetilde{W}^{m,p}(M)} \le (m+1)^{1-\frac{1}{p}} \|u\|_{W^{m,p}(M)}.$$

Proof. Set $x_j := \left(\int_M |\nabla^j u|_{\mathbf{g}}^p \, d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}}$. The inequalities $\|x\|_{\ell^p} \le \|x\|_{\ell^1} \le (m+1)^{1-\frac{1}{p}} \|x\|_{\ell^p}$, obtained by Hölder's inequality on $\mathbb{R}^{m+1}$, are exactly the two asserted inequalities. □

Compact Riemannian manifolds play a particularly important role in the study of Sobolev spaces, since they allow generalizations of the results previously established in Euclidean spaces. If M is closed, then $\operatorname{Int} M = M$ and $C_c^\infty(M) = C^\infty(M)$; consequently, $W_0^{m,p}(M) = W^{m,p}(M)$, and $C^\infty(M)$ is dense in this space. When M has boundary, the relationship between $W_0^{m,p}(M)$ and the trace will be studied in Section 9.4, in the more general setting of sections of vector bundles.

At times, a smooth manifold M may carry two Riemannian metrics, say $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, making it necessary to

distinguish the metric used in the Sobolev norm. In this case, we write $\|u\|_{W^{m,p}(M),\mathbf{g}}$ for the norm associated with $\mathbf{g}$ and $\|u\|_{W^{m,p}(M),\widetilde{\mathbf{g}}}$ for the norm associated with $\widetilde{\mathbf{g}}$.

In the compact case, other useful properties also hold; for example, the Sobolev spaces do not depend on the Riemannian metric. To prove this, we first establish the following results.

Lemma 7.1.8. Let $k,l \in \mathbb{N}_0$ and let M be a compact smooth manifold with or without boundary equipped with two Riemannian metrics $\mathbf{g}$ and $\widetilde{\mathbf{g}}$. Then there exist $c_{k,l}, C_{k,l} > 0$ such that for every $p \in M$ and $v \in T^{(k,l)}(T_pM)$,

$$c_{k,l}|v|_{\mathbf{g}}^2 \le |v|_{\widetilde{\mathbf{g}}}^2 \le C_{k,l}|v|_{\mathbf{g}}^2.$$

Here the norms are those induced on the tensor bundle. In the proof, we shall also write $\mathbf{g}_p(v,v)$ and $\widetilde{\mathbf{g}}_p(v,v)$ for the induced inner products evaluated at (v,v). For $(k,l) = (1,0)$, these are the original metrics on T_pM, and the conclusion is precisely the comparison in Definition 2.1.5.

Proof. Let d be the rank of $T^{(k,l)}(TM)$. If $M = \varnothing$ or $d = 0$, the assertion is immediate. By Proposition 1.2.11, Remark 1.2.10, and compactness of M, there are charts (U_i, φ_i) and compact regular coordinate balls and, if M has boundary, compact regular coordinate half-balls

$$\overline{B}_1, \dots, \overline{B}_N$$

covering M and satisfying $\overline{B}_i \subseteq U_i$. For each i, let

$$\Phi_i \colon T^{(k,l)}(TM) \restriction_{U_i} \longrightarrow U_i \times \mathbb{R}^d$$

be the trivialization induced by the tensor frame of the chart containing $\overline{B}_i$. If $p \in \overline{B}_i$ and $w \in \mathbb{R}^d$, set

$$Q_i(p,w) := \mathbf{g}_p\big(\Phi_i^{-1}(p,w), \Phi_i^{-1}(p,w)\big),$$
$$\widetilde{Q}_i(p,w) := \widetilde{\mathbf{g}}_p\big(\Phi_i^{-1}(p,w), \Phi_i^{-1}(p,w)\big).$$

Both functions are continuous and, for each p, are positive definite quadratic forms in w. Thus,

$$R_i(p,w) := \frac{\widetilde{Q}_i(p,w)}{Q_i(p,w)}$$

is continuous and strictly positive on

$$\overline{B}_i \times S^{d-1}, \qquad S^{d-1} := \{w \in \mathbb{R}^d \mid \|w\|_2 = 1\}.$$

This set is compact, so there exist

$$c_i := \min_{(p,w)\in \overline{B}_i \times S^{d-1}} R_i(p,w) > 0, \qquad C_i := \max_{(p,w)\in \overline{B}_i \times S^{d-1}} R_i(p,w) < \infty.$$

Homogeneity of the two quadratic forms implies

$$c_i Q_i(p,w) \le \widetilde{Q}_i(p,w) \le C_i Q_i(p,w)$$

for every $p \in \overline{B}_i$ and every $w \in \mathbb{R}^d$. Taking

$$c_{k,l} := \min_{1 \le i \le N} c_i > 0, \qquad C_{k,l} := \max_{1 \le i \le N} C_i < \infty,$$

and writing w for the components of $v \in T^{(k,l)}(T_pM)$ in a chart containing p gives the stated inequality. □

The preceding argument uses only continuity and positivity of the inner products on compact subsets of trivializations. It therefore also applies on any compact set $K \subseteq M$, with continuous metrics defined on a neighborhood of K: simply cover K by finitely many of the balls or half-balls used in the proof. The constants may depend on K and on the metrics. The next proposition specifies the bounds obtained when comparison constants on the tangent bundle are already known.

Proposition 7.1.9. (Comparison of dual norms, tensor norms, and volumes) Let $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ be two Riemannian metrics on a manifold M of dimension n, and let $A \subseteq M$. Suppose that $0 < c \le C < \infty$ and

$$c\,\mathbf{g}_p(v,v) \le \widetilde{\mathbf{g}}_p(v,v) \le C\,\mathbf{g}_p(v,v) \qquad (p \in A,\ v \in T_pM).$$

Then the dual metrics satisfy

$$C^{-1}\mathbf{g}_p^{-1}(\xi,\xi) \le \widetilde{\mathbf{g}}_p^{-1}(\xi,\xi) \le c^{-1}\mathbf{g}_p^{-1}(\xi,\xi) \qquad (\xi \in T_p^*M,\ p \in A). \tag{7.1}$$

Moreover, for $k,l \in \mathbb{N}_0$ and $\mathbf{T} \in T^{(k,l)}(T_pM)$,

$$c^{k/2}C^{-l/2}|\mathbf{T}|_{\mathbf{g}} \le |\mathbf{T}|_{\widetilde{\mathbf{g}}} \le C^{k/2}c^{-l/2}|\mathbf{T}|_{\mathbf{g}}. \tag{7.2}$$

In particular, a comparison $K^{-1}\mathbf{g} \le \widetilde{\mathbf{g}} \le K\mathbf{g}$, with $K \ge 1$, gives the constants $K^{-(k+l)/2}$ and $K^{(k+l)/2}$ for the norms of tensors of type (k,l).
The ratio of the volume elements is the positive function

$$\rho(p) = \frac{\sqrt{\det(\widetilde{\mathbf{g}}_p)}}{\sqrt{\det(\mathbf{g}_p)}},$$

where both determinants are computed in the same chart. This function is independent of the chart, satisfies $d\lambda_{\widetilde{\mathbf{g}}} = \rho\, d\lambda_{\mathbf{g}}$, and obeys

$$c^{n/2} \le \rho(p) \le C^{n/2} \qquad (p \in A). \tag{7.3}$$

Consequently, if A is measurable and $f : A \to [0,\infty]$ is measurable, then

$$c^{n/2}\int_A f\, d\lambda_{\mathbf{g}} \le \int_A f\, d\lambda_{\widetilde{\mathbf{g}}} \le C^{n/2}\int_A f\, d\lambda_{\mathbf{g}}.$$

Proof. Fix $p \in A$. There is a unique endomorphism $L_p : T_pM \to T_pM$ such that

$$\widetilde{\mathbf{g}}_p(v,w) = \mathbf{g}_p(L_pv, w) \qquad (v,w \in T_pM).$$

It is obtained by applying the inverse of the isomorphism $w \mapsto \mathbf{g}_p(w,\cdot)$ to the covector $\widetilde{\mathbf{g}}_p(v,\cdot)$. Symmetry of

$\widetilde{\mathbf{g}}_p$ implies that L_p is self-adjoint with respect to $\mathbf{g}_p$, and its positivity implies $\mathbf{g}_p(L_p v, v) > 0$ when $v \neq 0$. The spectral theorem provides a basis $e_1, \dots, e_n$ orthonormal for $\mathbf{g}_p$ and numbers $\lambda_1, \dots, \lambda_n > 0$ such that

$$L_p e_i = \lambda_i e_i, \qquad \mathbf{g}_p(e_i, e_j) = \delta_{ij}, \qquad \widetilde{\mathbf{g}}_p(e_i, e_j) = \lambda_i \delta_{ij}.$$

Applying the hypothesis to $v = e_i$ gives $c \le \lambda_i \le C$ for every i. This basis is chosen in the fixed fiber T_pM; the chosen bases need not vary smoothly with p.

Let $e^1, \dots, e^n$ be the dual basis. The matrices of the dual metrics are the inverses of the matrices of the tangent metrics. Thus, for $\xi = \sum_{i=1}^{n} \xi_i e^i$,

$$\mathbf{g}_p^{-1}(\xi, \xi) = \sum_{i=1}^{n} \xi_i^2, \qquad \widetilde{\mathbf{g}}_p^{-1}(\xi, \xi) = \sum_{i=1}^{n} \lambda_i^{-1} \xi_i^2.$$

Multiplying the inequalities $C^{-1} \le \lambda_i^{-1} \le c^{-1}$ by ξ_i^2 and summing proves (7.1). This explains why the lower and upper constants exchange roles when comparing dual metrics.

Now consider the basis tensors

$$E_{I,J} := e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l},$$

where $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ range over the indices from 1 to n. They are orthogonal for both tensor inner products, and

$$|E_{I,J}|_{\mathbf{g}}^2 = 1, \qquad |E_{I,J}|_{\widetilde{\mathbf{g}}}^2 = \lambda_{i_1} \cdots \lambda_{i_k} \lambda_{j_1}^{-1} \cdots \lambda_{j_l}^{-1}.$$

An empty product is interpreted as 1. Each weight in the last expression lies between $c^k C^{-l}$ and $C^k c^{-l}$. If $\mathbf{T} = \sum_{\substack{I \in \{1,\dots,n\}^k \\ J \in \{1,\dots,n\}^l}} T_J^I E_{I,J}$, orthogonality gives

$$|\mathbf{T}|_{\mathbf{g}}^2 = \sum_{\substack{I \in \{1,\dots,n\}^k \\ J \in \{1,\dots,n\}^l}} |T_J^I|^2, \qquad |\mathbf{T}|_{\widetilde{\mathbf{g}}}^2 = \sum_{\substack{I \in \{1,\dots,n\}^k \\ J \in \{1,\dots,n\}^l}} |T_J^I|^2 \lambda_{i_1} \cdots \lambda_{i_k} \lambda_{j_1}^{-1} \cdots \lambda_{j_l}^{-1}.$$

Multiplying the bounds for each weight by $|T_J^I|^2$ and summing yields

$$c^k C^{-l} |\mathbf{T}|_{\mathbf{g}}^2 \le |\mathbf{T}|_{\widetilde{\mathbf{g}}}^2 \le C^k c^{-l} |\mathbf{T}|_{\mathbf{g}}^2.$$

Taking square roots proves (7.2).

In the basis $e_1, \dots, e_n$, the ratio of determinants is $\lambda_1 \cdots \lambda_n$. A simultaneous change of basis multiplies both determinants by the same square of the determinant of the change-of-basis matrix, leaving their ratio unchanged. In particular,

$$\frac{\det(\widetilde{\mathbf{g}}_p)}{\det(\mathbf{g}_p)} = \prod_{i=1}^{n} \lambda_i, \qquad c^n \le \prod_{i=1}^{n} \lambda_i \le C^n.$$

The coordinate formula for Riemannian measure implies $d\lambda_{\widetilde{\mathbf{g}}} = \rho \, d\lambda_{\mathbf{g}}$, and the square roots of the last bounds prove (7.3). Finally,

$$\int_A f \, d\lambda_{\widetilde{\mathbf{g}}} = \int_A f \rho \, d\lambda_{\mathbf{g}}.$$

Multiply (7.3) by $f \geq 0$ and use monotonicity of the integral. The argument also holds when one of the integrals is infinite. □

This proposition compares two norms of the same tensor. In studying Sobolev norms, we must also compare the tensors $\nabla^j u$ and $\widetilde{\nabla}^j u$ when the connection changes. For functions, $\nabla u = \widetilde{\nabla} u = du$; from order two onward, covariant derivatives may differ. The following lemma compares the full collections of derivatives up to order m, including their lower-order terms.

Lemma 7.1.10. Let $m \geq 0$ and let M be a compact smooth manifold with or without boundary, with two Riemannian metrics $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, and let ∇ and $\widetilde{\nabla}$ be two connections on TM. Then there exist $c_m, C_m > 0$ such that

$$c_m \left(\sum_{j=0}^{m} |\nabla^j u|^2_{\mathbf{g}} \right) \leq \sum_{j=0}^{m} |\widetilde{\nabla}^j u|^2_{\widetilde{\mathbf{g}}} \leq C_m \left(\sum_{j=0}^{m} |\nabla^j u|^2_{\mathbf{g}} \right)$$

for every $u \in C^\infty(M)$.

Proof. If $M = \varnothing$, the assertion is immediate. Assume $M \neq \varnothing$. By Definition 1.7.10, we may consider the bundle

$$\mathscr{T}_m := \bigoplus_{j=0}^{m} T^{(0,j)}(TM), \qquad T^{(0,0)}(TM) := M \times \mathbb{R}.$$

For $u \in C^\infty(M)$, write

$$J^m_\nabla u := (u, \nabla u, \dots, \nabla^m u)$$

and, for each $p \in M$, define

$$\mathscr{D}_{\nabla,p} := \{ J^m_\nabla u(p) \mid u \in C^\infty(M) \} \subseteq (\mathscr{T}_m)_p.$$

Linearity of ∇^j implies that $\mathscr{D}_{\nabla,p}$ is a vector subspace of $(\mathscr{T}_m)_p$.
Let (U, φ) be a smooth chart, let $p \in U$, and set $x := \varphi(p)$ and

$$N := \sum_{j=0}^{m} \binom{n+j-1}{j} = \binom{n+m}{m}.$$

With the multi-index notation from Definition 1.1.10, define the linear map

$$F^\varphi_{\nabla,p} \colon \mathscr{D}_{\nabla,p} \longrightarrow \mathbb{R}^N, \qquad F^\varphi_{\nabla,p}(J^m_\nabla u(p)) := \big(D^\alpha (u \circ \varphi^{-1})(x) \big)_{|\alpha| \leq m}.$$

Propositions 7.1.1 and 7.1.2 give, for $j \in \{1, \dots, m\}$,

$$(\nabla^j u)_{i_1 \dots i_j} = \partial_{i_1} \cdots \partial_{i_j} (u \circ \varphi^{-1}) + \sum_{|\alpha| < j} A^\alpha_{i_1 \dots i_j} D^\alpha (u \circ \varphi^{-1}), \tag{7.4}$$

where the coefficients $A^\alpha_{i_1 \dots i_j}$ are smooth on U. By induction on j, the equality $J^m_\nabla u(p) = J^m_\nabla v(p)$ determines first $u(p) = v(p)$ and then all partial derivatives of $u \circ \varphi^{-1}$ and $v \circ \varphi^{-1}$ up to order m. Thus $F^\varphi_{\nabla,p}$ is well defined and injective.

Let $a = (a_\alpha)_{|\alpha|\le m} \in \mathbb{R}^N$. The polynomial

$$P_a(y) := \sum_{|\alpha|\le m} \frac{a_\alpha}{\alpha!}(y-x)^\alpha$$

satisfies $D^\alpha P_a(x) = a_\alpha$ for every $|\alpha| \le m$. Theorem 1.3.7, applied to $P_a \circ \varphi$, provides a function $u \in C^\infty(M)$ agreeing with $P_a \circ \varphi$ on a neighborhood of p. Then $F^\varphi_{\nabla,p}(J^m_\nabla u(p)) = a$, and $F^\varphi_{\nabla,p}$ is an isomorphism.
Let $(e_1,\dots,e_N)$ be the standard basis of $\mathbb{R}^N$. Substituting each e_ν as the prescribed partial jet in (7.4) shows that the components of

$$\sigma_\nu(q) := (F^\varphi_{\nabla,q})^{-1}(e_\nu), \qquad q \in U, \quad \nu \in \{1,\dots,N\},$$

in the coordinate tensor frames are smooth functions. Moreover, $\sigma_1(q),\dots,\sigma_N(q)$ form a basis of $\mathscr{D}_{\nabla,q}$ for each $q \in U$. Lemma 1.7.52 implies that

$$\mathscr{D}_\nabla := \bigcup_{p\in M} \mathscr{D}_{\nabla,p}$$

is a smooth vector subbundle of $\mathscr{T}_m$ of rank N. The same construction applied to $\widetilde{\nabla}$ produces the subbundle $\mathscr{D}_{\widetilde{\nabla}}$ and the isomorphisms $F^\varphi_{\widetilde{\nabla},p} : \mathscr{D}_{\widetilde{\nabla},p} \longrightarrow \mathbb{R}^N$.
Let $(\overline{B},\varphi)$ be a compact regular coordinate ball or half-ball whose coordinate map is the restriction of the preceding chart. For $p \in \overline{B}$ and $v \in \mathbb{R}^N$, define

$$Q(p,v) := \sum_{j=0}^m \left| \left((F^\varphi_{\nabla,p})^{-1}(v)\right)_j \right|^2_{\mathbf{g}},$$

$$\widetilde{Q}(p,v) := \sum_{j=0}^m \left| \left((F^\varphi_{\widetilde{\nabla},p})^{-1}(v)\right)_j \right|^2_{\widetilde{\mathbf{g}}}.$$

The families $\sigma_1,\dots,\sigma_N$ constructed above and their analogues for $\widetilde{\nabla}$ show that Q and $\widetilde{Q}$ are continuous on $\overline{B}\times\mathbb{R}^N$. For each p, both are positive definite quadratic forms in v. Thus the ratio

$$R(p,v) := \frac{\widetilde{Q}(p,v)}{Q(p,v)}$$

is continuous and strictly positive on $\overline{B}\times S^{N-1}$, where $S^{N-1} := \{v \in \mathbb{R}^N \mid \|v\|_2 = 1\}$. Compactness of $\overline{B}\times S^{N-1}$ provides numbers

$$c_{\overline{B},m} := \min_{(p,v)\in\overline{B}\times S^{N-1}} R(p,v) > 0, \qquad C_{\overline{B},m} := \max_{(p,v)\in\overline{B}\times S^{N-1}} R(p,v) < \infty.$$

Since Q and $\widetilde{Q}$ are homogeneous of degree two in v, it follows that

$$c_{\overline{B},m} Q(p,v) \le \widetilde{Q}(p,v) \le C_{\overline{B},m} Q(p,v) \tag{7.5}$$

for every $p \in \overline{B}$ and every $v \in \mathbb{R}^N$; for $v \neq 0$, apply the definitions of $c_{\overline{B},m}$ and $C_{\overline{B},m}$ to $v/\|v\|_2$, and for $v = 0$ the inequality is immediate.

For $p \in \overline{B}$ and $u \in C^\infty(M)$, the definition of the two isomorphisms gives

$$F^{\varphi}_{\nabla,p}(J^m_\nabla u(p)) = \left(D^\alpha(u\circ\varphi^{-1})(\varphi(p))\right)_{|\alpha|\le m} = F^{\varphi}_{\widetilde{\nabla},p}(J^m_{\widetilde{\nabla}} u(p)).$$

Substituting this vector into (7.5) yields

$$c_{\overline{B},m}\sum_{j=0}^m |\nabla^j u(p)|^2_{\mathbf{g}} \le \sum_{j=0}^m |\widetilde{\nabla}^j u(p)|^2_{\widetilde{\mathbf{g}}} \le C_{\overline{B},m}\sum_{j=0}^m |\nabla^j u(p)|^2_{\mathbf{g}}.$$

Compactness of M allows us to choose a finite cover $\overline{B}_1,\dots,\overline{B}_k$ by compact regular coordinate balls or half-balls. Taking

$$c_m := \min_{1\le i\le k} c_{\overline{B}_i,m} > 0, \qquad C_m := \max_{1\le i\le k} C_{\overline{B}_i,m} < \infty,$$

gives the two-sided inequality on all of M. □

We would also like a way to relate integrals with respect to two metrics when M is compact; this can be obtained using Lemma 7.1.8.

Lemma 7.1.11. Let M be a compact smooth manifold with or without boundary equipped with two Riemannian metrics $\mathbf{g}$ and $\widetilde{\mathbf{g}}$. Then there exist $C, c > 0$ such that for every measurable $f : M \longrightarrow [0,\infty]$,

$$c\int_M f\, d\lambda_{\mathbf{g}} \le \int_M f\, d\lambda_{\widetilde{\mathbf{g}}} \le C\int_M f\, d\lambda_{\mathbf{g}}.$$

Proof. Consider a finite family $(\overline{B}_i,\varphi_i)_{i=1}^k$ of compact regular coordinate balls and, where appropriate, half-balls whose relative interiors cover M (this is possible because M is compact). We have $\sqrt{\det(\mathbf{g})}, \sqrt{\det(\widetilde{\mathbf{g}})} > 0$ on $\overline{B}_i$ for every $i \in \{1,\dots,k\}$, since the matrices of both metrics are symmetric and positive definite. We may therefore define $h_i : \overline{B}_i \longrightarrow [0,\infty)$, $h_i(p) = \dfrac{\sqrt{\det(\widetilde{\mathbf{g}}_p)}}{\sqrt{\det(\mathbf{g}_p)}}$. Note that $h_i > 0$. Since $\overline{B}_i$ is compact, there exist $c_i = \min\limits_{p\in\overline{B}_i} h_i(p)$ and $C_i = \max\limits_{p\in\overline{B}_i} h_i(p)$, both positive. Then $c_i\sqrt{\det(\mathbf{g})} \le \sqrt{\det(\widetilde{\mathbf{g}})} \le C_i\sqrt{\det(\mathbf{g})}$ on $\overline{B}_i$. Let $(A_i)_{i=1}^k$ be a family of pairwise disjoint measurable sets as in Definition 2.15.5 for $A = M$. We have $M = \coprod_{i=1}^k A_i$. Let $c := \min\limits_{1\le i\le k} c_i$ and $C := \max\limits_{1\le i\le k} C_i$. We have $f = \sum_{i=1}^k f\mathbf{1}_{A_i}$. Moreover,

$$c_i\int_{A_i} f\, d\lambda_{\mathbf{g}} = c_i\int_{\varphi_i(A_i)} (f\circ\varphi_i^{-1})\sqrt{\det(\mathbf{g})\circ\varphi_i^{-1}}\, d\lambda_n$$

$$\le \int_{\varphi_i(A_i)} (f\circ\varphi_i^{-1})\sqrt{\det(\widetilde{\mathbf{g}})\circ\varphi_i^{-1}}\, d\lambda_n = \int_{A_i} f\, d\lambda_{\widetilde{\mathbf{g}}}.$$

Similarly,

$$\int_{A_i} f\, d\lambda_{\widetilde{\mathbf{g}}} \le C_i\int_{A_i} f\, d\lambda_{\mathbf{g}}.$$

Therefore,

$$c_i \int_M f \mathbf{1}_{A_i} d\lambda_{\mathbf{g}} \le \int_M f \mathbf{1}_{A_i} d\lambda_{\widetilde{\mathbf{g}}} \le C_i \int_M f \mathbf{1}_{A_i} d\lambda_{\mathbf{g}}.$$

Hence

$$c \int_M f \, d\lambda_{\mathbf{g}} = c \sum_{i=1}^{k} \int_M f \mathbf{1}_{A_i} d\lambda_{\mathbf{g}} \le \sum_{i=1}^{k} c_i \int_M f \mathbf{1}_{A_i} d\lambda_{\mathbf{g}} \le \sum_{i=1}^{k} \int_M f \mathbf{1}_{A_i} d\lambda_{\widetilde{\mathbf{g}}} = \int_M f \, d\lambda_{\widetilde{\mathbf{g}}}.$$

Similarly,

$$\int_M f \, d\lambda_{\widetilde{\mathbf{g}}} \le C \int_M f \, d\lambda_{\mathbf{g}}.$$

□

Remark 7.1.12. If $u, v, w \colon M \longrightarrow [0, \infty]$ are measurable and satisfy $u \le v \le w$, then Lemma 7.1.11 gives

$$c \int_M u \, d\lambda_{\mathbf{g}} \le \int_M u \, d\lambda_{\widetilde{\mathbf{g}}} \le \int_M v \, d\lambda_{\widetilde{\mathbf{g}}} \le \int_M w \, d\lambda_{\widetilde{\mathbf{g}}} \le C \int_M w \, d\lambda_{\mathbf{g}},$$

; that is,

$$c \int_M u \, d\lambda_{\mathbf{g}} \le \int_M v \, d\lambda_{\widetilde{\mathbf{g}}} \le C \int_M w \, d\lambda_{\mathbf{g}}.$$

Theorem 7.1.13. Let M be a compact smooth manifold with or without boundary, let $m \in \mathbb{N}_0$, and let $1 \le p < \infty$. If $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ are two Riemannian metrics on M, then the norms $\|\cdot\|_{W^{m,p}(M),\mathbf{g}}$ and $\|\cdot\|_{W^{m,p}(M),\widetilde{\mathbf{g}}}$ are equivalent. In particular, the two completions canonically define the same space $W^{m,p}(M)$.

Proof. Let ∇ and $\widetilde{\nabla}$ be the Levi–Civita connections of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively. For $u \in C^\infty(M)$ and $q \in M$, set

$$x(q) := \big(|u(q)|, |\nabla u(q)|_{\mathbf{g}}, \dots, |\nabla^m u(q)|_{\mathbf{g}}\big),$$

$$y(q) := \big(|u(q)|, |\widetilde{\nabla} u(q)|_{\widetilde{\mathbf{g}}}, \dots, |\widetilde{\nabla}^m u(q)|_{\widetilde{\mathbf{g}}}\big).$$

By Lemma 7.1.10, there exist $c_m, C_m > 0$ such that

$$\sqrt{c_m}\,\|x(q)\|_2 \le \|y(q)\|_2 \le \sqrt{C_m}\,\|x(q)\|_2 \qquad (q \in M).$$

Since all norms on $\mathbb{R}^{m+1}$ are equivalent, there exist $a_{m,p}, b_{m,p} > 0$ such that

$$a_{m,p}\|z\|_p \le \|z\|_2 \le b_{m,p}\|z\|_p \qquad (z \in \mathbb{R}^{m+1}).$$

Applying these two inequalities first to $x(q)$ and then to $y(q)$ yields, pointwise,

$$\frac{a_{m,p}\sqrt{c_m}}{b_{m,p}}\|x(q)\|_p \le \|y(q)\|_p \le \frac{b_{m,p}\sqrt{C_m}}{a_{m,p}}\|x(q)\|_p.$$

Raise this to the power p. Lemma 7.1.11, applied to the two Riemannian measures, allows us to integrate the outer expressions and gives constants $A, B > 0$, independent of u, for which

$$A\|u\|_{W^{m,p}(M),\mathbf{g}} \le \|u\|_{W^{m,p}(M),\widetilde{\mathbf{g}}} \le B\|u\|_{W^{m,p}(M),\mathbf{g}}.$$

Thus, the identity on $C^\infty(M)$ is bicontinuous for the two norms. By the universal property of completion, it extends uniquely to a bicontinuous isomorphism between the two completions, identifying each smooth function with itself. □

Example 7.1.14. (Dependence of the constants on derivatives of the metric) Comparison of metrics at order zero does not suffice to control the equivalence constants of all Sobolev norms. Consider the compact manifold with boundary $M = [0, 2\pi]$, with $\mathbf{g}_0 = (\mathbf{d}x)^2$, and, for each integer $N \geq 1$, the metric

$$\mathbf{g}_N = e^{2\sin(Nx)}(\mathbf{d}x)^2.$$

For every $x \in M$ and every $v \in T_xM$,

$$e^{-2}\mathbf{g}_0(v,v) \leq \mathbf{g}_N(v,v) \leq e^2\mathbf{g}_0(v,v).$$

The constants are independent of N. However, the only coefficient of the Levi–Civita connection of $\mathbf{g}_N$ is

$$\Gamma^1_{11}(\mathbf{g}_N) = \frac{1}{2}e^{-2\sin(Nx)}\frac{d}{dx}\left(e^{2\sin(Nx)}\right) = N\cos(Nx).$$

For the fixed smooth function $u(x) = x$, we have $u' = 1$, $u'' = 0$, and, by the coordinate formula for the Hessian,

$$(\nabla^{\mathbf{g}_N})^2 u = \left(u'' - \Gamma^1_{11}(\mathbf{g}_N)u'\right)(\mathbf{d}x)^2 = -N\cos(Nx)(\mathbf{d}x)^2.$$

Since $|(\mathbf{d}x)^2|^2_{\mathbf{g}_N} = e^{-4\sin(Nx)}$ and $d\lambda_{\mathbf{g}_N} = e^{\sin(Nx)}\,dx$, we obtain

$$\begin{aligned}\|(\nabla^{\mathbf{g}_N})^2 u\|^2_{L^2(M,\mathbf{g}_N)} &= N^2\int_0^{2\pi}\cos^2(Nx)e^{-3\sin(Nx)}\,dx \\ &\geq e^{-3}N^2\int_0^{2\pi}\cos^2(Nx)\,dx = \pi e^{-3}N^2.\end{aligned}$$

Therefore, $\|u\|_{W^{2,2}(M),\mathbf{g}_N}$ tends to infinity, while $\|u\|_{W^{2,2}(M),\mathbf{g}_0}$ remains fixed. Theorem 7.1.13 does give equivalence for each fixed pair $(\mathbf{g}_0, \mathbf{g}_N)$. What this example rules out is a common constant deduced solely from the bounds e^{-2} and e^2. The additional control of the connections and their derivatives appears in Lemma 7.1.10.

Remark 7.1.15. (Comparison on compact coordinate subsets) The comparisons used in the preceding proof are pointwise on the compact coordinate subsets: they are obtained by bounding the ratio of two positive quadratic forms on a compact sphere and the ratio of the volume elements. They therefore also hold on a coordinate ball or half-ball U whose closure is contained in an ambient chart, for smooth metrics and connections defined on that chart. It suffices to apply the bounds on $\overline{U}$ and integrate over U; $\overline{U}$ need not be a smooth manifold with boundary. This matters for half-balls, whose closures have a corner where the flat and spherical portions meet. In the following formulas, norms of smooth functions on $\overline{U}$ denote these same integrals: the coordinate boundary has zero n-dimensional measure.

7.2 Kato's inequality

We shall prove Rademacher's theorem on Riemannian manifolds in order to establish an important inequality known as *Kato's inequality* for sections of vector bundles. It will be useful in proving the Sobolev embeddings in the next section.

Recall Rademacher's theorem in Euclidean spaces.

Theorem 7.2.1. (Rademacher in Euclidean spaces) Let $U \subseteq \mathbb{R}^n$ be open and let $f\colon U \longrightarrow \mathbb{R}^m$ be a Lipschitz function. Then f is differentiable almost everywhere on U.

This theorem is proved in [80] and allows us to prove a version of Kato's inequality. We shall establish its counterpart on Riemannian manifolds to develop nonlinear analysis on Riemannian manifolds further.

Theorem 7.2.2. (Rademacher on Riemannian manifolds) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let $u\colon M \longrightarrow \mathbb{R}$ be a Lipschitz function. Then the first covariant derivative ∇u exists almost everywhere on M.

Proof. By Proposition 1.2.11, M has a countable basis of regular coordinate balls (and if M has boundary, some are regular coordinate half-balls). By the definition of regular coordinate balls (or half-balls), we may assume that there is a countable atlas $\{(B_j, \varphi_j) \mid j \in \mathbb{N}\}$ of regular coordinate balls (or half-balls) such that $\{(\overline{B}_j, \varphi_j) \mid j \in \mathbb{N}\}$ is a family of compact regular coordinate balls (or half-balls), and $M = \bigcup\limits_{j\in\mathbb{N}} B_j = \bigcup\limits_{j\in\mathbb{N}} \overline{B}_j$. Each $\overline{B}_j$ is a compact subset contained in an ambient chart; it need not be viewed as a smooth submanifold with boundary. For each $j \in \mathbb{N}$, let $C_j = \varphi_j(\overline{B}_j)$.

The set B_j admits two Riemannian metrics, $\mathbf{g}$ and $\boldsymbol{\delta} := \varphi_j^*\overline{\mathbf{g}}$, where $\overline{\mathbf{g}}$ denotes the usual metric on $\mathbb{R}^n$. By the pointwise comparison of Lemma 7.1.8 on the compact set $\overline{B}_j$, in the sense of Remark 7.1.15, there is $K_j > 0$ such that for any $p \in B_j$ and $v \in T_pM$, $\mathbf{g}_p(v,v) \leq K_j \boldsymbol{\delta}_p(v,v)$.

We shall prove that $h_j := u \circ \varphi_j^{-1}\colon C_j \longrightarrow \mathbb{R}$ is Lipschitz. Since u is Lipschitz, there is $K > 0$ such that $|u(p) - u(q)| \leq K d_{\mathbf{g}}(p,q)$ for any $p, q \in M$. Let $x = \varphi_j(p), y = \varphi_j(q) \in C_j$. Since C_j is a closed ball (or closed half-ball) in $\mathbb{R}^n$, it is convex, and hence the segment joining x to y remains in C_j. Let $\gamma\colon [0,1] \longrightarrow C_j$ be given by $\gamma(t) = (1-t)x + ty$. Then γ is a smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$. Thus $\widetilde{\gamma}_j := \varphi_j^{-1} \circ \gamma\colon [0,1] \longrightarrow \overline{B}_j$ is a smooth curve with $\widetilde{\gamma}_j(0) = p$ and $\widetilde{\gamma}_j(1) = q$.

We therefore have

$$
\begin{aligned}
|h_j(x) - h_j(y)| = |u(\varphi_j^{-1}(x)) - u(\varphi_j^{-1}(y))| &\leq K d_{\mathbf{g}}(p,q) \leq K L_{\mathbf{g}}(\widetilde{\gamma}_j) \\
&= K \int_0^1 |\widetilde{\gamma}_j'(t)|_{\mathbf{g}} dt = K \int_0^1 \sqrt{\mathbf{g}_{\widetilde{\gamma}_j(t)}(\widetilde{\gamma}_j'(t), \widetilde{\gamma}_j'(t))} dt \\
&\leq K\sqrt{K_j} \int_0^1 \sqrt{\boldsymbol{\delta}_{(\varphi_j^{-1}\circ\gamma)(t)}((\varphi_j^{-1}\circ\gamma)'(t), (\varphi_j^{-1}\circ\gamma)'(t))} dt \\
&= K\sqrt{K_j} \int_0^1 \sqrt{\overline{\mathbf{g}}_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt = K\sqrt{K_j} \int_0^1 \|\gamma'(t)\| dt
\end{aligned}
$$

$$= K\sqrt{K_j}\int_0^1 \|y-x\|dt = K\sqrt{K_j}\|x-y\|,$$

which proves that h_j is Lipschitz. By Rademacher's theorem (Theorem 7.2.1), the derivative of h_j exists almost everywhere on $\operatorname{int}(C_j)$. But C_j is a closed ball or a closed half-ball, so its Euclidean boundary has zero n-dimensional measure. Thus the derivative of h_j exists almost everywhere on C_j; that is, there exists $N_j \subseteq C_j$ such that $\lambda_n(N_j) = 0$ and dh_j exists on $C_j \setminus N_j$. Let $N_j' = \varphi_j^{-1}(N_j)$. Since

$$\lambda_{\mathbf{g}}(N_j') = \int_{\varphi_j(N_j')} (\varphi_j)_* \sqrt{\det(\mathbf{g})} d\lambda_n = \int_{N_j} \sqrt{\det(\mathbf{g})} \circ \varphi_j^{-1} d\lambda_n = 0,$$

we conclude that N_j' has zero Riemann–Lebesgue measure in M.

Let $p \in B_j \setminus N_j'$. Then $\varphi_j(p) \in C_j \setminus N_j$, so $dh_j|_{\varphi_j(p)}$ exists. Since φ_j is smooth, $d\varphi_j|_p$ exists. Because $u|_{B_j} = h_j \circ \varphi_j$, the chain rule shows that $du|_p$ exists. Since the first covariant derivative of a function agrees with its differential, $\nabla u(p)$ exists. Thus ∇u exists on $B_j \setminus N_j'$. But $N = \bigcup_{j\in\mathbb{N}} N_j'$ satisfies $\lambda_{\mathbf{g}}(N) = 0$, and moreover $\nabla u(p)$ exists for every $p \in \bigcup_{j\in\mathbb{N}} B_j \setminus \bigcup_{j\in\mathbb{N}} N_j' = M \setminus N$. Therefore, ∇u exists almost everywhere on M. □

We now show that if $\mathbf{u} \in \Gamma(\mathbf{E})$, then $\nabla|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ exists almost everywhere on M. First, we prove two results.

Proposition 7.2.3. Let $U \subseteq \mathbb{R}^n$ be a convex open subset, let $s \in \mathbb{N} \cup \{0\}$, and let $f : U \longrightarrow \mathbb{R}$ be a function of class C^{s+1}. If there is $M > 0$ such that $\|\nabla^{s+1} f(x)\|_F \leq M$ for every $x \in U$, then

$$\|\nabla^s f(x) - \nabla^s f(y)\|_F \leq M\|x-y\|$$

for any $x, y \in U$, where $\nabla^0 f = f$ and, for $s \geq 1$,

$$\nabla^s f(x) = \frac{\partial^s f}{\partial x_{i_1} \cdots \partial x_{i_s}}\bigg|_x \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_s}.$$

Proof. If $s = 0$, the following argument is interpreted with $\nabla^0 f = f$. Let $x, y \in U$. Define $\gamma : [0,1] \longrightarrow U$ by $\gamma(t) = x + t(y-x)$. Then $\gamma'(t) = y - x$. We have

$$\begin{aligned}
\nabla^s f(y) - \nabla^s f(x) &= \frac{\partial^s f}{\partial x_{i_1} \cdots \partial x_{i_s}}\bigg|_{\gamma(1)} \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_s} - \frac{\partial^s f}{\partial x_{i_1} \cdots \partial x_{i_s}}\bigg|_{\gamma(0)} \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_s} \\
&= \left(\frac{\partial^s f}{\partial x_{i_1} \cdots \partial x_{i_s}}(\gamma(1)) - \frac{\partial^s f}{\partial x_{i_1} \cdots \partial x_{i_s}}(\gamma(0))\right) \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_s} \\
&= \left(\int_0^1 \frac{d}{dt}\left(\frac{\partial^s f}{\partial x_{i_1} \cdots \partial x_{i_s}}(\gamma(t))\right) dt\right) \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_s} \\
&= \left(\int_0^1 \left\langle \nabla\left(\frac{\partial^s f}{\partial x_{i_1} \cdots \partial x_{i_s}}\right)_{\gamma(t)}, y - x\right\rangle_{\mathbb{R}^n} dt\right) \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_s}
\end{aligned}$$

$$= \left(\int_0^1 \sum_{j=1}^n \frac{\partial^{s+1} f}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t))(y^j - x^j) dt \right) \mathbf{d}x^{i_1} \otimes \cdots \otimes \mathbf{d}x^{i_s}.$$

Thus,

$$\|\nabla^s f(y) - \nabla^s f(x)\|_F = \sqrt{\sum_{i_1,\ldots,i_s=1}^n \left(\int_0^1 \sum_{j=1}^n \frac{\partial^{s+1} f}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t))(y^j - x^j) dt \right)^2}$$

$$\leq \sqrt{\sum_{i_1,\ldots,i_s=1}^n \int_0^1 \left(\sum_{j=1}^n \frac{\partial^{s+1} f}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t))(y^j - x^j) \right)^2 dt},$$

where the last inequality follows from Jensen's inequality (Proposition B.7.11) and convexity of $\varphi(x) = x^2$. But

$$\sqrt{\sum_{i_1,\ldots,i_s=1}^n \int_0^1 \left(\sum_{j=1}^n \frac{\partial^{s+1} f}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t))(y^j - x^j) \right)^2 dt}$$

$$= \sqrt{\sum_{i_1,\ldots,i_s=1}^n \int_0^1 \left\langle \left(\frac{\partial^{s+1} f}{\partial x_1 \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)), \ldots, \frac{\partial^{s+1} f}{\partial x_n \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)) \right), y - x \right\rangle_{\mathbb{R}^n}^2 dt}$$

$$\leq \sqrt{\sum_{i_1,\ldots,i_s=1}^n \int_0^1 \left\| \left(\frac{\partial^{s+1} f}{\partial x_1 \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)), \ldots, \frac{\partial^{s+1} f}{\partial x_n \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)) \right) \right\|^2 \|y - x\|^2 dt}$$

$$= \|x - y\| \sqrt{\sum_{i_1,\ldots,i_s=1}^n \int_0^1 \left\| \left(\frac{\partial^{s+1} f}{\partial x_1 \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)), \ldots, \frac{\partial^{s+1} f}{\partial x_n \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)) \right) \right\|^2 dt}$$

$$= \|x - y\| \sqrt{\int_0^1 \sum_{i_1,\ldots,i_s=1}^n \left\| \left(\frac{\partial^{s+1} f}{\partial x_1 \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)), \ldots, \frac{\partial^{s+1} f}{\partial x_n \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)) \right) \right\|^2 dt}$$

$$= \|x - y\| \sqrt{\int_0^1 \sum_{i_1,\ldots,i_s=1}^n \sum_{j=1}^n \left(\frac{\partial^{s+1} f}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_s}} (\gamma(t)) \right)^2 dt}$$

$$= \|x - y\| \sqrt{\int_0^1 \|\nabla^{s+1} f(\gamma(t))\|_F^2 dt} \leq \|x - y\| \sqrt{\int_0^1 M^2 dt} = M \|x - y\|.$$

□

Lemma 7.2.4. Let $U \subseteq \mathbb{R}^n$ be open with $\overline{B}_{\text{euc}}(0,2) \subseteq U$, and let $f : U \longrightarrow [0,\infty)$ be a function of class C^2. Then the function $\sqrt{f}$ is Lipschitz on $\overline{B}_{\text{euc}}(0,1)$.

Proof. Since f is of class C^2 on the compact set $\overline{B}_{\text{euc}}(0,2)$, its gradient and Hessian matrix are bounded on that set. Let

$$K := 1 + \sup_{z \in \overline{B}_{\text{euc}}(0,2)} \|\nabla f(z)\| + \sup_{z \in \overline{B}_{\text{euc}}(0,2)} \|\nabla^2 f(z)\|_F.$$

Then $K > 0$ and $\|\nabla f(z)\| \leq K$, $\|\nabla^2 f(z)\|_F \leq K$ for every $z \in \overline{B}_{\text{euc}}(0,2)$.

For $\varepsilon > 0$, consider $f_\varepsilon = f + \varepsilon$. Fix $x \in \overline{B}_{\mathrm{euc}}(0,1)$. If $\nabla f(x) = 0$, then $\nabla\sqrt{f_\varepsilon}(x) = 0$. Thus assume $\nabla f(x) \neq 0$ and define

$$v := \frac{\nabla f(x)}{\|\nabla f(x)\|}.$$

Then $\|v\| = 1$ and, since $x \in \overline{B}_{\mathrm{euc}}(0,1)$, we have $x - tv \in \overline{B}_{\mathrm{euc}}(0,2)$ for every $t \in [0,1]$.
Consider the function $g\colon [0,1] \longrightarrow (0,\infty)$ given by $g(t) := f_\varepsilon(x - tv)$. Then

$$g'(0) = -\langle \nabla f(x), v\rangle_{\mathbb{R}^n} = -\|\nabla f(x)\|.$$

Moreover, for every $t \in [0,1]$,

$$g''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x - tv) v_i v_j.$$

The Cauchy–Schwarz inequality from Theorem B.3.2, applied in $\mathbb{R}^{n^2}$, gives

$$|g''(t)| \leq \left(\sum_{i,j=1}^{n} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x - tv) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{n} |v_i v_j|^2 \right)^{\frac{1}{2}}$$

$$= \|\nabla^2 f(x - tv)\|_F \|v\|^2 \leq K.$$

By Taylor's formula with integral remainder from Theorem 1.4.9, applied in one dimension, for every $t \in [0,1]$,

$$g(t) = g(0) + g'(0)t + \int_0^t (t - r) g''(r)\, dr \leq g(0) - \|\nabla f(x)\| t + \frac{K}{2} t^2.$$

Let $t_0 := \frac{\|\nabla f(x)\|}{K}$. Since $\|\nabla f(x)\| \leq K$, we have $t_0 \in [0,1]$. Moreover, since $f \geq 0$ and $\varepsilon > 0$, we have $g(t_0) = f(x - t_0 v) + \varepsilon \geq \varepsilon > 0$. Therefore,

$$0 < g(t_0) \leq g(0) - \frac{\|\nabla f(x)\|^2}{K} + \frac{K}{2} \frac{\|\nabla f(x)\|^2}{K^2} = g(0) - \frac{\|\nabla f(x)\|^2}{2K}.$$

It follows that $\|\nabla f(x)\|^2 < 2K g(0) = 2K(f(x) + \varepsilon)$ and consequently

$$\|\nabla \sqrt{f_\varepsilon}(x)\| = \frac{\|\nabla f(x)\|}{2\sqrt{f(x) + \varepsilon}} \leq \sqrt{\frac{K}{2}}.$$

This bound holds for every $x \in \overline{B}_{\mathrm{euc}}(0,1)$, and the constant on the right-hand side is independent of x and ε. Since $\sqrt{f_\varepsilon}$ is of class C^1 on the convex open subset $B_{\mathrm{euc}}(0,1)$, Proposition 7.2.3, applied with $s = 0$, implies

$$\left| \sqrt{f(x) + \varepsilon} - \sqrt{f(y) + \varepsilon} \right| \leq \sqrt{\frac{K}{2}} \|x - y\|$$

for any $x, y \in B_{\mathrm{euc}}(0,1)$. By continuity, the inequality also holds for any $x, y \in \overline{B}_{\mathrm{euc}}(0,1)$. Finally, letting $\varepsilon \to 0$, we obtain

$$|\sqrt{f(x)} - \sqrt{f(y)}| \leq \sqrt{\frac{K}{2}} \|x - y\|$$

for any $x, y \in \overline{B}_{\mathrm{euc}}(0,1)$, so $\sqrt{f}$ is Lipschitz on $\overline{B}_{\mathrm{euc}}(0,1)$. □

Lemma 7.2.5. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. If $\mathbf{u} \in \Gamma(\mathbf{E})$, then for every integer $s \geq 0$, the function $|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ is locally Lipschitz on $\mathrm{Int}(M)$. In particular, $\nabla|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ exists almost everywhere on M.

Proof. Since ∂M has measure zero, it suffices to prove that $\nabla|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ exists almost everywhere on $\mathrm{Int}(M)$. Let (V, φ) be a strongly convex regular normal geodesic coordinate ball whose closure is contained in $\mathrm{Int}(M)$. Composing φ with an affine transformation of $\mathbb{R}^n$, we may assume $\overline{B}_{\mathrm{euc}}(0,3) \subseteq \varphi(V)$. Define $B := \varphi^{-1}(\overline{B}_{\mathrm{euc}}(0,1))$.
On V, we have

$$|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} = \sqrt{g^{i_1 j_1} \cdots g^{i_s j_s} h_{ab} (\nabla^s \mathbf{u})^a_{i_1 \cdots i_s} (\nabla^s \mathbf{u})^b_{j_1 \cdots j_s}} = \sqrt{\langle \nabla^s \mathbf{u}, \nabla^s \mathbf{u} \rangle_{\mathbf{g},\mathbf{h_E}}},$$

where, if $s = 0$, the factors corresponding to $\mathbf{g}^{-1}$ are absent. Since $\langle \nabla^s \mathbf{u}, \nabla^s \mathbf{u} \rangle_{\mathbf{g},\mathbf{h_E}} \circ \varphi^{-1}$ is smooth and $\varphi(V)$ contains $\overline{B}_{\mathrm{euc}}(0,2)$, Lemma 7.2.4 implies that

$$|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \circ \varphi^{-1} = \sqrt{\langle \nabla^s \mathbf{u}, \nabla^s \mathbf{u} \rangle_{\mathbf{g},\mathbf{h_E}} \circ \varphi^{-1}}$$

is Lipschitz on $\overline{B}_{\mathrm{euc}}(0,1)$. Thus there is $C > 0$ such that

$$\left| \left(|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \circ \varphi^{-1}\right)(x) - \left(|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \circ \varphi^{-1}\right)(y) \right| \leq C \|x - y\|$$

for any $x, y \in \overline{B}_{\mathrm{euc}}(0,1)$.
Let $p, q \in B$. Let $\boldsymbol{\delta} := \varphi^* \overline{\mathbf{g}}$ be the Euclidean metric transported to V. Since $\overline{V}$ is compact, Lemma 7.1.8, applied to the metrics $\mathbf{g}$ and $\boldsymbol{\delta}$, provides a constant $K > 0$ such that $|X|_{\boldsymbol{\delta}} \leq K |X|_{\mathbf{g}}$ for every $X \in TV$. Since V is strongly convex, there is a minimizing geodesic $\gamma \colon [0,1] \longrightarrow V$ joining p to q and satisfying $L_{\mathbf{g}}(\gamma) = d_{\mathbf{g}}(p,q)$. The curve $\varphi \circ \gamma$ joins $\varphi(p)$ to $\varphi(q)$, and therefore

$$\|\varphi(p) - \varphi(q)\| \leq L_{\overline{\mathbf{g}}}(\varphi \circ \gamma) = L_{\boldsymbol{\delta}}(\gamma) \leq K L_{\mathbf{g}}(\gamma) = K d_{\mathbf{g}}(p,q).$$

Consequently,

$$\begin{aligned} \left| |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}(p) - |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}(q) \right| &\leq C \|\varphi(p) - \varphi(q)\| \\ &\leq C K \, d_{\mathbf{g}}(p,q), \end{aligned}$$

so $|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ is Lipschitz on B.
Since every Riemannian manifold admits a cover by strongly convex normal geodesic coordinate balls, we conclude that $|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ is locally Lipschitz on $\mathrm{Int}(M)$.
By second countability of M, we may extract a countable subcover $(B_j)_{j \in \mathbb{N}}$ consisting of balls of the preceding type. Theorem 7.2.2 implies that $\nabla|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ exists almost everywhere on each B_j. Since the union is countable, it exists almost everywhere on $\mathrm{Int}(M)$ and, because ∂M has measure zero, also almost everywhere on M. □

The preceding results allow us to prove Kato's inequality for sections of vector bundles.

Theorem 7.2.6. (Kato's inequality) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection

$\nabla^{\mathbf{E}}$. For any integer $s \geq 0$ and any $\mathbf{u} \in \Gamma(\mathbf{E})$, we have

$$|\nabla|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}|_{\mathbf{g}} \leq |\nabla^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$$

almost everywhere on M.

Proof. If $|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \neq 0$ at a point, continuity gives a smooth chart (U,φ), with $\varphi = (x^1,\dots,x^n)$, on which this function is positive. Thus $|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ is classically differentiable on U, and the chain rule gives

$$\nabla|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} = \frac{\nabla\langle\nabla^s \mathbf{u}, \nabla^s \mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}}{2|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}}.$$

To compute $\nabla\langle\nabla^s \mathbf{u}, \nabla^s \mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}$, recall that

$$\left(\nabla\langle\nabla^s \mathbf{u}, \nabla^s \mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}\right)_i = \frac{\partial\langle\nabla^s \mathbf{u}, \nabla^s \mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}}{\partial x^i}.$$

Since the Levi–Civita connection is compatible with $\mathbf{g}$ and $\nabla^{\mathbf{E}}$ is compatible with $\mathbf{h_E}$, the induced connection on $T^{(0,s)}(TM)\otimes\mathbf{E}$ is compatible with the product metric $\langle\cdot,\cdot\rangle_{\mathbf{g},\mathbf{h_E}}$. Therefore,

$$\begin{aligned}\frac{\partial\langle\nabla^s \mathbf{u}, \nabla^s \mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}}{\partial x^i} &= \left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}} + \left\langle\nabla^s \mathbf{u}, \nabla_{\partial_i}(\nabla^s \mathbf{u})\right\rangle_{\mathbf{g},\mathbf{h_E}} \\ &= 2\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}}.\end{aligned}$$

Thus,

$$\begin{aligned}\nabla|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} = \frac{\nabla\langle\nabla^s \mathbf{u}, \nabla^s \mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}}{2|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}} &= \frac{2\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}} \mathbf{d}x^i}{2|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}} \\ &= \frac{\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}} \mathbf{d}x^i}{|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}}.\end{aligned}$$

Consider the square root of $\mathbf{g}^{-1}$, given by Proposition C.1.14, which is positive definite and symmetric, and denote it by (R^{ij}).

Then

$$\begin{aligned}\left|\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}} \mathbf{d}x^i\right|^2_{\mathbf{g}} &= \sum_{1\leq i,j\leq n} g^{ij}\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}}\left\langle\nabla_{\partial_j}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}} \\ &= \sum_{1\leq i,j\leq n}\sum_{m=1}^{n} R^{im}R^{mj}\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}}\left\langle\nabla_{\partial_j}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}} \\ &= \sum_{m=1}^{n}\sum_{1\leq i,j\leq n} R^{im}R^{jm}\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}}\left\langle\nabla_{\partial_j}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}} \\ &= \sum_{m=1}^{n}\left(\sum_{i=1}^{n} R^{im}\left\langle\nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u}\right\rangle_{\mathbf{g},\mathbf{h_E}}\right)^2\end{aligned}$$

$$= \sum_{m=1}^{n} \left(\left\langle \sum_{i=1}^{n} R^{im} \nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u} \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right)^2$$

$$= \sum_{m=1}^{n} \left| \left\langle \sum_{i=1}^{n} R^{im} \nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u} \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right|^2$$

$$\underset{\text{Cauchy–Schwarz}}{\leq} \sum_{m=1}^{n} \left| \sum_{i=1}^{n} R^{im} \nabla_{\partial_i}(\nabla^s \mathbf{u}) \right|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

$$= |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \sum_{m=1}^{n} \left\langle \sum_{i=1}^{n} R^{im} \nabla_{\partial_i}(\nabla^s \mathbf{u}), \sum_{i=1}^{n} R^{im} \nabla_{\partial_i}(\nabla^s \mathbf{u}) \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

$$= |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \sum_{m=1}^{n} \left\langle \sum_{i=1}^{n} R^{im} \nabla_{\partial_i}(\nabla^s \mathbf{u}), \sum_{j=1}^{n} R^{mj} \nabla_{\partial_j}(\nabla^s \mathbf{u}) \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

$$= |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \sum_{m=1}^{n} \left(\sum_{i=1}^{n} R^{im} \sum_{j=1}^{n} R^{mj} \left\langle \nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla_{\partial_j}(\nabla^s \mathbf{u}) \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right)$$

$$= |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \sum_{m=1}^{n} \left(\sum_{1 \leq i,j \leq n} R^{im} R^{mj} \left\langle \nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla_{\partial_j}(\nabla^s \mathbf{u}) \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right)$$

$$= |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \sum_{1 \leq i,j \leq n} \sum_{m=1}^{n} R^{im} R^{mj} \left\langle \nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla_{\partial_j}(\nabla^s \mathbf{u}) \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

$$= |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \sum_{1 \leq i,j \leq n} g^{ij} \left\langle \nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla_{\partial_j}(\nabla^s \mathbf{u}) \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

$$= |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} |\nabla(\nabla^s \mathbf{u})|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} = |\nabla^s \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} |\nabla^{s+1} \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}},$$

where we used bilinearity of the induced bundle metric and the Cauchy–Schwarz inequality from Theorem B.3.2 in each fiber of $T^{(0,s)}(TM) \otimes \mathbf{E}$. Thus,

$$|\nabla |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}|_\mathbf{g} = \frac{\left| \left\langle \nabla_{\partial_i}(\nabla^s \mathbf{u}), \nabla^s \mathbf{u} \right\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \mathbf{d}x^i \right|_\mathbf{g}}{|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}} \leq \frac{|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} |\nabla^{s+1} \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}}{|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}} = |\nabla^{s+1} \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Moreover, by Lemma 7.2.5, $\nabla |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$ exists almost everywhere on M. If $|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} = 0$ at a point of $\mathrm{Int}(M)$ where $\nabla |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$ exists, the nonnegative function $|\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$ has a global minimum at that point. Its differential therefore vanishes, and consequently $\nabla |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} = 0$. Hence

$$|\nabla |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}|_\mathbf{g} = 0 \leq |\nabla^{s+1} \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

We thus conclude that

$$|\nabla |\nabla^s \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}|_\mathbf{g} \leq |\nabla^{s+1} \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

almost everywhere on M. □

7.3 Sobolev embeddings on compact Riemannian manifolds

Localization in coordinate balls and half-balls allows the Euclidean embeddings to be transferred to a compact Riemannian manifold. We shall use Kato's inequality to pass from first order to higher orders.

Proposition 7.3.1. (Iteration of first-order embeddings) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. Let $0 \le m < k$ be integers, set $d := k-m$, and consider exponents $p_0,\dots,p_d \in [1,\infty)$. If the inclusions

$$W^{1,p_i}(M) \hookrightarrow L^{p_{i+1}}(M) \qquad (i \in \{0,\dots,d-1\})$$

are continuous, then so is

$$W^{k,p_0}(M) \hookrightarrow W^{m,p_d}(M).$$

Proof. We first show that a continuous embedding $W^{1,p}(M) \hookrightarrow L^q(M)$, with $p,q<\infty$, implies

$$W^{s,p}(M) \hookrightarrow W^{s-1,q}(M) \qquad \text{for every integer } s \ge 1.$$

Let $u \in C^{s,p}(M)$ and fix $0 \le j \le s-1$. Kato's inequality, Theorem 7.2.6, gives

$$\left|\nabla |\nabla^j u|_{\mathbf{g}}\right|_{\mathbf{g}} \le |\nabla^{j+1} u|_{\mathbf{g}} \qquad \text{almost everywhere.}$$

To work with the definition by completion, consider

$$f_\varepsilon := \sqrt{|\nabla^j u|_{\mathbf{g}}^2 + \varepsilon^2} - \varepsilon, \qquad \varepsilon > 0.$$

These functions are smooth and satisfy $0 \le f_\varepsilon \le |\nabla^j u|_{\mathbf{g}}$. Where $|\nabla^j u|_{\mathbf{g}} > 0$, we have

$$\nabla f_\varepsilon = \frac{|\nabla^j u|_{\mathbf{g}}}{\sqrt{|\nabla^j u|_{\mathbf{g}}^2 + \varepsilon^2}} \nabla |\nabla^j u|_{\mathbf{g}}.$$

Where $\nabla^j u = 0$, the differential of its squared norm is zero, so $\nabla f_\varepsilon = 0$. Consequently,

$$|\nabla f_\varepsilon|_{\mathbf{g}} \le |\nabla^{j+1} u|_{\mathbf{g}}.$$

Moreover, $f_\varepsilon \to |\nabla^j u|_{\mathbf{g}}$, and their differentials converge almost everywhere to the differential of that norm: at interior zeros where it exists, the differential vanishes because the point is a minimum. Dominated convergence, applied to the pth powers of the preceding bounds, gives convergence in L^p of the functions and their differentials. Thus (f_ε) is Cauchy in the norm defining $W^{1,p}(M)$ and represents the function $|\nabla^j u|_{\mathbf{g}}$ in its completion. We also obtain

$$\left\| |\nabla^j u|_{\mathbf{g}} \right\|_{W^{1,p}(M)} \le \|\nabla^j u\|_{L^p(M,T^{(0,j)}(TM))} + \|\nabla^{j+1} u\|_{L^p(M,T^{(0,j+1)}(TM))}.$$

Apply the assumed first-order embedding to this function. There is $C_{p,q} > 0$ such that

$$\|\nabla^j u\|_{L^q(M,T^{(0,j)}(TM))} \le C_{p,q} \left(\|\nabla^j u\|_{L^p(M,T^{(0,j)}(TM))} + \|\nabla^{j+1} u\|_{L^p(M,T^{(0,j+1)}(TM))} \right).$$

Summing for $j \in \{0,\dots,s-1\}$ and using Proposition 7.1.7 gives

$$\|u\|_{W^{s-1,q}(M)} \leq C_s \|u\|_{W^{s,p}(M)}.$$

Extension to the completions, Corollary B.2.37, provides the desired embedding. This extension agrees with the canonical inclusion: a smooth sequence converging in $L^p(M)$ to u and in $L^q(M)$ to v converges to both in L^1 on every compact set by Hölder's inequality, so $u = v$ almost everywhere.
For each $i \in \{0,\dots,d-1\}$, apply the preceding argument with $s = k-i$, $p = p_i$, and $q = p_{i+1}$. We obtain

$$\|u\|_{W^{k-i-1,p_{i+1}}(M)} \leq C_i \|u\|_{W^{k-i,p_i}(M)}.$$

The composition of these d embeddings gives

$$\|u\|_{W^{m,p_d}(M)} \leq \left(\prod_{i=0}^{d-1} C_i\right) \|u\|_{W^{k,p_0}(M)},$$

as required. □

Theorem 7.3.2. (Sobolev embedding with finite orders and exponents) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n, with or without boundary. Let $0 \leq m < k$ be integers and $1 \leq p,q < \infty$ satisfy

$$k - \frac{n}{p} \geq m - \frac{n}{q}.$$

Then the inclusion $W^{k,p}(M) \hookrightarrow W^{m,q}(M)$ is continuous.

Proof. If $n = 0$, compactness implies that M is a finite set, and the assertion follows from equivalence of norms in finite dimensions. Assume $n \geq 1$. We begin with first order. Fix $1 \leq p,q < \infty$ with $1 - \frac{n}{p} \geq -\frac{n}{q}$. Choose a finite cover $(U_i,\varphi_i)_{i=1}^N$ consisting of interior regular coordinate balls and regular coordinate half-balls centered on the boundary. Let $(\psi_i)_{i=1}^N$ be a subordinate smooth partition of unity with $\operatorname{supp}\psi_i \Subset U_i$, and set

$$\Omega_i := \varphi_i\big(U_i \cap \operatorname{Int}(M)\big), \qquad \boldsymbol{\delta}_i := \varphi_i^* \overline{\mathbf{g}},$$

where $\overline{\mathbf{g}}$ is the Euclidean metric.
Let $u \in C^\infty(M)$ and define $v_i := (\psi_i u) \circ \varphi_i^{-1}$ on Ω_i. If U_i is interior, $v_i \in C_c^\infty(\Omega_i)$, and Theorem 3.3.16, with total orders one and zero, gives

$$\|v_i\|_{L^q(\Omega_i)} \leq A_i \|v_i\|_{W^{1,p}(\Omega_i)}.$$

If U_i is a boundary chart, Ω_i is a half-ball. The support of v_i is separated from the spherical portion, so its extension by zero within $\mathbb{R}^n_+$ is smooth up to $x_n = 0$ and has support in the fixed compact set

$$K_i := \varphi_i(\operatorname{supp}\psi_i) \subseteq \overline{\mathbb{R}^n_+}.$$

Theorem 3.6.6, with total orders one and zero, gives the same estimate upon restriction to Ω_i.

Change of variables and the definition of the metric δ_i give

$$\int_{\Omega_i} |v_i|^q \, d\lambda_n = \int_{U_i} |\psi_i u|^q \, d\lambda_{\delta_i}, \qquad \int_{\Omega_i} |dv_i|^p_{\overline{\mathbf{g}}} \, d\lambda_n = \int_{U_i} |d(\psi_i u)|^p_{\delta_i} \, d\lambda_{\delta_i}.$$

Equivalence of norms on $\mathbb{R}^n$, Proposition B.1.2, compares $|dv_i|_{\overline{\mathbf{g}}}$ with $\big(\sum_{a=1}^{n} |D_a v_i|^p\big)^{1/p}$. On the other hand, Remark 7.1.15 allows comparison of the metrics δ_i and $\mathbf{g}$ and their measures on $\overline{U_i}$. Thus there exist $a_i, b_i > 0$, independent of u, such that

$$\|v_i\|_{W^{1,p}(\Omega_i)} \le a_i \|\psi_i u\|_{W^{1,p}(U_i),\mathbf{g}}, \qquad \|\psi_i u\|_{L^q(U_i),\mathbf{g}} \le b_i \|v_i\|_{L^q(\Omega_i)}.$$

The Euclidean estimate then implies

$$\|\psi_i u\|_{L^q(M)} \le a_i b_i A_i \|\psi_i u\|_{W^{1,p}(M)},$$

since $\psi_i u$ and its derivatives vanish outside U_i.
To bound the localization, use $d(\psi_i u) = \psi_i \, du + u \, d\psi_i$ and $0 \le \psi_i \le 1$. Since M is compact, $|d\psi_i|_{\mathbf{g}}$ is bounded and

$$\begin{aligned} \|\psi_i u\|_{W^{1,p}(M)} &\le \|\psi_i u\|_{L^p(M)} + \|d(\psi_i u)\|_{L^p(M,T^*M)} \\ &\le \big(1 + \||d\psi_i|_{\mathbf{g}}\|_{L^\infty(M)}\big) \|u\|_{L^p(M)} + \|du\|_{L^p(M,T^*M)} \\ &\le C_i \|u\|_{W^{1,p}(M)}. \end{aligned}$$

Summing over the finite cover and using $u = \sum_{i=1}^{N} \psi_i u$ gives

$$\|u\|_{L^q(M)} \le \sum_{i=1}^{N} \|\psi_i u\|_{L^q(M)} \le C \|u\|_{W^{1,p}(M)}.$$

The density defining $W^{1,p}(M)$ and Theorem B.2.36 extend this estimate to all of $W^{1,p}(M)$. The limit in L^q is identified with the limit in L^p by uniqueness of the limit in $L^1(M)$.
Now consider the orders $m < k$ and the exponents in the statement. Set $d := k - m$ and define

$$\frac{1}{p_i} := \frac{1}{p} - \frac{i}{d}\left(\frac{1}{p} - \frac{1}{q}\right), \qquad i \in \{0, \dots, d\}.$$

Each p_i is finite and belongs to $[1, \infty)$, since its reciprocal is a convex combination of $1/p$ and $1/q$. Moreover, $p_0 = p$, $p_d = q$, and

$$1 - \frac{n}{p_i} + \frac{n}{p_{i+1}} = 1 - \frac{n}{d}\left(\frac{1}{p} - \frac{1}{q}\right) \ge 0.$$

The first-order part gives $W^{1,p_i}(M) \hookrightarrow L^{p_{i+1}}(M)$ continuously for each $i \in \{0, \dots, d-1\}$. Proposition 7.3.1 then yields $W^{k,p}(M) \hookrightarrow W^{m,q}(M)$ continuously. □

Corollary 7.3.3. (Sobolev embedding and critical exponent) Let $(M, \mathbf{g})$ be a compact Riemannian manifold

of dimension n, with or without boundary. Let $j \geq 0$ and $d \geq 1$ be integers, and let $1 \leq p < n/d$. Then

$$W^{j+d,p}(M) \hookrightarrow W^{j,q}(M) \qquad \text{for every } 1 \leq q \leq p_d^* := \frac{np}{n-dp},$$

and these inclusions are continuous.

Proof. The condition $q \leq p_d^*$ is equivalent to $j + d - \frac{n}{p} \geq j - \frac{n}{q}$. Apply Theorem 7.3.2 with $k = j + d$ and $m = j$. □

7.4 The Rellich–Kondrashov theorem on compact Riemannian manifolds

The Rellich–Kondrashov theorem provides the compact counterpart of the preceding embeddings. The Euclidean version in Theorem 3.4.2 can be transferred to a compact Riemannian manifold $(M, \mathbf{g})$ by means of a finite cover by charts and a partition of unity; in boundary charts, we use the localized half-space version. The following result specifies the orders and exponents for which compactness holds, including the loss of one derivative with any finite exponent. In the range where a finite critical exponent exists, we shall recover the subcritical embeddings from Corollary 7.3.3 as a special case.

Theorem 7.4.1. (Rellich–Kondrashov) Let $(M, \mathbf{g})$ be a compact Riemannian manifold, with or without boundary, of dimension n. For any integers $j \geq 0$ and $m \geq 1$, and any $1 \leq p, q < \infty$ satisfying $j + m - \frac{n}{p} > j - \frac{n}{q}$, the inclusion $W^{j+m,p}(M) \subseteq W^{j,q}(M)$ is compact.

The proof uses the Leibniz rule for the tensor connection, Proposition 2.4.13, iterated while preserving the order of the covariant indices.

Proof. Since M is compact, there is a finite open cover $\{(U_i, \varphi_i) \mid i \in \{1, \dots, N\}\}$ consisting of regular coordinate balls contained in $\mathrm{Int}(M)$ and, when $\partial M \neq \varnothing$, regular coordinate half-balls centered on the boundary. Let $(\psi_i)_{i=1}^N$ be a partition of unity subordinate to this open cover. In both cases, φ_i extends to $\overline{U_i}$, which is compact. Consider again $\boldsymbol{\delta}_i = \varphi_i^* \overline{\mathbf{g}}$.

Let $(u_k)_{k \in \mathbb{N}} \subseteq W^{j+m,p}(M) \cap C^\infty(M)$ be a bounded sequence.

By Remark 7.1.15, the local estimates in Theorem 7.1.13 apply on each $\overline{U_i}$, even when a half-ball has a corner. Thus, for each $i \in \{1, \dots, N\}$, there exist $C_i, c_i > 0$ such that $c_i \|\psi_i u_k\|_{W^{j+m,p}(\overline{U_i}),\boldsymbol{\delta}_i} \leq \|\psi_i u_k\|_{W^{j+m,p}(\overline{U_i}),\mathbf{g}} \leq C_i \|\psi_i u_k\|_{W^{j+m,p}(\overline{U_i}),\boldsymbol{\delta}_i}$, where the constants are independent of the function. Set $v_{i,k} := (\psi_i u_k) \circ \varphi_i^{-1}$. Change of variables gives

$$\|\psi_i u_k\|_{W^{j+m,p}(U_i),\mathbf{g}} \geq c_i \left(\sum_{s=0}^{j+m} \int_{\varphi_i(U_i)} |\nabla^s v_{i,k}|_{\mathbf{g}}^p \, d\lambda_n \right)^{\frac{1}{p}}.$$

For each $s \in \{0, \dots, j+m\}$, Proposition B.1.2 provides $M_s, m_s > 0$ such that $m_s \|\cdot\|_p \leq \|\cdot\|_2 \leq M_s \|\cdot\|_p$ on $\mathbb{R}^{n^s}$, where n^s counts the ordered blocks of s indices. Each multi-index α with $|\alpha| = s$ occurs $s!/\alpha!$ times among

these blocks, with $\alpha! := \alpha_1! \cdots \alpha_n!$. Therefore,

$$\begin{aligned} |\nabla^s v_{i,k}|_{\mathbf{g}}^p &= \left(\sum_{|\alpha|=s} \frac{s!}{\alpha!} |D^\alpha v_{i,k}|^2 \right)^{\frac{p}{2}} \\ &\geq m_s^p \sum_{|\alpha|=s} |D^\alpha v_{i,k}|^p. \end{aligned}$$

Integrating and summing over the orders gives

$$\begin{aligned} \sum_{s=0}^{j+m} \int_{\varphi_i(U_i)} |\nabla^s v_{i,k}|_{\mathbf{g}}^p \, d\lambda_n &\geq \sum_{s=0}^{j+m} m_s^p \sum_{|\alpha|=s} \int_{\varphi_i(U_i)} |D^\alpha v_{i,k}|^p \, d\lambda_n \\ &\geq (m')^p \sum_{|\alpha|\leq j+m} \int_{\varphi_i(U_i)} |D^\alpha v_{i,k}|^p \, d\lambda_n, \end{aligned}$$

where $m' := \min_{0\leq s\leq j+m} m_s > 0$. Taking the pth root and using the definition of the Euclidean Sobolev norm, we conclude that

$$\|\psi_i u_k\|_{W^{j+m,p}(U_i),\mathbf{g}} \geq c_i m' \|v_{i,k}\|_{W^{j+m,p}(\varphi_i(U_i))}.$$

The iterated Leibniz rule expresses $\nabla^s(\psi_i u_k)$ as a sum of terms obtained by distributing the s derivatives between ψ_i and u_k. For each $r \in \{0,\dots,s\}$, there are $\binom{s}{r}$ terms with $s-r$ derivatives on ψ_i and r on u_k. These terms differ by permutations of the covariant indices, which preserve the tensor norm. Proposition 2.3.4 gives

$$|\mathbf{F}\otimes\mathbf{G}|_{\mathbf{g}} = |\mathbf{F}|_{\mathbf{g}}|\mathbf{G}|_{\mathbf{g}} = |\mathbf{G}\otimes\mathbf{F}|_{\mathbf{g}}.$$

Thus, for $0 \leq s \leq j+m$,

$$\begin{aligned} |\nabla^s(\psi_i u_k)|_{\mathbf{g}} &\leq \sum_{r=0}^{s} \binom{s}{r} |\nabla^{s-r}\psi_i|_{\mathbf{g}} |\nabla^r u_k|_{\mathbf{g}} \\ &\leq \sum_{r=0}^{s} \binom{s}{r} \| |\nabla^{s-r}\psi_i|_{\mathbf{g}} \|_{L^\infty(M)} |\nabla^r u_k|_{\mathbf{g}}. \end{aligned}$$

Let

$$A_i := \max_{0\leq s\leq j+m} \| |\nabla^s \psi_i|_{\mathbf{g}} \|_{L^\infty(M)}.$$

Compactness of M implies $A_i < \infty$. Integrating the pointwise estimate and applying Minkowski's inequality gives

$$\|\nabla^s(\psi_i u_k)\|_{L^p(M,T^{(0,s)}(TM))} \leq A_i \sum_{r=0}^{s} \binom{s}{r} \|\nabla^r u_k\|_{L^p(M,T^{(0,r)}(TM))}.$$

Now sum over s and use Proposition 7.1.7:

$$\begin{aligned} \|\psi_i u_k\|_{W^{j+m,p}(M)} &\leq A_i \sum_{s=0}^{j+m} \sum_{r=0}^{s} \binom{s}{r} \|\nabla^r u_k\|_{L^p(M,T^{(0,r)}(TM))} \\ &\leq C_i' \|u_k\|_{W^{j+m,p}(M)}. \end{aligned}$$

The constant C_i' is independent of k. Together with the preceding comparison, this gives

$$\frac{c_i m'}{C_i'}\|v_{i,k}\|_{W^{j+m,p}(\varphi_i(U_i))} \leq \|u_k\|_{W^{j+m,p}(M)}.$$

Thus $(v_{i,k})_k$ is bounded in the corresponding Euclidean space. If U_i is interior, the support of each $v_{i,k}$ is contained in the fixed compact set $\varphi_i(\operatorname{supp}\psi_i) \Subset \varphi_i(U_i)$. Theorem 3.4.2 provides a subsequence converging in $W^{j,q}(\varphi_i(U_i))$.

If U_i is a boundary chart, extend $v_{i,k}$ by zero outside the half-ball, but within $\mathbb{R}^n_+$. Lemma 3.6.8 preserves its Sobolev norm, and the extensions have support contained in the fixed compact set $K_i := \varphi_i(\operatorname{supp}\psi_i) \subseteq \overline{\mathbb{R}^n_+}$. Theorem 3.6.6 gives a subsequence converging in $W^{j,q}(\mathbb{R}^n_+)$ and, upon restriction, a Cauchy subsequence in the Sobolev space on the half-ball.

Since there are finitely many charts, extract successively a subsequence for U_1, then a subsequence of that one for U_2, and so on up to U_N. Relabeling the last subsequence, we may assume without loss of generality that $(\psi_i u_k \circ \varphi_i^{-1})_{k\in\mathbb{N}}$ is Cauchy in $W^{j,q}(\varphi_i(U_i))$ for every $i \in \{1,\dots,N\}$. We shall show that $(\psi_i u_k)_{k\in\mathbb{N}}$ is Cauchy in $W^{j,q}(U_i)$.

Repeating a procedure analogous to the preceding one, if $\widetilde{C_i}, \widetilde{c_i} > 0$ satisfy

$$\widetilde{c_i}\|\psi_i u_k\|_{W^{j,q}(\overline{U_i}),\boldsymbol{\delta}_i} \leq \|\psi_i u_k\|_{W^{j,q}(\overline{U_i}),\mathbf{g}} \leq \widetilde{C_i}\|\psi_i u_k\|_{W^{j,q}(\overline{U_i}),\boldsymbol{\delta}_i}$$

, where the constants are independent of the function, and $\widetilde{M'} = \max_{0\leq s\leq j}\left(\widetilde{M_s}(s!)^{1/q}\right)$, where $\widetilde{M_s}, \widetilde{m_s} > 0$ satisfy $\widetilde{m_s}\|\cdot\|_q \leq \|\cdot\|_2 \leq \widetilde{M_s}\|\cdot\|_q$ on $\mathbb{R}^{n^s}$ with $s \in \{0,\dots,j\}$, use $s!/\alpha! \leq s!$ to compare the sum over blocks with the sum over multi-indices. This includes $j = 0$, with $0! = 1$. Then

$$\|\psi_i u_k - \psi_i u_l\|_{W^{j,q}(U_i),\mathbf{g}} \leq \widetilde{C_i}\widetilde{M'}\|\psi_i u_k \circ \varphi_i^{-1} - \psi_i u_l \circ \varphi_i^{-1}\|_{W^{j,q}(\varphi_i(U_i)),\bar{\mathbf{g}}}.$$

Thus, the sequence $(\psi_i u_k)_{k\in\mathbb{N}}$ is Cauchy in $W^{j,q}(U_i)$. Finally, if $k, l \in \mathbb{N}$, then

$$\|u_k - u_l\|_{W^{j,q}(M),\mathbf{g}} = \left\|\sum_{i=1}^N \psi_i(u_k - u_l)\right\|_{W^{j,q}(M),\mathbf{g}} \leq \sum_{i=1}^N \|\psi_i(u_k - u_l)\|_{W^{j,q}(M),\mathbf{g}}$$

$$= \sum_{i=1}^N \|\psi_i(u_k - u_l)\|_{W^{j,q}(U_i),\mathbf{g}} = \sum_{i=1}^N \|\psi_i u_k - \psi_i u_l\|_{W^{j,q}(U_i),\mathbf{g}}.$$

Hence $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $W^{j,q}(M)$. We have proved that the restricted inclusion $I_0\colon C^{j+m,p}(M) \longrightarrow W^{j,q}(M)$, given by $I_0 u = u$, is compact. Moreover, I_0 is continuous: the compactness just proved maps its unit ball into a relatively compact, and hence bounded, subset of $W^{j,q}(M)$; linearity turns this bound into continuity. Since $C^{j+m,p}(M)$ is dense in $W^{j+m,p}(M)$ by definition and $W^{j,q}(M)$ is a Banach space, Theorem B.2.36 provides a unique compact linear extension $I\colon W^{j+m,p}(M) \longrightarrow W^{j,q}(M)$. This extension agrees with the canonical inclusion. Therefore, the inclusion $W^{j+m,p}(M) \subseteq W^{j,q}(M)$ is compact for all the orders and exponents in the statement. □

Corollary 7.4.2. (Total orders and loss of one derivative) Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension n, with possibly empty smooth boundary. If $m > k \geq 0$ are integers and $1 \leq p, q < \infty$ satisfy $m - \frac{n}{p} > k - \frac{n}{q}$, then $W^{m,p}(M) \hookrightarrow W^{k,q}(M)$ is compact. In particular,

$$W^{m,p}(M) \hookrightarrow W^{m-1,p}(M)$$

is compact for every $m \geq 1$ and every $1 \leq p < \infty$.

Proof. Apply the preceding theorem with $j = k$ and a gain of $m - k \geq 1$ derivatives. For the final embedding, take $k = m - 1$ and $q = p$; the strict inequality reduces to $1 > 0$. Thus this case includes $p = 1$ and dimensions $n = 1, 2$ without introducing a critical exponent. □

Corollary 7.4.3. (Subcritical formulation) Under the geometric assumptions of the preceding corollary, if $j \geq 0$, $a \geq 1$, and $ap < n$, the inclusion $W^{j+a,p}(M) \hookrightarrow W^{j,q}(M)$ is compact for $1 \leq q < np/(n - ap)$. At the endpoint $q = np/(n - ap)$, Corollary 7.3.3 gives continuity.

Proof. With $ap < n$, the inequality $j + a - \frac{n}{p} > j - \frac{n}{q}$ is equivalent to $q < np/(n - ap)$, so the general theorem applies. Equality of indices corresponds to the critical exponent in the cited continuous embedding theorem. □

8

Partial differential operators on vector bundles

Differential operators on manifolds naturally act between sections of vector bundles. The gradient, exterior derivative, connections, Hodge operators, and geometric Laplacians are examples of a common structure: their value at a point depends only on finitely many local derivatives of the section.

The intrinsic formulation will allow us to study the order of a differential operator, composition, and commutators with smooth functions without favoring coordinates. The formal adjoint will provide the integration by parts identity needed to define weak covariant derivatives and bundle-valued distributions later. The principal symbol and ellipticity will appear once these two theories have been constructed, so that their consequences can be expressed directly in terms of Sobolev sections and geometric distributions.

From local operators to the language of bundles

For a long time, differential operators were presented through coordinate formulas. Differential geometry showed that the essential feature was not the particular list of coefficients, but the local dependence on finitely many derivatives and the way those expressions change between trivializations. The language of bundles brings operators as seemingly different as the exterior derivative, connections, and Dirac operators together in one theory. Later, the principal symbol will retain precisely the highest-order part of this local information.

8.1 Motivation and basic properties

Before turning to the intrinsic definition, it is helpful to recall the Euclidean case. There, operators are written in terms of partial derivatives of the components; under a change of coordinates, those expressions transform consistently and determine a global map between spaces of sections. The abstract definition will isolate precisely this local dependence on finitely many derivatives.

Consider the Laplacian

$$\Delta\colon C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n), \qquad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

This operator can be interpreted as

$$\Delta\colon \Gamma(\underline{\mathbb{R}}) \longrightarrow \Gamma(\underline{\mathbb{R}}),$$

where $\underline{\mathbb{R}}$ denotes the trivial bundle of rank 1 over $\mathbb{R}^n$, that is,

$$\pi\colon \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n, \qquad \pi(x,v) = x.$$

Another example is the exterior derivative

$$d\colon \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n),$$

where a k-form $\omega \in \Omega^k(\mathbb{R}^n)$ has, in the standard coordinates, a smooth representative

$$\widetilde{\omega}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^{\binom{n}{k}}.$$

Thus, d can be written as an operator

$$d\colon C^\infty\left(\mathbb{R}^n, \mathbb{R}^{\binom{n}{k}}\right) \longrightarrow C^\infty\left(\mathbb{R}^n, \mathbb{R}^{\binom{n}{k+1}}\right).$$

Remark 8.1.1. In the case of the trivial bundle of rank m over $\mathbb{R}^n$, denoted by $\underline{\mathbb{R}}^m$,

$$\pi\colon \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n,$$

every smooth section σ has the form $\sigma(x) = (x, f(x))$ for a unique function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$. This gives a natural identification

$$\Gamma(\underline{\mathbb{R}}^m) \cong C^\infty(\mathbb{R}^n, \mathbb{R}^m).$$

More generally, if M is a smooth manifold with or without boundary and we consider the trivial bundle of rank m, again denoted by $\underline{\mathbb{R}}^m$,

$$\pi\colon M \times \mathbb{R}^m \longrightarrow M,$$

the map sending a section $\sigma(p) = (p, f(p))$ to its second component f is linear and bijective, with inverse $f \mapsto [p \mapsto (p, f(p))]$. Therefore,

$$\Gamma(\underline{\mathbb{R}}^m) \cong C^\infty(M, \mathbb{R}^m).$$

In this sense, the exterior derivative may be viewed as

$$d\colon \Gamma\left(\underline{\mathbb{R}}^{\binom{n}{k}}\right) \longrightarrow \Gamma\left(\underline{\mathbb{R}}^{\binom{n}{k+1}}\right).$$

These examples show that many classical differential operators can naturally be described as operators between spaces of sections of vector bundles. This setting can be generalized as follows:

Definition 8.1.2. Let M be a smooth manifold with or without boundary, and let $\pi_1\colon \mathbf{E} \longrightarrow M$, $\pi_2\colon \mathbf{F} \longrightarrow M$ be smooth vector bundles (real or complex) over M. We write

$$\mathbf{Op}(\mathbf{E}, \mathbf{F}) := \{P\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{F}) \mid P \text{ is } \mathbb{K}\text{-linear}\},$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

The spaces of smooth sections $\Gamma(\mathbf{E})$ and $\Gamma(\mathbf{F})$ are not only $\mathbb{K}$-vector spaces, but also $C^\infty(M)$-modules. Partial differential operators are elements of $\mathbf{Op}$ that interact in a special way with the $C^\infty(M)$-module structure of the spaces of sections.

An element $P \in \mathbf{Op}(\mathbf{E},\mathbf{F})$ is $C^\infty(M)$-linear if $P(f\mathbf{u}) = f(P\mathbf{u})$ for any $\mathbf{u} \in \Gamma(\mathbf{E})$. We define the commutator of $P \in \mathbf{Op}(\mathbf{E},\mathbf{F})$ and $f \in C^\infty(M)$ to be the element $[P,f] \in \mathbf{Op}(\mathbf{E},\mathbf{F})$ such that $[P,f](\mathbf{u}) = P(f\mathbf{u}) - f(P\mathbf{u})$ for every $\mathbf{u} \in \Gamma(\mathbf{E})$.

The operator P is $C^\infty(M)$–linear if and only if $[P,f] = 0$ for any $f \in C^\infty(M)$.

Each function $f \in C^\infty(M)$ defines a map

$$\mathrm{ad}(f)\colon \mathbf{Op}(\mathbf{E},\mathbf{F}) \longrightarrow \mathbf{Op}(\mathbf{E},\mathbf{F}), \qquad \mathrm{ad}(f)(P) = [P,f].$$

We define a partial differential operator recursively as follows:

Definition 8.1.3. Let M be a smooth manifold with or without boundary and let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth vector bundles over M.

(a) $\mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F}) := \{P \in \mathbf{Op}(\mathbf{E},\mathbf{F}) \mid [P,f] = 0, \forall f \in C^\infty(M)\}$.

(b) If $m > 0$,
$$\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}) := \{P \in \mathbf{Op}(\mathbf{E},\mathbf{F}) \mid [P,f] \in \mathbf{PDO}^{(m-1)}(\mathbf{E},\mathbf{F}), \forall f \in C^\infty(M)\}.$$

(c) $\mathbf{PDO}(\mathbf{E},\mathbf{F}) := \bigcup_{m=0}^{\infty} \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$.

In general, if $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, we say that P is a partial differential operator of order less than or equal to m.

In terms of the notation ad, this definition can be rewritten more explicitly:

Proposition 8.1.4. Let M be a smooth manifold with or without boundary and let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth vector bundles over M.

(a) $\mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F}) = \bigcap_{f \in C^\infty(M)} \ker(\mathrm{ad}(f))$.

(b) More generally, for $m \geq 1$,
$$\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}) = \bigcap_{f_0,\ldots,f_m \in C^\infty(M)} \ker\left(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)\right).$$

In particular, $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ if and only if

$$\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P) = 0, \qquad \forall f_0,\ldots,f_m \in C^\infty(M).$$

Proof. For the base case $m = 0$ we use the definition; note that $P \in \mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F})$ if and only if $[P,f] = 0$ for

every $f \in C^\infty(M)$, that is, $\mathrm{ad}(f)(P) = 0$ for every $f \in C^\infty(M)$, which is equivalent to

$$P \in \bigcap_{f \in C^\infty(M)} \ker(\mathrm{ad}(f)).$$

Now suppose that the characterization holds up to $m-1$, that is,

$$\mathbf{PDO}^{(m-1)}(\mathbf{E},\mathbf{F}) = \bigcap_{f_0,\dots,f_{m-1} \in C^\infty(M)} \ker\big(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_{m-1})\big).$$

We wish to show that the same description holds for m.
By definition,

$$P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}) \quad \Longleftrightarrow \quad [P, f_m] \in \mathbf{PDO}^{(m-1)}(\mathbf{E},\mathbf{F}) \text{ for every } f_m \in C^\infty(M).$$

Applying the induction hypothesis, this condition is equivalent to

$$\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_{m-1})([P, f_m]) = 0, \qquad \forall f_0,\dots,f_{m-1},f_m \in C^\infty(M).$$

But this holds if and only if

$$\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_{m-1})\,\mathrm{ad}(f_m)(P) = 0, \qquad \forall f_0,\dots,f_m \in C^\infty(M),$$

that is, if and only if

$$P \in \bigcap_{f_0,\dots,f_m \in C^\infty(M)} \ker\big(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)\big).$$

Therefore, the equivalence holds for m, completing the induction. □

Remark 8.1.5. In the literature, one commonly encounters the abbreviation

$$\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}) = \ker \mathrm{ad}^{m+1},$$

which is to be understood in the preceding sense: not as a power of a single operator, but as the intersection of the kernels of *all* possible compositions of $m+1$ operators of the form $\mathrm{ad}(f)$ with $f \in C^\infty(M)$.

Proposition 8.1.6. Let M be a smooth manifold with or without boundary and let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth vector bundles. Let $P, Q \in \mathbf{Op}(\mathbf{E},\mathbf{F})$, $f, g, h \in C^\infty(M)$, and $\alpha, \beta \in \mathbb{K}$. The commutator

$$[P, f](\mathbf{u}) := P(f\mathbf{u}) - fP(\mathbf{u}), \qquad \mathbf{u} \in \Gamma(\mathbf{E}),$$

satisfies the following properties:

(a) $\mathbb{K}$-linearity in the operator:

$$[\alpha P + \beta Q, f] = \alpha\,[P, f] + \beta\,[Q, f].$$

(b) $C^\infty(M)$-linearity in the operator:

$$[hP, f] = h\,[P, f].$$

(c) $\mathbb{K}$-linearity in the function:

$$[P, \alpha f + \beta g] = \alpha\,[P, f] + \beta\,[P, g].$$

(d) Leibniz rule: for every section $\mathbf{u} \in \Gamma(\mathbf{E})$,

$$[P, fg](\mathbf{u}) = [P, f](g\mathbf{u}) + f\,[P, g](\mathbf{u}).$$

If $P \in \mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F})$, this identity reduces to

$$[P, fg](\mathbf{u}) = g\,[P, f](\mathbf{u}) + f\,[P, g](\mathbf{u}).$$

(e) If $c \in \mathbb{K}$ is viewed as a constant function, then

$$[P, c] = 0.$$

(f) The operator P is $C^\infty(M)$-linear if and only if

$$[P, f] = 0 \qquad \forall f \in C^\infty(M).$$

Proof. Let $\mathbf{u} \in \Gamma(\mathbf{E})$.
(a) We have

$$[\alpha P + \beta Q, f](\mathbf{u}) = (\alpha P + \beta Q)(f\mathbf{u}) - f(\alpha P + \beta Q)(\mathbf{u}) = \alpha(P(f\mathbf{u}) - fP(\mathbf{u})) + \beta(Q(f\mathbf{u}) - fQ(\mathbf{u})),$$

which is $\alpha[P, f](\mathbf{u}) + \beta[Q, f](\mathbf{u})$.
(b) By definition,

$$[hP, f](\mathbf{u}) = (hP)(f\mathbf{u}) - f(hP)(\mathbf{u}) = hP(f\mathbf{u}) - fhP(\mathbf{u}) = h\big(P(f\mathbf{u}) - fP(\mathbf{u})\big) = h\,[P, f](\mathbf{u}).$$

(c) For $\alpha f + \beta g$,

$$[P, \alpha f + \beta g](\mathbf{u}) = P((\alpha f + \beta g)\mathbf{u}) - (\alpha f + \beta g)P(\mathbf{u}) = \alpha(P(f\mathbf{u}) - fP(\mathbf{u})) + \beta(P(g\mathbf{u}) - gP(\mathbf{u})).$$

(d) For fg,

$$[P, fg](\mathbf{u}) = P(fg\,\mathbf{u}) - fgP(\mathbf{u}).$$

We add and subtract $fP(g\mathbf{u})$ to obtain

$$[P, fg](\mathbf{u}) = \big(P(fg\mathbf{u}) - fP(g\mathbf{u})\big) + f\big(P(g\mathbf{u}) - gP(\mathbf{u})\big) = [P, f](g\mathbf{u}) + f\,[P, g](\mathbf{u}).$$

If $P \in \mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F})$, then $[P, f] \in \mathbf{PDO}^{(0)}(\mathbf{E}, \mathbf{F})$ by definition. In particular, $[[P, f], g] = 0$, that is,

$$[P, f](g\mathbf{u}) = g[P, f](\mathbf{u}).$$

Substituting this equality gives the formula for the first-order case.

(e) If c is constant, then

$$[P,c](\mathbf{u}) = P(c\mathbf{u}) - cP(\mathbf{u}) = cP(\mathbf{u}) - cP(\mathbf{u}) = 0.$$

(f) If P is $C^\infty(M)$-linear, then $P(f\mathbf{u}) = fP(\mathbf{u})$, and hence $[P,f] = 0$. Conversely, if $[P,f] = 0$ for every f, then

$$P(f\mathbf{u}) - fP(\mathbf{u}) = 0,$$

for every section $\mathbf{u}$, implying that P is $C^\infty(M)$-linear. □

Proposition 8.1.7. Let M be a smooth manifold with or without boundary and let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth vector bundles. For each $f \in C^\infty(M)$ define

$$\operatorname{ad}(f)\colon \mathbf{Op}(\mathbf{E},\mathbf{F}) \longrightarrow \mathbf{Op}(\mathbf{E},\mathbf{F}), \qquad \operatorname{ad}(f)(P) = [P,f].$$

Then $\operatorname{ad}(f)$ satisfies the following properties:

(a) $\mathbb{K}$-linearity:

$$\operatorname{ad}(f)(\alpha P + \beta Q) = \alpha \operatorname{ad}(f)(P) + \beta \operatorname{ad}(f)(Q).$$

(b) $C^\infty(M)$-linearity in the operator:

$$\operatorname{ad}(f)(hP + Q) = h \operatorname{ad}(f)(P) + \operatorname{ad}(f)(Q).$$

(c) $\mathbb{K}$-linearity in the function:

$$\operatorname{ad}(\alpha f + \beta g) = \alpha \operatorname{ad}(f) + \beta \operatorname{ad}(g).$$

(d) If c is constant, then

$$\operatorname{ad}(c) = 0.$$

Proof. (a) This follows from property (a) of Proposition 8.1.6:

$$\operatorname{ad}(f)(\alpha P + \beta Q) = [\alpha P + \beta Q, f] = \alpha[P,f] + \beta[Q,f].$$

(b) Using property (b) of Proposition 8.1.6,

$$\operatorname{ad}(f)(hP + Q) = [hP + Q, f] = [hP, f] + [Q,f] = h[P,f] + \operatorname{ad}(f)(Q).$$

(c) Using property (c) of Proposition 8.1.6,

$$\operatorname{ad}(\alpha f + \beta g)(P) = [P, \alpha f + \beta g] = \alpha[P,f] + \beta[P,g].$$

(d) If c is constant, property (e) of Proposition 8.1.6 implies

$$\operatorname{ad}(c)(P) = [P,c] = 0,$$

for every $P \in \mathbf{Op}(\mathbf{E},\mathbf{F})$. □

Remark 8.1.8. The preceding propositions imply that $\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ has the structure of an $C^\infty(M)$-module.

We give some simple examples of partial differential operators in the Euclidean case to familiarize the reader with this concept.

Example 8.1.9. Consider $M=\mathbb{R}^n$ and the trivial bundles $\mathbf{E}=\mathbf{F}=\underline{\mathbb{R}}$ of rank 1.

(a) The operator of multiplication by a smooth function is the most elementary example of a partial differential operator. Let M_f be defined by

$$(M_f u)(x) = f(x)u(x), \qquad f \in C^\infty(\mathbb{R}^n).$$

For any $g,u \in C^\infty(\mathbb{R}^n)$ we have

$$[M_f, g](u) = M_f(gu) - g(M_f u) = f\cdot(gu) - g\cdot(fu) = 0.$$

Consequently, $[M_f,g]=0$ for every $g\in C^\infty(\mathbb{R}^n)$, and therefore

$$M_f \in \mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F}).$$

(b) Now consider the partial derivative operator $P=\partial_i=\dfrac{\partial}{\partial x_i}$. For $u,f\in C^\infty(\mathbb{R}^n)$,

$$[\partial_i, f](u) = \partial_i(fu) - f\,\partial_i u.$$

Applying the product rule,

$$\partial_i(fu) = (\partial_i f)u + f\,\partial_i u,$$

gives

$$[\partial_i, f](u) = (\partial_i f)u.$$

That is, the commutator with f is simply multiplication by the function $\partial_i f$, an operator of order 0. Thus, $[\partial_i,f]\in\mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F})$, and by definition

$$\partial_i \in \mathbf{PDO}^{(1)}(\mathbf{E},\mathbf{F}).$$

(c) Finally, we analyze second-order partial derivatives. Let $P=\partial_i^2$. For $u\in C^\infty(\mathbb{R}^n)$,

$$[\partial_i^2, f](u) = \partial_i^2(fu) - f\,\partial_i^2 u.$$

Applying the product rule twice:

$$\partial_i^2(fu) = \partial_i\big((\partial_i f)u + f\,\partial_i u\big) = (\partial_i^2 f)u + 2(\partial_i f)\partial_i u + f\,\partial_i^2 u.$$

Subtracting $f\,\partial_i^2 u$ gives

$$[\partial_i^2, f](u) = (\partial_i^2 f)u + 2(\partial_i f)\partial_i u.$$

Taking a second commutator with another function $g \in C^\infty(\mathbb{R}^n)$,

$$[[\partial_i^2, f], g](u) = [M_{\partial_i^2 f} + 2(\partial_i f)\partial_i, g](u).$$

The first term $(\partial_i^2 f)\mathrm{Id}$ commutes with g, since it is multiplication by a smooth function. For the second,

$$[2(\partial_i f)\partial_i, g](u) = 2(\partial_i f)\big(\partial_i(gu) - g\partial_i u\big) = 2(\partial_i f)(\partial_i g)u,$$

which is multiplication by a smooth function, that is, an operator of order 0. Thus,

$[[\partial_i^2, f], g] \in \mathbf{PDO}^{(0)}(\mathbf{E}, \mathbf{F})$, y $\partial_i^2 \in \mathbf{PDO}^{(2)}(\mathbf{E}, \mathbf{F})$.

The example of multiplication by a smooth function generalizes as follows:

Example 8.1.10. Let M be a smooth manifold with or without boundary and let $\mathbf{E} \to M$ be a smooth vector bundle (real or complex). Fixing a function $f \in C^\infty(M)$, consider the operator

$$M_f : \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E}), \qquad M_f(\mathbf{u}) := f\mathbf{u}.$$

We wish to show that $M_f \in \mathbf{PDO}^{(0)}(\mathbf{E}, \mathbf{E})$. For every function $g \in C^\infty(M)$ and every section $\mathbf{u} \in \Gamma(\mathbf{E})$ we have

$$[M_f, g](\mathbf{u}) = M_f(g\mathbf{u}) - gM_f(\mathbf{u}) = f(g\mathbf{u}) - g(f\mathbf{u}) = (fg)\mathbf{u} - (gf)\mathbf{u} = (fg)\mathbf{u} - (fg)\mathbf{u} = 0.$$

Therefore, $[M_f, g] = 0$ for every $g \in C^\infty(M)$, meaning that

$$M_f \in \mathbf{PDO}^{(0)}(\mathbf{E}, \mathbf{E}).$$

Another example of a partial differential operator is the Laplace–Beltrami operator on a Riemannian manifold:

Example 8.1.11. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and consider the trivial bundle $\mathbf{E} = \mathbf{F} = \underline{\mathbb{R}}$. The Laplace–Beltrami operator is defined by

$$\Delta_{\mathbf{g}} u := \operatorname{div}(\operatorname{grad} u), \qquad u \in C^\infty(M).$$

We will show that $\Delta_{\mathbf{g}} \in \mathbf{PDO}^{(2)}(\mathbf{E}, \mathbf{F})$ but that $\Delta_{\mathbf{g}} \notin \mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F})$.
For every function $f \in C^\infty(M)$, Proposition 2.16.11 gives

$$\Delta_{\mathbf{g}}(fu) = f\Delta_{\mathbf{g}} u + 2\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + u\Delta_{\mathbf{g}} f,$$

and therefore

$$[\Delta_{\mathbf{g}}, f](u) = \Delta_{\mathbf{g}}(fu) - f\Delta_{\mathbf{g}} u = 2\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)u.$$

Now let $h \in C^\infty(M)$. By definition of the commutator, for every $u \in C^\infty(M)$ we have

$$[[\Delta_{\mathbf{g}}, f], h](u) = [\Delta_{\mathbf{g}}, f](hu) - h[\Delta_{\mathbf{g}}, f](u).$$

Using the identity

$$[\Delta_{\mathbf{g}}, f](v) = 2\langle \operatorname{grad} f, \operatorname{grad} v\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)\, v, \qquad v \in C^\infty(M),$$

we obtain

$$[\Delta_{\mathbf{g}}, f](hu) = 2\langle \operatorname{grad} f, \operatorname{grad}(hu)\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)hu$$

and

$$[\Delta_{\mathbf{g}}, f](u) = 2\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)\, u.$$

Therefore,

$$[[\Delta_{\mathbf{g}}, f], h](u) = \Big(2\langle \operatorname{grad} f, \operatorname{grad}(hu)\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)hu\Big) - h\Big(2\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)u\Big)$$

$$= 2\langle \operatorname{grad} f, \operatorname{grad}(hu)\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)hu - 2h\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} - h(\Delta_{\mathbf{g}} f)\, u.$$

We now use the Leibniz rule for the gradient (Proposition 2.16.3):

$$\operatorname{grad}(hu) = h \operatorname{grad} u + u \operatorname{grad} h,$$

so that

$$\langle \operatorname{grad} f, \operatorname{grad}(hu)\rangle_{\mathbf{g}} = \big\langle \operatorname{grad} f,\, h \operatorname{grad} u + u \operatorname{grad} h\big\rangle_{\mathbf{g}}$$

$$= h\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + u\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}},$$

where we have used the $C^\infty(M)$–bilinearity of the inner product induced by $\mathbf{g}$. Consequently,

$$2\langle \operatorname{grad} f, \operatorname{grad}(hu)\rangle_{\mathbf{g}} = 2h\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + 2u\,\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}}.$$

Substituting this expression into the formula for $[[\Delta_{\mathbf{g}}, f], h](u)$, we have

$$[[\Delta_{\mathbf{g}}, f], h](u) = \Big(2h\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + 2u\,\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}} + (\Delta_{\mathbf{g}} f)hu\Big)$$

$$-\Big(2h\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} + h(\Delta_{\mathbf{g}} f)\, u\Big).$$

We now simplify term by term. First,

$$2h\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} - 2h\langle \operatorname{grad} f, \operatorname{grad} u\rangle_{\mathbf{g}} = 0.$$

Then, since multiplication of functions is commutative,

$$(\Delta_{\mathbf{g}} f)hu - h(\Delta_{\mathbf{g}} f)\, u = 0.$$

Therefore, only

$$[[\Delta_{\mathbf{g}}, f], h](u) = 2u\,\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}}.$$

remains.

In other words, for every $u \in C^\infty(M)$ we have

$$[[\Delta_{\mathbf{g}}, f], h](u) = 2\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}}\, u,$$

that is,

$$[[\Delta_{\mathbf{g}}, f], h] = M_{2\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}}}.$$

This result is multiplication by the smooth function $2\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}}$, so

$$[[\Delta_{\mathbf{g}}, f], h] \in \mathbf{PDO}^{(0)}(\mathbf{E}, \mathbf{F}) \quad \text{for any } f, h \in C^\infty(M).$$

Thus,

$$[\Delta_{\mathbf{g}}, f] \in \mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F}) \quad \text{for every } f \in C^\infty(M),$$

and consequently

$$\Delta_{\mathbf{g}} \in \mathbf{PDO}^{(2)}(\mathbf{E}, \mathbf{F}).$$

To show that $\Delta_{\mathbf{g}} \notin \mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F})$, it suffices to note that if an operator belongs to $\mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F})$, then its second commutators with smooth functions would vanish. However, we have obtained

$$[[\Delta_{\mathbf{g}}, f], h] = M_{2\langle \operatorname{grad} f, \operatorname{grad} h\rangle_{\mathbf{g}}},$$

which is not the zero operator in general. Consequently,

$$\Delta_{\mathbf{g}} \notin \mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F}).$$

The preceding example shows that calling these partial differential operators of order less than or equal to m makes sense, since having one of order less than or equal to m does not necessarily imply that it has order less than or equal to $m-1$. Moreover, if it has order less than or equal to m, one can prove that it has order less than or equal to $m+1$, as will be done later in Lemma 8.1.18.

The covariant derivative defined on a smooth vector bundle is another example of a partial differential operator, in this case of order ≤ 1:

Proposition 8.1.12. Let M be a smooth manifold with or without boundary and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle over M. If

$$\nabla \colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(T^{(0,1)}(TM) \otimes \mathbf{E})$$

is a connection on $\mathbf{E}$, then $\nabla \in \mathbf{PDO}^{(1)}(\mathbf{E}, T^{(0,1)}(TM) \otimes \mathbf{E})$.

Proof. Let $f \in C^\infty(M)$ and $\mathbf{u} \in \Gamma(\mathbf{E})$. Then by Proposition 2.4.25,

$$[\nabla, f](\mathbf{u}) = \nabla(f\mathbf{u}) - f\nabla\mathbf{u} = df \otimes \mathbf{u} + f\nabla\mathbf{u} - f\nabla\mathbf{u} = df \otimes \mathbf{u}.$$

Define $M^{\otimes}_{df} \in \mathbf{Op}(\mathbf{E}, T^{(0,1)}(TM) \otimes \mathbf{E})$ by $M^{\otimes}_{df}(\mathbf{u}) = df \otimes \mathbf{u}$. Let $g \in C^\infty(M)$; then

$$[M^{\otimes}_{df}, g](\mathbf{u}) = M^{\otimes}_{df}(g\mathbf{u}) - gM^{\otimes}_{df}(\mathbf{u}) = df \otimes (g\mathbf{u}) - g(df \otimes \mathbf{u}) = g(df \otimes \mathbf{u}) - g(df \otimes \mathbf{u}) = 0,$$

, so $M^{\otimes}_{df} \in \mathbf{PDO}^{(0)}(\mathbf{E}, T^{(0,1)}(TM) \otimes \mathbf{E})$ and consequently $\nabla \in \mathbf{PDO}^{(1)}(\mathbf{E}, T^{(0,1)}(TM) \otimes \mathbf{E})$. □

To become more familiar with this concept, we study what partial differential operators of order 1 look like on $\underline{\mathbb{R}}$ viewed as a vector bundle over $\mathbb{R}^n$.

Example 8.1.13. Consider again the trivial bundle $\underline{\mathbb{R}}$ over $\mathbb{R}^n$. As we saw earlier, the sections of $\underline{\mathbb{R}}$ are precisely the functions in $C^\infty(\mathbb{R}^n)$. We wish to find $\mathbf{PDO}^{(1)}(\underline{\mathbb{R}},\underline{\mathbb{R}})$. Let $P \in \mathbf{PDO}^{(1)}(\underline{\mathbb{R}},\underline{\mathbb{R}})$ and $u, f \in C^\infty(\mathbb{R}^n)$. Then

$$[P,f]u = \sigma(f)u \quad \text{for some } \sigma(f) \in C^\infty(\mathbb{R}^n),$$

, since $[P,f]$ is $C^\infty(M)$-linear because $P \in \mathbf{PDO}^{(1)}(\underline{\mathbb{R}},\underline{\mathbb{R}})$, and we may take $\sigma(f) = [P,f](1)$. On the other hand, for any $f, g \in C^\infty(\mathbb{R}^n)$,

$$\sigma(fg)u = [P,fg]u = [P,f](gu) + f([P,g]u) = \sigma(f)gu + f\sigma(g)u.$$

Therefore, $\sigma(fg) = \sigma(f)g + f\sigma(g)$, so the map σ is a derivation on $C^\infty(M)$; hence there exists a smooth vector field $\mathbf{X} \in \mathfrak{X}(\mathbb{R}^n)$ such that

$$\sigma(f) = \mathbf{X} \cdot f, \quad \forall f \in C^\infty(\mathbb{R}^n).$$

Then for every $u \in C^\infty(\mathbb{R}^n)$,

$$P(u) = P(u \cdot 1) = [P,u] \cdot 1 + u \cdot P(1) = \mathbf{X} \cdot u + P(1) \cdot u,$$

where $P(1) \in C^\infty(\mathbb{R}^n)$.

Operators of order 0 can be characterized as follows:

Lemma 8.1.14. Let M be a smooth manifold with or without boundary and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles. If $P \in \mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F})$, then there exists a section $\mathbf{A} \in \Gamma(\mathrm{Hom}(\mathbf{E},\mathbf{F}))$ such that

$$P(\mathbf{u})(p) = \mathbf{A}(p)(\mathbf{u}(p)), \qquad \forall \mathbf{u} \in \Gamma(\mathbf{E}), p \in M.$$

Proof. By definition of a partial differential operator of order 0, the map $P\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{F})$ is $C^\infty(M)$-linear. Proposition 1.7.31 identifies maps with this property with sections of $\mathrm{Hom}(\mathbf{E},\mathbf{F})$: there is a unique section $\mathbf{A}$ such that $P(\mathbf{u}) = \mathbf{A}(\mathbf{u})$. Evaluating this equality at $p \in M$ gives $P(\mathbf{u})(p) = \mathbf{A}(p)(\mathbf{u}(p))$. □

As a consequence of this lemma, partial differential operators are local.

Proposition 8.1.15. Let M be a smooth manifold with or without boundary and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles. Every $L \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ is a local operator, that is, $\forall \mathbf{u} \in \Gamma(\mathbf{E})$,

$$\mathrm{supp}(L\mathbf{u}) \subseteq \mathrm{supp}(\mathbf{u}).$$

Proof. The case $m = 0$ follows from Lemma 8.1.14, since if $\mathbf{u}(p) = 0$ then $L(\mathbf{u})(p) = \mathbf{A}(p)(\mathbf{u}(p)) = \mathbf{A}(p)(0) = 0$, so

$$\mathrm{supp}(L\mathbf{u}) \subseteq \mathrm{supp}(\mathbf{u}).$$

Suppose the assertion holds for $m \geq 0$ and let $L \in \mathbf{PDO}^{(m+1)}(\mathbf{E},\mathbf{F})$.
For any $f \in C^\infty(M)$ we have

$$L(f\mathbf{u}) = [L,f]\mathbf{u} + fL(\mathbf{u}).$$

Since $[L,f] \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, the induction hypothesis gives

$$\begin{aligned} \operatorname{supp} L(f\mathbf{u}) &\subseteq \operatorname{supp}([L,f]\mathbf{u}) \cup \operatorname{supp}(fL(\mathbf{u})) \subseteq \operatorname{supp}(\mathbf{u}) \cup (\operatorname{supp}(f) \cap \operatorname{supp}(L\mathbf{u})) \\ &\subseteq \operatorname{supp}(\mathbf{u}) \cup \operatorname{supp}(f), \quad \forall f \in C^\infty(M). \end{aligned} \tag{8.1}$$

Under these hypotheses, we show that $\operatorname{supp}(L\mathbf{u}) \subseteq \operatorname{supp}(\mathbf{u})$. We argue by contradiction and suppose that there exists $p \in M$ such that $p \in \operatorname{supp}(L\mathbf{u})$ but $p \notin \operatorname{supp}(\mathbf{u})$. Since $\operatorname{supp}(\mathbf{u})$ is closed and M is a regular topological space (in fact, it is metrizable), there is an open set $U \subseteq M$ such that $p \in U$ and $\overline{U} \cap \operatorname{supp}(\mathbf{u}) = \varnothing$, that is, $\operatorname{supp}(\mathbf{u}) \subseteq M \setminus \overline{U}$, where $M \setminus \overline{U}$ is open. By Proposition 1.3.6, there exists $f \in C^\infty(M)$ such that $f \equiv 0$ on $\overline{U}$ and $f \equiv 1$ on $\operatorname{supp}(\mathbf{u})$. The identity $f\mathbf{u} = \mathbf{u}$ holds throughout M, since $f \equiv 1$ on $\operatorname{supp}(\mathbf{u})$, and if a point x lies outside $\operatorname{supp}(\mathbf{u})$, then $\mathbf{u}(x) = 0 = f(x)\mathbf{u}(x)$. Therefore,

$$\operatorname{supp}(L\mathbf{u}) = \operatorname{supp}(L(f\mathbf{u})),$$

, allowing us to conclude that $p \in \operatorname{supp}(L(f\mathbf{u}))$ since we had taken $p \in \operatorname{supp}(L\mathbf{u})$. But this is impossible, because (8.1) would imply $p \in \operatorname{supp}(\mathbf{u}) \cup \operatorname{supp}(f)$, whereas $p \notin \operatorname{supp}(\mathbf{u})$ and $p \notin \operatorname{supp}(f)$ (the latter holds because $f \equiv 0$ on $U \subseteq \overline{U}$, so $U \cap \{x \in M \mid f(x) \neq 0\} = \varnothing$, and thus $p \notin \overline{\{x \in M \mid f(x) \neq 0\}} = \operatorname{supp}(f)$). The contradiction arose from assuming that $p \notin \operatorname{supp}(\mathbf{u})$, so $\operatorname{supp}(L\mathbf{u}) \subseteq \operatorname{supp}(\mathbf{u})$, completing the proof. □

Remark 8.1.16. If $L \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ and $\mathbf{u},\mathbf{v} \in \Gamma(\mathbf{E})$ are such that there is an open set $U \subseteq M$ with $\mathbf{u} \equiv \mathbf{v}$ on U, and we define $\mathbf{w} := \mathbf{u} - \mathbf{v}$, this shows that $U \cap \operatorname{supp}(\mathbf{w}) = \varnothing$. Proposition 8.1.15 therefore implies that $\operatorname{supp}(L\mathbf{w}) \subseteq \operatorname{supp}(\mathbf{w})$, and consequently $U \cap \operatorname{supp}(L\mathbf{w}) = \varnothing$, from which we conclude that $L\mathbf{u} - L\mathbf{v} = L\mathbf{w} = 0$ on U; that is, $L\mathbf{u}$ and $L\mathbf{v}$ agree on U.
This justifies the word "local" in that proposition, since it tells us that the values of a partial differential operator are completely determined by the values of the section to which it is applied on an open subset of M.

We show that the composition of partial differential operators is again a partial differential operator. For this, we need the following lemmas:

Lemma 8.1.17. Let M be a smooth manifold with or without boundary. Let $\mathbf{E},\mathbf{F},\mathbf{G}$ be smooth vector bundles over M, let $f \in C^\infty(M)$, and let $P\colon \Gamma(\mathbf{F}) \longrightarrow \Gamma(\mathbf{G})$, $Q\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{F})$ be linear operators. Then, for every $\mathbf{u} \in \Gamma(\mathbf{E})$,

$$[P \circ Q, f](\mathbf{u}) = [P,f]\big(Q(\mathbf{u})\big) + P\big([Q,f](\mathbf{u})\big).$$

Proof. By definition,

$$[P \circ Q, f](\mathbf{u}) = (P \circ Q)(f\mathbf{u}) - f(P \circ Q)(\mathbf{u}) = P\big(Q(f\mathbf{u})\big) - fP\big(Q(\mathbf{u})\big).$$

Add and subtract $P\big(fQ(\mathbf{u})\big)$:

$$\begin{aligned} [P \circ Q, f](\mathbf{u}) &= \Big(P\big(Q(f\mathbf{u})\big) - P\big(fQ(\mathbf{u})\big)\Big) + \Big(P\big(fQ(\mathbf{u})\big) - fP\big(Q(\mathbf{u})\big)\Big) \\ &= P\big(Q(f\mathbf{u}) - fQ(\mathbf{u})\big) + \big(P(fQ(\mathbf{u})) - fP(Q(\mathbf{u}))\big) \\ &= P\big([Q,f](\mathbf{u})\big) + [P,f]\big(Q(\mathbf{u})\big). \end{aligned}$$

□

Also note that if $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, this should be interpreted as saying that P is a partial differential operator of order less than or equal to m. This tells us that if a partial differential operator has order less than or equal to m, then it has order less than or equal to $m+1$, justifying the initial choice of terminology.

Lemma 8.1.18. Let M be a smooth manifold with or without boundary, let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth vector bundles, and let $m \geq 0$. Then

$$\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}) \subseteq \mathbf{PDO}^{(m+1)}(\mathbf{E},\mathbf{F}).$$

Proof. We prove the assertion by induction on m. If $m = 0$ and $P \in \mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F})$, then for every $f \in C^\infty(M)$ $[P,f] = 0$, so by the recursive definition of partial differential operators we have $[P,f] = 0 \in \mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F})$. Suppose as the induction hypothesis that $\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}) \subseteq \mathbf{PDO}^{(m+1)}(\mathbf{E},\mathbf{F})$. If $P \in \mathbf{PDO}^{(m+1)}(\mathbf{E},\mathbf{F})$, then for every $f \in C^\infty(M)$, $[P,f] \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}) \subseteq \mathbf{PDO}^{(m+1)}(\mathbf{E},\mathbf{F})$, so $[P,f] \in \mathbf{PDO}^{(m+1)}(\mathbf{E},\mathbf{F})$ $\forall f \in C^\infty(M)$. This allows us to conclude that $P \in \mathbf{PDO}^{(m+2)}(\mathbf{E},\mathbf{F})$. □

We now examine their behavior under composition.

Proposition 8.1.19. Let M be a smooth manifold with or without boundary and let $\mathbf{E},\mathbf{F},\mathbf{G}$ be smooth real or complex vector bundles over M. If $P \in \mathbf{PDO}^{(m)}(\mathbf{F},\mathbf{G})$ and $Q \in \mathbf{PDO}^{(n)}(\mathbf{E},\mathbf{F})$, then $P \circ Q \in \mathbf{PDO}^{(m+n)}(\mathbf{E},\mathbf{G})$.

Proof. Let $k = m+n$. We proceed by induction on k.
For the base case $k = 0$, we have $m = n = 0$, so P,Q are $C^\infty(M)$–linear, and their composition is as well. Therefore, $P \circ Q \in \mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{G})$.
Now suppose as the induction hypothesis that the result holds for any $m,n \geq 0$ such that $m+n = k$; that is, if $P \in \mathbf{PDO}^{(m)}(\mathbf{F},\mathbf{G})$ and $Q \in \mathbf{PDO}^{(n)}(\mathbf{E},\mathbf{F})$ with $m+n = k$, then $P \circ Q \in \mathbf{PDO}^{(k)}(\mathbf{E},\mathbf{G})$. We must prove that the result holds for any $m,n \geq 0$ with $m+n = k+1$.
Let $P \in \mathbf{PDO}^{(m)}(\mathbf{F},\mathbf{G})$ and $Q \in \mathbf{PDO}^{(n)}(\mathbf{E},\mathbf{F})$ with $m+n = k+1$. For $f \in C^\infty(M)$, Lemma 8.1.17 states that

$$[P \circ Q, f] = [P,f] \circ Q + P \circ [Q,f].$$

If $m = 0$, the $C^\infty(M)$–linearity of P gives $[P,f] = 0$; if $m \geq 1$, then $[P,f] \in \mathbf{PDO}^{(m-1)}(\mathbf{F},\mathbf{G})$ and $(m-1)+n = k$, so the induction hypothesis gives $[P,f] \circ Q \in \mathbf{PDO}^{(k)}(\mathbf{E},\mathbf{G})$. For the second summand, if $n = 0$ then $[Q,f] = 0$; if $n \geq 1$, then $[Q,f] \in \mathbf{PDO}^{(n-1)}(\mathbf{E},\mathbf{F})$ and $m+(n-1) = k$, so $P \circ [Q,f] \in \mathbf{PDO}^{(k)}(\mathbf{E},\mathbf{G})$ by the same hypothesis. Consequently, $[P \circ Q, f] \in \mathbf{PDO}^{(k)}(\mathbf{E},\mathbf{G})$ for every $f \in C^\infty(M)$. The recursive definition implies that $P \circ Q \in \mathbf{PDO}^{(k+1)}(\mathbf{E},\mathbf{G}) = \mathbf{PDO}^{(m+n)}(\mathbf{E},\mathbf{G})$. □

As a corollary of the preceding results, the most common differential operators on $\mathbb{R}^n$ defined in terms of partial derivatives are partial differential operators in the sense we have defined.

Corollary 8.1.20. The operator

$$L = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$$

with $a_\alpha \in C^\infty(\mathbb{R}^n)$, is a partial differential operator of order less than or equal to m.

Proof. By Example 8.1.9, each partial derivative ∂_i is a first-order partial differential operator, so Proposition 8.1.19 ensures that L is a partial differential operator of order $\leq m$, since multiplication by a smooth function does not change the order of **PDO**. □

We examine what happens when the ad operators are permuted:

Lemma 8.1.21. Let M be a smooth manifold with or without boundary and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles over M. Then for any $P \in \mathbf{PDO}(\mathbf{E}, \mathbf{F})$ and any $f, g \in C^\infty(M)$

$$(\mathrm{ad}(f) \circ \mathrm{ad}(g))P = (\mathrm{ad}(g) \circ \mathrm{ad}(f))P.$$

Proof. Let $P \in \mathbf{PDO}(\mathbf{E}, \mathbf{F})$, $f, g \in C^\infty(M)$, and $\mathbf{u} \in \Gamma(\mathbf{E})$. Then

$$(\mathrm{ad}(f) \circ \mathrm{ad}(g))(P)(\mathbf{u}) = [[P, g], f](\mathbf{u}) = [P, g](f\mathbf{u}) - f[P, g](\mathbf{u}) = P(gfu) - gP(f\mathbf{u}) - fP(g\mathbf{u}) + fgP(\mathbf{u}).$$

On the other hand,

$$(\mathrm{ad}(g) \circ \mathrm{ad}(f))(P)(\mathbf{u}) = [[P, f], g](\mathbf{u}) = [P, f](g\mathbf{u}) - g[P, f](\mathbf{u}) = P(fgu) - fP(g\mathbf{u}) - gP(f\mathbf{u}) + gfP(\mathbf{u}).$$

Terms of the same color are equal, since we can interchange f and g, so $(\mathrm{ad}(f) \circ \mathrm{ad}(g))P = (\mathrm{ad}(g) \circ \mathrm{ad}(f))P$. □

Remark 8.1.22. The preceding lemma tells us that if $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$, then for any $f_1, \ldots, f_m \in C^\infty(M)$, the bundle homomorphism

$$\mathrm{ad}(f_1) \circ \mathrm{ad}(f_2) \cdots \circ \mathrm{ad}(f_m)P$$

is unchanged when $f_1, \ldots, f_m$ are permuted.

We next characterize iterated commutators using equality of the respective differentials of the functions; to do so, we first prove a lemma.

Lemma 8.1.23. Let M be a smooth manifold with or without boundary. For each $p \in M$, consider the ideals of the ring $C^\infty_p(M)$,

$$\mathbf{m}_p = \{[f] \in C^\infty_p(M) \mid f(p) = 0\}, \qquad \mathcal{I}_p := \{[f] \in C^\infty_p(M) \mid f(p) = 0,\ df(p) = 0\}.$$

Then $\mathcal{I}_p = \mathbf{m}_p^2$; that is, every smooth function f that vanishes at p together with its differential at p can be written, in a neighborhood of p, as

$$f = \sum_{j=1}^{k} g_j h_j, \qquad g_j, h_j \in \mathbf{m}_p.$$

Proof. Let $p \in M$. First observe that $\mathbf{m}_p^2 \subseteq \mathcal{I}_p$. If $g, h \in \mathbf{m}_p$, then $g(p) = h(p) = 0$, and the product rule gives

$$d(gh)(p) = g(p)\,dh(p) + h(p)\,dg(p) = 0.$$

Therefore, $gh \in \mathcal{I}_p$, and every finite combination of such products belongs to $\mathcal{I}_p$.
Now let $f \in \mathcal{I}_p$, so that $f(p) = 0$ and $df(p) = 0$. Take a smooth chart (U, φ) around p, chosen so that $\varphi(U)$ is convex, $\varphi(p) = 0 \in \mathbb{R}^n$, and write

$$\varphi = (x^1, \dots, x^n).$$

Consider the coordinate representation of f given by

$$\tilde{f} := f \circ \varphi^{-1} \colon \varphi(U) \longrightarrow \mathbb{R}.$$

The conditions $f(p) = 0$ and $df(p) = 0$ are equivalent to

$$\tilde{f}(0) = 0, \qquad \frac{\partial \tilde{f}}{\partial x^i}(0) = 0, \quad \forall i \in \{1, \dots, n\}.$$

Taylor's theorem (Theorem 1.4.9) for $k = 1$, applied at the point $a = 0$, gives

$$\tilde{f}(x) = \tilde{f}(0) + \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial x^i}(0)\, x^i + \sum_{|\alpha|=2} \frac{2}{\alpha!}\, x^\alpha \int_0^1 (1-t) D^\alpha \tilde{f}(tx)\, dt.$$

The first two terms vanish, so, passing from multi-index notation to the usual Leibniz notation, we obtain

$$\tilde{f}(x) = \sum_{i,j=1}^{n} x^i x^j R_{ij}(x),$$

where

$$R_{ij}(x) := \int_0^1 (1-t) \frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j}(tx)\, dt.$$

Each function R_{ij} is of class C^∞ in a neighborhood of 0 because $\tilde{f}$ is.
Define

$$\tilde{g}_i(x) := \sum_{j=1}^{n} x^j R_{ij}(x),$$

so that

$$\tilde{f}(x) = \sum_{i=1}^{n} x^i \tilde{g}_i(x).$$

Observe that, for each i,

$$\tilde{g}_i(0) = \sum_{j=1}^{n} 0 \cdot R_{ij}(0) = 0.$$

Now let $g_i := \tilde{g}_i \circ \varphi \in C^\infty(U)$. Then, for every $q \in U$ we have

$$f(q) = \sum_{i=1}^{n} x^i(q)\, g_i(q).$$

We can extend x^i, g_i smoothly to M using Theorem 1.3.7 so that this equality holds in a neighborhood $W \subseteq U$ of p.

Therefore,

$$[f] = \left[\sum_{i=1}^{n} x^i g_i\right] = \sum_{i=1}^{n} [x^i][g_i]$$

as germs of functions. Since $x^i(p) = 0$ and $g_i(p) = \tilde{g}_i(0) = 0$, we have $[x^i], [g_i] \in \mathbf{m}_p$ for every $i \in \{1, \dots, n\}$, and therefore

$$[f] \in \mathbf{m}_p^2.$$

We have proved that $\mathcal{I}_p \subseteq \mathbf{m}_p^2$, and together with the reverse inclusion we obtain

$$\mathcal{I}_p = \mathbf{m}_p^2.$$

□

Definition 8.1.24. Let M be a smooth manifold with or without boundary, let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$. For $p \in M$ and $f_1, \dots, f_m \in C^\infty(M)$ define

$$\operatorname{ad}(f_1) \cdots \operatorname{ad}(f_m)(P)|_p \colon \mathbf{E}_p \longrightarrow \mathbf{F}_p$$

by

$$\big(\operatorname{ad}(f_1) \cdots \operatorname{ad}(f_m)(P)|_p\big)(v) := \big(\operatorname{ad}(f_1) \cdots \operatorname{ad}(f_m)(P)\big)(\sigma)(p),$$

where $\sigma \in \Gamma(\mathbf{E})$ satisfies $\sigma(p) = v$. Such a section exists by Lemma 1.7.28, and the value does not depend on the choice of σ because the iterated operator has order zero.

Proposition 8.1.25. Let M be a smooth manifold with or without boundary, let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles, and let $m \geq 1$. Let also $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$, $f_i, g_i \in C^\infty(M)$ with $i \in \{1, \dots, m\}$ be such that for $p \in M$,

$$df_i(p) = dg_i(p) \in T_p^* M \qquad \forall i \in \{1, \dots, m\},$$

then

$$\operatorname{ad}(f_1) \circ \operatorname{ad}(f_2) \cdots \circ \operatorname{ad}(f_m) P|_p = \operatorname{ad}(g_1) \circ \operatorname{ad}(g_2) \cdots \circ \operatorname{ad}(g_m) P|_p.$$

Proof. By Proposition 8.1.7, $\operatorname{ad}(c) = 0$ for every constant function c. We may replace each g_i by

$$\widehat{g}_i := g_i + f_i(p) - g_i(p).$$

This replacement does not change $\operatorname{ad}(g_i)$ and simultaneously ensures

$$f_i(p) = \widehat{g}_i(p), \qquad df_i(p) = d\widehat{g}_i(p).$$

We again write g_i in place of $\widehat{g}_i$ and set $\varphi_i := f_i - g_i$. Then

$$\varphi_i(p) = 0, \qquad d\varphi_i(p) = 0.$$

By Lemma 8.1.23, the germ of φ_i belongs to $\mathbf{m}_p^2$.
Let $Q \in \mathbf{PDO}^{(1)}(\mathbf{E}, \mathbf{F})$ and let $\varphi \in C^\infty(M)$ with $\varphi(p) = 0$ and $d\varphi(p) = 0$. The same lemma provides, in a

neighborhood of p, functions α_ν, β_ν such that

$$\varphi = \sum_{\nu=1}^{q} \alpha_\nu \beta_\nu, \qquad \alpha_\nu(p) = \beta_\nu(p) = 0.$$

Theorem 1.3.7 allows us to choose global representatives of these germs. By Remark 8.1.16, equality in a neighborhood of p suffices to compute the value at p. The Leibniz rule in Proposition 8.1.6 gives

$$\operatorname{ad}(\varphi)(Q) = \sum_{\nu=1}^{q} \Big(\operatorname{ad}(\alpha_\nu)(Q) \circ M_{\beta_\nu} + M_{\alpha_\nu} \circ \operatorname{ad}(\beta_\nu)(Q) \Big), \tag{8.2}$$

where M_h denotes multiplication by h. Since Q has order at most one, the operators $\operatorname{ad}(\alpha_\nu)(Q)$ and $\operatorname{ad}(\beta_\nu)(Q)$ have order zero. If $\sigma \in \Gamma(\mathbf{E})$, evaluating (8.2) at p gives

$$\big(\operatorname{ad}(\alpha_\nu)(Q)\big)(\beta_\nu \sigma)(p) = 0, \qquad \alpha_\nu(p)\big(\operatorname{ad}(\beta_\nu)(Q)\sigma\big)(p) = 0.$$

Hence,

$$\operatorname{ad}(\varphi)(Q)|_p = 0.$$

For $r \in \{0, \dots, m\}$ define

$$T_r := \operatorname{ad}(g_1) \cdots \operatorname{ad}(g_r) \operatorname{ad}(f_{r+1}) \cdots \operatorname{ad}(f_m)(P),$$

with the convention that an empty product is the identity, and for $r \in \{1, \dots, m\}$ set

$$Q_r := \operatorname{ad}(g_1) \cdots \operatorname{ad}(g_{r-1}) \operatorname{ad}(f_{r+1}) \cdots \operatorname{ad}(f_m)(P).$$

Here exactly $m-1$ commutators have been applied to an operator of order at most m, so Q_r has order at most one. Moreover, the operators $\operatorname{ad}(h)$ commute with one another by Lemma 8.1.21. Also using linearity from Proposition 8.1.7,

$$T_{r-1} - T_r = \operatorname{ad}(f_r - g_r)(Q_r) = \operatorname{ad}(\varphi_r)(Q_r).$$

Equality $\operatorname{ad}(\varphi)(Q)|_p = 0$, applied to $\varphi = \varphi_r$ and $Q = Q_r$, shows that $(T_{r-1} - T_r)|_p = 0$ for every r. Summing telescopically,

$$(T_0 - T_m)|_p = \sum_{r=1}^{m} (T_{r-1} - T_r)|_p = 0.$$

Since T_0 and T_m are, respectively, the two iterated commutators in the statement, the required equality is proved.

□

We will characterize partial differential operators between the trivial vector bundles $\underline{\mathbb{K}}^p$ and $\underline{\mathbb{K}}^q$ explicitly in terms of their local coordinates, giving us a better idea of what a partial differential operator looks like in general.

The ideals in Lemma 8.1.23 and the use of Taylor's theorem in its proof suggest that these ideals are important in characterizing functions in terms of their derivatives. Just as the cotangent bundle captures information about the first derivatives of smooth functions, there are bundles called *jet bundles* that store information about higher-order derivatives of a function. Introducing and studying these bundles will make it easier to characterize partial differential operators locally and to work with related concepts both intrinsically and locally.

Definition 8.1.26. (Jets of order m of smooth functions) Let M be a smooth manifold with or without boundary and let $p \in M$. Denote the ring of germs of smooth functions at p by $C^\infty_p(M)$, and write

$$\mathbf{m}_p = \{[f] \in C^\infty_p(M) \mid f(p) = 0\}$$

which is a maximal ideal.
For $m \geq 0$ define the power

$$\mathbf{m}_p^{m+1} := \left\langle [f_1]\cdots[f_{m+1}] \mid [f_i] \in \mathbf{m}_p,\ i = 1,\dots,m+1 \right\rangle \subset C^\infty_p(M).$$

The *space of jets of order m of smooth functions at p* is defined as the quotient vector space

$$J^m_p(M) = C^\infty_p(M)/\mathbf{m}_p^{m+1}.$$

For $[f] \in C^\infty_p(M)$, denote its class in $J^m_p(M)$ by

$$j^m_p f,$$

and call it the *jet of order m* of f at p.

This definition can be characterized in terms of derivatives, showing the importance of jets.

Lemma 8.1.27. Let M be a smooth manifold with or without boundary. For each $p \in M$, consider the subsets of the ring $C^\infty_p(M)$

$$\mathcal{I}_p^{(m)} := \left\{[f] \in C^\infty_p(M) \mid D^\alpha f(p) = 0 \text{ for every } |\alpha| \leq m\right\}, \qquad \mathbf{m}_p^{m+1} := \left\langle [f_1]\cdots[f_{m+1}] \mid [f_i] \in \mathbf{m}_p \right\rangle.$$

Then $\mathcal{I}_p^{(m)} = \mathbf{m}_p^{m+1}$.

Proof. Let $p \in M$. First we show that $\mathbf{m}_p^{m+1} \subseteq \mathcal{I}_p^{(m)}$. If

$$[f] = [g_1]\cdots[g_{m+1}], \qquad [g_i] \in \mathbf{m}_p,$$

then $g_i(p) = 0$ for every $i \in \{1,\dots,m+1\}$. For any derivative of order $|\alpha| \leq m$, the Leibniz rule in Proposition 1.1.18 gives

$$D^\alpha(g_1\cdots g_{m+1})(p) = \sum_{\alpha_1+\cdots+\alpha_{m+1}=\alpha} \binom{\alpha}{\alpha_1,\dots,\alpha_{m+1}} D^{\alpha_1}g_1(p)\cdots D^{\alpha_{m+1}}g_{m+1}(p),$$

where

$$\binom{\alpha}{\alpha_1,\dots,\alpha_{m+1}} = \frac{\alpha!}{\alpha_1!\cdots\alpha_{m+1}!}$$

is the multinomial coefficient associated with the multi-indices $\alpha, \alpha_1, \dots, \alpha_{m+1}$. Since $|\alpha| \leq m$, in every tuple $\alpha_1,\dots,\alpha_{m+1}$ with sum α at least one of the multi-indices is zero. This implies that in each term of the Leibniz rule, at least one factor $g_i(p)$ is left undifferentiated; since $g_i(p) = 0$, all summands vanish. Therefore,

$$D^\alpha(g_1\cdots g_{m+1})(p) = 0, \qquad \forall\, |\alpha| \leq m,$$

which shows that $[f] \in \mathcal{I}_p^{(m)}$.
Now let $[f] \in \mathcal{I}_p^{(m)}$, so that

$$D^\alpha f(p) = 0, \qquad \forall |\alpha| \le m.$$

Take a smooth chart (U, φ) around p, with $\varphi(U)$ convex and $\varphi(p) = 0$, and write

$$\varphi = (x^1, \dots, x^n).$$

Let

$$\tilde{f} := f \circ \varphi^{-1} \colon \varphi(U) \longrightarrow \mathbb{R}.$$

Then

$$D^\alpha \tilde{f}(0) = 0, \qquad \forall |\alpha| \le m.$$

Taylor's theorem (Theorem 1.4.9) for $k = m$ at the point 0 gives

$$\tilde{f}(x) = \sum_{|\alpha| \le m} \frac{D^\alpha \tilde{f}(0)}{\alpha!} x^\alpha + \sum_{|\alpha| = m+1} \frac{m+1}{\alpha!} x^\alpha \int_0^1 (1-t)^m D^\alpha \tilde{f}(tx)\, dt.$$

The first summand vanishes by hypothesis, so

$$\tilde{f}(x) = \sum_{|\alpha| = m+1} x^\alpha \tilde{g}_\alpha(x),$$

where

$$\tilde{g}_\alpha(x) := \frac{m+1}{\alpha!} \int_0^1 (1-t)^m D^\alpha \tilde{f}(tx)\, dt,$$

and each $\tilde{g}_\alpha$ is of class C^∞ in a neighborhood of 0.
Let $g_\alpha := \tilde{g}_\alpha \circ \varphi \in C^\infty(U)$. Then, for every $q \in U$,

$$f(q) = \sum_{|\alpha| = m+1} x^\alpha(q)\, g_\alpha(q).$$

Using Theorem 1.3.7, extend x^i and g_α to global smooth functions without changing this equality in a neighborhood $W \subseteq U$ of p.
Thus,

$$[f] = \left[\sum_{|\alpha| = m+1} x^\alpha g_\alpha \right] = \sum_{|\alpha| = m+1} [x^\alpha][g_\alpha] \quad \text{in } C_p^\infty(M).$$

Observe that if $|\alpha| = m+1$, then

$$x^\alpha = (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$$

is a product of exactly $m+1$ coordinate functions, all in $\mathbf{m}_p$. Therefore,

$$[x^\alpha] \in \mathbf{m}_p^{m+1}.$$

Finally, since $\mathbf{m}_p^{m+1}$ is an ideal of $C_p^\infty(M)$, multiplying an element of $\mathbf{m}_p^{m+1}$ by any germ $[g_\alpha]$ again yields an

element of $\mathbf{m}_p^{m+1}$. It follows that

$$[f] \in \mathbf{m}_p^{m+1}.$$

This proves that $\mathcal{I}_p^{(m)} \subseteq \mathbf{m}_p^{m+1}$, and together with the reverse inclusion we obtain

$$\mathcal{I}_p^{(m)} = \mathbf{m}_p^{m+1}.$$

□

This allows us to characterize jets at a given point through derivatives and to determine the dimension of the corresponding fibers.

Corollary 8.1.28. Let M be a smooth manifold with or without boundary, let $p \in M$, and let $m \geq 0$. Fix a local chart centered at p and write D^α for derivatives in those coordinates. Then, for any $f, g \in C^\infty(M)$ we have

$$j_p^m f = j_p^m g \quad \Longleftrightarrow \quad D^\alpha f(p) = D^\alpha g(p), \qquad \forall\, |\alpha| \leq m.$$

In particular, the chosen chart determines an isomorphism between the jet space $J_p^m(M)$ and the vector space

$$\mathbb{K}^N, \qquad N := \binom{n+m}{m},$$

through the map

$$J_p^m(M) \longrightarrow \mathbb{K}^N, \qquad j_p^m f \longmapsto \left(D^\alpha f(p)\right)_{|\alpha| \leq m}.$$

Under this identification, $\dim J_p^m(M) = \binom{n+m}{m}$.

Proof. By definition, $j_p^m f = j_p^m g$ if and only if the germs $[f]$ and $[g]$ satisfy

$$[f] - [g] \in \mathbf{m}_p^{m+1}.$$

By Lemma 8.1.27, $\mathbf{m}_p^{m+1} = \mathcal{I}_p^{(m)}$, that is,

$$[f] - [g] \in \mathcal{I}_p^{(m)} \quad \Longleftrightarrow \quad D^\alpha (f-g)(p) = 0,\ \forall\, |\alpha| \leq m.$$

This is equivalent to

$$D^\alpha f(p) = D^\alpha g(p), \qquad \forall\, |\alpha| \leq m,$$

which proves the first assertion.

To establish the isomorphism, use the smooth chart (U, φ) fixed in the statement, with $\varphi(p) = 0$. Define

$$\Psi : J_p^m(M) \longrightarrow \mathbb{K}^N, \qquad \Psi\left(j_p^m f\right) = \left(D^\alpha f(p)\right)_{|\alpha| \leq m}$$

This is well defined by the preceding criterion: if $j_p^m f = j_p^m g$, then $D^\alpha f(p) = D^\alpha g(p)$ for every $|\alpha| \leq m$. It is also linear by linearity of differentiation.

To show that Ψ is injective, Lemma 8.1.27 gives:

$$j_p^m f = j_p^m 0 \iff [f] \in \mathbf{m}_p^{m+1} \iff D^\alpha f(p) = 0,\ \forall |\alpha| \le m \iff \Psi\left(j_p^m f\right) = 0.$$

To show that Ψ is surjective, given a collection of numbers $\{a_\alpha\}_{|\alpha|\le m}$, consider the polynomial

$$P(x) = \sum_{|\alpha|\le m} \frac{a_\alpha}{\alpha!} x^\alpha,$$

which satisfies $D^\alpha P(0) = a_\alpha$ for every multi-index $|\alpha| \le m$. Define $f := P \circ \varphi \colon U \longrightarrow \mathbb{K}$. Since $p \in U$ and $f \in C^\infty(U)$, Theorem 1.3.7 gives an open set $W \subseteq M$ such that $p \in W \subseteq \overline{W} \subseteq U$ and a function $\widetilde{f} \in C^\infty(M)$ such that $\widetilde{f}\restriction_W = f$. Since $D^\alpha f(p) = D^\alpha (f \circ \varphi^{-1})|_{\varphi(p)} = D^\alpha P(0)$ and $D^\alpha \widetilde{f}(p) = D^\alpha f(p)$, we then have

$$\left(D^\alpha \widetilde{f}(p)\right)_{|\alpha|\le m} = (a_\alpha)_{|\alpha|\le m}.$$

with $\widetilde{f} \in C^\infty(M)$.
Therefore, Ψ is a linear isomorphism and the dimension of $J_p^m(M)$ is the number of multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$, namely,

$$\dim J_p^m(M) = \binom{n+m}{m}.$$

□

To gain more practice with differential operators between vector bundles, we extend the concept of a partial derivative to local sections of smooth vector bundles.

Lemma 8.1.29. Let $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank r over a smooth manifold M with or without boundary, and let (U, φ) be a chart with local coordinates $\varphi = (x^1, \dots, x^n)$ such that U is the domain of a local trivialization

$$\tau_{\mathbf{E}} \colon \mathbf{E}\restriction_U \longrightarrow U \times \mathbb{K}^r.$$

For each multi-index $\beta \in \mathbb{N}_0^n$ define $D^\beta \colon \Gamma(\mathbf{E}\restriction_U) \longrightarrow \Gamma(\mathbf{E}\restriction_U)$ by

$$(D^\beta \mathbf{u})(p) := \tau_{\mathbf{E}}^{-1}\left(p, D^\beta\left((\pi_2 \circ \tau_{\mathbf{E}} \circ \mathbf{u})\right)(p)\right), \qquad \mathbf{u} \in \Gamma(\mathbf{E}\restriction_U),\ p \in U,$$

where D^β on the right-hand side is the partial derivative of order $|\beta|$ applied componentwise to the function

$$\pi_2 \circ \tau_{\mathbf{E}} \circ \mathbf{u} \colon U \longrightarrow \mathbb{K}^r.$$

Then, for every multi-index β with $|\beta| \ge 1$, we have

$$[D^\beta, f](\mathbf{u}) = \tau_{\mathbf{E}}^{-1}\left(p, \sum_{0<\gamma\le\beta} \binom{\beta}{\gamma} (D^\gamma f)(p)\, D^{\beta-\gamma}\left((\pi_2 \circ \tau_{\mathbf{E}} \circ \mathbf{u})\right)(p)\right),$$

for every $f \in C^\infty(U)$ and every $\mathbf{u} \in \Gamma(\mathbf{E}\restriction_U)$.

Proof. Let $\mathbf{u} \in \Gamma(\mathbf{E} \restriction_U)$. Its local representation is given by the function

$$\widetilde{u} := \pi_2 \circ \tau_{\mathbf{E}} \circ \mathbf{u} \colon U \longrightarrow \mathbb{K}^r,$$

so that

$$\tau_{\mathbf{E}}(\mathbf{u}(p)) = (p, \widetilde{u}(p)), \qquad p \in U.$$

In these coordinates, D^β acts componentwise on $\widetilde{u}$.

The commutator is defined by

$$[D^\beta, f]\mathbf{u} = D^\beta(f\mathbf{u}) \; - \; f\, D^\beta \mathbf{u}.$$

Applying $\pi_2 \circ \tau_{\mathbf{E}}$ gives

$$(\pi_2 \circ \tau_{\mathbf{E}} \circ [D^\beta, f]\mathbf{u})(p) = D^\beta\big(f\,\widetilde{u}\big)(p) - f(p)\, D^\beta \widetilde{u}(p). \tag{8.3}$$

By the Leibniz rule (Proposition 1.1.17), for every $p \in U$,

$$D^\beta(f\,\widetilde{u})(p) = \sum_{\gamma \le \beta} \binom{\beta}{\gamma} (D^\gamma f)(p)\, D^{\beta-\gamma}\widetilde{u}(p). \tag{8.4}$$

The term with $\gamma = 0$ is

$$(D^0 f)(p)\, D^\beta \widetilde{u}(p) = f(p)\, D^\beta \widetilde{u}(p).$$

Substituting (8.4) into (8.3) and subtracting the term $\gamma = 0$ gives

$$(\pi_2 \circ \tau_{\mathbf{E}} \circ [D^\beta, f]\mathbf{u})(p) = \sum_{0 < \gamma \le \beta} \binom{\beta}{\gamma} (D^\gamma f)(p)\, D^{\beta-\gamma}\widetilde{u}(p). \tag{8.5}$$

Finally, to recover the bundle section from its representation in $\mathbb{K}^r$, apply the inverse of the trivialization:

$$[D^\beta, f]\mathbf{u}(p) = \tau_{\mathbf{E}}^{-1}\Big(p, (\pi_2 \circ \tau_{\mathbf{E}} \circ [D^\beta, f]\mathbf{u})(p)\Big).$$

Substituting (8.5) gives

$$[D^\beta, f]\mathbf{u} = \tau_{\mathbf{E}}^{-1}\left(p, \sum_{0 < \gamma \le \beta} \binom{\beta}{\gamma} (D^\gamma f)(p)\, D^{\beta-\gamma}\widetilde{u}(p)\right),$$

which is the desired formula. □

We continue with a lemma that will help characterize PDOs locally using what we have already established about jets.

Lemma 8.1.30. (Vanishing on germs of order $\ge m{+}1$) Let M be a smooth manifold with or without boundary and let $p \in M$. Consider the trivial bundle

$$\underline{\mathbb{K}}^r = M \times \mathbb{K}^r, \qquad \underline{\mathbb{K}}^s = M \times \mathbb{K}^s,$$

and let

$$P \in \mathbf{PDO}^{(m)}\big(\underline{\mathbb{K}}^r, \underline{\mathbb{K}}^s\big).$$

Define the map

$$L_p\colon \ \big(C_p^\infty(M)\big)^r \longrightarrow \mathbb{K}^s, \qquad L_p([u]) := (P(u))(p),$$

which is well defined by locality (Remark 8.1.16).
We write

$$\mathbf{m}_p^{m+1}\big(C_p^\infty(M)\big)^r := \left\{ \sum_{j=1}^{N} [f_j][u_j] \;\middle|\; [f_j] \in \mathbf{m}_p^{m+1},\ [u_j] \in \big(C_p^\infty(M)\big)^r,\ N \in \mathbb{N} \right\},$$

Here we recall that $(C_p^\infty(M))^r$ has the structure of a module over the ring $C_p^\infty(M)$, and note that $\mathbf{m}_p^{m+1}\big(C_p^\infty(M)\big)^r$ is an $C_p^\infty(M)$–submodule of $(C_p^\infty(M))^r$, so the quotient

$$\big(C_p^\infty(M)\big)^r / \mathbf{m}_p^{m+1}\big(C_p^\infty(M)\big)^r$$

has the structure of an $C_p^\infty(M)$–module. Since the field $\mathbb{K}$ can be regarded as a subset of $C_p^\infty(M)$, this space also inherits an $\mathbb{K}$–vector space structure.
With this notation and these hypotheses,

$$L_p\big(\mathbf{m}_p^{m+1}(C_p^\infty(M))^r\big) = \{0\} \subset \mathbb{K}^s,$$

and therefore L_p induces a well-defined $\mathbb{K}$–linear map

$$\overline{L}_p\colon \ \big(C_p^\infty(M)\big)^r / \mathbf{m}_p^{m+1}\big(C_p^\infty(M)\big)^r \longrightarrow \mathbb{K}^s.$$

Proof. Let $[u] \in \big(C_p^\infty(M)\big)^r$. By $\mathbb{K}$–linearity of P, it suffices to prove that

$$L_p\big([f][u]\big) = 0 \qquad \text{for every } [f] \in \mathbf{m}_p^{m+1}.$$

By Lemma 8.1.27,

$$\mathbf{m}_p^{m+1} = \mathcal{I}_p^{(m)} = \big\{[f] \in C_p^\infty(M) \ \mid\ D^\alpha f(p) = 0,\ |\alpha| \le m\big\}.$$

Take a smooth chart (U, φ) with $p \in U$, $\varphi(U)$ convex, $\varphi(p) = 0$, and coordinate functions $x^1, \dots, x^n$. Let $\tilde{f} := f \circ \varphi^{-1}$. The conditions $D^\alpha f(p) = 0$ are equivalent to $D^\alpha \tilde{f}(0) = 0$ for every $|\alpha| \le m$.
Taylor's theorem gives the identity

$$\tilde{f}(x) = \sum_{|\alpha|=m+1} x^\alpha\, \tilde{g}_\alpha(x), \qquad \tilde{g}_\alpha \in C^\infty(\varphi(U)).$$

Let

$$g_\alpha := \tilde{g}_\alpha \circ \varphi \in C^\infty(U), \qquad |\alpha| = m+1.$$

Then on U

$$f = \sum_{|\alpha|=m+1} x^\alpha g_\alpha.$$

We now extend all these functions simultaneously: by Theorem 1.3.7 there exist an open set

$$p \in W \subseteq \overline{W} \subseteq U$$

and smooth functions defined on M

$$\tilde{x}^i,\ \tilde{g}_\alpha^M \in C^\infty(M)$$

such that

$$\tilde{x}^i \restriction_W = x^i \restriction_W, \qquad \tilde{g}_\alpha^M \restriction_W = g_\alpha \restriction_W .$$

Therefore, on W,

$$f = \sum_{|\alpha|=m+1} x^\alpha g_\alpha = \sum_{|\alpha|=m+1} \tilde{x}^\alpha \tilde{g}_\alpha^M,$$

and passing to germs at p gives

$$[f] = \sum_{|\alpha|=m+1} [\tilde{x}^\alpha][\tilde{g}_\alpha^M] \qquad \text{in } C_p^\infty(M).$$

Now let

$$v_\alpha := \tilde{g}_\alpha^M u \in (C^\infty(M))^r.$$

By linearity, it suffices to prove that

$$L_p\big([\tilde{x}^\alpha][v_\alpha]\big) = 0.$$

We write

$$\tilde{x}^\alpha = (\tilde{x}^1)^{\alpha_1} \cdots (\tilde{x}^n)^{\alpha_n}.$$

Define inductively, for $k \geq 1$,

$$\mathrm{ad}(f)^k(P) := \mathrm{ad}(f)\big(\mathrm{ad}(f)^{k-1}(P)\big), \qquad \mathrm{ad}(f)^0(P) := P.$$

Apply the identity

$$P(fw) = f\,P(w) + [P, f](w), \qquad f \in C^\infty(M),$$

successively $|\alpha| = m+1$ times, always using the global functions $\tilde{x}^i$. Upon evaluation at p, all terms retaining any factor $\tilde{x}^i$ vanish because $\tilde{x}^i(p) = 0$, and only the term accumulating a commutator at each step survives. Thus we obtain

$$P\big(\tilde{x}^\alpha v_\alpha\big)(p) = \Big(\mathrm{ad}(\tilde{x}^1)^{\alpha_1} \cdots \mathrm{ad}(\tilde{x}^n)^{\alpha_n} P\Big)\big(v_\alpha\big)(p).$$

But this is an iterated commutator of length

$$\alpha_1 + \cdots + \alpha_n = |\alpha| = m+1,$$

and vanishes by Proposition 8.1.4. In particular,

$$L_p\big([\tilde{x}^\alpha][v_\alpha]\big) = P\big(\tilde{x}^\alpha v_\alpha\big)(p) = 0.$$

Therefore,

$$L_p([f][u]) = 0 \qquad \forall\, [f] \in \mathbf{m}_p^{m+1}.$$

Now let

$$L_p \colon \left(C_p^\infty(M)\right)^r \longrightarrow \mathbb{K}^s.$$

Since

$$L_p\left(\mathbf{m}_p^{m+1}(C_p^\infty(M))^r\right) = \{0\},$$

if $[u]$ and $[w]$ represent the same class in the quotient, then $[u]-[w] \in \mathbf{m}_p^{m+1}(C_p^\infty(M))^r$ and therefore

$$L_p([u]) - L_p([w]) = L_p([u]-[w]) = 0.$$

We therefore define

$$\overline{L}_p([u]) := L_p([u]) = (P(u))(p).$$

The definition is independent of the representative, and the $\mathbb{K}$–linearity of L_p shows that $\overline{L}_p$ is $\mathbb{K}$–linear. □

Remark 8.1.31. Lemma 8.1.30 expresses the fact that a differential operator of order $\leq m$ between trivial bundles depends only on the m-jet of the germ evaluated at the point. Consequently, germs of order $\geq m+1$ are invisible to any differential operator of order $\leq m$ at the pointwise level, as the following example shows.

Example 8.1.32. Let $M = \mathbb{R}$, $p = 0$, and consider the operator

$$P = \frac{d^2}{dx^2} \in \mathbf{PDO}^{(2)}(\underline{\mathbb{R}}, \underline{\mathbb{R}}).$$

Let $f(x) = x^3$, so that

$$f(0) = f'(0) = f''(0) = 0, \text{ so that } [f] \in \mathbf{m}_0^3.$$

Let $u(x) = \sin x$. Then

$$P(fu) = \frac{d^2}{dx^2}(x^3 \sin x),$$

and we obtain

$$(x^3 \sin x)'' = 6x \sin x + 6x^2 \cos x - x^3 \sin x.$$

Evaluating at $p = 0$ gives

$$P(fu)(0) = 0.$$

Therefore,

$$L_0\left([f][u]\right) = (P(fu))(0) = 0,$$

which illustrates Lemma 8.1.30: every germ vanishing to order $m+1$ is invisible to a differential operator of order $\leq m$ when evaluated at the point.

Theorem 8.1.33. (Local characterization of partial differential operators in the trivial case) Let M be a

smooth manifold with or without boundary and consider the trivial bundles

$$\mathbf{E} = \underline{\mathbb{K}}^r = M \times \mathbb{K}^r, \qquad \mathbf{F} = \underline{\mathbb{K}}^s = M \times \mathbb{K}^s.$$

Let

$$P\colon C^\infty(M,\mathbb{K}^r) \longrightarrow C^\infty(M,\mathbb{K}^s)$$

be an $\mathbb{K}$–linear map. Fix $m \geq 0$. The following are equivalent:

(a) $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$.

(b) For every smooth chart (U,φ) with local coordinates $\varphi = (x^1,\dots,x^n)$ there exist smooth functions

$$a_\alpha\colon U \longrightarrow \mathrm{Hom}(\mathbb{K}^r,\mathbb{K}^s), \qquad |\alpha| \leq m,$$

such that, for every function $u \in C^\infty(M,\mathbb{K}^r)$ and every $p \in U$,

$$(Pu)(p) = \sum_{|\alpha|\leq m} a_\alpha(p)\, D^\alpha u(p),$$

where $D^\alpha u(p) \in \mathbb{K}^r$ denotes the vector of partial derivatives of order $|\alpha|$ of the components of u in the chart (U,φ).

Proof. $(b) \Rightarrow (a)$: Suppose that (b) holds. This means that, for each smooth chart (U,φ), there is a collection of smooth coefficients a_α such that, for every function $u \in C^\infty(M,\mathbb{K}^r)$, the restriction of Pu to U satisfies

$$(Pu)\restriction_U = \sum_{|\alpha|\leq m} a_\alpha\, D^\alpha(u\restriction_U).$$

Denote by P_U the differential operator defined on the open set U by this local expression:

$$P_U\colon C^\infty(U,\mathbb{K}^r) \longrightarrow C^\infty(U,\mathbb{K}^s), \qquad v \longmapsto \sum_{|\alpha|\leq m} a_\alpha\, D^\alpha v.$$

We know that, on the domain U, each term D^α is a differential operator of order $|\alpha|$ (Corollary 8.1.20), and multiplication by a_α has order 0. By Proposition 8.1.19, we conclude that P_U is a partial differential operator of order $\leq m$ on U; that is,

$$P_U \in \mathbf{PDO}^{(m)}(\mathbf{E}\restriction_U, \mathbf{F}\restriction_U).$$

To prove that the global operator P belongs to $\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, we use the recursive definition based on commutators and adjoints. We must check that for any $m+1$ global smooth functions $f_0,\dots,f_m \in C^\infty(M)$, the iterated commutator vanishes:

$$\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P) = 0.$$

Consider $u \in C^\infty(M,\mathbb{K}^r)$. Since we explicitly know the behavior of Pu on U, for any $f \in C^\infty(M)$ we have the identity:

$$(\mathrm{ad}(f)(P)u)\restriction_U = ([P,f]u)\restriction_U = P_U((fu)\restriction_U) - f\restriction_U P_U(u\restriction_U) = (\mathrm{ad}(f\restriction_U)(P_U))(u\restriction_U).$$

Applying this recursively to the iterated commutators gives, in each chart (U,φ):

$$\big(\big(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P)\big)(u)\big)\restriction_U = \big(\mathrm{ad}(f_0\restriction_U)\cdots\mathrm{ad}(f_m\restriction_U)(P_U)\big)(u\restriction_U).$$

Since P_U is a differential operator of order $\le m$ on U, its commutator with $m+1$ functions vanishes identically on U. Therefore,

$$\big(\big(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P)\big)(u)\big)\restriction_U = 0.$$

Since we can choose an open cover of M consisting of chart domains, and the function

$$\big(\big(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P)\big)(u)\big)\restriction_U$$

vanishes on each of those domains,

$$\big(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P)\big)(u) = 0 \quad \text{throughout } M.$$

Since u was arbitrary, the operator $\mathrm{ad}(f_0)\dots\mathrm{ad}(f_m)(P)$ is the zero operator. This proves that $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$.

$(a)\Rightarrow(b)$: Now suppose that $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$. In particular, by Remark 8.1.16, P is local. Since $\mathbf{E}$ and $\mathbf{F}$ are trivial bundles, a section of $\mathbf{E}$ can be viewed as a function $u\colon M\longrightarrow\mathbb{K}^r$ and a section of $\mathbf{F}$ as a function $v\colon M\longrightarrow\mathbb{K}^s$. For each $p\in M$ consider the map

$$L_p\colon\ (C^\infty_p(M))^r\longrightarrow\mathbb{K}^s,\qquad L_p([u]) := (Pu)(p),$$

where $[u]$ denotes the germ of u at p. Locality of P ensures that L_p is well defined.
By Lemma 8.1.30, applied to M, the point p, and the operator P, we have

$$L_p\big(\mathbf{m}_p^{m+1}(C^\infty_p(M))^r\big) = \{0\}.$$

Therefore, L_p induces a well-defined $\mathbb{K}$–linear map

$$\overline{L}_p\colon\ (C^\infty_p(M))^r\big/\mathbf{m}_p^{m+1}(C^\infty_p(M))^r\longrightarrow\mathbb{K}^s.$$

We now justify in some detail the structure of the quotient

$$(C^\infty_p(M))^r\big/\mathbf{m}_p^{m+1}(C^\infty_p(M))^r.$$

By definition of the submodule

$$\mathbf{m}_p^{m+1}(C^\infty_p(M))^r = \left\{\sum_{j=1}^{N}[f_j][u_j]\;\middle|\;[f_j]\in\mathbf{m}_p^{m+1},\ [u_j]\in(C^\infty_p(M))^r,\ N\in\mathbb{N}\right\},$$

two germs $[u],[v]\in(C^\infty_p(M))^r$, with coordinates

$$u=(u^1,\dots,u^r),\qquad v=(v^1,\dots,v^r),$$

represent the same class in the quotient if and only if their difference belongs to that submodule, that is, if

and only if

$$[u^j] - [v^j] \in \mathbf{m}_p^{m+1} \qquad \text{for each } j \in \{1,\dots,r\}.$$

In other words,

$$[u] = [v] \text{ in } (C_p^\infty(M))^r / \mathbf{m}_p^{m+1}(C_p^\infty(M))^r \iff [u^j] = [v^j] \text{ in } C_p^\infty(M)/\mathbf{m}_p^{m+1},\ \forall j \in \{1,\dots,r\}.$$

It follows that the quotient, viewed as a vector space,

$$(C_p^\infty(M))^r / \mathbf{m}_p^{m+1}(C_p^\infty(M))^r$$

is isomorphic to the product of vector spaces

$$\left(C_p^\infty(M)/\mathbf{m}_p^{m+1}\right)^r.$$

Now take a smooth chart (U,φ) with $p \in U$ and coordinates $\varphi = (x^1,\dots,x^n)$. By Definition 8.1.26, the space of jets of order m at p is

$$J_p^m(M) := C_p^\infty(M)/\mathbf{m}_p^{m+1},$$

and Corollary 8.1.28 states that each class $j_p^m f$ is determined exactly by the derivatives

$$\left(D^\alpha f(p)\right)_{|\alpha|\le m},$$

in coordinates $x^1,\dots,x^n$, and that the map

$$J_p^m(M) \longrightarrow \mathbb{K}^N, \qquad j_p^m f \longmapsto \left(D^\alpha f(p)\right)_{|\alpha|\le m},$$

is an isomorphism of vector spaces, where $N = \binom{n+m}{m}$.
Combining these observations, we obtain a chain of isomorphisms

$$(C_p^\infty(M))^r / \mathbf{m}_p^{m+1}(C_p^\infty(M))^r \ \cong\ \left(C_p^\infty(M)/\mathbf{m}_p^{m+1}\right)^r \ \cong\ \left(J_p^m(M)\right)^r \ \cong\ \mathbb{K}^{r\binom{n+m}{m}},$$

where the last identification sends the class of

$$u = (u^1,\dots,u^r)$$

to the finite collection of partial derivatives

$$\left(D^\alpha u^j(p)\right)_{|\alpha|\le m,\ 1\le j\le r}.$$

In particular, under this isomorphism the functional $\overline{L}_p$ corresponds to a linear transformation

$$\mathbb{K}^{r\binom{n+m}{m}} \longrightarrow \mathbb{K}^s,$$

uniquely determined by a collection of linear transformations

$$a_\alpha(p) \in \operatorname{Hom}(\mathbb{K}^r,\mathbb{K}^s), \qquad |\alpha| \le m,$$

such that, for every germ $[u] \in (C^\infty_p(M))^r$,

$$\overline{L}_p([u]) = \sum_{|\alpha| \le m} a_\alpha(p)\, D^\alpha u(p),$$

where $D^\alpha u(p) \in \mathbb{K}^r$ is the vector of partial derivatives of order $|\alpha|$ of the components of u in the chart (U, φ). By definition of $\overline{L}_p$ and L_p, if $u \in C^\infty(M, \mathbb{K}^r)$, the germ $[u]$ at p satisfies

$$(Pu)(p) = L_p([u]) = \overline{L}_p([u]) = \sum_{|\alpha| \le m} a_\alpha(p)\, D^\alpha u(p).$$

This proves the local formula in (b) in the chart (U, φ):

$$(Pu)(p) = \sum_{|\alpha| \le m} a_\alpha(p)\, D^\alpha u(p), \qquad p \in U.$$

It remains to check that the coefficients a_α depend smoothly on the point. This property is local. Fix $p_0 \in U$ and choose open sets $V \Subset W \Subset U$ with $p_0 \in V$, together with a function $\chi \in C^\infty_c(U)$ equal to one on W. For each vector in the standard basis $e_j \in \mathbb{K}^r$ and each multi-index γ with $|\gamma| \le m$, extend the function

$$u_{\gamma,j}(x) := \chi(x)(x^1)^{\gamma_1} \cdots (x^n)^{\gamma_n} e_j.$$

globally. On V, the cutoff function contributes to none of the derivatives. The formula already obtained, applied first to $u_{0,j}$, shows that $a_0(\cdot)e_j = P(u_{0,j})$ on V and therefore that a_0 is smooth there. Suppose by induction that a_β is smooth on V for every $|\beta| < \ell$. If $|\gamma| = \ell$, then

$$P(u_{\gamma,j}) = \gamma!\, a_\gamma e_j + \sum_{|\beta| < \ell} a_\beta D^\beta u_{\gamma,j} \qquad \text{on } V.$$

Indeed, the derivatives of order ℓ of the monomial x^γ vanish except for the one corresponding to γ, which equals $\gamma!$. Thus,

$$a_\gamma e_j = \frac{1}{\gamma!}\left(P(u_{\gamma,j}) - \sum_{|\beta| < \ell} a_\beta D^\beta u_{\gamma,j} \right) \qquad \text{on } V,$$

and the right-hand side is smooth by the induction hypothesis. Running through the standard basis and the orders $\ell = 0, \dots, m$, we conclude that all the a_α are smooth near p_0. Since p_0 was arbitrary, they are smooth throughout U. □

Finally, we give the local characterization of partial differential operators on a smooth real or complex vector bundle.

Theorem 8.1.34. (Local characterization of partial differential operators on vector bundles) Let M be a smooth manifold with or without boundary. Let $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ and $\pi_{\mathbf{F}} \colon \mathbf{F} \longrightarrow M$ be smooth vector bundles of ranks r and s (real or complex), and let $m \ge 0$. For an $\mathbb{K}$–linear map

$$P \colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{F}),$$

the following are equivalent:

(a) $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$.

(b) For every smooth chart (U,φ) with local trivializations

$$\tau_{\mathbf{E}}\colon \mathbf{E}\restriction_U \longrightarrow U\times\mathbb{K}^r, \qquad \tau_{\mathbf{F}}\colon \mathbf{F}\restriction_U \longrightarrow U\times\mathbb{K}^s,$$

there exist smooth functions

$$a_\alpha\colon U \longrightarrow \mathrm{Hom}(\mathbb{K}^r,\mathbb{K}^s), \qquad |\alpha|\le m,$$

such that, for every section $\mathbf{u}\in\Gamma(\mathbf{E})$ with local representation in U given by

$$\widetilde{u} := \pi_2\circ\tau_{\mathbf{E}}\circ(\mathbf{u}\restriction_U)\in C^\infty(U,\mathbb{K}^r),$$

we have, for each $p\in U$,

$$P(\mathbf{u})(p) = \tau_{\mathbf{F}}^{-1}\left(p,\ \sum_{|\alpha|\le m} a_\alpha(p)\,D^\alpha\widetilde{u}(p)\right).$$

Proof. Fix a smooth chart (U,φ) with coordinates $\varphi=(x^1,\dots,x^n)$ and local trivializations $\tau_{\mathbf{E}}$ and $\tau_{\mathbf{F}}$ defined on U. Given a global section $\mathbf{u}\in\Gamma(\mathbf{E})$, denote by $\widetilde{u}\in C^\infty(U,\mathbb{K}^r)$ the local representation of $\mathbf{u}$

$$\widetilde{u} := \pi_2\circ\tau_{\mathbf{E}}\circ(\mathbf{u}\restriction_U).$$

Given a global section $\mathbf{v}\in\Gamma(\mathbf{F})$, its representation in the trivialization of $\mathbf{F}$ is, by definition, $\widetilde{v} := \pi_2\circ\tau_{\mathbf{F}}\circ(\mathbf{v}\restriction_U)$.

$(b)\Rightarrow(a)$: Suppose that (b) holds. Define an auxiliary operator $\widetilde{P}_U$ on the space of vector-valued functions on the open set U by the explicit expression in the hypothesis:

$$\widetilde{P}_U\colon C^\infty(U,\mathbb{K}^r)\longrightarrow C^\infty(U,\mathbb{K}^s), \qquad \widetilde{P}_U(w)=\sum_{|\alpha|\le m} a_\alpha\,D^\alpha w.$$

By Theorem 8.1.33, the operator $\widetilde{P}_U$ is a partial differential operator of order $\le m$ between the trivial bundles over U. Hypothesis (b) states that the global operator P and this local operator $\widetilde{P}_U$ are related through the trivializations. Specifically, for every global section $\mathbf{u}\in\Gamma(\mathbf{E})$:

$$P(\mathbf{u})\restriction_U = \tau_{\mathbf{F}}^{-1}\left(\cdot\ ,\ \sum_{|\alpha|\le m} a_\alpha D^\alpha\widetilde{u}\right),$$

so, composing with $\tau_{\mathbf{F}}$ and then projecting onto the second coordinate,

$$\pi_2\circ\tau_{\mathbf{F}}\circ(P(\mathbf{u})\restriction_U) = \widetilde{P}_U(\widetilde{u}).$$

To check that $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, it suffices to prove that

$$\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P)=0, \qquad \forall\, f_0,\dots,f_m\in C^\infty(M).$$

Let $\mathbf{u} \in \Gamma(\mathbf{E})$ and fix arbitrary functions $f_0, \dots, f_m \in C^\infty(M)$. We work in the chart (U, φ) with the chosen trivializations.

Given $f \in C^\infty(M)$ and $\mathbf{u} \in \Gamma(\mathbf{E})$, first observe that the pointwise action of f passes to the local representation through the identity

$$\widetilde{f\mathbf{u}} = \pi_2 \circ \tau_{\mathbf{E}}\big((f\mathbf{u}) \restriction_U\big) = (f \restriction_U)\,\widetilde{u}. \tag{8.6}$$

This equality follows directly from linearity of the trivialization $\tau_{\mathbf{E}}$ on each fiber.

Now apply $\pi_2 \circ \tau_{\mathbf{F}}$ to the commutator $\mathrm{ad}(f)(P)$:

$$\begin{aligned}
\pi_2 \circ \tau_{\mathbf{F}} \circ \big(\mathrm{ad}(f)(P)(\mathbf{u})\big) \restriction_U &= \pi_2 \circ \tau_{\mathbf{F}}\Big(P(f\mathbf{u}) \restriction_U - (fP(\mathbf{u})) \restriction_U\Big) \\
&= \widetilde{P}_U\big(\widetilde{f\mathbf{u}}\big) - (f \restriction_U)\,\widetilde{P}_U(\widetilde{u}) \\
&\overset{(8.6)}{=} \widetilde{P}_U\big((f \restriction_U)\,\widetilde{u}\big) - (f \restriction_U)\,\widetilde{P}_U(\widetilde{u}) \\
&= \big[\widetilde{P}_U, f \restriction_U\big](\widetilde{u}) \\
&= \big(\mathrm{ad}(f \restriction_U)(\widetilde{P}_U)\big)(\widetilde{u}).
\end{aligned}$$

Repeating this argument $m+1$ times and using at each step identity (8.6) together with the compatibility between P and $\widetilde{P}_U$ provided by the trivializations gives

$$\pi_2 \circ \tau_{\mathbf{F}} \circ \big(\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P)(\mathbf{u})\big) \restriction_U = \big(\mathrm{ad}(f_0 \restriction_U)\cdots\mathrm{ad}(f_m \restriction_U)(\widetilde{P}_U)\big)(\widetilde{u}).$$

By Theorem 8.1.33, the operator $\widetilde{P}_U$ is a partial differential operator of order $\le m$ on the trivial bundle over U. By Proposition 8.1.4, every iterated commutator of length $m+1$ with smooth functions on U vanishes:

$$\mathrm{ad}(f_0 \restriction_U)\cdots\mathrm{ad}(f_m \restriction_U)(\widetilde{P}_U) = 0.$$

Therefore, the right-hand side of the preceding equality is 0 for every $\widetilde{u}$, and consequently

$$\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P)(\mathbf{u}) = 0 \qquad \text{on } U.$$

Since M admits an open cover by charts that are trivialization domains for $\mathbf{E}$ and $\mathbf{F}$, and the preceding identity holds on each of them, we conclude that

$$\mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)(P) = 0 \qquad \text{throughout } M.$$

This proves that $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$.

$(a) \Rightarrow (b)$:
Now suppose that $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$. First, P is a local operator (Remark 8.1.16). This means that if two global sections agree on an open set, their images under P agree on that open set. This locality allows us to define the version of the operator in each trivialization.
We define an operator $\widetilde{P}\colon C^\infty(U, \mathbb{K}^r) \longrightarrow C^\infty(U, \mathbb{K}^s)$ transferring the structure of P to U through a pointwise construction. Let $w \in C^\infty(U, \mathbb{K}^r)$ and $p \in U$. To evaluate $\widetilde{P}(w)$ at p, we need to transfer w to a global section

on which P can act. Choose an open set $V \subseteq M$ such that $p \in V$ and $\overline{V} \subseteq U$. By Proposition 1.3.6 (existence of cutoff functions), there exists a smooth function $\psi\colon M \longrightarrow \mathbb{R}$ such that $\psi \equiv 1$ on $\overline{V}$ and $\operatorname{supp}(\psi) \subseteq U$. Construct a global extension $\widehat{\mathbf{w}} \in \Gamma(\mathbf{E})$ defined by:

$$\widehat{\mathbf{w}}(q) := \begin{cases} \tau_{\mathbf{E}}^{-1}(q, \psi(q)w(q)) & \text{if } q \in U, \\ 0 & \text{if } q \in M \setminus U. \end{cases} \tag{8.7}$$

We then define the value of $\widetilde{P}(w)$ at the point p by:

$$\widetilde{P}(w)(p) := \pi_2 \circ \tau_{\mathbf{F}}\big(P(\widehat{\mathbf{w}})(p)\big).$$

By locality of P, this value is independent of the choice of cutoff function ψ and open set V, since any other extension would agree with $\widehat{\mathbf{w}}$ on a neighborhood of p. Therefore, $\widetilde{P}$ is well defined as a linear operator between sections of the trivial bundles.

To apply Theorem 8.1.33, we must prove that $\widetilde{P}$ is a partial differential operator of order $\leq m$ between the trivial bundles over U. That is, we must show that for any $g_0, \cdots, g_m \in C^\infty(U)$ we have:

$$\operatorname{ad}(g_0)\cdots\operatorname{ad}(g_m)(\widetilde{P}) = 0.$$

Let $p \in U$ be arbitrary. Choose an open set $V \subseteq U$ such that $\overline{V} \subseteq U$ and a smooth function $\psi\colon M \longrightarrow \mathbb{R}$ with $\psi \equiv 1$ on $\overline{V}$ and $\operatorname{supp}(\psi) \subseteq U$. This choice fixes a uniform procedure for extending scalar functions and sections defined on U.

Given a function $g \in C^\infty(U)$, define its global extension $\widehat{g} \in C^\infty(M, \mathbb{K})$ by

$$\widehat{g}(q) = \begin{cases} \psi(q)\, g(q), & q \in U, \\ 0, & q \in M \setminus U. \end{cases}$$

By construction, $\widehat{g} \restriction_V = g \restriction_V$ and $\operatorname{supp}(\widehat{g}) \subseteq U$. Likewise, given a vector-valued function $w \in C^\infty(U, \mathbb{K}^r)$, define its global extension $\widehat{\mathbf{w}} \in \Gamma(\mathbf{E})$ as in (8.7):

$$\widehat{\mathbf{w}}(q) = \begin{cases} \tau_{\mathbf{E}}^{-1}(q, \psi(q)\, w(q)), & q \in U, \\ 0, & q \in M \setminus U. \end{cases}$$

In particular, $\widehat{\mathbf{w}} \restriction_V = \tau_{\mathbf{E}}^{-1}(\cdot, w(\cdot))$.

With these extensions fixed, we relate the commutators of $\widetilde{P}$ to those of P. We begin with the case of a single commutator. Let $g \in C^\infty(U)$ and let $\widehat{g} \in C^\infty(M)$ be its global extension defined above. Let $w \in C^\infty(U, \mathbb{K}^r)$, and let $\widehat{\mathbf{w}} \in \Gamma(\mathbf{E})$ be its global extension. By definition of $\widetilde{P}$,

$$\widetilde{P}(w)(q) = \pi_2 \circ \tau_{\mathbf{F}}\big(P(\widehat{\mathbf{w}})(q)\big), \qquad q \in V.$$

Now fix $q \in V$. Since $\psi \equiv 1$ on V, we have

$$\widehat{g}(q) = g(q), \qquad \widehat{\mathbf{w}}(q) = \tau_{\mathbf{E}}^{-1}(q, w(q)).$$

Let $\widehat{\mathbf{w}}_g$ be the global extension of the vector-valued function gw, constructed using the same function ψ:

$$\widehat{\mathbf{w}}_g(q') = \begin{cases} \tau_{\mathbf{E}}^{-1}\big(q', \psi(q')\, g(q') w(q')\big), & q' \in U, \\ 0, & q' \notin U. \end{cases}$$

For $q' \in V$,

$$\widehat{\mathbf{w}}_g(q') = \tau_{\mathbf{E}}^{-1}\big(q', g(q') w(q')\big) = \widehat{g}(q')\, \widehat{\mathbf{w}}(q'),$$

where the last equality is scalar multiplication in the fiber.
Since $\widehat{\mathbf{w}}_g$ and $\widehat{g}\,\widehat{\mathbf{w}}$ agree on a neighborhood of q, locality of P implies

$$P(\widehat{\mathbf{w}}_g)(q) = P(\widehat{g}\,\widehat{\mathbf{w}})(q).$$

Therefore,

$$\begin{aligned} \big(\mathrm{ad}(g)(\widetilde{P})w\big)(q) &= \widetilde{P}(gw)(q) - g(q)\,\widetilde{P}(w)(q) \\ &= \pi_2 \circ \tau_{\mathbf{F}}\big(P(\widehat{\mathbf{w}}_g)(q)\big) - \widehat{g}(q)\,\pi_2 \circ \tau_{\mathbf{F}}\big(P(\widehat{\mathbf{w}})(q)\big) \\ &= \pi_2 \circ \tau_{\mathbf{F}}\big(P(\widehat{g}\,\widehat{\mathbf{w}})(q) - \widehat{g}(q)\,P(\widehat{\mathbf{w}})(q)\big) \\ &= \pi_2 \circ \tau_{\mathbf{F}}\big(\mathrm{ad}(\widehat{g})(P)(\widehat{\mathbf{w}})(q)\big). \end{aligned}$$

Thus, for every $q \in V$,

$$\big(\mathrm{ad}(g)(\widetilde{P})w\big)(q) = \pi_2 \circ \tau_{\mathbf{F}}\big(\mathrm{ad}(\widehat{g})(P)(\widehat{\mathbf{w}})(q)\big).$$

Applying the same reasoning inductively gives, for any $g_0, \dots, g_m \in C^\infty(U)$ and their extensions $\widehat{g}_0, \dots, \widehat{g}_m$,

$$\big(\mathrm{ad}(g_0)\cdots\mathrm{ad}(g_m)(\widetilde{P})w\big)(q) = \pi_2 \circ \tau_{\mathbf{F}}\big(\mathrm{ad}(\widehat{g}_0)\cdots\mathrm{ad}(\widehat{g}_m)(P)(\widehat{\mathbf{w}})\big)(q), \qquad q \in V.$$

Since by hypothesis $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, $\mathrm{ad}(\hat{g}_0)\cdots\mathrm{ad}(\widehat{g}_m)(P)$ is the zero operator on M. In particular, $\big(\mathrm{ad}(\widehat{g}_0)\cdots\mathrm{ad}(\widehat{g}_m)(P)(\widehat{\mathbf{w}})\big)(q) = 0$ and therefore $\big(\big(\mathrm{ad}(g_0)\cdots\mathrm{ad}(g_m)(\widetilde{P})\big)(w)\big)(q) = 0$. Taking $q = p$, $\big(\big(\mathrm{ad}(g_0)\cdots\mathrm{ad}(g_m)(\widetilde{P})\big)(w)\big)(p) = 0$, and using the fact that p and w were arbitrary, we conclude that

$$\mathrm{ad}(g_0)\cdots\mathrm{ad}(g_m)(\widetilde{P}) = 0 \quad \text{throughout } U.$$

Therefore, $\widetilde{P} \in \mathbf{PDO}^{(m)}(\underline{\mathbb{K}}^r, \underline{\mathbb{K}}^s)$, where $\underline{\mathbb{K}}^r$ and $\underline{\mathbb{K}}^s$ are the trivial bundles of ranks r and s, respectively, over U. Applying Theorem 8.1.33 to the operator $\widetilde{P}$ ensures the existence of smooth functions $a_\alpha \colon U \longrightarrow \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^s)$ such that, for every $w \in C^\infty(U, \mathbb{K}^r)$ and every $p \in U$:

$$\widetilde{P}(w)(p) = \sum_{|\alpha| \le m} a_\alpha(p)\, D^\alpha w(p).$$

Finally, consider a global section $\mathbf{u} \in \Gamma(\mathbf{E})$. Let $\widetilde{u} \in C^\infty(U, \mathbb{K}^r)$ be its local representation. For any $p \in U$, $\mathbf{u}$ itself serves as a global extension of its local behavior. By definition of $\widetilde{P}$, we have for every $p \in U$:

$$\pi_2 \circ \tau_{\mathbf{F}}\big(P(\mathbf{u})(p)\big) = \widetilde{P}(\widetilde{u})(p) = \sum_{|\alpha| \le m} a_\alpha(p)\, D^\alpha \widetilde{u}(p).$$

Applying $\tau_{\mathbf{F}}^{-1}(p,\cdot)$ to both sides gives

$$P(\mathbf{u})(p) = \tau_{\mathbf{F}}^{-1}\left(p,\ \sum_{|\alpha|\le m} a_\alpha(p)\, D^\alpha \widetilde{u}(p)\right).$$

□

Corollary 8.1.35. (Local expression of a PDO using local frames) Let M be a smooth manifold with or without boundary. Let $\pi_{\mathbf{E}}\colon \mathbf{E} \longrightarrow M$ and $\pi_{\mathbf{F}}\colon \mathbf{F} \longrightarrow M$ be smooth vector bundles of ranks r and s. Under the hypotheses of Theorem 8.1.34, for a continuous linear operator

$$P \in \mathbf{Op}(\mathbf{E},\mathbf{F})$$

and an integer $m \ge 0$, the following are equivalent:

(a) $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$.

(b) For every smooth chart (U,φ) and every pair of smooth local frames

$$\{\mathbf{e}_1,\dots,\mathbf{e}_r\} \text{ in } \mathbf{E}\restriction_U, \qquad \{\mathbf{f}_1,\dots,\mathbf{f}_s\} \text{ in } \mathbf{F}\restriction_U,$$

there exist smooth functions

$$(a_\alpha)_{ij}\colon U \longrightarrow \mathbb{K}, \qquad 1 \le i \le r,\ 1 \le j \le s,\ |\alpha| \le m,$$

such that every section written in components as

$$\mathbf{u} = \sum_{i=1}^{r} u^i \mathbf{e}_i, \qquad u^i \in C^\infty(U,\mathbb{K}),$$

satisfies the identity

$$P(\mathbf{u}) = \sum_{j=1}^{s}\left(\sum_{|\alpha|\le m}\sum_{i=1}^{r} (a_\alpha)_{ij}\, D^\alpha u^i\right)\mathbf{f}_j.$$

on U.

Proof. (a)⇒(b). Fix a chart (U,φ) and local frames $\{\mathbf{e}_i\}$ in $\mathbf{E}\restriction_U$ and $\{\widehat{\mathbf{e}}_j\}$ in $\mathbf{F}\restriction_U$. Recall that, by Proposition 1.7.18, a smooth local frame determines a smooth local trivialization

$$\boldsymbol{\tau}_{\mathbf{E}}\colon \mathbf{E}\restriction_U \longrightarrow U\times\mathbb{K}^r, \qquad \boldsymbol{\tau}_{\mathbf{F}}\colon \mathbf{F}\restriction_U \longrightarrow U\times\mathbb{K}^s,$$

characterized by

$$\boldsymbol{\tau}_{\mathbf{E}}(\mathbf{e}_i(p)) = (p, e_i^0), \qquad \boldsymbol{\tau}_{\mathbf{F}}(\widehat{\mathbf{e}}_j(p)) = (p, \widehat{e}_j^{\,0}),$$

where $\{e_i^0\}$ and $\{\widehat{e}_j^{\,0}\}$ are the standard bases of $\mathbb{K}^r$ and $\mathbb{K}^s$, respectively.

Applying Theorem 8.1.34 to these trivializations, there exist smooth maps

$$a_\alpha : U \longrightarrow \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^s), \qquad |\alpha| \le m,$$

such that, for the local representation

$$\widetilde{u} := \pi_2 \circ \boldsymbol{\tau}_{\mathbf{E}} \circ \mathbf{u} \in C^\infty(U, \mathbb{K}^r),$$

we have

$$\pi_2 \circ \boldsymbol{\tau}_{\mathbf{F}}\big(P(\mathbf{u})\big) = \sum_{|\alpha| \le m} a_\alpha \, D^\alpha \widetilde{u}.$$

To obtain the component expression, write the action of $a_\alpha(p)$ on the standard basis:

$$a_\alpha(p)(e_i^0) = \sum_{j=1}^{s} (a_\alpha)_{ij}(p)\, \widehat{e}_j^0.$$

This defines smooth functions $(a_\alpha)_{ij}$, which are precisely the matrix entries of $a_\alpha(p)$.

If $\mathbf{u} = \sum_{i=1}^{r} u^i \mathbf{e}_i$, then, for each $p \in U$,

$$\mathbf{u}(p) = \sum_{i=1}^{r} u^i(p)\, \mathbf{e}_i(p).$$

The trivialization induced by the local frame $\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ is given by

$$\boldsymbol{\tau}_{\mathbf{E}}(\mathbf{e}_i(p)) = (p, e_i^0),$$

where (e_i^0) is the standard basis of $\mathbb{K}^r$. By linearity on each fiber,

$$\boldsymbol{\tau}_{\mathbf{E}}(\mathbf{u}(p)) = \boldsymbol{\tau}_{\mathbf{E}}\left(\sum_{i=1}^{r} u^i(p)\mathbf{e}_i(p)\right) = \left(p, \sum_{i=1}^{r} u^i(p) e_i^0\right) = \big(p, (u^1(p), \dots, u^r(p))\big).$$

Thus, the local representation

$$\widetilde{u} = \pi_2 \circ \boldsymbol{\tau}_{\mathbf{E}} \circ \mathbf{u} : U \longrightarrow \mathbb{K}^r$$

is explicitly given by

$$\widetilde{u}(p) = (u^1(p), \dots, u^r(p)).$$

Therefore,

$$a_\alpha(p)\, D^\alpha \widetilde{u}(p) = \left(\sum_{i=1}^{r} (a_\alpha)_{i1}(p) D^\alpha u^i(p), \dots, \sum_{i=1}^{r} (a_\alpha)_{is}(p) D^\alpha u^i(p)\right).$$

The local representation of $P(\mathbf{u})$ is

$$\pi_2 \circ \boldsymbol{\tau}_{\mathbf{F}}\big(P(\mathbf{u})\big)(p) = \left(\sum_{|\alpha| \le m} \sum_{i=1}^{r} (a_\alpha)_{i1}(p) D^\alpha u^i(p), \dots, \sum_{|\alpha| \le m} \sum_{i=1}^{r} (a_\alpha)_{is}(p) D^\alpha u^i(p)\right).$$

Since $\boldsymbol{\tau}_{\mathbf{F}}(\mathbf{f}_j(p)) = (p, f_j^0)$, inverting $\boldsymbol{\tau}_{\mathbf{F}}$ gives

$$P(\mathbf{u})(p) = \sum_{j=1}^{s} \left(\sum_{|\alpha|\le m} \sum_{i=1}^{r} (a_\alpha)_{ij}(p) D^\alpha u^i(p) \right) \widehat{\mathbf{e}}_j(p),$$

and therefore on U

$$P(\mathbf{u}) = \sum_{j=1}^{s} \left(\sum_{|\alpha|\le m} \sum_{i=1}^{r} (a_\alpha)_{ij} D^\alpha u^i \right) \widehat{\mathbf{e}}_j.$$

(b)⇒(a). Suppose that (b) holds. We must prove that $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ in the sense of Theorem 8.1.34. Let (U,φ) be an arbitrary smooth chart of M and take arbitrary smooth local trivializations

$$\boldsymbol{\tau}_{\mathbf{E}} \colon \mathbf{E} \restriction_U \longrightarrow U \times \mathbb{K}^r, \qquad \boldsymbol{\tau}_{\mathbf{F}} \colon \mathbf{F} \restriction_U \longrightarrow U \times \mathbb{K}^s.$$

By Proposition 1.7.18, each of these trivializations induces a smooth local frame over U. More precisely, there exists a smooth local frame $\{\mathbf{e}_1,\dots,\mathbf{e}_r\}$ in $\mathbf{E} \restriction_U$ associated with $\boldsymbol{\tau}_{\mathbf{E}}$, characterized by

$$\mathbf{e}_i(p) = \boldsymbol{\tau}_{\mathbf{E}}^{-1}(p, e_i^0), \qquad 1 \le i \le r,$$

where $\{e_i^0\}$ is the standard basis of $\mathbb{K}^r$. The trivialization $\boldsymbol{\tau}_{\mathbf{F}}$ induces the smooth local frame $\{\widehat{\mathbf{e}}_1,\dots,\widehat{\mathbf{e}}_s\}$ in $\mathbf{F} \restriction_U$ given by

$$\widehat{\mathbf{e}}_j(p) = \boldsymbol{\tau}_{\mathbf{F}}^{-1}(p, \widehat{e}_j^0), \qquad 1 \le j \le s,$$

where $\{\widehat{e}_j^0\}$ is the standard basis of $\mathbb{K}^s$.
Since by hypothesis (b) holds for every pair of local frames, in particular for the frames $\{\mathbf{e}_i\}$ and $\{\widehat{\mathbf{e}}_j\}$ associated with these trivializations, there exist smooth functions

$$(a_\alpha)_{ij} \colon U \longrightarrow \mathbb{K}, \qquad 1 \le i \le r,\ 1 \le j \le s,\ |\alpha| \le m,$$

such that, for every section

$$\mathbf{u} = \sum_{i=1}^{r} u^i \mathbf{e}_i, \qquad u^i \in C^\infty(U,\mathbb{K}),$$

the identity

$$P(\mathbf{u}) = \sum_{j=1}^{s} \left(\sum_{|\alpha|\le m} \sum_{i=1}^{r} (a_\alpha)_{ij}\, D^\alpha u^i \right) \widehat{\mathbf{e}}_j.$$

holds on U.
Now define, for each multi-index α, a map

$$a_\alpha \colon U \longrightarrow \operatorname{Hom}(\mathbb{K}^r, \mathbb{K}^s)$$

as follows: for each $p \in U$ and each vector $v = (v^1,\dots,v^r) \in \mathbb{K}^r$, set

$$a_\alpha(p)(v) := \left(\sum_{i=1}^{r} (a_\alpha)_{i1}(p) v^i, \dots, \sum_{i=1}^{r} (a_\alpha)_{is}(p) v^i \right) \in \mathbb{K}^s.$$

In other words, the matrix of $a_\alpha(p)$ in the standard bases $\{e_i^0\}$ and $\{\widehat{e}_j^0\}$ is precisely

$$\big((a_\alpha)_{ij}(p)\big)_{1\le i\le r,\ 1\le j\le s}.$$

Since the functions $(a_\alpha)_{ij}$ are smooth, it follows that a_α is a smooth map with values in $\mathrm{Hom}(\mathbb{K}^r,\mathbb{K}^s)$.
We show that these a_α provide the local expression of P in the trivializations $\boldsymbol{\tau}_{\mathbf{E}}$ and $\boldsymbol{\tau}_{\mathbf{F}}$. Let $\mathbf{u}\in\Gamma(\mathbf{E})$ and write on U

$$\mathbf{u}=\sum_{i=1}^{r} u^i\mathbf{e}_i,\qquad u^i\in C^\infty(U,\mathbb{K}).$$

By construction of $\boldsymbol{\tau}_{\mathbf{E}}$ from the frame $\{\mathbf{e}_i\}$, the local representation of $\mathbf{u}$ is

$$\widetilde{u}:=\pi_2\circ\boldsymbol{\tau}_{\mathbf{E}}\circ\mathbf{u}\colon U\longrightarrow\mathbb{K}^r,\qquad \widetilde{u}(p)=(u^1(p),\dots,u^r(p)).$$

In particular,

$$D^\alpha\widetilde{u}(p)=\big(D^\alpha u^1(p),\dots,D^\alpha u^r(p)\big).$$

On the other hand, using the formula in (b) for $P(\mathbf{u})$ and applying $\boldsymbol{\tau}_{\mathbf{F}}$ gives

$$\boldsymbol{\tau}_{\mathbf{F}}\big(P(\mathbf{u})(p)\big)=\left(p,\ \sum_{j=1}^{s}\left(\sum_{|\alpha|\le m}\sum_{i=1}^{r}(a_\alpha)_{ij}(p)\,D^\alpha u^i(p)\right)\widehat{e}_j^0\right).$$

By definition of $a_\alpha(p)$, the vector in $\mathbb{K}^s$ whose jth component is

$$\sum_{|\alpha|\le m}\sum_{i=1}^{r}(a_\alpha)_{ij}(p)\,D^\alpha u^i(p)$$

is precisely

$$\sum_{|\alpha|\le m}a_\alpha(p)\,D^\alpha\widetilde{u}(p).$$

Thus,

$$\pi_2\circ\boldsymbol{\tau}_{\mathbf{F}}\big(P(\mathbf{u})\big)(p)=\sum_{|\alpha|\le m}a_\alpha(p)\,D^\alpha\widetilde{u}(p),\qquad\forall\,p\in U.$$

Since (U,φ) and the trivializations $\boldsymbol{\tau}_{\mathbf{E}},\boldsymbol{\tau}_{\mathbf{F}}$ were arbitrary, we have checked that, in every smooth chart and for every pair of local trivializations, there is a family of smooth maps

$$a_\alpha\colon U\longrightarrow\mathrm{Hom}(\mathbb{K}^r,\mathbb{K}^s),\qquad|\alpha|\le m,$$

providing the expression

$$\pi_2\circ\boldsymbol{\tau}_{\mathbf{F}}\big(P(\mathbf{u})\big)(p)=\sum_{|\alpha|\le m}a_\alpha(p)\,D^\alpha\big(\pi_2\circ\boldsymbol{\tau}_{\mathbf{E}}\circ\mathbf{u}\big)(p)$$

for every $\mathbf{u}\in\Gamma(\mathbf{E})$ and every $p\in U$. By Theorem 8.1.34, we conclude that

$$P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F}).$$

□

Proposition 8.1.36. (Conjugation of differential operators) Let M be a smooth manifold with or without boundary. Let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles and let

$$P\colon \Gamma(\mathbf{E})\longrightarrow\Gamma(\mathbf{F})$$

be a complex-linear operator. Define

$$\overline{P}\colon \Gamma(\overline{\mathbf{E}})\longrightarrow\Gamma(\overline{\mathbf{F}}),\qquad \overline{P}(\overline{\mathbf{u}}):=\overline{P\mathbf{u}}.$$

Then:

(a) $\overline{P}$ is well defined and complex-linear;

(b) if $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, then $\overline{P}\in\mathbf{PDO}^{(m)}(\overline{\mathbf{E}},\overline{\mathbf{F}})$;

(c) if, in a chart and with respect to local frames $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ and $(\mathbf{f}_1,\dots,\mathbf{f}_s)$, the expression for P is

$$(P\mathbf{u})^{\sim}=\sum_{|\alpha|\le m}A_\alpha D^\alpha\widetilde{u},\qquad A_\alpha\colon U\longrightarrow M_{s\times r}(\mathbb{C}),$$

then, with respect to the conjugate frames $(\overline{\mathbf{e}}_1,\dots,\overline{\mathbf{e}}_r)$ and $(\overline{\mathbf{f}}_1,\dots,\overline{\mathbf{f}}_s)$,

$$(\overline{P}\mathbf{z})^{\sim}=\sum_{|\alpha|\le m}\overline{A_\alpha}\,D^\alpha\widetilde{z};$$

(d) conjugation commutes with restrictions and satisfies

$$\overline{Q\circ P}=\overline{Q}\circ\overline{P},\qquad \overline{\mathrm{id}_{\mathbf{E}}}=\mathrm{id}_{\overline{\mathbf{E}}}.$$

In particular, if P is invertible, then $\overline{P^{-1}}=(\overline{P})^{-1}$.

Proof. Conjugation of sections is an antilinear bijection, so every section of $\overline{\mathbf{E}}$ is uniquely expressible as $\overline{\mathbf{u}}$. If $\lambda\in\mathbb{C}$, the scalar structure of $\overline{\mathbf{E}}$ gives $\lambda\overline{\mathbf{u}}=\overline{\overline{\lambda}\mathbf{u}}$; therefore,

$$\overline{P}(\lambda\overline{\mathbf{u}})=\overline{P(\overline{\lambda}\mathbf{u})}=\overline{\overline{\lambda}P\mathbf{u}}=\lambda\overline{P\mathbf{u}}.$$

Additivity is checked in the same way.

For the local formula, write $\mathbf{z}=\sum_{j=1}^{r}z^j\overline{\mathbf{e}}_j$. Then $\mathbf{z}=\overline{\mathbf{u}}$ for $\mathbf{u}=\sum_{j=1}^{r}\overline{z^j}\mathbf{e}_j$. Conjugating the local expression for $P\mathbf{u}$ gives

$$(\overline{P}\mathbf{z})^{\sim}=\overline{\sum_{|\alpha|\le m}A_\alpha D^\alpha(\widetilde{\overline{z}})}=\sum_{|\alpha|\le m}\overline{A_\alpha}D^\alpha\widetilde{z}.$$

The local characterization of differential operators then shows that $\overline{P}$ has order at most m. Compatibility with restrictions is immediate from the same formula. Finally, for every $\mathbf{u}\in\Gamma(\mathbf{E})$,

$$\overline{Q\circ P}(\overline{\mathbf{u}})=\overline{Q(P\mathbf{u})}=(\overline{Q}\circ\overline{P})(\overline{\mathbf{u}}),$$

and the assertions about identities and inverses follow from this equality. □

8.2 Formal Hermitian adjoints

Throughout the complex case, we use Hermitian metrics linear in the first variable and conjugate-linear in the second. All integrals are taken with respect to the Riemann–Lebesgue measure $\lambda_{\mathbf{g}}$ constructed in Definition 2.15.5; in particular, no orientation of M is required.

Definition 8.2.1. (Formal Hermitian adjoint) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E},\mathbf{F}\to M$ be real vector bundles with bundle metrics if $\mathbb{K}=\mathbb{R}$, or Hermitian bundles if $\mathbb{K}=\mathbb{C}$. For $P\in\mathbf{PDO}(\mathbf{E},\mathbf{F})$, an operator

$$P_h^*\in\mathbf{PDO}(\mathbf{F},\mathbf{E})$$

is called a *formal Hermitian adjoint* of P if

$$\int_M \mathbf{h}_{\mathbf{F}}(P\mathbf{u},\mathbf{v})\,d\lambda_{\mathbf{g}}=\int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},P_h^*\mathbf{v})\,d\lambda_{\mathbf{g}}$$

for every $\mathbf{u}\in\Gamma_c(\mathbf{E})$ and every $\mathbf{v}\in\Gamma_c(\mathbf{F})$. In the real case, this is the usual real notion.

Lemma 8.2.2. (Uniqueness) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E},\mathbf{F}\to M$ be real vector bundles with bundle metrics or Hermitian bundles. A differential operator $P\in\mathbf{PDO}(\mathbf{E},\mathbf{F})$ admits at most one formal Hermitian adjoint.

Proof. Let $S_1,S_2\in\mathbf{PDO}(\mathbf{F},\mathbf{E})$ be two adjoints and fix $\mathbf{v}\in\Gamma_c(\mathbf{F})$. Locality implies that $\mathbf{w}:=(S_1-S_2)\mathbf{v}$ belongs to $\Gamma_c(\mathbf{E})$. Taking $\mathbf{u}=\mathbf{w}$ in the difference of the two adjunction identities gives

$$0=\int_M \mathbf{h}_{\mathbf{E}}(\mathbf{w},\mathbf{w})\,d\lambda_{\mathbf{g}},$$

and, by positivity, $\mathbf{w}=0$. Thus, $S_1\mathbf{v}=S_2\mathbf{v}$ for every compactly supported section. Now given $\mathbf{v}\in\Gamma(\mathbf{F})$ and $x\in M$, choose $\chi\in C_c^\infty(M)$ equal to one on a neighborhood of x. By locality, $(S_1-S_2)\mathbf{v}=(S_1-S_2)(\chi\mathbf{v})$ near x, and the last expression is zero. Since x was arbitrary, $S_1=S_2$. □

Theorem 8.2.3. (Existence and local formula for the Hermitian adjoint) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E},\mathbf{F}\to M$ be real vector bundles with bundle metrics or Hermitian bundles. Let $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$. Then there exists a unique $P_h^*\in\mathbf{PDO}^{(m)}(\mathbf{F},\mathbf{E})$.
More precisely, let $\varphi\colon U\to\Omega\subseteq\mathbb{R}^n$ be a chart, with coordinate functions $(x^1,\dots,x^n)$, and choose orthonormal local frames (unitary in the complex case) $\mathbf{e}_1,\dots,\mathbf{e}_r$ and $\mathbf{f}_1,\dots,\mathbf{f}_s$ for $\mathbf{E}$ and $\mathbf{F}$, respectively. For a scalar function $a\in C^\infty(U,\mathbb{K})$ write

$$D^\alpha_\varphi a:=\left[D^\alpha(a\circ\varphi^{-1})\right]\circ\varphi.$$

If the functions $a^{\,j}_{\alpha i}\in C^\infty(U,\mathbb{K})$ are determined by

$$P\left(\sum_{i=1}^{r}u^i\mathbf{e}_i\right)=\sum_{j=1}^{s}\left(\sum_{i=1}^{r}\sum_{|\alpha|\le m}a^{\,j}_{\alpha i}D^\alpha_\varphi u^i\right)\mathbf{f}_j,\tag{8.8}$$

and $g_{ab} := \mathbf{g}(\partial_a, \partial_b)$ are the coordinate coefficients of the metric, then the expression for P_h^* with respect to the same frames is

$$P_h^*\left(\sum_{j=1}^{s} v^j \mathbf{f}_j\right) = \frac{1}{\sqrt{\det(\mathbf{g})}} \sum_{i=1}^{r} \left[\sum_{j=1}^{s} \sum_{|\alpha|\le m} (-1)^{|\alpha|} D_\varphi^\alpha \left(\sqrt{\det(\mathbf{g})}\, \overline{a_{\alpha i}^j}\, v^j\right)\right] \mathbf{e}_i. \tag{8.9}$$

In the real case, the bar does not alter the coefficient. If the coefficients are grouped into the matrix-valued map

$$A_\alpha = (a_{\alpha i}^j) \colon U \longrightarrow M_{s\times r}(\mathbb{K}),$$

its pointwise conjugate transpose is denoted by $A_\alpha^* = \overline{A_\alpha}^T$.

Proof. Fix a chart $\varphi \colon U \to \Omega$ and orthonormal local frames $\mathbf{e}_1, \dots, \mathbf{e}_r$ and $\mathbf{f}_1, \dots, \mathbf{f}_s$ for $\mathbf{E}$ and $\mathbf{F}$, respectively. The expression on the right-hand side of (8.9) defines an operator

$$Q_U : \Gamma(\mathbf{F}|_U) \longrightarrow \Gamma(\mathbf{E}|_U).$$

Indeed, $\det(\mathbf{g})$ is smooth and strictly positive. Expanding each term using the Leibniz rule for multi-indices, $Q_U \mathbf{v}$ depends on the coordinate derivatives of the functions v^j of order at most m. The local characterization in Corollary 8.1.35 therefore implies that $Q_U \in \mathbf{PDO}^{(m)}(\mathbf{F}|_U, \mathbf{E}|_U)$.
We prove the local adjunction identity. Let $\mathbf{u} \in \Gamma_c(\mathbf{E}|_U)$ and let $\mathbf{v} \in \Gamma(\mathbf{F}|_U)$, and write

$$\mathbf{u} = \sum_{i=1}^{r} u^i \mathbf{e}_i, \qquad \mathbf{v} = \sum_{j=1}^{s} v^j \mathbf{f}_j.$$

The image $\varphi(\operatorname{supp}\mathbf{u})$ is a compact set contained in Ω. Since the frames are orthonormal and the Hermitian metric is linear in the first variable, the local formula for the Riemann–Lebesgue measure and (8.8) give

$$\begin{aligned}
&\int_U \mathbf{h}_{\mathbf{F}}(P\mathbf{u}, \mathbf{v})\, d\lambda_{\mathbf{g}} \\
&\quad = \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{|\alpha|\le m} \int_\Omega \left(a_{\alpha i}^j \circ \varphi^{-1}\right) D^\alpha \left(u^i \circ \varphi^{-1}\right) \overline{\left(v^j \circ \varphi^{-1}\right)} \left(\sqrt{\det(\mathbf{g})} \circ \varphi^{-1}\right) d\lambda_n.
\end{aligned}$$

Fix i, j, and $\alpha = (\alpha_1, \dots, \alpha_n)$. We can integrate α_a times with respect to x^a, for each a, because $u^i \circ \varphi^{-1}$ has compact support in Ω. No boundary term appears, and each integration introduces a minus sign. Therefore,

$$\begin{aligned}
&\int_\Omega \left(a_{\alpha i}^j \circ \varphi^{-1}\right) D^\alpha \left(u^i \circ \varphi^{-1}\right) \overline{\left(v^j \circ \varphi^{-1}\right)} \left(\sqrt{\det(\mathbf{g})} \circ \varphi^{-1}\right) d\lambda_n \\
&\quad = (-1)^{|\alpha|} \int_\Omega \left(u^i \circ \varphi^{-1}\right) D^\alpha \left[\left(a_{\alpha i}^j\, \overline{v^j} \sqrt{\det(\mathbf{g})}\right) \circ \varphi^{-1}\right] d\lambda_n.
\end{aligned}$$

All coordinate derivatives are real and $\sqrt{\det(\mathbf{g})}$ is real. Consequently,

$$D^\alpha \left[\left(a_{\alpha i}^j\, \overline{v^j} \sqrt{\det(\mathbf{g})}\right) \circ \varphi^{-1}\right] = \overline{D^\alpha \left[\left(\overline{a_{\alpha i}^j}\, v^j \sqrt{\det(\mathbf{g})}\right) \circ \varphi^{-1}\right]}.$$

This equality includes the real case, in which conjugation is the identity. Substituting it, summing over i, j, and α, and inserting the factor $\bigl(\sqrt{\det(\mathbf{g})}\bigr)^{-1}\sqrt{\det(\mathbf{g})}$ gives

$$\begin{aligned}
&\int_U \mathbf{h}_{\mathbf{F}}(P\mathbf{u},\mathbf{v})\,d\lambda_{\mathbf{g}} \\
&\quad= \int_\Omega \sum_{i=1}^{r}\bigl(u^i\circ\varphi^{-1}\bigr)\,\overline{\left[\frac{1}{\sqrt{\det(\mathbf{g})}}\sum_{j=1}^{s}\sum_{|\alpha|\le m}(-1)^{|\alpha|}D_\varphi^\alpha\Bigl(\sqrt{\det(\mathbf{g})}\,\overline{a^j_{\alpha i}}\,v^j\Bigr)\right]}\circ\varphi^{-1}\,\bigl(\sqrt{\det(\mathbf{g})}\circ\varphi^{-1}\bigr)\,d\lambda_n \\
&\quad= \int_U \mathbf{h}_{\mathbf{E}}(\mathbf{u},Q_U\mathbf{v})\,d\lambda_{\mathbf{g}}.
\end{aligned}$$

We have proved a slightly stronger identity than required: only $\mathbf{u}$ needs to have compact support in U; the section $\mathbf{v}$ may be any smooth section over U.

We now show that the local operators are compatible. Let U and V be two coordinate domains with orthonormal frames as above, and let $W = U\cap V$. For $\mathbf{u}\in\Gamma_c(\mathbf{E}|_W)$ and $\mathbf{v}\in\Gamma_c(\mathbf{F}|_W)$, the two formulas already proved give

$$\int_W \mathbf{h}_{\mathbf{E}}\bigl(\mathbf{u},(Q_U|_W)\mathbf{v}\bigr)\,d\lambda_{\mathbf{g}} = \int_W \mathbf{h}_{\mathbf{F}}(P\mathbf{u},\mathbf{v})\,d\lambda_{\mathbf{g}} = \int_W \mathbf{h}_{\mathbf{E}}\bigl(\mathbf{u},(Q_V|_W)\mathbf{v}\bigr)\,d\lambda_{\mathbf{g}}.$$

Thus, $Q_U|_W$ and $Q_V|_W$ are formal adjoints of the same operator $P|_W$. Lemma 8.2.2, applied to the Riemannian manifold W, implies that $Q_U|_W = Q_V|_W$. In particular, the equality is not assumed from the coordinate expressions; it follows from the integral identity both satisfy.

We can cover M by a locally finite family $\{U_\ell\}_{\ell\in L}$ of coordinate domains over which $\mathbf{E}$ and $\mathbf{F}$ admit orthonormal frames. Compatibility on intersections allows us to define, for $\mathbf{v}\in\Gamma(\mathbf{F})$, a section $P_h^*\mathbf{v}$ by

$$(P_h^*\mathbf{v})|_{U_\ell} := Q_{U_\ell}(\mathbf{v}|_{U_\ell}).$$

The sections on the right-hand side agree on overlaps, so they define a unique global smooth section. Since the order of a differential operator can be checked locally, the formulas for the Q_{U_ℓ} show that

$$P_h^*\in\mathbf{PDO}^{(m)}(\mathbf{F},\mathbf{E}).$$

It remains to check the global identity. Let $\{\chi_\ell\}_{\ell\in L}$ be a smooth partition of unity subordinate to $\{U_\ell\}_{\ell\in L}$, and let $\mathbf{u}\in\Gamma_c(\mathbf{E})$ and $\mathbf{v}\in\Gamma_c(\mathbf{F})$. The family $\{\operatorname{supp}(\chi_\ell)\}_{\ell\in L}$ is locally finite. Thus each point of the compact set $\operatorname{supp}(\mathbf{u})$ has a neighborhood meeting only finitely many of those supports; a finite subcover by such neighborhoods shows that the set

$$L_{\mathbf{u}} := \{\ell\in L \mid \operatorname{supp}(\mathbf{u})\cap\operatorname{supp}(\chi_\ell)\neq\varnothing\}$$

is finite. Consequently,

$$\mathbf{u} = \sum_{\ell\in L_{\mathbf{u}}}\chi_\ell\mathbf{u}.$$

Moreover, $\chi_\ell\mathbf{u}$ has compact support contained in U_ℓ. Locality of P implies

$$\operatorname{supp}\bigl(P(\chi_\ell\mathbf{u})\bigr)\subseteq\operatorname{supp}(\chi_\ell\mathbf{u})\subseteq U_\ell.$$

We can apply on U_ℓ the local identity in which the first section has compact support and the second is $\mathbf{v}|_{U_\ell}$.

Thus,

$$\begin{aligned}\int_M \mathbf{h}_{\mathbf{F}}(P\mathbf{u},\mathbf{v})\,d\lambda_{\mathbf{g}} &= \sum_{\ell\in L_{\mathbf{u}}}\int_{U_\ell}\mathbf{h}_{\mathbf{F}}\big(P(\chi_\ell\mathbf{u}),\mathbf{v}\big)\,d\lambda_{\mathbf{g}}\\ &= \sum_{\ell\in L_{\mathbf{u}}}\int_{U_\ell}\mathbf{h}_{\mathbf{E}}\big(\chi_\ell\mathbf{u},P_h^*\mathbf{v}\big)\,d\lambda_{\mathbf{g}}\\ &= \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},P_h^*\mathbf{v})\,d\lambda_{\mathbf{g}}.\end{aligned}$$

Therefore, P_h^* is a formal Hermitian adjoint of P. Its uniqueness is that of Lemma 8.2.2. □

Remark 8.2.4. (Formal adjoint in the presence of a boundary) If M has smooth boundary and the coefficients of P, the bundle metrics, and the base metric are smooth up to it, the same local formula defines a differential operator P_h^* smooth up to the boundary. Indeed, in a boundary chart the determinant of the metric remains positive, and all derivatives of the coefficients in (8.9) are smooth up to the flat part. On overlaps, the expressions agree first in the interior by the uniqueness already proved, and then on the boundary by continuity. The identity characterizing this formal adjoint is required for test sections with compact support contained in $\operatorname{Int}(M)$.

When the sections reach the boundary, integration by parts may produce additional terms, such as those we will calculate for covariant derivatives. The formal differential operator is determined before boundary conditions are chosen. Its realization as an unbounded operator on L^2, and the domain of its Hilbert space adjoint, require specifying a domain of sections; they are not identified by the differential formula alone.

Proposition 8.2.5. (Rules for the formal Hermitian adjoint) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $\mathbf{E},\mathbf{F},\mathbf{G}\to M$ be real vector bundles with bundle metrics or Hermitian bundles, and let $P\in\mathbf{PDO}(\mathbf{E},\mathbf{F})$ and $Q\in\mathbf{PDO}(\mathbf{F},\mathbf{G})$. Then:

(a) $(Q\circ P)_h^* = P_h^*\circ Q_h^*$;

(b) for $R\in\mathbf{PDO}(\mathbf{E},\mathbf{F})$ and $\lambda\in\mathbb{K}$,

$$(P+\lambda R)_h^* = P_h^* + \overline{\lambda}\,R_h^*,$$

where $\overline{\lambda}=\lambda$ in the real case;

(c) if M_f denotes multiplication by $f\in C^\infty(M,\mathbb{K})$, then $(M_f)_h^* = M_{\overline{f}}$;

(d) for every open set $U\subseteq M$,

$$(P|_U)^*_{h|_U} = (P_h^*)|_U;$$

(e) if $\mathbf{E},\mathbf{F}$ are complex and $\overline{\mathbf{E}},\overline{\mathbf{F}}$ are equipped with the conjugate metrics, then

$$(\overline{P})^*_{\overline{h}} = \overline{P_h^*}.$$

Proof. The first identity follows by successively transferring Q and P in the integral; supports remain compact by locality. For the second, linearity in the first variable and conjugate-linearity in the second give

$$\mathbf{h}_{\mathbf{E}}(\mathbf{u},\overline{\lambda}R_h^*\mathbf{v}) = \lambda\mathbf{h}_{\mathbf{E}}(\mathbf{u},R_h^*\mathbf{v}).$$

The third is proved in the same way. The fourth follows by applying the integral identity to compactly supported sections in U and using uniqueness.
For the last assertion, write the conjugate metrics as $\mathbf{h}_{\overline{\mathbf{E}}}(\overline{\mathbf{u}},\overline{\mathbf{w}}) = \overline{\mathbf{h}_{\mathbf{E}}(\mathbf{u},\mathbf{w})}$ and $\mathbf{h}_{\overline{\mathbf{F}}}(\overline{\mathbf{v}},\overline{\mathbf{z}}) = \overline{\mathbf{h}_{\mathbf{F}}(\mathbf{v},\mathbf{z})}$. Then

$$\begin{aligned}\int_M \mathbf{h}_{\overline{\mathbf{F}}}(\overline{P}\,\overline{\mathbf{u}},\overline{\mathbf{v}})\,d\lambda_{\mathbf{g}} &= \overline{\int_M \mathbf{h}_{\mathbf{F}}(P\mathbf{u},\mathbf{v})\,d\lambda_{\mathbf{g}}}\\ &= \overline{\int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},P_h^*\mathbf{v})\,d\lambda_{\mathbf{g}}}\\ &= \int_M \mathbf{h}_{\overline{\mathbf{E}}}(\overline{\mathbf{u}},\overline{P_h^*}\,\overline{\mathbf{v}})\,d\lambda_{\mathbf{g}},\end{aligned}$$

and uniqueness completes the proof. □

Exercise 8.2.6. Check the rule $(P+\lambda R)_h^* = P_h^* + \overline{\lambda} R_h^*$ directly and explain why the bar is indispensable when $\mathbb{K} = \mathbb{C}$.

Definition 8.2.7. (Formally self-adjoint operator) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E} \to M$ be a real vector bundle with a bundle metric or a Hermitian bundle. An operator $P \in \mathbf{PDO}(\mathbf{E},\mathbf{E})$ is *formally self-adjoint* if $P = P_h^*$.

We conclude this section with examples illustrating the concept of a formal Hermitian adjoint. We begin with the directional covariant derivative.

Proposition 8.2.8. (Formal adjoint of $\nabla_{\mathbf{X}}$) Let $(M,\mathbf{g})$ be a Riemannian manifold (with or without boundary ∂M) and let ∇ be the Levi–Civita connection. Let $\mathbf{X} \in \mathfrak{X}(M)$ be a fixed smooth vector field. Then the operator

$$\nabla_{\mathbf{X}} \colon \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \qquad \mathbf{Y} \longmapsto \nabla_{\mathbf{X}}\mathbf{Y},$$

is a partial differential operator of order 1, that is,

$$\nabla_{\mathbf{X}} \in \mathbf{PDO}^{(1)}(TM,TM).$$

Moreover, for any vector fields $\mathbf{Y},\mathbf{Z} \in \mathfrak{X}_c(M)$ we have

$$\int_M \mathbf{g}(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z})\,d\lambda_{\mathbf{g}} = -\int_M \mathbf{g}\big(\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\big)\,d\lambda_{\mathbf{g}} - \int_M (\operatorname{div}\mathbf{X})\,\mathbf{g}(\mathbf{Y},\mathbf{Z})\,d\lambda_{\mathbf{g}} + \int_{\partial M} \mathbf{g}(\mathbf{Y},\mathbf{Z})\,\mathbf{g}(\mathbf{X},\boldsymbol{\nu})\,d\lambda_{\tilde{\mathbf{g}}}, \tag{8.10}$$

where, if $\partial M \neq \varnothing$, $\boldsymbol{\nu}$ is the outward unit normal to ∂M.

Proof. First we check that $\nabla_{\mathbf{X}}$ is a partial differential operator of order 1 in the commutator sense.
Let $f \in C^\infty(M)$ and $\mathbf{Y} \in \mathfrak{X}(M)$. Using the Leibniz rule for the connection gives

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = \mathbf{X}(f)\,\mathbf{Y} + f\,\nabla_{\mathbf{X}}\mathbf{Y}.$$

Therefore,

$$[\nabla_{\mathbf{X}}, f]\mathbf{Y} = \nabla_{\mathbf{X}}(f\mathbf{Y}) - f\,\nabla_{\mathbf{X}}\mathbf{Y} = \mathbf{X}(f)\mathbf{Y}.$$

This depends only on $\mathbf{Y}$ and not on its derivatives, so the operator $[\nabla_{\mathbf{X}}, f]$ is $C^\infty(M)$-linear. Consequently, $\nabla_{\mathbf{X}} \in \mathbf{PDO}^{(1)}(TM, TM)$.

We now prove the integration by parts identity. Let $\mathbf{Y}, \mathbf{Z} \in \mathfrak{X}_c(M)$ and define

$$f := \mathbf{g}(\mathbf{Y}, \mathbf{Z}), \qquad \mathbf{V} := f\mathbf{X}.$$

By the Leibniz rule for divergence (Proposition 2.16.6),

$$\operatorname{div}(f\mathbf{X}) = \mathbf{X}(f) + f \operatorname{div}\mathbf{X}.$$

Since the Levi–Civita connection is compatible with the metric,

$$\mathbf{X}(f) = \mathbf{X}\big(\mathbf{g}(\mathbf{Y}, \mathbf{Z})\big) = \mathbf{g}(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}) + \mathbf{g}(\mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}).$$

Consequently,

$$\operatorname{div}(f\mathbf{X}) = \mathbf{g}(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}) + \mathbf{g}(\mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}) + (\operatorname{div}\mathbf{X})\mathbf{g}(\mathbf{Y}, \mathbf{Z}).$$

Integrate over M and apply the divergence theorem:

$$\int_M \operatorname{div}(f\mathbf{X})\, d\lambda_{\mathbf{g}} = \int_{\partial M} \mathbf{g}(f\mathbf{X}, \boldsymbol{\nu})\, d\lambda_{\tilde{\mathbf{g}}} = \int_{\partial M} \mathbf{g}(\mathbf{Y}, \mathbf{Z})\, \mathbf{g}(\mathbf{X}, \boldsymbol{\nu})\, d\lambda_{\tilde{\mathbf{g}}}.$$

Rearranging the terms gives (8.10). □

Remark 8.2.9. (Formal adjoint of $\nabla_{\mathbf{X}}$) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{X} \in \mathfrak{X}(M)$. For $\mathbf{Y}, \mathbf{Z} \in \mathfrak{X}_c(M)$, identity (8.10) takes the form

$$\int_M \mathbf{g}(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z})\, d\lambda_{\mathbf{g}} = \int_M \mathbf{g}\big(\mathbf{Y}, -\nabla_{\mathbf{X}}\mathbf{Z} - (\operatorname{div}\mathbf{X})\mathbf{Z}\big)\, d\lambda_{\mathbf{g}}.$$

In this sense, the formal adjoint of the partial differential operator

$$\nabla_{\mathbf{X}} \colon \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

is given by

$$\nabla_{\mathbf{X}}^* \mathbf{Z} = -\nabla_{\mathbf{X}}\mathbf{Z} - (\operatorname{div}\mathbf{X})\mathbf{Z}.$$

This operator should be distinguished from the global formal adjoint

$$\nabla^* \colon \Gamma(T^*M \otimes TM) \longrightarrow \mathfrak{X}(M),$$

corresponding to the covariant derivative viewed as a first-order differential operator between vector bundles. Later we will derive an integration by parts formula allowing us to compute ∇^* explicitly.

Having introduced the formal adjoint of the covariant derivative, we can define a second-order operator fundamental to Riemannian geometry.

Definition 8.2.10. (Bochner Laplacian) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $\pi_{\mathbf{E}} : \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. We define the Bochner Laplacian of a section $\mathbf{u} \in \Gamma(\mathbf{E})$ by

$$\Delta_B \mathbf{u} := \nabla^{\mathbf{E}*} \nabla^{\mathbf{E}} \mathbf{u},$$

where $\nabla^{\mathbf{E}*}$ denotes the formal adjoint of

$$\nabla^{\mathbf{E}} : \Gamma(\mathbf{E}) \longrightarrow \Gamma(T^*M \otimes \mathbf{E}).$$

Remark 8.2.11. The Bochner Laplacian is formally self-adjoint, as shown by the identity

$$(\nabla^{\mathbf{E}*} \nabla^{\mathbf{E}})^* = (\nabla^{\mathbf{E}})^* (\nabla^{\mathbf{E}*})^* = \nabla^{\mathbf{E}*} \nabla^{\mathbf{E}},$$

since $(\nabla^{\mathbf{E}*})^* = \nabla^{\mathbf{E}}$.

Besides the Bochner Laplacian, there are other important Laplacians, such as the Hodge–de Rham Laplacian, defined on spaces of differential forms:

Definition 8.2.12. (Space of differential forms) Let M be a smooth manifold with or without boundary of dimension n. Recall that the space of differential k-forms is defined by

$$\Omega^k(M) := \Gamma\left(\Lambda^k T^*M\right), \qquad k \in \{0, \dots, n\}.$$

We also define the graded space of all forms as the finite direct sum of $C^\infty(M)$-modules

$$\Omega^\bullet(M) := \bigoplus_{k=0}^{n} \Omega^k(M).$$

Now let $(M, \mathbf{g})$ be an oriented Riemannian manifold without boundary. The Riemannian metric induces a pointwise inner product on each exterior bundle $\Lambda^k T^*M$, denoted by $\langle \cdot, \cdot \rangle_{\mathbf{g}}$, and the Riemannian measure $d\lambda_{\mathbf{g}}$ allows us to define the global inner product

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{L^2(M, \Lambda^k T^*M)} := \int_M \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}, \qquad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^k_c(M).$$

Recall that the exterior derivative

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

is a partial differential operator of order 1.

If $f \in C^\infty(M)$ and $\boldsymbol{\alpha} \in \Omega^k(M)$, the Leibniz rule gives

$$d(f\boldsymbol{\alpha}) = df \wedge \boldsymbol{\alpha} + f \, d\boldsymbol{\alpha},$$

and therefore

$$[d, f]\boldsymbol{\alpha} = d(f\boldsymbol{\alpha}) - f \, d\boldsymbol{\alpha} = df \wedge \boldsymbol{\alpha}.$$

This depends only on α and not on its derivatives, so the operator $[d, f]$ is $C^\infty(M)$-linear. Consequently,

$$d \in \mathbf{PDO}^{(1)}\big(\Lambda^k T^*M, \Lambda^{k+1} T^*M\big).$$

Definition 8.2.13. (Codifferential) Let $(M, \mathbf{g})$ be an oriented Riemannian manifold without boundary of dimension n. The formal adjoint of the exterior derivative

$$d \colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

is denoted by

$$\delta \colon \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$$

and is called the codifferential.
In terms of the Hodge star operator $*$, it is given on $\Omega^{k+1}(M)$ by

$$\delta = (-1)^{k+1} \, *^{-1} \, d \, *.$$

This formal adjoint is determined by the rule

$$\int_M \langle d\alpha, \beta \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M \langle \alpha, \delta\beta \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}},$$

for $\alpha \in \Omega^k_c(M)$ and $\beta \in \Omega^{k+1}_c(M)$.

Definition 8.2.14. (Hodge–de Rham Laplacian) Let $(M, \mathbf{g})$ be an oriented Riemannian manifold without boundary. We define the Hodge–de Rham Laplacian to be the second-order operator

$$\Delta := d\delta + \delta d \colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M).$$

Proposition 8.2.15. Let $(M, \mathbf{g})$ be an oriented Riemannian manifold without boundary. The Hodge–de Rham Laplacian Δ is formally self-adjoint.

Proof. We have

$$(d\delta)^* = \delta^* d^* = d\delta, \qquad (\delta d)^* = d^* \delta^* = \delta d,$$

since $\delta = d^*$ and $(d^*)^* = d$. Therefore,

$$\Delta^* = (d\delta + \delta d)^* = d\delta + \delta d = \Delta.$$

□

The classical exterior differential operator extends naturally to the setting of vector bundles.

Let M be a smooth manifold with or without boundary of dimension n. The space of smooth differential k-forms is defined by

$$\Omega^k(M) := \Gamma\Big(\Lambda^k T^*M\Big),$$

that is, the space of smooth sections of the bundle $\Lambda^k T^*M$. The full space of forms is defined as the direct sum of

$C^\infty(M)$-modules

$$\Omega^\bullet(M) := \bigoplus_{k=0}^{n} \Omega^k(M).$$

If $(x^1, \dots, x^n)$ are local coordinates, we use the notation of increasing multi-indices

$$I = (i_1, \dots, i_k), \qquad 1 \le i_1 < \cdots < i_k \le n,$$

and write

$$\mathbf{d}x^I := \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k}.$$

With this notation, a k-form can locally be expressed as

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_I \, \mathbf{d}x^I,$$

and the classical exterior derivative takes the form

$$d\left(\sum_{1 \le i_1 < \cdots < i_k \le n} \omega_I \, \mathbf{d}x^I \right) = \sum_{1 \le i_1 < \cdots < i_k \le n} d\omega_I \wedge \mathbf{d}x^I.$$

Now let $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank r. This allows us to generalize the concept of a differential form by considering forms with values in $\mathbf{E}$.

Definition 8.2.16. (Bundle-valued differential forms) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let $k \ge 0$. Define

$$\Omega^k(M, \mathbf{E}) := \Gamma\left(\Lambda^k T^*M \otimes \mathbf{E}\right).$$

The full space is defined as the direct sum of $C^\infty(M)$-modules

$$\Omega^\bullet(M, \mathbf{E}) := \bigoplus_{k=0}^{n} \Omega^k(M, \mathbf{E}).$$

Observe that if $\mathbf{E} = M \times \mathbb{R}$ is the trivial bundle, the classical case is recovered:

$$\Omega^k(M, \mathbf{E}) \cong \Omega^k(M).$$

To define an exterior derivative in this setting, a connection on the bundle $\mathbf{E}$ is needed.

Definition 8.2.17. (Induced connection) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let $\nabla^{\mathbf{E}}$ be a connection on $\mathbf{E}$. Also fix an auxiliary connection ∇^M on TM, and use the same notation for its induced connections on the tensor bundles constructed from TM and T^*M. The product connection ∇ on $\Lambda^k T^*M \otimes \mathbf{E}$ is defined for each vector field $\mathbf{X} \in \mathfrak{X}(M)$ by the Leibniz rule

$$\nabla_{\mathbf{X}}(\boldsymbol{\alpha} \otimes \mathbf{s}) = \nabla^M_{\mathbf{X}} \boldsymbol{\alpha} \otimes \mathbf{s} + \boldsymbol{\alpha} \otimes \nabla^{\mathbf{E}}_{\mathbf{X}} \mathbf{s},$$

for $\boldsymbol{\alpha} \in \Omega^k(M)$ and $\mathbf{s} \in \Gamma(\mathbf{E})$.

The alternation operator will relate the product connection to the exterior derivative defined next.

Definition 8.2.18. (Alternation operator) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let

$$\mathbf{T} \in \Gamma\left(T^{(0,k+1)}(TM) \otimes \mathbf{E}\right).$$

Define

$$\operatorname{Alt}(\mathbf{T}) \in \Gamma\left(\Lambda^{k+1} T^*M \otimes \mathbf{E}\right)$$

by

$$\operatorname{Alt}(\mathbf{T})(\mathbf{X}_1, \dots, \mathbf{X}_{k+1}) := \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma)\, \mathbf{T}(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k+1)}),$$

for vector fields $\mathbf{X}_1, \dots, \mathbf{X}_{k+1} \in \mathfrak{X}(M)$.

Definition 8.2.19. (Bundle-valued exterior derivative) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle over $\mathbb{K}$, and let $\nabla^{\mathbf{E}}$ be a connection on $\mathbf{E}$. For each $k \geq 0$ define

$$d^{\mathbf{E}} \colon \Omega^k(M, \mathbf{E}) \longrightarrow \Omega^{k+1}(M, \mathbf{E})$$

to be the $\mathbb{K}$-linear operator determined locally by the rule

$$d^{\mathbf{E}}(\boldsymbol{\alpha} \otimes \mathbf{s}) = d\boldsymbol{\alpha} \otimes \mathbf{s} + (-1)^k \boldsymbol{\alpha} \wedge \nabla^{\mathbf{E}} \mathbf{s},$$

for every smooth scalar k-form $\boldsymbol{\alpha}$ and every smooth section $\mathbf{s}$ of $\mathbf{E}$ defined on the same open set. Here the wedge product with an $\mathbf{E}$-valued form is obtained by linearity from

$$\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \otimes \mathbf{s}) = (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \otimes \mathbf{s}.$$

If $\mathbb{K} = \mathbb{C}$, scalar forms may be complex and d extends by $\mathbb{C}$-linearity. In degree zero, we recover the connection, $d^{\mathbf{E}}\mathbf{s} = \nabla^{\mathbf{E}}\mathbf{s}$, and in degree $n = \dim M$ the operator takes values in $\Omega^{n+1}(M, \mathbf{E}) = \{0\}$.

We check that this rule determines a global operator without choosing a connection on TM. For a smooth scalar function f, the Leibniz rules for d and $\nabla^{\mathbf{E}}$ give

$$d(f\boldsymbol{\alpha}) \otimes \mathbf{s} + (-1)^k (f\boldsymbol{\alpha}) \wedge \nabla^{\mathbf{E}} \mathbf{s} = f\, d\boldsymbol{\alpha} \otimes \mathbf{s} + (df \wedge \boldsymbol{\alpha}) \otimes \mathbf{s} + (-1)^k f \boldsymbol{\alpha} \wedge \nabla^{\mathbf{E}} \mathbf{s},$$

$$d\boldsymbol{\alpha} \otimes (f\mathbf{s}) + (-1)^k \boldsymbol{\alpha} \wedge \nabla^{\mathbf{E}}(f\mathbf{s}) = f\, d\boldsymbol{\alpha} \otimes \mathbf{s} + (-1)^k (\boldsymbol{\alpha} \wedge df) \otimes \mathbf{s} + (-1)^k f \boldsymbol{\alpha} \wedge \nabla^{\mathbf{E}} \mathbf{s}.$$

Since $df \wedge \boldsymbol{\alpha} = (-1)^k \boldsymbol{\alpha} \wedge df$, the two expressions agree. The rule therefore respects the relation $(f\boldsymbol{\alpha}) \otimes \mathbf{s} = \boldsymbol{\alpha} \otimes (f\mathbf{s})$.

On an open set U with a frame $(\mathbf{e}_a)_{a=1}^r$, every form $\boldsymbol{\omega} \in \Omega^k(U, \mathbf{E})$ has a unique expression $\boldsymbol{\omega} = \sum_{a=1}^{r} \boldsymbol{\alpha}^a \otimes \mathbf{e}_a$. Define its derivative by applying the rule to each summand. If $\mathbf{s} = \sum_{a=1}^{r} f^a \mathbf{e}_a$, the preceding identity and additivity show that the operator thus defined satisfies the same rule on $\boldsymbol{\alpha} \otimes \mathbf{s}$. When we change to another frame, each of its elements is a smooth combination of the $\mathbf{e}_a$; applying that rule to each term of the new representation therefore gives the same result. The construction also commutes with restrictions, because d and $\nabla^{\mathbf{E}}$ do. The local operators consequently agree on intersections and define a unique global smooth form $d^{\mathbf{E}}\boldsymbol{\omega}$. This construction proves the

stated existence, uniqueness, and $\mathbb{K}$-linearity, without assuming that the bundle admits a global frame.

In local coordinates, if $(\mathbf{e}_a)_{a=1}^r$ is a frame for $\mathbf{E}$ and

$$\boldsymbol{\omega} = \sum_{1\le i_1<\cdots<i_k\le n} \sum_{a=1}^{r} \omega_I^a \, \mathbf{d}x^I \otimes \mathbf{e}_a,$$

then

$$d^{\mathbf{E}}\boldsymbol{\omega} = \sum_{1\le i_1<\cdots<i_k\le n} \sum_{a=1}^{r} \left(d\omega_I^a + \sum_{b=1}^{r} \theta_b^a \, \omega_I^b \right) \wedge \mathbf{d}x^I \otimes \mathbf{e}_a,$$

where the connection 1-forms are given by

$$\nabla^{\mathbf{E}} \mathbf{e}_b = \sum_{a=1}^{r} \theta_b^a \otimes \mathbf{e}_a.$$

Indeed, $d(\mathbf{d}x^I) = 0$ and $(-1)^k \mathbf{d}x^I \wedge \theta_b^a = \theta_b^a \wedge \mathbf{d}x^I$; applying the definition to each term of $\boldsymbol{\omega}$ gives exactly this expression.

We now relate this definition to alternation. If the auxiliary connection ∇^M is torsion-free and ∇ is the product connection of Definition 8.2.17, then

$$d^{\mathbf{E}}\boldsymbol{\omega} = (k+1)\,\mathrm{Alt}(\nabla\boldsymbol{\omega}).$$

To prove this, first observe that $\nabla\boldsymbol{\omega}$ is alternating in its last k entries. In the definition of Alt, grouping permutations by the first entry leaves $k!$ equal summands in each group. Hence,

$$(k+1)\,\mathrm{Alt}(\nabla\boldsymbol{\omega})(\mathbf{X}_0,\dots,\mathbf{X}_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{\mathbf{X}_i}\boldsymbol{\omega})(\mathbf{X}_0,\dots,\widehat{\mathbf{X}_i},\dots,\mathbf{X}_k),$$

where the hat indicates omission of that entry.

For a scalar form $\boldsymbol{\alpha}$, this sum is $d\boldsymbol{\alpha}$ when ∇^M is torsion-free. We make the cancellation justifying this explicit. In coordinates, write $\nabla^M_{\partial_\ell}\partial_i = \sum_{j=1}^{n} \Gamma_{\ell i}^j \partial_j$; since $[\partial_\ell, \partial_i] = 0$, vanishing torsion gives $\Gamma_{\ell i}^j = \Gamma_{i\ell}^j$. The dual connection satisfies

$$\nabla^M_{\partial_\ell} \mathbf{d}x^j = -\sum_{i=1}^{n} \Gamma_{\ell i}^j \mathbf{d}x^i.$$

Upon differentiating $\boldsymbol{\alpha} = \sum_{I\in\mathcal{J}_{n,k}} \alpha_I \mathbf{d}x^I$, where $\mathcal{J}_{n,k} := \{(i_1,\dots,i_k) \in \{1,\dots,n\}^k \mid i_1 < \cdots < i_k\}$, the terms differentiating the coefficients produce $\sum_{I\in\mathcal{J}_{n,k}} d\alpha_I \wedge \mathbf{d}x^I$. In the terms differentiating one of the factors $\mathbf{d}x^j$, alternation combines, up to the sign of the position of that factor, the sum

$$\sum_{\ell,i=1}^{n} \Gamma_{\ell i}^j \, \mathbf{d}x^\ell \wedge \mathbf{d}x^i = \frac{1}{2} \sum_{\ell,i=1}^{n} (\Gamma_{\ell i}^j - \Gamma_{i\ell}^j)\, \mathbf{d}x^\ell \wedge \mathbf{d}x^i = 0.$$

Therefore,

$$(k+1)\,\mathrm{Alt}(\nabla^M\boldsymbol{\alpha}) = \sum_{I\in\mathcal{J}_{n,k}} d\alpha_I \wedge \mathbf{d}x^I = d\boldsymbol{\alpha}.$$

In degree zero, there are no covector factors to differentiate, and the same identity is simply $\nabla^M f = df$.

Now apply the product connection to $\alpha \otimes \mathbf{s}$, and write locally $\nabla^{\mathbf{E}}\mathbf{s} = \sum_{a=1}^{r} \eta^a \otimes \mathbf{e}_a$. By the preceding argument, the terms differentiating α give $d\alpha \otimes \mathbf{s}$. By the alternating sum formula, those differentiating $\mathbf{s}$ give

$$\sum_{a=1}^{r} (\eta^a \wedge \alpha) \otimes \mathbf{e}_a = (-1)^k \alpha \wedge \nabla^{\mathbf{E}}\mathbf{s}.$$

Consequently,

$$(k+1)\operatorname{Alt}\big(\nabla(\alpha \otimes \mathbf{s})\big) = d\alpha \otimes \mathbf{s} + (-1)^k \alpha \wedge \nabla^{\mathbf{E}}\mathbf{s} = d^{\mathbf{E}}(\alpha \otimes \mathbf{s}).$$

Every $\mathbf{E}$-valued form is locally a finite sum of these tensors, so the identity holds for any $\omega \in \Omega^k(M, \mathbf{E})$. In particular, the result of alternation does not depend on the chosen auxiliary torsion-free connection. For example, one may take the Levi–Civita connection of an auxiliary Riemannian metric, but that metric is not part of the data defining $d^{\mathbf{E}}$. The vanishing torsion condition is imposed only on ∇^M, not on the bundle connection $\nabla^{\mathbf{E}}$.

Proposition 8.2.20. Let M be a smooth manifold with or without boundary and let $\mathbf{E} \to M$ be a smooth vector bundle with a connection. The operator

$$d^{\mathbf{E}} \colon \Omega^k(M, \mathbf{E}) \longrightarrow \Omega^{k+1}(M, \mathbf{E})$$

is a partial differential operator of order at most 1.

Proof. The rule defining $d^{\mathbf{E}}$ and the Leibniz rule for d imply, first on simple tensors and then by linearity, that

$$d^{\mathbf{E}}(f\omega) = df \wedge \omega + f\, d^{\mathbf{E}}\omega.$$

Thus, $[d^{\mathbf{E}}, f](\omega) = df \wedge \omega$. This commutator is $C^\infty(M)$-linear and hence has order 0. The recursive definition of a differential operator shows that $d^{\mathbf{E}} \in \mathbf{PDO}^{(1)}$, as desired. □

The composition of $d^{\mathbf{E}}$ with itself is controlled by the curvature of the connection.

Definition 8.2.21. (Curvature of a connection) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let $\nabla^{\mathbf{E}}$ be a connection on $\mathbf{E}$. We define its curvature to be the tensor field

$$\mathbf{R}^{\mathbf{E}} \in \Gamma\big(\Lambda^2 T^*M \otimes \operatorname{End}(\mathbf{E})\big)$$

given by

$$\mathbf{R}^{\mathbf{E}}(\mathbf{X}, \mathbf{Y}) := \nabla^{\mathbf{E}}_{\mathbf{X}} \nabla^{\mathbf{E}}_{\mathbf{Y}} - \nabla^{\mathbf{E}}_{\mathbf{Y}} \nabla^{\mathbf{E}}_{\mathbf{X}} - \nabla^{\mathbf{E}}_{[\mathbf{X},\mathbf{Y}]},$$

for vector fields $\mathbf{X}, \mathbf{Y}$.

Lemma 8.2.22. (Covariant derivative of curvature) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. The product connection on $\Lambda^2 T^*M \otimes \operatorname{End}(\mathbf{E})$ determines the covariant derivative of $\mathbf{R}^{\mathbf{E}}$. For any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(M)$,

$$(\nabla_{\mathbf{Z}} \mathbf{R}^{\mathbf{E}})(\mathbf{X}, \mathbf{Y}) = \nabla^{\operatorname{End}(\mathbf{E})}_{\mathbf{Z}}\big(\mathbf{R}^{\mathbf{E}}(\mathbf{X}, \mathbf{Y})\big) - \mathbf{R}^{\mathbf{E}}(\nabla^M_{\mathbf{Z}}\mathbf{X}, \mathbf{Y}) - \mathbf{R}^{\mathbf{E}}(\mathbf{X}, \nabla^M_{\mathbf{Z}}\mathbf{Y}). \tag{8.11}$$

Consequently, for every section $\mathbf{v} \in \Gamma(\mathbf{E})$,

$$\begin{aligned}\nabla^{\mathbf{E}}_{\mathbf{Z}}\big(\mathbf{R}^{\mathbf{E}}(\mathbf{X},\mathbf{Y})\mathbf{v}\big) &= (\nabla_{\mathbf{Z}}\mathbf{R}^{\mathbf{E}})(\mathbf{X},\mathbf{Y})\mathbf{v} + \mathbf{R}^{\mathbf{E}}(\nabla^{M}_{\mathbf{Z}}\mathbf{X},\mathbf{Y})\mathbf{v} \\ &\quad + \mathbf{R}^{\mathbf{E}}(\mathbf{X},\nabla^{M}_{\mathbf{Z}}\mathbf{Y})\mathbf{v} + \mathbf{R}^{\mathbf{E}}(\mathbf{X},\mathbf{Y})\nabla^{\mathbf{E}}_{\mathbf{Z}}\mathbf{v}.\end{aligned} \tag{8.12}$$

Proof. The first identity is the formula for the induced connection on an 2-covariant tensor with values in $\operatorname{End}(\mathbf{E})$. For the second, apply the Leibniz rule to the parallel evaluation

$$\operatorname{End}(\mathbf{E}) \otimes \mathbf{E} \longrightarrow \mathbf{E}, \qquad (A,\mathbf{v}) \longmapsto A\mathbf{v},$$

and then substitute (8.11). □

Proposition 8.2.23. (Curvature of tensor bundles and the Ricci identity) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary with Levi–Civita connection ∇^{M}, and let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. For $q \in \mathbb{N}_0$, set

$$\mathbf{F}_q := T^{(0,q)}(TM) \otimes \mathbf{E}$$

and equip $\mathbf{F}_q$ with the induced product connection. If $\mathbf{T} \in \Gamma(\mathbf{F}_q)$, then, for any $\mathbf{X},\mathbf{Y},\mathbf{Z}_1,\dots,\mathbf{Z}_q \in \mathfrak{X}(M)$,

$$\begin{aligned}\big(\mathbf{R}^{\mathbf{F}_q}(\mathbf{X},\mathbf{Y})\mathbf{T}\big)(\mathbf{Z}_1,\dots,\mathbf{Z}_q) &= \mathbf{R}^{\mathbf{E}}(\mathbf{X},\mathbf{Y})\big(\mathbf{T}(\mathbf{Z}_1,\dots,\mathbf{Z}_q)\big) \\ &\quad - \sum_{a=1}^{q} \mathbf{T}(\mathbf{Z}_1,\dots,\mathbf{R}^{M}(\mathbf{X},\mathbf{Y})\mathbf{Z}_a,\dots,\mathbf{Z}_q).\end{aligned} \tag{8.13}$$

When $q = 0$, the sum is understood to be empty and $\mathbf{R}^{\mathbf{F}_0} = \mathbf{R}^{\mathbf{E}}$.
Moreover, the first two entries of the second covariant derivative satisfy the Ricci identity

$$\begin{aligned}&(\nabla^2\mathbf{T})(\mathbf{X},\mathbf{Y},\mathbf{Z}_1,\dots,\mathbf{Z}_q) - (\nabla^2\mathbf{T})(\mathbf{Y},\mathbf{X},\mathbf{Z}_1,\dots,\mathbf{Z}_q) \\ &\quad = \big(\mathbf{R}^{\mathbf{F}_q}(\mathbf{X},\mathbf{Y})\mathbf{T}\big)(\mathbf{Z}_1,\dots,\mathbf{Z}_q).\end{aligned} \tag{8.14}$$

In particular, if $\mathbf{A} \in \Gamma(T^*M \otimes \mathbf{E})$, then

$$\big(\mathbf{R}^{T^*M\otimes\mathbf{E}}(\mathbf{X},\mathbf{Y})\mathbf{A}\big)(\mathbf{Z}) = \mathbf{R}^{\mathbf{E}}(\mathbf{X},\mathbf{Y})\mathbf{A}(\mathbf{Z}) - \mathbf{A}\big(\mathbf{R}^{M}(\mathbf{X},\mathbf{Y})\mathbf{Z}\big). \tag{8.15}$$

Proof. We begin with the curvature of a tensor product. If $\mathbf{G},\mathbf{H} \to M$ are bundles with connections and $\mathbf{s} \in \Gamma(\mathbf{G})$, $\mathbf{t} \in \Gamma(\mathbf{H})$, applying the Leibniz rule twice to the definition of curvature gives

$$\mathbf{R}^{\mathbf{G}\otimes\mathbf{H}}(\mathbf{X},\mathbf{Y})(\mathbf{s}\otimes\mathbf{t}) = \mathbf{R}^{\mathbf{G}}(\mathbf{X},\mathbf{Y})\mathbf{s}\otimes\mathbf{t} + \mathbf{s}\otimes\mathbf{R}^{\mathbf{H}}(\mathbf{X},\mathbf{Y})\mathbf{t}.$$

The mixed terms cancel in pairs. Since simple tensor sections locally generate $\Gamma(\mathbf{G}\otimes\mathbf{H})$, the identity holds for every section.
For the induced connection on the dual, the definition and the Leibniz rule for evaluation imply

$$\big(\mathbf{R}^{\mathbf{G}^*}(\mathbf{X},\mathbf{Y})\boldsymbol{\alpha}\big)(\mathbf{s}) = -\boldsymbol{\alpha}\big(\mathbf{R}^{\mathbf{G}}(\mathbf{X},\mathbf{Y})\mathbf{s}\big).$$

Apply these two formulas to each factor of $T^{(0,q)}(TM) = (T^*M)^{\otimes q}$. The curvature $\mathbf{R}^{\mathbf{E}}$ acts on the factor $\mathbf{E}$,

whereas the curvature of each factor T^*M acts with a negative sign on the corresponding entry. This gives exactly (8.13).

We now prove (8.14). By the formula for the second covariant derivative in Proposition 2.4.29, now applied to the section $\mathbf{T}$ of the bundle $\mathbf{F}_q$, we have

$$(\nabla^2\mathbf{T})(\mathbf{X},\mathbf{Y}) = \nabla^{\mathbf{F}_q}_{\mathbf{X}}\nabla^{\mathbf{F}_q}_{\mathbf{Y}}\mathbf{T} - \nabla^{\mathbf{F}_q}_{\nabla^M_{\mathbf{X}}\mathbf{Y}}\mathbf{T}.$$

Therefore,

$$\begin{aligned}(\nabla^2\mathbf{T})(\mathbf{X},\mathbf{Y}) - (\nabla^2\mathbf{T})(\mathbf{Y},\mathbf{X}) = {}& \nabla^{\mathbf{F}_q}_{\mathbf{X}}\nabla^{\mathbf{F}_q}_{\mathbf{Y}}\mathbf{T} \\ & - \nabla^{\mathbf{F}_q}_{\mathbf{Y}}\nabla^{\mathbf{F}_q}_{\mathbf{X}}\mathbf{T} \\ & - \nabla^{\mathbf{F}_q}_{\nabla^M_{\mathbf{X}}\mathbf{Y}-\nabla^M_{\mathbf{Y}}\mathbf{X}}\mathbf{T}.\end{aligned}$$

The Levi–Civita connection is torsion-free, so $\nabla^M_{\mathbf{X}}\mathbf{Y} - \nabla^M_{\mathbf{Y}}\mathbf{X} = [\mathbf{X},\mathbf{Y}]$. Consequently, the right-hand side is

$$\nabla^{\mathbf{F}_q}_{\mathbf{X}}\nabla^{\mathbf{F}_q}_{\mathbf{Y}}\mathbf{T} - \nabla^{\mathbf{F}_q}_{\mathbf{Y}}\nabla^{\mathbf{F}_q}_{\mathbf{X}}\mathbf{T} - \nabla^{\mathbf{F}_q}_{[\mathbf{X},\mathbf{Y}]}\mathbf{T} = \mathbf{R}^{\mathbf{F}_q}(\mathbf{X},\mathbf{Y})\mathbf{T}.$$

Evaluating this equality on $\mathbf{Z}_1,\dots,\mathbf{Z}_q$ proves (8.14). Finally, (8.15) is the case $q = 1$ of (8.13). □

Proposition 8.2.24. (Skew-adjointness of the curvature of a metric connection) Let M be a smooth manifold with or without boundary, let $(\mathbf{E},\mathbf{h}_\mathbf{E}) \to M$ be a vector bundle with a bundle metric, and let $\nabla^\mathbf{E}$ be a connection compatible with $\mathbf{h}_\mathbf{E}$. Then, for any $\mathbf{X},\mathbf{Y} \in \mathfrak{X}(M)$ and $\mathbf{u},\mathbf{v} \in \Gamma(\mathbf{E})$,

$$\Big\langle \mathbf{R}^\mathbf{E}(\mathbf{X},\mathbf{Y})\mathbf{u},\mathbf{v}\Big\rangle_{\mathbf{h}_\mathbf{E}} + \Big\langle \mathbf{u},\mathbf{R}^\mathbf{E}(\mathbf{X},\mathbf{Y})\mathbf{v}\Big\rangle_{\mathbf{h}_\mathbf{E}} = 0. \tag{8.16}$$

In particular, $\mathbf{R}^\mathbf{E}(\mathbf{X},\mathbf{Y})\colon \mathbf{E}_x \longrightarrow \mathbf{E}_x$ is skew-adjoint on each fiber.

Proof. Metric compatibility gives

$$\mathbf{X}\langle \mathbf{u},\mathbf{v}\rangle_{\mathbf{h}_\mathbf{E}} = \langle \nabla^\mathbf{E}_\mathbf{X}\mathbf{u},\mathbf{v}\rangle_{\mathbf{h}_\mathbf{E}} + \langle \mathbf{u},\nabla^\mathbf{E}_\mathbf{X}\mathbf{v}\rangle_{\mathbf{h}_\mathbf{E}}.$$

Apply this identity successively with $\mathbf{X}$ and $\mathbf{Y}$, subtract the two resulting expressions, and subtract the derivative in the direction $[\mathbf{X},\mathbf{Y}]$. The mixed terms cancel, and the left-hand side is

$$\big(\mathbf{XY} - \mathbf{YX} - [\mathbf{X},\mathbf{Y}]\big)\langle \mathbf{u},\mathbf{v}\rangle_{\mathbf{h}_\mathbf{E}} = 0.$$

The right-hand side is precisely

$$\langle \mathbf{R}^\mathbf{E}(\mathbf{X},\mathbf{Y})\mathbf{u},\mathbf{v}\rangle_{\mathbf{h}_\mathbf{E}} + \langle \mathbf{u},\mathbf{R}^\mathbf{E}(\mathbf{X},\mathbf{Y})\mathbf{v}\rangle_{\mathbf{h}_\mathbf{E}}.$$

This proves (8.16). □

The expression for curvature in a synchronous frame admits an integral formula that will be useful when studying bundles over manifolds of bounded geometry.

Proposition 8.2.25. (Curvature formula in a synchronous frame) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $p \in M$, let U be a normal neighborhood of p, and let $\mathbf{e} = (\mathbf{e}_1,\dots,\mathbf{e}_r)$ be a synchronous frame centered at p. In normal coordinates $x = (x^1,\dots,x^n)$, write

$$\nabla^{\mathbf{E}} = d + \mathbf{A}, \qquad \mathbf{A} = \sum_{\mu=1}^{n} A_\mu \, \mathbf{d}x^\mu,$$

and represent the curvature by

$$\mathbf{R}^{\mathbf{E}} = \frac{1}{2} \sum_{\nu,\mu=1}^{n} F_{\nu\mu} \, \mathbf{d}x^\nu \wedge \mathbf{d}x^\mu.$$

Then, for each $\mu \in \{1,\dots,n\}$ and each x in the chart domain,

$$A_\mu(x) = \int_0^1 t \sum_{\nu=1}^{n} x^\nu F_{\nu\mu}(tx)\, dt. \tag{8.17}$$

Proof. If $\mathbf{u} = \sum_{a=1}^{r} u^a \mathbf{e}_a$ and $\widehat{u} = (u^1,\dots,u^r)^T$ is its column of components, then

$$(\nabla^{\mathbf{E}}_{\partial_\mu} \mathbf{u})^\wedge = \partial_\mu \widehat{u} + A_\mu \widehat{u}.$$

Expanding the definition of $\mathbf{R}^{\mathbf{E}}$ on coordinate vector fields gives

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\nu, A_\mu]. \tag{8.18}$$

By Proposition 2.7.17, the frame satisfies the radial condition

$$\sum_{\nu=1}^{n} x^\nu A_\nu(x) = 0.$$

Differentiating it with respect to x^μ gives

$$\sum_{\nu=1}^{n} x^\nu \partial_\mu A_\nu(x) = -A_\mu(x),$$

and the same condition makes $\sum_{\nu=1}^{n} x^\nu [A_\nu, A_\mu]$ vanish. Contracting (8.18) with the radial vector field gives

$$\sum_{\nu=1}^{n} x^\nu F_{\nu\mu}(x) = \sum_{\nu=1}^{n} x^\nu \partial_\nu A_\mu(x) + A_\mu(x).$$

Applied at the point tx, this identity is equivalent to

$$\frac{d}{dt}\big[t A_\mu(tx)\big] = t \sum_{\nu=1}^{n} x^\nu F_{\nu\mu}(tx).$$

Since $tA_\mu(tx)$ tends to zero as $t \to 0^+$, integration from 0 to 1 proves (8.17). □

Proposition 8.2.26. (Uniform bounds for families of synchronous frames) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary. Let $r_0 > 0$ and let

$$(\widehat{U}_i, \varphi_i, p_i)_{i \in I}$$

be a family of normal neighborhoods centered at p_i such that $\varphi_i(\widehat{U}_i) = B_{\mathrm{euc}}(0, 3r_0)$. Set

$$U_i := \varphi_i^{-1}\big(B_{\mathrm{euc}}(0, 2r_0)\big).$$

Suppose that the metric matrices in these charts are uniformly elliptic and that, for every multi-index α, their derivatives of order $|\alpha|$ are bounded by a constant independent of i. Also suppose that all derivatives of the coordinate changes $\varphi_j \circ \varphi_i^{-1}$ and their inverses are uniformly bounded on overlaps.
Let $\mathbf{E} \to M$ be a vector bundle of rank $r_{\mathbf{E}}$, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. On $\Lambda^2 T^*M \otimes \mathrm{End}(\mathbf{E})$ we use the product connection induced by the Levi–Civita connection and $\nabla^{\mathbf{E}}$. Suppose that, for each $k \in \mathbb{N}_0$, there exists $C_k^{\mathbf{E}} > 0$ such that

$$\sup_{i \in I} \sup_{x \in \widehat{U}_i} \Big| \big(\nabla^{\Lambda^2 T^*M \otimes \mathrm{End}(\mathbf{E})}\big)^k \mathbf{R}^{\mathbf{E}}(x) \Big|_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} \leq C_k^{\mathbf{E}}.$$

On each $\widehat{U}_i$, construct a synchronous frame $\mathbf{e}_i = (\mathbf{e}_{i,1}, \dots, \mathbf{e}_{i,r_{\mathbf{E}}})$ from an orthonormal basis of $\mathbf{E}_{p_i}$ and write, in coordinates φ_i,

$$\nabla^{\mathbf{E}} = d + \mathbf{A}_i, \qquad \mathbf{A}_i = \sum_{\mu=1}^{n} A_{i,\mu}\, \mathbf{d}x^\mu.$$

Then, for every multi-index α, there exists $C_\alpha > 0$, independent of i, such that

$$\sup_{x \in B_{\mathrm{euc}}(0, 2r_0)} \big\| D^\alpha A_{i,\mu}(x) \big\|_{\mathrm{op}, \mathbb{K}^{r_{\mathbf{E}}}} \leq C_\alpha, \qquad \mu \in \{1, \dots, n\}. \tag{8.19}$$

where the Euclidean operator norm is used on the matrices written in the orthonormal synchronous frame. If G_{ij} is the transition matrix determined on $U_i \cap U_j$ by $u_i = G_{ij} u_j$, then G_{ij}, G_{ij}^{-1}, and all their derivatives, expressed in either of the two charts, satisfy bounds independent of i and j.

Proof. The frames $\mathbf{e}_i$ are smooth and orthonormal by Proposition 2.7.17. Represent the curvature in the frame $\mathbf{e}_i$ by

$$\mathbf{R}^{\mathbf{E}} = \frac{1}{2} \sum_{\nu, \mu = 1}^{n} F_{i, \nu\mu}\, \mathbf{d}x^\nu \wedge \mathbf{d}x^\mu.$$

Formula (8.17) applies in each chart and gives

$$A_{i,\mu}(x) = \int_0^1 t \sum_{\nu=1}^{n} x^\nu F_{i,\nu\mu}(tx)\, dt. \tag{8.20}$$

Uniform equivalence between $\mathbf{g}$ and the Euclidean metric, together with the bound on $\mathbf{R}^{\mathbf{E}}$, controls the components $F_{i,\nu\mu}$ at order zero. The preceding formula then provides the uniform bound for $\mathbf{A}_i$ at order

zero.

To obtain the higher orders, let $\mathbf{T}$ be a tensor with values in $\operatorname{End}(\mathbf{E})$. In coordinates φ_i and the frame $\mathbf{e}_i$, the product connection satisfies schematically

$$\partial \mathbf{T} = \nabla \mathbf{T} + \Gamma_i * \mathbf{T} + \mathbf{A}_i * \mathbf{T} - \mathbf{T} * \mathbf{A}_i, \tag{8.21}$$

where $*$ represents a finite sum of tensor products and contractions. The hypotheses on the metric uniformly bound the Christoffel symbols and all their derivatives. Suppose the ordinary derivatives of the matrices $F_{i,\nu\mu}$ and $A_{i,\mu}$ are controlled up to order $k-1$. Differentiating (8.21) and using the Leibniz rule expresses the derivatives of order k of $F_{i,\nu\mu}$ in terms of the components of $(\nabla^{\mathbf{E}})^j \mathbf{R}^{\mathbf{E}}$, with $j \le k$, and quantities already controlled. Differentiating (8.20) under the integral sign yields the order k bounds for $A_{i,\mu}$. Induction proves (8.19).

On an overlap, set $y = \varphi_j \circ \varphi_i^{-1}(x)$. Compatibility of the two local expressions for the connection and the equality $u_i = G_{ij} u_j$ imply

$$\partial_{x^\mu} G_{ij}(x) = \sum_{\nu=1}^{n} G_{ij}(x) A_{j,\nu}(y) \frac{\partial y^\nu}{\partial x^\mu}(x) - A_{i,\mu}(x) G_{ij}(x). \tag{8.22}$$

The matrices G_{ij} are orthogonal or unitary, so $\|G_{ij}\|_{\mathrm{op},\mathbb{K}^{r_{\mathbf{E}}}} = \|G_{ij}^{-1}\|_{\mathrm{op},\mathbb{K}^{r_{\mathbf{E}}}} = 1$. The preceding equation and the bounds already proved control the first derivative. After applying D^α to (8.22), the Leibniz rule expresses the derivatives of order $|\alpha|+1$ of G_{ij} in terms of derivatives of order at most $|\alpha|$ of G_{ij} and uniformly bounded quantities. An induction completes the estimate for all derivatives. For the inverse, use $G_{ij}^{-1} = G_{ij}^{*}$. □

Curvature acts naturally on bundle-valued forms through the exterior product in the differential-form factor and composition in the bundle factor.

Proposition 8.2.27. Let M be a smooth manifold with or without boundary and let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. For every $\omega \in \Omega^k(M, \mathbf{E})$ we have

$$(d^{\mathbf{E}})^2 \omega = \mathbf{R}^{\mathbf{E}} \wedge \omega.$$

In particular, $(d^{\mathbf{E}})^2 = 0$ as a graded operator on $\Omega^\bullet(M, \mathbf{E})$ if and only if the connection is flat.

Proof. The identity is local. Fix a frame $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_r)$ for $\mathbf{E}$ and write

$$\nabla^{\mathbf{E}} = d + \mathbf{A}, \qquad \mathbf{A} = (\theta^a{}_b)_{a,b=1}^{r}.$$

A form $\omega \in \Omega^k(M, \mathbf{E})$ is represented in this frame by the column $\widehat{\omega} = (\omega^1, \dots, \omega^r)^T$ of scalar k-forms. The local formula for $d^{\mathbf{E}}$ is equivalent to

$$\widehat{d^{\mathbf{E}} \omega} = d\widehat{\omega} + \mathbf{A} \wedge \widehat{\omega}.$$

Apply $d^{\mathbf{E}}$ again. Since $\mathbf{A}$ has degree one, the graded Leibniz rule gives

$$\begin{aligned}\widehat{(d^{\mathbf{E}})^2\omega} &= d\big(d\widehat{\omega} + \mathbf{A}\wedge\widehat{\omega}\big) + \mathbf{A}\wedge\big(d\widehat{\omega} + \mathbf{A}\wedge\widehat{\omega}\big)\\ &= d\mathbf{A}\wedge\widehat{\omega} - \mathbf{A}\wedge d\widehat{\omega} + \mathbf{A}\wedge d\widehat{\omega} + \mathbf{A}\wedge\mathbf{A}\wedge\widehat{\omega}\\ &= \big(d\mathbf{A} + \mathbf{A}\wedge\mathbf{A}\big)\wedge\widehat{\omega}.\end{aligned}$$

By the local curvature formula,

$$\widehat{\mathbf{R}^{\mathbf{E}}} = d\mathbf{A} + \mathbf{A}\wedge\mathbf{A};$$

in components, this equality is exactly

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\nu, A_\mu].$$

Therefore,

$$(d^{\mathbf{E}})^2\omega = \mathbf{R}^{\mathbf{E}}\wedge\omega.$$

If the connection is flat, the right-hand side vanishes in every degree. Conversely, suppose that $(d^{\mathbf{E}})^2 = 0$ as a graded operator. In particular, it vanishes on $\Omega^0(M,\mathbf{E}) = \Gamma(\mathbf{E})$. Given $p\in M$, $v\in\mathbf{E}_p$, and $X, Y\in T_pM$, choose a smooth section $\mathbf{s}$ with $\mathbf{s}(p) = v$ and smooth extensions of X, Y. Then

$$0 = \big((d^{\mathbf{E}})^2\mathbf{s}\big)_p(X,Y) = \mathbf{R}^{\mathbf{E}}_p(X,Y)v.$$

Since p, v, X, Y are arbitrary, $\mathbf{R}^{\mathbf{E}} = 0$; that is, the connection is flat. □

This construction generalizes the Hodge–de Rham exterior operator to bundle-valued forms. It defines a complex only when the connection is flat; if $\mathbf{R}^{\mathbf{E}}\neq 0$, the preceding identity measures exactly the defect. Later, introducing the formal adjoint of $d^{\mathbf{E}}$ will lead to the $\mathbf{E}$-valued Hodge Laplacian, which is related to the Bochner Laplacian through Weitzenböck-type identities.

Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary and let $\pi_{\mathbf{E}}\colon\mathbf{E}\longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Recall that we have already defined the operator

$$d^{\mathbf{E}}\colon\Omega^k(M,\mathbf{E})\longrightarrow\Omega^{k+1}(M,\mathbf{E}).$$

The Riemannian metric and bundle metric naturally induce a pointwise inner product on $\Lambda^k T^*M\otimes\mathbf{E}$. Integrating with respect to the Riemannian measure gives a global inner product

$$\langle\alpha,\beta\rangle_{L^2(M,\Lambda^kT^*M\otimes\mathbf{E})} := \int_M \langle\alpha,\beta\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}}, \qquad \alpha,\beta\in\Omega^k_c(M,\mathbf{E}).$$

Definition 8.2.28. (Formal adjoint of $d^{\mathbf{E}}$) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E}\to M$ be a vector bundle with a bundle metric and a compatible connection. We define the formal adjoint of

$$d^{\mathbf{E}}\colon\Omega^k(M,\mathbf{E})\longrightarrow\Omega^{k+1}(M,\mathbf{E})$$

to be the operator

$$\delta^{\mathbf{E}}\colon\Omega^{k+1}(M,\mathbf{E})\longrightarrow\Omega^k(M,\mathbf{E})$$

characterized by the identity

$$\int_M \langle d^{\mathbf{E}}\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} = \int_M \langle \boldsymbol{\alpha}, \delta^{\mathbf{E}}\boldsymbol{\beta}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}}$$

for all

$$\boldsymbol{\alpha} \in \Omega_c^k(M,\mathbf{E}), \qquad \boldsymbol{\beta} \in \Omega_c^{k+1}(M,\mathbf{E}).$$

Observe that this operator exists and is unique by Theorem 8.2.3.

If M is also oriented, the formal adjoint can be expressed through the Hodge operator. We briefly recall this construction.

Definition 8.2.29. (Hodge operator) Let $(M,\mathbf{g})$ be an oriented Riemannian manifold with or without boundary and let $n = \dim M$. The Hodge operator is the bundle isomorphism

$$*\colon \Lambda^k T^*M \longrightarrow \Lambda^{n-k} T^*M$$

characterized by the identity

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle_{\mathbf{g}}\, dV_{\mathbf{g}},$$

for all $\alpha, \beta \in \Lambda^k T_x^*M$, where $dV_{\mathbf{g}}$ denotes the Riemannian volume form. This operator extends naturally to $\mathbf{E}$-valued forms by

$$*(\alpha \otimes s) := (*\alpha) \otimes s.$$

With this notation, we obtain the classical expression for the formal adjoint.

Proposition 8.2.30. Let $(M,\mathbf{g})$ be an oriented Riemannian manifold without boundary and let $\mathbf{E} \to M$ be a vector bundle with a bundle metric and a compatible connection. Then

$$\delta^{\mathbf{E}} = (-1)^{k+1}\, *^{-1}\, d^{\mathbf{E}} *$$

on $\Omega^{k+1}(M,\mathbf{E})$.

Proof. Let $\boldsymbol{\alpha} \in \Omega_c^k(M,\mathbf{E})$ and $\boldsymbol{\beta} \in \Omega_c^{k+1}(M,\mathbf{E})$. The bundle metric induces a pairing between $\mathbf{E}$-valued forms: in an orthonormal local frame, if $\boldsymbol{\eta} = \sum_{a=1}^{r} \eta^a \otimes \mathbf{e}_a$ and $\boldsymbol{\zeta} = \sum_{a=1}^{r} \zeta^a \otimes \mathbf{e}_a$, set

$$\mathbf{h}_{\mathbf{E}}(\boldsymbol{\eta} \wedge \boldsymbol{\zeta}) := \sum_{a=1}^{r} \eta^a \wedge \overline{\zeta^a},$$

omitting conjugation in the real case. Unitary invariance of this expression shows that it is independent of the frame. Compatibility of $\nabla^{\mathbf{E}}$ with $\mathbf{h}_{\mathbf{E}}$ and the graded Leibniz rule give

$$\begin{aligned} d\,\mathbf{h}_{\mathbf{E}}\big(\boldsymbol{\alpha} \wedge *\boldsymbol{\beta}\big) &= \mathbf{h}_{\mathbf{E}}\big(d^{\mathbf{E}}\boldsymbol{\alpha} \wedge *\boldsymbol{\beta}\big) \\ &\quad + (-1)^k \mathbf{h}_{\mathbf{E}}\big(\boldsymbol{\alpha} \wedge d^{\mathbf{E}}(*\boldsymbol{\beta})\big). \end{aligned}$$

The form on the left-hand side has compact support. Since M is oriented and has no boundary, Stokes'

theorem implies

$$\int_M \mathbf{h}_{\mathbf{E}}\left(d^{\mathbf{E}}\boldsymbol{\alpha}\wedge *\boldsymbol{\beta}\right) = (-1)^{k+1}\int_M \mathbf{h}_{\mathbf{E}}\left(\boldsymbol{\alpha}\wedge d^{\mathbf{E}}(*\boldsymbol{\beta})\right).$$

By definition of the Hodge operator,

$$\mathbf{h}_{\mathbf{E}}\left(d^{\mathbf{E}}\boldsymbol{\alpha}\wedge *\boldsymbol{\beta}\right) = \langle d^{\mathbf{E}}\boldsymbol{\alpha},\boldsymbol{\beta}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, dV_{\mathbf{g}},$$

whereas

$$\mathbf{h}_{\mathbf{E}}\left(\boldsymbol{\alpha}\wedge d^{\mathbf{E}}(*\boldsymbol{\beta})\right) = \left\langle \boldsymbol{\alpha}, *^{-1}d^{\mathbf{E}}(*\boldsymbol{\beta})\right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} dV_{\mathbf{g}}.$$

Therefore,

$$\int_M \langle d^{\mathbf{E}}\boldsymbol{\alpha},\boldsymbol{\beta}\rangle\, d\lambda_{\mathbf{g}} = \int_M \left\langle \boldsymbol{\alpha}, (-1)^{k+1} *^{-1} d^{\mathbf{E}} * \boldsymbol{\beta}\right\rangle d\lambda_{\mathbf{g}}.$$

Uniqueness of the formal adjoint implies that $\delta^{\mathbf{E}} = (-1)^{k+1} *^{-1} d^{\mathbf{E}} *$ on $\Omega^{k+1}(M,\mathbf{E})$. □

From these operators, we can define the bundle-valued Hodge Laplacian.

Definition 8.2.31. (**E**-valued Hodge Laplacian) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E}\to M$ be a vector bundle with a bundle metric and a compatible connection. Define the operator

$$\Delta^{\mathbf{E}}_{\mathbf{H}}\colon \Omega^k(M,\mathbf{E}) \longrightarrow \Omega^k(M,\mathbf{E})$$

by

$$\Delta^{\mathbf{E}}_{\mathbf{H}} := d^{\mathbf{E}}\delta^{\mathbf{E}} + \delta^{\mathbf{E}}d^{\mathbf{E}}.$$

This operator is formally self-adjoint and has order 2.

On the other hand, recall that the connection induces a covariant derivative on $\Lambda^k T^*M\otimes\mathbf{E}$, allowing us to define the Bochner Laplacian on bundle-valued forms.

Definition 8.2.32. (Bochner Laplacian on bundle-valued forms) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E}\to M$ be a vector bundle with a bundle metric and a compatible connection. Define

$$\Delta^{\mathbf{E}}_{B} := \nabla^*\nabla,$$

where

$$\nabla\colon \Omega^k(M,\mathbf{E}) \longrightarrow \Gamma(T^*M\otimes\Lambda^k T^*M\otimes\mathbf{E})$$

is the induced connection and ∇^* its formal adjoint.

These two operators are related by a fundamental identity involving the curvature of the connection and the Riemannian curvature.

Theorem 8.2.33. (Weitzenböck identity) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and let $\mathbf{E}\to M$ be a vector bundle with compatible connection $\nabla^{\mathbf{E}}$. Then, on $\Omega^k(M,\mathbf{E})$, we have

$$\Delta^{\mathbf{E}}_{\mathbf{H}} = \Delta^{\mathbf{E}}_{B} - \sum_{i,j=1}^{n} \varepsilon(\mathbf{e}^i)\iota(\mathbf{e}_j)\mathbf{R}^{\Lambda^k T^*M\otimes\mathbf{E}}(\mathbf{e}_i,\mathbf{e}_j), \tag{8.23}$$

where $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ is any orthonormal local frame and $\varepsilon(\mathbf{e}^i)$ denotes exterior multiplication. The last summand is a zeroth-order endomorphism and includes both the Riemannian curvature on $\Lambda^k T^*M$ and $\mathbf{R}^{\mathbf{E}}$ on the factor $\mathbf{E}$.

Proof. Fix $p \in M$ and choose a frame orthonormal and synchronous at p; thus $\nabla \mathbf{e}_i(p) = 0$ and $[\mathbf{e}_i, \mathbf{e}_j](p) = 0$. At that point,

$$d^{\mathbf{E}} = \sum_{i=1}^{n} \varepsilon(\mathbf{e}^i)\nabla_{\mathbf{e}_i}, \qquad \delta^{\mathbf{E}} = -\sum_{j=1}^{n} \iota(\mathbf{e}_j)\nabla_{\mathbf{e}_j}.$$

Using $\iota(\mathbf{e}_j)\varepsilon(\mathbf{e}^i) + \varepsilon(\mathbf{e}^i)\iota(\mathbf{e}_j) = \delta_{ij} I$ and grouping compositions in the order in which they appear gives

$$\begin{aligned} d^{\mathbf{E}}\delta^{\mathbf{E}} + \delta^{\mathbf{E}} d^{\mathbf{E}} = &- \sum_{i=1}^{n} \nabla_{\mathbf{e}_i}\nabla_{\mathbf{e}_i} \\ &- \sum_{i,j=1}^{n} \varepsilon(\mathbf{e}^i)\iota(\mathbf{e}_j)\big(\nabla_{\mathbf{e}_i}\nabla_{\mathbf{e}_j} - \nabla_{\mathbf{e}_j}\nabla_{\mathbf{e}_i}\big). \end{aligned}$$

The first summand is $\nabla^*\nabla$ at p, and the expression in parentheses is $\mathbf{R}^{\Lambda^k T^*M \otimes \mathbf{E}}(\mathbf{e}_i, \mathbf{e}_j)$. The resulting identity is tensorial; since p was arbitrary, it holds globally. □

In particular, the Hodge Laplacian and the Bochner Laplacian differ only by the algebraic curvature term in (8.23). For $k = 1$ and $\mathbf{E}$ trivial, that term is $\mathrm{Ric}^\sharp$; for $k = 0$ it vanishes. If the connection is not flat, one must also distinguish

$$(d^{\mathbf{E}} + \delta^{\mathbf{E}})^2 = \Delta_{\mathbf{H}}^{\mathbf{E}} + (d^{\mathbf{E}})^2 + (\delta^{\mathbf{E}})^2;$$

therefore, the square of the odd operator agrees with $\Delta_{\mathbf{H}}^{\mathbf{E}}$ only in the flat case.

This identity relates the analytical properties of solutions of differential equations to the curvature of the manifold and the bundle.

8.3 Operators associated with Clifford connections

Section 2.17 constructed Clifford modules, multiplication c, and compatible connections without yet associating an operator with them. Now that the general theory of differential operators on bundles has been defined, we can perform the contraction producing the Dirac operator. No symbols or pseudodifferential operators are used in this section.

Definition 8.3.1. (Dirac-type operator) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary and let $(\mathbf{S}, c, \nabla^{\mathbf{S}})$ be a Hermitian Clifford module with a Clifford connection, in the sense of Definition 2.17.14. Clifford contraction is the bundle homomorphism

$$C\colon T^*M \otimes \mathbf{S} \longrightarrow \mathbf{S}, \qquad C(\xi \otimes s) = c(\xi)s.$$

The associated *Dirac-type operator* is

$$D := C \circ \nabla^{\mathbf{S}} \colon \Gamma(\mathbf{S}) \longrightarrow \Gamma(\mathbf{S}). \tag{8.24}$$

When $\mathbf{S} = \Sigma M$ is the spinor bundle of a spin structure, it is called the *spin Dirac operator*.

Theorem 8.3.2. (Differential nature and Green's identity) Let $(\mathbf{S}, c, \nabla^{\mathbf{S}})$ be as in the preceding definition and let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an orthonormal local frame, with coframe $(\mathbf{e}^1, \dots, \mathbf{e}^n)$. Then

$$D\mathbf{s} = \sum_{j=1}^{n} c(\mathbf{e}^j) \nabla^{\mathbf{S}}_{\mathbf{e}_j} \mathbf{s}. \tag{8.25}$$

The expression is independent of the frame and defines a differential operator of order at most one. More precisely,

$$[D, f]\mathbf{s} = c(df)\mathbf{s}, \qquad f \in C^\infty(M), \quad \mathbf{s} \in \Gamma(\mathbf{S}). \tag{8.26}$$

If $\dim M > 0$, its order is exactly one. For $\mathbf{s}, \mathbf{t} \in \Gamma_c(\mathbf{S})$, Green's identity holds:

$$\begin{aligned} &\int_M \big(\langle D\mathbf{s}, \mathbf{t} \rangle_{\mathbf{S}} - \langle \mathbf{s}, D\mathbf{t} \rangle_{\mathbf{S}} \big)\, d\lambda_{\mathbf{g}} \\ &\qquad = \int_{\partial M} \langle c(\boldsymbol{\nu}^\flat)\mathbf{s}, \mathbf{t} \rangle_{\mathbf{S}}\, d\lambda_{\tilde{\mathbf{g}}}, \end{aligned} \tag{8.27}$$

where the right-hand side is interpreted as zero if the boundary is empty and $\boldsymbol{\nu}$ is the outward unit normal otherwise. In particular, on a manifold without boundary, $D_h^* = D$.

Proof. The local formula follows by writing $\nabla^{\mathbf{S}}\mathbf{s} = \sum_{j=1}^{n} \mathbf{e}^j \otimes \nabla^{\mathbf{S}}_{\mathbf{e}_j}\mathbf{s}$. The connection is an intrinsically defined differential operator on sections, whereas C is a bundle homomorphism. Their composition therefore does not depend on the frame. The Leibniz rule gives

$$D(f\mathbf{s}) = c(df)\mathbf{s} + f D\mathbf{s},$$

which proves (8.26) and, by the commutator characterization, shows that $D \in \mathbf{PDO}^{(1)}(\mathbf{S}, \mathbf{S})$. If $\dim M > 0$, given $x \in M$ and $0 \neq \xi \in T^*_x M$, there locally exists f with $df_x = \xi$; since $c(\xi)^2 = -|\xi|^2 I$, the commutator does not vanish at x, so D does not have order zero.

For the integral identity, define the complex vector field $\mathbf{V}$ by

$$\mathbf{g}(\mathbf{V}, \mathbf{X}) = \langle c(\mathbf{X}^\flat)\mathbf{s}, \mathbf{t} \rangle_{\mathbf{S}}.$$

At a point, choose an orthonormal synchronous frame. Compatibility of the Clifford connection and the identity $c(\xi)^* = -c(\xi)$, using the convention that the Hermitian product is linear in the first variable, give

$$\operatorname{div} \mathbf{V} = \langle D\mathbf{s}, \mathbf{t} \rangle_{\mathbf{S}} - \langle \mathbf{s}, D\mathbf{t} \rangle_{\mathbf{S}}.$$

The divergence theorem yields (8.27). If $\partial M = \varnothing$, the equality is precisely the formal symmetry of D. □

Proposition 8.3.3. (Grading and the Hodge–de Rham example) If $\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-$ is a graded Clifford module,

then D is odd and is written as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad D^\pm \colon \Gamma(\mathbf{S}^\pm) \longrightarrow \Gamma(\mathbf{S}^\mp).$$

In the absence of a boundary, $(D^+)_h^* = D^-$. Under the same hypothesis, for the module $\mathbf{S} = \Lambda^\bullet T^*M \otimes \mathbb{C}$ of Example 2.17.15,

$$D = d + \delta, \qquad D^2 = \Delta_{\mathbf{H}}. \tag{8.28}$$

Proof. The connection preserves the grading and multiplication by a covector reverses it, so the first assertion is immediate. The relation between the blocks follows from formal symmetry. For forms, in an orthonormal synchronous frame,

$$d = \sum_{j=1}^n \varepsilon(\mathbf{e}^j)\nabla_{\mathbf{e}_j}, \qquad \delta = -\sum_{j=1}^n \iota_{\mathbf{e}_j}\nabla_{\mathbf{e}_j}.$$

Since $c(\mathbf{e}^j) = \varepsilon(\mathbf{e}^j) - \iota_{\mathbf{e}_j}$, we obtain $D = d + \delta$; the identities $d^2 = \delta^2 = 0$ give the second equality. □

8.3.1 Bochner and Lichnerowicz formulas

For the rest of this section, we assume that M has no boundary, in accordance with the global definition of the formal adjoint and the connection Laplacian. The resulting identities are also local identities of differential expressions in the interior of a manifold with boundary.

Recall that the curvature of $\nabla^{\mathbf{S}}$ is

$$\mathbf{R}^{\mathbf{S}}(\mathbf{X}, \mathbf{Y}) = [\nabla^{\mathbf{S}}_{\mathbf{X}}, \nabla^{\mathbf{S}}_{\mathbf{Y}}] - \nabla^{\mathbf{S}}_{[\mathbf{X},\mathbf{Y}]}.$$

Proposition 8.3.4. (Bochner formula for a Clifford module) For every Clifford module with connection,

$$D^2 = (\nabla^{\mathbf{S}})^*\nabla^{\mathbf{S}} + \frac{1}{2}\sum_{i,j=1}^n c(\mathbf{e}^i)c(\mathbf{e}^j)\mathbf{R}^{\mathbf{S}}(\mathbf{e}_i, \mathbf{e}_j). \tag{8.29}$$

Here the connection Laplacian is given locally by

$$(\nabla^{\mathbf{S}})^*\nabla^{\mathbf{S}} = -\sum_{i=1}^n \left(\nabla^{\mathbf{S}}_{\mathbf{e}_i}\nabla^{\mathbf{S}}_{\mathbf{e}_i} - \nabla^{\mathbf{S}}_{\nabla_{\mathbf{e}_i}\mathbf{e}_i}\right). \tag{8.30}$$

Proof. Fix $x \in M$ and choose an orthonormal synchronous frame at x. Clifford compatibility implies $\nabla_{\mathbf{e}_i} c(\mathbf{e}^j) = 0$ at that point. Hence,

$$D^2 = \sum_{i,j=1}^n c(\mathbf{e}^i)c(\mathbf{e}^j)\nabla^{\mathbf{S}}_{\mathbf{e}_i}\nabla^{\mathbf{S}}_{\mathbf{e}_j}.$$

The diagonal terms are $-\sum_{i=1}^{n} \nabla^{\mathbf{S}}_{\mathbf{e}_i}\nabla^{\mathbf{S}}_{\mathbf{e}_i}$. Grouping the terms with $i<j$ and using $c(\mathbf{e}^i)c(\mathbf{e}^j) = -c(\mathbf{e}^j)c(\mathbf{e}^i)$ leaves

$$\sum_{1\le i<j\le n} c(\mathbf{e}^i)c(\mathbf{e}^j)[\nabla^{\mathbf{S}}_{\mathbf{e}_i}, \nabla^{\mathbf{S}}_{\mathbf{e}_j}],$$

which is the curvature term in (8.29) because $[\mathbf{e}_i, \mathbf{e}_j](x) = 0$. In the same frame, (8.30) agrees with the diagonal term. The equality is intrinsic and x was arbitrary. □

Lemma 8.3.5. (Spinor curvature and its contraction) On the spinor bundle of a spin manifold, we have

$$\mathbf{R}^{\Sigma}(\mathbf{X},\mathbf{Y}) = \frac{1}{4}\sum_{a,b=1}^{n} \mathbf{g}\big(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{e}_a, \mathbf{e}_b\big)c(\mathbf{e}^a)c(\mathbf{e}^b), \tag{8.31}$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n} c(\mathbf{e}^i)c(\mathbf{e}^j)\mathbf{R}^{\Sigma}(\mathbf{e}_i,\mathbf{e}_j) = \frac{1}{4}R_{\mathbf{g}} I. \tag{8.32}$$

Proof. The local formula (2.115) identifies the spinor connection form with the image of the Levi–Civita form under $A = (A_{ab}) \mapsto \frac{1}{4}\sum_{a,b=1}^{n} A_{ab}c(\mathbf{e}^a)c(\mathbf{e}^b)$. Applying the same homomorphism to the curvature form gives (8.31).

Set $R_{ijab} = \mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_a,\mathbf{e}_b)$. Substituting the first formula into the left-hand side of the second produces

$$\frac{1}{8}\sum_{i,j,a,b=1}^{n} R_{ijab}c(\mathbf{e}^i)c(\mathbf{e}^j)c(\mathbf{e}^a)c(\mathbf{e}^b).$$

The antisymmetries of R_{ijab}, symmetry under interchange of pairs, and the first Bianchi identity cancel the terms with distinct indices. Using $c(\mathbf{e}^i)^2 = -I$, the remaining terms reduce to

$$\frac{1}{4}\sum_{i,j=1}^{n} \mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_j,\mathbf{e}_i)I = \frac{1}{4}R_{\mathbf{g}} I,$$

by definition of scalar curvature. □

Theorem 8.3.6. (Lichnerowicz formula) The spin Dirac operator satisfies

$$D^2 = (\nabla^{\Sigma})^*\nabla^{\Sigma} + \frac{1}{4}R_{\mathbf{g}} I. \tag{8.33}$$

If M is closed and $R_{\mathbf{g}} > 0$, then $\ker D = \{0\}$.

Proof. The identity follows from Proposition 8.3.4 and Lemma 8.3.5. If $D\psi = 0$, formal symmetry and (8.33) give

$$0 = \|D\psi\|^2_{L^2(M,\Sigma)} = \|\nabla^{\Sigma}\psi\|^2_{L^2(M,T^*M\otimes\Sigma)} + \frac{1}{4}\int_M R_{\mathbf{g}}|\psi|^2\, d\lambda_{\mathbf{g}}.$$

Both summands are nonnegative, and the second can vanish only if $\psi = 0$. □

8.3.2 Twisted operators and Spin^c

Definition 8.3.7. (Twisted Dirac operator) Let M be spin and let $\mathbf{E} \to M$ be a Hermitian bundle with unitary connection $\nabla^{\mathbf{E}}$. On $\mathbf{S} = \Sigma M \otimes \mathbf{E}$ consider

$$c_{\mathbf{E}}(\xi) = c(\xi) \otimes I_{\mathbf{E}}, \qquad \nabla^{\mathbf{S}} = \nabla^{\Sigma} \otimes I + I \otimes \nabla^{\mathbf{E}}.$$

This is a Clifford module with connection. Its associated operator is the *twisted Dirac operator*

$$D_{\mathbf{E}} = \sum_{j=1}^{n} c(\mathbf{e}^j) \otimes I_{\mathbf{E}} \nabla^{\mathbf{S}}_{\mathbf{e}_j}. \tag{8.34}$$

In even dimensions, it is odd and determines the blocks $D^{\pm}_{\mathbf{E}}$.

Theorem 8.3.8. (Weitzenböck formula for the twisted Dirac operator) With the preceding notation,

$$D^2_{\mathbf{E}} = (\nabla^{\mathbf{S}})^* \nabla^{\mathbf{S}} + \frac{1}{4} R_{\mathbf{g}} I + \frac{1}{2} \sum_{i,j=1}^{n} c(\mathbf{e}^i) c(\mathbf{e}^j) \otimes \mathbf{R}^{\mathbf{E}}(\mathbf{e}_i, \mathbf{e}_j). \tag{8.35}$$

Proof. The curvature of the product connection is

$$\mathbf{R}^{\mathbf{S}} = \mathbf{R}^{\Sigma} \otimes I + I \otimes \mathbf{R}^{\mathbf{E}}.$$

Substitute this identity into (8.29); the spinor part is contracted using (8.32), and the $\mathbf{E}$ part produces the last summand. □

Proposition 8.3.9. (Spin^c operators) Let M be a Spin^c manifold with determinant line $\mathbf{L}$, as in Definition 2.17.17, and choose a unitary connection on $\mathbf{L}$. The induced Clifford connection on $\Sigma^c M$ defines an operator $D_{\mathbf{L}} = c \circ \nabla^{\Sigma^c}$. If $F_{\mathbf{L}} \in \Omega^2(M; i\mathbb{R})$ is the curvature of the connection and

$$c(F_{\mathbf{L}}) := \frac{1}{2} \sum_{i,j=1}^{n} F_{\mathbf{L}}(\mathbf{e}_i, \mathbf{e}_j) c(\mathbf{e}^i) c(\mathbf{e}^j),$$

then

$$D^2_{\mathbf{L}} = (\nabla^{\Sigma^c})^* \nabla^{\Sigma^c} + \frac{1}{4} R_{\mathbf{g}} I + \frac{1}{2} c(F_{\mathbf{L}}). \tag{8.36}$$

In even dimensions, $D_{\mathbf{L}}$ interchanges $\Sigma^{c,+} M$ and $\Sigma^{c,-} M$.

Proof. The infinitesimal homomorphism of Spin$^c(n)$ on spinors is the spin representation on the $\mathfrak{so}(n)$ component and half the representation of the determinant line on the $i\mathbb{R}$ component. Hence,

$$\mathbf{R}^{\Sigma^c}(\mathbf{X}, \mathbf{Y}) = \mathbf{R}^{\Sigma}(\mathbf{X}, \mathbf{Y}) + \frac{1}{2} F_{\mathbf{L}}(\mathbf{X}, \mathbf{Y}) I$$

in any local spinor trivialization. The general Bochner formula and (8.32) give (8.36). The graded assertion follows from the fact that the connection preserves chirality and Clifford multiplication is odd. □

9

Sobolev spaces on vector bundles: an intrinsic approach

From functions to sections

Passing from functions to sections does not merely repeat the Euclidean theory component by component. Components depend on the frame, and ordinary derivatives of them produce additional terms under a change of trivialization. A connection organizes these terms and makes it possible to define weak derivatives taking values in a specific tensor bundle. The integration-by-parts formula will bring these two perspectives together: it is computed locally, yet expresses an intrinsic identity and allows the regularity of a section to be recognized without choosing a global frame.

9.1 Weak covariant derivatives and Sobolev spaces on vector bundles

The preceding construction of Sobolev spaces on manifolds started with smooth sections and completed that space in a covariant norm. There is another description, closer to the Euclidean case: require a section in L^p to possess weak covariant derivatives through the order under consideration. This chapter develops that formulation on a vector bundle and proves its compatibility with the construction by completion.

For compactly supported smooth sections, the identity

$$\int_M \langle \nabla^{\mathbf{E}} \mathbf{u}, \mathbf{\Phi} \rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}} = \int_M \langle \mathbf{u}, (\nabla^{\mathbf{E}})^* \mathbf{\Phi} \rangle_{\mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}}$$

transfers the derivative from $\mathbf{u}$ to a test section. The right-hand side still makes sense for $\mathbf{u} \in L^p(M, \mathbf{E})$ provided $(\nabla^{\mathbf{E}})^* \mathbf{\Phi}$ belongs to the appropriate dual space. This observation leads to the weak covariant derivative and allows elliptic systems on bundles to be formulated without choosing global components.

We work with a smooth vector bundle $\mathbf{E} \longrightarrow M$ equipped with a bundle metric and a smooth connection $\nabla^{\mathbf{E}}$. On tensor factors constructed from TM and T^*M, we use the Levi–Civita connection of $\mathbf{g}$. The definitions of weak covariant derivative and $W^{m,p}(M, \mathbf{E})$ do not require $\nabla^{\mathbf{E}}$ to be compatible with the bundle metric; whenever compatibility is needed for an integration-by-parts formula, an identification of the formal adjoint, or a subsequent

estimate, it will be explicitly included in the hypotheses. The notation $W^{m,p}(M,\mathbf{E})$ denotes the space of sections of $L^p(M,\mathbf{E})$ whose weak covariant derivatives through order m belong to the corresponding L^p spaces of $\mathbf{E}$-valued tensors. This choice retains the notation used in the Euclidean case and treats functions, differential forms, tensor fields, and sections of general bundles uniformly.

To formulate the definition, we first need to construct iterated covariant derivatives and the induced bundle metrics on $T^{(0,s)}(TM)\otimes\mathbf{E}$. We then prove completeness, density of smooth sections, and equivalence with local descriptions. These results give an intrinsic version of the Meyers–Serrin theorem and prepare the embedding, trace, and compactness theorems.

Definition 9.1.1. Let M be a smooth manifold with or without boundary, let $\mathbf{E}\to M$ be a smooth vector bundle with a connection $\nabla^{\mathbf{E}}$, and let ∇^{TM} be a connection on TM. For $s\in\mathbb{N}$, define the sth covariant derivative of $\mathbf{u}\in\Gamma(\mathbf{E})$ as the section $\nabla^s\mathbf{u}\in\Gamma\big(T^{(0,s)}(TM)\otimes\mathbf{E}\big)$ defined recursively as follows: $\nabla^0\mathbf{u}:=\mathbf{u}$ and, for $s\geq 0$,

$$\nabla^{s+1}\mathbf{u}:=\nabla^{T^{(0,s)}(TM)\otimes\mathbf{E}}(\nabla^s\mathbf{u}),$$

where we identify $T^*M\otimes(T^{(0,s)}(TM)\otimes\mathbf{E})$ with $T^{(0,s+1)}(TM)\otimes\mathbf{E}$ by placing the new factor T^*M first. The connection on the right-hand side is the product connection induced by ∇^{TM} and $\nabla^{\mathbf{E}}$.
Schematically,

$$\begin{array}{ccc}
\Gamma(\mathbf{E}) & \xrightarrow{\ \nabla^{\mathbf{E}}\ } & \Gamma\big(T^{(0,1)}(TM)\otimes\mathbf{E}\big) \\
 & & \Big\downarrow{\scriptstyle \nabla^{T^{(0,1)}(TM)\otimes\mathbf{E}}} \\
\Gamma\big(T^{(0,3)}(TM)\otimes\mathbf{E}\big) & \xleftarrow[\ \nabla^{T^{(0,2)}(TM)\otimes\mathbf{E}}\ ]{} & \Gamma\big(T^{(0,2)}(TM)\otimes\mathbf{E}\big) \\
\Big\downarrow{\scriptstyle \nabla^{T^{(0,3)}(TM)\otimes\mathbf{E}}} & & \\
\cdots & \xrightarrow{\ \nabla^{T^{(0,s-1)}(TM)\otimes\mathbf{E}}\ } & \Gamma\big(T^{(0,s)}(TM)\otimes\mathbf{E}\big)
\end{array}$$

Equivalently,

$$(\nabla^{s+1}\mathbf{u})(\mathbf{X}_0,\dots,\mathbf{X}_s):=\big(\nabla_{\mathbf{X}_0}(\nabla^s\mathbf{u})\big)(\mathbf{X}_1,\dots,\mathbf{X}_s).$$

In a chart $(x^1,\dots,x^n)$, with $\partial_i=\partial/\partial x^i$, write

$$(\nabla^s\mathbf{u})_{i_1\cdots i_s}:=(\nabla^s\mathbf{u})(\partial_{i_1},\dots,\partial_{i_s}).$$

In particular,

$$(\nabla^2\mathbf{u})_{ij}=\nabla^{\mathbf{E}}_{\partial_i}\big(\nabla^{\mathbf{E}}_{\partial_j}\mathbf{u}\big)-\Gamma^k_{ij}\nabla^{\mathbf{E}}_{\partial_k}\mathbf{u}. \tag{9.1}$$

This is a component of the tensor $\nabla^2\mathbf{u}$, rather than the mere directional composition: it includes the Christoffel term. At higher orders, we will primarily use $\nabla^s\mathbf{u}$ and its components $(\nabla^s\mathbf{u})_{i_1\cdots i_s}$, according to the same recursion.

The section $\nabla^s\mathbf{u}$ takes values in $T^{(0,s)}(TM)\otimes\mathbf{E}$. To measure it pointwise, we need the bundle metric induced by the Riemannian metric on M and the metric on $\mathbf{E}$.

The same construction defines a bundle metric on every bundle $T^{(k,l)}(TM)\otimes\mathbf{E}$.

Proposition 9.1.2. Let $(M,\mathbf{g})$ be a Riemannian manifold (with or without boundary), and let $\mathbf{E}\to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$. Then there is a unique smooth bundle metric

$\langle\cdot,\cdot\rangle_{\mathbf{g},\mathbf{h_E}}$ on each $\mathbf{E}$-valued tensor bundle, $T^{(k,l)}(TM)\otimes\mathbf{E}$, such that, for simple tensor fields

$$\boldsymbol{\alpha}_1\otimes\cdots\otimes\boldsymbol{\alpha}_k\otimes\boldsymbol{\omega}_1\otimes\cdots\otimes\boldsymbol{\omega}_l\otimes\mathbf{u},\quad \boldsymbol{\beta}_1\otimes\cdots\otimes\boldsymbol{\beta}_k\otimes\boldsymbol{\eta}_1\otimes\cdots\otimes\boldsymbol{\eta}_l\otimes\mathbf{v},$$

with $\boldsymbol{\alpha}_r,\boldsymbol{\beta}_r\in\mathfrak{X}(M)$, $\boldsymbol{\omega}_s,\boldsymbol{\eta}_s\in\Omega^1(M)$, and $\mathbf{u},\mathbf{v}\in\Gamma(\mathbf{E})$, we have

$$\left\langle\boldsymbol{\alpha}_1\otimes\cdots\otimes\boldsymbol{\alpha}_k\otimes\boldsymbol{\omega}_1\otimes\cdots\otimes\boldsymbol{\omega}_l\otimes\mathbf{u},\boldsymbol{\beta}_1\otimes\cdots\otimes\boldsymbol{\beta}_k\otimes\boldsymbol{\eta}_1\otimes\cdots\otimes\boldsymbol{\eta}_l\otimes\mathbf{v}\right\rangle_{\mathbf{g},\mathbf{h_E}}=\left(\prod_{s=1}^{k}\langle\boldsymbol{\alpha}_s,\boldsymbol{\beta}_s\rangle_{\mathbf{g}}\right)\left(\prod_{t=1}^{l}\langle\boldsymbol{\omega}_t,\boldsymbol{\eta}_t\rangle_{\mathbf{g}}\right)\langle\mathbf{u},\mathbf{v}\rangle_{\mathbf{h_E}}.$$

Moreover, if $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ is a smooth local frame of TM, with dual coframe $(\varepsilon^1,\dots,\varepsilon^n)$, $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ is a local frame of $\mathbf{E}$, all defined on $U\subseteq M$, and

$$\mathbf{F}=F^{i_1\dots i_k a}_{j_1\dots j_l}\mathbf{E}_{i_1}\otimes\cdots\otimes\mathbf{E}_{i_k}\otimes\varepsilon^{j_1}\otimes\cdots\otimes\varepsilon^{j_l}\otimes\mathbf{e}_a,$$

$$\mathbf{G}=G^{r_1\dots r_k b}_{s_1\dots s_l}\mathbf{E}_{r_1}\otimes\cdots\otimes\mathbf{E}_{r_k}\otimes\varepsilon^{s_1}\otimes\cdots\otimes\varepsilon^{s_l}\otimes\mathbf{e}_b,$$

then

$$\langle\mathbf{F},\mathbf{G}\rangle_{\mathbf{g},\mathbf{h_E}}=g_{i_1r_1}\cdots g_{i_kr_k}g^{j_1s_1}\cdots g^{j_ls_l}h_{ab}F^{i_1\dots i_k a}_{j_1\dots j_l}\overline{G^{r_1\dots r_k b}_{s_1\dots s_l}}.$$

In the real case, the bar is omitted. In the complex case, it appears because the Hermitian convention fixed in Definition 2.5.1 is linear in the first argument and antilinear in the second. When no confusion is possible, we write simply $\langle\mathbf{F},\mathbf{G}\rangle_{\mathbf{g},\mathbf{h_E}}=\langle\mathbf{F},\mathbf{G}\rangle_{\mathbf{h_E}}$. We use the notation $|\mathbf{F}|_{\mathbf{h_E}}:=\sqrt{\langle\mathbf{F},\mathbf{F}\rangle_{\mathbf{h_E}}}$.

Proof. At each point $p\in M$, the metric $\mathbf{g}_p$ induces the dual inner product $\mathbf{g}_p^{-1}$ on T_p^*M. The tensor product of k copies of $\mathbf{g}_p$, l copies of $\mathbf{g}_p^{-1}$, and one copy of $(\mathbf{h_E})_p$ defines a positive-definite form on

$$T_p^{(k,l)}(T_pM)\otimes\mathbf{E}_p.$$

The formula on simple tensors is precisely the definition of this tensor product. Since simple tensors span the fiber, sesquilinearity determines the form uniquely on the entire fiber.
In the frames of the statement, the inner-product matrices of the factors are, respectively,

$$(g_{ir}),\qquad (g^{js}),\qquad (h_{ab}).$$

Expanding $\mathbf{F}$ and $\mathbf{G}$ in the tensor basis and using linearity of the Hermitian product in the first variable and antilinearity in the second gives

$$\langle\mathbf{F},\mathbf{G}\rangle_{\mathbf{g},\mathbf{h_E}}=g_{i_1r_1}\cdots g_{i_kr_k}g^{j_1s_1}\cdots g^{j_ls_l}h_{ab}F^{i_1\dots i_k a}_{j_1\dots j_l}\overline{G^{r_1\dots r_k b}_{s_1\dots s_l}}.$$

In the real case, the same calculation is bilinear and involves no conjugation. The coefficients g_{ir}, g^{js}, and h_{ab} are smooth in every local frame; hence the resulting family of forms varies smoothly with p. Their definition through tensor products shows that expressions in different frames agree on overlaps. This gives a global smooth bundle metric, and the fiberwise uniqueness already proved completes the proof. □

With these definitions and conventions, we can define what it means for a section to be an L^p-section.

Definition 9.1.3. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $1 \le p \le \infty$, and let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$. We say that a function $\mathbf{u}\colon M \longrightarrow \mathbf{E}$ is an L^p-section if

(a) $\mathbf{u}$ is measurable, that is, for every open subset $U \subseteq \mathbf{E}$ we have $\mathbf{u}^{-1}(U) \in \mathcal{L}_M$.

(b) $(\pi \circ \mathbf{u})(x) = x$ for almost every $x \in M$.

(c) $|\mathbf{u}|_{\mathbf{h_E}} \in L^p(M, d\lambda_{\mathbf{g}})$.

Write

$$\widetilde{L}^p(\mathbf{E}) := \{\mathbf{u}\colon M \longrightarrow \mathbf{E} \mid \mathbf{u} \text{ is an } L^p \text{ section of } \mathbf{E}\}.$$

Define an equivalence relation $\sim$ on $\widetilde{L}^p(\mathbf{E})$ by

$$\mathbf{u} \sim \mathbf{v} \quad \Longleftrightarrow \quad \mathbf{u}(x) = \mathbf{v}(x) \text{ for almost every } x \in M.$$

Finally, define $L^p(M, \mathbf{E}) := \widetilde{L}^p(\mathbf{E}) \big/ \sim$. Likewise, define

$$L^p_{\mathrm{loc}}(M, \mathbf{E}) := \left\{ \mathbf{u}\colon M \longrightarrow \mathbf{E} \;\middle|\; \begin{array}{l} \mathbf{u} \text{ is measurable, } (\pi \circ \mathbf{u})(x) = x \text{ for almost every } x \in M, \\ \forall K \subseteq M \text{ compact, } \displaystyle\int_K |\mathbf{u}|^p_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} < \infty \end{array} \right\} \Big/ \sim,$$

where $\sim$ again denotes equality almost everywhere on M.

These spaces can be equipped with norms or seminorms as appropriate:

Definition 9.1.4. (L^p norms and local seminorms) Let $(M, \mathbf{g})$ be a smooth Riemannian manifold with or without boundary, let $1 \le p \le \infty$, and let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$.

(a) *Global case.*

(a) If $1 \le p < \infty$ and $\mathbf{u} \in L^p(M, \mathbf{E})$, define

$$\|\mathbf{u}\|_{L^p(M,\mathbf{E})} := \left(\int_M |\mathbf{u}|^p_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}}.$$

(b) If $p = \infty$ and $\mathbf{u} \in L^\infty(M, \mathbf{E})$, define

$$\|\mathbf{u}\|_{L^\infty(M,\mathbf{E})} := \operatorname*{ess\,sup}_{x \in M} |\mathbf{u}(x)|_{\mathbf{h_E}}.$$

(b) *Local case.*

(a) If $1 \le p < \infty$ and $\mathbf{u} \in L^p_{\mathrm{loc}}(M, \mathbf{E})$, for each compact set $K \subseteq M$ define the seminorm

$$\|\mathbf{u}\|_{L^p(K,\mathbf{E})} := \left(\int_K |\mathbf{u}|^p_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}}.$$

(b) If $p=\infty$ and $\mathbf{u}\in L^{\infty}_{\mathrm{loc}}(M,\mathbf{E})$, for each compact set $K\subseteq M$ define

$$\|\mathbf{u}\|_{L^{\infty}(K,\mathbf{E})}:=\operatorname*{ess\,sup}_{x\in K}|\mathbf{u}(x)|_{\mathbf{h_E}}.$$

The spaces $L^{p}_{\mathrm{loc}}(M,\mathbf{E})$ are equipped with the family of seminorms $\{\|\cdot\|_{L^p(K,\mathbf{E})}\}_{K\subseteq M\text{ compact}}$.

Our definition of weak derivative uses the formal Hermitian adjoint developed in the preceding chapter. Throughout this chapter, $(\nabla^s)^*_h$ denotes that adjoint; when a previously established formula abbreviates it as $(\nabla^s)^*$, the abbreviation has exactly the same meaning and never denotes the bilinear transpose $(\nabla^s)'$.

Definition 9.1.5. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E}$ be a smooth vector bundle over M equipped with a bundle metric $\mathbf{h_E}$ and a smooth connection $\nabla^{\mathbf{E}}$. In constructing ∇^s, use the product connection induced by the Levi–Civita connection of $\mathbf{g}$ and by $\nabla^{\mathbf{E}}$. Let $\mathbf{u}\in L^1_{\mathrm{loc}}(M,\mathbf{E})$. We say that $\mathbf{v}\in L^1_{\mathrm{loc}}(M,T^{(0,s)}(TM)\otimes\mathbf{E})$ is the sth weak covariant derivative of $\mathbf{u}$ if, for every $\boldsymbol{\sigma}\in\Gamma_c(T^{(0,s)}(TM)\otimes\mathbf{E})$,

$$\int_M\langle\mathbf{v},\boldsymbol{\sigma}\rangle_{\mathbf{g},\mathbf{h_E}}\,d\lambda_{\mathbf{g}}=\int_M\langle\mathbf{u},(\nabla^s)^*_h\boldsymbol{\sigma}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}.$$

The weak derivative of a section $\mathbf{u}\in L^1_{\mathrm{loc}}(M,\mathbf{E})$ will be unique, as a consequence of a result analogous to Proposition B.6.20.

Lemma 9.1.6. (Localization) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, $\mathbf{E}$ a smooth vector bundle with bundle metric $\mathbf{h_E}$, and $\mathbf{v}\in L^1_{\mathrm{loc}}(M,\mathbf{E})$ such that

$$\int_M\langle\mathbf{v},\boldsymbol{\sigma}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}=0,\qquad\forall\boldsymbol{\sigma}\in\Gamma_c(\mathbf{E}).$$

Then $\mathbf{v}\equiv 0$ almost everywhere on M.

Proof. Consider an open subset $U\subseteq M$ such that (U,ψ) is a smooth chart of M and U is also the domain of a smooth local orthonormal frame of $\mathbf{E}$ over U, say $(\mathbf{e}_1,\dots,\mathbf{e}_r)$.
Given a section $\boldsymbol{\sigma}\in\Gamma_c(\mathbf{E}\restriction_U)$, extend it by zero outside U to obtain $\widetilde{\boldsymbol{\sigma}}\in\Gamma_c(\mathbf{E})$. Applying the hypothesis gives

$$0=\int_M\langle\mathbf{v},\widetilde{\boldsymbol{\sigma}}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}=\int_U\langle\mathbf{v},\boldsymbol{\sigma}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}.$$

Since $\mathbf{v}\in L^1_{\mathrm{loc}}(M,\mathbf{E})$, we can write $\mathbf{v}=\sum_{i=1}^{r}v^i\mathbf{e}_i$ with $v^i\in L^1_{\mathrm{loc}}(U)$ and $\boldsymbol{\sigma}=\sum_{j=1}^{r}\sigma^j\mathbf{e}_j$ with $\sigma^j\in C^\infty_c(U)$. Orthonormality of the frame and the convention of linearity in the first variable give

$$0=\int_U\sum_{i=1}^{r}v^i\overline{\sigma^i}\,d\lambda_{\mathbf{g}}.$$

Now fix $j\in\{1,\dots,r\}$ and take $\boldsymbol{\sigma}:=f\mathbf{e}_j$ with $f\in C^\infty_c(U)$, obtaining

$$\int_U v^j\overline{f}\,d\lambda_{\mathbf{g}}=0,\qquad\forall f\in C^\infty_c(U).$$

Passing to local coordinates gives

$$\int_{\psi(U)} (v^j \circ \psi^{-1})\overline{(f \circ \psi^{-1})}\big(\sqrt{\det(\mathbf{g})} \circ \psi^{-1}\big) d\lambda_n = 0, \qquad \forall f \in C_c^\infty(U).$$

Given arbitrary $\varphi \in C_c^\infty(\psi(U))$, define

$$f := \left(\overline{\frac{\varphi}{\sqrt{\det(\mathbf{g})} \circ \psi^{-1}}}\right) \circ \psi \in C_c^\infty(U).$$

The function is well defined and smooth because $\sqrt{\det(\mathbf{g})} \circ \psi^{-1}$ is smooth and strictly positive. Moreover,

$$\operatorname{supp}(f) \subseteq \psi^{-1}\left(\operatorname{supp}\left(\frac{\varphi}{\sqrt{\det(\mathbf{g})} \circ \psi^{-1}}\right)\right),$$

, and the set on the right is compact: the support inside ψ^{-1} is compact in $\psi(U)$, and ψ is a homeomorphism. Substituting this choice yields

$$\int_{\psi(U)} (v^j \circ \psi^{-1})\varphi \, d\lambda_n = 0, \qquad \forall \varphi \in C_c^\infty(\psi(U)).$$

By Proposition B.6.20, we conclude that $v^j \circ \psi^{-1} = 0$ almost everywhere on $\psi(U)$, and hence $v^j = 0$ almost everywhere on U. Since $j \in \{1, \dots, r\}$ was arbitrary, it follows that $\mathbf{v} = 0$ almost everywhere on U.
Finally, since M is Lindelöf, it can be covered by countably many open subsets of this kind, on each of which $\mathbf{v} = 0$ almost everywhere. It follows that $\mathbf{v} \equiv 0$ almost everywhere on M. □

Corollary 9.1.7. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with a bundle metric. If $\mathbf{u} \in L^1_{\mathrm{loc}}(M, \mathbf{E})$ has an sth weak covariant derivative, it is unique, and we denote it by $\nabla^s_w \mathbf{u}$, where w stands for "weak."

Proof. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are two sth weak covariant derivatives of $\mathbf{u}$. For every $\sigma \in \Gamma_c(T^{(0,s)}(TM) \otimes \mathbf{E})$, their two defining identities have the same right-hand side. Subtraction gives

$$\int_M \langle \mathbf{v}_1 - \mathbf{v}_2, \sigma \rangle_{\mathbf{g}, \mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} = 0.$$

Lemma 9.1.6, applied to the bundle $T^{(0,s)}(TM) \otimes \mathbf{E}$ with its induced bundle metric, implies that $\mathbf{v}_1 - \mathbf{v}_2 = 0$ almost everywhere. Therefore $\mathbf{v}_1 = \mathbf{v}_2$ almost everywhere. □

The weak covariant derivative is a local operator, just like its classical counterpart:

Proposition 9.1.8. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and $\mathbf{E} \longrightarrow M$ a smooth vector bundle with bundle metric $\mathbf{h}_\mathbf{E}$. Let $\mathbf{u} \in L^1_{\mathrm{loc}}(M, \mathbf{E})$ be such that $\mathbf{u}$ has an sth weak covariant derivative $\nabla^s_w \mathbf{u} \in L^1_{\mathrm{loc}}(M, T^{(0,s)}(TM) \otimes \mathbf{E})$. Let $U \subseteq M$ be open. If $\mathbf{u} \equiv 0$ almost everywhere on U, then $\nabla^s_w \mathbf{u} \equiv 0$ almost everywhere on U.

Proof. Let $\sigma \in \Gamma_c(T^{(0,s)}(TU)\otimes \mathbf{E}\restriction_U)$. We can define $\widetilde{\sigma} \in \Gamma_c(T^{(0,s)}(TM)\otimes\mathbf{E})$ by

$$\widetilde{\sigma}(p) = \begin{cases} \sigma(p) & \text{if } p \in U \\ 0_{\mathbf{E}_p} & \text{if } p \notin U \end{cases}.$$

Since U is open and $\operatorname{supp}(\sigma) \subseteq U$, $\widetilde{\sigma}$ is smooth, so $\widetilde{\sigma} \in \Gamma_c(T^{(0,s)}(TM)\otimes\mathbf{E})$. By the definition of weak covariant derivative, we have

$$\int_M \langle \nabla^s_w \mathbf{u}, \widetilde{\sigma}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g} = \int_M \langle \mathbf{u}, (\nabla^s)^*\widetilde{\sigma}\rangle_{\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g}.$$

Since $\operatorname{supp}(\widetilde{\sigma}) = \operatorname{supp}(\sigma) \subseteq U$ and by Proposition 8.1.15 ($(\nabla^s)^*$ is a local operator), we have $\operatorname{supp}((\nabla^s)^*\widetilde{\sigma}) \subseteq \operatorname{supp}(\widetilde{\sigma}) \subseteq U$. Then

$$\int_U \langle \nabla^s_w \mathbf{u}, \widetilde{\sigma}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g} = \int_M \langle \nabla^s_w \mathbf{u}, \widetilde{\sigma}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g} = \int_M \langle \mathbf{u}, (\nabla^s)^*\widetilde{\sigma}\rangle_{\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g} = \int_U \langle \mathbf{u}, (\nabla^s)^*\widetilde{\sigma}\rangle_{\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g} = 0,$$

where the last equality holds because $\mathbf{u} \equiv 0$ on U. Therefore Lemma 9.1.6 gives $\nabla^s_w \mathbf{u} \equiv 0$ on U. □

With these definitions, we can define Sobolev spaces by weak covariant derivatives on any smooth vector bundle.

Definition 9.1.9. (Global and local Sobolev spaces) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\pi\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a smooth connection $\nabla^\mathbf{E}$. Let $m \in \mathbb{N}_0$ and $1 \le p \le \infty$.

(a) *Global Sobolev spaces.* Define

$$W^{m,p}(M,\mathbf{E}) := \left\{ \mathbf{u} \in L^1_{\mathrm{loc}}(M,\mathbf{E}) \;\middle|\; \nabla^s_w \mathbf{u} \in L^p\big(M, T^{(0,s)}(TM)\otimes\mathbf{E}\big) \text{ for every } s \in \{0,\dots,m\} \right\}.$$

The space $W^{m,p}(M,\mathbf{E})$ is a Banach space with norm

$$\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} := \begin{cases} \left(\displaystyle\sum_{s=0}^m \int_M |\nabla^s_w \mathbf{u}|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \displaystyle\max_{0\le s\le m} \operatorname*{ess\,sup}_{x\in M} |\nabla^s_w \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, & p = \infty. \end{cases}$$

(b) *Local Sobolev spaces.* Define

$$W^{m,p}_{\mathrm{loc}}(M,\mathbf{E}) := \left\{ \mathbf{u} \in L^1_{\mathrm{loc}}(M,\mathbf{E}) \;\middle|\; \nabla^s_w \mathbf{u} \in L^p_{\mathrm{loc}}\big(M, T^{(0,s)}(TM)\otimes\mathbf{E}\big) \text{ for every } s \in \{0,\dots,m\} \right\}.$$

For each compact set $K \subseteq M$, equip $W^{m,p}_{\mathrm{loc}}(M,\mathbf{E})$ with the local *seminorm*

$$\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E};K)} := \begin{cases} \left(\displaystyle\sum_{s=0}^m \int_K |\nabla^s_w \mathbf{u}|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \displaystyle\max_{0\le s\le m} \operatorname*{ess\,sup}_{x\in K} |\nabla^s_w \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, & p = \infty. \end{cases}$$

The space $W^{m,p}_{\mathrm{loc}}(M,\mathbf{E})$ is equipped with the family of seminorms $\{\|\cdot\|_{W^{m,p}(M,\mathbf{E};K)}\}_{K\subseteq M \text{ compact}}$.

Exercise 9.1.10. Under the hypotheses of the preceding definition, where M is a smooth manifold with or without boundary, prove that $(W^{m,p}(M,\mathbf{E}),\|\cdot\|_{W^{m,p}(M,\mathbf{E})})$ is a normed space.

Remark 9.1.11. In the trivial case $\mathbf{E} = M\times\mathbb{R}$, there is a natural bundle isomorphism

$$T^{(0,s)}(TM)\otimes(M\times\mathbb{R}) \cong T^{(0,s)}(TM),$$

given fiberwise by $v\otimes\alpha \mapsto \alpha v$. (More explicitly, considering $v_p \in T^{(0,s)}(T_pM)$, $(p,\alpha)\in\{p\}\times\mathbb{R},$, the assignment is $v_p\otimes(p,\alpha)\mapsto \alpha v_p \in T^{(0,s)}(T_pM)$.) Under this identification, the Sobolev spaces reduce to the usual spaces for functions

$$W^{m,p}(M) := \Big\{u\in L^1_{\mathrm{loc}}(M)\,|\,\nabla^s_w u\in L^p\big(M,T^{(0,s)}(TM)\big)\text{for every } s\in\{0,\dots,m\}\Big\}.$$

9.1.1 Fundamental properties: uniform convexity, reflexivity, and separability

We now prove the fundamental properties of Sobolev spaces on vector bundles, which will be analogous to their Euclidean counterparts. First, we show that $W^{m,p}(M,\mathbf{E})$ is a Banach space. The following lemma is particularly useful for this purpose:

Lemma 9.1.12. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E}\longrightarrow M$ be a smooth vector bundle with a bundle metric $\mathbf{h_E}$ and a connection $\nabla^{\mathbf{E}}$. Let $1\le p\le\infty$. If $(\mathbf{u}_k)_{k\in\mathbb{N}}\subseteq W^{m,p}(M,\mathbf{E})$ is such that $\mathbf{u}_k\to\mathbf{u}$ in $L^p(M,\mathbf{E})$ and $\nabla^j_w\mathbf{u}_k\to\mathbf{v}_j$ in $L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})$ as $k\to\infty$ for every $j\in\{1,\dots,m\}$, then $\mathbf{u}\in W^{m,p}(M,\mathbf{E})$ and $\nabla^j_w\mathbf{u}=\mathbf{v}_j$ for every $j\in\{1,\dots,m\}$.

Proof. Let $j\in\{1,\dots,m\}$ be arbitrary and let $\boldsymbol{\sigma}\in\Gamma_c\big(T^{(0,j)}(TM)\otimes\mathbf{E}\big)$. Since $\mathbf{u}_k\in W^{m,p}(M,\mathbf{E})$, the definition of weak covariant derivative of order j gives

$$\int_M\langle\nabla^j_w\mathbf{u}_k,\boldsymbol{\sigma}\rangle_{\mathbf{g},\mathbf{h_E}}\,d\lambda_{\mathbf{g}} = \int_M\langle\mathbf{u}_k,(\nabla^j)^*\boldsymbol{\sigma}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}.$$

By Hölder's inequality (Proposition B.7.9), with p' conjugate to p, we have

$$\left|\int_M\langle\mathbf{u}_k-\mathbf{u},(\nabla^j)^*\boldsymbol{\sigma}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}\right| \le \|\mathbf{u}_k-\mathbf{u}\|_{L^p(M,\mathbf{E})}\,\|(\nabla^j)^*\boldsymbol{\sigma}\|_{L^{p'}(M,\mathbf{E})},$$

and, since $\mathbf{u}_k\to\mathbf{u}$ in $L^p(M,\mathbf{E})$, it follows that

$$\int_M\langle\mathbf{u}_k,(\nabla^j)^*\boldsymbol{\sigma}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}\longrightarrow\int_M\langle\mathbf{u},(\nabla^j)^*\boldsymbol{\sigma}\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}.$$

Likewise, Hölder's inequality (Proposition B.7.9) gives

$$\left|\int_M\langle\nabla^j_w\mathbf{u}_k-\mathbf{v}_j,\boldsymbol{\sigma}\rangle_{\mathbf{g},\mathbf{h_E}}\,d\lambda_{\mathbf{g}}\right| \le \|\nabla^j_w\mathbf{u}_k-\mathbf{v}_j\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\,\|\boldsymbol{\sigma}\|_{L^{p'}(M,T^{(0,j)}(TM)\otimes\mathbf{E})},$$

and, since $\nabla_w^j \mathbf{u}_k \to \mathbf{v}_j$ in $L^p(M, T^{(0,j)}(TM)\otimes\mathbf{E})$, we obtain

$$\int_M \langle \nabla_w^j \mathbf{u}_k, \sigma\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \longrightarrow \int_M \langle \mathbf{v}_j, \sigma\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Passing to the limit in the initial identity yields

$$\int_M \langle \mathbf{v}_j, \sigma\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \int_M \langle \mathbf{u}, (\nabla^j)^*\sigma\rangle_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \qquad \forall\, \sigma \in \Gamma_c\big(T^{(0,j)}(TM)\otimes\mathbf{E}\big).$$

By the definition of weak covariant derivative of order j, this implies that $\mathbf{u}$ admits a weak covariant derivative of order j and that $\nabla_w^j \mathbf{u} = \mathbf{v}_j$.
Since $j \in \{1,\dots,m\}$ was arbitrary, we conclude that $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$ and $\nabla_w^j \mathbf{u} = \mathbf{v}_j$ for every $j \in \{1,\dots,m\}$. □

To show that the spaces $W^{m,p}(M,\mathbf{E})$ are Banach, we will, as the reader may expect, use the preceding lemma; first, however, we need to show that $L^p(M,\mathbf{E})$ is a Banach space.

Theorem 9.1.13. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\pi_{\mathbf{E}}\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$. If $1 \le p \le \infty$, then $L^p(M,\mathbf{E})$ is a Banach space.

Proof. Let $(\mathbf{s}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(M,\mathbf{E})$.
First consider the case $1 \le p < \infty$. There is a subsequence $(\mathbf{s}_{n_k})_{k\in\mathbb{N}}$ such that

$$\|\mathbf{s}_{n_{k+1}} - \mathbf{s}_{n_k}\|_{L^p(M,\mathbf{E})} < \frac{1}{2^k} \qquad \text{for every } k \in \mathbb{N}.$$

Define

$$G_K(x) := \sum_{k=1}^{K} |\mathbf{s}_{n_{k+1}}(x) - \mathbf{s}_{n_k}(x)|_{\mathbf{h_E}}.$$

Then (G_K) is increasing and, by Minkowski's inequality in $L^p(M)$ (Theorem B.7.3),

$$\|G_K\|_{L^p(M)} \le \sum_{k=1}^{K} \|\mathbf{s}_{n_{k+1}} - \mathbf{s}_{n_k}\|_{L^p(M,\mathbf{E})} < 1.$$

By Theorem B.6.11, there exists $G \in L^p(M)$ such that $G_K \to G$ pointwise and $G(x) < \infty$ almost everywhere.
For each k, there exists a null set $Z_k \subset M$ such that $\pi_{\mathbf{E}}(\mathbf{s}_{n_k}(x)) = x$ for every $x \in M \setminus Z_k$. Define

$$Z := \{x \in M \mid G(x) = \infty\} \cup \bigcup_{k=1}^{\infty} Z_k.$$

Then Z is a null set, and for every $x \in M \setminus Z$ we have $G(x) < \infty$ and $\pi_{\mathbf{E}}(\mathbf{s}_{n_k}(x)) = x$ for every k.
For $x \in M \setminus Z$, the series

$$\sum_{k=1}^{\infty} (\mathbf{s}_{n_{k+1}}(x) - \mathbf{s}_{n_k}(x))$$

converges absolutely in the fiber $\mathbf{E}_x$, since the sum of the norms is $G(x) < \infty$. Since $\mathbf{E}_x$ is finite dimensional, it is complete, and the series converges to a well-defined vector in $\mathbf{E}_x$.

Define

$$\mathbf{s}(x) := \begin{cases} \mathbf{s}_{n_1}(x) + \displaystyle\sum_{k=1}^{\infty} (\mathbf{s}_{n_{k+1}}(x) - \mathbf{s}_{n_k}(x)), & x \in M \setminus Z, \\ 0_x, & x \in Z, \end{cases}$$

where 0_x denotes the zero vector of the fiber $\mathbf{E}_x$.
For $x \in M \setminus Z$, the sum telescopes, giving

$$\mathbf{s}_{n_{k+1}}(x) \to \mathbf{s}(x) \qquad \text{a.e. on } M.$$

Each $\mathbf{s}_{n_k}$ is measurable, so $\mathbf{s}$ is measurable as the almost-everywhere pointwise limit of measurable functions. Moreover, by construction $\pi_{\mathbf{E}}(\mathbf{s}(x)) = x$ for every $x \in M$, so $\mathbf{s}$ is a measurable section of $\mathbf{E}$.
Let $\varepsilon > 0$. Since $(\mathbf{s}_n)$ is Cauchy, there exists N such that

$$\|\mathbf{s}_n - \mathbf{s}_m\|_{L^p(M,\mathbf{E})} < \varepsilon \qquad \text{for every } n, m \geq N.$$

For fixed $n \geq N$, we have

$$|\mathbf{s}_n(x) - \mathbf{s}_{n_k}(x)|^p_{\mathbf{h}_\mathbf{E}} \to |\mathbf{s}_n(x) - \mathbf{s}(x)|^p_{\mathbf{h}_\mathbf{E}} \qquad \text{a.e.}$$

Since $n_k \longrightarrow \infty$, there exists $k_0 \in \mathbb{N}$ such that $n_k \geq N$ for every $k \geq k_0$. Applying Fatou's lemma (Lemma B.6.12), since the functions are nonnegative and measurable,

$$\int_M |\mathbf{s}_n - \mathbf{s}|^p_{\mathbf{h}_\mathbf{E}} \, d\lambda_{\mathbf{g}} \leq \liminf_{k\to\infty} \int_M |\mathbf{s}_n - \mathbf{s}_{n_k}|^p_{\mathbf{h}_\mathbf{E}} \, d\lambda_{\mathbf{g}} \leq \varepsilon^p.$$

Therefore $\|\mathbf{s}_n - \mathbf{s}\|_{L^p(M,\mathbf{E})} \leq \varepsilon$ for every $n \geq N$, and we conclude that $\mathbf{s}_n \to \mathbf{s}$ in $L^p(M,\mathbf{E})$. Taking a fixed index n_0,

$$\|\mathbf{s}\|_{L^p(M,\mathbf{E})} \leq \|\mathbf{s}_{n_0}\|_{L^p(M,\mathbf{E})} + \|\mathbf{s} - \mathbf{s}_{n_0}\|_{L^p(M,\mathbf{E})} < \infty,$$

so $\mathbf{s} \in L^p(M,\mathbf{E})$.
Now consider the case $p = \infty$. By the definition of the essential supremum norm, for each pair $n, m \in \mathbb{N}$ there is a null set $Z_{n,m} \subset M$ such that

$$|\mathbf{s}_n(x) - \mathbf{s}_m(x)|_{\mathbf{h}_\mathbf{E}} \leq \|\mathbf{s}_n - \mathbf{s}_m\|_{L^\infty(M,\mathbf{E})} \qquad \text{for every } x \in M \setminus Z_{n,m}.$$

In addition, for each $n \in \mathbb{N}$ there is a null set $W_n \subset M$ such that $\pi_{\mathbf{E}}(\mathbf{s}_n(x)) = x$ for every $x \in M \setminus W_n$. Define

$$Z := \left(\bigcup_{n,m=1}^{\infty} Z_{n,m} \right) \cup \left(\bigcup_{n=1}^{\infty} W_n \right).$$

Then Z is a null set and, for every $x \in M \setminus Z$ and every $n, m \in \mathbb{N}$,

$$|\mathbf{s}_n(x) - \mathbf{s}_m(x)|_{\mathbf{h}_\mathbf{E}} \leq \|\mathbf{s}_n - \mathbf{s}_m\|_{L^\infty(M,\mathbf{E})}.$$

Since $(\mathbf{s}_n)$ is Cauchy in $L^\infty(M,\mathbf{E})$, the preceding inequality implies that, for each $x \in M \setminus Z$, the sequence

$(\mathbf{s}_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbf{E}_x$. Since $\mathbf{E}_x$ is finite dimensional, it is complete, and there exists $\mathbf{s}(x) \in \mathbf{E}_x$ such that

$$\mathbf{s}_n(x) \longrightarrow \mathbf{s}(x) \qquad \text{as } n \to \infty.$$

Define

$$\mathbf{s}(x) := \begin{cases} \lim\limits_{n\to\infty} \mathbf{s}_n(x), & x \in M \setminus Z, \\ 0_x, & x \in Z, \end{cases}$$

where 0_x denotes the zero vector of the fiber $\mathbf{E}_x$. By construction, $\pi_{\mathbf{E}}(\mathbf{s}(x)) = x$ for every $x \in M \setminus Z$, and also for $x \in Z$ because $\pi_{\mathbf{E}}(0_x) = x$. Moreover, $\mathbf{s}$ is measurable as the almost-everywhere pointwise limit of the measurable sections $\mathbf{s}_n$.
Let $\varepsilon > 0$. Since $(\mathbf{s}_n)$ is Cauchy in $L^\infty(M,\mathbf{E})$, there exists $N \in \mathbb{N}$ such that

$$\|\mathbf{s}_n - \mathbf{s}_m\|_{L^\infty(M,\mathbf{E})} < \varepsilon \qquad \text{for every } n, m \geq N.$$

By the construction of Z, for every $x \in M \setminus Z$ and every $n, m \geq N$ we have

$$|\mathbf{s}_n(x) - \mathbf{s}_m(x)|_{\mathbf{h}_\mathbf{E}} \leq \|\mathbf{s}_n - \mathbf{s}_m\|_{L^\infty(M,\mathbf{E})} < \varepsilon.$$

For fixed $n \geq N$, taking the limit as $m \to \infty$ and using $\mathbf{s}_m(x) \to \mathbf{s}(x)$ for $x \in M \setminus Z$ gives

$$|\mathbf{s}_n(x) - \mathbf{s}(x)|_{\mathbf{h}_\mathbf{E}} \leq \varepsilon \qquad \text{for every } x \in M \setminus Z.$$

Therefore

$$\|\mathbf{s}_n - \mathbf{s}\|_{L^\infty(M,\mathbf{E})} \leq \varepsilon \qquad \text{for every } n \geq N,$$

, and we conclude that $\mathbf{s}_n \to \mathbf{s}$ in $L^\infty(M,\mathbf{E})$. In particular, taking $n = N$ gives $\|\mathbf{s} - \mathbf{s}_N\|_{L^\infty(M,\mathbf{E})} \leq \varepsilon$, and since $\mathbf{s}_N \in L^\infty(M,\mathbf{E})$, we deduce that $\mathbf{s} \in L^\infty(M,\mathbf{E})$.

We conclude that every Cauchy sequence in $L^p(M,\mathbf{E})$ converges to an element of $L^p(M,\mathbf{E})$. Therefore $L^p(M,\mathbf{E})$ is complete for every $1 \leq p \leq \infty$. □

A section together with its weak covariant derivatives can be viewed as an element of a product of L^p spaces. The resulting embedding is isometric and transfers functional properties of the product to the Sobolev space.

Theorem 9.1.14. (Isometric embedding into L^p) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle with a bundle metric $\mathbf{h}_\mathbf{E}$ and a smooth connection $\nabla^\mathbf{E}$. Let $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. Then the operator $J_m \colon W^{m,p}(M,\mathbf{E}) \subseteq L^p(M,\mathbf{E}) \longrightarrow \prod_{j=0}^{m} L^p(M, T^{(0,j)}(TM) \otimes \mathbf{E})$ given by $J_m(\mathbf{u}) = \left(\nabla^j_w \mathbf{u}\right)_{j=0}^{m}$ is an isometry onto its image. Moreover, $J_m(W^{m,p}(M,\mathbf{E}))$ is closed, and therefore $W^{m,p}(M,\mathbf{E})$ is a Banach space.

Proof. By the definition of the Sobolev norm, if $1 \le p < \infty$ we have

$$\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} = \left(\sum_{j=0}^{m} \|\nabla_w^j \mathbf{u}\|^p_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})} \right)^{\frac{1}{p}},$$

whereas for $p = \infty$ we have

$$\|\mathbf{u}\|_{W^{m,\infty}(M,\mathbf{E})} = \max_{0\le j\le m} \|\nabla_w^j \mathbf{u}\|_{L^\infty(M,T^{(0,j)}(TM)\otimes\mathbf{E})}.$$

Equip the product $\prod_{j=0}^{m} L^p(M, T^{(0,j)}(TM)\otimes\mathbf{E})$ with the ℓ^p norm if $1 \le p < \infty$ and the maximum norm if $p = \infty$. Then, for every $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$,

$$\|J_m(\mathbf{u})\|_{\ell^p} = \left\| (\nabla_w^j \mathbf{u})_{j=0}^m \right\|_{\ell^p} = \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})},$$

showing that J_m is an isometry onto its image.
Now prove that $J_m(W^{m,p}(M,\mathbf{E}))$ is closed. Let

$$J_m(\mathbf{u}_k) = \left(\nabla_w^j \mathbf{u}_k\right)_{j=0}^m \in J_m(W^{m,p}(M,\mathbf{E}))$$

be a sequence such that

$$J_m(\mathbf{u}_k) \longrightarrow (\mathbf{w}_j)_{j=0}^m \quad \text{in} \quad \prod_{j=0}^{m} L^p(M, T^{(0,j)}(TM)\otimes\mathbf{E}).$$

In particular,

$$\mathbf{u}_k \to \mathbf{w}_0 \text{ in } L^p(M,\mathbf{E}), \qquad \nabla_w^j \mathbf{u}_k \to \mathbf{w}_j \text{ in } L^p\left(M, T^{(0,j)}(TM)\otimes\mathbf{E}\right), \quad j = 1,\dots,m.$$

Applying Lemma 9.1.12 gives $\mathbf{w}_0 \in W^{m,p}(M,\mathbf{E})$ and

$$\nabla_w^j \mathbf{w}_0 = \mathbf{w}_j, \qquad j = 1,\dots,m.$$

Therefore

$$(\mathbf{w}_j)_{j=0}^m = J_m(\mathbf{w}_0) \in J_m(W^{m,p}(M,\mathbf{E})),$$

proving that $J_m(W^{m,p}(M,\mathbf{E}))$ is closed.
Finally, since $\prod_{j=0}^{m} L^p(M, T^{(0,j)}(TM) \otimes \mathbf{E})$ is a Banach space and $J_m(W^{m,p}(M,\mathbf{E}))$ is a closed subspace, $J_m(W^{m,p}(M,\mathbf{E}))$ is a Banach space. Since $J_m\colon W^{m,p}(M,\mathbf{E}) \longrightarrow J_m(W^{m,p}(M,\mathbf{E}))$ is an isometry onto its image, we conclude that $W^{m,p}(M,\mathbf{E})$ is a Banach space. □

Among the properties of Sobolev spaces are, in certain cases, uniform convexity, reflexivity, and separability. We record uniform convexity in the following definition:

Definition 9.1.15. (Uniform convexity) Let $(X, \|\cdot\|)$ be a normed space. We say that X is *uniformly convex* if

for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $x, y \in X$ with

$$\|x\| = \|y\| = 1 \quad \text{and} \quad \|x - y\| \geq \varepsilon,$$

, we have

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

To establish uniform convexity, we first recall an important theorem of functional analysis, the Milman–Pettis theorem:

Theorem 9.1.16. (Milman–Pettis) Every uniformly convex Banach space is reflexive.

The following inequalities are named after James A. Clarkson [39], who studied bounds of this kind for norms on L^p spaces. In the special case of a Hilbert space, its geometric structure considerably simplifies the argument, allowing these inequalities to be deduced directly from the parallelogram law.

Lemma 9.1.17. (Clarkson-type inequalities in Hilbert spaces) Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\| \cdot \|$ its induced norm.

(a) If $2 \leq p < \infty$, then for any $u, v \in H$,

$$\left\| \frac{u+v}{2} \right\|^p + \left\| \frac{u-v}{2} \right\|^p \leq \frac{\|u\|^p + \|v\|^p}{2}.$$

(b) If $1 < p \leq 2$ and q is the conjugate exponent of p, then for any $u, v \in H$,

$$\left\| \frac{u+v}{2} \right\|^q + \left\| \frac{u-v}{2} \right\|^q \leq \left(\frac{\|u\|^p + \|v\|^p}{2} \right)^{\frac{q}{p}}.$$

The scalar inequality used below is [39, Theorem 2].

Proof. For any $u, v \in H$, the parallelogram law implies

$$\left\| \frac{u+v}{2} \right\|^2 + \left\| \frac{u-v}{2} \right\|^2 = \frac{\|u\|^2 + \|v\|^2}{2}.$$

Define

$$A = \left\| \frac{u+v}{2} \right\|^2, \qquad B = \left\| \frac{u-v}{2} \right\|^2,$$

so that

$$A + B = \frac{\|u\|^2 + \|v\|^2}{2}.$$

First prove (a). Since $p \geq 2$, we have $\frac{p}{2} \geq 1$. For $A, B \geq 0$, this gives

$$A^{\frac{p}{2}} + B^{\frac{p}{2}} \leq (A + B)^{\frac{p}{2}}.$$

On the other hand, convexity of the function $t \mapsto t^{\frac{p}{2}}$ on $[0,\infty)$ yields

$$(A+B)^{\frac{p}{2}} = \left(\frac{\|u\|^2+\|v\|^2}{2}\right)^{\frac{p}{2}} \leq \frac{\left(\|u\|^2\right)^{\frac{p}{2}}+\left(\|v\|^2\right)^{\frac{p}{2}}}{2} = \frac{\|u\|^p+\|v\|^p}{2}.$$

Substituting A and B gives the result.

Finally, prove (b). Let $x = \|u\|$ and $y = \|v\|$. By the parallelogram law, the sum of our constants is

$$A+B = \frac{x^2+y^2}{2}.$$

The triangle inequality gives the following upper and lower bounds for A:

$$\left|\frac{x-y}{2}\right| = \frac{|\|u\|-\|v\||}{2} \leq \left\|\frac{u+v}{2}\right\| \leq \frac{\|u\|+\|v\|}{2} = \frac{x+y}{2}.$$

Squaring shows that A belongs to the closed interval

$$I = \left[\left(\frac{x-y}{2}\right)^2, \left(\frac{x+y}{2}\right)^2\right].$$

To bound the quantity $A^{\frac{q}{2}}+B^{\frac{q}{2}}$, consider the real-variable function $f : I \longrightarrow \mathbb{R}$ defined by

$$f(t) = t^{\frac{q}{2}} + \left(\frac{x^2+y^2}{2} - t\right)^{\frac{q}{2}}.$$

Since $1 < p \leq 2$, the conjugate exponent satisfies $q \geq 2$, so $r = \frac{q}{2} \geq 1$. Computing the derivatives of f gives:

$$f'(t) = rt^{r-1} - r\left(\frac{x^2+y^2}{2} - t\right)^{r-1},$$
$$f''(t) = r(r-1)t^{r-2} + r(r-1)\left(\frac{x^2+y^2}{2} - t\right)^{r-2}.$$

Since $r \geq 1$, it follows that $r(r-1) \geq 0$. Moreover, on the domain under consideration, both t and $\left(\frac{x^2+y^2}{2} - t\right)$ are nonnegative, so $f''(t) \geq 0$ for every $t \in I$. Therefore f is convex.
The maximum of a convex function on a closed interval is always attained at at least one endpoint. Evaluating f at the endpoints of I gives the same value in both cases:

$$f\left(\left(\frac{x+y}{2}\right)^2\right) = f\left(\left(\frac{x-y}{2}\right)^2\right) = \left(\frac{x+y}{2}\right)^q + \left|\frac{x-y}{2}\right|^q.$$

Since the constant $A \in I$, evaluation at $t = A$ yields

$$A^{\frac{q}{2}}+B^{\frac{q}{2}} = f(A) \leq \left(\frac{x+y}{2}\right)^q + \left|\frac{x-y}{2}\right|^q.$$

This step geometrically reduces the vector problem to one involving nonnegative real scalars. Applying the

cited scalar inequality to the real numbers $x, y \geq 0$ and $1 < p \leq 2$ gives

$$\left(\frac{x+y}{2}\right)^q + \left|\frac{x-y}{2}\right|^q \leq \left(\frac{x^p+y^p}{2}\right)^{\frac{q}{p}}.$$

Combining the inequalities yields the desired result. □

Proposition 9.1.18. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with bundle metric $\mathbf{h_E}$. If $1 < p < \infty$, then $L^p(M, \mathbf{E})$ is uniformly convex.

Proof. Let $\mathbf{u}, \mathbf{v} \in L^p(M, \mathbf{E})$ be such that

$$\|\mathbf{u}\|_{L^p(M,\mathbf{E})} = \|\mathbf{v}\|_{L^p(M,\mathbf{E})} = 1, \qquad \|\mathbf{u} - \mathbf{v}\|_{L^p(M,\mathbf{E})} \geq \varepsilon,$$

with $\varepsilon > 0$. For each $x \in M$, the fiber $(\mathbf{E}_x, \mathbf{h_E}(x))$ is a Hilbert space, so Lemma 9.1.17 applies pointwise, giving two cases:

Case $2 \leq p < \infty$. Applying Lemma 9.1.17 (a) to $\mathbf{u}(x), \mathbf{v}(x) \in \mathbf{E}_x$ and then integrating with respect to $\lambda_\mathbf{g}$ gives

$$\int_M \left|\frac{\mathbf{u}+\mathbf{v}}{2}\right|^p_{\mathbf{h_E}} d\lambda_\mathbf{g} + \int_M \left|\frac{\mathbf{u}-\mathbf{v}}{2}\right|^p_{\mathbf{h_E}} d\lambda_\mathbf{g} \leq \frac{1}{2}\int_M \left(|\mathbf{u}|^p_{\mathbf{h_E}} + |\mathbf{v}|^p_{\mathbf{h_E}}\right) d\lambda_\mathbf{g}.$$

By the definition of the norm on $L^p(M, \mathbf{E})$, this is equivalent to

$$\left\|\frac{\mathbf{u}+\mathbf{v}}{2}\right\|^p_{L^p(M,\mathbf{E})} + \left\|\frac{\mathbf{u}-\mathbf{v}}{2}\right\|^p_{L^p(M,\mathbf{E})} \leq \frac{\|\mathbf{u}\|^p_{L^p(M,\mathbf{E})} + \|\mathbf{v}\|^p_{L^p(M,\mathbf{E})}}{2} = 1.$$

Moreover,

$$\left\|\frac{\mathbf{u}-\mathbf{v}}{2}\right\|^p_{L^p(M,\mathbf{E})} = \frac{1}{2^p}\|\mathbf{u}-\mathbf{v}\|^p_{L^p(M,\mathbf{E})} \geq \left(\frac{\varepsilon}{2}\right)^p.$$

Substituting into the preceding inequality,

$$\left\|\frac{\mathbf{u}+\mathbf{v}}{2}\right\|^p_{L^p(M,\mathbf{E})} \leq 1 - \left(\frac{\varepsilon}{2}\right)^p.$$

Taking pth roots,

$$\left\|\frac{\mathbf{u}+\mathbf{v}}{2}\right\|_{L^p(M,\mathbf{E})} \leq \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}$$

Defining

$$\delta(\varepsilon) := 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}} > 0,$$

gives the result.

Case $1 < p < 2$. Let q be the conjugate exponent of p. Applying Lemma 9.1.17 (b) to $\mathbf{u}(x), \mathbf{v}(x) \in \mathbf{E}_x$ gives the pointwise inequality

$$\left|\frac{\mathbf{u}+\mathbf{v}}{2}\right|^q_{\mathbf{h_E}} + \left|\frac{\mathbf{u}-\mathbf{v}}{2}\right|^q_{\mathbf{h_E}} \leq \left(\frac{|\mathbf{u}|^p_{\mathbf{h_E}} + |\mathbf{v}|^p_{\mathbf{h_E}}}{2}\right)^{\frac{q}{p}}.$$

Raising both sides to the power $\frac{p}{q}$ and integrating with respect to $\lambda_{\mathbf{g}}$ yields

$$\int_M \left(\left| \frac{\mathbf{u}+\mathbf{v}}{2} \right|_{\mathbf{h_E}}^q + \left| \frac{\mathbf{u}-\mathbf{v}}{2} \right|_{\mathbf{h_E}}^q \right)^{\frac{p}{q}} d\lambda_{\mathbf{g}} \leq \frac{1}{2} \int_M \left(|\mathbf{u}|_{\mathbf{h_E}}^p + |\mathbf{v}|_{\mathbf{h_E}}^p \right) d\lambda_{\mathbf{g}}.$$

Since $\|\mathbf{u}\|_{L^p(M,\mathbf{E})} = \|\mathbf{v}\|_{L^p(M,\mathbf{E})} = 1$, the right-hand side is exactly 1. On the other hand, since $p < 2$, the exponent satisfies $\frac{p}{q} = p - 1 < 1$. Applying the reverse Minkowski inequality of Proposition B.7.4 with $r = \frac{p}{q}$ bounds the left-hand side from below, giving:

$$\left(\int_M \left| \frac{\mathbf{u}+\mathbf{v}}{2} \right|_{\mathbf{h_E}}^p d\lambda_{\mathbf{g}} \right)^{\frac{q}{p}} + \left(\int_M \left| \frac{\mathbf{u}-\mathbf{v}}{2} \right|_{\mathbf{h_E}}^p d\lambda_{\mathbf{g}} \right)^{\frac{q}{p}} \leq \left(\int_M \left(\left| \frac{\mathbf{u}+\mathbf{v}}{2} \right|_{\mathbf{h_E}}^q + \left| \frac{\mathbf{u}-\mathbf{v}}{2} \right|_{\mathbf{h_E}}^q \right)^{\frac{p}{q}} d\lambda_{\mathbf{g}} \right)^{\frac{q}{p}}.$$

Since the integral on the right is bounded by 1 and $1^{\frac{q}{p}} = 1$, in terms of the norm on $L^p(M,\mathbf{E})$ this reads

$$\left\| \frac{\mathbf{u}+\mathbf{v}}{2} \right\|_{L^p(M,\mathbf{E})}^q + \left\| \frac{\mathbf{u}-\mathbf{v}}{2} \right\|_{L^p(M,\mathbf{E})}^q \leq 1.$$

As in the preceding case, we have

$$\left\| \frac{\mathbf{u}-\mathbf{v}}{2} \right\|_{L^p(M,\mathbf{E})}^q \geq \left(\frac{\varepsilon}{2} \right)^q,$$

which implies

$$\left\| \frac{\mathbf{u}+\mathbf{v}}{2} \right\|_{L^p(M,\mathbf{E})}^q \leq 1 - \left(\frac{\varepsilon}{2} \right)^q.$$

Taking qth roots, we conclude

$$\left\| \frac{\mathbf{u}+\mathbf{v}}{2} \right\|_{L^p(M,\mathbf{E})} \leq \left(1 - \left(\frac{\varepsilon}{2} \right)^q \right)^{\frac{1}{q}} = 1 - \delta(\varepsilon),$$

with the explicit choice

$$\delta(\varepsilon) := 1 - \left(1 - \left(\frac{\varepsilon}{2} \right)^q \right)^{\frac{1}{q}} > 0.$$

In both cases, we have found $\delta(\varepsilon) > 0$ depending only on ε and p. This proves that $L^p(M,\mathbf{E})$ is uniformly convex. □

We therefore have the following:

Corollary 9.1.19. (Uniform convexity and reflexivity of $W^{m,p}(M,\mathbf{E})$) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$ and a connection $\nabla^{\mathbf{E}}$. If $1 < p < \infty$, the Sobolev space $W^{m,p}(M,\mathbf{E})$ is uniformly convex. In particular, by Theorem 9.1.16, $W^{m,p}(M,\mathbf{E})$ is reflexive.

Stability of uniform convexity under ℓ^p products is [39, Theorem 1].

Proof. By Proposition 9.1.18, for each $j \in \{0,\dots,m\}$, $L^p(M, T^{(0,j)}(TM) \otimes \mathbf{E})$ is uniformly convex. Thus the product $\prod_{j=0}^{m} L^p(M, T^{(0,j)}(TM) \otimes \mathbf{E})$ equipped with the norm ℓ^p is uniformly convex. Uniform convexity passes to subspaces and is preserved by isometries, so Theorem 9.1.14 implies that $W^{m,p}(M,\mathbf{E})$ is uniformly con-

vex. □

Separability will be an important property, and the spaces L^p inherit it from a property of measure spaces with the same name:

Definition 9.1.20. (Separable measure space) Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $(X, \mathcal{A}, \mu)$ is a *separable measure space* if there exists a countable family of sets $\mathcal{C} \subseteq \mathcal{A}$ such that the σ-algebra $\mathcal{A}$ is contained in the completion of the σ-algebra generated by $\mathcal{C}$ (denoted by $\overline{\sigma(\mathcal{C})}^{\mu}$). That is:

$$\mathcal{A} \subseteq \overline{\sigma(\mathcal{C})}^{\mu}.$$

In the special case where $\mathcal{A}$ is already complete, this becomes the equality:

$$\mathcal{A} = \overline{\sigma(\mathcal{C})}^{\mu}.$$

For a proof of the following lemma, see [22].

Lemma 9.1.21. (Separability of $L^p(X, \mu)$) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $1 \leq p < \infty$. If the measure space $(X, \mathcal{A}, \mu)$ is separable, then $L^p(X, \mathcal{A}, \mu)$ is separable.

This tells us that $L^p(M)$ is separable for $1 \leq p < \infty$ because $(M, \mathcal{L}_M, \lambda_g)$ is a separable measure space:

Proposition 9.1.22. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $\lambda_{\mathbf{g}}$ be the associated Riemann–Lebesgue measure. Then the measure space $(M, \mathcal{L}_M, \lambda_{\mathbf{g}})$ is separable and σ-finite.

Proof. Since M is a smooth manifold, it is a second-countable topological space. In particular, its topology has a countable base of open subsets $(U_k)_{k \in \mathbb{N}}$ on M.
Let $\mathcal{B}$ be the Borel σ-algebra of M. Then

$$\mathcal{B} = \sigma\big(\{U_k \mid k \in \mathbb{N}\}\big).$$

Moreover, by Lemma 2.15.11, $\mathcal{L}_M$ agrees with the completion of $\mathcal{B}$ with respect to $\lambda_{\mathbf{g}}$, that is, $\mathcal{L}_M = \overline{\mathcal{B}}^{\lambda_{\mathbf{g}}}$, and $\lambda_{\mathbf{g}}$ is σ-finite.
Therefore

$$\mathcal{L}_M = \overline{\sigma(\{U_k \mid k \in \mathbb{N}\})}^{\lambda_{\mathbf{g}}},$$

showing that $(M, \mathcal{L}_M, \lambda_{\mathbf{g}})$ is a separable measure space. □

We use this to prove separability of $L^p(M, \mathbf{E})$:

Theorem 9.1.23. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with bundle metric $\mathbf{h}_{\mathbf{E}}$. If $1 \leq p < \infty$, then $L^p(M, \mathbf{E})$ is separable.

Proof. Let $(U_k, \varphi_k)_{k \in \mathbb{N}}$ be a countable atlas of M. Refining it by bundle trivialization domains, we may assume that for every k there is a local $\mathbf{h}_{\mathbf{E}}$-orthonormal frame $(\mathbf{e}_{1,k}, \ldots, \mathbf{e}_{r,k})$ of $\mathbf{E}$ over U_k. Let $(\psi_k)_{k \in \mathbb{N}}$ be a smooth partition of unity subordinate to $(U_k)_{k \in \mathbb{N}}$.

For each $k \in \mathbb{N}$, since $L^p(U_k)$ is separable, fix a countable dense subset $\mathcal{D}_k \subseteq L^p(U_k)$. Define

$$\mathcal{D} := \left\{ \sum_{k=1}^{N} \psi_k \left(\sum_{j=1}^{r} f_{j,k}\, \widetilde{\mathbf{e}}_{j,k} \right) \;\middle|\; N \in \mathbb{N},\ \ f_{j,k} \in \mathcal{D}_k \right\},$$

where $\widetilde{\mathbf{e}}_{j,k}$ denotes the extension of $\mathbf{e}_{j,k}$ by zero outside U_k. Then $\mathcal{D}$ is countable and $\mathcal{D} \subseteq L^p(M, \mathbf{E})$.
Let us show that $\mathcal{D}$ is dense in $L^p(M, \mathbf{E})$. Let $\mathbf{s} \in L^p(M, \mathbf{E})$ and let $\varepsilon > 0$. For $N \in \mathbb{N}$, define

$$\mathbf{s}_N := \sum_{k=1}^{N} \psi_k \mathbf{s}.$$

Writing $w_N := \sum_{k>N} \psi_k$, we have $0 \le w_N \le 1$ and $w_N(x) \to 0$ for every $x \in M$ (since $\sum_{k=1}^{\infty} \psi_k(x) = 1$). Moreover,

$$|\mathbf{s} - \mathbf{s}_N|_{\mathbf{h_E}} = |w_N \mathbf{s}|_{\mathbf{h_E}} = w_N\, |\mathbf{s}|_{\mathbf{h_E}} \le |\mathbf{s}|_{\mathbf{h_E}}.$$

Therefore

$$\|\mathbf{s} - \mathbf{s}_N\|^p_{L^p(M,\mathbf{E})} = \int_M |\mathbf{s} - \mathbf{s}_N|^p_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \int_M w_N^p\, |\mathbf{s}|^p_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \longrightarrow 0,$$

by Theorem B.6.14, since $w_N^p |\mathbf{s}|^p_{\mathbf{h_E}} \to 0$ pointwise and

$$0 \le w_N^p |\mathbf{s}|^p_{\mathbf{h_E}} \le |\mathbf{s}|^p_{\mathbf{h_E}} \in L^1(M).$$

In particular, there exists $N \in \mathbb{N}$ such that

$$\|\mathbf{s} - \mathbf{s}_N\|_{L^p(M,\mathbf{E})} < \frac{\varepsilon}{2}.$$

Now fix $k \in \{1, \dots, N\}$. On U_k, write

$$\mathbf{s} \restriction_{U_k} = \sum_{j=1}^{r} a_{j,k}\, \mathbf{e}_{j,k}, \qquad a_{j,k} \text{ measurable on } U_k.$$

Since the frame is $\mathbf{h_E}$-orthonormal, for almost every $x \in U_k$ we have

$$|\mathbf{s}(x)|^2_{\mathbf{h_E}} = \sum_{j=1}^{r} |a_{j,k}(x)|^2,$$

and in particular $|a_{j,k}(x)| \le |\mathbf{s}(x)|_{\mathbf{h_E}}$. Hence

$$\|a_{j,k}\|^p_{L^p(U_k)} = \int_{U_k} |a_{j,k}|^p\, d\lambda_{\mathbf{g}} \le \int_{U_k} |\mathbf{s}|^p_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le \int_M |\mathbf{s}|^p_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \|\mathbf{s}\|^p_{L^p(M,\mathbf{E})} < \infty,$$

, and therefore $a_{j,k} \in L^p(U_k)$ for every $j \in \{1, \dots, r\}$.
Since $\mathcal{D}_k$ is dense in $L^p(U_k)$, choose $f_{j,k} \in \mathcal{D}_k$ such that

$$\|a_{j,k} - f_{j,k}\|_{L^p(U_k)} < \frac{\varepsilon}{2Nr}, \qquad j \in \{1, \dots, r\}.$$

Define

$$u := \sum_{k=1}^{N} \psi_k \left(\sum_{j=1}^{r} f_{j,k} \widetilde{\mathbf{e}}_{j,k} \right) \in \mathcal{D}.$$

For each $k \in \{1, \dots, N\}$, using $\psi_k \le 1$ and orthonormality of the frame, for almost every $x \in U_k$ we have

$$\left| \psi_k(x) \sum_{j=1}^{r} (a_{j,k}(x) - f_{j,k}(x)) \mathbf{e}_{j,k}(x) \right|_{\mathbf{h_E}}^p = |\psi_k(x)|^p \left(\sum_{j=1}^{r} |a_{j,k}(x) - f_{j,k}(x)|^2 \right)^{\frac{p}{2}} \le \left(\sum_{j=1}^{r} |a_{j,k}(x) - f_{j,k}(x)| \right)^p.$$

Applying the inequality

$$\left(\sum_{j=1}^{r} b_j \right)^p \le r^{p-1} \sum_{j=1}^{r} b_j^p, \qquad b_j \ge 0,$$

gives

$$\left| \psi_k \sum_{j=1}^{r} (a_{j,k} - f_{j,k}) \mathbf{e}_{j,k} \right|_{\mathbf{h_E}}^p \le r^{p-1} \sum_{j=1}^{r} |a_{j,k} - f_{j,k}|^p \quad \text{in } U_k.$$

Integrating and using $\operatorname{supp}(\psi_k) \subset U_k$ yields

$$\left\| \psi_k \sum_{j=1}^{r} (a_{j,k} - f_{j,k}) \widetilde{\mathbf{e}}_{j,k} \right\|_{L^p(M,\mathbf{E})}^p \le r^{p-1} \sum_{j=1}^{r} \|a_{j,k} - f_{j,k}\|_{L^p(U_k)}^p.$$

In particular,

$$\left\| \psi_k \sum_{j=1}^{r} (a_{j,k} - f_{j,k}) \widetilde{\mathbf{e}}_{j,k} \right\|_{L^p(M,\mathbf{E})} \le r^{1-\frac{1}{p}} \left(\sum_{j=1}^{r} \|a_{j,k} - f_{j,k}\|_{L^p(U_k)}^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{2N}.$$

Summing over $k \in \{1, \dots, N\}$,

$$\|\mathbf{s}_N - u\|_{L^p(M,\mathbf{E})} \le \sum_{k=1}^{N} \left\| \psi_k \sum_{j=1}^{r} (a_{j,k} - f_{j,k}) \widetilde{\mathbf{e}}_{j,k} \right\|_{L^p(M,\mathbf{E})} < \sum_{k=1}^{N} \frac{\varepsilon}{2N} = \frac{\varepsilon}{2}.$$

Finally,

$$\|\mathbf{s} - u\|_{L^p(M,\mathbf{E})} \le \|\mathbf{s} - \mathbf{s}_N\|_{L^p(M,\mathbf{E})} + \|\mathbf{s}_N - u\|_{L^p(M,\mathbf{E})} < \varepsilon.$$

Since $\mathbf{s}$ and ε were arbitrary, $\mathcal{D}$ is dense in $L^p(M,\mathbf{E})$. Consequently, $L^p(M,\mathbf{E})$ is separable. □

Corollary 9.1.24. (Separability of $W^{m,p}(M,\mathbf{E})$) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$ and a connection $\nabla^{\mathbf{E}}$. If $1 \le p < \infty$, the Sobolev space $W^{m,p}(M,\mathbf{E})$ is separable.

Proof. By Theorem 9.1.14, the operator

$$J_m \colon W^{m,p}(M,\mathbf{E}) \longrightarrow \prod_{j=0}^{m} L^p\left(M, T^{(0,j)}(TM) \otimes \mathbf{E}\right), \qquad J_m(u) = \left((\nabla^{\mathbf{E}})^j u\right)_{j=0}^m,$$

is an isometry onto its image, and $J_m\big(W^{m,p}(M,\mathbf{E})\big)$ is a closed subspace.
By Theorem 9.1.23, for each $j \in \{0,\dots,m\}$ the space $L^p\big(M, T^{(0,j)}(TM)\otimes\mathbf{E}\big)$ is separable. Consequently, the finite product $\prod_{j=0}^{m} L^p\big(M, T^{(0,j)}(TM)\otimes\mathbf{E}\big)$ is separable.
Since every subspace of a separable metric space is separable, $J_m\big(W^{m,p}(M,\mathbf{E})\big)$ is separable. Finally, since J_m is an isometry onto its image (and hence a homeomorphism onto its image), we conclude that $W^{m,p}(M,\mathbf{E})$ is separable. □

We now study the dual of $L^1(M,\mathbf{E})$. Besides extending classical L^1–L^∞ duality to sections, this identification gives a concrete description of the weak topology of $L^1(M,\mathbf{E})$ through the bundle inner product and integration.

Proposition 9.1.25. (Dual of $L^1(M,\mathbf{E})$) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E}\to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_E}$. Then the map

$$J_{\mathbf{E}}\colon L^\infty(M,\mathbf{E}) \longrightarrow (L^1(M,\mathbf{E}))', \qquad J_{\mathbf{E}}(\mathbf{v})(\mathbf{u}) := \int_M \langle \mathbf{u},\mathbf{v}\rangle_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}},$$

is an isometric isomorphism. In particular, $(L^1(M,\mathbf{E}))' \cong L^\infty(M,\mathbf{E})$ and $\sigma(L^1(M,\mathbf{E}),L^\infty(M,\mathbf{E}))$ agrees with $\sigma(L^1(M,\mathbf{E}),(L^1(M,\mathbf{E}))')$.

Proof. Riemann–Lebesgue measure is σ-finite. For $\mathbf{v}\in L^\infty(M,\mathbf{E})$, Hölder's inequality (Proposition B.7.9) implies $|J_{\mathbf{E}}(\mathbf{v})(\mathbf{u})| \le \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}\|\mathbf{u}\|_{L^1(M,\mathbf{E})}$ for every $\mathbf{u}\in L^1(M,\mathbf{E})$, so $J_{\mathbf{E}}$ is well defined and $\|J_{\mathbf{E}}(\mathbf{v})\| \le \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$. Let us show that $J_{\mathbf{E}}$ is surjective. Let $\Lambda\in(L^1(M,\mathbf{E}))'$. Take a countable open cover $(U_j)_{j\in\mathbb{N}}$ of M, with a local trivialization of $\mathbf{E}$ defined over each U_j, and fix a smooth local orthonormal frame $(\mathbf{e}_{j,1},\dots,\mathbf{e}_{j,r})$ on each of these open subsets. Define the disjoint Borel sets $A_1 := U_1$ and

$$A_j := U_j \setminus \bigcup_{i=1}^{j-1} U_i, \qquad j\ge 2.$$

Then $M = \bigcup_{j\in\mathbb{N}} A_j$. For $j\in\mathbb{N}$ and $a\in\{1,\dots,r\}$, define

$$\Lambda_{j,a}\colon L^1(A_j)\longrightarrow\mathbb{R}, \qquad \Lambda_{j,a}(f) := \Lambda(\widetilde{f\mathbf{e}_{j,a}}),$$

where the tilde denotes extension by zero outside A_j. Since Λ is continuous,

$$|\Lambda_{j,a}(f)| = |\Lambda(\widetilde{f\mathbf{e}_{j,a}})| \le \|\Lambda\|\,\|\widetilde{f\mathbf{e}_{j,a}}\|_{L^1(M,\mathbf{E})}.$$

Moreover, since the frame is orthonormal, $|\mathbf{e}_{j,a}|_{\mathbf{h_E}} = 1$ on A_j, so

$$\|\widetilde{f\mathbf{e}_{j,a}}\|_{L^1(M,\mathbf{E})} = \int_{A_j} |f\mathbf{e}_{j,a}|_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \int_{A_j} |f|\, d\lambda_{\mathbf{g}} = \|f\|_{L^1(A_j)}.$$

Consequently, $|\Lambda_{j,a}(f)| \le \|\Lambda\|\|f\|_{L^1(A_j)}$, and hence $\|\Lambda_{j,a}\| \le \|\Lambda\|$. By Theorem B.7.7, applied to $L^1(A_j)$, there

exists a unique $v_j^a \in L^\infty(A_j)$ such that

$$\Lambda_{j,a}(f) = \int_{A_j} f v_j^a \, d\lambda_{\mathbf{g}}, \qquad \|v_j^a\|_{L^\infty(A_j)} = \|\Lambda_{j,a}\| \le \|\Lambda\|.$$

Define a measurable section $\mathbf{v}$ by $\mathbf{v} \restriction_{A_j} := \sum_{a=1}^{r} v_j^a \mathbf{e}_{j,a}$. Since the frames are orthonormal, $|\mathbf{v}|_{\mathbf{h_E}}^2 = \sum_{a=1}^{r} |v_j^a|^2 \le r\|\Lambda\|^2$ almost everywhere on A_j, giving $|\mathbf{v}|_{\mathbf{h_E}} \le \sqrt{r}\|\Lambda\|$ almost everywhere and therefore $\mathbf{v} \in L^\infty(M, \mathbf{E})$.

For $\mathbf{u} \in L^1(M, \mathbf{E})$, write $u_j^a := \langle \mathbf{u}, \mathbf{e}_{j,a} \rangle_{\mathbf{h_E}}$ on A_j and define $\mathbf{u}_N := \sum_{j=1}^{N} \mathbf{1}_{A_j} \mathbf{u}$. Since

$$\|\mathbf{u} - \mathbf{u}_N\|_{L^1(M,\mathbf{E})} = \sum_{j>N} \int_{A_j} |\mathbf{u}|_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}} \longrightarrow 0,$$

we have $\mathbf{u}_N \to \mathbf{u}$ in $L^1(M, \mathbf{E})$. Moreover, $\mathbf{u}_N = \sum_{j=1}^{N} \sum_{a=1}^{r} \widetilde{u_j^a e_{j,a}}$.

By linearity of Λ and the definition of $\Lambda_{j,a}$,

$$\Lambda(\mathbf{u}_N) = \sum_{j=1}^{N} \sum_{a=1}^{r} \Lambda\big(\widetilde{u_j^a e_{j,a}}\big) = \sum_{j=1}^{N} \sum_{a=1}^{r} \Lambda_{j,a}(u_j^a) = \sum_{j=1}^{N} \sum_{a=1}^{r} \int_{A_j} u_j^a v_j^a \, d\lambda_{\mathbf{g}}.$$

Since the frames are orthonormal, $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{h_E}} = \sum_{a=1}^{r} u_j^a v_j^a$ on each A_j, so

$$\sum_{j=1}^{N} \sum_{a=1}^{r} \int_{A_j} u_j^a v_j^a \, d\lambda_{\mathbf{g}} = \int_M \mathbf{1}_{\bigcup_{j=1}^{N} A_j} \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}}.$$

Moreover,

$$\mathbf{1}_{\bigcup_{j=1}^{N} A_j} \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{h_E}} \longrightarrow \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{h_E}} \quad \text{almost everywhere,}$$

and

$$\left| \mathbf{1}_{\bigcup_{j=1}^{N} A_j} \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{h_E}} \right| \le |\mathbf{u}|_{\mathbf{h_E}} |\mathbf{v}|_{\mathbf{h_E}} \in L^1(M),$$

since $\mathbf{u} \in L^1(M, \mathbf{E})$ and $\mathbf{v} \in L^\infty(M, \mathbf{E})$. By Theorem B.6.14 and continuity of Λ,

$$\Lambda(\mathbf{u}) = \lim_{N\to\infty} \Lambda(\mathbf{u}_N) = \lim_{N\to\infty} \sum_{j=1}^{N} \sum_{a=1}^{r} \int_{A_j} u_j^a v_j^a \, d\lambda_{\mathbf{g}} = \int_M \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}}.$$

Consequently, $\Lambda = J_{\mathbf{E}}(\mathbf{v})$.

Finally, prove that $J_{\mathbf{E}}$ is isometric. We already know that $\|J_{\mathbf{E}}(\mathbf{v})\| \le \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$. Suppose that $\mathbf{v} \neq 0$ and let $0 < \varepsilon < \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$. By the definition of essential supremum, the set

$$A_\varepsilon := \left\{ x \in M \;\middle|\; |\mathbf{v}(x)|_{\mathbf{h_E}} > \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})} - \varepsilon \right\}$$

has positive measure, since otherwise $|\mathbf{v}|_{\mathbf{h_E}} \le \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})} - \varepsilon$ almost everywhere, contradicting the definition of $\|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$. Since $\lambda_{\mathbf{g}}$ is σ-finite, there exist measurable sets $(C_k)_{k\in\mathbb{N}}$ such that $M = \bigcup_{k\in\mathbb{N}} C_k$ and $\lambda_{\mathbf{g}}(C_k) < \infty$. Since $A_\varepsilon = \bigcup_{k\in\mathbb{N}} (A_\varepsilon \cap C_k)$ has positive measure, there exists $k \in \mathbb{N}$ such that $B_\varepsilon := A_\varepsilon \cap C_k$ satisfies $0 < \lambda_{\mathbf{g}}(B_\varepsilon) < \infty$.

Define

$$\mathbf{u}_\varepsilon(x) := \begin{cases} \dfrac{\mathbf{v}(x)}{\lambda_{\mathbf{g}}(B_\varepsilon)|\mathbf{v}(x)|_{\mathbf{h_E}}}, & x \in B_\varepsilon, \\ 0_x, & x \notin B_\varepsilon. \end{cases}$$

Since $B_\varepsilon \subseteq A_\varepsilon$ and $\varepsilon < \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$, we have $|\mathbf{v}|_{\mathbf{h_E}} > 0$ on B_ε. Moreover,

$$\|\mathbf{u}_\varepsilon\|_{L^1(M,\mathbf{E})} = \frac{1}{\lambda_{\mathbf{g}}(B_\varepsilon)} \int_{B_\varepsilon} 1 \, d\lambda_{\mathbf{g}} = 1,$$

and

$$J_{\mathbf{E}}(\mathbf{v})(\mathbf{u}_\varepsilon) = \frac{1}{\lambda_{\mathbf{g}}(B_\varepsilon)} \int_{B_\varepsilon} |\mathbf{v}|_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}} > \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})} - \varepsilon.$$

By the definition of the operator norm,

$$\|J_{\mathbf{E}}(\mathbf{v})\| = \sup_{\substack{\mathbf{u}\in L^1(M,\mathbf{E}) \\ \|\mathbf{u}\|_{L^1(M,\mathbf{E})}\le 1}} |J_{\mathbf{E}}(\mathbf{v})(\mathbf{u})| \ge |J_{\mathbf{E}}(\mathbf{v})(\mathbf{u}_\varepsilon)| > \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})} - \varepsilon.$$

Since this inequality holds for every $0 < \varepsilon < \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$, letting $\varepsilon \to 0$ gives $\|J_{\mathbf{E}}(\mathbf{v})\| \ge \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$. Consequently, $\|J_{\mathbf{E}}(\mathbf{v})\| = \|\mathbf{v}\|_{L^\infty(M,\mathbf{E})}$, and $J_{\mathbf{E}}$ is an isometric isomorphism. □

For a smooth *real* vector bundle $\mathbf{E} \to M$ of finite rank equipped with a bundle metric $\mathbf{h_E}$, consider on $L^1(M,\mathbf{E}) \times L^\infty(M,\mathbf{E})$ the duality

$$\mathfrak{b}_{\mathbf{E}}(u,\mathbf{v}) := \int_M \langle u, \mathbf{v}\rangle_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}}.$$

By Proposition 9.1.25, the pairing $\mathfrak{b}_{\mathbf{E}}$ identifies $L^\infty(M,\mathbf{E})$ isometrically with $(L^1(M,\mathbf{E}))'$. Consequently,

$$\sigma(L^1(M,\mathbf{E}), L^\infty(M,\mathbf{E})) = \sigma(L^1(M,\mathbf{E}), (L^1(M,\mathbf{E}))'),$$

that is, $\sigma(L^1(M,\mathbf{E}), L^\infty(M,\mathbf{E}))$ agrees with the weak topology of $L^1(M,\mathbf{E})$. Thus a net $(u_\gamma)_\gamma$ converges to u in this topology if and only if

$$\int_M \langle u_\gamma, \mathbf{v}\rangle_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}} \longrightarrow \int_M \langle u, \mathbf{v}\rangle_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}}$$

for every $\mathbf{v} \in L^\infty(M,\mathbf{E})$.

If $\mathbf{E}$ is complex Hermitian, the complex-linear formulation replaces the second factor with $L^\infty(M,\overline{\mathbf{E}})$ and uses $\mathfrak{b}_{\mathbf{E}}(u,\overline{v}) = \int_M \mathbf{h_E}(u,v) \, d\lambda_{\mathbf{g}}$; this version, for every $1 \le p < \infty$, is proved in Corollary B.7.8 of Appendix B.

9.1.2 Compatibility with restrictions and local expressions for covariant derivatives on vector bundles

Another important property of Sobolev sections is that restriction to an open subset gives a Sobolev section on that open subset.

Proposition 9.1.26. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a smooth connection $\nabla^{\mathbf{E}}$. If $U\subseteq M$ is open, $\mathbf{u}\in W^{m,p}(M,\mathbf{E})$, and $s\in\{0,\dots,m\}$, then $\mathbf{u}\restriction_U\in W^{m,p}(U,\mathbf{E}\restriction_U)$ and $\nabla^s_w(\mathbf{u}\restriction_U)=(\nabla^s_w\mathbf{u})\restriction_U$ almost everywhere on U.

Proof. Let $s\leq m$. We must prove that $\mathbf{u}\restriction_U$ has a weak covariant derivative of order s on $L^p(U,T^{(0,s)}(TU)\otimes\mathbf{E}\restriction_U)$. To this end, take $\sigma\in\Gamma_c(T^{(0,s)}(TU)\otimes\mathbf{E}\restriction_U)$. We may extend σ to $\widetilde{\sigma}\in\Gamma_c(T^{(0,s)}(TM)\otimes\mathbf{E})$ as in Lemma 9.1.6. On the other hand, since $\mathbf{u}\in W^{m,p}(M,\mathbf{E})$, there exists $\nabla^s_w\mathbf{u}$, the sth weak covariant derivative of $\mathbf{u}$. It satisfies

$$\int_M\langle\nabla^s_w\mathbf{u},\widetilde{\sigma}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}d\lambda_{\mathbf{g}}=\int_M\langle\mathbf{u},(\nabla^s)^*\widetilde{\sigma}\rangle_{\mathbf{h}_{\mathbf{E}}}d\lambda_{\mathbf{g}}.$$

Since $\sigma\equiv\widetilde{\sigma}$ on U, which is open, Remark 8.1.16 gives $(\nabla^s)^*\widetilde{\sigma}\equiv(\nabla^s)^*\sigma$ on U. Then

$$\int_U\langle(\nabla^s_w\mathbf{u})\restriction_U,\sigma\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}d\lambda_{\mathbf{g}}=\int_M\langle\nabla^s_w\mathbf{u},\widetilde{\sigma}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}d\lambda_{\mathbf{g}}=\int_M\langle\mathbf{u},(\nabla^s)^*\widetilde{\sigma}\rangle_{\mathbf{h}_{\mathbf{E}}}d\lambda_{\mathbf{g}}=\int_U\langle\mathbf{u}\restriction_U,(\nabla^s)^*\sigma\rangle_{\mathbf{h}_{\mathbf{E}}}d\lambda_{\mathbf{g}},$$

where the last equality holds because Proposition 8.1.15 gives $\operatorname{supp}((\nabla^s)^*\widetilde{\sigma})\subseteq\operatorname{supp}(\widetilde{\sigma})=\operatorname{supp}(\sigma)\subseteq U$ and $(\nabla^s)^*\widetilde{\sigma}\equiv(\nabla^s)^*\sigma$ on U. Thus $(\nabla^s_w\mathbf{u})\restriction_U$ satisfies the identity characterizing the sth weak covariant derivative of $\mathbf{u}\restriction_U$. By uniqueness of the weak covariant derivative, $\nabla^s_w(\mathbf{u}\restriction_U)=(\nabla^s_w\mathbf{u})\restriction_U$ almost everywhere on U. Since $s\leq m$ was arbitrary, $\mathbf{u}\restriction_U\in W^{m,p}(U,\mathbf{E}\restriction_U)$. □

Studying local properties of Sobolev spaces is important because it gives a way to work with them computationally. To achieve this, the reader should first become thoroughly familiar with coordinate expressions for higher-order covariant derivatives on smooth vector bundles. Such expressions are common in the literature in certain cases, but higher orders are less often treated because of their complexity. In this section, we calculate these local expressions in detail.

Lemma 9.1.27. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary with a connection ∇ on TM, and let $\mathbf{E}$ be a smooth vector bundle with a connection $\nabla^{\mathbf{E}}$. Also denote by ∇ the induced connection on the tensor bundle $T^{(k,l)}(TM)\otimes\mathbf{E}$ characterized by Proposition 2.4.27. Let $U\subseteq M$ be an open subset on which $(\mathbf{E}_1,\dots,\mathbf{E}_n)$ is a local frame of TM over U, $(\varepsilon^1,\dots,\varepsilon^n)$ its associated dual coframe, and $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ a local frame of $\mathbf{E}$ over U. If $\mathbf{F}\in\Gamma(T^{(k,l)}(TM)\otimes\mathbf{E})$ is expressed locally as

$$\mathbf{F}=F^{i_1\dots i_k a}_{j_1\dots j_l}\mathbf{E}_{i_1}\otimes\cdots\otimes\mathbf{E}_{i_k}\otimes\varepsilon^{j_1}\otimes\cdots\otimes\varepsilon^{j_l}\otimes\mathbf{e}_a,$$

then the components of $\nabla\mathbf{F}$ in these frames are

$$(\nabla_{\mathbf{E}_c}\mathbf{F})^{i_1\dots i_k a}_{j_1\dots j_l}=\mathbf{E}_c\big(F^{i_1\dots i_k a}_{j_1\dots j_l}\big)+\sum_{t=1}^{k}\Gamma^{i_t}_{cs}F^{i_1\dots s\dots i_k a}_{j_1\dots j_l}-\sum_{t=1}^{l}\Gamma^{s}_{cj_t}F^{i_1\dots i_k a}_{j_1\dots s\dots j_l}+\sum_{b=1}^{r}(A_c)^a_b F^{i_1\dots i_k b}_{j_1\dots j_l},$$

where the connection coefficients are given by $\nabla_{\mathbf{E}_c}\mathbf{E}_j=\Gamma^s_{cj}\mathbf{E}_s$ and $\nabla^{\mathbf{E}}_{\mathbf{E}_c}\mathbf{e}_b=(A_c)^a_b\mathbf{e}_a$.

Proof. The covariant derivative of $\mathbf{F}$ in the direction $\mathbf{E}_c$ is

$$\nabla_{\mathbf{E}_c}\mathbf{F} = \nabla_{\mathbf{E}_c}\left(\sum_{a=1}^{r}\mathbf{F}^{(a)}\otimes\mathbf{e}_a\right),$$

where $\mathbf{F}^{(a)} \in \Gamma(T^{(k,l)}(TM))$ is the tensor field with components $(\mathbf{F}^{(a)})^{i_1\dots i_k}_{j_1\dots j_l} = F^{i_1\dots i_k a}_{j_1\dots j_l}$. By Proposition 2.4.27, applying the Leibniz rule gives

$$\nabla_{\mathbf{E}_c}\mathbf{F} = \sum_{a=1}^{r}\left((\nabla_{\mathbf{E}_c}\mathbf{F}^{(a)})\otimes\mathbf{e}_a + \mathbf{F}^{(a)}\otimes\nabla^{\mathbf{E}}_{\mathbf{E}_c}\mathbf{e}_a\right). \tag{9.2}$$

By Proposition 2.4.15, the coefficients of the first term in (9.2) are

$$(\nabla_{\mathbf{E}_c}\mathbf{F}^{(a)})^{i_1\dots i_k}_{j_1\dots j_l} = \mathbf{E}_c(F^{i_1\dots i_k a}_{j_1\dots j_l}) + \sum_{t=1}^{k}\Gamma^{i_t}_{cs}F^{i_1\dots s\dots i_k a}_{j_1\dots j_l} - \sum_{t=1}^{l}\Gamma^{s}_{cj_t}F^{i_1\dots i_k a}_{j_1\dots s\dots j_l}.$$

For the second term in (9.2), substitute $\nabla^{\mathbf{E}}_{\mathbf{E}_c}\mathbf{e}_b = (A_c)^a_b\mathbf{e}_a$. Changing the summation index for the local frame $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ to b in (9.2) yields

$$\sum_{b=1}^{r}\mathbf{F}^{(b)}\otimes\nabla^{\mathbf{E}}_{\mathbf{E}_c}\mathbf{e}_b = \sum_{b=1}^{r}\mathbf{F}^{(b)}\otimes\left(\sum_{a=1}^{r}(A_c)^a_b\mathbf{e}_a\right) = \sum_{a=1}^{r}\left(\sum_{b=1}^{r}(A_c)^a_b\mathbf{F}^{(b)}\right)\otimes\mathbf{e}_a.$$

The component of this term corresponding to $\mathbf{e}_a$ is $\displaystyle\sum_{b=1}^{r}(A_c)^a_b F^{i_1\dots i_k b}_{j_1\dots j_l}$, completing the proof. □

This gives the following local expression for higher-order derivatives:

Lemma 9.1.28. (Local expression for the sth covariant derivative) Let M be a smooth manifold with or without boundary equipped with a connection ∇ on TM, and let $\pi_{\mathbf{E}}\colon \mathbf{E}\longrightarrow M$ be a smooth vector bundle of rank r equipped with a connection $\nabla^{\mathbf{E}}$. Let $(U,x^1,\dots,x^n)$ be a smooth chart of M and $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ a local frame of $\mathbf{E}$ over U. Denote the connection coefficients by

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n}\Gamma^k_{ij}\partial_k, \qquad \nabla^{\mathbf{E}}_{\partial_i}\mathbf{e}_b = \sum_{a=1}^{r}(A_i)^a_b\mathbf{e}_a.$$

If $\mathbf{u}\in\Gamma(\mathbf{E})$ has coordinate expression $\mathbf{u} = \displaystyle\sum_{b=1}^{r}u^b\mathbf{e}_b$ on U, then for every integer $s\geq 1$, the components of the sth covariant derivative $\nabla^s\mathbf{u}\in\Gamma(T^{(0,s)}(TM)\otimes\mathbf{E})$ are given by

$$(\nabla^s\mathbf{u})^a_{i_1\dots i_s} = \partial_{i_1}\cdots\partial_{i_s}u^a + \sum_{b=1}^{r}\sum_{|\beta|\leq s-1}\left(B_\beta\right)^a_{b\,i_1\dots i_s}\partial^\beta u^b, \tag{9.3}$$

where the coefficients $\left(B_\beta\right)^a_{b\,i_1\dots i_s}$ are smooth functions on U depending only on the connection coefficients Γ^k_{ij}, $(A_i)^a_b$, and their partial derivatives through order $s-1$.

Proof. We prove the assertion by induction on s. For $s = 1$, the expression for the connection on $\mathbf{E}$ in the chosen frame gives

$$(\nabla \mathbf{u})^a_i = \partial_i u^a + \sum_{b=1}^{r} (A_i)^a{}_b u^b.$$

This is (9.3) with $\big(B_0^{(1)}\big)^a_{b,i} = (A_i)^a{}_b$.

Now suppose that, for some $s \geq 1$, there exist smooth coefficients $B_\beta^{(s)}$, with $|\beta| \leq s-1$, for which

$$(\nabla^s \mathbf{u})^a_{i_2 \dots i_{s+1}} = \partial_{i_2} \cdots \partial_{i_{s+1}} u^a + \sum_{b=1}^{r} \sum_{|\beta| \leq s-1} \big(B_\beta^{(s)}\big)^a_{b,i_2 \dots i_{s+1}} \partial^\beta u^b. \tag{9.4}$$

As part of the induction hypothesis, also suppose these coefficients depend only on Γ^k_{ij}, $(A_i)^a{}_b$, and their partial derivatives of order at most $s-1$.

Set $\mathbf{G} := \nabla^s \mathbf{u}$ and abbreviate $I = (i_2, \dots, i_{s+1})$. For $t \in \{2, \dots, s+1\}$, denote by $I_t(d)$ the list obtained from I by replacing i_t with d. The formula for the induced connection on $T^{(0,s)}(TM) \otimes \mathbf{E}$ gives

$$(\nabla \mathbf{G})^a_{i_1 I} = \partial_{i_1} G^a_I - \sum_{t=2}^{s+1} \sum_{d=1}^{n} \Gamma^d_{i_1 i_t} G^a_{I_t(d)} + \sum_{c=1}^{r} (A_{i_1})^a{}_c G^c_I. \tag{9.5}$$

Substitute (9.4) into each of the three terms of (9.5). In the first, the Leibniz rule gives

$$\partial_{i_1} G^a_I = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{s+1}} u^a + \sum_{b=1}^{r} \sum_{|\beta| \leq s-1} \partial_{i_1} \Big(\big(B_\beta^{(s)}\big)^a_{b,I} \Big) \partial^\beta u^b + \sum_{b=1}^{r} \sum_{|\beta| \leq s-1} \big(B_\beta^{(s)}\big)^a_{b,I} \partial^{\beta + e_{i_1}} u^b,$$

where e_{i_1} is the multi-index whose only nonzero component is the i_1th. The second term becomes

$$-\sum_{t=2}^{s+1} \sum_{d=1}^{n} \Gamma^d_{i_1 i_t} G^a_{I_t(d)} = -\sum_{t=2}^{s+1} \sum_{d=1}^{n} \Gamma^d_{i_1 i_t} \partial_{i_2} \cdots \partial_{i_{t-1}} \partial_d \partial_{i_{t+1}} \cdots \partial_{i_{s+1}} u^a - \sum_{t=2}^{s+1} \sum_{d=1}^{n} \sum_{b=1}^{r} \sum_{|\beta| \leq s-1} \Gamma^d_{i_1 i_t} \big(B_\beta^{(s)}\big)^a_{b,I_t(d)} \partial^\beta u^b.$$

Finally, the term corresponding to the connection on $\mathbf{E}$ is

$$\sum_{c=1}^{r} (A_{i_1})^a{}_c G^c_I = \sum_{c=1}^{r} (A_{i_1})^a{}_c \partial_{i_2} \cdots \partial_{i_{s+1}} u^c + \sum_{c=1}^{r} \sum_{b=1}^{r} \sum_{|\beta| \leq s-1} (A_{i_1})^a{}_c \big(B_\beta^{(s)}\big)^c_{b,I} \partial^\beta u^b.$$

These three equalities contain exactly one term with $s+1$ derivatives of a component of $\mathbf{u}$, namely

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{s+1}} u^a.$$

The terms in the third line of the first expansion contain derivatives of order $|\beta| + 1 \leq s$; the first summand of the second expansion and the first of the third contain derivatives of order s; and all remaining summands contain derivatives of order at most $s-1$.

For fixed $a, b, i_1, \dots, i_{s+1}$ and a multi-index γ with $|\gamma| \leq s$, define $\big(B_\gamma^{(s+1)}\big)^a_{b,i_1 \dots i_{s+1}}$ as the sum of the coefficients multiplying $\partial^\gamma u^b$ in the preceding three expansions. This is a finite sum of products of Γ^k_{ij}, $(A_i)^a{}_b$, $B_\beta^{(s)}$, and $\partial_{i_1} B_\beta^{(s)}$. By the induction hypothesis, $B_\beta^{(s)}$ depends on the connection coefficients and their derivatives

through order $s-1$, while $\partial_{i_1} B_\beta^{(s)}$ depends on them through order s. Consequently, $B_\gamma^{(s+1)}$ is smooth and depends only on Γ_{ij}^k, $(A_i)^a{}_b$, and their partial derivatives of order at most s. With this definition, we obtain

$$(\nabla^{s+1}\mathbf{u})^a_{i_1\dots i_{s+1}} = \partial_{i_1}\cdots\partial_{i_{s+1}}u^a + \sum_{b=1}^{r}\sum_{|\gamma|\le s}\big(B_\gamma^{(s+1)}\big)^a_{b,i_1\dots i_{s+1}}\partial^\gamma u^b,$$

which is the assertion for $s+1$ and completes the induction. □

9.1.3 The integration-by-parts formula for vector bundles and the formal adjoint

One of the most important examples of a formal adjoint is that of the sth covariant derivative. This adjoint can be computed globally and explicitly from an integration-by-parts formula. This is natural, since the concept of formal adjoint is motivated precisely by the integration-by-parts formulas that gave rise to weak derivatives in the Euclidean case.

In this section, we prove an original result found in [41] that made possible an intrinsic, unified treatment of Sobolev spaces. Its derivation was initially motivated by [33], which sought to establish the Meyers–Serrin theorem on manifolds in certain cases by giving explicit expressions for first- and second-order weak covariant derivatives. Subsequent analysis of these ideas culminated in the work mentioned first. The result is as follows:

Theorem 9.1.29. (Integration-by-parts formula for higher-order covariant derivatives on vector bundles) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary and Levi–Civita connection ∇. Let $\mathbf{E}\longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a compatible connection $\nabla^\mathbf{E}$. Also denote by ∇ the induced connection on the tensor bundle $T^{(k,l)}(TM)\otimes\mathbf{E}$ characterized by Proposition 2.4.27. If $s\ge 1$, $\mathbf{F}\in\Gamma_c\big(T^{(k,l)}(TM)\otimes\mathbf{E}\big)$, and $\mathbf{G}\in\Gamma_c\big(T^{(k,l+s)}(TM)\otimes\mathbf{E}\big)$, then

$$\int_M \langle\nabla^s\mathbf{F},\mathbf{G}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\,d\lambda_\mathbf{g} = (-1)^s\int_M\big\langle\mathbf{F},(\mathrm{tr}_\mathbf{g}\circ\nabla)^s\mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\,d\lambda_\mathbf{g}$$

$$+\sum_{j=0}^{s-1}(-1)^{s-1-j}\int_{\partial M}\Big\langle\nabla^j\mathbf{F},\iota_{\boldsymbol{\nu}}\big((\mathrm{tr}_\mathbf{g}\circ\nabla)^{s-1-j}\mathbf{G}\big)\Big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\,d\lambda_{\widetilde{\mathbf{g}}},$$

where $\boldsymbol{\nu}$ is the outward unit normal to ∂M and $\iota_{\boldsymbol{\nu}}$ denotes contraction of $\boldsymbol{\nu}$ in the first covariant index. More precisely, if

$$\mathbf{H}\in\Gamma\big(T^{(k,m+1)}(TM)\otimes\mathbf{E}\big),$$

then $\iota_{\boldsymbol{\nu}}\mathbf{H}\in\Gamma\big((T^{(k,m)}(TM)\otimes\mathbf{E})\restriction_{\partial M}\big)$ and, in local coordinates on the boundary,

$$\big(\iota_{\boldsymbol{\nu}}\mathbf{H}\big)^{i_1\dots i_k a}_{j_1\dots j_m} := \nu^q\,H^{i_1\dots i_k a}_{qj_1\dots j_m},$$

where $\boldsymbol{\nu}=\nu^q\partial_q$ in those coordinates. If $\partial M=\varnothing$, the sum of integrals over ∂M is interpreted as zero, and no choice of normal is involved.

Here $\mathrm{tr}_\mathbf{g}$ denotes contraction, using $\mathbf{g}$, of the first two covariant indices of $\nabla\mathbf{H}$. More precisely, if $m\ge 1$ and $\mathbf{H}\in\Gamma\big(T^{(k,m)}(TM)\otimes\mathbf{E}\big)$, then $\nabla\mathbf{H}\in\Gamma\big(T^{(k,m+1)}(TM)\otimes\mathbf{E}\big)$ and, in local coordinates,

$$\big(\mathrm{tr}_\mathbf{g}(\nabla\mathbf{H})\big)^{i_1\dots i_k a}_{j_1\dots j_{m-1}} := g^{pq}\,(\nabla\mathbf{H})^{i_1\dots i_k a}_{pqj_1\dots j_{m-1}},$$

that is, the new derivative index is contracted with the first covariant index inherited from $\mathbf{H}$.

Before proving the integration-by-parts formula, we recover several classical identities as special cases of Theorem 9.1.29. These examples will clarify the meaning of each contraction and each boundary term.

Example 9.1.30. (Classical integration-by-parts formulas as special cases of Theorem 9.1.29) We show how several classical integration-by-parts formulas arise as special cases of Theorem 9.1.29. In every part we take $s = 1$ and $k = l = 0$, so that

$$\mathbf{F} \in \Gamma_c(\mathbf{E}), \qquad \mathbf{G} \in \Gamma_c(T^*M \otimes \mathbf{E}).$$

When $\mathbf{E}$ is trivial of rank 1, we identify $\Gamma_c(\mathbf{E})$ with $C_c^\infty(M)$ and $\Gamma_c(T^*M \otimes \mathbf{E})$ with $\Gamma_c(T^*M)$.
Recall that, if $(M, \mathbf{g})$ is a Riemannian manifold, the musical isomorphism $\flat\colon T_xM \longrightarrow T_x^*M$ defined by $X^\flat = \mathbf{g}(\mathbf{X}, \cdot)$ is a linear isomorphism whose inverse is $\sharp\colon T_x^*M \longrightarrow T_xM$. Moreover, these isomorphisms preserve the inner product in the sense that, for $\alpha, \beta \in T_x^*M$ and $\mathbf{X}, Y \in T_xM$, we have

$$\langle \alpha, \beta \rangle_{\mathbf{g}} = \mathbf{g}(\alpha^\sharp, \beta^\sharp), \qquad \mathbf{g}(\mathbf{X}, Y) = \langle X^\flat, Y^\flat \rangle_{\mathbf{g}}.$$

If $\mathbf{E}$ is a vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$, recall that the inner product induced on $T^*M \otimes \mathbf{E}$ by $\mathbf{g}$ and $\mathbf{h}_{\mathbf{E}}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}$.
For simple tensor fields it is defined by

$$\langle \alpha \otimes s, \beta \otimes t \rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} = \langle \alpha, \beta \rangle_{\mathbf{g}} \, \mathbf{h}_{\mathbf{E}}(s, t).$$

When $\mathbf{E} = M \times \mathbb{R}$ is trivial of rank 1 with the usual metric $\mathbf{h}_{\mathbf{E}}(a, b) = ab$, there is a canonical identification

$$T^*M \otimes \mathbf{E} \cong T^*M, \qquad \alpha \otimes a \longleftrightarrow a\,\alpha.$$

Under this identification we have

$$\langle \alpha \otimes a, \beta \otimes b \rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} = ab \, \langle \alpha, \beta \rangle_{\mathbf{g}} = \langle a\alpha, b\beta \rangle_{\mathbf{g}},$$

so the induced product $\langle \cdot, \cdot \rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}$ agrees with the usual Riemannian inner product on T^*M. Consequently, when $\mathbf{E}$ is trivial of rank 1, we omit the distinction between the two metrics.

a) Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary. We consider $M = \overline{\Omega}$ with the Euclidean metric

$$\overline{\mathbf{g}} = \sum_{i=1}^{n} \mathbf{d}x^i \otimes \mathbf{d}x^i, \qquad (\overline{g}_{ij}) = \delta_{ij}, \quad (\overline{g}^{ij}) = \delta^{ij}.$$

We take the trivial bundle $\mathbf{E} = M \times \mathbb{R}$ with the usual bundle metric $\mathbf{h}_{\mathbf{E}}(a, b) = ab$ and the flat connection induced by the Euclidean Levi–Civita connection.

We apply Theorem 9.1.29. The corresponding identity is

$$\int_M \langle \nabla \mathbf{F}, \mathbf{G} \rangle_{\overline{\mathbf{g}}, \mathbf{h}_{\mathbf{E}}} \, d\lambda_n = -\int_M \langle \mathbf{F}, (\mathrm{tr}_{\overline{\mathbf{g}}} \circ \nabla)\mathbf{G} \rangle_{\mathbf{h}_{\mathbf{E}}} \, d\lambda_n + \int_{\partial M} \langle \mathbf{F}, \iota_{\boldsymbol{\nu}} \mathbf{G} \rangle_{\mathbf{h}_{\mathbf{E}}} \, dS.$$

Since $\mathbf{E}$ is trivial of rank 1, this identity reduces to

$$\int_M \langle \nabla \mathbf{F}, \mathbf{G} \rangle_{\overline{\mathbf{g}}} \, d\lambda_n = -\int_M \mathbf{F} \, (\mathrm{tr}_{\overline{\mathbf{g}}} \circ \nabla) \mathbf{G} \, d\lambda_n + \int_{\partial M} \mathbf{F} \, \iota_{\boldsymbol{\nu}} \mathbf{G} \, dS.$$

We take $u, v \in C^\infty(M)$ and choose

$$\mathbf{F} = u, \qquad \mathbf{G} = \nabla v.$$

In general, for a smooth function v on a Riemannian manifold, the covariant derivative agrees with the differential:

$$\nabla v = dv = (\partial_i v) \, \mathbf{d}x^i.$$

Thus, in Cartesian coordinates,

$$(\nabla v)_i = \partial_i v.$$

Moreover,

$$\langle \nabla u, \nabla v \rangle_{\overline{\mathbf{g}}} = \overline{g}^{ij} (\partial_i u)(\partial_j v) = \sum_{i=1}^{n} (\partial_i u)(\partial_i v).$$

The general expression for the covariant derivative of a 1-form is

$$(\nabla H)_{ij} = \partial_i H_j - \Gamma_{ij}^k H_k.$$

Applying this to $H = \nabla v$ gives

$$(\nabla(\nabla v))_{ij} = \partial_i \partial_j v - \Gamma_{ij}^k \partial_k v.$$

Since the Euclidean Levi–Civita connection satisfies $\Gamma_{ij}^k = 0$, we obtain

$$(\mathrm{tr}_{\overline{\mathbf{g}}} \circ \nabla)(\nabla v) = \delta^{ij} \partial_i \partial_j v = \Delta v.$$

Moreover,

$$\iota_{\boldsymbol{\nu}} (\nabla v) = \partial_{\boldsymbol{\nu}} v.$$

Therefore,

$$\int_\Omega \nabla u \cdot \nabla v \, d\lambda_n = -\int_\Omega u \, \Delta v \, d\lambda_n + \int_{\partial\Omega} u \, \partial_{\boldsymbol{\nu}} v \, dS.$$

b) Under the preceding hypotheses, subtracting the identities obtained for (u, v) and (v, u) gives

$$\int_\Omega \left(u \, \Delta v - v \, \Delta u \right) d\lambda_n = \int_{\partial\Omega} \left(u \, \partial_{\boldsymbol{\nu}} v - v \, \partial_{\boldsymbol{\nu}} u \right) dS,$$

which is known as Green's second identity.

c) Now let $(M, \mathbf{g})$ be a Riemannian manifold with boundary. Again take $\mathbf{E} = M \times \mathbb{R}$. For $u, v \in C_c^\infty(M)$, take $\mathbf{F} = u$ and $\mathbf{G} = \nabla v$, with ∇ the Levi-Civita connection.

The abstract identity is

$$\int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}, \mathbf{h}_\mathbf{E}} \, d\lambda_{\mathbf{g}} = -\int_M \langle u, (\mathrm{tr}_{\mathbf{g}} \circ \nabla)(\nabla v) \rangle_{\mathbf{h}_\mathbf{E}} \, d\lambda_{\mathbf{g}} + \int_{\partial M} \langle u, \iota_{\boldsymbol{\nu}} (\nabla v) \rangle_{\mathbf{h}_\mathbf{E}} \, d\lambda_{\widetilde{\mathbf{g}}}.$$

Since $\mathbf{E}$ is trivial of rank 1, it reduces to

$$\int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = -\int_M u \, (\mathrm{tr}_{\mathbf{g}} \circ \nabla)(\nabla v) \, d\lambda_{\mathbf{g}} + \int_{\partial M} u \, \iota_{\boldsymbol{\nu}}(\nabla v) \, d\lambda_{\tilde{\mathbf{g}}}.$$

In local coordinates,

$$(\nabla v)_j = \partial_j v, \qquad (\nabla(\nabla v))_{ij} = \partial_i \partial_j v - \Gamma_{ij}^k \partial_k v.$$

Therefore,

$$(\mathrm{tr}_{\mathbf{g}} \circ \nabla)(\nabla v) = g^{ij} \left(\partial_i \partial_j v - \Gamma_{ij}^k \partial_k v \right).$$

Let us verify that this expression agrees with the coordinate expression for the Laplace–Beltrami Laplacian,

$$\Delta_{\mathbf{g}} v = \frac{1}{\sqrt{\det(\mathbf{g})}} \, \partial_i \left(\sqrt{\det(\mathbf{g})} \, g^{ij} \partial_j v \right).$$

By the definition of the metric trace,

$$(\mathrm{tr}_{\mathbf{g}} \circ \nabla)(\nabla v) = g^{ij} (\nabla(\nabla v))_{ij},$$

and since $(\nabla v)_j = \partial_j v$, we have

$$(\nabla(\nabla v))_{ij} = \partial_i \partial_j v - \Gamma_{ij}^k \partial_k v.$$

Therefore,

$$(\mathrm{tr}_{\mathbf{g}} \circ \nabla)(\nabla v) = g^{ij} \partial_i \partial_j v - g^{ij} \Gamma_{ij}^k \partial_k v. \tag{9.6}$$

On the other hand,

$$\Delta_{\mathbf{g}} v = \frac{1}{\sqrt{\det(\mathbf{g})}} \, \partial_i \big(\sqrt{\det(\mathbf{g})} \, g^{ij} \partial_j v \big).$$

Applying the Leibniz rule,

$$\Delta_{\mathbf{g}} v = \frac{1}{\sqrt{\det(\mathbf{g})}} \Big(\partial_i \big(\sqrt{\det(\mathbf{g})} \, g^{ij} \big) \, \partial_j v + \sqrt{\det(\mathbf{g})} \, g^{ij} \partial_i \partial_j v \Big)$$

$$= g^{ij} \partial_i \partial_j v + \frac{1}{\sqrt{\det(\mathbf{g})}} \, \partial_i \big(\sqrt{\det(\mathbf{g})} \, g^{ij} \big) \, \partial_j v.$$

We now expand the remaining term using the Leibniz rule once more:

$$\partial_i \left(\sqrt{\det(\mathbf{g})} \, g^{ij} \right) = \left(\partial_i \sqrt{\det(\mathbf{g})} \right) g^{ij} + \sqrt{\det(\mathbf{g})} \, \partial_i g^{ij}.$$

Dividing by $\sqrt{\det(\mathbf{g})}$ gives

$$\frac{1}{\sqrt{\det(\mathbf{g})}} \partial_i \left(\sqrt{\det(\mathbf{g})} \, g^{ij} \right) = g^{ij} \frac{\partial_i \sqrt{\det(\mathbf{g})}}{\sqrt{\det(\mathbf{g})}} + \partial_i g^{ij}.$$

By Remark 2.14.7,

$$\frac{\partial_i \sqrt{\det(\mathbf{g})}}{\sqrt{\det(\mathbf{g})}} = \Gamma_{ik}^k,$$

and, by compatibility of the connection with the metric (see 2.4.17),

$$\partial_i g^{ij} + \Gamma^k_{ik} g^{ij} = -g^{ik}\Gamma^j_{ik},$$

we conclude that

$$\frac{1}{\sqrt{\det(\mathbf{g})}}\partial_i\left(\sqrt{\det(\mathbf{g})}\, g^{ij}\right) = -g^{ik}\Gamma^j_{ik}.$$

Substitution into the expression for $\Delta_{\mathbf{g}} v$ yields

$$\Delta_{\mathbf{g}} v = g^{ij}\partial_i\partial_j v - g^{ik}\Gamma^j_{ik}\partial_j v.$$

Renaming the dummy index $j \mapsto k$ gives

$$\Delta_{\mathbf{g}} v = g^{ij}\partial_i\partial_j v - g^{ij}\Gamma^k_{ij}\partial_k v,$$

which agrees with (9.6). Therefore,

$$(\mathrm{tr}_{\mathbf{g}} \circ \nabla)(\nabla v) = \Delta_{\mathbf{g}} v.$$

Moreover,

$$\langle \nabla u, \nabla v\rangle_{\mathbf{g}} = g^{ij}(\partial_i u)(\partial_j v) = \mathbf{g}(\mathrm{grad}_{\mathbf{g}}\, u, \mathrm{grad}_{\mathbf{g}}\, v).$$

Finally,

$$\iota_{\boldsymbol{\nu}}(\nabla v) = \nu^j \partial_j v = \mathbf{g}(\mathrm{grad}_{\mathbf{g}}\, v, \boldsymbol{\nu}) = \partial_{\boldsymbol{\nu}} v.$$

Substituting into the theorem gives

$$\int_M \mathbf{g}(\mathrm{grad}_{\mathbf{g}}\, u, \mathrm{grad}_{\mathbf{g}}\, v)\, d\lambda_{\mathbf{g}} = -\int_M u\, \Delta_{\mathbf{g}} v\, d\lambda_{\mathbf{g}} + \int_{\partial M} u\, \partial_{\boldsymbol{\nu}} v\, d\lambda_{\tilde{\mathbf{g}}}.$$

d) Let $\mathbf{X} \in \Gamma_c(TM)$ and let $\boldsymbol{\alpha} = \mathbf{X}^\flat$. We apply Theorem 9.1.29 with $\mathbf{F} \equiv 1$ and $\mathbf{G} = \boldsymbol{\alpha}$. Since $\nabla(1) = 0$, we obtain

$$0 = -\int_M (\mathrm{tr}_{\mathbf{g}} \circ \nabla)\boldsymbol{\alpha}\, d\lambda_{\mathbf{g}} + \int_{\partial M} \iota_{\boldsymbol{\nu}}\boldsymbol{\alpha}\, d\lambda_{\tilde{\mathbf{g}}}.$$

In local coordinates we have $(\nabla\boldsymbol{\alpha})_{ij} = \partial_i\alpha_j - \Gamma^k_{ij}\alpha_k$, and hence

$$(\mathrm{tr}_{\mathbf{g}} \circ \nabla)\boldsymbol{\alpha} = g^{ij}(\partial_i\alpha_j - \Gamma^k_{ij}\alpha_k).$$

Writing $\alpha_j = g_{jk}X^k$ and applying the Leibniz rule,

$$\partial_i\alpha_j = \partial_i(g_{jk}X^k) = (\partial_i g_{jk})X^k + g_{jk}\partial_i X^k.$$

Substituting into the preceding expression gives

$$(\mathrm{tr}_{\mathbf{g}} \circ \nabla)\boldsymbol{\alpha} = g^{ij}(\partial_i g_{jk})X^k + g^{ij}g_{jk}\partial_i X^k - g^{ij}\Gamma^m_{ij}g_{mk}X^k.$$

Since $g^{ij}g_{jk} = \delta^i_k$, the second term is $\partial_i X^i$.

To simplify the remaining terms, we use Corollary 2.4.16, which in coordinates states that

$$\partial_i g_{jk} = \Gamma^s_{ij} g_{sk} + \Gamma^s_{ik} g_{js}.$$

Multiplying by g^{ij} gives

$$g^{ij}\partial_i g_{jk} = g^{ij}\Gamma^s_{ij} g_{sk} + g^{ij}\Gamma^s_{ik} g_{js}.$$

The second term satisfies $g^{ij}\Gamma^s_{ik} g_{js} = \Gamma^s_{ik}\delta^i_s = \Gamma^i_{ik}$, and therefore

$$g^{ij}(\partial_i g_{jk}) - g^{ij}\Gamma^s_{ij} g_{sk} = \Gamma^i_{ik}.$$

Substitution into the expression for $(\mathrm{tr}_{\mathbf{g}} \circ \nabla)\boldsymbol{\alpha}$ shows that

$$(\mathrm{tr}_{\mathbf{g}} \circ \nabla)\boldsymbol{\alpha} = \partial_i X^i + \Gamma^i_{ik} X^k = \mathrm{div}_{\mathbf{g}} \mathbf{X}.$$

On the other hand, $\iota_{\boldsymbol{\nu}}\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\nu}) = \mathbf{g}(\mathbf{X}, \boldsymbol{\nu})$. Substituting into the initial identity finally gives

$$\int_M \mathrm{div}_{\mathbf{g}} \mathbf{X}\, d\lambda_{\mathbf{g}} = \int_{\partial M} \mathbf{g}(\mathbf{X}, \boldsymbol{\nu})\, d\lambda_{\tilde{\mathbf{g}}}.$$

The preceding examples illustrate the structure that the general formula must preserve. We now turn to its proof.

Proof of Theorem 9.1.29. We proceed by induction on s. The general idea is to "transfer" derivatives from $\mathbf{F}$ to $\mathbf{G}$ by covariant integration by parts, keeping track of the boundary terms that arise at each step.

Case $s = 1$. Fix a point $p \in \mathrm{Int}(M)$ and choose normal coordinates centered at p, with associated coordinate frame $(\partial_1, \dots, \partial_n)$. In these coordinates,

$$g_{ij}(p) = \delta_{ij}, \qquad \Gamma^k_{ij}(p) = 0,$$

and, in particular, $\nabla_{\partial_i}\partial_j(p) = 0$ for all indices $i, j \in \{1, \dots, n\}$.
Now let $\mathbf{F} \in \Gamma_c\big(T^{(k,l)}(TM) \otimes \mathbf{E}\big)$ and $\mathbf{G} \in \Gamma_c\big(T^{(k,l+1)}(TM) \otimes \mathbf{E}\big)$. Define a map

$$\Phi \colon \mathfrak{X}(M) \longrightarrow C^\infty(M, \mathbb{K}), \qquad \Phi(\mathbf{Y}) := \big\langle \mathbf{F}, \iota_{\mathbf{Y}}\mathbf{G} \big\rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}.$$

The map $\mathbf{Y} \mapsto \iota_{\mathbf{Y}}\mathbf{G}$ is $C^\infty(M, \mathbb{R})$–linear in $\mathbf{Y}$, by the multilinearity of the tensor field $\mathbf{G}$ in its arguments. Consequently, Φ defines a smooth real 1–form if $\mathbf{E}$ is real and a smooth complex 1–form if $\mathbf{E}$ is complex. In the latter case, all the following arguments take place in the complexified bundles, and the divergence theorem is applied separately to the real and imaginary parts. Thus consider $\mathbf{X} := \Phi^\sharp$, which defines a smooth vector field, complexified in the complex case, characterized (according to Definition 2.1.10) by

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathbf{g}} = \Phi(\mathbf{Y}) = \big\langle \mathbf{F}, \iota_{\mathbf{Y}}\mathbf{G} \big\rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}, \qquad \forall\, \mathbf{Y} \in \mathfrak{X}(M).$$

In particular, for each $c \in \{1, \dots, n\}$,

$$\langle \mathbf{X}, \partial_c \rangle_{\mathbf{g}} = \big\langle \mathbf{F}, \iota_{\partial_c}\mathbf{G} \big\rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}. \tag{9.7}$$

At the point p, the divergence of $\mathbf{X}$ is expressed as

$$\operatorname{div}_{\mathbf{g}} \mathbf{X}(p) = \sum_{c=1}^{n} \left\langle \nabla_{\partial_c} \mathbf{X}, \partial_c \right\rangle_{\mathbf{g}}(p).$$

Compatibility of ∇ with $\mathbf{g}$ tells us that

$$\left\langle \nabla_{\partial_c} \mathbf{X}, \partial_c \right\rangle_{\mathbf{g}} = \partial_c \left(\langle \mathbf{X}, \partial_c \rangle_{\mathbf{g}} \right) - \left\langle \mathbf{X}, \nabla_{\partial_c} \partial_c \right\rangle_{\mathbf{g}}.$$

Evaluating at p and using $\nabla_{\partial_c} \partial_c(p) = 0$ (since $\Gamma_{cc}^{k}(p) = 0$), we find that

$$\operatorname{div}_{\mathbf{g}} \mathbf{X}(p) = \sum_{c=1}^{n} \partial_c \left(\langle \mathbf{X}, \partial_c \rangle_{\mathbf{g}} \right)(p). \tag{9.8}$$

Substituting (9.7) into (9.8) gives

$$\operatorname{div}_{\mathbf{g}} \mathbf{X}(p) = \sum_{c=1}^{n} \partial_c \left(\left\langle \mathbf{F}, \iota_{\partial_c} \mathbf{G} \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} \right)(p). \tag{9.9}$$

The induced connection on $T^{(k,l)}(TM) \otimes \mathbf{E}$ (which we continue to denote by ∇) is compatible with the inner product $\langle \cdot, \cdot \rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}$. Consequently, for each c,

$$\partial_c \left(\left\langle \mathbf{F}, \iota_{\partial_c} \mathbf{G} \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} \right) = \left\langle \nabla_{\partial_c} \mathbf{F}, \iota_{\partial_c} \mathbf{G} \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} + \left\langle \mathbf{F}, \nabla_{\partial_c} \left(\iota_{\partial_c} \mathbf{G} \right) \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}. \tag{9.10}$$

Evaluating (9.10) at p and substituting into (9.9) gives

$$\operatorname{div}_{\mathbf{g}} \mathbf{X}(p) = \sum_{c=1}^{n} \left(\left\langle \nabla_{\partial_c} \mathbf{F}, \iota_{\partial_c} \mathbf{G} \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}(p) + \left\langle \mathbf{F}, \nabla_{\partial_c} \left(\iota_{\partial_c} \mathbf{G} \right) \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}(p) \right). \tag{9.11}$$

For the second term we use the identity (the Leibniz rule for contraction)

$$\nabla_{\partial_c} \left(\iota_{\partial_c} \mathbf{G} \right) = \iota_{\partial_c} \left(\nabla_{\partial_c} \mathbf{G} \right) + \iota_{\nabla_{\partial_c} \partial_c} \mathbf{G},$$

which follows from Proposition 2.4.23. Evaluating at p and using $\nabla_{\partial_c} \partial_c(p) = 0$ gives

$$\nabla_{\partial_c} \left(\iota_{\partial_c} \mathbf{G} \right)(p) = \iota_{\partial_c} \left(\nabla_{\partial_c} \mathbf{G} \right)(p).$$

Substituting this into (9.11) yields

$$\operatorname{div}_{\mathbf{g}} \mathbf{X}(p) = \sum_{c=1}^{n} \left(\left\langle \nabla_{\partial_c} \mathbf{F}, \iota_{\partial_c} \mathbf{G} \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}(p) + \left\langle \mathbf{F}, \iota_{\partial_c} (\nabla_{\partial_c} \mathbf{G}) \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}(p) \right). \tag{9.12}$$

In the chosen normal coordinates we have

$$\nabla \mathbf{F} = \sum_{c=1}^{n} \mathbf{d}x^c \otimes \nabla_{\partial_c} \mathbf{F}, \qquad \iota_{\partial_c} \mathbf{G} = \mathbf{G}(\partial_c, \cdot),$$

because the derivative index and the inserted index both occupy the first position. The definition of the tensor product metric then implies

$$\sum_{c=1}^{n} \big\langle \nabla_{\partial_c} \mathbf{F}, \iota_{\partial_c} \mathbf{G} \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} = \langle \nabla \mathbf{F}, \mathbf{G} \rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \quad \text{en } p.$$

Likewise,

$$\operatorname{tr}_\mathbf{g}(\nabla \mathbf{G}) = \sum_{c=1}^{n} (\nabla_{\partial_c} \mathbf{G})(\partial_c, \cdot) = \sum_{c=1}^{n} \iota_{\partial_c} (\nabla_{\partial_c} \mathbf{G}) \quad \text{en } p,$$

and the second sum in (9.12) is $\langle \mathbf{F}, \operatorname{tr}_\mathbf{g}(\nabla \mathbf{G}) \rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}(p)$.

Combining the two identifications gives

$$\operatorname{div}_\mathbf{g} \mathbf{X}(p) = \big\langle \nabla \mathbf{F}, \mathbf{G} \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}(p) + \big\langle \mathbf{F}, \operatorname{tr}_\mathbf{g}(\nabla \mathbf{G}) \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}(p).$$

Since $p \in \operatorname{Int}(M)$ is arbitrary, the pointwise identity holds on $\operatorname{Int}(M)$. Both sides are smooth functions up to the boundary, and $\operatorname{Int}(M)$ is dense in M; by continuity, the identity also holds on ∂M. It therefore holds throughout M:

$$\operatorname{div}_\mathbf{g} \mathbf{X} = \langle \nabla \mathbf{F}, \mathbf{G} \rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} + \langle \mathbf{F}, \operatorname{tr}_\mathbf{g}(\nabla \mathbf{G}) \rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Integrating over M and applying Theorem 2.14.21 (the divergence theorem) gives

$$\int_M \operatorname{div}_\mathbf{g} \mathbf{X} \, d\lambda_\mathbf{g} = \int_{\partial M} \langle \mathbf{X}, \boldsymbol{\nu} \rangle_\mathbf{g} \, d\lambda_{\tilde{\mathbf{g}}}.$$

The field $\mathbf{X}$ has compact support, since it vanishes wherever $\mathbf{F}$ or $\mathbf{G}$ vanishes; thus the divergence theorem also applies when M is noncompact. If $\partial M = \varnothing$, the right-hand side is zero by the convention in the statement. On the boundary, by the definition of $\mathbf{X}$ (and of Φ),

$$\langle \mathbf{X}, \boldsymbol{\nu} \rangle_\mathbf{g} = (\mathbf{X}^\flat)(\boldsymbol{\nu}) = \Phi(\boldsymbol{\nu}) = \big\langle \mathbf{F}, \iota_{\boldsymbol{\nu}} \mathbf{G} \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Combining this with the preceding pointwise identity gives

$$\int_M \big\langle \nabla \mathbf{F}, \mathbf{G} \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} = -\int_M \big\langle \mathbf{F}, \operatorname{tr}_\mathbf{g}(\nabla \mathbf{G}) \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} + \int_{\partial M} \big\langle \mathbf{F}, \iota_{\boldsymbol{\nu}} \mathbf{G} \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_{\tilde{\mathbf{g}}}.$$

This proves the case $s = 1$.

We now verify the induction step. Suppose that the identity in the statement holds for some integer $s \geq 1$, that is, for every

$$\mathbf{F} \in \Gamma_c\big(T^{(k,l)}(TM) \otimes \mathbf{E}\big), \quad \mathbf{H} \in \Gamma_c\big(T^{(k,l+s)}(TM) \otimes \mathbf{E}\big),$$

we have

$$\int_M \langle \nabla^s \mathbf{F}, \mathbf{H} \rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} = (-1)^s \int_M \big\langle \mathbf{F}, (\operatorname{tr}_\mathbf{g} \circ \nabla)^s \mathbf{H} \big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g}$$
$$+ \sum_{j=0}^{s-1} (-1)^{s-1-j} \int_{\partial M} \Big\langle \nabla^j \mathbf{F}, \iota_{\boldsymbol{\nu}} \big((\operatorname{tr}_\mathbf{g} \circ \nabla)^{s-1-j} \mathbf{H} \big) \Big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_{\tilde{\mathbf{g}}}.$$

Our goal is to show that the same identity holds for $s+1$. Thus let

$$\mathbf{G} \in \Gamma_c\big(T^{(k,l+s+1)}(TM) \otimes \mathbf{E}\big).$$

We apply the case $s = 1$ to the pair of fields $\nabla^s \mathbf{F}$ and $\mathbf{G}$. This gives

$$\int_M \langle \nabla^{s+1}\mathbf{F}, \mathbf{G}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} = -\int_M \big\langle \nabla^s \mathbf{F}, \mathrm{tr}_\mathbf{g}(\nabla \mathbf{G})\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} + \int_{\partial M} \big\langle \nabla^s \mathbf{F},\, \iota_{\boldsymbol{\nu}} \mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_{\tilde{\mathbf{g}}}.$$

Now define

$$\mathbf{H} := \mathrm{tr}_\mathbf{g}(\nabla \mathbf{G}) \in \Gamma_c\big(T^{(k,l+s)}(TM) \otimes \mathbf{E}\big).$$

Applying the induction hypothesis to the pair $(\mathbf{F}, \mathbf{H})$ gives

$$\int_M \langle \nabla^s \mathbf{F}, \mathbf{H}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} = (-1)^s \int_M \big\langle \mathbf{F}, (\mathrm{tr}_\mathbf{g} \circ \nabla)^s \mathbf{H}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}$$

$$+ \sum_{j=0}^{s-1} (-1)^{s-1-j} \int_{\partial M} \Big\langle \nabla^j \mathbf{F},\, \iota_{\boldsymbol{\nu}} \big((\mathrm{tr}_\mathbf{g} \circ \nabla)^{s-1-j} \mathbf{H}\big)\Big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_{\tilde{\mathbf{g}}}.$$

Since $\mathbf{H} = \mathrm{tr}_\mathbf{g}(\nabla \mathbf{G})$, we have

$$(\mathrm{tr}_\mathbf{g} \circ \nabla)^s \mathbf{H} = (\mathrm{tr}_\mathbf{g} \circ \nabla)^{s+1} \mathbf{G}, \qquad (\mathrm{tr}_\mathbf{g} \circ \nabla)^{s-1-j} \mathbf{H} = (\mathrm{tr}_\mathbf{g} \circ \nabla)^{s-j} \mathbf{G},$$

and therefore

$$\int_M \big\langle \nabla^s \mathbf{F}, \mathrm{tr}_\mathbf{g}(\nabla \mathbf{G})\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} = (-1)^s \int_M \big\langle \mathbf{F}, (\mathrm{tr}_\mathbf{g} \circ \nabla)^{s+1} \mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}$$

$$+ \sum_{j=0}^{s-1} (-1)^{s-1-j} \int_{\partial M} \Big\langle \nabla^j \mathbf{F},\, \iota_{\boldsymbol{\nu}} \big((\mathrm{tr}_\mathbf{g} \circ \nabla)^{s-j} \mathbf{G}\big)\Big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_{\tilde{\mathbf{g}}}.$$

Substituting this identity into the expression for $\displaystyle\int_M \langle \nabla^{s+1}\mathbf{F}, \mathbf{G}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}$ gives

$$\int_M \langle \nabla^{s+1}\mathbf{F}, \mathbf{G}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} = (-1)^{s+1} \int_M \big\langle \mathbf{F}, (\mathrm{tr}_\mathbf{g} \circ \nabla)^{s+1} \mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}$$

$$+ \sum_{j=0}^{s-1} (-1)^{s-j} \int_{\partial M} \Big\langle \nabla^j \mathbf{F},\, \iota_{\boldsymbol{\nu}} \big((\mathrm{tr}_\mathbf{g} \circ \nabla)^{s-j} \mathbf{G}\big)\Big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_{\tilde{\mathbf{g}}} + \int_{\partial M} \big\langle \nabla^s \mathbf{F},\, \iota_{\boldsymbol{\nu}} \mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_{\tilde{\mathbf{g}}}.$$

The last boundary term already has the desired form if we interpret $(\mathrm{tr}_\mathbf{g} \circ \nabla)^0$ as the identity:

$$\int_{\partial M} \big\langle \nabla^s \mathbf{F},\, \iota_{\boldsymbol{\nu}} \mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_{\tilde{\mathbf{g}}} = (-1)^{s-s} \int_{\partial M} \Big\langle \nabla^s \mathbf{F},\, \iota_{\boldsymbol{\nu}} \big((\mathrm{tr}_\mathbf{g} \circ \nabla)^{s-s} \mathbf{G}\big)\Big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_{\tilde{\mathbf{g}}}.$$

We can therefore incorporate this term into the preceding sum by taking $j = s$, obtaining

$$\int_M \langle \nabla^{s+1}\mathbf{F}, \mathbf{G}\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} = (-1)^{s+1} \int_M \big\langle \mathbf{F}, (\mathrm{tr}_\mathbf{g} \circ \nabla)^{s+1} \mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}$$

$$+\sum_{j=0}^{s}(-1)^{s-j}\int_{\partial M}\left\langle \nabla^{j}\mathbf{F},\, \iota_{\boldsymbol{\nu}}\big((\mathrm{tr}_{\mathbf{g}}\circ\nabla)^{s-j}\mathbf{G}\big)\right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} d\lambda_{\tilde{\mathbf{g}}},$$

which is precisely the formula in the statement for $s+1$. The result then follows by induction on s. □

As a corollary, we can extend the usual Green's formula to the Bochner Laplacian:

Corollary 9.1.31. (Green's formula for the Bochner Laplacian) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\pi_{\mathbf{E}}\colon \mathbf{E}\longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $\boldsymbol{\nu}$ be the outward unit normal to ∂M.
Then, for any $\mathbf{u},\mathbf{v}\in\Gamma_c(\mathbf{E})$, we have

$$\int_M \langle\nabla^{\mathbf{E}}\mathbf{u},\nabla^{\mathbf{E}}\mathbf{v}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} = \int_M \langle\mathbf{u},\Delta_B\mathbf{v}\rangle_{\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} + \int_{\partial M}\left\langle \mathbf{u},\nabla^{\mathbf{E}}_{\boldsymbol{\nu}}\mathbf{v}\right\rangle_{\mathbf{h}_{\mathbf{E}}} d\lambda_{\tilde{\mathbf{g}}},$$

where, by definition,

$$\Delta_B := (\nabla^{\mathbf{E}})^{*}_{h}\nabla^{\mathbf{E}}.$$

When $\partial M\neq\varnothing$, we use the formal differential adjoint up to the boundary from Remark 8.2.4; no domain for L^2 with boundary conditions has yet been chosen.

Proof. We apply Theorem 9.1.29 with $s=1$ and $k=l=0$, taking

$$\mathbf{F}=\mathbf{u}\in\Gamma_c(\mathbf{E}),\qquad \mathbf{G}=\nabla^{\mathbf{E}}\mathbf{v}\in\Gamma_c(T^{*}M\otimes\mathbf{E}).$$

The identity in the theorem takes the form

$$\int_M \langle\nabla^{\mathbf{E}}\mathbf{u},\nabla^{\mathbf{E}}\mathbf{v}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} = -\int_M \left\langle \mathbf{u},(\mathrm{tr}_{\mathbf{g}}\circ\nabla)(\nabla^{\mathbf{E}}\mathbf{v})\right\rangle_{\mathbf{h}_{\mathbf{E}}} d\lambda_{\mathbf{g}} + \int_{\partial M}\left\langle \mathbf{u},\,\iota_{\boldsymbol{\nu}}(\nabla^{\mathbf{E}}\mathbf{v})\right\rangle_{\mathbf{h}_{\mathbf{E}}} d\lambda_{\tilde{\mathbf{g}}}.$$

By the convention adopted in the definition of the Bochner Laplacian,

$$\Delta_B = -(\mathrm{tr}_{\mathbf{g}}\circ\nabla)\circ\nabla^{\mathbf{E}},$$

and the interior term can therefore be rewritten as

$$-\left\langle \mathbf{u},(\mathrm{tr}_{\mathbf{g}}\circ\nabla)(\nabla^{\mathbf{E}}\mathbf{v})\right\rangle_{\mathbf{h}_{\mathbf{E}}} = \langle\mathbf{u},\Delta_B\mathbf{v}\rangle_{\mathbf{h}_{\mathbf{E}}}.$$

It remains only to identify the boundary term. In local coordinates $(x^1,\dots,x^n)$ we write $\boldsymbol{\nu}=\nu^i\partial_i$. By the definition of interior multiplication $\iota_{\boldsymbol{\nu}}$ in the first covariant index, $\iota_{\boldsymbol{\nu}}(\nabla^{\mathbf{E}}\mathbf{v})=\nu^i(\nabla^{\mathbf{E}}\mathbf{v})_i$. Moreover, $(\nabla^{\mathbf{E}}\mathbf{v})_i=\nabla^{\mathbf{E}}_{\partial_i}\mathbf{v}$. Substitution gives $\iota_{\boldsymbol{\nu}}(\nabla^{\mathbf{E}}\mathbf{v})=\nu^i\nabla^{\mathbf{E}}_{\partial_i}\mathbf{v}$. Finally, using the $C^\infty(M)$-linearity of the connection in the vector field,

$$\nabla^{\mathbf{E}}_{\boldsymbol{\nu}}\mathbf{v}=\nabla^{\mathbf{E}}_{\nu^i\partial_i}\mathbf{v}=\nu^i\nabla^{\mathbf{E}}_{\partial_i}\mathbf{v},$$

we conclude that $\iota_{\boldsymbol{\nu}}(\nabla^{\mathbf{E}}\mathbf{v})=\nabla^{\mathbf{E}}_{\boldsymbol{\nu}}\mathbf{v}$. Substituting into the preceding formula gives the stated identity. □

The following result is another corollary of this integration-by-parts formula:

Corollary 9.1.32. (Local expression for the formal adjoint of ∇^s) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a vector bundle with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. For $s\geq 1$, the formal adjoint of

$$\nabla^s\colon \Gamma(\mathbf{E})\longrightarrow \Gamma\big(T^{(0,s)}(TM)\otimes\mathbf{E}\big)$$

is given by

$$(\nabla^s)^* = (-1)^s(\mathrm{tr}_{\mathbf{g}}\circ\nabla)^s.$$

Moreover, in a chart $(U,x^1,\dots,x^n)$, if $\mathbf{G}\in\Gamma\big(T^{(0,s)}(TM)\otimes\mathbf{E}\big)$ has components $G^a_{j_1\dots j_s}$, then

$$\big((\nabla^s)^*\mathbf{G}\big)^a = (-1)^s\left[\frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(\sqrt{\det(\mathbf{g})}\,g^{i_1j_1}\cdots g^{i_sj_s}\,G^a_{j_1\dots j_s}\Big)+\sum_{b=1}^{r}\sum_{|\beta|\leq s-1}\big(C_\beta\big)_b^{\,a\,j_1\dots j_s}\,\partial^\beta\big(G^b_{j_1\dots j_s}\big)\right], \tag{9.13}$$

where the coefficients $\big(C_\beta\big)_b^{\,a\,j_1\dots j_s}$ are smooth functions on U that depend only on g_{ij}, the connection coefficients Γ^k_{ij} and $(A_i)^a_b$, and their partial derivatives of orders up to $s-1$.

Proof. Since M has no boundary, the boundary terms in Theorem 9.1.29 vanish, and therefore

$$\int_M\langle\nabla^s\mathbf{F},\mathbf{G}\rangle_{\mathbf{g},\mathbf{h_E}}\,d\lambda_{\mathbf{g}} = (-1)^s\int_M\big\langle\mathbf{F},(\mathrm{tr}_{\mathbf{g}}\circ\nabla)^s\mathbf{G}\big\rangle_{\mathbf{g},\mathbf{h_E}}\,d\lambda_{\mathbf{g}},\qquad\forall\,\mathbf{F}\in\Gamma_c(\mathbf{E}),\ \forall\,\mathbf{G}\in\Gamma_c\big(T^{(0,s)}(TM)\otimes\mathbf{E}\big).$$

By Corollary 8.2.3 (existence and uniqueness of the formal adjoint),

$$(\nabla^s)^* = (-1)^s(\mathrm{tr}_{\mathbf{g}}\circ\nabla)^s.$$

It remains to verify the local formula (9.13). Fix a chart $(U,x^1,\dots,x^n)$ and a local frame $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ of $\mathbf{E}\restriction_U$. Let $m\geq 0$ and $\mathbf{H}\in\Gamma\big(T^{(0,m+1)}(TM)\otimes\mathbf{E}\big)$ with components $H^a_{qj_1\dots j_m}$. By definition,

$$\big((\mathrm{tr}_{\mathbf{g}}\circ\nabla)\mathbf{H}\big)^a_{j_1\dots j_m} = g^{pq}\,(\nabla_{\partial_p}\mathbf{H})^a_{qj_1\dots j_m}.$$

Applying Lemma 9.1.27 (with $k=0$, $l=m+1$) gives

$$\begin{aligned}\big((\mathrm{tr}_{\mathbf{g}}\circ\nabla)\mathbf{H}\big)^a_{j_1\dots j_m} &= g^{pq}\partial_p\big(H^a_{qj_1\dots j_m}\big)-\Gamma^\alpha_{pq}\,g^{pq}\,H^a_{\alpha j_1\dots j_m}\\ &\quad-\sum_{\ell=1}^{m}\Gamma^\alpha_{pj_\ell}\,g^{pq}\,H^a_{qj_1\dots j_{\ell-1}\alpha j_{\ell+1}\dots j_m}+\sum_{b=1}^{r}(A_p)^a_b\,g^{pq}H^b_{qj_1\dots j_m}.\end{aligned} \tag{9.14}$$

Using Remark 2.14.7, which tells us that $\Gamma^\alpha_{\alpha p}=\dfrac{1}{\sqrt{\det(\mathbf{g})}}\partial_p\Big(\sqrt{\det(\mathbf{g})}\Big)$ and the identity in Corollary 2.4.17, which can be written as $\partial_p(g^{pq})=-\Gamma^p_{p\alpha}g^{\alpha q}-\Gamma^q_{p\alpha}g^{p\alpha}$, we obtain the equality

$$g^{pq}\partial_p\big(Y_q\big)-\Gamma^\alpha_{pq}g^{pq}Y_\alpha=\frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_p\Big(\sqrt{\det(\mathbf{g})}\,g^{pq}Y_q\Big),\qquad\forall(Y_q)_{q=1}^n. \tag{9.15}$$

To verify it, we expand the right-hand side:

$$\frac{1}{\sqrt{\det(\mathbf{g})}}\partial_p\left(\sqrt{\det(\mathbf{g})}g^{pq}Y_q\right) = \frac{1}{\sqrt{\det(\mathbf{g})}}\left(\partial_p\left(\sqrt{\det(\mathbf{g})}\right)g^{pq}Y_q + \cancel{\sqrt{\det(\mathbf{g})}}\partial_p(g^{pq})Y_q + \cancel{\sqrt{\det(\mathbf{g})}}g^{pq}\partial_p(Y_q)\right)$$

$$= \Gamma^{\alpha}_{\alpha p}g^{pq}Y_q + \partial_p(g^{pq})Y_q + g^{pq}\partial_p(Y_q)$$

$$= \underbrace{\Gamma^{p}_{p\alpha}g^{\alpha q}Y_q}_{p\leftrightarrow\alpha} - \Gamma^{p}_{p\alpha}g^{\alpha q}Y_q - \Gamma^{q}_{p\alpha}g^{p\alpha}Y_q + g^{pq}\partial_p(Y_q)$$

$$= g^{pq}\partial_p(Y_q) - \underbrace{\Gamma^{q}_{p\alpha}g^{p\alpha}Y_q}_{\alpha\leftrightarrow q}$$

$$= g^{pq}\partial_p(Y_q) - \Gamma^{\alpha}_{pq}g^{pq}Y_\alpha.$$

Applying (9.15) with $Y_q = H^a_{qj_1\dots j_m}$, we can rewrite equation (9.14) as

$$((\mathrm{tr}_{\mathbf{g}}\circ\nabla)\mathbf{H})^a_{j_1\dots j_m} = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_p\left(\sqrt{\det(\mathbf{g})}g^{pq}H^a_{qj_1\dots j_m}\right) - \sum_{\ell=1}^{m}\Gamma^{\alpha}_{pj_\ell}g^{pq}H^a_{qj_1\dots j_{\ell-1}\alpha j_{\ell+1}\dots j_m} + \sum_{b=1}^{r}(A_p)^a_b g^{pq}H^b_{qj_1\dots j_m}. \tag{9.16}$$

We prove by induction on $s\geq 1$ that, for every $\mathbf{G}\in\Gamma\left(T^{(0,s)}(TM)\otimes\mathbf{E}\right)$ with components $G^a_{j_1\dots j_s}$, we have

$$\left((\mathrm{tr}_{\mathbf{g}}\circ\nabla)^s\mathbf{G}\right)^a = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_{i_1}\cdots\partial_{i_s}\left(\sqrt{\det(\mathbf{g})}g^{i_1j_1}\cdots g^{i_sj_s}G^a_{j_1\dots j_s}\right) + \sum_{b=1}^{r}\sum_{|\beta|\leq s-1}\left(C_\beta\right)^{aj_1\dots j_s}_b\partial^\beta\left(G^b_{j_1\dots j_s}\right). \tag{9.17}$$

For $s=1$, taking $m=0$ in (9.16) gives

$$((\mathrm{tr}_{\mathbf{g}}\circ\nabla)\mathbf{G})^a = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_p\left(\sqrt{\det(\mathbf{g})}g^{pq}G^a_q\right) + \sum_{b=1}^{r}(A_p)^a_b g^{pq}G^b_q,$$

which has the form (9.17) with $|\beta|=0$. Now suppose that (9.17) holds for some $s\geq 1$, and let us prove the case $s+1$. Let $\mathbf{G}\in\Gamma\left(T^{(0,s+1)}(TM)\otimes\mathbf{E}\right)$ and define

$$\mathbf{H} := (\mathrm{tr}_{\mathbf{g}}\circ\nabla)\mathbf{G}\in\Gamma\left(T^{(0,s)}(TM)\otimes\mathbf{E}\right).$$

By the induction hypothesis applied to $\mathbf{H}$, we have

$$\left((\mathrm{tr}_{\mathbf{g}}\circ\nabla)^s\mathbf{H}\right)^a = \frac{1}{\sqrt{\det(\mathbf{g})}}\partial_{i_1}\cdots\partial_{i_s}\left(\sqrt{\det(\mathbf{g})}g^{i_1j_1}\cdots g^{i_sj_s}H^a_{j_1\dots j_s}\right) + \sum_{b=1}^{r}\sum_{|\beta|\leq s-1}\left(C_\beta\right)^{aj_1\dots j_s}_b\partial^\beta\left(H^b_{j_1\dots j_s}\right). \tag{9.18}$$

Since, by definition,

$$(\mathrm{tr}_{\mathbf{g}}\circ\nabla)^s\mathbf{H} = (\mathrm{tr}_{\mathbf{g}}\circ\nabla)^{s+1}\mathbf{G},$$

it suffices to expand the components $H^a_{j_1\dots j_s}$ explicitly in terms of $\mathbf{G}$. Using (9.16) with $m = s$ gives

$$\begin{aligned} H^a_{j_1\dots j_s} = \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_p\Big(\sqrt{\det(\mathbf{g})}\,g^{pq}G^a_{qj_1\dots j_s}\Big) - \sum_{\ell=1}^{s}\Gamma^{\alpha}_{pj_\ell}\,g^{pq}\,G^a_{qj_1\dots j_{\ell-1}\alpha j_{\ell+1}\dots j_s} \\ + \sum_{b=1}^{r}(A_p)^a_b\,g^{pq}G^b_{qj_1\dots j_s}. \end{aligned} \tag{9.19}$$

Substituting (9.19) into the leading term of (9.18) and separating the summands gives

$$\frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(\sqrt{\det(\mathbf{g})}\,g^{i_1j_1}\cdots g^{i_sj_s}\,H^a_{j_1\dots j_s}\Big) = \text{(I)} + \text{(II)} + \text{(III)}. \tag{9.20}$$

where

$$\text{(I)} := \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(g^{i_1j_1}\cdots g^{i_sj_s}\partial_p\big(\sqrt{\det(\mathbf{g})}\,g^{pq}G^a_{qj_1\dots j_s}\big)\Big),$$

$$\text{(II)} := -\frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(\sqrt{\det(\mathbf{g})}\,g^{i_1j_1}\cdots g^{i_sj_s}\sum_{\ell=1}^{s}\Gamma^{\alpha}_{pj_\ell}\,g^{pq}\,G^a_{qj_1\dots j_{\ell-1}\alpha j_{\ell+1}\dots j_s}\Big),$$

$$\text{(III)} := \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(\sqrt{\det(\mathbf{g})}\,g^{i_1j_1}\cdots g^{i_sj_s}\sum_{b=1}^{r}(A_p)^a_b\,g^{pq}G^b_{qj_1\dots j_s}\Big).$$

For the term (I), we write

$$\text{(I)} = \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(\big(g^{i_1j_1}\cdots g^{i_sj_s}\big)\,\partial_p\big(\sqrt{\det(\mathbf{g})}\,g^{pq}G^a_{qj_1\dots j_s}\big)\Big).$$

We move the factor $g^{i_1j_1}\cdots g^{i_sj_s}$ inside ∂_p using the identity

$$\big(g^{i_1j_1}\cdots g^{i_sj_s}\big)\,\partial_p(f) = \partial_p\Big(\big(g^{i_1j_1}\cdots g^{i_sj_s}\big)f\Big) - \partial_p\big(g^{i_1j_1}\cdots g^{i_sj_s}\big)\,f, \qquad \forall\, f\in C^\infty(U).$$

Applying it to $f = \sqrt{\det(\mathbf{g})}\,g^{pq}G^a_{qj_1\dots j_s}$ gives

$$\begin{aligned} \text{(I)} = \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\partial_p\Big(\sqrt{\det(\mathbf{g})}\,g^{pq}g^{i_1j_1}\cdots g^{i_sj_s}\,G^a_{qj_1\dots j_s}\Big) \\ - \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(\partial_p\big(g^{i_1j_1}\cdots g^{i_sj_s}\big)\,\sqrt{\det(\mathbf{g})}\,g^{pq}G^a_{qj_1\dots j_s}\Big). \end{aligned} \tag{9.21}$$

The first term in (9.21) is the leading term of order $s+1$. In the second term, the factor $\partial_p(g^{i_1j_1}\cdots g^{i_sj_s})$ already contains the derivative separated from the leading term. Only the s outer derivatives $\partial_{i_1}\cdots\partial_{i_s}$ remain to be distributed between the coefficients and $\mathbf{G}$ by the Leibniz rule; even when they all act on $\mathbf{G}$, the order of differentiation of its components is at most s. Consequently, there are smooth coefficients $\big(B_\beta\big)^{a\,j_1\dots j_sq}_c$ with $|\beta|\le s$ such that

$$\text{(I)} = \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\partial_p\Big(\sqrt{\det(\mathbf{g})}\,g^{pq}g^{i_1j_1}\cdots g^{i_sj_s}\,G^a_{qj_1\dots j_s}\Big) + \sum_{c=1}^{r}\sum_{|\beta|\le s}\big(B_\beta\big)^{a\,j_1\dots j_sq}_c\,\partial^\beta\big(G^c_{qj_1\dots j_s}\big). \tag{9.22}$$

Now consider (II). Here no additional derivative ∂_p acts on $\mathbf{G}$. Thus, expanding the operator $\partial_{i_1}\cdots\partial_{i_s}$ acting

on the product

$$\sqrt{\det(\mathbf{g})}\, g^{i_1 j_1}\cdots g^{i_s j_s}\,\Gamma^{\alpha}_{p j_\ell}\, g^{pq}\, G^{a}_{q j_1\ldots j_{\ell-1}\alpha j_{\ell+1}\ldots j_s},$$

by the Leibniz rule yields only terms in which at most s derivatives act on the components of $\mathbf{G}$. Consequently, there are smooth coefficients $\big(\widetilde{B}_\beta\big)_c^{a\,j_1\ldots j_s q}$ with $|\beta|\le s$ such that

$$\text{(II)} = \sum_{c=1}^{r}\sum_{|\beta|\le s}\big(\widetilde{B}_\beta\big)_c^{a\,j_1\ldots j_s q}\,\partial^\beta\big(G^{c}_{q j_1\ldots j_s}\big). \tag{9.23}$$

In (III) the dependence on $\mathbf{G}$ is linear and contains no additional derivatives of $\mathbf{G}$. By the Leibniz formula, each of the s outer derivatives can act on that factor at most once more; thus only derivatives $\partial^\beta\mathbf{G}$ with $|\beta|\le s$ arise. Therefore, there are smooth coefficients $\big(\widehat{B}_\beta\big)_c^{a\,j_1\ldots j_s q}$ with $|\beta|\le s$ such that

$$\text{(III)} = \sum_{c=1}^{r}\sum_{|\beta|\le s}\big(\widehat{B}_\beta\big)_c^{a\,j_1\ldots j_s q}\,\partial^\beta\big(G^{c}_{q j_1\ldots j_s}\big). \tag{9.24}$$

Substituting (9.22), (9.23), and (9.24) into (9.20) gives

$$\begin{aligned}\frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\Big(\sqrt{\det(\mathbf{g})}\,g^{i_1 j_1}\cdots g^{i_s j_s}\,H^{a}_{j_1\ldots j_s}\Big) &= \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_p\partial_{i_1}\cdots\partial_{i_s}\Big(\sqrt{\det(\mathbf{g})}\,g^{pq}g^{i_1 j_1}\cdots g^{i_s j_s}\,G^{a}_{q j_1\ldots j_s}\Big)\\ &\quad+\sum_{c=1}^{r}\sum_{|\beta|\le s}\big(D_\beta\big)_c^{a\,j_1\ldots j_s q}\,\partial^\beta\big(G^{c}_{q j_1\ldots j_s}\big),\end{aligned} \tag{9.25}$$

where

$$\big(D_\beta\big)_c^{a\,j_1\ldots j_s q} := \big(B_\beta\big)_c^{a\,j_1\ldots j_s q} + \big(\widetilde{B}_\beta\big)_c^{a\,j_1\ldots j_s q} + \big(\widehat{B}_\beta\big)_c^{a\,j_1\ldots j_s q},\qquad |\beta|\le s.$$

On the other hand, for the lower-order term in (9.18),

$$\sum_{b=1}^{r}\sum_{|\beta|\le s-1}\big(C_\beta\big)_b^{a\,j_1\ldots j_s}\,\partial^\beta\big(H^{b}_{j_1\ldots j_s}\big),$$

substituting (9.19) into $H^{b}_{j_1\ldots j_s}$ shows that each $\partial^\beta(H^{b}_{j_1\ldots j_s})$ is a linear combination (with smooth coefficients) of terms $\partial^\gamma(G^{c}_{q j_1\ldots j_s})$ with $|\gamma|\le|\beta|+1\le s$. Consequently, there are smooth coefficients $\big(\overline{D}_\beta\big)_c^{a\,j_1\ldots j_s q}$ with $|\beta|\le s$ such that

$$\sum_{b=1}^{r}\sum_{|\beta|\le s-1}\big(C_\beta\big)_b^{a\,j_1\ldots j_s}\,\partial^\beta\big(H^{b}_{j_1\ldots j_s}\big) = \sum_{c=1}^{r}\sum_{|\beta|\le s}\big(\overline{D}_\beta\big)_c^{a\,j_1\ldots j_s q}\,\partial^\beta\big(G^{c}_{q j_1\ldots j_s}\big). \tag{9.26}$$

Combining (9.18), (9.25), and (9.26), we conclude that there are smooth coefficients $\big(C_\beta\big)_c^{a\,j_1\ldots j_s q}$ with $|\beta|\le s$ such that

$$\big((\mathrm{tr}_{\mathbf{g}}\circ\nabla)^{s+1}\mathbf{G}\big)^{a} = \frac{1}{\sqrt{\det(\mathbf{g})}}\,\partial_{i_1}\cdots\partial_{i_s}\partial_p\Big(\sqrt{\det(\mathbf{g})}\,g^{pq}g^{i_1 j_1}\cdots g^{i_s j_s}\,G^{a}_{q j_1\ldots j_s}\Big) + \sum_{c=1}^{r}\sum_{|\beta|\le s}\big(C_\beta\big)_c^{a\,j_1\ldots j_s q}\,\partial^\beta\big(G^{c}_{q j_1\ldots j_s}\big).$$

Finally, we rename $p=i_1$ and $q=j_1$, simultaneously replacing the previous indices (i_ℓ, j_ℓ) by $(i_{\ell+1}, j_{\ell+1})$ for $\ell\in\{1,\ldots,s\}$. The partial derivatives commute, so their outer order can be rearranged, while the indices of $\mathbf{G}$ remain in their original order. This gives exactly the form of (9.17) for $s+1$, completing the induction.

□

Let us consider some examples illustrating the preceding result.

Example 9.1.33. Let $\Omega \subseteq \mathbb{R}^n$ be open, and regard $M := \Omega$ as a smooth manifold without boundary, equipped with the Euclidean metric $\overline{\mathbf{g}}$. In the trivial line bundle

$$\mathbf{E} := M \times \mathbb{R} \longrightarrow M$$

, let $\mathbf{e}(p) := (p, 1)$ be the constant unit section. Every section has a unique expression $\mathbf{u} = u\mathbf{e}$, with $u \in C^\infty(\Omega)$; the trivial metric and connection satisfy

$$\mathbf{h}_{\mathbf{E}}(u\mathbf{e}, v\mathbf{e}) = uv, \qquad \nabla^{\mathbf{E}}_{\partial_i}(u\mathbf{e}) = (\partial_i u)\mathbf{e}.$$

For each $s \geq 0$, the bundle isomorphism

$$I_s \colon T^{(0,s)}(TM) \otimes \mathbf{E} \longrightarrow T^{(0,s)}(TM), \qquad I_s\big(\xi \otimes \lambda \mathbf{e}(p)\big) := \lambda \xi,$$

identifies $\nabla^s \mathbf{u}$ with the Euclidean derivative $D^s u$. In particular,

$$\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} = \left(\sum_{s=0}^{m} \|D^s u\|^p_{L^p(\Omega)} \right)^{1/p}$$

if $1 \leq p < \infty$, with the usual modification using maxima when $p = \infty$. The same identification in the weak derivative identity shows that

$$\mathbf{u} \in W^{m,p}(M, \mathbf{E}) \quad \Longleftrightarrow \quad u \in W^{m,p}(\Omega).$$

Thus the bundle definition recovers precisely the scalar Sobolev space for the trivial rank-one bundle.

The formal adjoint allows us to define certain nonlinear operators important in the study of PDEs, such as the p–Laplacian, in the setting of vector bundles.

Definition 9.1.34. (The scalar and vector bundle p-Laplacian) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $p > 1$.

(a) If $u \in C^2(M)$, the *scalar p-Laplacian* of u is defined by

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}_{\mathbf{g}} \nabla u\big).$$

Equivalently, using the formal adjoint of the covariant derivative,

$$\Delta_p u = -\nabla^*\big(|\nabla u|^{p-2}_{\mathbf{g}} \nabla u\big).$$

When $p = 2$, this recovers the Laplace–Beltrami Laplacian $\Delta_2 u = \Delta_{\mathbf{g}} u$.

(b) Now let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible

connection $\nabla^{\mathbf{E}}$. If $\mathbf{u} \in \Gamma^2(\mathbf{E})$, the *p-Laplacian associated with* $\nabla^{\mathbf{E}}$ is defined by

$$\Delta_p^{\mathbf{E}}\mathbf{u} := -\left(\nabla^{\mathbf{E}}\right)^* \left(|\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}^{p-2}\nabla^{\mathbf{E}}\mathbf{u}\right).$$

In both definitions we adopt $|\xi|^{p-2}\xi = 0$ when $\xi = 0$. If $1 < p < 2$, even for data of class C^2, this flux may fail to be of class C^1 at its zeros. In that case, the divergence and adjoint in the preceding formulas are interpreted weakly: for a test section $\boldsymbol{\varphi}$ supported in the interior, the value of $\Delta_p^{\mathbf{E}}\mathbf{u}$ is determined by

$$\langle \Delta_p^{\mathbf{E}}\mathbf{u}, \boldsymbol{\varphi}\rangle = -\int_M \left\langle |\nabla^{\mathbf{E}}\mathbf{u}|^{p-2}\nabla^{\mathbf{E}}\mathbf{u}, \nabla^{\mathbf{E}}\boldsymbol{\varphi}\right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} d\lambda_{\mathbf{g}}.$$

The scalar case is obtained using the trivial bundle. The flux is continuous and hence locally integrable, so this identity is always defined. When the flux is of class C^1, it recovers the classical expression, and the following boundary formula applies. For $p \geq 2$, data of class C^2 ensure this regularity of the flux.

Our integration-by-parts formula also yields further integration-by-parts formulas, such as the following:

Proposition 9.1.35. (Integration-by-parts formula for the p-Laplacian) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let $p > 1$.
In the scalar case, let $u \in C^2(M)$ be such that $|\nabla u|_{\mathbf{g}}^{p-2}\nabla u$ is of class C^1. Then, for every $\varphi \in C_c^\infty(M)$,

$$\int_M |\nabla u|_{\mathbf{g}}^{p-2}\langle \nabla u, \nabla\varphi\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} = -\int_M \varphi\, \Delta_p u\, d\lambda_{\mathbf{g}} + \int_{\partial M} \varphi\, |\nabla u|_{\mathbf{g}}^{p-2}\partial_\nu u\, d\lambda_{\tilde{\mathbf{g}}}.$$

More generally, let $\pi_{\mathbf{E}}\colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. If $\mathbf{u} \in \Gamma^2(\mathbf{E})$ is such that $|\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}^{p-2}\nabla^{\mathbf{E}}\mathbf{u}$ is of class C^1, then, for every $\boldsymbol{\varphi} \in \Gamma_c(\mathbf{E})$,

$$\int_M |\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}^{p-2}\langle \nabla^{\mathbf{E}}\mathbf{u}, \nabla^{\mathbf{E}}\boldsymbol{\varphi}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} = -\int_M \langle \Delta_p^{\mathbf{E}}\mathbf{u}, \boldsymbol{\varphi}\rangle_{\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} + \int_{\partial M} \left\langle \iota_\nu\left(|\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}^{p-2}\nabla^{\mathbf{E}}\mathbf{u}\right), \boldsymbol{\varphi}\right\rangle_{\mathbf{h}_{\mathbf{E}}} d\lambda_{\tilde{\mathbf{g}}}.$$

In particular, if $\boldsymbol{\varphi}$ has compact support contained in $\mathrm{Int}(M)$, the boundary term vanishes.

Proof. We first prove the bundle identity. Define $\mathbf{A}_{\mathbf{u}} := |\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}^{p-2}\nabla^{\mathbf{E}}\mathbf{u}$, so that $\mathbf{A}_{\mathbf{u}} \in \Gamma^1(T^*M \otimes \mathbf{E})$ by the regularity hypothesis. The calculation in the case $s = 1$ of Green's formula uses only continuous first derivatives, so it remains valid with this regularity. We also multiply $\mathbf{A}_{\mathbf{u}}$ by a compactly supported smooth function equal to one on a neighborhood of $\operatorname{supp}\boldsymbol{\varphi}$; this leaves every integral unchanged and satisfies the support condition. We apply Theorem 9.1.29 with $s = 1$, $k = l = 0$, $\mathbf{F} = \boldsymbol{\varphi}$, and $\mathbf{G} = \mathbf{A}_{\mathbf{u}}$. Then

$$\int_M \langle \nabla^{\mathbf{E}}\boldsymbol{\varphi}, \mathbf{A}_{\mathbf{u}}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} = -\int_M \left\langle \boldsymbol{\varphi}, (\mathrm{tr}_{\mathbf{g}} \circ \nabla)\mathbf{A}_{\mathbf{u}}\right\rangle_{\mathbf{h}_{\mathbf{E}}} d\lambda_{\mathbf{g}} + \int_{\partial M} \langle \boldsymbol{\varphi}, \iota_\nu \mathbf{A}_{\mathbf{u}}\rangle_{\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\tilde{\mathbf{g}}}.$$

Since $(\nabla^{\mathbf{E}})^*\mathbf{A}_{\mathbf{u}} = -(\mathrm{tr}_{\mathbf{g}} \circ \nabla)\mathbf{A}_{\mathbf{u}}$ and, by definition, $\Delta_p^{\mathbf{E}}\mathbf{u} = -(\nabla^{\mathbf{E}})^*\mathbf{A}_{\mathbf{u}}$, we conjugate the preceding identity in the complex case, or use symmetry in the real case. The interior term then becomes $-\langle \Delta_p^{\mathbf{E}}\mathbf{u}, \boldsymbol{\varphi}\rangle_{\mathbf{h}_{\mathbf{E}}}$. Moreover, with the bar acting trivially in the real case, $\overline{\langle \nabla^{\mathbf{E}}\boldsymbol{\varphi}, \mathbf{A}_{\mathbf{u}}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}} = |\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}^{p-2}\langle \nabla^{\mathbf{E}}\mathbf{u}, \nabla^{\mathbf{E}}\boldsymbol{\varphi}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}$. Substitution of these two identities gives the bundle formula.
The scalar case follows by taking $\mathbf{E} = M \times \mathbb{R}$ with its trivial metric and connection and identifying a section

with its component function u. In this case $\mathbf{A}_u = |\nabla u|_{\mathbf{g}}^{p-2}\nabla u$ and

$$\iota_{\boldsymbol{\nu}}\mathbf{A}_u = \langle \mathbf{A}_u, \boldsymbol{\nu}\rangle_{\mathbf{g}} = |\nabla u|_{\mathbf{g}}^{p-2}\langle \nabla u, \boldsymbol{\nu}\rangle_{\mathbf{g}} = |\nabla u|_{\mathbf{g}}^{p-2}\partial_{\boldsymbol{\nu}} u.$$

Thus the preceding identity becomes exactly the scalar formula. □

The integration-by-parts formula also allows us to compare the usual covariant norm on H^m with a norm defined by powers of the Laplacian. This equivalence is a recurring tool in geometric analysis, since it translates control of derivatives into spectral control. The next section establishes the result with the hypotheses and constants needed in the vector bundle setting.

9.1.4 Equivalence of norms: an application of the integration-by-parts formula in vector bundles

In the Euclidean case, the integration-by-parts formula is fundamental to Sobolev space theory and nonlinear analysis. In this section we use it to prove, on a closed manifold M, the equivalence of the usual norm on $H^m(M) := W^{m,2}(M)$ and a norm defined by powers of the Laplacian. We begin with the corresponding identity on Euclidean domains:

Theorem 9.1.36. (An equivalent norm on $H_0^m(\Omega)$ in terms of the Laplacian) Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $m \in \mathbb{N}$. For every function $u \in H_0^m(\Omega)$, the standard Sobolev norm of order m, defined using the partial derivative operator,

$$\|u\|_{H^m(\Omega)}^2 := \sum_{j=0}^{m} \int_\Omega |\nabla^j u|^2 \, d\lambda_n,$$

is exactly equal to the norm structured in powers of the Laplacian operator $\Delta = \sum_{i=1}^{n} \partial_i^2$, given by

$$\|u\|_{\Delta}^2 := \sum_{\substack{j=0 \\ j \text{ even}}}^{m} \int_\Omega |\Delta^{\frac{j}{2}} u|^2 \, d\lambda_n + \sum_{\substack{j=1 \\ j \text{ odd}}}^{m} \int_\Omega \left|\nabla \Delta^{\frac{j-1}{2}} u\right|^2 d\lambda_n.$$

Moreover, equality holds term by term for each order of differentiation $j \in \{0, \dots, m\}$.

Proof. First let $u \in H_0^m(\Omega)$. By the definition of this space, there is a sequence $(u_\nu)_{\nu\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ such that $u_\nu \to u$ in $H^m(\Omega)$. In particular, for every multi-index α with $|\alpha| \le m$,

$$D^\alpha u_\nu \longrightarrow D^\alpha u \qquad \text{in } L^2(\Omega).$$

If $j = 2k \le m$, the operator Δ^k is a finite linear combination, with constant coefficients, of partial derivatives of order j. If $j = 2k+1 \le m$, each component of $\nabla\Delta^k$ is likewise a finite linear combination of partial derivatives of order j. Therefore,

$$\Delta^k u_\nu \longrightarrow \Delta^k u \quad \text{and} \quad \nabla\Delta^k u_\nu \longrightarrow \nabla\Delta^k u$$

in L^2 whenever the corresponding order does not exceed m. Thus, once the term-by-term equalities have

been proved for u_ν, we can pass to the limit in each of them. It therefore suffices to establish them for a function $u \in C_c^\infty(\Omega)$. For such a function, all integrations by parts have no boundary terms. Moreover, partial derivatives commute, and all norms are taken with respect to the Euclidean metric, by contraction of indices.

We proceed by induction on j and prove, for every $u \in C_c^\infty(\Omega)$, the equality corresponding to order j.

For $j = 0$ and $j = 1$ the assertion follows immediately from the definition.

Now consider the fundamental case $j = 2$. Observe that

$$\int_\Omega |\nabla^2 u|^2 \, d\lambda_n = \int_\Omega \sum_{p,q=1}^{n} (\partial_p \partial_q u)^2 \, d\lambda_n.$$

We first integrate by parts with respect to the variable x^q and then with respect to x^p. Since u has compact support in Ω and partial derivatives commute, we obtain

$$\int_\Omega (\partial_p \partial_q u)^2 \, d\lambda_n = -\int_\Omega (\partial_p u)\, \partial_p \partial_q^2 u \, d\lambda_n = \int_\Omega (\partial_p^2 u)(\partial_q^2 u) \, d\lambda_n.$$

Summing over p, q gives

$$\int_\Omega |\nabla^2 u|^2 \, d\lambda_n = \int_\Omega \left(\sum_{p=1}^{n} \partial_p^2 u \right) \left(\sum_{q=1}^{n} \partial_q^2 u \right) d\lambda_n = \int_\Omega |\Delta u|^2 \, d\lambda_n.$$

Now suppose that the assertion holds for all orders less than $j \geq 3$. For each list $(i_1, \dots, i_{j-2})$, the function $\partial_{i_1} \cdots \partial_{i_{j-2}} u$ belongs to $C_c^\infty(\Omega)$. Applying the identity already proved for two derivatives to each of these functions, and summing over all indices, gives, in tensor notation,

$$\int_\Omega |\nabla^j u|^2 \, d\lambda_n = \int_\Omega |\Delta(\nabla^{j-2} u)|^2 \, d\lambda_n.$$

Since the Laplacian commutes with partial derivatives in $\mathbb{R}^n$, we have

$$\Delta(\nabla^{j-2} u) = \nabla^{j-2}(\Delta u),$$

and therefore

$$\int_\Omega |\nabla^j u|^2 \, d\lambda_n = \int_\Omega |\nabla^{j-2}(\Delta u)|^2 \, d\lambda_n.$$

The function $v := \Delta u$ also belongs to $C_c^\infty(\Omega)$. We can therefore apply the induction hypothesis to it at order $j - 2$.

If j is even, then $j - 2$ is even, and we obtain

$$\int_\Omega |\nabla^{j-2}(\Delta u)|^2 = \int_\Omega |\Delta^{\frac{j-2}{2}}(\Delta u)|^2 = \int_\Omega |\Delta^{\frac{j}{2}} u|^2.$$

If j is odd, then $j - 2$ is odd, and we obtain

$$\int_\Omega |\nabla^{j-2}(\Delta u)|^2 = \int_\Omega \left|\nabla \Delta^{\frac{j-3}{2}}(\Delta u)\right|^2 = \int_\Omega \left|\nabla \Delta^{\frac{j-1}{2}} u\right|^2.$$

In both cases this recovers exactly the corresponding term in the structured norm. This proves the term-by-term identities for functions in $C_c^\infty(\Omega)$. Applying them to the sequence (u_ν) chosen at the beginning and using the convergence in L^2 established there, we pass to the limit on both sides of each identity. This gives the same equalities for every $u \in H_0^m(\Omega)$; summing them for $0 \le j \le m$ yields equality of the two norms. □

To extend this result to Riemannian manifolds, we first introduce notation that simplifies the manipulation of tensor contractions, together with an auxiliary analytic result.

Notation 9.1.37. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let

$$\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M, \qquad \pi_{\mathbf{F}} \colon \mathbf{F} \longrightarrow M$$

be smooth vector bundles of finite rank, equipped with smooth bundle metrics $\mathbf{h}_{\mathbf{E}}$ and $\mathbf{h}_{\mathbf{F}}$.
Let $\mathbf{G}_1, \dots, \mathbf{G}_r, \mathbf{H}$ be vector bundles obtained from $\mathbf{E}$, $\mathbf{F}$, their duals, and the tangent and cotangent bundles of M by finitely many tensor products. Let also

$$\mathbf{u}_1 \in \Gamma(\mathbf{G}_1), \dots, \mathbf{u}_r \in \Gamma(\mathbf{G}_r).$$

We say that a section $\mathbf{T} \in \Gamma(\mathbf{H})$ has the form

$$\mathbf{u}_1 * \cdots * \mathbf{u}_r$$

if there is a smooth bundle homomorphism

$$\Phi \colon \mathbf{G}_1 \otimes \cdots \otimes \mathbf{G}_r \longrightarrow \mathbf{H}$$

obtained by composition and finite linear combination of the following natural operations:

(a) tensor products;

(b) permutations of tensor factors;

(c) contractions between a bundle and its dual;

(d) the bundle isomorphisms induced by the bundle metrics $\mathbf{h}_{\mathbf{E}}$, $\mathbf{h}_{\mathbf{F}}$ and the Riemannian metric $\mathbf{g}$, together with their inverses.

and such that

$$\mathbf{T} = \Phi(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r).$$

In particular, if $\mathbf{u} \in \Gamma(\mathbf{G})$ and $\mathbf{v} \in \Gamma(\mathbf{K})$, the expression

$$\mathbf{u} * \mathbf{v}$$

denotes any section obtained from $\mathbf{u} \otimes \mathbf{v}$ by a finite combination of the preceding operations. Different occurrences of the symbol $*$ may correspond to different bundle homomorphisms.

Let us consider an example illustrating how this notation works.

Example 9.1.38. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary. Consider

$$\mathbf{R} \in \Gamma(T^{(0,4)}(TM)), \qquad \boldsymbol{\omega} \in \Gamma(T^{(0,1)}(TM)).$$

There is a canonical bundle isomorphism

$$T^{(0,4)}(TM) \otimes T^{(0,1)}(TM) \cong T^{(0,5)}(TM).$$

Define a bundle homomorphism

$$\Phi \colon T^{(0,5)}(TM) \longrightarrow T^{(0,1)}(TM)$$

by metric contractions with $\mathbf{g}^{-1} \in \Gamma(T^{(2,0)}(TM))$. More explicitly, in local coordinates, if

$$\mathbf{A} \in \Gamma(T^{(0,5)}(TM)),$$

and A_{abcde} are its components, then the components of $\Phi(\mathbf{A})$ are

$$\Phi(\mathbf{A})_b = g^{ae} g^{cd} A_{abcde}.$$

This homomorphism is obtained from tensor products, permutations, and contractions, so it is among those allowed in Notation 9.1.37.
Evaluating at $\mathbf{A} = \mathbf{R} \otimes \boldsymbol{\omega}$ gives

$$\Phi(\mathbf{R} \otimes \boldsymbol{\omega})_i = g^{jk} g^{lm} R_{jilm} \omega_k.$$

Consequently,

$$\mathbf{T} = \Phi(\mathbf{R} \otimes \boldsymbol{\omega})$$

is a term of the type $\mathbf{R} * \boldsymbol{\omega}$.

Lemma 9.1.39. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let

$$\mathbf{E}_1, \dots, \mathbf{E}_r, \mathbf{H} \longrightarrow M$$

be smooth vector bundles of finite rank, equipped with smooth bundle metrics.
Let

$$\Phi \colon \mathbf{E}_1 \otimes \cdots \otimes \mathbf{E}_r \longrightarrow \mathbf{H}$$

be a smooth bundle homomorphism obtained by composition and finite linear combination of tensor products, permutations of factors, contractions, and the isomorphisms induced by the bundle metrics and the Riemannian metric $\mathbf{g}$, together with their inverses.
Then, for every compact set $K \subseteq M$, there is a constant $C_K > 0$ such that, for every $x \in K$ and all

$$v_i \in (\mathbf{E}_i)_x, \qquad i = 1, \dots, r,$$

, we have

$$\left|\Phi_x(v_1 \otimes \cdots \otimes v_r)\right|_{\mathbf{h}_\mathbf{H}} \leq C_K |v_1|_{\mathbf{h}_{\mathbf{E}_1}} \cdots |v_r|_{\mathbf{h}_{\mathbf{E}_r}}.$$

In particular, if $\mathbf{u}_i \in \Gamma(\mathbf{E}_i)$ for $i = 1,\dots,r$, then

$$|\Phi(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r)(x)|_{\mathbf{h}_\mathbf{H}} \leq C_K\, |\mathbf{u}_1(x)|_{\mathbf{h}_{\mathbf{E}_1}} \cdots |\mathbf{u}_r(x)|_{\mathbf{h}_{\mathbf{E}_r}}, \qquad \forall x \in K.$$

Consequently, if $T = \mathbf{u}_1 * \cdots * \mathbf{u}_r$, then

$$|T(x)|_{\mathbf{h}_\mathbf{H}} \leq C_K\, |\mathbf{u}_1(x)|_{\mathbf{h}_{\mathbf{E}_1}} \cdots |\mathbf{u}_r(x)|_{\mathbf{h}_{\mathbf{E}_r}}, \qquad \forall x \in K.$$

Proof. Fix $x \in M$. Consider the map

$$\widetilde{\Phi}_x \colon (\mathbf{E}_1)_x \times \cdots \times (\mathbf{E}_r)_x \longrightarrow \mathbf{H}_x, \qquad \widetilde{\Phi}_x(v_1,\dots,v_r) := \Phi_x(v_1 \otimes \cdots \otimes v_r).$$

Let

$$\theta_x : (\mathbf{E}_1)_x \times \cdots \times (\mathbf{E}_r)_x \longrightarrow (\mathbf{E}_1)_x \otimes \cdots \otimes (\mathbf{E}_r)_x, \qquad \theta_x(v_1,\dots,v_r) = v_1 \otimes \cdots \otimes v_r.$$

The map θ_x is multilinear. On the other hand, since Φ is a bundle homomorphism,

$$\Phi_x \colon (\mathbf{E}_1)_x \otimes \cdots \otimes (\mathbf{E}_r)_x \longrightarrow \mathbf{H}_x$$

is linear. Consequently,

$$\widetilde{\Phi}_x = \Phi_x \circ \theta_x$$

is multilinear.

Denote by $|\cdot|_{\mathbf{h}_{\mathbf{E}_i,x}}$ the norm on $(\mathbf{E}_i)_x$ induced by the bundle metric $\mathbf{h}_{\mathbf{E}_i}$, and by $|\cdot|_{\mathbf{h}_{\mathbf{H},x}}$ the norm on $\mathbf{H}_x$ induced by the bundle metric of $\mathbf{H}$. Since the spaces involved are finite-dimensional, every multilinear map between them is continuous. Thus there is a constant $C(x) > 0$ such that

$$|\Phi_x(v_1 \otimes \cdots \otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}} \leq C(x)\, |v_1|_{\mathbf{h}_{\mathbf{E}_1,x}} \cdots |v_r|_{\mathbf{h}_{\mathbf{E}_r,x}}$$

for any $v_i \in (\mathbf{E}_i)_x$, with $i \in \{1,\dots,r\}$.

It remains to show that this constant can be chosen uniformly on compact sets. Let $p \in K$. Choose open sets $U_p, V_p \subseteq M$ such that

$$p \in U_p, \qquad \overline{U_p} \subseteq V_p, \qquad \overline{U_p} \text{ is compact},$$

and such that all the bundles

$$\mathbf{E}_1,\dots,\mathbf{E}_r,\mathbf{H}$$

admit smooth local frames over V_p. Fix local frames on V_p,

$$e_1^{(i)},\dots,e_{m_i}^{(i)} \quad \text{for } \mathbf{E}_i, \qquad \varepsilon_1,\dots,\varepsilon_\ell \quad \text{for } \mathbf{H}.$$

In these frames there are smooth functions

$$\Phi^b_{a_1\cdots a_r} \colon V_p \longrightarrow \mathbb{R}$$

such that

$$\Phi_x\big(e_{a_1}^{(1)}(x)\otimes\cdots\otimes e_{a_r}^{(r)}(x)\big)=\sum_{b=1}^{\ell}\Phi^{b}_{a_1\cdots a_r}(x)\,\varepsilon_b(x).$$

Since the frames and these functions are defined on a neighborhood of $\overline{U_p}$, their continuity and the compactness of $\overline{U_p}$ give $A_p>0$ such that

$$|\Phi^{b}_{a_1\cdots a_r}(x)|\le A_p\qquad\forall x\in\overline{U_p}.$$

Now let $x\in\overline{U_p}$ and, for each $i\in\{1,\dots,r\}$, write

$$v_i=\sum_{a_i=1}^{m_i}v_i^{a_i}\,e_{a_i}^{(i)}(x).$$

Then

$$\Phi_x(v_1\otimes\cdots\otimes v_r)=\sum_{\substack{1\le b\le\ell\\1\le a_i\le m_i}}\Phi^{b}_{a_1\cdots a_r}(x)\,v_1^{a_1}\cdots v_r^{a_r}\,\varepsilon_b(x).$$

By the triangle inequality,

$$|\Phi_x(v_1\otimes\cdots\otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}}\le\sum_{\substack{1\le b\le\ell\\1\le a_i\le m_i}}|\Phi^{b}_{a_1\cdots a_r}(x)||v_1^{a_1}|\cdots|v_r^{a_r}||\varepsilon_b(x)|_{\mathbf{h}_{\mathbf{H},x}}.$$

Using the bound $|\Phi^{b}_{a_1\cdots a_r}(x)|\le A_p$ gives

$$|\Phi_x(v_1\otimes\cdots\otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}}\le A_p\sum_{\substack{1\le b\le\ell\\1\le a_i\le m_i}}|v_1^{a_1}|\cdots|v_r^{a_r}||\varepsilon_b(x)|_{\mathbf{h}_{\mathbf{H},x}}.$$

Since $\overline{U_p}$ is compact and the sections ε_b are smooth, there is $B_p>0$ such that

$$|\varepsilon_b(x)|_{\mathbf{h}_{\mathbf{H},x}}\le B_p\qquad\forall x\in\overline{U_p},\ \forall b.$$

Therefore,

$$|\Phi_x(v_1\otimes\cdots\otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}}\le A_pB_p\sum_{\substack{1\le b\le\ell\\1\le a_i\le m_i}}|v_1^{a_1}|\cdots|v_r^{a_r}|.$$

Now,

$$\sum_{\substack{1\le b\le\ell\\1\le a_i\le m_i}}|v_1^{a_1}|\cdots|v_r^{a_r}|=\sum_{b=1}^{\ell}\sum_{a_1=1}^{m_1}\cdots\sum_{a_r=1}^{m_r}|v_1^{a_1}|\cdots|v_r^{a_r}|.$$

Since the inner expression is independent of b, we obtain

$$\sum_{\substack{1\le b\le\ell\\1\le a_i\le m_i}}|v_1^{a_1}|\cdots|v_r^{a_r}|=\ell\sum_{a_1=1}^{m_1}\cdots\sum_{a_r=1}^{m_r}|v_1^{a_1}|\cdots|v_r^{a_r}|.$$

Moreover, by distributivity,

$$\sum_{a_1=1}^{m_1}\cdots\sum_{a_r=1}^{m_r}|v_1^{a_1}|\cdots|v_r^{a_r}| = \left(\sum_{a_1=1}^{m_1}|v_1^{a_1}|\right)\cdots\left(\sum_{a_r=1}^{m_r}|v_r^{a_r}|\right).$$

Consequently,

$$|\Phi_x(v_1\otimes\cdots\otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}} \le A_p B_p\,\ell\prod_{i=1}^{r}\left(\sum_{a_i=1}^{m_i}|v_i^{a_i}|\right).$$

For each $i\in\{1,\dots,r\}$, the local frame $\mathbf{e}_1^{(i)},\dots,\mathbf{e}_{m_i}^{(i)}$ induces on each fiber $(\mathbf{E}_i)_x$, with $x\in\overline{U_p}$, the norm

$$\|v_i\|_{(i),x} := \sum_{a_i=1}^{m_i}|v_i^{a_i}|.$$

The matrix of $\mathbf{h}_{\mathbf{E}_i}$ in this frame and its inverse depend smoothly on x. Their coefficients are bounded on $\overline{U_p}$, and their positive minors are bounded away from zero. The compactness of $\overline{U_p}$ and of the unit sphere in each component space gives constants $0<d_{p,i}\le D_{p,i}<+\infty$, independent of x, such that

$$d_{p,i}|v_i|_{\mathbf{h}_{\mathbf{E}_i},x} \le \|v_i\|_{(i),x} \le D_{p,i}|v_i|_{\mathbf{h}_{\mathbf{E}_i},x} \qquad \forall x\in\overline{U_p},\ \forall v_i\in(\mathbf{E}_i)_x.$$

Substituting these estimates into the preceding inequality gives

$$|\Phi_x(v_1\otimes\cdots\otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}} \le A_pB_p\,\ell\,D_{p,1}\cdots D_{p,r}\,|v_1|_{\mathbf{h}_{\mathbf{E}_1,x}}\cdots|v_r|_{\mathbf{h}_{\mathbf{E}_r,x}}$$

for every $x\in\overline{U_p}$. Defining

$$C_p := A_pB_p\,\ell\,D_{p,1}\cdots D_{p,r},$$

yields

$$|\Phi_x(v_1\otimes\cdots\otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}} \le C_p\,|v_1|_{\mathbf{h}_{\mathbf{E}_1,x}}\cdots|v_r|_{\mathbf{h}_{\mathbf{E}_r,x}} \qquad \forall x\in\overline{U_p}.$$

The open sets U_p, with $p\in K$, cover K. By compactness there is a finite subcover

$$U_{p_1},\dots,U_{p_N}.$$

Defining

$$C_K := \max\{C_{p_1},\dots,C_{p_N}\},$$

gives

$$|\Phi_x(v_1\otimes\cdots\otimes v_r)|_{\mathbf{h}_{\mathbf{H},x}} \le C_K\,|v_1|_{\mathbf{h}_{\mathbf{E}_1,x}}\cdots|v_r|_{\mathbf{h}_{\mathbf{E}_r,x}} \qquad \forall x\in K.$$

The assertion for sections follows by taking $v_i=\mathbf{u}_i(x)$. □

Proposition 9.1.40. (Leibniz rule for the operator $*$) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary. Let $\mathbf{E}_1,\dots,\mathbf{E}_r,\mathbf{F}\longrightarrow M$ be smooth vector bundles with bundle metrics (Hermitian in the complex case) and connections compatible with them. We use the Levi–Civita connection on the tangent and

cotangent factors, and the induced connections on all products, duals, and homomorphism bundles. Let

$$\mathbf{\Phi}\colon \mathbf{E}_1 \otimes \cdots \otimes \mathbf{E}_r \longrightarrow \mathbf{F}$$

be a smooth bundle homomorphism obtained using tensor products, permutations, contractions, and the isomorphisms induced by the bundle metrics and the Riemannian metric.
If $\mathbf{u}_i \in \Gamma(\mathbf{E}_i)$ for $i = 1, \dots, r$, define

$$\mathbf{T} = \mathbf{\Phi}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r) = \mathbf{u}_1 * \cdots * \mathbf{u}_r.$$

Then, for every $\mathbf{X} \in \mathfrak{X}(M)$,

$$\nabla_{\mathbf{X}}\mathbf{T} = \sum_{i=1}^{r} \mathbf{u}_1 * \cdots * (\nabla_{\mathbf{X}}\mathbf{u}_i) * \cdots * \mathbf{u}_r.$$

Proof. By definition, $\mathbf{T} = \mathbf{\Phi}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r)$. We regard $\mathbf{\Phi}$ as a section of $\operatorname{Hom}(\mathbf{E}_1 \otimes \cdots \otimes \mathbf{E}_r, \mathbf{F})$, with the connection induced by those on $\mathbf{E}_1, \dots, \mathbf{E}_r$ and $\mathbf{F}$.
By the definition of this connection, for every $\mathbf{S} \in \Gamma(\mathbf{E}_1 \otimes \cdots \otimes \mathbf{E}_r)$ we have

$$(\nabla_{\mathbf{X}}\mathbf{\Phi})(\mathbf{S}) = \nabla_{\mathbf{X}}(\mathbf{\Phi}(\mathbf{S})) - \mathbf{\Phi}(\nabla_{\mathbf{X}}\mathbf{S}).$$

Applying this to $\mathbf{S} = \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r$ gives

$$\nabla_{\mathbf{X}}\mathbf{T} = (\nabla_{\mathbf{X}}\mathbf{\Phi})(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r) + \mathbf{\Phi}\big(\nabla_{\mathbf{X}}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r)\big).$$

By construction, $\mathbf{\Phi}$ is a combination of tensor products, permutations, contractions, and isomorphisms induced by the metrics. Compatibility of the connections ensures that the musical identifications are parallel; contractions, permutations, and tensor products are also parallel for the induced connections. Each operation therefore commutes with the connection in the sense of Definition 2.4.33. It follows that $\mathbf{\Phi}$ is parallel, that is, $\nabla_{\mathbf{X}}\mathbf{\Phi} = 0$.
Consequently,

$$\nabla_{\mathbf{X}}\mathbf{T} = \mathbf{\Phi}\big(\nabla_{\mathbf{X}}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r)\big).$$

Applying the Leibniz rule for the tensor product,

$$\nabla_{\mathbf{X}}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r) = \sum_{i=1}^{r} \mathbf{u}_1 \otimes \cdots \otimes (\nabla_{\mathbf{X}}\mathbf{u}_i) \otimes \cdots \otimes \mathbf{u}_r,$$

and substituting gives

$$\nabla_{\mathbf{X}}\mathbf{T} = \sum_{i=1}^{r} \mathbf{\Phi}\big(\mathbf{u}_1 \otimes \cdots \otimes (\nabla_{\mathbf{X}}\mathbf{u}_i) \otimes \cdots \otimes \mathbf{u}_r\big).$$

This is equivalent to

$$\nabla_{\mathbf{X}}\mathbf{T} = \sum_{i=1}^{r} \mathbf{u}_1 * \cdots * (\nabla_{\mathbf{X}}\mathbf{u}_i) * \cdots * \mathbf{u}_r,$$

as required. □

To study equivalence of norms, we need to control the commutator of covariant derivatives with the Laplacian. Recall the curvature $\mathbf{R}^{\mathbf{E}}$ of the connection, introduced in Definition 8.2.21. The product connection on tensor bundles combines $\mathbf{R}^{\mathbf{E}}$ with the Riemannian curvature; both therefore appear in the following commutator.

Lemma 9.1.41. (Commutator of the Bochner Laplacian with covariant derivatives) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. For $\ell\in\mathbb{N}_0$, set

$$\mathbf{E}_\ell := T^{(0,\ell)}(TM)\otimes\mathbf{E},$$

with the induced product metric and connection, and denote the covariant derivative by $\nabla_\ell\colon\Gamma(\mathbf{E}_\ell)\to\Gamma(\mathbf{E}_{\ell+1})$. For $m\in\mathbb{N}$ we write

$$\nabla^m := \nabla_{\ell+m-1}\circ\cdots\circ\nabla_\ell,$$

so that each new derivative index occupies the first covariant factor. Let

$$\Delta_{B,\ell} := (\nabla_\ell)^*_h\nabla_\ell = -\operatorname{tr}_{\mathbf{g}}(\nabla_\ell^2)$$

be the Bochner Laplacian on $\mathbf{E}_\ell$. Then, for every $\mathbf{u}\in\Gamma(\mathbf{E}_\ell)$,

$$\Delta_{B,\ell+m}(\nabla^m\mathbf{u})-\nabla^m(\Delta_{B,\ell}\mathbf{u}) = \sum_{k=0}^{m}\nabla^k\operatorname{Rm}*\nabla^{m-k}\mathbf{u}. \tag{9.27}$$

Here Rm represents either $\mathbf{R}^M$ or $\mathbf{R}^{\mathbf{E}}$, depending on the summand, with their induced connections, and $*$ denotes a finite sum of tensor products, permutations, and universal contractions. The coefficients and the number of terms depend only on m, ℓ, $\dim M$, and the rank of $\mathbf{E}$.

Proof. We first prove the identity for a single derivative. Let $\mathbf{F}\to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{F}}$ and a compatible connection $\nabla^{\mathbf{F}}$, curvature $\mathbf{R}^{\mathbf{F}}$, and Bochner Laplacian

$$\Delta_B^{\mathbf{F}} := (\nabla^{\mathbf{F}})^*\nabla^{\mathbf{F}} = -\operatorname{tr}_{\mathbf{g}}\big((\nabla^{\mathbf{F}})^2\big).$$

We claim that, for each $\mathbf{v}\in\Gamma(\mathbf{F})$, each $x\in M$, each $\mathbf{X}\in T_xM$, and any orthonormal basis $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ of T_xM, we have

$$\big(\nabla^{\mathbf{F}}\Delta_B^{\mathbf{F}}\mathbf{v}-\Delta_B^{T^*M\otimes\mathbf{F}}(\nabla^{\mathbf{F}}\mathbf{v})\big)(\mathbf{X}) = \nabla^{\mathbf{F}}_{\operatorname{Ric}^\sharp\mathbf{X}}\mathbf{v}+2\sum_{i=1}^{n}\mathbf{R}^{\mathbf{F}}(\mathbf{e}_i,\mathbf{X})\nabla^{\mathbf{F}}_{\mathbf{e}_i}\mathbf{v}+\sum_{i=1}^{n}(\nabla_{\mathbf{e}_i}\mathbf{R}^{\mathbf{F}})(\mathbf{e}_i,\mathbf{X})\mathbf{v}. \tag{9.28}$$

All terms on the right-hand side are evaluated at x. The sums are tensor contractions and thus do not depend on the chosen orthonormal basis.

First suppose that $x\in\operatorname{Int}M$. Both sides of (9.28) belong to $T^*_xM\otimes\mathbf{F}_x$; we therefore fix an arbitrary vector $\mathbf{X}\in T_xM$ and compare their values at $\mathbf{X}$. Extend $\mathbf{X},\mathbf{e}_1,\dots,\mathbf{e}_n$ to local fields, denoted in the same way, such that

$$\nabla^M_{\mathbf{Y}}\mathbf{X}(x)=0,\qquad \nabla^M_{\mathbf{Y}}\mathbf{e}_i(x)=0$$

for every $\mathbf{Y}\in T_xM$ and every i. These extensions are obtained by taking constant components in normal coordinates centered at x.

Set $\mathbf{A}:=\nabla^{\mathbf{F}}\mathbf{v}\in\Gamma(T^*M\otimes\mathbf{F})$. According to the convention in Proposition 2.4.29, the first two arguments

of $\nabla^3\mathbf{v} = \nabla^2\mathbf{A}$ are the added derivative indices, and the third is the original argument of $\mathbf{A}$. Since the Levi–Civita connection commutes with metric contractions by Proposition 2.4.37, at the point x we have

$$\begin{aligned}(\nabla^{\mathbf{F}}\Delta_B^{\mathbf{F}}\mathbf{v})(\mathbf{X}) &= -\sum_{i=1}^{n}(\nabla^3\mathbf{v})(\mathbf{X},\mathbf{e}_i,\mathbf{e}_i),\\ \big(\Delta_B^{T^*M\otimes\mathbf{F}}\mathbf{A}\big)(\mathbf{X}) &= -\sum_{i=1}^{n}(\nabla^3\mathbf{v})(\mathbf{e}_i,\mathbf{e}_i,\mathbf{X}).\end{aligned} \tag{9.29}$$

The first equality follows by differentiating the tensor $-\operatorname{tr}_{\mathbf{g}}\nabla^2\mathbf{v}$, rather than a formula valid only at x. Subtracting the two equalities in (9.29) gives

$$\big(\nabla^{\mathbf{F}}\Delta_B^{\mathbf{F}}\mathbf{v} - \Delta_B^{T^*M\otimes\mathbf{F}}\mathbf{A}\big)(\mathbf{X}) = \sum_{i=1}^{n}\big[(\nabla^3\mathbf{v})(\mathbf{e}_i,\mathbf{e}_i,\mathbf{X}) - (\nabla^3\mathbf{v})(\mathbf{X},\mathbf{e}_i,\mathbf{e}_i)\big]. \tag{9.30}$$

We interchange the indices in two steps. The Ricci identity in Proposition 8.2.23, first applied to $\mathbf{v}$ and then covariantly differentiated in the direction $\mathbf{e}_i$, gives

$$(\nabla^3\mathbf{v})(\mathbf{e}_i,\mathbf{e}_i,\mathbf{X}) - (\nabla^3\mathbf{v})(\mathbf{e}_i,\mathbf{X},\mathbf{e}_i) = (\nabla_{\mathbf{e}_i}\mathbf{R}^{\mathbf{F}})(\mathbf{e}_i,\mathbf{X})\mathbf{v} + \mathbf{R}^{\mathbf{F}}(\mathbf{e}_i,\mathbf{X})\nabla^{\mathbf{F}}_{\mathbf{e}_i}\mathbf{v}. \tag{9.31}$$

Indeed, the terms in which the derivative acts on $\mathbf{e}_i$ or $\mathbf{X}$ vanish at x by the choice of extensions.
We now apply the same Ricci identity to the section $\mathbf{A} = \nabla^{\mathbf{F}}\mathbf{v}$ of the bundle $T^*M\otimes\mathbf{F}$. By the product curvature formula (8.15),

$$(\nabla^3\mathbf{v})(\mathbf{e}_i,\mathbf{X},\mathbf{e}_i) - (\nabla^3\mathbf{v})(\mathbf{X},\mathbf{e}_i,\mathbf{e}_i) = \mathbf{R}^{\mathbf{F}}(\mathbf{e}_i,\mathbf{X})\nabla^{\mathbf{F}}_{\mathbf{e}_i}\mathbf{v} - \nabla^{\mathbf{F}}_{\mathbf{R}^M(\mathbf{e}_i,\mathbf{X})\mathbf{e}_i}\mathbf{v}. \tag{9.32}$$

Adding (9.31) and (9.32), then summing over i and using (9.30), gives

$$\big(\nabla^{\mathbf{F}}\Delta_B^{\mathbf{F}}\mathbf{v} - \Delta_B^{T^*M\otimes\mathbf{F}}(\nabla^{\mathbf{F}}\mathbf{v})\big)(\mathbf{X}) = \sum_{i=1}^{n}(\nabla_{\mathbf{e}_i}\mathbf{R}^{\mathbf{F}})(\mathbf{e}_i,\mathbf{X})\mathbf{v} + 2\sum_{i=1}^{n}\mathbf{R}^{\mathbf{F}}(\mathbf{e}_i,\mathbf{X})\nabla^{\mathbf{F}}_{\mathbf{e}_i}\mathbf{v} - \nabla^{\mathbf{F}}_{\sum_{i=1}^{n}\mathbf{R}^M(\mathbf{e}_i,\mathbf{X})\mathbf{e}_i}\mathbf{v}.$$

Proposition 2.10.9 states that $-\sum_{i=1}^{n}\mathbf{R}^M(\mathbf{e}_i,\mathbf{X})\mathbf{e}_i = \operatorname{Ric}^\sharp\mathbf{X}$, and we therefore obtain (9.28). Since x and $\mathbf{X}$ were arbitrary, the identity holds on $\operatorname{Int}M$. If M has boundary, both sides are smooth sections up to ∂M and $\operatorname{Int}M$ is dense in M; by continuity, the identity also holds on the boundary.
As a check on the sign, if $\mathbf{F} = M\times\mathbb{R}$ has the trivial connection, the preceding formula reduces to

$$d\Delta_B f - \Delta_B^{T^*M}(df) = df\circ\operatorname{Ric}^\sharp.$$

We now apply the identity to $\mathbf{F} = \mathbf{E}_\ell = T^{(0,\ell)}(TM)\otimes\mathbf{E}$. By (8.13), the curvature of $\mathbf{E}_\ell$ is the sum of the action of $\mathbf{R}^{\mathbf{E}}$ on the factor $\mathbf{E}$ and the actions of $\mathbf{R}^M$ on the covariant factors. Moving (9.28) to the other side and absorbing the signs and universal contractions into the symbol $*$ gives

$$\Delta_{B,\ell+1}(\nabla\mathbf{u}) - \nabla(\Delta_{B,\ell}\mathbf{u}) = \operatorname{Rm}*\nabla\mathbf{u} + \nabla\operatorname{Rm}*\mathbf{u}. \tag{9.33}$$

This is the case $m = 1$.

Suppose that (9.27) holds for some $m \geq 1$. Then

$$\Delta_{B,\ell+m+1}(\nabla^{m+1}\mathbf{u}) - \nabla^{m+1}(\Delta_{B,\ell}\mathbf{u}) = \big(\Delta_{B,\ell+m+1}\nabla - \nabla\Delta_{B,\ell+m}\big)(\nabla^m\mathbf{u}) + \nabla\big(\Delta_{B,\ell+m}(\nabla^m\mathbf{u}) - \nabla^m(\Delta_{B,\ell}\mathbf{u})\big).$$

By (9.33), the first term is a universal sum of terms of the types $\mathrm{Rm}*\nabla^{m+1}\mathbf{u}$ and $\nabla\mathrm{Rm}*\nabla^m\mathbf{u}$. For the second we use the induction hypothesis and the Leibniz rule in Proposition 9.1.40:

$$\nabla\big(\Delta_{B,\ell+m}(\nabla^m\mathbf{u}) - \nabla^m(\Delta_{B,\ell}\mathbf{u})\big) = \sum_{k=0}^{m}\Big(\nabla^{k+1}\mathrm{Rm}*\nabla^{m-k}\mathbf{u} + \nabla^k\mathrm{Rm}*\nabla^{m+1-k}\mathbf{u}\Big).$$

All the resulting terms have the form $\nabla^q\mathrm{Rm}*\nabla^{m+1-q}\mathbf{u}$ with $0 \leq q \leq m+1$. Regrouping, for each q, the universal contractions that occur, we conclude that

$$\Delta_{B,\ell+m+1}(\nabla^{m+1}\mathbf{u}) - \nabla^{m+1}(\Delta_{B,\ell}\mathbf{u}) = \sum_{q=0}^{m+1}\nabla^q\mathrm{Rm}*\nabla^{m+1-q}\mathbf{u}.$$

This completes the induction. □

Theorem 9.1.42. (Equivalence of norms on $H^m(M,\mathbf{E})$ in terms of the Bochner Laplacian) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, and let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $m \in \mathbb{N}$. For $\mathbf{u} \in \Gamma(\mathbf{E})$ define

$$\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})} := \sum_{j=0}^{m}\int_M |\nabla^j\mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}},$$

and

$$\|\mathbf{u}\|^2_{\Delta_B} := \sum_{\substack{j=0\\ j\text{ even}}}^{m}\int_M |\Delta_B^{\frac{j}{2}}\mathbf{u}|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} + \sum_{\substack{j=1\\ j\text{ odd}}}^{m}\int_M \big|\nabla\Delta_B^{\frac{j-1}{2}}\mathbf{u}\big|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Denote by $H^m(M,\mathbf{E})$ the completion of $\Gamma(\mathbf{E})$ with respect to the first norm. Then there are constants $c, C > 0$ such that

$$c\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})} \leq \|\mathbf{u}\|^2_{\Delta_B} \leq C\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})}, \qquad \forall\mathbf{u} \in \Gamma(\mathbf{E}).$$

Consequently, the second norm determines the same completion. For an element of $H^m(M,\mathbf{E})$, each term containing a power of Δ_B is understood as the limit in L^2 of the corresponding term of a smooth Cauchy sequence.

Proof. Since M is closed, it is compact and $\partial M = \varnothing$. We prove the inequalities on $\Gamma(\mathbf{E})$; the assertion about completions will then follow by continuity.
Throughout what follows, all integrations by parts have no boundary terms.

We first prove that

$$\|\mathbf{u}\|^2_{\Delta_B} \leq C\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})}.$$

By definition,

$$\Delta_B = \nabla^*\nabla = -\mathrm{tr}_{\mathbf{g}}(\nabla^2).$$

Since the connection is compatible with the metric, covariant differentiation commutes with contraction. Define the contractions

$$C_k^{\mathrm{even}}(T) := g^{i_1 i_2} \cdots g^{i_{2k-1} i_{2k}} T_{i_1 \cdots i_{2k}},$$
$$\big(C_k^{\mathrm{odd}}(S)\big)_a := g^{i_1 i_2} \cdots g^{i_{2k-1} i_{2k}} S_{a i_1 \cdots i_{2k}}.$$

The first contracts the pairs $(1,2),(3,4),\ldots$; the second leaves the first index free and contracts $(2,3),(4,5),\ldots$. Induction on k gives

$$\Delta_B^k \mathbf{u} = (-1)^k C_k^{\mathrm{even}}(\nabla^{2k}\mathbf{u}).$$

Set $\Phi_k := (-1)^k C_k^{\mathrm{even}}$. Since Φ_k is obtained using contractions and linear combinations, Lemma 9.1.39 implies that there is a constant $C_k > 0$ such that

$$|\Delta_B^k \mathbf{u}|_{\mathbf{h_E}} \le C_k |\nabla^{2k}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}.$$

Squaring and integrating gives

$$\int_M |\Delta_B^k \mathbf{u}|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le C_k^2 \int_M |\nabla^{2k}\mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Since $\nabla \mathbf{g} = 0$, differentiating each factor $\mathbf{g}^{-1}$ used in the contraction gives zero; by the Leibniz rule,

$$\nabla \Delta_B^k \mathbf{u} = (-1)^k \nabla\big(C_k^{\mathrm{even}}(\nabla^{2k}\mathbf{u})\big) = (-1)^k C_k^{\mathrm{odd}}(\nabla^{2k+1}\mathbf{u}).$$

We therefore define

$$\Psi_k \colon T^{(0,2k+1)}(TM) \otimes \mathbf{E} \longrightarrow T^*M \otimes \mathbf{E}, \qquad \Psi_k(S) := (-1)^k C_k^{\mathrm{odd}}(S).$$

Thus

$$\nabla \Delta_B^k \mathbf{u} = \Psi_k(\nabla^{2k+1}\mathbf{u}),$$

and, again by Lemma 9.1.39, there is $C_k' > 0$ such that

$$|\nabla \Delta_B^k \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \le C_k' |\nabla^{2k+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}.$$

Squaring and integrating gives

$$\int_M |\nabla \Delta_B^k \mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le (C_k')^2 \int_M |\nabla^{2k+1}\mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Consequently, each even summand of $\|\mathbf{u}\|^2_{\Delta_B}$ is bounded by the corresponding even-order summand of $\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})}$, and each odd summand of $\|\mathbf{u}\|^2_{\Delta_B}$ is bounded by the corresponding odd-order summand of $\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})}$. Adding all the terms, we conclude that there is a constant $C > 0$ such that

$$\|\mathbf{u}\|^2_{\Delta_B} \le C \|\mathbf{u}\|^2_{H^m(M,\mathbf{E})}.$$

We now prove that

$$\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})} \le C\|\mathbf{u}\|^2_{\Delta_B}.$$

We formulate the inductive estimate for all smooth sections. For $0 \le j \le m$ and $\mathbf{v} \in \Gamma(\mathbf{E})$, introduce the truncated quantity

$$\mathcal{N}_j(\mathbf{v})^2 := \sum_{\substack{0\le r\le j\\ r \text{ even}}} \|\Delta_B^{\frac{r}{2}}\mathbf{v}\|^2_{L^2(M,\mathbf{E})} + \sum_{\substack{1\le r\le j\\ r \text{ odd}}} \|\nabla^{\mathbf{E}}\Delta_B^{\frac{r-1}{2}}\mathbf{v}\|^2_{L^2(M,T^*M\otimes\mathbf{E})}. \tag{9.34}$$

Thus $\mathcal{N}_m(\mathbf{u}) = \|\mathbf{u}\|_{\Delta_B}$. We prove the following quantified assertion by induction: for each $j \in \{0,\dots,m\}$ there is $C_j > 0$ such that, for every smooth section $\mathbf{v} \in \Gamma(\mathbf{E})$,

$$\int_M |\nabla^j\mathbf{v}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le C_j\mathcal{N}_j(\mathbf{v})^2.$$

Since the estimate is formulated for every smooth section, at the step of order $j-2$ we may take $\mathbf{v} = \Delta_B\mathbf{u}$.
For $j = 0$ and every $\mathbf{v} \in \Gamma(\mathbf{E})$ we have

$$\int_M |\mathbf{v}|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \mathcal{N}_0(\mathbf{v})^2,$$

since the truncated sum contains only the term of order zero.
For $j = 1$ and every $\mathbf{v} \in \Gamma(\mathbf{E})$ we have

$$\int_M |\nabla\mathbf{v}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le \mathcal{N}_1(\mathbf{v})^2,$$

since this is the odd term of order one.
Now consider the case $j = 2$. Applying Corollary 9.1.31 to the bundle $T^*M \otimes \mathbf{E}$ gives

$$\int_M |\nabla^2\mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \int_M \langle\nabla\mathbf{u}, \Delta_{B,1}(\nabla\mathbf{u})\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

By Lemma 9.1.41 with $m = 1$,

$$\Delta_{B,1}(\nabla\mathbf{u}) = \nabla(\Delta_{B,0}\mathbf{u}) + \mathrm{Rm}*\nabla\mathbf{u} + \nabla\mathrm{Rm}*\mathbf{u}.$$

Substituting,

$$\int_M |\nabla^2\mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \int_M \langle\nabla\mathbf{u}, \nabla(\Delta_B\mathbf{u})\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} + \int_M \langle\nabla\mathbf{u}, \mathrm{Rm}*\nabla\mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} + \int_M \langle\nabla\mathbf{u}, \nabla\mathrm{Rm}*\mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Applying Corollary 9.1.31 once more, now to the bundle $\mathbf{E}$, gives

$$\int_M \langle\nabla\mathbf{u}, \nabla(\Delta_B\mathbf{u})\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \int_M |\Delta_B\mathbf{u}|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Moreover, since M is compact, the curvature tensor is uniformly bounded and, by Lemma 9.1.39, there is a constant $C > 0$ such that

$$|\mathrm{Rm}*\nabla\mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \le C|\nabla\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}.$$

Consequently,

$$\left|\int_M \langle\nabla\mathbf{u}, \mathrm{Rm}*\nabla\mathbf{u}\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}\right| \le C\int_M |\nabla\mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Likewise, the compactness of M gives a uniform bound on $\nabla \operatorname{Rm}$, so that

$$|\nabla \operatorname{Rm} * \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \le C|\mathbf{u}|_{\mathbf{h}_\mathbf{E}}.$$

By the Cauchy–Schwarz inequality,

$$\left| \int_M \langle \nabla \mathbf{u}, \nabla \operatorname{Rm} * \mathbf{u} \rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right| \le C \|\nabla \mathbf{u}\|_{L^2(M,T^*M\otimes\mathbf{E})} \|\mathbf{u}\|_{L^2(M,\mathbf{E})}.$$

The two factors correspond to lower-order terms appearing in $\|\mathbf{u}\|_{\Delta_B}$. The elementary inequality $2ab \le a^2 + b^2$, applied with $a = \|\nabla \mathbf{u}\|_{L^2(M,T^*M\otimes\mathbf{E})}$ and $b = \|\mathbf{u}\|_{L^2(M,\mathbf{E})}$, gives

$$\left| \int_M \langle \nabla \mathbf{u}, \nabla \operatorname{Rm} * \mathbf{u} \rangle_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right| \le C \|\nabla \mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})} + C \|\mathbf{u}\|^2_{L^2(M,\mathbf{E})}.$$

Consequently,

$$\int_M |\nabla^2 \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \le \int_M |\Delta_B \mathbf{u}|^2_{\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} + C \int_M |\nabla \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} + C \int_M |\mathbf{u}|^2_{\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g}.$$

The three terms on the right-hand side appear, respectively, in $\mathcal{N}_2(\mathbf{u})^2$. Therefore,

$$\int_M |\nabla^2 \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \le C_2 \mathcal{N}_2(\mathbf{u})^2.$$

Now let $j \ge 3$ and suppose that, for every integer $l < j$ and every $\mathbf{v} \in \Gamma(\mathbf{E})$, we have already proved that

$$\int_M |\nabla^l \mathbf{v}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \le C_l \mathcal{N}_l(\mathbf{v})^2.$$

We want to prove the same estimate for j.

Write $\nabla^j \mathbf{u} = \nabla^2(\nabla^{j-2}\mathbf{u})$. The argument for the case $j = 2$ depends only on compatibility of the connection and the uniform bounds on the curvature and its first derivative. It therefore applies to the bundle $T^{(0,j-2)}(TM) \otimes \mathbf{E}$ and the section $\nabla^{j-2}\mathbf{u}$. The curvature of this product bundle and its first derivative are sums of contractions of $\mathbf{R}^M$, $\mathbf{R}^\mathbf{E}$, and their derivatives; all are bounded because M is compact. This yields

$$\begin{aligned} \int_M |\nabla^j \mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \le \int_M |\Delta_{B,j-2}(\nabla^{j-2}\mathbf{u})|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} + C \int_M |\nabla^{j-1}\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \\ + C \int_M |\nabla^{j-2}\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g}. \end{aligned}$$

By the induction hypothesis, the last two terms are bounded by $C\mathcal{N}_j(\mathbf{u})^2$, since $\mathcal{N}_l(\mathbf{u}) \le \mathcal{N}_j(\mathbf{u})$ for $l < j$. Thus it suffices to estimate the first.

Applying Lemma 9.1.41 with $m = j - 2$ gives

$$\Delta_{B,j-2}(\nabla^{j-2}\mathbf{u}) = \nabla^{j-2}(\Delta_B \mathbf{u}) + \sum_{k=0}^{j-2} \nabla^k \operatorname{Rm} * \nabla^{j-2-k}\mathbf{u}.$$

Squaring, using the inequality

$$|A_0 + \cdots + A_N|^2 \le C \sum_{r=0}^{N} |A_r|^2,$$

, and integrating, we obtain

$$\int_M |\Delta_{B,j-2}(\nabla^{j-2}\mathbf{u})|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le C \int_M |\nabla^{j-2}(\Delta_B \mathbf{u})|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} + C \sum_{k=0}^{j-2} \int_M |\nabla^k \operatorname{Rm} * \nabla^{j-2-k}\mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Since M is compact, all covariant derivatives of Rm are uniformly bounded. By Lemma 9.1.39, for each k there is a constant $C_k > 0$ such that

$$|\nabla^k \operatorname{Rm} * \nabla^{j-2-k}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \le C_k |\nabla^{j-2-k}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}.$$

Therefore,

$$\int_M |\Delta_{B,j-2}(\nabla^{j-2}\mathbf{u})|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le C \int_M |\nabla^{j-2}(\Delta_B \mathbf{u})|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} + C \sum_{l=0}^{j-2} \int_M |\nabla^l \mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

By the induction hypothesis, the lower-order sum is controlled by $\mathcal{N}_j(\mathbf{u})^2$. Consequently,

$$\int_M |\nabla^j \mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le C \int_M |\nabla^{j-2}(\Delta_B \mathbf{u})|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} + C\mathcal{N}_j(\mathbf{u})^2.$$

We now apply the inductive assertion, which holds for every smooth section, to $\mathbf{v} = \Delta_B \mathbf{u}$ at order $j-2$. Since $j-2 < j$, we obtain

$$\int_M |\nabla^{j-2}(\Delta_B \mathbf{u})|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le C \left(\sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{j-2} \int_M \left|\Delta_B^{\frac{\ell}{2}}(\Delta_B \mathbf{u})\right|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} + \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{j-2} \int_M \left|\nabla \Delta_B^{\frac{\ell-1}{2}}(\Delta_B \mathbf{u})\right|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \right).$$

We now rewrite each of these sums.
For the even sum,

$$\Delta_B^{\frac{\ell}{2}}(\Delta_B \mathbf{u}) = \Delta_B^{\frac{\ell}{2}+1}\mathbf{u}.$$

If ℓ ranges over the even integers between 0 and $j-2$, then $r = \ell + 2$ ranges over the even integers between 2 and j. Therefore,

$$\sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{j-2} \int_M \left|\Delta_B^{\frac{\ell}{2}}(\Delta_B \mathbf{u})\right|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \sum_{\substack{r=2 \\ r \text{ even}}}^{j} \int_M \left|\Delta_B^{\frac{r}{2}}\mathbf{u}\right|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

For the odd sum,

$$\nabla \Delta_B^{\frac{\ell-1}{2}}(\Delta_B \mathbf{u}) = \nabla \Delta_B^{\frac{\ell+1}{2}}\mathbf{u}.$$

If ℓ ranges over the odd integers between 1 and $j-2$, then $s = \ell + 2$ ranges over the odd integers between 3 and j. Thus

$$\sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{j-2} \int_M \left|\nabla \Delta_B^{\frac{\ell-1}{2}}(\Delta_B \mathbf{u})\right|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \sum_{\substack{s=3 \\ s \text{ odd}}}^{j} \int_M \left|\nabla \Delta_B^{\frac{s-1}{2}}\mathbf{u}\right|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

Substituting these identities into the preceding estimate gives

$$\int_M |\nabla^{j-2}(\Delta_B \mathbf{u})|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} \le C\left(\sum_{\substack{r=2\\ r\ \text{even}}}^{j}\int_M \left|\Delta_B^{\frac{r}{2}}\mathbf{u}\right|^2_{\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g} + \sum_{\substack{s=3\\ s\ \text{odd}}}^{j}\int_M \left|\nabla\Delta_B^{\frac{s-1}{2}}\mathbf{u}\right|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}} d\lambda_\mathbf{g}\right).$$

Each of these terms appears in the definition of $\mathcal{N}_j(\mathbf{u})^2$. Equivalently, $\mathcal{N}_{j-2}(\Delta_B\mathbf{u}) \le \mathcal{N}_j(\mathbf{u})$: the even powers and odd terms shift by exactly two orders. Consequently,

$$\int_M |\nabla^{j-2}(\Delta_B \mathbf{u})|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} \le C\mathcal{N}_j(\mathbf{u})^2.$$

Returning to the previous estimate for $\displaystyle\int_M |\nabla^j\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}$, we conclude that

$$\int_M |\nabla^j\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} \le C_j\mathcal{N}_j(\mathbf{u})^2.$$

This completes the induction step. Finally, $\mathcal{N}_j(\mathbf{u}) \le \mathcal{N}_m(\mathbf{u}) = \|\mathbf{u}\|_{\Delta_B}$ for $j \le m$. Summing over $j \in \{0,\dots,m\}$ gives

$$\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})} = \sum_{j=0}^{m}\int_M |\nabla^j\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g} \le C\|\mathbf{u}\|^2_{\Delta_B}.$$

Together with the inequality proved at the beginning,

$$\|\mathbf{u}\|^2_{\Delta_B} \le C\|\mathbf{u}\|^2_{H^m(M,\mathbf{E})},$$

this gives the two stated inequalities with positive constants. If $(\mathbf{u}_\nu)$ is Cauchy in the norm of H^m, the forward inequality shows that each sequence $\Delta_B^k\mathbf{u}_\nu$ and $\nabla\Delta_B^k\mathbf{u}_\nu$ appearing in the structured norm is Cauchy in the corresponding space L^2. We define these terms in the completion by their limits. The reverse inequality proves that this procedure adds no new elements and is independent of the representative sequence. This justifies the final assertion in the statement. □

9.1.5 Local characterization of Sobolev spaces in vector bundles

In this subsection we show how to characterize Sobolev sections through their component functions. To this end, we first prove an analogue of Lemma 7.1.8 for arbitrary vector bundles:

Lemma 9.1.43. Let M be a smooth manifold with or without boundary, let $K \subseteq M$ be compact, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle of finite rank. Let $\mathbf{h}$ and $\widetilde{\mathbf{h}}$ be two smooth bundle metrics on $\mathbf{E}$. Then there are constants $c, C > 0$ such that, for every $p \in K$ and every $v \in \mathbf{E}_p$,

$$c\,\mathbf{h}_p(v,v) \le \widetilde{\mathbf{h}}_p(v,v) \le C\,\mathbf{h}_p(v,v).$$

Equivalently, $\sqrt{c}\,|v|_\mathbf{h} \le |v|_{\widetilde{\mathbf{h}}} \le \sqrt{C}\,|v|_\mathbf{h}$ in the same fibers. In particular, if M is compact, we may take $K = M$.

Proof. If $K = \varnothing$ or the bundle has rank zero, it suffices to take $c = C = 1$. Suppose that $K \neq \varnothing$ and the rank is $r \geq 1$. Let $p \in M$. Consider

$$S_p := \{v \in \mathbf{E}_p \mid \mathbf{h}_p(v, v) = 1\}.$$

This set is closed and bounded in a finite-dimensional normed vector space, and is therefore compact. Moreover, the function $f_p : S_p \longrightarrow \mathbb{R}$, given by $f_p(v) = \widetilde{\mathbf{h}}_p(v, v)$, is continuous and hence attains its maximum. Thus the function $f : M \longrightarrow \mathbb{R}$ defined by

$$f(p) := \max_{v \in S_p} f_p(v)$$

is well defined. We prove its continuity by identifying the spheres in nearby fibers with a fixed sphere. On a neighborhood U of any point $p_* \in M$, the Gram–Schmidt process applied to a smooth frame gives a frame $\mathbf{e}_1, \ldots, \mathbf{e}_r$ orthonormal for $\mathbf{h}$. If the bundle is real, set $\mathbb{K} = \mathbb{R}$; if it is complex with Hermitian metrics, set $\mathbb{K} = \mathbb{C}$. Define

$$S := \left\{ w \in \mathbb{K}^r \;\middle|\; \sum_{a=1}^{r} |w^a|^2 = 1 \right\}, \qquad F(q, w) := \widetilde{\mathbf{h}}_q \left(\sum_{a=1}^{r} w^a \mathbf{e}_a(q), \sum_{a=1}^{r} w^a \mathbf{e}_a(q) \right).$$

For $q \in U$, the map $w \mapsto \sum_{a=1}^{r} w^a \mathbf{e}_a(q)$ identifies S with S_q. Hence $f(q) = \max_{w \in S} F(q, w)$. Let $H_{ab}(q) = \widetilde{\mathbf{h}}_q(\mathbf{e}_a(q), \mathbf{e}_b(q))$ be the continuous coefficients of the second metric in that frame. The Cauchy–Schwarz inequality gives $\sum_{a=1}^{r} |w^a| \leq \sqrt{r}$ for $w \in S$. Consequently,

$$\begin{aligned} \sup_{w \in S} |F(q, w) - F(p_*, w)| &\leq \max_{1 \leq a, b \leq r} |H_{ab}(q) - H_{ab}(p_*)| \sup_{w \in S} \left(\sum_{a=1}^{r} |w^a| \right)^2 \\ &\leq r \max_{1 \leq a, b \leq r} |H_{ab}(q) - H_{ab}(p_*)| \longrightarrow 0 \qquad (q \to p_*). \end{aligned}$$

If two functions on S differ by at most ε at every point, their maxima differ by at most ε: first use $F(q, w) \leq F(p_*, w) + \varepsilon$, and then the inequality with the two points interchanged. Thus

$$|f(q) - f(p_*)| \leq \sup_{w \in S} |F(q, w) - F(p_*, w)|,$$

which proves continuity of f on p_*. Since the point was arbitrary, f is continuous on M. The compactness of K gives $p_0 \in K$ such that $f(p) \leq f(p_0) =: C$ for every $p \in K$.
Moreover,

$$C = f(p_0) = \max_{v \in S_{p_0}} \widetilde{\mathbf{h}}_{p_0}(v, v) > 0,$$

since $v \neq 0$ for every $v \in S_{p_0}$. Thus, for every $p \in K$ and every $v \in S_p$, we have

$$\widetilde{\mathbf{h}}_p(v, v) \leq C.$$

Define $m(p) := \min_{v \in S_p} \widetilde{\mathbf{h}}_p(v, v)$. In the preceding frame, $m(q) = \min_{w \in S} F(q, w)$. The same inequalities between

$F(q,w)$ and $F(p_*,w)$, now taking minima, give

$$|m(q)-m(p_*)| \le \sup_{w\in S} |F(q,w)-F(p_*,w)| \longrightarrow 0.$$

Thus m is also continuous. Each $m(p)$ is positive because the minimum is attained at a nonzero vector and $\widetilde{\mathbf{h}}_p$ is positive definite. The compactness of K implies that m attains a minimum there, which remains positive: $c := \min_{p\in K} m(p) > 0$. Therefore, for every $p \in K$ and every $v \in S_p$, we have

$$c \le \widetilde{\mathbf{h}}_p(v,v).$$

Now let $p \in K$ and $v \in \mathbf{E}_p \setminus \{0\}$. Then $\dfrac{v}{|v|_{\mathbf{h}_p}} \in S_p$, so that

$$c \le \widetilde{\mathbf{h}}_p\left(\frac{v}{|v|_{\mathbf{h}_p}}, \frac{v}{|v|_{\mathbf{h}_p}}\right) \le C.$$

This implies that

$$c\,|v|^2_{\mathbf{h}_p} \le \widetilde{\mathbf{h}}_p(v,v) \le C\,|v|^2_{\mathbf{h}_p},$$

that is,

$$c\,\mathbf{h}_p(v,v) \le \widetilde{\mathbf{h}}_p(v,v) \le C\,\mathbf{h}_p(v,v).$$

The case $v = 0$ is immediate, since all the preceding quantities equal 0. □

With this in mind, we are ready to prove the following lemma, which allows us to work locally with these globally and intrinsically defined Sobolev spaces. Unlike the abstract definition of $W^{m,p}$, this lemma assumes that the connection is compatible with the bundle metric, because the proof uses the explicit form of the formal adjoint obtained from the integration-by-parts formula.

Lemma 9.1.44. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary. Let $m \in \mathbb{N}_0$ and $1 \le p < \infty$, let (U,φ) be a regular coordinate ball in M, and let $\pi_{\mathbf{E}}\colon \mathbf{E} \longrightarrow M$ be a smooth real vector bundle of rank r equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $\mathbf{F} \in L^1_{\mathrm{loc}}(M,\mathbf{E})$ and let $\{\mathbf{e}_1,\dots,\mathbf{e}_r\}$ be a local frame of $\mathbf{E}$ defined on a neighborhood of $\overline{U}$. Suppose that $\mathbf{F} = \sum_{a=1}^{r} F^a \mathbf{e}_a$ on U. Then

$$\mathbf{F}\restriction_U \in W^{m,p}(U,\mathbf{E}\restriction_U) \quad \text{if and only if} \quad F^a \circ \varphi^{-1} \in W^{m,p}(\varphi(U))$$

for every $1 \le a \le r$.
Moreover, there are constants $C_m, c_m > 0$ (independent of $\mathbf{F}$) such that

$$c_m \sum_{a=1}^{r} \left\|F^a \circ \varphi^{-1}\right\|_{W^{m,p}(\varphi(U))} \le \|\mathbf{F}\|_{W^{m,p}(U,\mathbf{E}\restriction_U)} \le C_m \sum_{a=1}^{r} \left\|F^a \circ \varphi^{-1}\right\|_{W^{m,p}(\varphi(U))}$$

for every $\mathbf{F} \in W^{m,p}(U,\mathbf{E}\restriction_U)$.

Proof. The case $m = 0$ follows directly from uniform equivalence of the fiber norms over the compact set $\overline{U}$ and equivalence of the volume elements. Thus suppose that $m \geq 1$. Fix $s \geq 1$ and denote by $I_s = (i_1, \ldots, i_s)$, $J_s = (j_1, \ldots, j_s)$, etc., the ordered tuples or blocks of indices in $\{1, \ldots, n\}$. We adopt the Einstein summation convention for these blocks. Define

$$\partial_{I_s} := \partial_{i_1} \cdots \partial_{i_s}, \quad g^{I_s J_s} := \prod_{l=1}^{s} g^{i_l j_l}, \quad g_{J_s T_s} := \prod_{l=1}^{s} g_{j_l t_l}.$$

Also, let $\beta_{I_s} \in \mathbb{N}_0^n$ be the multi-index associated with the block I_s such that $\partial^{\beta_{I_s}} \equiv \partial_{I_s}$.
Since $\nabla^{\mathbf{E}}$ is compatible with $\mathbf{h}_{\mathbf{E}}$, we may use the expression for the adjoint obtained in Corollary 9.1.32. Set $h_{ab} := \mathbf{h}_{\mathbf{E}}(\mathbf{e}_a, \mathbf{e}_b)$ and $(h^{ab}) := (h_{ab})^{-1}$. Written in the chosen frame, with the leading term separated from the lower-order terms, that formula gives

$$\left((\nabla^s)^* \boldsymbol{\Psi}\right)^a = (-1)^s \frac{h^{ab}}{\sqrt{\det(\mathbf{g})}} \partial_{I_s} \left(\sqrt{\det(\mathbf{g})} h_{bc} g^{I_s J_s} \Psi^c_{J_s} \right) + \sum_{|\beta| \leq s-1} (Q_\beta)_c^{a J_s} \partial^\beta \Psi^c_{J_s},$$

with smooth coefficients Q_β; the fiber indices b, c range over $\{1, \ldots, r\}$. The metric factors arise when both inner products are written in components before integrating by parts. Expanding by the Leibniz rule and using $h^{ab} h_{bc} = \delta^a_c$, the term of order s is $(-1)^s g^{I_s J_s} \partial_{I_s} \Psi^a_{J_s}$. All the remaining terms have order at most $s-1$. Moving $\sqrt{\det(\mathbf{g})}$ inside the leading derivative changes only terms of this lower order. Thus the local expression used below has the indicated leading term and smooth lower-order coefficients D_β; metric compatibility is precisely the hypothesis that allows us to identify the formal adjoint with the formula in Corollary 9.1.32.

($\Rightarrow$) First suppose that $\mathbf{F} \restriction_U \in W^{m,p}(U, \mathbf{E} \restriction_U)$. Then, for each $s \leq m$, the sth weak covariant derivative of $\mathbf{F} \restriction_U$ exists; denote it by $\mathbf{G} \in L^p\left(U, T^{(0,s)}(TU) \otimes \mathbf{E} \restriction_U\right)$. By definition, for every $\boldsymbol{\Psi} \in \Gamma_c\left(T^{(0,s)}(TU) \otimes \mathbf{E} \restriction_U\right)$ we have

$$\int_U \langle \mathbf{F}, (\nabla^s)^* \boldsymbol{\Psi} \rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}} = \int_U \langle \mathbf{G}, \boldsymbol{\Psi} \rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}}.$$

In local coordinates, writing $\mathbf{F} = F^a \mathbf{e}_a$ and $\boldsymbol{\Psi} = \Psi^c_{J_s} \mathbf{d}x^{J_s} \otimes \mathbf{e}_c$, this identity reads

$$\int_U h_{bc} F^b \left((\nabla^s)^* \boldsymbol{\Psi}\right)^c d\lambda_{\mathbf{g}} = \int_U g^{I_s J_s} h_{bc} G^b_{I_s} \Psi^c_{J_s} \, d\lambda_{\mathbf{g}}.$$

Now fix a target block $T_s = (t_1, \ldots, t_s)$ and a fiber index a. Given $\psi \in C_c^\infty(\varphi(U))$, define the test tensor field $\boldsymbol{\Psi}$ by its components:

$$\Psi^c_{J_s} = \frac{1}{\sqrt{\det(\mathbf{g})}} h^{ca} g_{J_s T_s} (\psi \circ \varphi). \tag{9.35}$$

The coordinate open set U, regarded as a manifold, has no boundary, and the test sections have compact support in it. Thus Theorem 8.2.3, in the local form described at the beginning of this proof, applies on U, and the formal adjoint acts on $\boldsymbol{\Psi}$ as

$$\left((\nabla^s)^* \boldsymbol{\Psi}\right)^c = (-1)^s \left[\frac{1}{\sqrt{\det(\mathbf{g})}} \partial_{I_s} \left(\sqrt{\det(\mathbf{g})} g^{I_s J_s} \Psi^c_{J_s} \right) + \sum_{|\beta| \leq s-1} \left(D_\beta\right)_d^{c J_s} \partial^\beta \Psi^d_{J_s} \right].$$

Substituting (9.35) into this expression and using $\frac{\sqrt{\det(\mathbf{g})}}{\sqrt{\det(\mathbf{g})}} = 1$, $g^{I_s J_s} g_{J_s T_s} = \delta^{I_s}_{T_s}$, we obtain

$$\big((\nabla^s)^*\boldsymbol{\Psi}\big)^c = (-1)^s \left[\frac{1}{\sqrt{\det(\mathbf{g})}} \partial_{T_s}\big(h^{ca}(\psi\circ\varphi)\big) + \sum_{|\beta|\le s-1} \big(D_\beta\big)^{cJ_s}_{d} \partial^\beta \left(\frac{h^{da} g_{J_s T_s}}{\sqrt{\det(\mathbf{g})}} (\psi\circ\varphi) \right) \right].$$

Let $I_1 := \int_U \langle \mathbf{F}, (\nabla^s)^*\boldsymbol{\Psi}\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}$ and $I_2 := \int_U \langle \mathbf{G}, \boldsymbol{\Psi}\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}$. Substituting (9.35) into I_2 and noting that $g^{I_s J_s} g_{J_s T_s} = \delta^{I_s}_{T_s}$ gives

$$I_2 = \int_U \frac{1}{\sqrt{\det(\mathbf{g})}} \delta^{I_s}_{T_s} \delta^a_b G^b_{I_s} (\psi\circ\varphi)\, d\lambda_{\mathbf{g}} = \int_{\varphi(U)} (G^a_{T_s}\circ\varphi^{-1})\psi\, d\lambda_n. \tag{9.36}$$

For I_1, we use the Leibniz formula in ∂_{T_s}:

$$\partial_{T_s}\big(h^{ca}(\psi\circ\varphi)\big) = h^{ca}\partial_{T_s}(\psi\circ\varphi) + \sum_{\alpha<\beta_{T_s}} \binom{\beta_{T_s}}{\alpha} \partial^\alpha(\psi\circ\varphi)\partial^{\beta_{T_s}-\alpha}(h^{ca}).$$

Substituting back into the integral I_1 gives

$$I_1 = (-1)^s \int_U \frac{1}{\sqrt{\det(\mathbf{g})}} F^a \partial_{T_s}(\psi\circ\varphi)\, d\lambda_{\mathbf{g}}$$

$$+(-1)^s \sum_{\alpha<\beta_{T_s}} \int_U \frac{1}{\sqrt{\det(\mathbf{g})}} h_{bc} F^b \binom{\beta_{T_s}}{\alpha} \partial^\alpha(\psi\circ\varphi)\partial^{\beta_{T_s}-\alpha}(h^{ca})\, d\lambda_{\mathbf{g}}$$

$$+ \sum_{|\beta|\le s-1} (-1)^s \int_U h_{bc} F^b \big(D_\beta\big)^{cJ_s}_{d} \partial^\beta \left(\frac{h^{da} g_{J_s T_s}}{\sqrt{\det(\mathbf{g})}} (\psi\circ\varphi) \right) d\lambda_{\mathbf{g}}.$$

We now proceed by induction on s. For $s = 1$, we have $T_1 = (t_1)$, $J_1 = (j_1)$, so the expression reduces to

$$I_1 = -\int_{\varphi(U)} (F^a\circ\varphi^{-1})\partial_{t_1}\psi\, d\lambda_n - \int_{\varphi(U)} ((h_{bc}F^b)\circ\varphi^{-1})\partial_{t_1}\left(h^{ca}\circ\varphi^{-1}\right)\psi\, d\lambda_n$$

$$- \int_{\varphi(U)} \left(\left(h_{bc}F^b (D_0)^{cj_1}_{d} h^{da} g_{j_1 t_1} \right)\circ\varphi^{-1} \right)\psi\, d\lambda_n.$$

Define

$$H^a_{t_1} := h_{bc}F^b\partial_{t_1}(h^{ca}) + h_{bc}F^b (D_0)^{cj_1}_{d} h^{da} g_{j_1 t_1}.$$

Since $\mathbf{F}\restriction_U \in W^{m,p}(U, \mathbf{E}\restriction_U)$ implies $F^b\circ\varphi^{-1} \in L^p(\varphi(U))$, and the metric coefficients are smooth on the compact set $\overline{\varphi(U)}$, it follows that $H^a_{t_1}\circ\varphi^{-1} \in L^p(\varphi(U))$. Since $I_1 = I_2$, we obtain

$$-\int_{\varphi(U)} (F^a\circ\varphi^{-1})\partial_{t_1}\psi\, d\lambda_n = \int_{\varphi(U)} \big((G^a_{t_1}\circ\varphi^{-1}) + (H^a_{t_1}\circ\varphi^{-1})\big)\psi\, d\lambda_n,$$

which shows that $F^a\circ\varphi^{-1} \in W^{1,p}(\varphi(U))$.

Now suppose that the assertion holds through order $s-1$ for $s \le m$. The induction hypothesis ensures that the components of $\mathbf{F}$ on U have weak derivatives of order $\le s-1$, so we may integrate by parts in the weak sense in the lower-order terms.

First, consider the terms arising from the Leibniz expansion in I_1:

$$(-1)^s \sum_{\alpha<\beta_{T_s}} \int_U \frac{h_{bc}F^b}{\sqrt{\det(\mathbf{g})}} \binom{\beta_{T_s}}{\alpha} \partial^\alpha(\psi\circ\varphi)\partial^{\beta_{T_s}-\alpha}(h^{ca})\, d\lambda_{\mathbf{g}}$$

$$= (-1)^s \sum_{\alpha<\beta_{T_s}} \int_{\varphi(U)} \left[\left(h_{bc}F^b \binom{\beta_{T_s}}{\alpha} \partial^{\beta_{T_s}-\alpha}(h^{ca}) \right) \circ \varphi^{-1} \right] \partial^\alpha \psi \, d\lambda_n.$$

Since $|\alpha| < |\beta_{I_s}| = s$, we have $|\alpha| \le s-1$. Integrating by parts (in the weak sense) $|\alpha|$ times transfers the derivatives from ψ to the coefficient (which involves F^b), so the preceding term equals

$$\sum_{\alpha<\beta_{T_s}} (-1)^{s+|\alpha|} \int_{\varphi(U)} \partial^\alpha \left[\left(h_{bc}F^b \binom{\beta_{T_s}}{\alpha} \partial^{\beta_{T_s}-\alpha}(h^{ca}) \right) \circ \varphi^{-1} \right] \psi \, d\lambda_n.$$

Second, consider the terms in I_1 involving the connection coefficients (D_β). Using the same change of variables and recalling that $|\beta| \le s-1$, we obtain

$$\sum_{|\beta|\le s-1} (-1)^s \int_U h_{bc}F^b (D_\beta)_d^{cJ_s} \partial^\beta \left(\frac{h^{da}g_{J_sT_s}}{\sqrt{\det(\mathbf{g})}}(\psi\circ\varphi) \right) d\lambda_{\mathbf{g}}$$

$$= (-1)^s \sum_{|\beta|\le s-1} \int_{\varphi(U)} \left[\left(h_{bc}F^b (D_\beta)_d^{cJ_s} \sqrt{\det(\mathbf{g})} \right) \circ \varphi^{-1} \right] \partial^\beta \left(\left(\frac{h^{da}g_{J_sT_s}}{\sqrt{\det(\mathbf{g})}} \circ \varphi^{-1} \right) \psi \right) d\lambda_n.$$

Integrating by parts $|\beta|$ times gives

$$= (-1)^s \sum_{|\beta|\le s-1} (-1)^{|\beta|} \int_{\varphi(U)} \partial^\beta \left[\left(h_{bc}F^b (D_\beta)_d^{cJ_s} \sqrt{\det(\mathbf{g})} \right) \circ \varphi^{-1} \right] \left(\frac{h^{da}g_{J_sT_s}}{\sqrt{\det(\mathbf{g})}} \circ \varphi^{-1} \right) \psi \, d\lambda_n.$$

Now define $\widetilde{G}^a_{T_s}$ on $\varphi(U)$ by grouping $G^a_{T_s}\circ\varphi^{-1}$ (which comes from the integral I_2) and moving to the right-hand side the terms computed above from I_1 (introducing an extra factor -1, which changes to $(-1)^{s+1}$):

$$\widetilde{G}^a_{T_s} := G^a_{T_s}\circ\varphi^{-1} + \sum_{\alpha<\beta_{T_s}} (-1)^{s+1+|\alpha|} \partial^\alpha \left[\left(h_{bc}F^b \binom{\beta_{T_s}}{\alpha} \partial^{\beta_{T_s}-\alpha}(h^{ca}) \right) \circ \varphi^{-1} \right]$$

$$+ \sum_{|\beta|\le s-1} (-1)^{s+1+|\beta|} \partial^\beta \left[\left(h_{bc}F^b (D_\beta)_d^{cJ_s} \sqrt{\det(\mathbf{g})} \right) \circ \varphi^{-1} \right] \left(\frac{h^{da}g_{J_sT_s}}{\sqrt{\det(\mathbf{g})}} \circ \varphi^{-1} \right).$$

The induction hypothesis states that $F^b\circ\varphi^{-1} \in W^{s-1,p}(\varphi(U))$ for every $b \in \{1,\dots,r\}$. Since $\widetilde{G}^a_{T_s}$ consists of smooth bounded functions (defined on $\overline{\varphi(U)}$) and weak derivatives of $F^b\circ\varphi^{-1}$ of order at most $s-1$, we conclude that $\widetilde{G}^a_{T_s} \in L^p(\varphi(U))$. Using the equality $I_1 = I_2$ and the definition of $\widetilde{G}$ gives

$$(-1)^s \int_{\varphi(U)} (F^a\circ\varphi^{-1}) \partial_{T_s}\psi \, d\lambda_n = \int_{\varphi(U)} \widetilde{G}^a_{T_s} \psi \, d\lambda_n.$$

This shows that $F^a\circ\varphi^{-1}$ has a weak derivative $\partial_{T_s}(F^a\circ\varphi^{-1}) = \widetilde{G}^a_{T_s} \in L^p(\varphi(U))$. Therefore, $F^a\circ\varphi^{-1} \in W^{s,p}(\varphi(U))$.

Since $s \le m$ was arbitrary, we conclude that $F^a \circ \varphi^{-1} \in W^{m,p}(\varphi(U))$ for every $a \in \{1,\dots,r\}$.

($\Leftarrow$) Now suppose that $F^a \circ \varphi^{-1} \in W^{m,p}(\varphi(U))$ for every $a \in \{1,\dots,r\}$. Fix $s \le m$ and take $\boldsymbol{\Psi} \in \Gamma_c\big(T^{(0,s)}(TU) \otimes \mathbf{E}\restriction_U\big)$. We have

$$\int_U \langle \mathbf{F}, (\nabla^s)^* \boldsymbol{\Psi} \rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \int_U h_{ab} F^a \big((\nabla^s)^* \boldsymbol{\Psi}\big)^b\, d\lambda_{\mathbf{g}}.$$

By the local expression for the formal adjoint justified at the beginning of this proof, the adjoint has the coordinate form

$$\big((\nabla^s)^* \boldsymbol{\Psi}\big)^b = (-1)^s \left[\frac{1}{\sqrt{\det(\mathbf{g})}} \partial_{I_s} \left(\sqrt{\det(\mathbf{g})}\, g^{I_s J_s} \Psi^b_{J_s} \right) + \sum_{|\beta| \le s-1} \big(D_\beta\big)_c^{b J_s}\, \partial^\beta \Psi^c_{J_s} \right].$$

The integral can then be written as a sum of two terms, I_1 and I_2. Expanding in coordinates gives

$$\int_U \langle \mathbf{F}, (\nabla^s)^* \boldsymbol{\Psi} \rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} = \underbrace{(-1)^s \int_U h_{ab} F^a \frac{1}{\sqrt{\det(\mathbf{g})}} \partial_{I_s} \left(\sqrt{\det(\mathbf{g})}\, g^{I_s J_s} \Psi^b_{J_s} \right) d\lambda_{\mathbf{g}}}_{I_1}$$

$$\underbrace{+ (-1)^s \sum_{|\beta| \le s-1} \int_U h_{ab} F^a \big(D_\beta\big)_c^{b J_s}\, \partial^\beta \Psi^c_{J_s}\, d\lambda_{\mathbf{g}}}_{I_2}.$$

Changing the domain of integration to $\varphi(U)$ and writing the compositions explicitly gives

$$I_1 = (-1)^s \int_{\varphi(U)} \left[(h_{ab} F^a) \circ \varphi^{-1} \right] \partial_{I_s} \left(\left(\sqrt{\det(\mathbf{g})}\, g^{I_s J_s} \Psi^b_{J_s} \right) \circ \varphi^{-1} \right) d\lambda_n,$$

$$I_2 = (-1)^s \sum_{|\beta| \le s-1} \int_{\varphi(U)} \left[\left(h_{ab} F^a (D_\beta)_c^{b J_s} \sqrt{\det(\mathbf{g})} \right) \circ \varphi^{-1} \right] \partial^\beta (\Psi^c_{J_s} \circ \varphi^{-1})\, d\lambda_n.$$

We now integrate by parts in the weak sense in I_1 (which is valid because $F^a \circ \varphi^{-1} \in W^{m,p}(\varphi(U))$ and $s \le m$):

$$I_1 = \int_{\varphi(U)} \partial_{I_s} \left[\left(h_{ab} \circ \varphi^{-1} \right) (F^a \circ \varphi^{-1}) \right] \left(\sqrt{\det(\mathbf{g})}\, g^{I_s J_s} \Psi^b_{J_s} \circ \varphi^{-1} \right) d\lambda_n.$$

Applying the Leibniz rule to the bracketed term gives

$$I_1 = \int_{\varphi(U)} \left[\left(h_{ab} g^{I_s J_s} \sqrt{\det(\mathbf{g})} \circ \varphi^{-1} \right) \partial_{I_s} (F^a \circ \varphi^{-1}) \right] (\Psi^b_{J_s} \circ \varphi^{-1})\, d\lambda_n$$

$$+ \int_{\varphi(U)} \left[\sum_{\substack{I_s \\ \alpha < \beta_{I_s}}} \binom{\beta_{I_s}}{\alpha} \partial^{\beta_{I_s} - \alpha} \left(h_{ab} \circ \varphi^{-1} \right) \partial^\alpha (F^a \circ \varphi^{-1}) \left(\left(g^{I_s J_s} \sqrt{\det(\mathbf{g})} \Psi^b_{J_s} \right) \circ \varphi^{-1} \right) \right] d\lambda_n.$$

Define the coefficient functions $(B_\gamma)_{abI_s}$ on $\varphi(U)$ for each $|\gamma| \le s-1$ by

$$(B_\gamma)_{ab}^{J_s} := \sum_{\substack{I_s \\ \gamma < \beta_{I_s}}} \binom{\beta_{I_s}}{\gamma} \partial^{\beta_{I_s}-\gamma}\left(h_{ab}\circ\varphi^{-1}\right)\left(\left(g^{I_sJ_s}\sqrt{\det(\mathbf{g})}\right)\circ\varphi^{-1}\right).$$

Substituting this definition into I_1 gives

$$I_1 = \int_{\varphi(U)}\left[\left(h_{ab}g^{I_sJ_s}\sqrt{\det(\mathbf{g})}\circ\varphi^{-1}\right)\partial_{I_s}(F^a\circ\varphi^{-1}) + \sum_{|\gamma|\le s-1}(B_\gamma)_{ab}^{J_s}\partial^\gamma(F^a\circ\varphi^{-1})\right](\Psi_{J_s}^b\circ\varphi^{-1})\,d\lambda_n.$$

For I_2, integrating by parts $|\beta|$ times gives

$$I_2 = \sum_{|\beta|\le s-1}(-1)^{s+|\beta|}\int_{\varphi(U)}\partial^\beta\left[\left(\left(h_{ab}(D_\beta)_c^{bJ_s}\sqrt{\det(\mathbf{g})}\right)\circ\varphi^{-1}\right)(F^a\circ\varphi^{-1})\right](\Psi_{J_s}^c\circ\varphi^{-1})\,d\lambda_n.$$

Applying the Leibniz rule once more to expand the derivative ∂^β of the product gives

$$I_2 = \int_{\varphi(U)}\sum_{|\beta|\le s-1}(-1)^{s+|\beta|}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^{\beta-\gamma}\left(\left(h_{ab}(D_\beta)_c^{bJ_s}\sqrt{\det(\mathbf{g})}\right)\circ\varphi^{-1}\right)\partial^\gamma(F^a\circ\varphi^{-1})(\Psi_{J_s}^c\circ\varphi^{-1})\,d\lambda_n.$$

Define the coefficient functions $(C_\gamma)_{ac}^{J_s}$ on $\varphi(U)$ for each $|\gamma| \le s-1$ by

$$(C_\gamma)_{ac}^{J_s} := \sum_{\substack{|\beta|\le s-1 \\ \gamma\le\beta}}(-1)^{s+|\beta|}\binom{\beta}{\gamma}\partial^{\beta-\gamma}\left(\left(h_{ab}(D_\beta)_c^{bJ_s}\sqrt{\det(\mathbf{g})}\right)\circ\varphi^{-1}\right).$$

Substituting this definition into I_2 gives

$$I_2 = \int_{\varphi(U)}\sum_{|\gamma|\le s-1}(C_\gamma)_{ac}^{J_s}\partial^\gamma(F^a\circ\varphi^{-1})(\Psi_{J_s}^c\circ\varphi^{-1})\,d\lambda_n.$$

We now combine both results. To unify the notation, in I_1 we rename the fiber index $b \to c$ (and use h_{ac}). In this way,

$$\int_U \langle \mathbf{F}, (\nabla^s)^*\mathbf{\Psi}\rangle\,d\lambda_{\mathbf{g}} =$$

$$\int_{\varphi(U)}\left[\left(h_{ac}g^{I_sJ_s}\sqrt{\det(\mathbf{g})}\circ\varphi^{-1}\right)\partial_{I_s}(F^a\circ\varphi^{-1}) + \sum_{|\gamma|\le s-1}\left((B_\gamma)_{ac}^{J_s}+(C_\gamma)_{ac}^{J_s}\right)\partial^\gamma(F^a\circ\varphi^{-1})\right](\Psi_{J_s}^c\circ\varphi^{-1})\,d\lambda_n.$$

Now define the correction coefficients $(\widehat{D}_\gamma)$ on U by taking the *pullback* of B and C and inverting the metric weights:

$$\left(\widehat{D}_\gamma\right)_{bI_s}^a := \frac{g_{I_s\tilde{J}_s}h^{a\tilde{c}}}{\sqrt{\det(\mathbf{g})}}\left[\left((B_\gamma)_{b\tilde{c}}^{\tilde{J}_s}\circ\varphi\right)+\left((C_\gamma)_{b\tilde{c}}^{\tilde{J}_s}\circ\varphi\right)\right].$$

Then define $\mathbf{G} \in L^p(U, T^{(0,s)}(TU) \otimes \mathbf{E} \restriction_U)$ in coordinates by

$$G^a_{I_s} = \partial_{I_s} F^a + \sum_{|\gamma| \le s-1} (\widehat{D}_\gamma)^a_{b\, I_s} \partial^\gamma F^b. \tag{9.37}$$

Let us show that $\mathbf{G}$ is the desired weak covariant derivative of order s:

$$\int_U \langle \mathbf{G}, \mathbf{\Psi} \rangle_{\mathbf{g}, \mathbf{h_E}} \, d\lambda_{\mathbf{g}} = \int_U h_{ac} g^{I_s J_s} G^a_{I_s} \Psi^c_{J_s} \, d\lambda_{\mathbf{g}}$$

$$= \int_U h_{ac} g^{I_s J_s} \left(\partial_{I_s} F^a + \sum_{|\gamma| \le s-1} (\widehat{D}_\gamma)^a_{b\, I_s} \partial^\gamma F^b \right) \Psi^c_{J_s} \, d\lambda_{\mathbf{g}}.$$

The first term agrees with the principal part. For the lower-order terms, we substitute the definition of $\widehat{D}_\gamma$:

$$\int_U \sum_{|\gamma| \le s-1} h_{ac} g^{I_s J_s} \left(\frac{g_{I_s \tilde{J}_s} h^{a\tilde{c}}}{\sqrt{\det(\mathbf{g})}} \left[\left((B_\gamma)^{\tilde{J}_s}_{b\tilde{c}} \circ \varphi \right) + \left((C_\gamma)^{\tilde{J}_s}_{b\tilde{c}} \circ \varphi \right) \right] \right) \partial^\gamma F^b \Psi^c_{J_s} \, d\lambda_{\mathbf{g}}$$

$$= \int_U \sum_{|\gamma| \le s-1} \frac{1}{\sqrt{\det(\mathbf{g})}} (h_{ac} h^{a\tilde{c}}) (g^{I_s J_s} g_{I_s \tilde{J}_s}) \left[\left((B_\gamma)^{\tilde{J}_s}_{b\tilde{c}} \circ \varphi \right) + \left((C_\gamma)^{\tilde{J}_s}_{b\tilde{c}} \circ \varphi \right) \right] \partial^\gamma F^b \Psi^c_{J_s} \, d\lambda_{\mathbf{g}}.$$

Using the identities $h_{ac} h^{a\tilde{c}} = \delta^{\tilde{c}}_c$ and $g^{I_s J_s} g_{I_s \tilde{J}_s} = \delta^{J_s}_{\tilde{J}_s}$ gives

$$= \int_U \sum_{|\gamma| \le s-1} \frac{1}{\sqrt{\det(\mathbf{g})}} \left[\left((B_\gamma)^{J_s}_{bc} \circ \varphi \right) + \left((C_\gamma)^{J_s}_{bc} \circ \varphi \right) \right] \partial^\gamma F^b \Psi^c_{J_s} \, d\lambda_{\mathbf{g}}.$$

$$= \int_{\varphi(U)} \sum_{|\gamma| \le s-1} \left[(B_\gamma)^{J_s}_{bc} + (C_\gamma)^{J_s}_{bc} \right] \partial^\gamma (F^b \circ \varphi^{-1}) (\Psi^c_{J_s} \circ \varphi^{-1}) \, d\lambda_n.$$

This expression agrees exactly with the expansion of $\int_U \langle \mathbf{F}, (\nabla^s)^* \mathbf{\Psi} \rangle \, d\lambda_{\mathbf{g}}$. Therefore, $\mathbf{G}$ is the sth weak covariant derivative of $\mathbf{F}$ on U. Moreover, $\mathbf{G} \in L^p(U, T^{(0,s)}(TU) \otimes \mathbf{E} \restriction_U)$, since each coordinate component is a linear combination of weak derivatives of the components of $\mathbf{F}$ (through order s) multiplied by smooth bounded functions (bounded on U because U is a regular coordinate ball). Since $s \le m$ was arbitrary, we conclude that $\mathbf{F} \restriction_U \in W^{m,p}\left(U, \mathbf{E} \restriction_U\right)$.

Finally, we prove both norm inequalities quantitatively. Set $n := \dim M$, $\Omega := \varphi(U)$, and $f^a := F^a \circ \varphi^{-1}$ for $a \in \{1, \dots, r\}$. Recall that

$$\|\mathbf{F}\|_{W^{m,p}(U, \mathbf{E} \restriction_U)} = \left(\sum_{s=0}^{m} \int_U |\nabla^s_w \mathbf{F}|^p_{\mathbf{g}, \mathbf{h_E}} \, d\lambda_{\mathbf{g}} \right)^{1/p}.$$

At order zero we understand $\nabla^0_w \mathbf{F} = \mathbf{F}$.

Define the bundle metric $\overline{\mathbf{h}}_{\mathbf{E}}$ by making the frame $\mathbf{e}_1, \dots, \mathbf{e}_r$ orthonormal. Explicitly, if $v = \sum_{a=1}^{r} v^a \mathbf{e}_a(x)$ and

$w = \sum_{a=1}^{r} w^a \mathbf{e}_a(x)$, set

$$\overline{\mathbf{h}}_{\mathbf{E},x}(v,w) = \sum_{a=1}^{r} v^a w^a.$$

Also let $\delta := \varphi^*\overline{\mathbf{g}}$, where $\overline{\mathbf{g}}$ is the Euclidean metric on Ω. The chart and frame are defined on a neighborhood of $\overline{U}$, so these auxiliary metrics are as well. For each $q \in \{0,\dots,m\}$ define

$$\|\mathbf{F}\|_{\mathrm{aux},q,p} := \left(\sum_{s=0}^{q} \int_U |\nabla_w^s \mathbf{F}|^p_{\delta,\overline{\mathbf{h}}_{\mathbf{E}}}\, d\lambda_\delta \right)^{1/p}.$$

The covariant derivatives in this expression remain the original ones: only the fiber norm of each tensor and the measure have changed.

For each $s \in \{0,\dots,m\}$ we apply Lemma 9.1.43 to the two induced metrics on $T^{(0,s)}(TM)\otimes\mathbf{E}$ over the compact set $\overline{U}$. There are numbers $a_s, b_s > 0$ such that, for every $x \in \overline{U}$ and every $\mathbf{T} \in T^{(0,s)}_x(TM)\otimes\mathbf{E}_x$,

$$a_s|\mathbf{T}|_{\delta,\overline{\mathbf{h}}_{\mathbf{E}}} \le |\mathbf{T}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} \le b_s|\mathbf{T}|_{\delta,\overline{\mathbf{h}}_{\mathbf{E}}}.$$

Taking $a_* := \min_{0\le s\le m} a_s > 0$ and $b_* := \max_{0\le s\le m} b_s < \infty$ gives constants valid simultaneously for all orders under consideration. On the other hand, in the coordinates of φ,

$$d\lambda_{\mathbf{g}} = \rho\, d\lambda_\delta, \qquad \rho := \sqrt{\det(\mathbf{g})}.$$

The function ρ is smooth and strictly positive on a neighborhood of $\overline{U}$. By compactness there are $\rho_-, \rho_+ > 0$ such that $\rho_- \le \rho(x) \le \rho_+$ for every $x \in \overline{U}$. Thus, for each $q \in \{0,\dots,m\}$,

$$\begin{aligned} a_*^p \rho_- \sum_{s=0}^{q} \int_U |\nabla_w^s \mathbf{F}|^p_{\delta,\overline{\mathbf{h}}_{\mathbf{E}}}\, d\lambda_\delta &\le \sum_{s=0}^{q} \int_U |\nabla_w^s \mathbf{F}|^p_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} \\ &\le b_*^p \rho_+ \sum_{s=0}^{q} \int_U |\nabla_w^s \mathbf{F}|^p_{\delta,\overline{\mathbf{h}}_{\mathbf{E}}}\, d\lambda_\delta. \end{aligned}$$

With $c := a_*\rho_-^{1/p}$ and $C := b_*\rho_+^{1/p}$, taking pth roots and specializing to $q = m$ gives

$$c\|\mathbf{F}\|_{\mathrm{aux},m,p} \le \|\mathbf{F}\|_{W^{m,p}(U,\mathbf{E}\restriction_U)} \le C\|\mathbf{F}\|_{\mathrm{aux},m,p}. \tag{9.38}$$

The same comparison holds at each order $q \le m$ with these constants.

We now pass from covariant derivatives to partial derivatives. For $s \in \{1,\dots,m\}$ write $\mathcal{I}_s := \{1,\dots,n\}^s$; its elements are ordered blocks $I_s = (i_1,\dots,i_s)$. For $s = 0$ set $\mathcal{I}_0 := \{\varnothing\}$ and $\partial_\varnothing f^a := f^a$. Formula (9.37) reads

$$\big((\nabla_w^s \mathbf{F})^a_{I_s}\big) \circ \varphi^{-1} = \partial_{I_s} f^a + \sum_{b=1}^{r} \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|\le s-1}} A^{abI_s}_{s,\gamma}\, \partial^\gamma f^b, \tag{9.39}$$

where

$$A^{abI_s}_{s,\gamma} := \big((\widehat{D}_\gamma)^a_{bI_s}\big) \circ \varphi^{-1}.$$

Each coefficient extends smoothly to a neighborhood of $\overline{\Omega}$ and, in particular, belongs to $L^\infty(\Omega)$. If $m \geq 1$, define

$$K := \max_{\substack{1\leq s\leq m,\ 1\leq a,b\leq r \\ I_s\in\mathcal{I}_s,\ \gamma\in\mathbb{N}_0^n,\ |\gamma|\leq s-1}} \|A_{s,\gamma}^{abI_s}\|_{L^\infty(\Omega)} < \infty.$$

The maximum contains finitely many coefficients. If $m = 0$, set $K := 0$, since there are no lower-order terms. Denote by

$$G_{s,I_s}^a := \big((\nabla_w^s \mathbf{F})_{I_s}^a\big) \circ \varphi^{-1}, \qquad Q_s := \left(\sum_{a=1}^r \sum_{I_s\in\mathcal{I}_s} |G_{s,I_s}^a|^2\right)^{1/2}.$$

In particular, $G_{0,\varnothing}^a = f^a$. The coordinate frame tensors, together with the frame $\mathbf{e}_a$, are orthonormal for the auxiliary metrics. Moreover, $\varphi_*\lambda_\delta = \lambda_n$. These two observations give the exact equalities

$$\begin{aligned}\|\mathbf{F}\|_{\mathrm{aux},m,p} &= \left(\sum_{s=0}^m \int_\Omega Q_s(x)^p \, d\lambda_n(x)\right)^{1/p} \\ &= \left[\sum_{s=0}^m \int_\Omega \left(\sum_{a=1}^r \sum_{I_s\in\mathcal{I}_s} \left|\partial_{I_s} f^a + \sum_{b=1}^r \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|\leq s-1}} A_{s,\gamma}^{abI_s} \partial^\gamma f^b\right|^2\right)^{p/2} d\lambda_n\right]^{1/p}.\end{aligned}$$

In the summand $s = 0$, the inner sum of lower-order terms is zero, and the expression inside the bars is simply f^a.

For the upper bound, fix $s \in \{1,\dots,m\}$. The inequality $|A+B|^2 \leq 2|A|^2 + 2|B|^2$ and the bound K on the coefficients imply, almost everywhere on Ω,

$$\begin{aligned}Q_s^2 &\leq 2\sum_{a=1}^r \sum_{I_s\in\mathcal{I}_s} |\partial_{I_s} f^a|^2 \\ &\quad + 2rn^s K^2 \left(\sum_{b=1}^r \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|\leq s-1}} |\partial^\gamma f^b|\right)^2 \\ &= 2\sum_{a=1}^r \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=s}} \frac{s!}{\alpha!} |\partial^\alpha f^a|^2 + 2rn^s K^2 \left(\sum_{b=1}^r \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|\leq s-1}} |\partial^\gamma f^b|\right)^2.\end{aligned}$$

The factor rn^s counts the pairs (a, I_s). The factor $s!/\alpha!$ counts the ordered blocks producing the same derivative ∂^α; distributional partial derivatives commute, so this grouping is valid for weak derivatives as well. Since $s!/\alpha! \leq s!$ and the Euclidean norm of a finite family is at most the sum of the absolute values of its components, it follows that

$$Q_s \leq \sqrt{2}\big(\sqrt{s!} + \sqrt{rn^s}\,K\big) \sum_{b=1}^r \sum_{\substack{\beta\in\mathbb{N}_0^n \\ |\beta|\leq m}} |\partial^\beta f^b|.$$

At order zero the same bound holds with the factor 1. Set

$$B_0 := 1, \qquad B_s := \sqrt{2}\big(\sqrt{s!} + \sqrt{r n^s}\, K\big)\ (1 \le s \le m), \qquad B := \max_{0 \le s \le m} B_s.$$

There are $N_m := \binom{n+m}{m}$ multi-indices of order at most m. Applying the finite-sum inequality

$$\left(\sum_{j=1}^{N} t_j\right)^p \le N^{p-1} \sum_{j=1}^{N} t_j^p \qquad (t_j \ge 0,\ N \in \mathbb{N}),$$

with $N = rN_m$, and then using the definition of the Euclidean Sobolev norm, gives

$$\begin{aligned} \|\mathbf{F}\|_{\mathrm{aux},m,p}^p &\le (m+1) B^p (rN_m)^{p-1} \sum_{a=1}^{r} \sum_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta| \le m}} \int_\Omega |\partial^\beta f^a|^p \, d\lambda_n \\ &= (m+1) B^p (rN_m)^{p-1} \sum_{a=1}^{r} \|f^a\|_{W^{m,p}(\Omega)}^p. \end{aligned}$$

Thus, with $C_{\mathrm{aux},m} := (m+1)^{1/p} B (rN_m)^{1-1/p}$, taking roots and using $(\sum_{a=1}^{r} t_a^p)^{1/p} \le \sum_{a=1}^{r} t_a$ yields

$$\|\mathbf{F}\|_{\mathrm{aux},m,p} \le C_{\mathrm{aux},m} \sum_{a=1}^{r} \|f^a\|_{W^{m,p}(\Omega)}. \tag{9.40}$$

For the lower bound, we proceed by induction on the order. For $q \in \{0, \dots, m\}$, write

$$S_q := \sum_{a=1}^{r} \|f^a\|_{W^{q,p}(\Omega)}, \qquad U_q := \|\mathbf{F}\|_{\mathrm{aux},q,p}.$$

The inductive assertion is that there are $K_q > 0$, independent of $\mathbf{F}$, such that

$$S_q \le K_q U_q. \tag{9.41}$$

For $q = 0$, the pointwise inequality $|f^a| \le Q_0$ gives $\|f^a\|_{L^p(\Omega)} \le U_0$ for each $a \in \{1, \dots, r\}$. Summing gives $S_0 \le r U_0$, so we may take $K_0 := r$.

Suppose the result has been proved for $q = s - 1$, with $1 \le s \le m$. For each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = s$, choose a block $I(\alpha) \in \mathcal{I}_s$ containing the index j exactly α_j times for each $j \in \{1, \dots, n\}$. Solving in (9.39) gives

$$\partial^\alpha f^a = G^a_{s,I(\alpha)} - \sum_{b=1}^{r} \sum_{\substack{\gamma \in \mathbb{N}_0^n \\ |\gamma| \le s-1}} A_{s,\gamma}^{abI(\alpha)} \partial^\gamma f^b.$$

Since $|G^{a}_{s,I(\alpha)}| \le Q_s$, the triangle inequality in $L^p(\Omega)$ and the estimate for bounded multipliers give

$$\|\partial^{\alpha} f^{a}\|_{L^p(\Omega)} \le U_s + K \sum_{b=1}^{r} \sum_{\substack{\gamma \in \mathbb{N}_0^n \\ |\gamma| \le s-1}} \|\partial^{\gamma} f^{b}\|_{L^p(\Omega)}.$$

For each component b, Hölder's inequality for the finite sum of its $N_{s-1} = \binom{n+s-1}{s-1}$ derivatives gives

$$\begin{aligned} \sum_{\substack{\gamma \in \mathbb{N}_0^n \\ |\gamma| \le s-1}} \|\partial^{\gamma} f^{b}\|_{L^p(\Omega)} &\le N_{s-1}^{1-1/p} \left(\sum_{\substack{\gamma \in \mathbb{N}_0^n \\ |\gamma| \le s-1}} \|\partial^{\gamma} f^{b}\|_{L^p(\Omega)}^{p} \right)^{1/p} \\ &= N_{s-1}^{1-1/p} \|f^{b}\|_{W^{s-1,p}(\Omega)}. \end{aligned}$$

For $p = 1$ this inequality is an equality. There are $d_s := \binom{n+s-1}{s}$ multi-indices of order exactly s. Thus, summing the preceding bound over the r components and these d_s multi-indices gives

$$\sum_{a=1}^{r} \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| = s}} \|\partial^{\alpha} f^{a}\|_{L^p(\Omega)} \le r d_s U_s + r d_s K N_{s-1}^{1-1/p} S_{s-1}.$$

The definition of the Sobolev norm also implies

$$\begin{aligned} S_s &\le S_{s-1} + \sum_{a=1}^{r} \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| = s}} \|\partial^{\alpha} f^{a}\|_{L^p(\Omega)} \\ &\le r d_s U_s + \bigl(1 + r d_s K N_{s-1}^{1-1/p}\bigr) S_{s-1}. \end{aligned}$$

By the induction hypothesis, $S_{s-1} \le K_{s-1} U_{s-1} \le K_{s-1} U_s$; the second inequality follows because the auxiliary norm of order s contains all the terms of the norm of order $s-1$. Consequently, (9.41) holds for $q = s$ with

$$K_s := r d_s + \bigl(1 + r d_s K N_{s-1}^{1-1/p}\bigr) K_{s-1}.$$

The induction reaches $s = m$ and proves

$$K_m^{-1} \sum_{a=1}^{r} \|f^{a}\|_{W^{m,p}(\Omega)} \le \|\mathbf{F}\|_{\mathrm{aux},m,p}. \tag{9.42}$$

Finally, combine (9.40), (9.42), and (9.38). Taking $c_m := c/K_m$ and $C_m := C C_{\mathrm{aux},m}$ gives

$$\begin{aligned} c_m \sum_{a=1}^{r} \|F^{a} \circ \varphi^{-1}\|_{W^{m,p}(\varphi(U))} &\le \|\mathbf{F}\|_{W^{m,p}(U, \mathbf{E}\restriction_U)}, \\ \|\mathbf{F}\|_{W^{m,p}(U, \mathbf{E}\restriction_U)} &\le C_m \sum_{a=1}^{r} \|F^{a} \circ \varphi^{-1}\|_{W^{m,p}(\varphi(U))}. \end{aligned}$$

The constants depend on m, p, n, r, the fixed chart and frame, the comparison bounds for the metrics and

volume on $\overline{U}$, and the $L^\infty(\Omega)$ norms of the coefficients appearing through order m. None depends on $\mathbf{F}$. The argument includes $m = 0$ under the indicated conventions and does not use approximation by smooth sections.

□

Before formulating the result for sections, we recall the Euclidean version to be used in local trivializations.

Definition 9.1.45. Let $\Omega \subseteq \mathbb{R}^n$ be open, let $m \in \mathbb{N}_0$, and let $1 \le p < \infty$. Define $C^{m,p}(\Omega) := \{u \in C^\infty(\Omega) \mid \|u\|_{W^{m,p}(\Omega)} < \infty\}$ and denote its completion with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ by $H^{m,p}(\Omega)$.

Theorem 9.1.46. (Meyers–Serrin) Let $\Omega \subseteq \mathbb{R}^n$ be open, $m \in \mathbb{N}_0$, and $1 \le p < \infty$. Then $H^{m,p}(\Omega) = W^{m,p}(\Omega)$.

A proof can be found in [102, 1].

9.1.6 The Meyers–Serrin approximation theorem in vector bundles

The bundle version of Meyers–Serrin compares the definition using weak covariant derivatives with the completion of smooth sections. To formulate it, we first introduce the space $H^{m,p}(M,\mathbf{E})$, analogous to the construction in Definition 7.1.3.

Definition 9.1.47. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$ and a connection $\nabla^{\mathbf{E}}$, and let $m \in \mathbb{N}_0$ and $1 \le p < \infty$. Define

$$C^{m,p}(M,\mathbf{E}) := \{\mathbf{u} \in \Gamma(\mathbf{E}) \mid \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} < \infty\},$$

where $\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} := \left(\sum_{s=0}^{m} \int_M |\nabla^s \mathbf{u}|^p_{\mathbf{g},\mathbf{h_E}} \, d\lambda_{\mathbf{g}}\right)^{\frac{1}{p}}$.

Define

$$H^{m,p}(M,\mathbf{E}) := \overline{C^{m,p}(M,\mathbf{E})}^{\|\cdot\|_{W^{m,p}(M,\mathbf{E})}},$$

which is the completion of $C^{m,p}(M,\mathbf{E})$ with respect to the Sobolev norm $\|\cdot\|_{W^{m,p}(M,\mathbf{E})}$.

As in the Euclidean case of Theorem 9.1.46, both constructions agree. The equality $H = W$ thus extends from functions to sections of a vector bundle.

Theorem 9.1.48. (Meyers–Serrin for vector bundles) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $\pi_{\mathbf{E}} \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$, and let $m \in \mathbb{N}_0$ and $1 \le p < \infty$. Then

$$H^{m,p}(M,\mathbf{E}) = W^{m,p}(M,\mathbf{E}).$$

Proof. By Proposition 1.2.11, there is a countable locally finite open cover of M by regular coordinate balls $(U_\alpha, \varphi_\alpha)_{\alpha\in\mathbb{N}}$. Refining it if necessary, we may require that $\overline{U_\alpha}$ lie in the domain of a local frame of $\mathbf{E}$. Let $(\psi_\alpha)_{\alpha\in\mathbb{N}}$ be a smooth partition of unity subordinate to this cover.

Now consider a section $\mathbf{F} \in W^{m,p}(M,\mathbf{E})$. We can write

$$\mathbf{F} = \sum_{\alpha=1}^{\infty} \psi_\alpha \mathbf{F},$$

where the sum is locally finite. For each α, the product $\psi_\alpha \mathbf{F} \restriction_{U_\alpha}$ belongs to $W^{m,p}(U_\alpha, \mathbf{E} \restriction_{U_\alpha})$, since ψ_α is smooth and has compact support contained in U_α.
Fix α and work in the chart $(U_\alpha, \varphi_\alpha)$, together with one of these local frames $(\mathbf{e}_1,\dots,\mathbf{e}_r)$, defined on a neighborhood of $\overline{U_\alpha}$. Applying Lemma 9.1.44 to $(\psi_\alpha \mathbf{F}) \restriction_{U_\alpha}$ gives, for every fiber index $a \in \{1,\dots,r\}$,

$$\big(\psi_\alpha \mathbf{F}\big)^a \circ \varphi_\alpha^{-1} \in W^{m,p}(\varphi_\alpha(U_\alpha)).$$

Moreover, there are constants $C_{m,\alpha} > 0$ such that, for every section $\mathbf{H} \in W^{m,p}(U_\alpha, \mathbf{E} \restriction_{U_\alpha})$,

$$\|\mathbf{H}\|_{W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} \le C_{m,\alpha} \sum_{a=1}^{r} \Big\| H^a \circ \varphi_\alpha^{-1} \Big\|_{W^{m,p}(\varphi_\alpha(U_\alpha))}.$$

Write $\Omega_\alpha := \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$. The support of each component $\big(\psi_\alpha \mathbf{F}\big)^a \circ \varphi_\alpha^{-1}$ is a compact subset of Ω_α. The classical Meyers–Serrin theorem on domains in $\mathbb{R}^n$, Theorem 9.1.46, provides smooth approximants in $W^{m,p}(\Omega_\alpha)$. To obtain compactly supported approximants, choose $\chi_\alpha \in C_c^\infty(\Omega_\alpha)$ equal to one on a neighborhood of the preceding support and multiply by χ_α. The Leibniz rule shows directly that multiplication by χ_α is continuous on $W^{m,p}(\Omega_\alpha)$. Thus, for each $\alpha \in \mathbb{N}$ and each component $a \in \{1,\dots,r\}$, there is a sequence

$$(\widetilde{G}_{\alpha,k})^a \in C_c^\infty(\Omega_\alpha), \qquad k \in \mathbb{N},$$

such that

$$\sum_{a=1}^{r} \Big\| (\widetilde{G}_{\alpha,k})^a - \big(\psi_\alpha \mathbf{F}\big)^a \circ \varphi_\alpha^{-1} \Big\|_{W^{m,p}(\Omega_\alpha)} \xrightarrow[k\to\infty]{} 0.$$

From these smooth functions we define a smooth section $\mathbf{G}_{\alpha,k} \in \Gamma(\mathbf{E} \restriction_{U_\alpha})$ by

$$\mathbf{G}_{\alpha,k} = \sum_{a=1}^{r} \Big[(\widetilde{G}_{\alpha,k})^a \circ \varphi_\alpha \Big]\, \mathbf{e}_a.$$

The estimate in Lemma 9.1.44 then gives

$$\|\mathbf{G}_{\alpha,k} - (\psi_\alpha \mathbf{F}) \restriction_{U_\alpha} \|_{W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} \xrightarrow[k\to\infty]{} 0.$$

Extending each $\mathbf{G}_{\alpha,k}$ by zero outside U_α, we may regard it as a global compactly supported section, that is, $\mathbf{G}_{\alpha,k} \in \Gamma_c(\mathbf{E})$, and moreover

$$\|\mathbf{G}_{\alpha,k} - (\psi_\alpha \mathbf{F}) \restriction_{U_\alpha} \|_{W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} = \left(\sum_{s=0}^{m} \int_{U_\alpha} |\nabla_w^s (\mathbf{G}_{\alpha,k} - (\psi_\alpha \mathbf{F}) \restriction_{U_\alpha})|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right)^{\frac{1}{p}}.$$

Since $\operatorname{supp}(\mathbf{G}_{\alpha,k} - \psi_\alpha \mathbf{F}) \subseteq U_\alpha$ and ∇^s_w is a local operator (Proposition 9.1.8), the preceding expression equals

$$\left(\sum_{s=0}^{m} \int_M |\nabla^s_w (\mathbf{G}_{\alpha,k} - \psi_\alpha \mathbf{F})|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right)^{\frac{1}{p}}.$$

Consequently,

$$\|\mathbf{G}_{\alpha,k} - \psi_\alpha \mathbf{F}\|_{W^{m,p}(M,\mathbf{E})} \xrightarrow[k\to\infty]{} 0.$$

Now fix $\alpha \in \mathbb{N}$ and an integer $q \in \mathbb{N}$. By the preceding convergence, there is $k_{\alpha,q} \in \mathbb{N}$ such that

$$\|\psi_\alpha \mathbf{F} - \mathbf{G}_{\alpha,k_{\alpha,q}}\|_{W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} < \frac{1}{2^{\alpha+q}}.$$

Since $\operatorname{supp}(\mathbf{G}_{\alpha,k}) \subseteq U_\alpha$ for $\alpha, k \in \mathbb{N}$ and the family $(U_\alpha)_{\alpha\in\mathbb{N}}$ is locally finite, the sum

$$\mathbf{G}_q := \sum_{\alpha=1}^{\infty} \mathbf{G}_{\alpha,k_{\alpha,q}}$$

is locally finite and therefore defines a global smooth section $\mathbf{G}_q \in \Gamma(\mathbf{E})$.
Let us now show that $\mathbf{G}_q$ approximates $\mathbf{F}$ in the Sobolev norm. Using the triangle inequality gives

$$\|\mathbf{F} - \mathbf{G}_q\|_{W^{m,p}(M,\mathbf{E})} \leq \sum_{\alpha=1}^{\infty} \|\psi_\alpha \mathbf{F} - \mathbf{G}_{\alpha,k_{\alpha,q}}\|_{W^{m,p}(M,\mathbf{E})} \leq \sum_{\alpha=1}^{\infty} \frac{1}{2^{\alpha+q}} = \frac{1}{2^q}.$$

In particular,

$$\|\mathbf{G}_q\|_{W^{m,p}(M,\mathbf{E})} \leq \|\mathbf{F}\|_{W^{m,p}(M,\mathbf{E})} + \|\mathbf{F} - \mathbf{G}_q\|_{W^{m,p}(M,\mathbf{E})} \leq \|\mathbf{F}\|_{W^{m,p}(M,\mathbf{E})} + 2^{-q} < \infty,$$

so that $\mathbf{G}_q \in W^{m,p}(M,\mathbf{E}) \cap \Gamma(\mathbf{E})$ for every $q \in \mathbb{N}$.
Since $2^{-q} \to 0$ as $q \to \infty$, we conclude that $\mathbf{G}_q \to \mathbf{F}$ in $W^{m,p}(M,\mathbf{E})$. That is, every section in $W^{m,p}(M,\mathbf{E})$ can be approximated in the Sobolev norm by smooth sections, showing that

$$H^{m,p}(M,\mathbf{E}) = W^{m,p}(M,\mathbf{E}).$$

□

Theorem 9.1.49. (Chain rule for Lipschitz functions) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $1 \leq p < \infty$, and let $u \in W^{1,p}_{\text{loc}}(M)$ be a real-valued function. If $\Phi\colon \mathbb{R} \longrightarrow \mathbb{R}$ is L-Lipschitz, then $\Phi \circ u \in W^{1,p}_{\text{loc}}(M)$ and

$$|d(\Phi \circ u)|_\mathbf{g} \leq L |du|_\mathbf{g}$$

almost everywhere. On the set of points x for which Φ is differentiable at $u(x)$, we have

$$d(\Phi \circ u)_x = \Phi'(u(x))\, du_x$$

almost everywhere. Moreover, $du = 0$ almost everywhere on each level set $\{u = c\}$.

If $u \in W^{1,p}(M)$ and $\Phi \circ u \in L^p(M)$, then $\Phi \circ u \in W^{1,p}(M)$. This integrability condition holds, in particular, when $\Phi(0) = 0$ or when $\lambda_{\mathbf{g}}(M) < \infty$.
For $u, v \in W^{1,p}_{\mathrm{loc}}(M)$ we obtain the formulas

$$d(u-c)_+ = \mathbf{1}_{\{u>c\}}\, du, \qquad du_- = -\mathbf{1}_{\{u<0\}}\, du,$$

and

$$d(u-v)_+ = \mathbf{1}_{\{u>v\}}(du - dv)$$

almost everywhere, where $t_+ := \max\{t, 0\}$ and $u_- := (-u)_+$.

Proof. We begin with a function $\Psi \in C^1(\mathbb{R})$ whose derivative is bounded. Let $U \Subset M$ be a regular coordinate ball. Since $u \in W^{1,p}_{\mathrm{loc}}(M)$, the restriction $u|_U$ belongs to $W^{1,p}(U)$. Apply the Meyers–Serrin theorem for vector bundles, Theorem 9.1.48, to the trivial real bundle over U. This gives a sequence $(u_j) \subseteq C^\infty(U) \cap W^{1,p}(U)$ such that $u_j \to u$ in $W^{1,p}(U)$. After passing to a subsequence, we may also assume that $u_j \to u$ almost everywhere on U. Since Ψ is Lipschitz, $\Psi(u_j) \to \Psi(u)$ in $L^p(U)$. For the differentials we have

$$d(\Psi \circ u_j) = \Psi'(u_j)\, du_j.$$

The inequality

$$\begin{aligned}\|\Psi'(u_j)du_j - \Psi'(u)du\|_{L^p(U,T^*U)} \le{}& \|\Psi'\|_{L^\infty(\mathbb{R})}\|du_j - du\|_{L^p(U,T^*U)} \\ &+ \|(\Psi'(u_j) - \Psi'(u))du\|_{L^p(U,T^*U)}\end{aligned}$$

shows that the left-hand side tends to zero. Indeed, the first term converges to zero by approximation in $W^{1,p}(U)$, and the second by the dominated convergence theorem B.6.14, since $\Psi'(u_j) \to \Psi'(u)$ almost everywhere and $|\Psi'(u_j) - \Psi'(u)|^p |du|^p_{\mathbf{g}} \le 2^p \|\Psi'\|^p_{L^\infty(\mathbb{R})} |du|^p_{\mathbf{g}}$. Closedness of the weak differential, Lemma 9.1.12, then implies that

$$d(\Psi \circ u) = \Psi'(u)\, du$$

on U. Since the regular coordinate balls with compact closure cover M, the identity holds locally throughout the manifold.
Before treating an arbitrary Lipschitz function, we prove the assertion about level sets. Fix $c \in \mathbb{R}$ and choose $\eta \in C^\infty_c(\mathbb{R})$ such that $0 \le \eta \le 1$, $\eta = 1$ on $[-1, 1]$, and $\eta = 0$ outside $[-2, 2]$. For $\varepsilon > 0$ define

$$\Psi_\varepsilon(t) := \int_c^t \eta\left(\frac{s-c}{\varepsilon}\right) ds.$$

We have $|\Psi_\varepsilon| \le 2\varepsilon$ and $\Psi_\varepsilon'(t) = \eta\left(\frac{t-c}{\varepsilon}\right)$. By what we have already proved,

$$d(\Psi_\varepsilon \circ u) = \eta\left(\frac{u-c}{\varepsilon}\right) du.$$

Fix a regular coordinate ball $U \Subset M$. Since $|\Psi_\varepsilon \circ u| \le 2\varepsilon$, we have $\Psi_\varepsilon \circ u \to 0$ in $L^p(U)$. On the other hand,

$$\eta\left(\frac{u-c}{\varepsilon}\right) du \longrightarrow \mathbf{1}_{\{u=c\}}\, du$$

in $L^p(U, T^*U)$ by the dominated convergence theorem B.6.14. Indeed, the scalar factor converges pointwise to $\mathbf{1}_{\{u=c\}}$ and its absolute value is bounded by one. Apply Lemma 9.1.12 to any sequence $\varepsilon_j \to 0^+$. The functions converge to zero and their differentials converge to $\mathbf{1}_{\{u=c\}} du$, so the latter must be the weak differential of the zero function. Consequently, $\mathbf{1}_{\{u=c\}} du = 0$ almost everywhere on U. Since these balls cover M, the assertion holds throughout the manifold.

Now let Φ be an L-Lipschitz function. Take the one-dimensional mollifier from Theorem B.9.1 and set $\Phi_j := \Phi * \rho_{\frac{1}{j}}$. Then $\Phi_j \in C^\infty(\mathbb{R})$,

$$\|\Phi_j - \Phi\|_{L^\infty(\mathbb{R})} \longrightarrow 0, \qquad |\Phi_j'| \le L.$$

In particular, $\Phi_j(u) \to \Phi(u)$ in $L^p(Q)$ for every $Q \Subset M$. By the first part of the proof,

$$d(\Phi_j \circ u) = \Phi_j'(u)\, du.$$

Again fix a regular coordinate ball $U \Subset M$ and write $a_j := \Phi_j'(u)$. The sequence (a_j) lies in the closed ball of radius L in $L^\infty(U)$. Since $L^1(U)$ is separable by Lemma 9.1.21, Corollary B.11.30 allows us to pass to a subsequence, without changing notation, such that $a_j \rightharpoonup^* a$ in $L^\infty(U)$ for some function a with $|a| \le L$ almost everywhere. Let $\omega \in \Gamma_c(T^*U)$. Since $\langle du, \omega\rangle_{\mathbf{g}} \in L^1(U)$, weak-$*$ convergence gives

$$\int_U a_j \langle du, \omega\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} \longrightarrow \int_U a \langle du, \omega\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}.$$

For each j, the defining identity for the weak differential reads

$$\int_U a_j \langle du, \omega\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} = \int_U \Phi_j(u) d^*\omega\, d\lambda_{\mathbf{g}}.$$

The right-hand side converges to $\displaystyle\int_U \Phi(u) d^*\omega\, d\lambda_{\mathbf{g}}$, since $\Phi_j(u) \to \Phi(u)$ in $L^p(U)$ and $d^*\omega$ is smooth with compact support. Therefore,

$$d(\Phi \circ u) = a\, du$$

on U. In particular, $|d(\Phi \circ u)|_{\mathbf{g}} \le L |du|_{\mathbf{g}}$ almost everywhere. On intersections of two balls, the differentials thus obtained agree by uniqueness of the weak derivative. They therefore define the weak differential of $\Phi \circ u$ throughout M.

It remains to identify the coefficient a at values where Φ is differentiable. If Φ is differentiable at t, then $\Phi_j'(t) \to \Phi'(t)$. To see this directly, write

$$\Phi_j'(t) = j \int_{\mathbb{R}} \Phi\left(t - \frac{s}{j}\right) \rho'(s)\, ds.$$

Write $\Phi(t+h) = \Phi(t) + h\Phi'(t) + r(h)$. For each $\varepsilon > 0$ there is $\delta > 0$ such that $|r(h)| \le \varepsilon |h|$ if $|h| < \delta$. Moreover,

$$\Phi\left(t - \frac{s}{j}\right) = \Phi(t) - \frac{s}{j}\Phi'(t) + r\left(-\frac{s}{j}\right).$$

The remainder, multiplied by j, converges uniformly to zero for $s \in \operatorname{supp}\rho$. Indeed, if this support is

contained in $[-R,R]$ and $j > \frac{R}{\delta}$, then

$$\sup_{s\in[-R,R]} j\left|r\left(-\frac{s}{j}\right)\right| \le R\varepsilon.$$

Since $\int_{\mathbb{R}} \rho'(s)\,ds = 0$ and $\int_{\mathbb{R}} s\rho'(s)\,ds = -1$, the preceding formula converges to $\Phi'(t)$. Let $D \subseteq U$ be the measurable set of points x for which Φ is differentiable at $u(x)$. Then $a_j(x) \to \Phi'(u(x))$ on D. For each $\omega \in \Gamma_c(T^*U)$, the function $\mathbf{1}_D\langle du, \omega\rangle_{\mathbf{g}}$ belongs to $L^1(U)$. Weak-$*$ convergence and, on the other hand, the dominated convergence theorem B.6.14 on D give

$$\int_D a\langle du, \omega\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} = \int_D \Phi'(u)\langle du, \omega\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}.$$

Since ω is arbitrary, the localization principle implies that

$$a\,du = \Phi'(u)\,du$$

almost everywhere there. This proves the stated chain rule formula.

If $u \in W^{1,p}(M)$ and $\Phi(u) \in L^p(M)$, the bound on the differential shows that $d(\Phi(u)) \in L^p(M, T^*M)$, so that $\Phi(u) \in W^{1,p}(M)$. If $\Phi(0) = 0$, then $|\Phi(u)| \le L|u|$. If M has finite volume, use $|\Phi(u)| \le |\Phi(0)| + L|u|$. In both cases this gives the required integrability.

Finally, the function $t \mapsto (t-c)_+$ is differentiable outside c, with derivative equal to $\mathbf{1}_{(c,\infty)}$. The formula already proved holds outside $\{u = c\}$, and on this set $du = 0$. Therefore, $d(u-c)_+ = \mathbf{1}_{\{u>c\}}du$. The identity for u_- follows by applying this formula to $-u$, and the last identity follows by applying it to $u - v$. □

9.2 Sobolev embeddings and the Rellich–Kondrashov theorem for vector bundles

Sobolev embeddings are a basic tool for studying partial differential equations on bundles. On a compact manifold, a finite cover by regular coordinate balls reduces the problem to Euclidean embeddings for the components, after which the sections can be reconstructed using a partition of unity. Lemma 9.1.44 ensures that this procedure preserves Sobolev regularity exactly.

Theorem 9.2.1. (Sobolev embedding for vector bundles) Let $(M, \mathbf{g})$ be a closed Riemannian manifold of dimension n, and let $\mathbf{E} \longrightarrow M$ be a smooth real or complex vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$ (Hermitian in the complex case) and a compatible connection $\nabla^{\mathbf{E}}$. Let $j \ge 0$ and $m \ge 1$ be integers and $1 \le p, q < \infty$ such that

$$j + m - \frac{n}{p} \ge j - \frac{n}{q}.$$

Then the inclusion

$$W^{j+m,p}(M, \mathbf{E}) \hookrightarrow W^{j,q}(M, \mathbf{E})$$

is continuous.

Proof. Fix the orders and exponents in the statement. Choose a finite cover $(U_\alpha,\varphi_\alpha)_{\alpha=1}^N$ by regular coordinate balls and, on each U_α, a frame $(\mathbf{e}_{1,\alpha},\dots,\mathbf{e}_{r,\alpha})$ defined on a neighborhood of $\overline{U_\alpha}$. Take a partition of unity $(\psi_\alpha)_{\alpha=1}^N$ subordinate to this cover. Since M is compact, we may and do assume that

$$K_\alpha := \operatorname{supp}(\psi_\alpha) \Subset U_\alpha \qquad \text{for every } \alpha \in \{1,\dots,N\}.$$

Set $\Omega_\alpha := \varphi_\alpha(U_\alpha)$.
Compatibility of $\nabla^{\mathbf{E}}$ with $\mathbf{h}_{\mathbf{E}}$ allows us to apply Lemma 9.1.44 and its local comparison estimates. For complex components, we apply the Euclidean estimates to their real and imaginary parts. Theorem 3.3.16, with total orders $j+m$ and j, gives, for each α, a constant $C_\alpha > 0$ such that

$$\|v\|_{W^{j,q}(\Omega_\alpha)} \le C_\alpha \|v\|_{W^{j+m,p}(\Omega_\alpha)} \qquad \text{for every } v \in W_0^{j+m,p}(\Omega_\alpha). \tag{9.43}$$

The formulation with $W_0^{j+m,p}$ suffices: all functions appearing below have compact support in Ω_α.
Let $\mathbf{F} \in W^{j+m,p}(M,\mathbf{E})$ and write

$$\mathbf{F} = \sum_{a=1}^{r} F_\alpha^a \mathbf{e}_{a,\alpha} \qquad \text{in } U_\alpha.$$

The localized section $\psi_\alpha \mathbf{F}$ has support contained in K_α. By Lemma 9.1.44, the functions

$$v_\alpha^a := (\psi_\alpha F_\alpha^a) \circ \varphi_\alpha^{-1}$$

belong to $W^{j+m,p}(\Omega_\alpha)$. Moreover,

$$\operatorname{supp}(v_\alpha^a) \subseteq \varphi_\alpha(K_\alpha) \Subset \Omega_\alpha,$$

so that $v_\alpha^a \in W_0^{j+m,p}(\Omega_\alpha)$. Estimate (9.43) and the local comparison of norms give

$$\|(\psi_\alpha \mathbf{F}) \restriction_{U_\alpha}\|_{W^{j,q}(U_\alpha,\mathbf{E})} \le C'_\alpha \|(\psi_\alpha \mathbf{F}) \restriction_{U_\alpha}\|_{W^{j+m,p}(U_\alpha,\mathbf{E})} \tag{9.44}$$

for a constant $C'_\alpha > 0$ independent of $\mathbf{F}$.
Let us make the global reconstruction of each localized section precise. For $s \in \{0,\dots,j\}$, let

$$G_{\alpha,s} := \nabla_w^s\big((\psi_\alpha \mathbf{F}) \restriction_{U_\alpha}\big).$$

Locality of the weak covariant derivative implies the inclusion

$$\operatorname{supp}_{U_\alpha}(G_{\alpha,s}) \subseteq \operatorname{supp}_{U_\alpha}\big((\psi_\alpha \mathbf{F}) \restriction_{U_\alpha}\big) \subseteq K_\alpha;$$

equality of supports is neither needed nor valid in general. Denote by $\widetilde{G}_{\alpha,s}$ the extension of $G_{\alpha,s}$ by zero to M. Since $K_\alpha \Subset U_\alpha$, this extension belongs to $L^q(M, T^{(0,s)}(TM)\otimes\mathbf{E})$.
Let us show that $\widetilde{G}_{\alpha,s}$ is the sth weak covariant derivative of $\psi_\alpha \mathbf{F}$. Choose $\eta_\alpha \in C_c^\infty(U_\alpha)$ equal to one on a neighborhood of K_α. For $\mathbf{\Phi} \in \Gamma_c^\infty(T^{(0,s)}(TM)\otimes\mathbf{E})$, the section $\eta_\alpha \mathbf{\Phi} \restriction_{U_\alpha}$ is a test function on U_α. The

definition of $G_{\alpha,s}$ and locality of $(\nabla^s)^*$ give

$$\begin{aligned}\int_M \langle \widetilde{G}_{\alpha,s}, \mathbf{\Phi}\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} &= \int_{U_\alpha} \langle G_{\alpha,s}, \eta_\alpha \mathbf{\Phi}\rangle_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \\ &= \int_{U_\alpha} \left\langle \psi_\alpha \mathbf{F}, (\nabla^s)^*(\eta_\alpha \mathbf{\Phi})\right\rangle_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \\ &= \int_M \langle \psi_\alpha \mathbf{F}, (\nabla^s)^* \mathbf{\Phi}\rangle_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.\end{aligned}$$

In the last equality we used $\eta_\alpha = 1$ on a neighborhood of the support of $\psi_\alpha \mathbf{F}$. Therefore, $\psi_\alpha \mathbf{F} \in W^{j,q}(M,\mathbf{E})$ and its global norm agrees with the norm obtained by integrating over U_α.

Finally, we need a bound for localization in the source space. For smooth sections, the iterated Leibniz rule gives, for $0 \le s \le j+m$,

$$|\nabla^s(\psi_\alpha \mathbf{F})|_{\mathbf{g},\mathbf{h_E}} \le A_s \sum_{t=0}^{s} |\nabla^{s-t}\psi_\alpha|_{\mathbf{g}} |\nabla^t \mathbf{F}|_{\mathbf{g},\mathbf{h_E}}.$$

The derivatives of ψ_α are bounded because M is compact. Summing the L^p norms gives a constant $A_\alpha > 0$ such that

$$\|\psi_\alpha \mathbf{F}\|_{W^{j+m,p}(M,\mathbf{E})} \le A_\alpha \|\mathbf{F}\|_{W^{j+m,p}(M,\mathbf{E})}$$

for every smooth section $\mathbf{F}$. The Meyers–Serrin theorem and passage to the limit in distributions extend the inequality to every $\mathbf{F} \in W^{j+m,p}(M,\mathbf{E})$. Since the cover is finite, there is $A > 0$ such that

$$\|\psi_\alpha \mathbf{F}\|_{W^{j+m,p}(M,\mathbf{E})} \le A \|\mathbf{F}\|_{W^{j+m,p}(M,\mathbf{E})} \tag{9.45}$$

for every $\alpha \in \{1,\dots,N\}$.

Using $\mathbf{F} = \sum_{\alpha=1}^{N} \psi_\alpha \mathbf{F}$, the triangle inequality, (9.44), and (9.45) gives

$$\begin{aligned}\|\mathbf{F}\|_{W^{j,q}(M,\mathbf{E})} &\le \sum_{\alpha=1}^{N} \|\psi_\alpha \mathbf{F}\|_{W^{j,q}(M,\mathbf{E})} \\ &\le \sum_{\alpha=1}^{N} C'_\alpha \|\psi_\alpha \mathbf{F}\|_{W^{j+m,p}(M,\mathbf{E})} \\ &\le C \|\mathbf{F}\|_{W^{j+m,p}(M,\mathbf{E})}.\end{aligned}$$

This proves continuity of the embedding. □

Theorem 9.2.2. (Rellich–Kondrashov in vector bundles) Let $(M,\mathbf{g})$ be a compact Riemannian manifold without boundary of dimension n, and let $\mathbf{E} \to M$ be a smooth real or complex vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$ (Hermitian in the complex case) and a compatible connection $\nabla^{\mathbf{E}}$. Let $j \in \mathbb{N}_0$, let $m \ge 1$ be an integer, and let $1 \le p, q < \infty$ satisfy

$$j + m - \frac{n}{p} > j - \frac{n}{q}.$$

Then the inclusion

$$W^{j+m,p}(M,\mathbf{E}) \hookrightarrow W^{j,q}(M,\mathbf{E})$$

is compact.

Proof. Let $(\mathbf{F}_\nu)_{\nu\in\mathbb{N}}$ be a bounded sequence in $W^{j+m,p}(M,\mathbf{E})$. Compatibility of $\nabla^{\mathbf{E}}$ with $\mathbf{h}_{\mathbf{E}}$ allows us to apply the local comparison of norms in Lemma 9.1.44. In the complex case, consider the underlying real bundle with metric $\operatorname{Re}\mathbf{h}_{\mathbf{E}}$ and the induced real connection; its pointwise norm agrees with the Hermitian norm, and its components are the real and imaginary parts of the complex components. Retain the cover, frames, and functions ψ_α from the preceding proof. By (9.45), for each α the sequence $(\psi_\alpha \mathbf{F}_\nu)_\nu$ is bounded in $W^{j+m,p}(M,\mathbf{E})$. Lemma 9.1.44 then shows that, for each pair (α,a), the sequence

$$v^a_{\alpha,\nu} := \big(\psi_\alpha(\mathbf{F}_\nu)^a_\alpha\big)\circ\varphi_\alpha^{-1}$$

is bounded in $W^{j+m,p}(\Omega_\alpha)$. Moreover,

$$\operatorname{supp}(v^a_{\alpha,\nu}) \subseteq \varphi_\alpha(K_\alpha) \Subset \Omega_\alpha$$

for every ν. In particular, $v^a_{\alpha,\nu} \in W^{j+m,p}_0(\Omega_\alpha)$.
For each pair (α,a), Theorem 3.4.2 gives a subsequence converging in $W^{j,q}(\Omega_\alpha)$. Since there are only Nr pairs, a finite diagonal extraction produces a subsequence, which we do not relabel, such that $(v^a_{\alpha,\nu})_\nu$ is Cauchy in $W^{j,q}(\Omega_\alpha)$ for every α and every a.
The local comparison of norms and reconstruction by extension by zero used in the proof of the preceding theorem give, for $\nu,\mu\in\mathbb{N}$,

$$\begin{aligned}\|\mathbf{F}_\nu-\mathbf{F}_\mu\|_{W^{j,q}(M,\mathbf{E})} &\le \sum_{\alpha=1}^{N}\|\psi_\alpha(\mathbf{F}_\nu-\mathbf{F}_\mu)\|_{W^{j,q}(M,\mathbf{E})}\\ &\le C\sum_{\alpha=1}^{N}\sum_{a=1}^{r}\|v^a_{\alpha,\nu}-v^a_{\alpha,\mu}\|_{W^{j,q}(\Omega_\alpha)}.\end{aligned}$$

The right-hand side tends to zero as $\nu,\mu\to\infty$. The subsequence is therefore Cauchy in the Banach space $W^{j,q}(M,\mathbf{E})$ and converges in it. This proves compactness. □

Corollary 9.2.3. (Embeddings and the critical exponent) Under the geometric hypotheses of the preceding theorems, let $j\ge 0$ and $m\ge 1$ be integers, and let $1\le p<n/m$. Then

$$W^{j+m,p}(M,\mathbf{E}) \hookrightarrow W^{j,q}(M,\mathbf{E})$$

continuously for $1\le q\le p^*_m := \dfrac{np}{n-mp}$, and compactly for $1\le q<p^*_m$.

Proof. Since $mp<n$, the condition $j+m-\dfrac{n}{p}\ge j-\dfrac{n}{q}$ is equivalent to $q\le p^*_m$, and the strict inequality is equivalent to $q<p^*_m$. Apply Theorems 9.2.1 and 9.2.2, respectively. □

9.2.1 Embeddings into spaces of Hölder sections

Solutions of many variational problems are initially obtained as elements of a Sobolev space. The localization principle for vector bundles in Lemma 9.1.6 allows us to convert the variational identity into a local equation in the weak sense. This step alone, however, does not yield classical regularity. The next question is when the weak solution admits a differentiable representative and with what continuity its derivatives vary. As in the Euclidean case, the embedding does not replace estimates specific to the equation: once those estimates give a bound in $W^{m,p}(M,\mathbf{E})$, spaces of Hölder sections translate this Sobolev regularity into quantitative classical regularity.

Before defining these spaces, we must specify how to compare vectors belonging to different fibers. We use the notation for parallel transport fixed in Definition 2.7.13. Throughout this subsection the manifolds have no boundary. Initially we assume that M is connected, because the Riemannian distance $d_{\mathbf{g}}$ was defined under this hypothesis. If M is disconnected, the definitions and results apply componentwise. In particular, connectedness imposes no restriction for compact manifolds, which have finitely many connected components.

Definition 9.2.4. (Fiberwise separation by parallel transport) Let $(M,\mathbf{g})$ be a connected Riemannian manifold without boundary such that $\operatorname{inj}(M)>0$. Let $\mathbf{F}\longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{F}}$ and a compatible connection $\nabla^{\mathbf{F}}$. Fix

$$\rho_M := \min\{1, \operatorname{inj}(M)\}.$$

If $x,y\in M$ satisfy $0<d_{\mathbf{g}}(x,y)<\rho_M$, denote by $\gamma_{xy}\colon[0,1]\longrightarrow M$ the unique minimizing geodesic parametrized with constant speed satisfying

$$\gamma_{xy}(0)=x, \qquad \gamma_{xy}(1)=y.$$

For $\xi\in\mathbf{F}_x$ and $\eta\in\mathbf{F}_y$, define

$$|\xi-\eta|_{\mathbf{h}_{\mathbf{F}};x,y} := \left|\xi - P^{\mathbf{F}}_{\gamma_{xy},1\to 0}(\eta)\right|_{\mathbf{h}_{\mathbf{F}}(x)}. \tag{9.46}$$

Since $\nabla^{\mathbf{F}}$ is compatible with $\mathbf{h}_{\mathbf{F}}$, parallel transport is an isometry. Proposition 2.7.14 then gives the equivalent expression

$$|\xi-\eta|_{\mathbf{h}_{\mathbf{F}};x,y} = \left|P^{\mathbf{F}}_{\gamma_{xy},0\to 1}(\xi)-\eta\right|_{\mathbf{h}_{\mathbf{F}}(y)}.$$

When $x=y$, set

$$|\xi-\eta|_{\mathbf{h}_{\mathbf{F}};x,x} := |\xi-\eta|_{\mathbf{h}_{\mathbf{F}}(x)}.$$

If $\mathbf{T}\in\Gamma^0(\mathbf{F})$ and $S\subseteq M$ satisfies $\operatorname{diam}_{\mathbf{g}}(S)<\rho_M$, define the fiberwise oscillation of $\mathbf{T}$ on S by

$$\operatorname{osc}_{\mathbf{F}}(\mathbf{T};S) := \sup_{x,y\in S} |\mathbf{T}(x)-\mathbf{T}(y)|_{\mathbf{h}_{\mathbf{F}};x,y}.$$

For $0<\alpha<1$, define

$$[\mathbf{T}]_{\alpha,\mathbf{F}} := \sup_{\substack{x,y\in M\\ 0<d_{\mathbf{g}}(x,y)<\rho_M}} \frac{|\mathbf{T}(x)-\mathbf{T}(y)|_{\mathbf{h}_{\mathbf{F}};x,y}}{d_{\mathbf{g}}(x,y)^{\alpha}}. \tag{9.47}$$

If $U\subseteq M$, write $[\mathbf{T}]_{\alpha,\mathbf{F};U}$ when the supremum is taken only over pairs $x,y\in U$.

Remark 9.2.5. (Choice of radius) The bound by 1 in the definition of ρ_M merely normalizes the scale. If $0 < \rho_1 \le \rho_2 \le \mathrm{inj}(M)$ and $[\mathbf{T}]_{\alpha,\mathbf{F};\rho_j}$ denotes the seminorm obtained using radius ρ_j, then

$$[\mathbf{T}]_{\alpha,\mathbf{F};\rho_1} \le [\mathbf{T}]_{\alpha,\mathbf{F};\rho_2}$$

and

$$[\mathbf{T}]_{\alpha,\mathbf{F};\rho_2} \le [\mathbf{T}]_{\alpha,\mathbf{F};\rho_1} + 2\rho_1^{-\alpha}\|\mathbf{T}\|_{L^\infty(M,\mathbf{F})}.$$

The second inequality follows by separating pairs at distance less than ρ_1 from those whose distance belongs to $[\rho_1,\rho_2)$. Thus, after adding the uniform norm, any two positive radii less than or equal to $\mathrm{inj}(M)$ produce equivalent norms.
One may also use the formulation in terms of balls

$$[\mathbf{T}]^{\mathrm{bol}}_{\alpha,\mathbf{F}} := \sup_{\substack{x\in M\\ 0<r<\frac{\rho_M}{2}}} \frac{\mathrm{osc}_{\mathbf{F}}(\mathbf{T}; B_{\mathbf{g}}(x,r))}{r^\alpha}.$$

If $x,y \in B_{\mathbf{g}}(z,r)$, then $d_{\mathbf{g}}(x,y) < 2r$, and therefore

$$[\mathbf{T}]^{\mathrm{bol}}_{\alpha,\mathbf{F}} \le 2^\alpha [\mathbf{T}]_{\alpha,\mathbf{F}}.$$

Conversely, if $d_{\mathbf{g}}(x,y) < \frac{\rho_M}{2}$, take $d_{\mathbf{g}}(x,y) < r < \frac{\rho_M}{2}$ and use $y \in B_{\mathbf{g}}(x,r)$. Letting $r \to d_{\mathbf{g}}(x,y)^+$ gives

$$\frac{|\mathbf{T}(x)-\mathbf{T}(y)|_{\mathbf{h}_{\mathbf{F}};x,y}}{d_{\mathbf{g}}(x,y)^\alpha} \le [\mathbf{T}]^{\mathrm{bol}}_{\alpha,\mathbf{F}}.$$

For $\frac{\rho_M}{2} \le d_{\mathbf{g}}(x,y) < \rho_M$, we have

$$\frac{|\mathbf{T}(x)-\mathbf{T}(y)|_{\mathbf{h}_{\mathbf{F}};x,y}}{d_{\mathbf{g}}(x,y)^\alpha} \le 2^{\alpha+1}\rho_M^{-\alpha}\|\mathbf{T}\|_{L^\infty(M,\mathbf{F})}.$$

Thus the quantities

$$\|\mathbf{T}\|_{L^\infty(M,\mathbf{F})} + [\mathbf{T}]_{\alpha,\mathbf{F}} \qquad \text{and} \qquad \|\mathbf{T}\|_{L^\infty(M,\mathbf{F})} + [\mathbf{T}]^{\mathrm{bol}}_{\alpha,\mathbf{F}}$$

are equivalent. The factor $\frac{1}{2}$ ensures that any two points of $B_{\mathbf{g}}(x,r)$ lie at distance less than ρ_M. Likewise, uniqueness of the minimizing geodesic requires $d_{\mathbf{g}}(x,y) < \mathrm{inj}(M)$; the non-strict inequality is insufficient.

Definition 9.2.6. (Hölder sections and Hölder sections up to the closure) Let $(M,\mathbf{g})$ be a connected Riemannian manifold without boundary with $\mathrm{inj}(M) > 0$, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. For $j \in \mathbb{N}_0$ set

$$\mathbf{E}_j := T^{(0,j)}(TM) \otimes \mathbf{E},$$

with the convention $\mathbf{E}_0 = \mathbf{E}$. On $\mathbf{E}_j$ consider the bundle metric induced by $\mathbf{g}$ and $\mathbf{h}_{\mathbf{E}}$, and the product connection induced by the Levi–Civita connection and $\nabla^{\mathbf{E}}$.

Let $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. For $\mathbf{u} \in \Gamma^k(\mathbf{E})$ define

$$\|\nabla^j \mathbf{u}\|_{L^\infty(M,\mathbf{E}_j)} := \sup_{x\in M} |\nabla^j \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, \qquad 0 \le j \le k,$$

and

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} := \sum_{j=0}^{k} \|\nabla^j \mathbf{u}\|_{L^\infty(M,\mathbf{E}_j)} + [\nabla^k \mathbf{u}]_{\alpha,\mathbf{E}_k}. \tag{9.48}$$

The space of Hölder sections of order (k,α) is

$$\Gamma^{k,\alpha}(\mathbf{E}) := \left\{ \mathbf{u} \in \Gamma^k(\mathbf{E}) \,\middle|\, \|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} < \infty \right\}.$$

Now let $U \Subset M$ be open. For $\mathbf{u} \in \Gamma^k(\mathbf{E}\restriction_U)$ set

$$\mathcal{E}^{k,\alpha}_{\overline{U}}(\mathbf{u}) := \left\{ \widetilde{\mathbf{u}} \in \Gamma^{k,\alpha}(\mathbf{E}) \,\middle|\, \widetilde{\mathbf{u}}\restriction_U = \mathbf{u} \right\}.$$

Define

$$\Gamma^{k,\alpha}(\overline{U},\mathbf{E}) := \left\{ \mathbf{u} \in \Gamma^k(\mathbf{E}\restriction_U) \,\middle|\, \mathcal{E}^{k,\alpha}_{\overline{U}}(\mathbf{u}) \neq \varnothing \right\}$$

and equip this space with the extension norm

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\overline{U},\mathbf{E})} := \inf_{\widetilde{\mathbf{u}} \in \mathcal{E}^{k,\alpha}_{\overline{U}}(\mathbf{u})} \|\widetilde{\mathbf{u}}\|_{\Gamma^{k,\alpha}(\mathbf{E})}. \tag{9.49}$$

The second argument records the ambient bundle. We are not regarding $\overline{U}$ as a smooth manifold. Since U is dense in $\overline{U}$, an extension uniquely determines the values of $\mathbf{u}$ and its covariant derivatives through order k on the closure.

If M has finitely many connected components $M_1,\dots,M_N$, apply the definition on each component and write

$$\Gamma^{k,\alpha}(\mathbf{E}) := \left\{ \mathbf{u} \in \Gamma^k(\mathbf{E}) \,\middle|\, \mathbf{u}\restriction_{M_\nu} \in \Gamma^{k,\alpha}(\mathbf{E}\restriction_{M_\nu}) \text{ for every } \nu \in \{1,\dots,N\} \right\},$$

with the norm

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} := \max_{1\le\nu\le N} \|\mathbf{u}\restriction_{M_\nu}\|_{\Gamma^{k,\alpha}(\mathbf{E}\restriction_{M_\nu})}.$$

If

$$\|\mathbf{u}\|_{\Sigma} := \sum_{\nu=1}^{N} \|\mathbf{u}\restriction_{M_\nu}\|_{\Gamma^{k,\alpha}(\mathbf{E}\restriction_{M_\nu})},$$

then

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le \|\mathbf{u}\|_{\Sigma} \le N \|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})}.$$

This is the convention used for disconnected compact manifolds.

Remark 9.2.7. (Local extensions of sections) The global extension condition in the preceding definition is equivalent to the existence of an extension to an open neighborhood of $\overline{U}$. More precisely, it suffices that there be open sets

$$\overline{U} \subseteq V_0 \Subset V_1 \Subset V$$

and a section $\widehat{\mathbf{u}} \in \Gamma^k(\mathbf{E}\restriction_V)$ extending $\mathbf{u}$ and satisfying

$$\sum_{j=0}^{k} \|\nabla^j \widehat{\mathbf{u}}\|_{L^\infty(V_1,\mathbf{E}_j)} + [\nabla^k \widehat{\mathbf{u}}]_{\alpha,\mathbf{E}_k;V_1} < \infty.$$

Indeed, choose a cutoff function $\chi \in C_c^\infty(V_1)$ equal to one on a neighborhood of $\overline{V_0}$. The extension of $\chi\widehat{\mathbf{u}}$ by zero outside V_1, denoted by $\widetilde{\chi\widehat{\mathbf{u}}}$, belongs to $\Gamma^{k,\alpha}(\mathbf{E})$ and extends $\mathbf{u}$. The presence of V_1 ensures that all terms in the Leibniz rule are controlled on a compact subset of the extension domain. This is the analogue for sections of the convention used in $C^{k,\alpha}(\overline{\Omega})$.
The term $\|\nabla^k \mathbf{u}\|_{L^\infty(M,T^{(0,k)}(TM)\otimes\mathbf{E})}$ cannot be omitted from (9.48). The Hölder seminorm does not control the size of $\nabla^k \mathbf{u}$ and vanishes, for example, on parallel tensors.

The local comparison between the geometric norm and the scalar components uses two facts: uniform equivalence of the metric with the Euclidean metric on a regular coordinate ball, and control of parallel transport along short geodesics.

Lemma 9.2.8. (Local comparison of the Hölder seminorm) Let $(M,\mathbf{g})$ be a connected Riemannian manifold without boundary with $\operatorname{inj}(M) > 0$. Let (U,φ) be a regular coordinate ball in M. Let $\mathbf{F} \longrightarrow M$ be a smooth vector bundle of rank N, equipped with a bundle metric $\mathbf{h}_\mathbf{F}$ and a compatible connection $\nabla^\mathbf{F}$, and let $(\mathbf{f}_1,\dots,\mathbf{f}_N)$ be a local frame of $\mathbf{F}$ defined on a neighborhood of $\overline{U}$. If

$$\mathbf{T} = \sum_{A=1}^{N} T^A \mathbf{f}_A \qquad \text{on } U,$$

then

$$\mathbf{T} \in \Gamma^{0,\alpha}(\overline{U},\mathbf{F}) \quad \Longleftrightarrow \quad T^A \circ \varphi^{-1} \in C^{0,\alpha}(\overline{\varphi(U)})$$

for every $A \in \{1,\dots,N\}$. Moreover, there are constants $c_\alpha, C_\alpha > 0$, independent of $\mathbf{T}$, such that

$$\begin{aligned} c_\alpha \sum_{A=1}^{N} \|T^A \circ \varphi^{-1}\|_{C^{0,\alpha}(\overline{\varphi(U)})} &\le \|\mathbf{T}\|_{\Gamma^{0,\alpha}(\overline{U},\mathbf{F})} \\ &\le C_\alpha \sum_{A=1}^{N} \|T^A \circ \varphi^{-1}\|_{C^{0,\alpha}(\overline{\varphi(U)})}. \end{aligned} \tag{9.50}$$

Proof. Let U' be a larger regular coordinate ball, so that $\overline{U} \subseteq U'$ and the chart and frame are defined on U'. Choose an open set V with

$$\overline{U} \subseteq V \Subset U'.$$

Since $\overline{V}$ is compact, there is $0 < \delta < \rho_M$ such that every minimizing geodesic of length less than δ with endpoints in $\overline{V}$ remains in U'. Decreasing δ if necessary, the Euclidean segment joining $\varphi(x)$ to $\varphi(y)$ remains in $\varphi(U')$ whenever $x,y \in \overline{V}$ and $d_\mathbf{g}(x,y) < \delta$.

For $x, y \in \overline{V}$ with $d_{\mathbf{g}}(x,y) < \delta$, define the matrix $G(x,y)$ by

$$P^{\mathbf{F}}_{\gamma_{xy},1\to 0}\big(\mathbf{f}_B(y)\big) = \sum_{A=1}^{N} G^A_{\ B}(x,y)\mathbf{f}_A(x). \tag{9.51}$$

Let $R(t)$ be the matrix of the transport $P^{\mathbf{F}}_{\gamma_{xy},t\to 0}$ in the frame. By Proposition 2.7.18, $R(0) = I$ and R satisfies a linear system whose coefficients have norm bounded by $C|\dot\gamma_{xy}(t)|_{\mathbf{g}} = C d_{\mathbf{g}}(x,y)$. Grönwall's lemma 1.8.19 gives

$$\|R(t)\|_{\mathrm{op}(\mathbb{K}^N)} \le e^{C d_{\mathbf{g}}(x,y)t}$$

and, integrating the equation,

$$\|G(x,y) - I\|_{\mathrm{op}(\mathbb{K}^N)} = \|R(1) - R(0)\|_{\mathrm{op}(\mathbb{K}^N)} \le C d_{\mathbf{g}}(x,y). \tag{9.52}$$

Here and below the constant may increase, but is independent of x and y.
Uniform equivalence of $\mathbf{g}$ and the Euclidean metric in the chart gives constants $c_0, C_0 > 0$ such that

$$c_0|\varphi(x) - \varphi(y)| \le d_{\mathbf{g}}(x,y) \le C_0|\varphi(x) - \varphi(y)| \tag{9.53}$$

for the preceding pairs. The first inequality follows by estimating the length of the minimizing geodesic in coordinates. For the second, estimate the length of the curve whose coordinate expression is the segment joining $\varphi(x)$ to $\varphi(y)$.
For $z \in U'$, let

$$\mathcal{F}_z \colon \mathbb{K}^N \longrightarrow \mathbf{F}_z, \qquad \mathcal{F}_z(c^1,\dots,c^N) := \sum_{A=1}^{N} c^A \mathbf{f}_A(z).$$

If $\mathbf{S} = \sum_{A=1}^{N} S^A \mathbf{f}_A$ and $s(z) := (S^1(z),\dots,S^N(z))$, then (9.51) implies

$$\mathbf{S}(x) - P^{\mathbf{F}}_{\gamma_{xy},1\to 0}(\mathbf{S}(y)) = \mathcal{F}_x\big(s(x) - G(x,y)s(y)\big).$$

Therefore,

$$s(x) - G(x,y)s(y) = s(x) - s(y) + (I - G(x,y))s(y).$$

Uniform equivalence of the Euclidean norm of the components and $\mathbf{h}_{\mathbf{F}}$, together with (9.52) and (9.53), gives

$$\begin{aligned} &\|\mathbf{S}\|_{L^\infty(\overline{U},\mathbf{F})} + [\mathbf{S}]_{\alpha,\mathbf{F};\overline{U}} \\ &\qquad \le C \sum_{A=1}^{N} \Big(\|S^A \circ \varphi^{-1}\|_{C^0(\overline{\varphi(U)})} + [S^A \circ \varphi^{-1}]_{C^{0,\alpha}(\overline{\varphi(U)})} \Big). \end{aligned} \tag{9.54}$$

For the reverse inequality, solve for

$$s(x) - s(y) = \big(s(x) - G(x,y)s(y)\big) + (G(x,y) - I)s(y)$$

and repeat the same estimates. Pairs with $d_{\mathbf{g}}(x,y) \ge \delta$ are controlled by the uniform norms. This proves equivalence of the intrinsic norms on $\overline{U}$.

We turn to the extension norms. Let $\widetilde{\mathbf{T}} \in \mathcal{E}^{0,\alpha}_{\overline{U}}(\mathbf{T})$. Choose a cutoff function $\chi \in C^\infty_c(U')$ equal to one on a neighborhood of $\overline{U}$. The components of $\chi\widetilde{\mathbf{T}}$ in the frame, transported by φ and extended by zero outside $\varphi(U')$, are global extensions of $T^A \circ \varphi^{-1}$. Estimate (9.54), applied on a compact set containing $\operatorname{supp}\chi$, and the product rule give

$$\sum_{A=1}^{N} \|T^A \circ \varphi^{-1}\|_{C^{0,\alpha}(\overline{\varphi(U)})} \le C\|\widetilde{\mathbf{T}}\|_{\Gamma^{0,\alpha}(\mathbf{F})}.$$

Taking the infimum over $\widetilde{\mathbf{T}}$ yields the first inequality in (9.50).
Conversely, choose extensions $\widetilde{t^A} \in C^{0,\alpha}(\mathbb{R}^n)$ of $T^A \circ \varphi^{-1}$. On U' define

$$\widehat{\mathbf{T}} := \chi \sum_{A=1}^{N} (\widetilde{t^A} \circ \varphi)\mathbf{f}_A$$

and denote its extension by zero outside U' by $\widetilde{\widehat{\mathbf{T}}}$. This section belongs to $\Gamma^{0,\alpha}(\mathbf{F})$, extends $\mathbf{T}$, and satisfies

$$\|\widetilde{\widehat{\mathbf{T}}}\|_{\Gamma^{0,\alpha}(\mathbf{F})} \le C \sum_{A=1}^{N} \|\widetilde{t^A}\|_{C^{0,\alpha}(\mathbb{R}^n)}.$$

Taking the infimum first over the scalar extensions and then in the definition of the norm of $\mathbf{T}$ gives the second inequality. The two constructions also prove equivalence of membership. □

Lemma 9.2.9. (Multiplication by smooth functions in spaces of Hölder sections) Let $(M,\mathbf{g})$ be a connected Riemannian manifold without boundary with $\operatorname{inj}(M) > 0$, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric and a compatible connection. Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$, and $\chi \in C^\infty_c(M)$. Then there is a constant $C_{\chi,k,\alpha} > 0$ such that

$$\|\chi\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C_{\chi,k,\alpha}\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})}$$

for every $\mathbf{u} \in \Gamma^{k,\alpha}(\mathbf{E})$.

Proof. We first record two elementary estimates. Let $\mathbf{F} \to M$ be a bundle with a compatible connection, and let $\mathbf{S} \in \Gamma^1(\mathbf{F})$. For $x, y \in M$ with $0 < d_{\mathbf{g}}(x,y) < \rho_M$, Lemma 2.7.21, applied to $\gamma_{xy}\colon [0,1] \longrightarrow M$, gives

$$P^{\mathbf{F}}_{\gamma_{xy},1\to 0}(\mathbf{S}(y)) - \mathbf{S}(x) = \int_0^1 P^{\mathbf{F}}_{\gamma_{xy},t\to 0}\big(\nabla^{\mathbf{F}}_{\dot\gamma_{xy}(t)}\mathbf{S}\big)\,dt.$$

Since $|\dot\gamma_{xy}(t)|_{\mathbf{g}} = d_{\mathbf{g}}(x,y)$ and parallel transport is an isometry,

$$|\mathbf{S}(x) - \mathbf{S}(y)|_{\mathbf{h}_{\mathbf{F}};x,y} \le d_{\mathbf{g}}(x,y)\big\||\nabla^{\mathbf{F}}\mathbf{S}|_{\mathbf{g},\mathbf{h}_{\mathbf{F}}}\big\|_{L^\infty(M)}. \tag{9.55}$$

In particular, if $\rho_M \le 1$, a section with bounded derivative is Hölder continuous with any exponent $0 < \alpha < 1$ for the pairs appearing in (9.47).
Now let $\mathbf{F}_1, \mathbf{F}_2 \to M$ be vector bundles with bundle metrics (Hermitian in the complex case) and compatible connections, $\mathbf{A} \in \Gamma^0(\mathbf{F}_1)$ and $\mathbf{B} \in \Gamma^0(\mathbf{F}_2)$. The product connection satisfies the Leibniz rule, so uniqueness of

parallel transport implies

$$P^{\mathbf{F}_1\otimes\mathbf{F}_2}_{\gamma_{xy},1\to0} = P^{\mathbf{F}_1}_{\gamma_{xy},1\to0}\otimes P^{\mathbf{F}_2}_{\gamma_{xy},1\to0}.$$

Adding and subtracting $P^{\mathbf{F}_1}_{\gamma_{xy},1\to0}(\mathbf{A}(y))\otimes\mathbf{B}(x)$ gives

$$[\mathbf{A}\otimes\mathbf{B}]_{\alpha,\mathbf{F}_1\otimes\mathbf{F}_2} \le \|\mathbf{B}\|_{L^\infty(M,\mathbf{F}_2)}[\mathbf{A}]_{\alpha,\mathbf{F}_1} + \|\mathbf{A}\|_{L^\infty(M,\mathbf{F}_1)}[\mathbf{B}]_{\alpha,\mathbf{F}_2}. \tag{9.56}$$

The same estimate holds after any tensor contraction of uniformly bounded norm.

The iterated Leibniz rule distributes the j derivatives between χ and $\mathbf{u}$. More precisely, for each $j\in\{0,\dots,k\}$ there are integers $N_{j,\ell}$, universal coefficients $C_{j,\ell,\nu}$, and permutations $\tau_{j,\ell,\nu}$ of the j covariant indices such that

$$\nabla^j(\chi\mathbf{u}) = \sum_{\ell=0}^{j}\sum_{\nu=1}^{N_{j,\ell}} C_{j,\ell,\nu}\,\tau_{j,\ell,\nu}\big(\nabla^{j-\ell}\chi\otimes\nabla^\ell\mathbf{u}\big).$$

Here a sum with $N_{j,\ell}=0$ is understood to be empty. Since χ has compact support, its covariant derivatives of every order, and the derivatives of those sections, are uniformly bounded. The preceding formula therefore controls the uniform norms of $\nabla^j(\chi\mathbf{u})$.

For the seminorm of $\nabla^k(\chi\mathbf{u})$, if $\ell<k$, the section $\nabla^\ell\mathbf{u}$ has bounded derivative $\nabla^{\ell+1}\mathbf{u}$. By (9.55), both $\nabla^\ell\mathbf{u}$ and $\nabla^{k-\ell}\chi$ are Hölder continuous with exponent α. Estimate (9.56) controls the corresponding term. For $\ell=k$, apply the scalar version of the same estimate to the product $\chi\nabla^k\mathbf{u}$; the Hölder seminorm of $\nabla^k\mathbf{u}$ is part of (9.48). Summing the terms gives the stated inequality. □

We can now obtain the classical analogue of Lemma 9.1.44. The proof is based on the triangular coordinate expression for covariant derivatives.

Lemma 9.2.10. (Local characterization of Hölder sections) Let $(M,\mathbf{g})$ be a connected Riemannian manifold without boundary with $\operatorname{inj}(M)>0$. Let (U,φ) be a regular coordinate ball in M, and let $\mathbf{E}\longrightarrow M$ be a smooth vector bundle of rank r, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ be a local frame of $\mathbf{E}$ defined on a neighborhood of $\overline{U}$, and write

$$\mathbf{u} = \sum_{a=1}^{r} u^a\mathbf{e}_a \qquad \text{on } U.$$

Let $k\in\mathbb{N}_0$ and $0<\alpha<1$. Then

$$\mathbf{u}\in\Gamma^{k,\alpha}(\overline{U},\mathbf{E}) \iff u^a\circ\varphi^{-1}\in C^{k,\alpha}(\overline{\varphi(U)}) \quad \text{for every } a\in\{1,\dots,r\}.$$

Moreover, there are constants $c_{k,\alpha}, C_{k,\alpha}>0$, independent of $\mathbf{u}$, such that

$$\begin{aligned} c_{k,\alpha}\sum_{a=1}^{r}\|u^a\circ\varphi^{-1}\|_{C^{k,\alpha}(\overline{\varphi(U)})} &\le \|\mathbf{u}\|_{\Gamma^{k,\alpha}(\overline{U},\mathbf{E})} \\ &\le C_{k,\alpha}\sum_{a=1}^{r}\|u^a\circ\varphi^{-1}\|_{C^{k,\alpha}(\overline{\varphi(U)})}. \end{aligned} \tag{9.57}$$

Proof. The case $k = 0$ is Lemma 9.2.8. Suppose that $k \geq 1$. Let U' be a larger regular coordinate ball, so that $\overline{U} \subseteq U'$ and the chart and frame are defined on U'. In the chosen chart and frame, Lemma 9.1.28 gives, for each $s \in \{1, \dots, k\}$,

$$(\nabla^s \mathbf{u})^a_{i_1 \dots i_s} = \partial_{i_1} \cdots \partial_{i_s} u^a + \sum_{b=1}^{r} \sum_{|\beta| \leq s-1} (B_{s,\beta})^a_{b\, i_1 \dots i_s} \partial^\beta u^b. \tag{9.58}$$

The coefficients $B_{s,\beta}$ are smooth on U'. Their derivatives of the orders appearing below are therefore uniformly bounded on each compact subset of U'.

We first prove the left inequality in (9.57). Let $\widetilde{\mathbf{u}} \in \mathcal{E}^{k,\alpha}_{\overline{U}}(\mathbf{u})$. Choose a cutoff function $\chi \in C^\infty_c(U')$ equal to one on a neighborhood of $\overline{U}$ and set $\mathbf{v} := \chi \widetilde{\mathbf{u}}$. If $\mathbf{v} = \sum_{a=1}^{r} v^a \mathbf{e}_a$ on U', denote by $\widetilde{v}^a$ the extension of $v^a \circ \varphi^{-1}$ by zero outside $\varphi(U')$. Each $\widetilde{v}^a$ belongs to $C^k(\mathbb{R}^n)$ and extends $u^a \circ \varphi^{-1}$.

Solving for the principal part in (9.58) gives

$$\partial_{i_1} \cdots \partial_{i_s} v^a = (\nabla^s \mathbf{v})^a_{i_1 \dots i_s} - \sum_{b=1}^{r} \sum_{|\beta| \leq s-1} (B_{s,\beta})^a_{b\, i_1 \dots i_s} \partial^\beta v^b. \tag{9.59}$$

Uniform equivalence of the fiber norms and induction on s give

$$\sum_{a=1}^{r} \sum_{|\beta| \leq k} \|D^\beta \widetilde{v}^a\|_{L^\infty(\mathbb{R}^n)} \leq C \sum_{s=0}^{k} \|\nabla^s \mathbf{v}\|_{L^\infty(M, \mathbf{E}_s)}. \tag{9.60}$$

Let us prove this estimate by induction. For $0 \leq s \leq k$, set

$$S_s := \sum_{a=1}^{r} \sum_{|\beta| \leq s} \|D^\beta \widetilde{v}^a\|_{L^\infty(\mathbb{R}^n)}, \qquad T_s := \sum_{q=0}^{s} \|\nabla^q \mathbf{v}\|_{L^\infty(M, \mathbf{E}_q)}.$$

Since $\mathbf{v}$ has compact support in the domain of the frame, extension by zero does not change the suprema of its components. Uniform equivalence of the norm of a section and the norms of its components gives $S_0 \leq C_0 T_0$.

Suppose that $S_{s-1} \leq C_{s-1} T_{s-1}$ for some $1 \leq s \leq k$. Taking absolute values in (9.59), using the uniform bounds on the coefficients $B_{s,\beta}$ on the compact set containing the support of $\mathbf{v}$, and summing over a and all blocks $(i_1, \dots, i_s)$ gives

$$\sum_{a=1}^{r} \sum_{|\gamma| = s} \|D^\gamma \widetilde{v}^a\|_{L^\infty(\mathbb{R}^n)} \leq C'_s \left(\|\nabla^s \mathbf{v}\|_{L^\infty(M, \mathbf{E}_s)} + S_{s-1} \right).$$

Here passing from ordered blocks of indices to multi-indices changes only the constant, since both families are finite. Consequently,

$$\begin{aligned} S_s &\leq S_{s-1} + C'_s \left(\|\nabla^s \mathbf{v}\|_{L^\infty(M, \mathbf{E}_s)} + S_{s-1} \right) \\ &\leq \left((1 + C'_s) C_{s-1} + C'_s \right) T_s. \end{aligned}$$

This proves the induction step. Taking the maximum of the constants appearing for $0 \leq s \leq k$ yields (9.60). It remains to control the Hölder seminorms of the derivatives of order k. The components of $\nabla^k \mathbf{v}$ are Hölder continuous in coordinates by Lemma 9.2.8, applied to the bundle $\mathbf{E}_k = T^{(0,k)}(TM) \otimes \mathbf{E}$ over an intermediate ball containing the support of χ. If $|\beta| \leq k - 1$, estimate (9.60) controls the first derivatives of $D^\beta \widetilde{v}^b$. By the

mean value theorem, these functions are Lipschitz and, since their supports lie in a fixed compact set, they are Hölder continuous with exponent α on $\mathbb{R}^n$. The inequality

$$[f\mathbf{g}]_{C^{0,\alpha}(K)} \le \|f\|_{C^0(K)}[\mathbf{g}]_{C^{0,\alpha}(K)} + \|\mathbf{g}\|_{C^0(K)}[f]_{C^{0,\alpha}(K)}$$

applied to the lower-order terms in (9.59) gives

$$\sum_{a=1}^{r} [\widetilde{v}^a]_{C^{k,\alpha}(\mathbb{R}^n)} \le C\|\mathbf{v}\|_{\Gamma^{k,\alpha}(\mathbf{E})}.$$

Together with (9.60), this implies

$$\sum_{a=1}^{r} \|u^a \circ \varphi^{-1}\|_{C^{k,\alpha}(\overline{\varphi(U)})} \le C\|\widetilde{\mathbf{u}}\|_{\Gamma^{k,\alpha}(\mathbf{E})},$$

where we use Lemma 9.2.9 to control $\mathbf{v} = \chi\widetilde{\mathbf{u}}$. Taking the infimum over all extensions $\widetilde{\mathbf{u}}$ proves the first inequality in (9.57).

We prove the reverse inequality. For each $a \in \{1,\dots,r\}$ choose an extension $w^a \in C^{k,\alpha}(\mathbb{R}^n)$ of $u^a \circ \varphi^{-1}$. Let $\eta \in C_c^\infty(\varphi(U'))$ be a cutoff function equal to one on a neighborhood of $\overline{\varphi(U)}$. On U' define

$$\mathbf{w} := \sum_{a=1}^{r} \big((\eta w^a) \circ \varphi\big)\mathbf{e}_a$$

and denote its extension by zero outside U' by $\widetilde{\mathbf{w}}$. The presence of η ensures that $\widetilde{\mathbf{w}}$ is a global section of class C^k and that $\widetilde{\mathbf{w}}\restriction_U = \mathbf{u}$.

The direct formula (9.58), the Leibniz rule, and Lemma 9.2.8, again applied to the bundle $\mathbf{E}_k$, give

$$\|\widetilde{\mathbf{w}}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C \sum_{a=1}^{r} \|w^a\|_{C^{k,\alpha}(\mathbb{R}^n)}.$$

The terms of order k are controlled by the seminorms of the w^a. The lower-order terms have an additional bounded derivative and are controlled by the mean value theorem, exactly as in the first part. Taking the infimum first over the scalar extensions w^a and then in the definition of the norm of $\mathbf{u}$ gives the second inequality in (9.57). The two constructions also prove equivalence of membership. □

The following localized version of the norm will allow us to combine estimates obtained in different charts.

Proposition 9.2.11. (Equivalence of the localized Hölder norm) Let $(M,\mathbf{g})$ be a compact connected Riemannian manifold without boundary. Let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric and a compatible connection. Consider regular coordinate balls $(U_i,\varphi_i)_{i=1}^N$ covering M and smooth cutoff functions $(\psi_i)_{i=1}^N$ such that

$$0 \le \psi_i \le 1, \qquad \operatorname{supp}(\psi_i) \Subset U_i, \qquad \sum_{i=1}^{N} \psi_i = 1.$$

For each i, let $(\mathbf{e}_{i,1},\dots,\mathbf{e}_{i,r})$ be a local frame of $\mathbf{E}$ defined on a neighborhood of $\overline{U_i}$, and write

$$\psi_i\mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a} \qquad \text{on } U_i.$$

If $k \in \mathbb{N}_0$ and $0 < \alpha < 1$, there are constants $c, C > 0$ such that

$$c\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le \sum_{i=1}^{N}\sum_{a=1}^{r} \|u_i^a \circ \varphi_i^{-1}\|_{C^{k,\alpha}(\overline{\varphi_i(U_i)})} \le C\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \tag{9.61}$$

for every $\mathbf{u} \in \Gamma^{k,\alpha}(\mathbf{E})$.

Proof. Lemma 9.2.9 gives a constant $C_0 > 0$, independent of i, such that

$$\|\psi_i\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C_0\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})}.$$

Uniformity follows because only a finite family of cutoff functions is involved. Lemma 9.2.10, applied to $\psi_i\mathbf{u}\restriction_{U_i}$, gives the second inequality in (9.61).

We prove the first. Set $K_i := \varphi_i(\operatorname{supp}\psi_i) \Subset \varphi_i(U_i)$ and choose $\eta_i \in C_c^\infty(\varphi_i(U_i))$ such that $\eta_i = 1$ on a neighborhood of K_i. For each a take an extension $v_i^a \in C^{k,\alpha}(\mathbb{R}^n)$ of $u_i^a \circ \varphi_i^{-1}$. The function $\eta_i v_i^a$, extended by zero outside $\varphi_i(U_i)$, remains an extension of that component. On U_i reconstruct the section

$$\sum_{a=1}^{r} \big((\eta_i v_i^a) \circ \varphi_i\big)\mathbf{e}_{i,a}$$

and denote its extension by zero outside U_i by $\widetilde{\mathbf{u}}_i$. By construction, $\widetilde{\mathbf{u}}_i = \psi_i\mathbf{u}$. The local comparison and product rule give

$$\|\psi_i\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C_i \sum_{a=1}^{r} \|v_i^a\|_{C^{k,\alpha}(\mathbb{R}^n)}.$$

Taking the infimum over the extensions v_i^a, summing over i, and using $\mathbf{u} = \sum_{i=1}^{N} \psi_i\mathbf{u}$ gives

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C \sum_{i=1}^{N}\sum_{a=1}^{r} \|u_i^a \circ \varphi_i^{-1}\|_{C^{k,\alpha}(\overline{\varphi_i(U_i)})}.$$

This completes the proof. □

Corollary 9.2.12. (Completeness of spaces of Hölder sections) Let $(M, \mathbf{g})$ be a compact connected Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with a bundle metric and a compatible connection. Let $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. Then $\Gamma^{k,\alpha}(\mathbf{E})$ is a Banach space.

Proof. Let $(\mathbf{u}_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $\Gamma^{k,\alpha}(\mathbf{E})$. For each $i \in \{1,\dots,N\}$ write

$$\psi_i \mathbf{u}_j = \sum_{a=1}^{r} u_{i,j}^a \mathbf{e}_{i,a} \qquad \text{on } U_i.$$

For each pair (i,a), the components of $\psi_i\mathbf{u}_j$ form a Cauchy sequence in $C^{k,\alpha}(\overline{\varphi_i(U_i)})$ by (9.61). Proposition 3.5.4 gives a limit f_i^a in that space. Since the extension norm controls the uniform norm on the closure, f_i^a vanishes outside the fixed compact set $K_i = \varphi_i(\operatorname{supp}\psi_i)$, as do all the components of the sequence. Choose global extensions of the f_i^a, multiply them by the functions η_i from the preceding proof, and reconstruct global sections $\mathbf{v}_i$ supported in U_i. The estimate obtained in that proof gives

$$\|\psi_i\mathbf{u}_j - \mathbf{v}_i\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C_i \sum_{a=1}^{r} \|(u_{i,j}^a \circ \varphi_i^{-1}) - f_i^a\|_{C^{k,\alpha}(\overline{\varphi_i(U_i)})},$$

and the right-hand side converges to zero. Define

$$\mathbf{u} := \sum_{i=1}^{N} \mathbf{v}_i.$$

Since $\mathbf{u}_j = \sum_{i=1}^{N} \psi_i\mathbf{u}_j$ and the sum is finite,

$$\|\mathbf{u}_j - \mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le \sum_{i=1}^{N} \|\psi_i\mathbf{u}_j - \mathbf{v}_i\|_{\Gamma^{k,\alpha}(\mathbf{E})} \longrightarrow 0.$$

Therefore, $\mathbf{u} \in \Gamma^{k,\alpha}(\mathbf{E})$ and the space is complete. □

We can now transfer the Euclidean embeddings to bundle sections.

Theorem 9.2.13. (Sobolev–Morrey embedding for vector bundles) Let $(M,\mathbf{g})$ be a compact Riemannian manifold without boundary of dimension n, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a compatible connection $\nabla^\mathbf{E}$. Let $m \in \mathbb{N}$, $1 \le p < \infty$, $k \in \mathbb{N}_0$, and $0 < \alpha < 1$. Suppose that

$$k + \alpha \le m - \frac{n}{p}. \tag{9.62}$$

Then each element of $W^{m,p}(M,\mathbf{E})$ admits a unique representative in $\Gamma^{k,\alpha}(\mathbf{E})$, and the map assigning this representative defines a continuous linear embedding

$$W^{m,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{k,\alpha}(\mathbf{E}).$$

In particular, there is a constant $C > 0$ such that

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \tag{9.63}$$

for every $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$.

Proof. We may work on each connected component of M, and therefore assume that M is connected. Choose regular coordinate balls, cutoff functions, and local frames as in Proposition 9.2.11. For $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$ write

$$\psi_i \mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a} \qquad \text{on } U_i.$$

The Leibniz estimate used in the proof of Theorem 9.2.1, in particular estimate (9.45) with $j = 0$, gives a constant $C_0 > 0$ such that

$$\|\psi_i \mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le C_0 \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \tag{9.64}$$

for every $i \in \{1,\dots,N\}$.
Lemma 9.1.44 shows that $u_i^a \circ \varphi_i^{-1} \in W^{m,p}(\varphi_i(U_i))$ and that

$$\sum_{a=1}^{r} \|u_i^a \circ \varphi_i^{-1}\|_{W^{m,p}(\varphi_i(U_i))} \le C_i \|\psi_i \mathbf{u}\|_{W^{m,p}(U_i,\mathbf{E}\restriction_{U_i})}. \tag{9.65}$$

Since $\operatorname{supp}(\psi_i) \Subset U_i$, each component has compact support contained in $\varphi_i(U_i)$ and therefore belongs to $W_0^{m,p}(\varphi_i(U_i))$. Theorem 3.5.7 and (9.62) give

$$\|u_i^a \circ \varphi_i^{-1}\|_{C^{k,\alpha}(\overline{\varphi_i(U_i)})} \le C_i' \|u_i^a \circ \varphi_i^{-1}\|_{W^{m,p}(\varphi_i(U_i))}.$$

Take the Euclidean representatives of the components, reconstruct the section $\psi_i \mathbf{u}$ on U_i, and extend it by zero outside U_i. The sum of these sections represents $\mathbf{u}$, since it agrees almost everywhere with $\sum_{i=1}^{N} \psi_i \mathbf{u} = \mathbf{u}$. Proposition 9.2.11, the preceding estimates, and finiteness of the cover give

$$\begin{aligned}
\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} &\le C \sum_{i=1}^{N} \sum_{a=1}^{r} \|u_i^a \circ \varphi_i^{-1}\|_{C^{k,\alpha}(\overline{\varphi_i(U_i)})} \\
&\le C \sum_{i=1}^{N} \sum_{a=1}^{r} \|u_i^a \circ \varphi_i^{-1}\|_{W^{m,p}(\varphi_i(U_i))} \\
&\le C \sum_{i=1}^{N} \|\psi_i \mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le C' \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.
\end{aligned}$$

This proves (9.63).
Uniqueness of the representative follows as in the Euclidean case. If a continuous section vanishes almost everywhere but is nonzero at some point, its norm is positive on an open neighborhood of that point, which has positive Riemann–Lebesgue measure. Thus the section must vanish throughout M. □

Theorem 9.2.14. (Rellich–Kondrashov into Hölder sections) Let $(M,\mathbf{g})$ be a compact Riemannian manifold without boundary of dimension n, and let $\mathbf{E} \to M$ be a smooth vector bundle with a bundle metric and a compatible connection. Let $m \in \mathbb{N}$, $1 \le p < \infty$, $k \in \mathbb{N}_0$, and $0 < \alpha < 1$. If

$$k + \alpha < m - \frac{n}{p}, \tag{9.66}$$

then the embedding

$$W^{m,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{k,\alpha}(\mathbf{E})$$

is compact.

Proof. Let $(\mathbf{u}_j)_{j\in\mathbb{N}}$ be a bounded sequence in $W^{m,p}(M,\mathbf{E})$. Use the cover, cutoff functions, and frames from the preceding proof. For each $i \in \{1,\dots,N\}$ write

$$\psi_i \mathbf{u}_j = \sum_{a=1}^{r} u_{i,j}^a \mathbf{e}_{i,a} \qquad \text{on } U_i.$$

Estimates (9.64) and (9.65) show that, for each pair (i,a), the sequence

$$\left(u_{i,j}^a \circ \varphi_i^{-1}\right)_{j\in\mathbb{N}}$$

is bounded in $W_0^{m,p}(\varphi_i(U_i))$. By Theorem 3.5.10, each of these sequences has a subsequence converging in $C^{k,\alpha}(\overline{\varphi_i(U_i)})$.

Only the Nr pairs of indices are involved. Extracting successive subsequences and relabeling the last one, we may assume that

$$u_{i,j}^a \circ \varphi_i^{-1} \quad \text{converges in} \quad C^{k,\alpha}(\overline{\varphi_i(U_i)})$$

for every $i \in \{1,\dots,N\}$ and every $a \in \{1,\dots,r\}$. In particular, the localized components form Cauchy sequences in those spaces. The first inequality in (9.61), applied to $\mathbf{u}_j - \mathbf{u}_\ell$, gives

$$\|\mathbf{u}_j - \mathbf{u}_\ell\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C \sum_{i=1}^{N} \sum_{a=1}^{r} \|(u_{i,j}^a - u_{i,\ell}^a) \circ \varphi_i^{-1}\|_{C^{k,\alpha}(\overline{\varphi_i(U_i)})}.$$

The right-hand side converges to zero as $j,\ell \to \infty$. Thus the subsequence is Cauchy in $\Gamma^{k,\alpha}(\mathbf{E})$ and converges in that space by Corollary 9.2.12. This proves compactness of the embedding. □

Corollary 9.2.15. (Regularity exponents for sections) Let $(M,\mathbf{g})$ be a compact Riemannian manifold without boundary of dimension n, and let $\mathbf{E} \to M$ be a smooth vector bundle with a bundle metric and a compatible connection. Let $m \in \mathbb{N}$ and $1 \le p < \infty$. Let

$$\sigma := m - \frac{n}{p} > 0.$$

Then the following assertions hold.

(a) If $\sigma = \ell + \theta$, with $\ell \in \mathbb{N}_0$ and $0 < \theta < 1$, then

$$W^{m,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{\ell,\theta}(\mathbf{E})$$

continuously, and the embedding into $\Gamma^{\ell,\beta}(\mathbf{E})$ is compact for every $0 < \beta < \theta$.

(b) If $\sigma = s \in \mathbb{N}$, then for every $0 < \alpha < 1$ the embedding

$$W^{m,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{s-1,\alpha}(\mathbf{E})$$

is compact and, in particular, continuous.

Proof. If $\sigma = \ell + \theta$ with $0 < \theta < 1$, Theorem 9.2.13 applied to $k = \ell$ and $\alpha = \theta$ gives the continuous embedding, since

$$\ell + \theta = m - \frac{n}{p}.$$

If $0 < \beta < \theta$, then

$$\ell + \beta < m - \frac{n}{p},$$

and Theorem 9.2.14 gives compactness of the embedding into $\Gamma^{\ell,\beta}(\mathbf{E})$.
If $\sigma = s \in \mathbb{N}$ and $0 < \alpha < 1$, we have

$$(s-1) + \alpha < s = m - \frac{n}{p}.$$

The same Rellich–Kondrashov theorem, with $k = s - 1$, gives the compact embedding into $\Gamma^{s-1,\alpha}(\mathbf{E})$. Every compact linear operator between normed spaces is continuous, which also proves the final assertion. □

9.2.2 Independence of Sobolev spaces in the compact case

As in the setting of Riemannian manifolds, Sobolev spaces on vector bundles over a compact Riemannian manifold are independent of the choice of metric and connection. To prove this, we establish some useful preliminary results. In particular, we formulate an analogue of Lemma 7.1.10 in the vector bundle setting.

Lemma 9.2.16. Let M be a smooth manifold with or without boundary, let $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ be two Riemannian metrics on M, let ∇^{TM} and $\widetilde{\nabla}^{TM}$ be two connections on TM, let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle of finite rank, let $\mathbf{h}$ and $\widetilde{\mathbf{h}}$ be two smooth bundle metrics on $\mathbf{E}$, and let $\nabla^{\mathbf{E}}$ and $\widetilde{\nabla}^{\mathbf{E}}$ be two connections on $\mathbf{E}$.
Let $K \subseteq M$ be compact and $m \geq 0$. Then there are constants $c_m, C_m > 0$ such that

$$c_m \sum_{j=0}^{m} |\nabla^j \mathbf{u}(x)|^2_{\mathbf{g},\mathbf{h}} \leq \sum_{j=0}^{m} |\widetilde{\nabla}^j \mathbf{u}(x)|^2_{\widetilde{\mathbf{g}},\widetilde{\mathbf{h}}} \leq C_m \sum_{j=0}^{m} |\nabla^j \mathbf{u}(x)|^2_{\mathbf{g},\mathbf{h}}, \qquad \forall x \in K,$$

for every section $\mathbf{u} \in \Gamma^m(\mathbf{E})$.

Proof. If $K = \varnothing$, the assertion is immediate. Suppose that $K \neq \varnothing$. The connections ∇^{TM} and $\nabla^{\mathbf{E}}$ induce connections on $T^{(0,j)}(TM) \otimes \mathbf{E}$, also denoted by ∇. By Definition 1.7.10, we may consider

$$\mathscr{T}_m(\mathbf{E}) := \bigoplus_{j=0}^{m} T^{(0,j)}(TM) \otimes \mathbf{E}.$$

For $\mathbf{u} \in \Gamma(\mathbf{E})$ write

$$J^m_\nabla \mathbf{u} := (\mathbf{u}, \nabla\mathbf{u}, \ldots, \nabla^m \mathbf{u})$$

and, for each $p \in M$, define

$$\mathscr{D}_{\nabla,p}(\mathbf{E}) := \{J^m_\nabla \mathbf{u}(p) \mid \mathbf{u} \in \Gamma(\mathbf{E})\} \subseteq \mathscr{T}_m(\mathbf{E})_p.$$

Each $\mathscr{D}_{\nabla,p}(\mathbf{E})$ is a vector subspace by linearity of covariant derivatives.
Let (U, φ) be a smooth chart, let $(\mathbf{e}_1, \ldots, \mathbf{e}_r)$ be a smooth local frame of $\mathbf{E}$ on U, let $p \in U$, and set $x := \varphi(p)$

and $N := \binom{n+m}{m}$. If $\mathbf{u} = \sum_{a=1}^{r} u^a \mathbf{e}_a$, define the linear map

$$F^{\varphi,\mathbf{e}}_{\nabla,p} \colon \mathscr{D}_{\nabla,p}(\mathbf{E}) \longrightarrow \mathbb{K}^{rN}, \qquad F^{\varphi,\mathbf{e}}_{\nabla,p}(J^m_\nabla \mathbf{u}(p)) := \big(D^\alpha(u^a \circ \varphi^{-1})(x)\big)_{1\le a\le r,\ |\alpha|\le m},$$

using the multi-index notation from Definition 1.1.10. Lemma 9.1.28, in particular formula (9.3), gives

$$(\nabla^j \mathbf{u})^a_{i_1\dots i_j} = \partial_{i_1}\cdots\partial_{i_j}(u^a \circ \varphi^{-1}) + \sum_{b=1}^{r}\sum_{|\alpha|<j} A^{ab,\alpha}_{i_1\dots i_j} D^\alpha(u^b \circ \varphi^{-1}), \tag{9.67}$$

with smooth coefficients on U. Induction on j in (9.67) shows that the covariant jet determines all partial derivatives of the components through order m. Consequently, $F^{\varphi,\mathbf{e}}_{\nabla,p}$ is well defined and injective.
Let $a = (a^b_\alpha)_{1\le b\le r,\ |\alpha|\le m} \in \mathbb{K}^{rN}$. Proposition 1.3.6 gives a function $\chi \in C^\infty_c(U)$ equal to one on a neighborhood of p. The section

$$\mathbf{v}(q) := \chi(q) \sum_{b=1}^{r} \left(\sum_{|\alpha|\le m} \frac{a^b_\alpha}{\alpha!} (\varphi(q) - x)^\alpha \right) \mathbf{e}_b(q), \qquad q \in U,$$

extended by zero outside U belongs to $\Gamma(\mathbf{E})$ and satisfies $D^\alpha(v^b \circ \varphi^{-1})(x) = a^b_\alpha$. Thus $F^{\varphi,\mathbf{e}}_{\nabla,p}$ is surjective and hence an isomorphism.
Let $(e_1,\dots,e_{rN})$ be the standard basis of $\mathbb{K}^{rN}$. Expressing the covariant jets in terms of the prescribed partial jets using (9.67) shows that the components of the local sections

$$\boldsymbol{\sigma}_\nu(q) := (F^{\varphi,\mathbf{e}}_{\nabla,q})^{-1}(e_\nu), \qquad q \in U, \quad \nu \in \{1,\dots,rN\},$$

are smooth. Their values form a basis of $\mathscr{D}_{\nabla,q}(\mathbf{E})$ for every $q \in U$. By Lemma 1.7.52,

$$\mathscr{D}_\nabla(\mathbf{E}) := \bigcup_{p\in M} \mathscr{D}_{\nabla,p}(\mathbf{E})$$

is a smooth vector subbundle of $\mathscr{T}_m(\mathbf{E})$ of rank rN. The connections $\widetilde{\nabla}^{TM}$ and $\widetilde{\nabla}^{\mathbf{E}}$ similarly produce the subbundle $\mathscr{D}_{\widetilde{\nabla}}(\mathbf{E})$ and the isomorphisms $F^{\varphi,\mathbf{e}}_{\widetilde{\nabla},p}$.
The compact set K is covered by finitely many compact regular coordinate balls and, if M has boundary, compact regular coordinate half-balls

$$\overline{B}_1,\dots,\overline{B}_\ell,$$

each contained in the domain of a chart φ_i and a local frame $\mathbf{e}_i = (\mathbf{e}_{1,i},\dots,\mathbf{e}_{r,i})$ of $\mathbf{E}$. For $p \in \overline{B}_i$ and $v \in \mathbb{K}^{rN}$ set

$$Q_i(p,v) := \sum_{j=0}^{m} \left| \big((F^{\varphi_i,\mathbf{e}_i}_{\nabla,p})^{-1}(v)\big)_j \right|^2_{\mathbf{g},\mathbf{h}},$$

$$\widetilde{Q}_i(p,v) := \sum_{j=0}^{m} \left| \big((F^{\varphi_i,\mathbf{e}_i}_{\widetilde{\nabla},p})^{-1}(v)\big)_j \right|^2_{\widetilde{\mathbf{g}},\widetilde{\mathbf{h}}}.$$

The functions Q_i and $\widetilde{Q}_i$ are continuous and, for each p, are positive definite quadratic forms in v. If

$$\mathbb{S}^{rN}_{\mathbb{K}} := \{v \in \mathbb{K}^{rN} \mid \|v\|_2 = 1\},$$

then

$$R_i(p,v) := \frac{\widetilde{Q}_i(p,v)}{Q_i(p,v)}$$

is continuous and strictly positive on the compact set $\overline{B}_i \times \mathbb{S}^{rN}_{\mathbb{K}}$. Define

$$c_{i,m} := \min_{(p,v)\in\overline{B}_i\times\mathbb{S}^{rN}_{\mathbb{K}}} R_i(p,v) > 0, \qquad C_{i,m} := \max_{(p,v)\in\overline{B}_i\times\mathbb{S}^{rN}_{\mathbb{K}}} R_i(p,v) < \infty.$$

Homogeneity of Q_i and $\widetilde{Q}_i$ implies that

$$c_{i,m} Q_i(p,v) \le \widetilde{Q}_i(p,v) \le C_{i,m} Q_i(p,v) \tag{9.68}$$

for every $p \in \overline{B}_i$ and every $v \in \mathbb{K}^{rN}$.
Let $\mathbf{u} \in \Gamma^m(\mathbf{E})$ and let $p \in \overline{B}_i$. The preceding polynomial construction gives a smooth section whose partial derivative jet at p agrees with that of $\mathbf{u}$. Formula (9.3) then shows that the covariant jets of the two sections agree at p for either pair of connections. Thus

$$\begin{aligned} F^{\varphi_i,\mathbf{e}_i}_{\nabla,p}(J^m_\nabla \mathbf{u}(p)) &= \big(D^\alpha(u^a_i \circ \varphi_i^{-1})(\varphi_i(p))\big)_{1\le a\le r,\ |\alpha|\le m} \\ &= F^{\varphi_i,\mathbf{e}_i}_{\widetilde{\nabla},p}(J^m_{\widetilde{\nabla}} \mathbf{u}(p)). \end{aligned}$$

Applying (9.68) to this vector gives

$$c_{i,m} \sum_{j=0}^{m} |\nabla^j \mathbf{u}(p)|^2_{\mathbf{g},\mathbf{h}} \le \sum_{j=0}^{m} |\widetilde{\nabla}^j \mathbf{u}(p)|^2_{\widetilde{\mathbf{g}},\widetilde{\mathbf{h}}} \le C_{i,m} \sum_{j=0}^{m} |\nabla^j \mathbf{u}(p)|^2_{\mathbf{g},\mathbf{h}}.$$

Taking

$$c_m := \min_{1\le i\le \ell} c_{i,m} > 0, \qquad C_m := \max_{1\le i\le \ell} C_{i,m} < \infty,$$

yields the inequality throughout K. □

9.3 Interpolation: the Gagliardo–Nirenberg inequality

In this section we study the *Gagliardo–Nirenberg interpolation* inequality on manifolds (and in vector bundles). The inequality states the following:

Theorem 9.3.1. (Gagliardo–Nirenberg inequality for vector bundles) Let $(M,\mathbf{g})$ be a compact Riemannian manifold without boundary of dimension n, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a compatible connection $\nabla^\mathbf{E}$. Let $m \ge 1$, $0 \le j \le m$, and $1 \le p < \infty$. Then there is a constant $C > 0$, depending only on $M, \mathbf{g}, \mathbf{E}, \mathbf{h}_\mathbf{E}, \nabla^\mathbf{E}, m, j$ and p, such that

$$\|\nabla^j \mathbf{F}\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})} \le C \|\mathbf{F}\|^{1-\frac{j}{m}}_{L^p(M,\mathbf{E})} \Big(\|\mathbf{F}\|_{L^p(M,\mathbf{E})} + \|\nabla^m \mathbf{F}\|_{L^p(M,T^{(0,m)}(TM)\otimes\mathbf{E})}\Big)^{\frac{j}{m}},$$

for every $\mathbf{F} \in W^{m,p}(M,\mathbf{E})$. Here we interpret $\nabla^0 \mathbf{F} = \mathbf{F}$, so that $T^{(0,0)}(TM)\otimes\mathbf{E} = \mathbf{E}$.

Proof. The cases $j = 0$ and $j = m$ are immediate. If $j = 0$, the inequality reduces to $\|\mathbf{F}\|_{L^p(M,\mathbf{E})} \le C\|\mathbf{F}\|_{L^p(M,\mathbf{E})}$; if $j = m$, it reduces to

$$\|\nabla^m \mathbf{F}\|_{L^p(M,T^{(0,m)}(TM)\otimes\mathbf{E})} \le C\left(\|\mathbf{F}\|_{L^p(M,\mathbf{E})} + \|\nabla^m \mathbf{F}\|_{L^p(M,T^{(0,m)}(TM)\otimes\mathbf{E})}\right).$$

Thus suppose that $0 < j < m$.

We first prove the inequality for smooth sections. Let $\mathbf{F} \in \Gamma(\mathbf{E})$. Since M is compact, we may take a finite cover $(U_\alpha, \varphi_\alpha)_{\alpha=1}^N$ of M by regular coordinate balls such that $\overline{U_\alpha}$ lies in the domain of a smooth local frame $(\mathbf{e}_{1,\alpha}, \dots, \mathbf{e}_{r,\alpha})$ of $\mathbf{E}$. Also take a partition of unity $(\psi_\alpha)_{\alpha=1}^N$ subordinate to this cover, with $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for every α.

For each $\alpha \in \{1, \dots, N\}$ write

$$(\psi_\alpha \mathbf{F}) \restriction_{U_\alpha} = \sum_{a=1}^{r} F_\alpha^a \mathbf{e}_{a,\alpha},$$

and denote $\Omega_\alpha := \varphi_\alpha(U_\alpha)$. Since $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$, we have $\operatorname{supp}(F_\alpha^a \circ \varphi_\alpha^{-1}) \Subset \Omega_\alpha$ for every $a \in \{1, \dots, r\}$. Define the extension of $F_\alpha^a \circ \varphi_\alpha^{-1}$ by zero to $\mathbb{R}^n$ by

$$\widetilde{F_\alpha^a \circ \varphi_\alpha^{-1}}(y) := \begin{cases} F_\alpha^a \circ \varphi_\alpha^{-1}(y), & y \in \Omega_\alpha, \\ 0, & y \in \mathbb{R}^n \setminus \Omega_\alpha. \end{cases}$$

Since $F_\alpha^a \circ \varphi_\alpha^{-1}$ is smooth and has compact support contained in Ω_α, its extension by zero belongs to $C_c^\infty(\mathbb{R}^n)$. Moreover, for every multi-index β, we have

$$D^\beta \widetilde{F_\alpha^a \circ \varphi_\alpha^{-1}} = \widetilde{D^\beta (F_\alpha^a \circ \varphi_\alpha^{-1})}.$$

Consequently, for each integer $s \ge 0$,

$$\sum_{|\beta|=s} \|D^\beta \widetilde{F_\alpha^a \circ \varphi_\alpha^{-1}}\|_{L^p(\mathbb{R}^n)} = \sum_{|\beta|=s} \|D^\beta (F_\alpha^a \circ \varphi_\alpha^{-1})\|_{L^p(\Omega_\alpha)}.$$

Applying the Euclidean Gagliardo–Nirenberg inequality on $\mathbb{R}^n$ in its classical form

$$\sum_{|\beta|=j} \|D^\beta v\|_{L^p(\mathbb{R}^n)} \le A \|v\|_{L^p(\mathbb{R}^n)}^{1-\frac{j}{m}} \left(\sum_{|\gamma|=m} \|D^\gamma v\|_{L^p(\mathbb{R}^n)} \right)^{\frac{j}{m}}, \qquad v \in C_c^\infty(\mathbb{R}^n),$$

gives, for $v = \widetilde{F_\alpha^a \circ \varphi_\alpha^{-1}}$,

$$\sum_{|\beta|=j} \|D^\beta (F_\alpha^a \circ \varphi_\alpha^{-1})\|_{L^p(\Omega_\alpha)} \le A \|F_\alpha^a \circ \varphi_\alpha^{-1}\|_{L^p(\Omega_\alpha)}^{1-\frac{j}{m}} \left(\sum_{|\gamma|=m} \|D^\gamma (F_\alpha^a \circ \varphi_\alpha^{-1})\|_{L^p(\Omega_\alpha)} \right)^{\frac{j}{m}}.$$

Summing over a and using Hölder's inequality from Proposition B.7.9 for the finite counting measure, with

exponents $\frac{m}{m-j}$ and $\frac{m}{j}$, gives

$$\sum_{a=1}^{r}\sum_{|\beta|=j}\|D^{\beta}(F_{\alpha}^{a}\circ\varphi_{\alpha}^{-1})\|_{L^{p}(\Omega_{\alpha})}\leq A\left(\sum_{a=1}^{r}\|F_{\alpha}^{a}\circ\varphi_{\alpha}^{-1}\|_{L^{p}(\Omega_{\alpha})}\right)^{1-\frac{j}{m}}\left(\sum_{a=1}^{r}\sum_{|\gamma|=m}\|D^{\gamma}(F_{\alpha}^{a}\circ\varphi_{\alpha}^{-1})\|_{L^{p}(\Omega_{\alpha})}\right)^{\frac{j}{m}}.$$

We now relate the Euclidean partial derivatives of the components to the covariant derivatives of $\psi_{\alpha}\mathbf{F}$. By Lemma 9.1.28, for each $s\in\{1,\dots,m\}$, the components of $\nabla^{s}(\psi_{\alpha}\mathbf{F})$ in the chart $(U_{\alpha},\varphi_{\alpha})$ and frame $(\mathbf{e}_{1,\alpha},\dots,\mathbf{e}_{r,\alpha})$ are given by

$$(\nabla^{s}(\psi_{\alpha}\mathbf{F}))^{a}_{i_{1}\dots i_{s}}=\partial_{i_{1}}\cdots\partial_{i_{s}}F_{\alpha}^{a}+\sum_{b=1}^{r}\sum_{|\beta|\leq s-1}(B_{\alpha,\beta})^{a}_{b\,i_{1}\dots i_{s}}\partial^{\beta}F_{\alpha}^{b},$$

where the coefficients $(B_{\alpha,\beta})^{a}_{b\,i_{1}\dots i_{s}}$ are smooth on U_{α}. Since $\operatorname{supp}(\psi_{\alpha}\mathbf{F})\subseteq U_{\alpha}$, all calculations take place on a compact subset of U_{α}, and therefore those coefficients, as well as the coefficients converting Riemannian measure to Euclidean measure in coordinates, are uniformly bounded. This triangular formula shows that there are constants $P_{\alpha,s},Q_{\alpha,s}>0$, independent of $\mathbf{F}$, such that

$$\|\nabla^{s}(\psi_{\alpha}\mathbf{F})\|_{L^{p}(M,T^{(0,s)}(TM)\otimes\mathbf{E})}\leq P_{\alpha,s}\sum_{a=1}^{r}\sum_{|\beta|\leq s}\|D^{\beta}(F_{\alpha}^{a}\circ\varphi_{\alpha}^{-1})\|_{L^{p}(\Omega_{\alpha})},$$

and

$$\sum_{a=1}^{r}\sum_{|\beta|\leq s}\|D^{\beta}(F_{\alpha}^{a}\circ\varphi_{\alpha}^{-1})\|_{L^{p}(\Omega_{\alpha})}\leq Q_{\alpha,s}\sum_{\ell=0}^{s}\|\nabla^{\ell}(\psi_{\alpha}\mathbf{F})\|_{L^{p}(M,T^{(0,\ell)}(TM)\otimes\mathbf{E})}.$$

In particular, using these estimates with $s=j$, also using the preceding Euclidean inequality for the orders $1,\dots,j$, and noting that order 0 corresponds to the L^{p} norm, we obtain a constant $R_{\alpha}>0$, independent of $\mathbf{F}$, such that

$$\|\nabla^{j}(\psi_{\alpha}\mathbf{F})\|_{L^{p}(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\leq R_{\alpha}\|\psi_{\alpha}\mathbf{F}\|^{1-\frac{j}{m}}_{L^{p}(M,\mathbf{E})}\left(\sum_{\ell=0}^{m}\|\nabla^{\ell}(\psi_{\alpha}\mathbf{F})\|_{L^{p}(M,T^{(0,\ell)}(TM)\otimes\mathbf{E})}\right)^{\frac{j}{m}}.$$

Since the cover is finite, taking $R:=\max\{R_{\alpha}\mid 1\leq\alpha\leq N\}$ gives the same estimate with R in place of R_{α}.

We now sum over α. Since $\mathbf{F}=\sum_{\alpha=1}^{N}\psi_{\alpha}\mathbf{F}$, the triangle inequality gives

$$\|\nabla^{j}\mathbf{F}\|_{L^{p}(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\leq\sum_{\alpha=1}^{N}\|\nabla^{j}(\psi_{\alpha}\mathbf{F})\|_{L^{p}(M,T^{(0,j)}(TM)\otimes\mathbf{E})}.$$

By the preceding estimate and Hölder's inequality from Proposition B.7.9 for the finite counting measure,

$$\|\nabla^{j}\mathbf{F}\|_{L^{p}(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\leq R\left(\sum_{\alpha=1}^{N}\|\psi_{\alpha}\mathbf{F}\|_{L^{p}(M,\mathbf{E})}\right)^{1-\frac{j}{m}}\left(\sum_{\alpha=1}^{N}\sum_{\ell=0}^{m}\|\nabla^{\ell}(\psi_{\alpha}\mathbf{F})\|_{L^{p}(M,T^{(0,\ell)}(TM)\otimes\mathbf{E})}\right)^{\frac{j}{m}}.$$

It remains to control the terms involving $\psi_{\alpha}\mathbf{F}$ by $\mathbf{F}$. For the zeroth-order term, since each ψ_{α} is smooth and

the family is finite, there is $D_0 > 0$, independent of $\mathbf{F}$, such that

$$\sum_{\alpha=1}^{N} \|\psi_\alpha \mathbf{F}\|_{L^p(M,\mathbf{E})} \leq D_0 \|\mathbf{F}\|_{L^p(M,\mathbf{E})}.$$

On the other hand, for each $s \leq m$, the Leibniz rule for the induced connection on $T^{(0,s)}(TM) \otimes \mathbf{E}$ gives

$$|\nabla^s(\psi_\alpha \mathbf{F})|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \leq \sum_{t=0}^{s} \binom{s}{t} |\nabla^{s-t}\psi_\alpha|_{\mathbf{g}} |\nabla^t \mathbf{F}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Since M is compact and the family $\{\psi_\alpha\}_{\alpha=1}^N$ is finite, the quantities $\| |\nabla^{s-t}\psi_\alpha|_{\mathbf{g}} \|_{L^\infty(M)}$, with $0 \leq t \leq s \leq m$ and $1 \leq \alpha \leq N$, are uniformly bounded. Consequently, there is $D_m > 0$, independent of $\mathbf{F}$, such that

$$\sum_{\alpha=1}^{N} \sum_{\ell=0}^{m} \|\nabla^\ell(\psi_\alpha \mathbf{F})\|_{L^p(M,T^{(0,\ell)}(TM)\otimes\mathbf{E})} \leq D_m \sum_{\ell=0}^{m} \|\nabla^\ell \mathbf{F}\|_{L^p(M,T^{(0,\ell)}(TM)\otimes\mathbf{E})}.$$

Substituting these estimates gives

$$\|\nabla^j \mathbf{F}\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})} \leq C_1 \|\mathbf{F}\|_{L^p(M,\mathbf{E})}^{1-\frac{j}{m}} \left(\sum_{\ell=0}^{m} \|\nabla^\ell \mathbf{F}\|_{L^p(M,T^{(0,\ell)}(TM)\otimes\mathbf{E})} \right)^{\frac{j}{m}}.$$

Finally, we use equivalence of the Sobolev norm with the norm given by the zeroth-order term and the covariant derivative of highest order; that is, there is $C_2 > 0$ such that

$$\sum_{\ell=0}^{m} \|\nabla^\ell \mathbf{F}\|_{L^p(M,T^{(0,\ell)}(TM)\otimes\mathbf{E})} \leq C_2 \left(\|\mathbf{F}\|_{L^p(M,\mathbf{E})} + \|\nabla^m \mathbf{F}\|_{L^p(M,T^{(0,m)}(TM)\otimes\mathbf{E})} \right).$$

This equivalence follows, for example, by applying Ehrling's lemma to the compact inclusions $W^{m,p}(M,\mathbf{E}) \hookrightarrow W^{\ell,p}(M,\mathbf{E}) \hookrightarrow L^p(M,\mathbf{E})$, for $0 < \ell < m$, and absorbing the intermediate terms. Therefore,

$$\|\nabla^j \mathbf{F}\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})} \leq C \|\mathbf{F}\|_{L^p(M,\mathbf{E})}^{1-\frac{j}{m}} \left(\|\mathbf{F}\|_{L^p(M,\mathbf{E})} + \|\nabla^m \mathbf{F}\|_{L^p(M,T^{(0,m)}(TM)\otimes\mathbf{E})} \right)^{\frac{j}{m}}$$

for every $\mathbf{F} \in \Gamma(\mathbf{E})$.

Now let $\mathbf{F} \in W^{m,p}(M,\mathbf{E})$. By Theorem 9.1.48, there is a sequence $(\mathbf{F}_k)_{k\in\mathbb{N}} \subseteq \Gamma(\mathbf{E}) \cap W^{m,p}(M,\mathbf{E})$ such that $\mathbf{F}_k \to \mathbf{F}$ in $W^{m,p}(M,\mathbf{E})$. In particular, $\mathbf{F}_k \to \mathbf{F}$ in $L^p(M,\mathbf{E})$, $\nabla^j \mathbf{F}_k \to \nabla_w^j \mathbf{F}$ in $L^p(M, T^{(0,j)}(TM)\otimes\mathbf{E})$, and $\nabla^m \mathbf{F}_k \to \nabla_w^m \mathbf{F}$ in $L^p(M, T^{(0,m)}(TM)\otimes\mathbf{E})$. Applying the inequality already proved to $\mathbf{F}_k$ gives

$$\|\nabla^j \mathbf{F}_k\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})} \leq C \|\mathbf{F}_k\|_{L^p(M,\mathbf{E})}^{1-\frac{j}{m}} \left(\|\mathbf{F}_k\|_{L^p(M,\mathbf{E})} + \|\nabla^m \mathbf{F}_k\|_{L^p(M,T^{(0,m)}(TM)\otimes\mathbf{E})} \right)^{\frac{j}{m}}.$$

Letting $k \to \infty$, we conclude that

$$\|\nabla^j \mathbf{F}\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})} \leq C \|\mathbf{F}\|_{L^p(M,\mathbf{E})}^{1-\frac{j}{m}} \left(\|\mathbf{F}\|_{L^p(M,\mathbf{E})} + \|\nabla_w^m \mathbf{F}\|_{L^p(M,T^{(0,m)}(TM)\otimes\mathbf{E})} \right)^{\frac{j}{m}}.$$

This completes the proof. □

9.4 Sobolev spaces on manifolds with boundary

In many applications, boundary conditions arise naturally when formulating a PDE. Sobolev spaces allow us to pass from a differential problem to its weak formulation; however, when the manifold has boundary, integration-by-parts formulas produce additional terms on ∂M. The identity with test sections supported in the interior determines the formal differential adjoint, as explained in Remark 8.2.4. Boundary conditions enter when choosing a domain for the realization of the operator in L^2 and determining the domain of its Hilbert space adjoint.

For this purpose, introduce the notation

$$\Gamma_{c,\mathrm{Int}(M)}(\mathbf{E}) := \{\mathbf{u} \in \Gamma(\mathbf{E}) | \mathrm{supp}(\mathbf{u}) \Subset \mathrm{Int}(M)\}.$$

That is, $\Gamma_{c,\mathrm{Int}(M)}(\mathbf{E})$ denotes the space of smooth sections whose compact support lies in the interior of M.

Under this convention, integration-by-parts formulas produce no boundary terms when the test sections belong to $\Gamma_{c,\mathrm{Int}(M)}$. For example, in Theorem 9.1.29, if we take $\mathbf{F} \in \Gamma(T^{(k,l)}(TM) \otimes \mathbf{E})$ and $\mathbf{G} \in \Gamma_{c,\mathrm{Int}(M)}(T^{(k,l+s)}(TM) \otimes \mathbf{E})$, the boundary term vanishes and the formula reduces to

$$\int_M \langle \nabla^s \mathbf{F}, \mathbf{G} \rangle_{\mathbf{g},\mathbf{h_E}} \, d\lambda_{\mathbf{g}} = (-1)^s \int_M \left\langle \mathbf{F}, (\mathrm{tr}_{\mathbf{g}} \circ \nabla)^s \mathbf{G} \right\rangle_{\mathbf{g},\mathbf{h_E}} d\lambda_{\mathbf{g}} + \sum_{j=0}^{s-1} (-1)^{s-1-j} \int_{\partial M} \left\langle \nabla^j \mathbf{F}, \iota_{\nu}\left((\mathrm{tr}_{\mathbf{g}} \circ \nabla)^{s-1-j} \mathbf{G}\right) \right\rangle_{\mathbf{g},\mathbf{h_E}} d\lambda_{\tilde{\mathbf{g}}}.$$

0

Thus, when defining weak derivatives on a manifold with boundary, we work with test sections belonging to $\Gamma_{c,\mathrm{Int}(M)}$. In this way, boundary conditions are not implicitly incorporated into the weak definition; they will be introduced later through the trace operator.

We next give the definitions needed to work with these Sobolev spaces.

Definition 9.4.1. (Interior formal adjoint) Let $(M, \mathbf{g})$ be a Riemannian manifold with nonempty boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles equipped with bundle metrics $\mathbf{h_E}$ and $\mathbf{h_F}$, respectively. Let $P \in \mathbf{PDO}(\mathbf{E}, \mathbf{F})$. We say that $Q \in \mathbf{PDO}(\mathbf{F}, \mathbf{E})$ is an *interior formal adjoint* of P if, for every $\mathbf{u} \in \Gamma_{c,\mathrm{Int}(M)}(\mathbf{E})$ and every $\mathbf{v} \in \Gamma_{c,\mathrm{Int}(M)}(\mathbf{F})$, we have

$$\int_M \langle P\mathbf{u}, \mathbf{v} \rangle_{\mathbf{h_F}} \, d\lambda_{\mathbf{g}} = \int_M \langle \mathbf{u}, Q\mathbf{v} \rangle_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}}.$$

In this case we write $Q = P_h^*$; the symbol P' is reserved for the complex–bilinear formal transpose.

The adjective "interior" recalls that the preceding identity is tested only against sections whose supports do not meet the boundary. Thus the operator P_h^* describes the formal Hermitian adjoint determined by the differential part of P, without yet choosing boundary conditions.

Theorem 9.4.2. (Existence and uniqueness of the interior formal adjoint) Let $(M, \mathbf{g})$ be a Riemannian manifold with nonempty boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles with bundle metrics $\mathbf{h_E}, \mathbf{h_F}$. If $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$, then P admits a unique interior formal adjoint

$$P_h^* \in \mathbf{PDO}^{(m)}(\mathbf{F}, \mathbf{E}).$$

Moreover, let (U, φ) be a chart, and let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ and $\mathbf{f}_1, \ldots, \mathbf{f}_s$ be local orthonormal frames of $\mathbf{E}$ and $\mathbf{F}$, re-

spectively, defined on U. Set

$$D^{\alpha}_{\varphi} a := \left[D^{\alpha}(a \circ \varphi^{-1}) \right] \circ \varphi, \qquad a \in C^{\infty}(U, \mathbb{K}).$$

If

$$P\left(\sum_{i=1}^{r} u^{i} \mathbf{e}_{i} \right) = \sum_{j=1}^{s} \left(\sum_{i=1}^{r} \sum_{|\alpha| \le m} a^{j}_{\alpha i} D^{\alpha}_{\varphi} u^{i} \right) \mathbf{f}_{j},$$

then

$$P^{*}_{h}\left(\sum_{j=1}^{s} v^{j} \mathbf{f}_{j} \right) = \frac{1}{\sqrt{\det(\mathbf{g})}} \sum_{i=1}^{r} \left[\sum_{j=1}^{s} \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^{\alpha}_{\varphi}\left(\sqrt{\det(\mathbf{g})}\, \overline{a^{j}_{\alpha i}}\, v^{j} \right) \right] \mathbf{e}_{i},$$

where $g_{ab} = \mathbf{g}(\partial_a, \partial_b)$. In the real case the bar leaves $a^{j}_{\alpha i}$ unchanged.

Proof. Apply Theorem 8.2.3 to the manifold without boundary $\operatorname{Int}(M)$ and the operator restricted to its interior. The local expression proved there depends smoothly on the coefficients of P, the bundle metrics, and the coordinate weight $\sqrt{\det(\mathbf{g})}$ of the Riemannian measure; since all these data are smooth up to ∂M, its coefficients extend smoothly to each boundary chart and define the operator displayed above throughout M. For test sections supported in $\operatorname{Int}(M)$, the integral identity is precisely that of the cited theorem and produces no boundary terms.

If Q_1, Q_2 satisfy the interior identity, the uniqueness lemma of Chapter 8 applied on $\operatorname{Int}(M)$ gives $Q_1 = Q_2$ there. Their local coefficients are smooth up to the boundary and agree on the dense open interior; by continuity they also agree on the boundary. Therefore, $Q_1 = Q_2$ on M. □

Definition 9.4.3. (Weak covariant derivative on manifolds with boundary) Let $(M, \mathbf{g})$ be a Riemannian manifold with nonempty boundary and $\mathbf{E} \to M$ a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a smooth connection $\nabla^{\mathbf{E}}$. In constructing ∇^{s}, we use the product connection induced by Levi–Civita and $\nabla^{\mathbf{E}}$. Let $s \ge 0$ and let $\mathbf{u} \in L^{1}_{\mathrm{loc}}(M, \mathbf{E})$. We say that a section $\mathbf{v} \in L^{1}_{\mathrm{loc}}(M, T^{(0,s)}(TM) \otimes \mathbf{E})$ is the sth weak covariant derivative of $\mathbf{u}$ if, for every $\sigma \in \Gamma_{c,\operatorname{Int}(M)}(T^{(0,s)}(TM) \otimes \mathbf{E})$, we have

$$\int_{M} \langle \mathbf{v}, \sigma \rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}} = \int_{M} \langle \mathbf{u}, (\nabla^{s})^{*}_{h} \sigma \rangle_{\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}}.$$

In this case we write $\mathbf{v} = \nabla^{s}_{w} \mathbf{u}$. For $s = 0$ we adopt the convention $\nabla^{0}_{w} \mathbf{u} = \mathbf{u}$.

In the preceding definition, $(\nabla^{s})^{*}_{h}$ denotes the interior formal Hermitian adjoint of ∇^{s}, not its bilinear transpose. This choice allows us to define weak derivatives without implicitly introducing boundary conditions.

Definition 9.4.4. (Global and local Sobolev spaces on manifolds with boundary) Let $(M, \mathbf{g})$ be a Riemannian manifold with nonempty boundary, and let $\pi \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a smooth connection $\nabla^{\mathbf{E}}$. Let $m \in \mathbb{N}_0$ and $1 \le p \le \infty$.

(a) *Global Sobolev spaces.* Define $W^{m,p}(M, \mathbf{E})$ to be the set of sections $\mathbf{u} \in L^{1}_{\mathrm{loc}}(M, \mathbf{E})$ such that, for every $s \in \{0, \dots, m\}$, the weak covariant derivative $\nabla^{s}_{w} \mathbf{u}$ exists and belongs to $L^{p}(M, T^{(0,s)}(TM) \otimes \mathbf{E})$. Equip

this space with the norm

$$\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} := \begin{cases} \left(\displaystyle\sum_{s=0}^{m} \int_M |\nabla_w^s \mathbf{u}|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \displaystyle\max_{0\le s\le m} \operatorname*{ess\,sup}_{x\in M} |\nabla_w^s \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, & p = \infty. \end{cases}$$

With this norm, $W^{m,p}(M,\mathbf{E})$ is a Banach space.

(b) *Local Sobolev spaces.* Define $W^{m,p}_{\mathrm{loc}}(M,\mathbf{E})$ to be the set of sections $\mathbf{u} \in L^1_{\mathrm{loc}}(M,\mathbf{E})$ such that, for every $s \in \{0,\dots,m\}$, the weak covariant derivative $\nabla_w^s \mathbf{u}$ exists and belongs to $L^p_{\mathrm{loc}}(M, T^{(0,s)}(TM) \otimes \mathbf{E})$.

If $K \subseteq M$ is compact, define the local seminorm

$$\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E};K)} := \begin{cases} \left(\displaystyle\sum_{s=0}^{m} \int_K |\nabla_w^s \mathbf{u}|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \displaystyle\max_{0\le s\le m} \operatorname*{ess\,sup}_{x\in K} |\nabla_w^s \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, & p = \infty. \end{cases}$$

We consider $W^{m,p}_{\mathrm{loc}}(M,\mathbf{E})$ with the family of seminorms $\{\|\cdot\|_{W^{m,p}(M,\mathbf{E};K)}\}_{K\subseteq M \text{ compact}}$.

Restriction to the interior identifies these spaces $W^{m,p}(M,\mathbf{E})$ isometrically with the spaces defined weakly on $\mathrm{Int}(M)$ using the restricted metric and connection. Indeed, the test sections are exactly those with compact support in the interior, and ∂M has zero n-dimensional Riemannian measure, so the global norms agree. The values of a measurable section on ∂M do not affect its equivalence class. In particular, Lemma 9.1.12 and completeness of Sobolev spaces also apply here for every $1 \le p \le \infty$, without a reflexivity hypothesis.

We give a local characterization of Sobolev spaces in the boundary case, analogous to Lemma 9.1.44, that will allow us to work with them. For this characterization we impose metric compatibility, since the proof uses the explicit formal adjoint formula obtained by integration by parts.

Lemma 9.4.5. Let $m \in \mathbb{N}_0$ and $1 \le p < \infty$, let (U,φ) be a regular coordinate half-ball in a Riemannian manifold M with nonempty boundary, and let $\pi_\mathbf{E} \colon \mathbf{E} \longrightarrow M$ be a smooth vector bundle of rank r, equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a compatible connection $\nabla^\mathbf{E}$. Write

$$B_U^+ := \varphi(U \cap \mathrm{Int}(M)) = \varphi(U) \cap \{x_n > 0\}.$$

Let $\mathbf{F} \in L^1_{\mathrm{loc}}(M,\mathbf{E})$ and let $\{\mathbf{e}_1,\dots,\mathbf{e}_r\}$ be a local frame of $\mathbf{E}$ defined on a neighborhood of $\overline{U}$. Suppose that $\mathbf{F} = \displaystyle\sum_{a=1}^{r} F^a \mathbf{e}_a$ on U. Then

$$\mathbf{F}\restriction_U \in W^{m,p}(U,\mathbf{E}\restriction_U) \quad \text{if and only if} \quad F^a \circ \varphi^{-1} \in W^{m,p}(B_U^+)$$

for every $1 \le a \le r$.

Moreover, there are constants $C_m, c_m > 0$, independent of $\mathbf{F}$, such that

$$c_m \sum_{a=1}^{r} \left\| F^a \circ \varphi^{-1} \right\|_{W^{m,p}(B_U^+)} \leq \|\mathbf{F}\|_{W^{m,p}(U,\mathbf{E}\restriction_U)} \leq C_m \sum_{a=1}^{r} \left\| F^a \circ \varphi^{-1} \right\|_{W^{m,p}(B_U^+)}$$

for every $\mathbf{F} \in W^{m,p}(U, \mathbf{E}\restriction_U)$.

Proof. We repeat the argument of Lemma 9.1.44. First we verify that the coordinate test functions have compact support in the corresponding Euclidean open set; this verification allows all the integrations by parts used in that proof to be carried out without boundary terms.
Let (U, φ) be a regular coordinate half-ball, and recall that $B_U^+ = \varphi(U \cap \mathrm{Int}(M))$. Fix $s \geq 0$ and take a test section

$$\boldsymbol{\Psi} \in \Gamma_{c,\mathrm{Int}(M)}\big(T^{(0,s)}(TU) \otimes \mathbf{E}\restriction_U\big).$$

By definition, $\mathrm{supp}(\boldsymbol{\Psi})$ is compact and contained in $U \cap \mathrm{Int}(M)$.
Write locally

$$\boldsymbol{\Psi} = \sum_{\substack{1 \leq c \leq r \\ J_s \in \{1,\dots,n\}^s}} \Psi^c_{J_s}\, \mathbf{d}x^{J_s} \otimes \mathbf{e}_c,$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ is the local frame of $\mathbf{E}$ over U and J_s ranges over blocks of indices of length s. For each $x \in U$, the family $\{\mathbf{d}x^{J_s} \otimes \mathbf{e}_c\}_{c,J_s}$ is a basis of the vector space $T^{(0,s)}_x M \otimes \mathbf{E}_x$. By pointwise linear independence, we have

$$\boldsymbol{\Psi}(x) = 0 \quad \text{if and only if} \quad \Psi^c_{J_s}(x) = 0 \text{ for every } c, J_s.$$

Thus the set where $\boldsymbol{\Psi}$ is nonzero agrees with the finite union of the sets where some component $\Psi^c_{J_s}$ is nonzero. Taking closures in U gives

$$\mathrm{supp}(\boldsymbol{\Psi}) = \bigcup_{c,J_s} \mathrm{supp}(\Psi^c_{J_s}).$$

In particular, for every pair of indices (c, J_s),

$$\mathrm{supp}(\Psi^c_{J_s}) \subseteq \mathrm{supp}(\boldsymbol{\Psi}) \Subset U \cap \mathrm{Int}(M).$$

Now compose with the chart. Since $\varphi\colon U \longrightarrow \varphi(U)$ is a homeomorphism, for each component we have

$$\mathrm{supp}(\Psi^c_{J_s} \circ \varphi^{-1}) = \varphi\big(\mathrm{supp}(\Psi^c_{J_s})\big).$$

Moreover,

$$\varphi\big(\mathrm{supp}(\Psi^c_{J_s})\big) \subseteq \varphi(\mathrm{supp}(\boldsymbol{\Psi})) \Subset \varphi(U \cap \mathrm{Int}(M)) = B_U^+.$$

Thus

$$\Psi^c_{J_s} \circ \varphi^{-1} \in C_c^\infty(B_U^+)$$

for every c, J_s.
Consequently, after passing to coordinates, the test functions that arise have compact support contained in the open set $B_U^+ \subseteq \mathbb{R}^n$. For this reason, all integrations by parts involved in the weak definition take place inside a Euclidean open set, without boundary terms. The local identities are therefore the same as for a regular coordinate ball, with the coordinate domain replaced by B_U^+.

The coordinate identities in the proof of Lemma 9.1.44 depend only on these integrations by parts and the bounded smoothness, on $\overline{U}$, of the coefficients of the metric, connection, and frame. Thus, with $\varphi(U)$ replaced by B_U^+, the induction constructed there gives

$$\mathbf{F}\restriction_U \in W^{m,p}(U,\mathbf{E}\restriction_U) \quad \text{if and only if} \quad F^a\circ\varphi^{-1}\in W^{m,p}(B_U^+)$$

for every $1\le a\le r$. Equivalence of norms is proved exactly as in the cited lemma. □

Theorem 9.4.6. (Meyers–Serrin on open sets with uniformly Lipschitz boundary) Let $\Omega\subseteq\mathbb{R}^n$ be an open set with uniformly Lipschitz continuous boundary in the sense of Definition 3.6.1. Let $m\in\mathbb{N}$ and $1\le p<\infty$. Then the restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ form a dense subspace of $W^{m,p}(\Omega)$.

Proof. Let $u\in W^{m,p}(\Omega)$. By Theorem 3.6.2, there is $v:=\mathcal{E}u\in W^{m,p}(\mathbb{R}^n)$ such that $v\restriction_\Omega = u$ almost everywhere. We prove explicitly that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$.
Choose $\chi\in C_c^\infty(\mathbb{R}^n)$ with $0\le\chi\le1$, $\chi=1$ on $B_{\mathrm{euc}}(0,1)$, and $\operatorname{supp}\chi\subset B_{\mathrm{euc}}(0,2)$, and set $\chi_R(x):=\chi(x/R)$ and $v_R:=\chi_R v$. For $|\alpha|\le m$, the weak Leibniz rule gives

$$D^\alpha v_R=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta\chi_R\,D^{\alpha-\beta}v.$$

The term with $\beta=0$ converges to $D^\alpha v$ in L^p by dominated convergence. If $\beta\neq0$, then

$$\|D^\beta\chi_R\,D^{\alpha-\beta}v\|_{L^p(\mathbb{R}^n)}\le R^{-|\beta|}\|D^\beta\chi\|_{L^\infty(\mathbb{R}^n)}\|D^{\alpha-\beta}v\|_{L^p(\mathbb{R}^n)}\longrightarrow0.$$

Since the sum has finitely many terms, $v_R\to v$ in $W^{m,p}(\mathbb{R}^n)$.
Now fix R. If ρ_ε is the mollifier from Theorem B.9.1, then $v_{R,\varepsilon}:=\rho_\varepsilon*v_R$ belongs to $C_c^\infty(\mathbb{R}^n)$ and

$$D^\alpha v_{R,\varepsilon}=\rho_\varepsilon*D^\alpha v_R,\qquad |\alpha|\le m.$$

Continuity of translations in L^p and Minkowski's integral inequality imply $\rho_\varepsilon*D^\alpha v_R\to D^\alpha v_R$ in L^p for each $|\alpha|\le m$. Therefore, $v_{R,\varepsilon}\to v_R$ in $W^{m,p}(\mathbb{R}^n)$. A diagonal choice gives $(\varphi_j)\subset C_c^\infty(\mathbb{R}^n)$ with $\varphi_j\to v$ in $W^{m,p}(\mathbb{R}^n)$. Since restriction to Ω does not increase any derivative norm,

$$\|\varphi_j\restriction_\Omega-u\|_{W^{m,p}(\Omega)}\le\|\varphi_j-v\|_{W^{m,p}(\mathbb{R}^n)}\longrightarrow0.$$

□

To discuss boundary conditions, we need to define an operator known as the *trace operator*. This operator is first constructed on smooth sections and then extended by density. For this reason, before introducing it, we must verify that the Meyers–Serrin theorem remains valid on manifolds with boundary.

The Euclidean Meyers–Serrin theorem for open sets with uniformly Lipschitz boundary, Theorem 9.4.6, allows approximation in boundary charts by restrictions of smooth functions defined on $\mathbb{R}^n$.

Definition 9.4.7. Let $(M,\mathbf{g})$ be a Riemannian manifold with nonempty boundary and $\mathbf{E}\longrightarrow M$ a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a smooth connection $\nabla^\mathbf{E}$. Let $m\in\mathbb{N}_0$ and let $1\le p<\infty$.

Define

$$C^{m,p}(M,\mathbf{E}) := \left\{ \mathbf{u} \in \Gamma(\mathbf{E}) \middle| \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} < \infty \right\},$$

where, for a smooth section $\mathbf{u}$, the Sobolev norm is given by

$$\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} := \left(\sum_{s=0}^{m} \int_M |\nabla^s \mathbf{u}|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right)^{\frac{1}{p}}.$$

Define

$$H^{m,p}(M,\mathbf{E}) := \overline{C^{m,p}(M,\mathbf{E})}^{\|\cdot\|_{W^{m,p}(M,\mathbf{E})}}.$$

That is, $H^{m,p}(M,\mathbf{E})$ is the completion of the smooth sections with finite Sobolev norm.

As in the case without boundary, this definition agrees with the weak definition of Sobolev spaces. The technical difference is that, when working locally near ∂M, we use regular coordinate half-balls and pass to the Euclidean open set $\varphi(U \cap \mathrm{Int}(M))$.

Theorem 9.4.8. (Meyers–Serrin for vector bundles on manifolds with boundary) Let $(M,\mathbf{g})$ be a Riemannian manifold with nonempty boundary, and let $\pi_\mathbf{E} : \mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a compatible connection $\nabla^\mathbf{E}$. If $m \in \mathbb{N}$ and $1 \leq p < \infty$, then $H^{m,p}(M,\mathbf{E}) = W^{m,p}(M,\mathbf{E})$.

Proof. Since $C^{m,p}(M,\mathbf{E}) \subseteq W^{m,p}(M,\mathbf{E})$ and $W^{m,p}(M,\mathbf{E})$ is complete, we directly have $H^{m,p}(M,\mathbf{E}) \subseteq W^{m,p}(M,\mathbf{E})$. We prove the reverse inclusion.

By the analogue of Proposition 1.2.11 for manifolds with boundary, there is a countable locally finite open cover of M consisting of regular coordinate balls contained in $\mathrm{Int}(M)$ and regular coordinate half-balls near ∂M. Denote this cover by $(U_\alpha, \varphi_\alpha)_{\alpha\in\mathbb{N}}$. Refining if necessary, we may assume that $\overline{U_\alpha}$ lies in the domain of a local frame $(\mathbf{e}_{1,\alpha},\dots,\mathbf{e}_{r,\alpha})$ of $\mathbf{E}$. Take a smooth partition of unity $(\psi_\alpha)_{\alpha\in\mathbb{N}}$ subordinate to this cover, with $\mathrm{supp}(\psi_\alpha) \Subset U_\alpha$ for every α.

Let $\mathbf{F} \in W^{m,p}(M,\mathbf{E})$. Since the partition of unity is locally finite, we have $\mathbf{F} = \sum_{\alpha=1}^{\infty} \psi_\alpha \mathbf{F}$. Fix $\alpha \in \mathbb{N}$. Write locally $\mathbf{F} = \sum_{a=1}^{r} F^a_\alpha \mathbf{e}_{a,\alpha}$ on U_α.

If U_α is a regular coordinate ball, set $\Omega_\alpha := \varphi_\alpha(U_\alpha)$. If U_α is a regular coordinate half-ball, set $\Omega_\alpha := \varphi_\alpha(U_\alpha \cap \mathrm{Int}(M))$. In the first case, Ω_α is an open ball in $\mathbb{R}^n$. In the second, Ω_α is the interior of a half-ball and, in particular, is an open subset of $\mathbb{R}^n$ with continuous boundary.

By Lemma 9.1.44 if U_α is a regular coordinate ball, and by Lemma 9.4.5 if U_α is a regular coordinate half-ball, we have $F^a_\alpha \circ \varphi_\alpha^{-1} \in W^{m,p}(\Omega_\alpha)$ for every $a \in \{1,\dots,r\}$.

We now show that $(\psi_\alpha \mathbf{F}) \restriction_{U_\alpha} \in W^{m,p}(U_\alpha, \mathbf{E} \restriction_{U_\alpha})$. Its components are $(\psi_\alpha \mathbf{F})^a_\alpha = \psi_\alpha F^a_\alpha$, and composition with the chart gives

$$(\psi_\alpha \mathbf{F})^a_\alpha \circ \varphi_\alpha^{-1} = (\psi_\alpha \circ \varphi_\alpha^{-1})(F^a_\alpha \circ \varphi_\alpha^{-1}).$$

Since $\psi_\alpha \circ \varphi_\alpha^{-1}$ is smooth and its derivatives through order m are bounded on Ω_α, the Leibniz formula implies that $(\psi_\alpha \mathbf{F})^a_\alpha \circ \varphi_\alpha^{-1} \in W^{m,p}(\Omega_\alpha)$. For every multi-index β with $|\beta| \leq m$,

$$\partial^\beta \Big((\psi_\alpha \circ \varphi_\alpha^{-1})(F^a_\alpha \circ \varphi_\alpha^{-1}) \Big) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma (\psi_\alpha \circ \varphi_\alpha^{-1}) \partial^{\beta-\gamma} (F^a_\alpha \circ \varphi_\alpha^{-1}),$$

and each term on the right-hand side belongs to $L^p(\Omega_\alpha)$. Applying the appropriate local lemma once more, we conclude that $(\psi_\alpha \mathbf{F})\restriction_{U_\alpha} \in W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})$.

Moreover, by local equivalence of norms, there is a constant $C_{m,\alpha} > 0$ such that, for every section $\mathbf{H} \in W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})$,

$$\|\mathbf{H}\|_{W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} \leq C_{m,\alpha} \sum_{a=1}^{r} \|H_\alpha^a \circ \varphi_\alpha^{-1}\|_{W^{m,p}(\Omega_\alpha)}.$$

We now approximate the components of $(\psi_\alpha \mathbf{F})\restriction_{U_\alpha}$. Here we distinguish the two types of charts.

If U_α is a regular coordinate ball, then $\Omega_\alpha = \varphi_\alpha(U_\alpha)$ is an open ball and $\operatorname{supp}((\psi_\alpha \mathbf{F})_\alpha^a \circ \varphi_\alpha^{-1}) \Subset \Omega_\alpha$, since $\operatorname{supp}(\psi_\alpha) \Subset U_\alpha$. By the Euclidean Meyers–Serrin theorem on Euclidean open sets, we may take functions $(\widetilde{G}_{\alpha,k})^a \in C_c^\infty(\Omega_\alpha)$ such that

$$\sum_{a=1}^{r} \left\| (\widetilde{G}_{\alpha,k})^a - (\psi_\alpha \mathbf{F})_\alpha^a \circ \varphi_\alpha^{-1} \right\|_{W^{m,p}(\Omega_\alpha)} \xrightarrow[k\to\infty]{} 0.$$

If U_α is a regular coordinate half-ball, use Theorem 9.4.6. Since Ω_α has uniformly Lipschitz boundary, for each component there is a sequence $(\widetilde{G}_{\alpha,k})^a \in C_c^\infty(\mathbb{R}^n)$ such that

$$\sum_{a=1}^{r} \left\| (\widetilde{G}_{\alpha,k})^a \restriction_{\Omega_\alpha} - (\psi_\alpha \mathbf{F})_\alpha^a \circ \varphi_\alpha^{-1} \right\|_{W^{m,p}(\Omega_\alpha)} \xrightarrow[k\to\infty]{} 0.$$

In this case we do not require the approximating functions to have compact support contained in Ω_α; we use only that they are restrictions to Ω_α of smooth functions defined on $\mathbb{R}^n$.

To return to the manifold and extend by zero outside U_α, choose $\chi_\alpha \in C_c^\infty(U_\alpha)$ such that $\chi_\alpha \equiv 1$ on a neighborhood of $\operatorname{supp}(\psi_\alpha)$. In this boundary case, replace $(\widetilde{G}_{\alpha,k})^a \restriction_{\Omega_\alpha}$ by $(\chi_\alpha \circ \varphi_\alpha^{-1})(\widetilde{G}_{\alpha,k})^a \restriction_{\Omega_\alpha}$.

This modification preserves convergence. To see this, write $\eta_\alpha := \chi_\alpha \circ \varphi_\alpha^{-1}$ and $f_\alpha^a := (\psi_\alpha \mathbf{F})_\alpha^a \circ \varphi_\alpha^{-1}$. Since $\chi_\alpha \equiv 1$ on a neighborhood of $\operatorname{supp}(\psi_\alpha)$, we have $\eta_\alpha f_\alpha^a = f_\alpha^a$. Therefore,

$$\eta_\alpha (\widetilde{G}_{\alpha,k})^a \restriction_{\Omega_\alpha} - f_\alpha^a = \eta_\alpha \left((\widetilde{G}_{\alpha,k})^a \restriction_{\Omega_\alpha} - f_\alpha^a \right).$$

Denote $h_{\alpha,k}^a := (\widetilde{G}_{\alpha,k})^a \restriction_{\Omega_\alpha} - f_\alpha^a$. By construction, $h_{\alpha,k}^a \to 0$ in $W^{m,p}(\Omega_\alpha)$. Let us show that also $\eta_\alpha h_{\alpha,k}^a \to 0$ in $W^{m,p}(\Omega_\alpha)$.

Let β be a multi-index with $|\beta| \leq m$. By the Leibniz formula,

$$\partial^\beta (\eta_\alpha h_{\alpha,k}^a) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (\partial^\gamma \eta_\alpha) \partial^{\beta-\gamma} h_{\alpha,k}^a.$$

Since η_α is smooth and supported in $\varphi_\alpha(U_\alpha)$, its derivatives of order $\leq m$ are bounded on Ω_α. Thus, for each β there is a constant $C_\beta > 0$, independent of k, such that

$$\|\partial^\beta (\eta_\alpha h_{\alpha,k}^a)\|_{L^p(\Omega_\alpha)} \leq C_\beta \sum_{\gamma \leq \beta} \|\partial^{\beta-\gamma} h_{\alpha,k}^a\|_{L^p(\Omega_\alpha)}.$$

Summing over all multi-indices $|\beta| \leq m$ gives a constant $C > 0$, independent of k, such that

$$\|\eta_\alpha h_{\alpha,k}^a\|_{W^{m,p}(\Omega_\alpha)} \leq C \|h_{\alpha,k}^a\|_{W^{m,p}(\Omega_\alpha)}.$$

Since $h^a_{\alpha,k} \to 0$ in $W^{m,p}(\Omega_\alpha)$, we conclude that $\eta_\alpha(\widetilde{G}_{\alpha,k})^a \restriction_{\Omega_\alpha} \to f^a_\alpha$ in $W^{m,p}(\Omega_\alpha)$. Thus multiplying the approximating functions by $\chi_\alpha \circ \varphi_\alpha^{-1}$ does not alter the limit.
After this replacement, the supports of the approximating functions in the chart lie in the image of a compact subset of U_α.
In either case, we have constructed smooth functions $(\widetilde{G}_{\alpha,k})^a$ on the corresponding coordinate domain, with the support needed to return to the manifold. Define a smooth section on U_α by

$$\mathbf{G}_{\alpha,k} = \sum_{a=1}^{r} \Big[(\widetilde{G}_{\alpha,k})^a \circ \varphi_\alpha\Big]\mathbf{e}_{a,\alpha}.$$

In a boundary chart, it is understood that we use the approximating functions already multiplied by $\chi_\alpha \circ \varphi_\alpha^{-1}$. By the choice of supports, $\mathbf{G}_{\alpha,k}$ extends by zero outside U_α to define a global smooth section of $\mathbf{E}$.
By the preceding local estimate,

$$\|\mathbf{G}_{\alpha,k} - (\psi_\alpha \mathbf{F})\restriction_{U_\alpha}\|_{W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} \xrightarrow[k\to\infty]{} 0.$$

Since $\operatorname{supp}(\mathbf{G}_{\alpha,k} - \psi_\alpha \mathbf{F}) \subseteq U_\alpha$ and ∇^s_w is a local operator for each $s \le m$, we have

$$\|\mathbf{G}_{\alpha,k} - (\psi_\alpha \mathbf{F})\restriction_{U_\alpha}\|_{W^{m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} = \|\mathbf{G}_{\alpha,k} - \psi_\alpha \mathbf{F}\|_{W^{m,p}(M,\mathbf{E})}.$$

Consequently, $\|\mathbf{G}_{\alpha,k} - \psi_\alpha \mathbf{F}\|_{W^{m,p}(M,\mathbf{E})} \to 0$ as $k \to \infty$.
Now fix $q \in \mathbb{N}$. By the preceding convergence, for each $\alpha \in \mathbb{N}$ there is $k_{\alpha,q} \in \mathbb{N}$ such that $\|\psi_\alpha \mathbf{F} - \mathbf{G}_{\alpha,k_{\alpha,q}}\|_{W^{m,p}(M,\mathbf{E})} < 2^{-(\alpha+q)}$. Since $\operatorname{supp}(\mathbf{G}_{\alpha,k_{\alpha,q}}) \subseteq U_\alpha$ and the family $(U_\alpha)_{\alpha\in\mathbb{N}}$ is locally finite, the sum $\mathbf{G}_q := \sum_{\alpha=1}^{\infty} \mathbf{G}_{\alpha,k_{\alpha,q}}$ is locally finite. It therefore defines a global smooth section $\mathbf{G}_q \in \Gamma(\mathbf{E})$.
Finally, $\mathbf{F} - \mathbf{G}_q = \sum_{\alpha=1}^{\infty} (\psi_\alpha \mathbf{F} - \mathbf{G}_{\alpha,k_{\alpha,q}})$, where the sum is locally finite. Using the triangle inequality,

$$\|\mathbf{F} - \mathbf{G}_q\|_{W^{m,p}(M,\mathbf{E})} \le \sum_{\alpha=1}^{\infty} \|\psi_\alpha \mathbf{F} - \mathbf{G}_{\alpha,k_{\alpha,q}}\|_{W^{m,p}(M,\mathbf{E})} \le \sum_{\alpha=1}^{\infty} \frac{1}{2^{\alpha+q}} = \frac{1}{2^q}.$$

Thus $\mathbf{G}_q \to \mathbf{F}$ in $W^{m,p}(M,\mathbf{E})$ as $q \to \infty$.
Moreover,

$$\|\mathbf{G}_q\|_{W^{m,p}(M,\mathbf{E})} \le \|\mathbf{F}\|_{W^{m,p}(M,\mathbf{E})} + \|\mathbf{F} - \mathbf{G}_q\|_{W^{m,p}(M,\mathbf{E})} \le \|\mathbf{F}\|_{W^{m,p}(M,\mathbf{E})} + 2^{-q} < \infty.$$

Therefore, $\mathbf{G}_q \in H^{m,p}(M,\mathbf{E})$ for every $q \in \mathbb{N}$.
We have proved that every section in $W^{m,p}(M,\mathbf{E})$ can be approximated in the Sobolev norm by smooth sections with finite Sobolev norm. This shows that $W^{m,p}(M,\mathbf{E}) \subseteq H^{m,p}(M,\mathbf{E})$. Together with the initial inclusion, we conclude that $H^{m,p}(M,\mathbf{E}) = W^{m,p}(M,\mathbf{E})$. □

Lemma 9.4.9. (Continuity of multiplication by smooth functions) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a compatible connection $\nabla^\mathbf{E}$. Let $m \in \mathbb{N}$ and $1 \le p < \infty$. Suppose that $\psi \in C^\infty(M)$ satisfies

$$\|\nabla^q \psi\|_{L^\infty(M, T^{(0,q)}(TM))} < \infty \qquad \text{for every } 0 \le q \le m.$$

Then the operator

$$\mathbf{u} \longmapsto \psi\mathbf{u}$$

is linear and continuous from $W^{m,p}(M,\mathbf{E})$ to itself. In particular, there is a constant $C_{\psi,m} > 0$ such that

$$\|\psi\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \leq C_{\psi,m}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$$

for every $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$.
In particular, if M is compact, every function $\psi \in C^\infty(M)$ defines a continuous multiplication operator on $W^{m,p}(M,\mathbf{E})$. More generally, on a noncompact manifold it suffices to take $\psi \in C^\infty_c(M)$.

Proof. We first prove the estimate for smooth sections. Let $\mathbf{u} \in \Gamma(\mathbf{E}) \cap W^{m,p}(M,\mathbf{E})$. For each $0 \leq s \leq m$, the iterated Leibniz rule for the induced connection shows that $\nabla^s(\psi\mathbf{u})$ is a finite sum of terms constructed from the covariant derivatives of ψ and $\mathbf{u}$, whose orders sum to s. Consequently, there is a constant $A_s > 0$, depending only on s and the tensor conventions, such that

$$|\nabla^s(\psi\mathbf{u})|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \leq A_s \sum_{t=0}^{s} |\nabla^{s-t}\psi|_{\mathbf{g}}\,|\nabla^t\mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Using the boundedness hypothesis on the derivatives of ψ gives

$$|\nabla^s(\psi\mathbf{u})|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \leq A_s \sum_{t=0}^{s} \|\nabla^{s-t}\psi\|_{L^\infty(M,T^{(0,s-t)}(TM))}|\nabla^t\mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Taking the $L^p\big(M, T^{(0,s)}(TM)\otimes\mathbf{E}\big)$ norm and using the triangle inequality,

$$\|\nabla^s(\psi\mathbf{u})\|_{L^p(M,T^{(0,s)}(TM)\otimes\mathbf{E})} \leq A_s \sum_{t=0}^{s} \|\nabla^{s-t}\psi\|_{L^\infty(M,T^{(0,s-t)}(TM))}\|\nabla^t\mathbf{u}\|_{L^p(M,T^{(0,t)}(TM)\otimes\mathbf{E})}.$$

Summing over $s \in \{0,\dots,m\}$ gives a constant $C_{\psi,m} > 0$ such that

$$\|\psi\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \leq C_{\psi,m}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$$

for every $\mathbf{u} \in \Gamma(\mathbf{E}) \cap W^{m,p}(M,\mathbf{E})$.
Now take $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$. By the Meyers–Serrin theorem for vector bundles (Theorem 9.4.8), there is a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}} \subseteq \Gamma(\mathbf{E}) \cap W^{m,p}(M,\mathbf{E})$ such that $\mathbf{u}_k \to \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. By the estimate already proved,

$$\|\psi\mathbf{u}_k - \psi\mathbf{u}_\ell\|_{W^{m,p}(M,\mathbf{E})} \leq C_{\psi,m}\|\mathbf{u}_k - \mathbf{u}_\ell\|_{W^{m,p}(M,\mathbf{E})}.$$

Thus $(\psi\mathbf{u}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $W^{m,p}(M,\mathbf{E})$. Define $\psi\mathbf{u}$ as its limit in $W^{m,p}(M,\mathbf{E})$. This definition is independent of the approximating sequence, since, if $\mathbf{u}_k \to \mathbf{u}$ and $\mathbf{v}_k \to \mathbf{u}$, then

$$\|\psi\mathbf{u}_k - \psi\mathbf{v}_k\|_{W^{m,p}(M,\mathbf{E})} \leq C_{\psi,m}\|\mathbf{u}_k - \mathbf{v}_k\|_{W^{m,p}(M,\mathbf{E})} \longrightarrow 0.$$

Moreover, passing to the limit in

$$\|\psi \mathbf{u}_k\|_{W^{m,p}(M,\mathbf{E})} \le C_{\psi,m}\|\mathbf{u}_k\|_{W^{m,p}(M,\mathbf{E})},$$

gives

$$\|\psi \mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le C_{\psi,m}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

This proves continuity of the multiplication operator. □

Definition 9.4.10. (Sobolev spaces with zero trace) Let $(M,\mathbf{g})$ be a Riemannian manifold with nonempty boundary and $\mathbf{E}\to M$ a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a smooth connection $\nabla^\mathbf{E}$. Let $m\in\mathbb{N}$ and $1\le p<\infty$.
Define the Sobolev space with zero trace by

$$W_0^{m,p}(M,\mathbf{E}) := \overline{\Gamma_{c,\mathrm{Int}(M)}(\mathbf{E})}^{\|\cdot\|_{W^{m,p}(M,\mathbf{E})}}.$$

That is, $W_0^{m,p}(M,\mathbf{E})$ is the closure, in the Sobolev norm, of the space of smooth sections with compact support contained in the interior of M.

Before formulating the trace theorem, we must fix the connection to be used on the bundle restricted to the boundary. If $\mathbf{u}$ is a smooth section of $\mathbf{E}$, then $\mathbf{u}\restriction_{\partial M}$ is a smooth section of $\mathbf{E}\restriction_{\partial M}$. To measure the Sobolev regularity of this restriction, it does not suffice to regard $\mathbf{E}\restriction_{\partial M}$ simply as a vector bundle: we also need a connection on it. The natural connection is obtained by restricting the connection of $\mathbf{E}$ to directions tangent to the boundary.

Lemma 9.4.11. (Induced connection on the restricted bundle) Let $(M,\mathbf{g})$ be a Riemannian manifold with nonempty boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle equipped with a connection $\nabla^\mathbf{E}$. Then there is a connection on the restricted bundle $\mathbf{E}\restriction_{\partial M}\to\partial M$, denoted by $\nabla^{\mathbf{E}\restriction_{\partial M}}$, defined as follows.
If $\mathbf{X}\in\mathfrak{X}(\partial M)$ and $\mathbf{u}\in\Gamma(\mathbf{E}\restriction_{\partial M})$, take smooth extensions $\widetilde{\mathbf{X}}\in\mathfrak{X}(M)$ and $\widetilde{\mathbf{u}}\in\Gamma(\mathbf{E})$ such that $\widetilde{\mathbf{X}}\restriction_{\partial M}=\mathbf{X}$ and $\widetilde{\mathbf{u}}\restriction_{\partial M}=\mathbf{u}$, and define

$$\nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{X}}\mathbf{u} := (\nabla^\mathbf{E}_{\widetilde{\mathbf{X}}}\widetilde{\mathbf{u}})\restriction_{\partial M}.$$

This definition is independent of the chosen extensions. Moreover, if $\nabla^\mathbf{E}$ is compatible with the bundle metric $\mathbf{h}_\mathbf{E}$, then $\nabla^{\mathbf{E}\restriction_{\partial M}}$ is compatible with the restricted bundle metric $\mathbf{h}_\mathbf{E}\restriction_{\mathbf{E}\restriction_{\partial M}}$.

Proof. We first prove independence of the chosen extensions. Let $\widetilde{\mathbf{X}}_1,\widetilde{\mathbf{X}}_2\in\mathfrak{X}(M)$ be two extensions of $\mathbf{X}$ and let $\widetilde{\mathbf{u}}_1,\widetilde{\mathbf{u}}_2\in\Gamma(\mathbf{E})$ be two extensions of $\mathbf{u}$. We want to prove that

$$(\nabla^\mathbf{E}_{\widetilde{\mathbf{X}}_1}\widetilde{\mathbf{u}}_1)\restriction_{\partial M} = (\nabla^\mathbf{E}_{\widetilde{\mathbf{X}}_2}\widetilde{\mathbf{u}}_2)\restriction_{\partial M}.$$

By linearity,

$$\nabla^\mathbf{E}_{\widetilde{\mathbf{X}}_1}\widetilde{\mathbf{u}}_1 - \nabla^\mathbf{E}_{\widetilde{\mathbf{X}}_2}\widetilde{\mathbf{u}}_2 = \nabla^\mathbf{E}_{\widetilde{\mathbf{X}}_1-\widetilde{\mathbf{X}}_2}\widetilde{\mathbf{u}}_1 + \nabla^\mathbf{E}_{\widetilde{\mathbf{X}}_2}(\widetilde{\mathbf{u}}_1-\widetilde{\mathbf{u}}_2).$$

The first term vanishes upon restriction to ∂M, since $\widetilde{\mathbf{X}}_1-\widetilde{\mathbf{X}}_2$ vanishes on ∂M and the connection is $C^\infty(M)$-linear in its vector argument. For each $x\in\partial M$, the value of $\nabla^\mathbf{E}_{\widetilde{\mathbf{X}}_1-\widetilde{\mathbf{X}}_2}\widetilde{\mathbf{u}}_1$ at x depends only on $(\widetilde{\mathbf{X}}_1-\widetilde{\mathbf{X}}_2)_x$, which is zero.
We now show that the second term also vanishes upon restriction to ∂M. Set $\mathbf{w}:=\widetilde{\mathbf{u}}_1-\widetilde{\mathbf{u}}_2$. Then $\mathbf{w}\restriction_{\partial M}=0$.

Since the argument is local, take a local frame $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ of $\mathbf{E}$ near a point of ∂M and write $\mathbf{w} = \sum_{a=1}^{r} w^a \mathbf{e}_a$. Since $\mathbf{w}$ vanishes on ∂M and the vectors $\mathbf{e}_1(x),\dots,\mathbf{e}_r(x)$ form a basis of $\mathbf{E}_x$ for each x, we have $w^a \restriction_{\partial M} = 0$ for every a.

Now, using the Leibniz rule for the connection,

$$\nabla^{\mathbf{E}}_{\widetilde{\mathbf{X}}_2} \mathbf{w} = \sum_{a=1}^{r} \widetilde{\mathbf{X}}_2(w^a)\mathbf{e}_a + \sum_{a=1}^{r} w^a \nabla^{\mathbf{E}}_{\widetilde{\mathbf{X}}_2} \mathbf{e}_a.$$

Upon restriction to ∂M, the second summand vanishes because $w^a \restriction_{\partial M} = 0$. For the first summand, use $\widetilde{\mathbf{X}}_2 \restriction_{\partial M} = \mathbf{X}$ and the fact that $\mathbf{X}$ is tangent to ∂M. Since $w^a \restriction_{\partial M} = 0$, we have $\mathbf{X}(w^a \restriction_{\partial M}) = 0$, and therefore $\widetilde{\mathbf{X}}_2(w^a) \restriction_{\partial M} = 0$. Thus $(\nabla^{\mathbf{E}}_{\widetilde{\mathbf{X}}_2} \mathbf{w}) \restriction_{\partial M} = 0$. This proves that $\nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{X}} \mathbf{u}$ is well defined.

We now prove that $\nabla^{\mathbf{E}\restriction_{\partial M}}$ is a connection. Let $f \in C^\infty(\partial M)$, $\mathbf{X},\mathbf{Y} \in \mathfrak{X}(\partial M)$, and $\mathbf{u},\mathbf{v} \in \Gamma(\mathbf{E}\restriction_{\partial M})$. Take smooth extensions of these objects to a neighborhood of ∂M. Since $\nabla^{\mathbf{E}}$ is $C^\infty(M)$-linear in its vector argument, we obtain $\nabla^{\mathbf{E}\restriction_{\partial M}}_{f\mathbf{X}+\mathbf{Y}} \mathbf{u} = f\nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{X}} \mathbf{u} + \nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{Y}} \mathbf{u}$. Likewise, using the Leibniz rule in the section argument,

$$\nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{X}}(f\mathbf{u}) = \mathbf{X}(f)\mathbf{u} + f\nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{X}} \mathbf{u}.$$

Therefore, $\nabla^{\mathbf{E}\restriction_{\partial M}}$ is a connection on $\mathbf{E}\restriction_{\partial M}$.

Finally, suppose that $\nabla^{\mathbf{E}}$ is compatible with $\mathbf{h}_{\mathbf{E}}$. Let $\mathbf{X} \in \mathfrak{X}(\partial M)$ and $\mathbf{u},\mathbf{v} \in \Gamma(\mathbf{E}\restriction_{\partial M})$. Take extensions $\widetilde{\mathbf{X}},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}}$ as before. By compatibility of $\nabla^{\mathbf{E}}$ with $\mathbf{h}_{\mathbf{E}}$,

$$\widetilde{\mathbf{X}}\big(\mathbf{h}_{\mathbf{E}}(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})\big) = \mathbf{h}_{\mathbf{E}}(\nabla^{\mathbf{E}}_{\widetilde{\mathbf{X}}}\widetilde{\mathbf{u}},\widetilde{\mathbf{v}}) + \mathbf{h}_{\mathbf{E}}(\widetilde{\mathbf{u}},\nabla^{\mathbf{E}}_{\widetilde{\mathbf{X}}}\widetilde{\mathbf{v}}).$$

Upon restriction to ∂M, the left-hand side becomes $\mathbf{X}\big(\mathbf{h}_{\mathbf{E}\restriction_{\partial M}}(\mathbf{u},\mathbf{v})\big)$, while the right-hand side becomes

$$\mathbf{h}_{\mathbf{E}\restriction_{\partial M}}(\nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{X}}\mathbf{u},\mathbf{v}) + \mathbf{h}_{\mathbf{E}\restriction_{\partial M}}(\mathbf{u},\nabla^{\mathbf{E}\restriction_{\partial M}}_{\mathbf{X}}\mathbf{v}).$$

Therefore, $\nabla^{\mathbf{E}\restriction_{\partial M}}$ is compatible with the restricted bundle metric. □

Henceforth, when we write $W^{m,p}(\partial M,\mathbf{E}\restriction_{\partial M})$, we understand that this space is constructed on the Riemannian manifold ∂M using the restricted bundle $\mathbf{E}\restriction_{\partial M}$, the restricted bundle metric $\mathbf{h}_{\mathbf{E}\restriction_{\partial M}}$, and the induced connection $\nabla^{\mathbf{E}\restriction_{\partial M}}$.

9.4.1 The trace operator

In a boundary chart, restriction of a section to ∂M is expressed through the trace of its components on the flat boundary. We use Theorem 3.7.1, Lemma 3.6.8, and Corollary 3.7.3, proved in the section on Euclidean traces in Chapter 3. In particular, components localized in a half-ball are extended by zero within $\mathbb{R}^n_+$ before taking the trace. We retain the convention fixed there for writing $\gamma_0 D^\beta v$ on the flat face of a half-ball.

Before stating the global result, recall that, when we write $W^{k,p}(\partial M,\mathbf{E}\restriction_{\partial M})$, we use the induced connection on the restricted bundle $\mathbf{E}\restriction_{\partial M} \to \partial M$, constructed in Lemma 9.4.11. This matters because the trace of a section in $\mathbf{E}$ will take values in a Sobolev space on $\mathbf{E}\restriction_{\partial M}$.

Theorem 9.4.12. (Integer-order trace theorem for vector bundles) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $m\geq 1$ and $1\leq p<\infty$. Then the restriction operator

$$\gamma_0\colon \Gamma(\mathbf{E})\longrightarrow \Gamma(\mathbf{E}\restriction_{\partial M}),\qquad \gamma_0(\mathbf{u})=\mathbf{u}\restriction_{\partial M},$$

admits a unique continuous linear extension

$$\mathbf{Tr}\colon W^{m,p}(M,\mathbf{E})\longrightarrow W^{m-1,p}(\partial M,\mathbf{E}\restriction_{\partial M}).$$

In particular, there is a constant $C>0$ such that

$$\|\mathbf{Tr}(\mathbf{u})\|_{W^{m-1,p}(\partial M,\mathbf{E}\restriction_{\partial M})}\leq C\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$$

for every $\mathbf{u}\in W^{m,p}(M,\mathbf{E})$.

Proof. We first prove the estimate for smooth sections. Since ∂M is compact, take a finite family of regular coordinate half-balls $(U_\alpha,\varphi_\alpha)_{\alpha=1}^N$ covering ∂M. Write $B_\alpha^+:=\varphi_\alpha(U_\alpha\cap\operatorname{Int}(M))$ and $B'_\alpha:=\varphi_\alpha(U_\alpha\cap\partial M)$. Refining if necessary, we may assume that $\overline{U_\alpha}$ lies in the domain of a smooth local frame $(\mathbf{e}_{1,\alpha},\dots,\mathbf{e}_{r,\alpha})$ trivializing $\mathbf{E}$. Take a smooth partition of unity $(\psi_\alpha)_{\alpha=1}^N$, defined on a neighborhood of ∂M, subordinate to the charts U_α and such that $\operatorname{supp}\psi_\alpha\Subset U_\alpha$.

Let $\mathbf{u}\in\Gamma(\mathbf{E})$. On U_α write $\mathbf{u}=\sum_{a=1}^r u_\alpha^a\mathbf{e}_{a,\alpha}$. Then $\psi_\alpha\mathbf{u}=\sum_{a=1}^r\psi_\alpha u_\alpha^a\mathbf{e}_{a,\alpha}$. For each component, define $f_\alpha^a:=(\psi_\alpha u_\alpha^a)\circ\varphi_\alpha^{-1}$ on B_α^+. Since $\operatorname{supp}\psi_\alpha\Subset U_\alpha$, the function f_α^a vanishes on a neighborhood of the spherical part of B_α^+. Let $\widetilde{f}_\alpha^a$ be its extension by zero to $\mathbb{R}^n_+$. By Lemma 3.6.8, $\widetilde{f}_\alpha^a\in W^{m,p}(\mathbb{R}^n_+)$, and moreover $\|\widetilde{f}_\alpha^a\|_{W^{m,p}(\mathbb{R}^n_+)}=\|f_\alpha^a\|_{W^{m,p}(B_\alpha^+)}$. Applying Theorem 3.7.1 to $\widetilde{f}_\alpha^a$, and using continuity of restriction from $W^{m-1,p}(\mathbb{R}^{n-1})$ to $W^{m-1,p}(B'_\alpha)$, gives

$$\left\|(\gamma_0\widetilde{f}_\alpha^a)\restriction_{B'_\alpha}\right\|_{W^{m-1,p}(B'_\alpha)}\leq C_\alpha\|f_\alpha^a\|_{W^{m,p}(B_\alpha^+)}.$$

For smooth $\mathbf{u}$, $(\gamma_0\widetilde{f}_\alpha^a)\restriction_{B'_\alpha}$ agrees with $(f_\alpha^a)\restriction_{B'_\alpha}$, since $\widetilde{f}_\alpha^a=f_\alpha^a$ near B'_α within the half-space. Summing over a and using local equivalence of norms in the regular coordinate half-ball gives

$$\sum_{a=1}^r\left\|(\gamma_0\widetilde{f}_\alpha^a)\restriction_{B'_\alpha}\right\|_{W^{m-1,p}(B'_\alpha)}\leq C'_\alpha\|\psi_\alpha\mathbf{u}\|_{W^{m,p}(U_\alpha,\mathbf{E}\restriction_{U_\alpha})}.$$

On the other hand, applying local equivalence of norms on the manifold without boundary ∂M to the restricted bundle $\mathbf{E}\restriction_{\partial M}\to\partial M$ gives

$$\|(\psi_\alpha\mathbf{u})\restriction_{\partial M}\|_{W^{m-1,p}(U_\alpha\cap\partial M,\mathbf{E}\restriction_{U_\alpha\cap\partial M})}\leq C''_\alpha\sum_{a=1}^r\left\|(\gamma_0\widetilde{f}_\alpha^a)\restriction_{B'_\alpha}\right\|_{W^{m-1,p}(B'_\alpha)}.$$

Therefore,

$$\|(\psi_\alpha\mathbf{u})\restriction_{\partial M}\|_{W^{m-1,p}(U_\alpha\cap\partial M,\mathbf{E}\restriction_{U_\alpha\cap\partial M})}\leq C'''_\alpha\|\psi_\alpha\mathbf{u}\|_{W^{m,p}(U_\alpha,\mathbf{E}\restriction_{U_\alpha})}.$$

Now using Lemma 9.4.9, applied to the fixed function ψ_α, there is a constant $C''''_\alpha>0$ such that

$\|\psi_\alpha \mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \leq C''''_\alpha \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$. Since restricting the global norm to U_α gives a value no greater than the global norm, we conclude that

$$\|(\psi_\alpha \mathbf{u}) \restriction_{\partial M}\|_{W^{m-1,p}(U_\alpha \cap \partial M, \mathbf{E}\restriction_{U_\alpha \cap \partial M})} \leq C''''_\alpha \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

Since there are finitely many charts and $\sum_{\alpha=1}^{N} \psi_\alpha = 1$ on a neighborhood of ∂M, we have $\mathbf{u} \restriction_{\partial M} = \sum_{\alpha=1}^{N} (\psi_\alpha \mathbf{u}) \restriction_{\partial M}$. Using the triangle inequality and local equivalence of norms on $\mathbf{E} \restriction_{\partial M}$ gives a constant $C > 0$ such that

$$\|\mathbf{u} \restriction_{\partial M}\|_{W^{m-1,p}(\partial M, \mathbf{E}\restriction_{\partial M})} \leq C \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$$

for every $\mathbf{u} \in \Gamma(\mathbf{E})$.

The preceding estimate shows that restriction is a continuous linear operator from $\Gamma(\mathbf{E}) \cap W^{m,p}(M,\mathbf{E})$ to $W^{m-1,p}(\partial M, \mathbf{E} \restriction_{\partial M})$. Theorem 9.4.8 states that its domain is dense in $W^{m,p}(M,\mathbf{E})$, while the target space is Banach. Theorem B.2.36 therefore gives a unique continuous linear extension

$$\mathbf{Tr}\colon W^{m,p}(M,\mathbf{E}) \longrightarrow W^{m-1,p}(\partial M, \mathbf{E} \restriction_{\partial M})$$

and preserves the estimate

$$\|\mathbf{Tr}(\mathbf{u})\|_{W^{m-1,p}(\partial M, \mathbf{E}\restriction_{\partial M})} \leq C \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

□

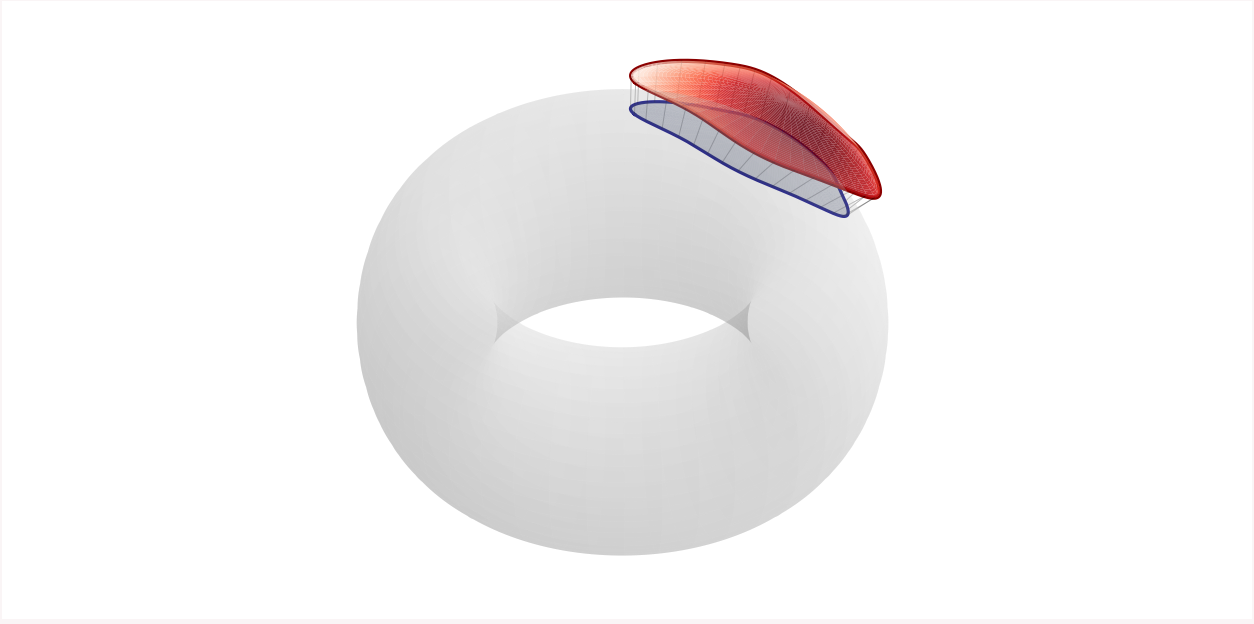


Figure 9.1. The trace theorem generalizes the idea of restricting a smooth function to the boundary of a manifold. Here $\Omega \subseteq T^2$ is an open subset of the torus whose closure $\overline{\Omega}$ is a manifold with boundary. The blue region represents Ω, while its boundary $\partial\Omega$ is shown in dark blue. The graph of the function $u\colon \overline{\Omega} \longrightarrow \mathbb{R}$ is shown in red and its trace on $\partial\Omega$ in dark red.

9.4.2 Intrinsic Sobolev–Slobodeckij spaces and the exact trace

The integer-order trace theorem gives an estimate in $W^{m-1,p}(\partial M, \mathbf{E}\restriction_{\partial M})$, but this regularity does not fully describe the image. Restriction to a hypersurface loses exactly $\frac{1}{p}$ derivatives. To formulate this assertion without choosing charts or frames, we compare values of a section in different fibers by parallel transport for the induced connection on the boundary.

The construction is presented first on a closed manifold of positive dimension, since ∂M is such a manifold. Let N be a compact Riemannian manifold without boundary, and let $\mathbf{F} \longrightarrow N$ be a smooth vector bundle with a bundle metric and a compatible connection. If N is disconnected, the following definitions apply to the restrictions of these data to each component; compactness ensures that there are only finitely many. For $s > 0$ and $1 \leq p < \infty$, if $N = N_1 \sqcup \cdots \sqcup N_q$, the global norm is defined by

$$\|\mathbf{v}\|_{W^{s,p}(N,\mathbf{F})} := \left(\sum_{\ell=1}^{q} \|\mathbf{v}\restriction_{N_\ell}\|^p_{W^{s,p}(N_\ell, \mathbf{F}\restriction_{N_\ell})} \right)^{\frac{1}{p}}.$$

In dimension zero, set $W^{s,p}(N,\mathbf{F}) := L^p(N,\mathbf{F})$ for every $s > 0$; this is then a finite sum of fibers, with no difference seminorm.

Definition 9.4.13. (Intrinsic Gagliardo seminorm for sections) Let $(N, \mathbf{h})$ be a compact connected Riemannian manifold without boundary of dimension $d \geq 1$, and let $\mathbf{F} \longrightarrow N$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{F}$ and a compatible connection $\nabla^\mathbf{F}$. Set

$$\rho_N := \min\{1, \mathrm{inj}(N)\}.$$

The compact manifold N is complete, and Proposition 2.13.15 shows that the pointwise injectivity radius is continuous and positive. Its minimum on N is therefore strictly positive. For $0 < \sigma < 1$, $1 \leq p < \infty$, and $\mathbf{v} \in L^p(N,\mathbf{F})$ define

$$[\mathbf{v}]_{W^{\sigma,p}(N,\mathbf{F})} := \left(\int_N \int_{\{y \in N \,|\, 0 < d_\mathbf{h}(x,y) < \rho_N\}} \frac{|\mathbf{v}(x) - \mathbf{v}(y)|^p_{\mathbf{h}_\mathbf{F};x,y}}{d_\mathbf{h}(x,y)^{d+\sigma p}} \, d\lambda_\mathbf{h}(y) \, d\lambda_\mathbf{h}(x) \right)^{\frac{1}{p}}. \tag{9.69}$$

The fiberwise separation in the numerator is that of Definition 9.2.4; thus

$$|\mathbf{v}(x) - \mathbf{v}(y)|_{\mathbf{h}_\mathbf{F};x,y} = \left| \mathbf{v}(x) - P^\mathbf{F}_{\gamma_{xy},1\to 0}\mathbf{v}(y) \right|_{\mathbf{h}_\mathbf{F}(x)},$$

where γ_{xy} is the unique short minimizing geodesic from x to y. Transport depends smoothly on (x,y) in the region $0 < d_\mathbf{h}(x,y) < \rho_N$, so the integrand is measurable. Changing a representative on a null set $Z \subseteq N$ changes the integrand only on $(Z \times N) \cup (N \times Z)$, which has zero product measure by Theorem B.6.17. Define

$$W^{\sigma,p}(N,\mathbf{F}) := \left\{ \mathbf{v} \in L^p(N,\mathbf{F}) \,\middle|\, [\mathbf{v}]_{W^{\sigma,p}(N,\mathbf{F})} < \infty \right\}$$

and equip it with the norm

$$\|\mathbf{v}\|_{W^{\sigma,p}(N,\mathbf{F})} := \|\mathbf{v}\|_{L^p(N,\mathbf{F})} + [\mathbf{v}]_{W^{\sigma,p}(N,\mathbf{F})}.$$

If $s = k + \sigma$, with $k \in \mathbb{N}_0$ and $0 < \sigma < 1$, set

$$\mathbf{F}_k := T^{(0,k)}(TN) \otimes \mathbf{F}$$

with its product metric and connection, and define

$$W^{s,p}(N,\mathbf{F}) := \left\{ \mathbf{v} \in W^{k,p}(N,\mathbf{F}) \,\middle|\, [\nabla_w^k \mathbf{v}]_{W^{\sigma,p}(N,\mathbf{F}_k)} < \infty \right\}, \tag{9.70}$$

with

$$\|\mathbf{v}\|_{W^{s,p}(N,\mathbf{F})} := \|\mathbf{v}\|_{W^{k,p}(N,\mathbf{F})} + [\nabla_w^k \mathbf{v}]_{W^{\sigma,p}(N,\mathbf{F}_k)}. \tag{9.71}$$

For $s \in \mathbb{N}_0$, retain the preceding intrinsic definition of $W^{s,p}(N,\mathbf{F})$ using weak covariant derivatives.

The seminorm of order $0 < \sigma < 1$ is the version, for a fixed smooth connection, of the covariant fractional spaces considered in [127, Definition 6.4]; higher orders are obtained by applying it to $\nabla_w^k \mathbf{v}$ with the product connection. Its relationship with Euclidean theory is established by the following localized characterization.

Lemma 9.4.14. (Independence of the comparison radius) Let $(N,\mathbf{h})$ be a compact connected Riemannian manifold without boundary of dimension $d \geq 1$, and let $\mathbf{F} \to N$ be a smooth vector bundle with a bundle metric and a compatible connection. Let $0 < \sigma < 1$, $1 \leq p < \infty$, and $0 < r_1 \leq r_2 \leq \operatorname{inj}(N)$. Denote by $[\mathbf{v}]_{\sigma,p;r_i}$ the expression in (9.69) with r_i in place of ρ_N. There are constants $c, C > 0$, independent of v, such that

$$c\left(\|\mathbf{v}\|_{L^p(N,\mathbf{F})} + [\mathbf{v}]_{\sigma,p;r_1}\right) \leq \|\mathbf{v}\|_{L^p(N,\mathbf{F})} + [\mathbf{v}]_{\sigma,p;r_2} \leq C\left(\|\mathbf{v}\|_{L^p(N,\mathbf{F})} + [\mathbf{v}]_{\sigma,p;r_1}\right).$$

Proof. The first inequality follows from inclusion of the regions of integration. In the region $r_1 \leq d_{\mathbf{h}}(x,y) < r_2$, the isometry property of parallel transport and the inequality $|a-b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ give

$$\frac{|\mathbf{v}(x) - \mathbf{v}(y)|^p_{\mathbf{h}_\mathbf{F};x,y}}{d_{\mathbf{h}}(x,y)^{d+\sigma p}} \leq 2^{p-1} r_1^{-d-\sigma p} \left(|\mathbf{v}(x)|^p_{\mathbf{h}_\mathbf{F}} + |\mathbf{v}(y)|^p_{\mathbf{h}_\mathbf{F}}\right).$$

Integrating over $N \times N$ and using $\lambda_{\mathbf{h}}(N) < \infty$ bounds the contribution of this region by $C\|\mathbf{v}\|^p_{L^p(N,\mathbf{F})}$. This proves the second inequality. □

The following Euclidean fact will control the lower-order terms that arise when a covariant derivative is expressed in a frame.

Lemma 9.4.15. (Fractional control by one integer-order derivative) Let $0 < \sigma < 1$ and $1 \leq p < \infty$. There is $C > 0$ such that

$$\|f\|_{W^{\sigma,p}(\mathbb{R}^d)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^d)}$$

for every $f \in W^{1,p}(\mathbb{R}^d)$. Moreover, if $a \in C_c^\infty(\mathbb{R}^d)$, then

$$\|af\|_{W^{\sigma,p}(\mathbb{R}^d)} \leq C_a\|f\|_{W^{\sigma,p}(\mathbb{R}^d)}.$$

Proof. By Proposition 6.2.2 and Lemma 6.2.4, for $\|z\| < 1$ we have

$$\|\tau_z f - f\|^p_{L^p(\mathbb{R}^d)} \le \|z\|^p \|Df\|^p_{L^p(\mathbb{R}^d,\mathbb{R}^d)},$$

while for $\|z\| \ge 1$ we use $\|\tau_z f - f\|^p_{L^p(\mathbb{R}^d)} \le 2^p \|f\|^p_{L^p(\mathbb{R}^d)}$. The resulting radial integrals are

$$\int_0^1 r^{p(1-\sigma)-1}\, dr < \infty, \qquad \int_1^\infty r^{-\sigma p-1}\, dr < \infty.$$

This proves the first assertion.
For the second, write

$$a(x)f(x) - a(y)f(y) = a(x)(f(x) - f(y)) + (a(x) - a(y))f(y).$$

The first term is controlled by $\|a\|_{L^\infty(\mathbb{R}^d)}[f]_{W^{\sigma,p}(\mathbb{R}^d)}$. For the second, use

$$|a(x) - a(y)| \le C_a \min\{1, \|x - y\|\};$$

the same separation into $\|x-y\| < 1$ and $\|x-y\| \ge 1$ gives a bound by $C_a\|f\|_{L^p(\mathbb{R}^d)}$. Together with the estimate L^p, this yields the result. □

Lemma 9.4.16. (Local comparison of transport and components) Let $(N, \mathbf{h})$ be a compact connected Riemannian manifold without boundary, and let $\mathbf{F} \to N$ be a smooth vector bundle with a bundle metric and a compatible connection. Let (U, φ) be a regular coordinate ball in N. Suppose that there is a geodesically convex normal ball U' such that $\overline{U} \subseteq U'$, $\operatorname{diam}_{\mathbf{h}}(U') < \rho_N$, and both φ and the smooth frame $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ of $\mathbf{F}$ extend to U'. Write $\mathbf{v} = \sum_{a=1}^{r} v^a \mathbf{e}_a$ on U. If $0 < \sigma < 1$ and $1 \le p < \infty$, there are constants $c, C > 0$, independent of $\mathbf{v}$, such that

$$\begin{aligned} c \sum_{a=1}^{r} \|v^a \circ \varphi^{-1}\|_{W^{\sigma,p}(\varphi(U))} &\le \|\mathbf{v}\|_{L^p(U,\mathbf{F})} + [\mathbf{v}]_{W^{\sigma,p}(U,\mathbf{F})} \\ &\le C \sum_{a=1}^{r} \|v^a \circ \varphi^{-1}\|_{W^{\sigma,p}(\varphi(U))}, \end{aligned} \tag{9.72}$$

where the middle seminorm integrates only over pairs of points in U.

Proof. Set $\Omega := \varphi(U)$. In the chart and frame, transport from y to x is represented by a matrix $G(x, y)$. If $c_{xy} := \varphi \circ \gamma_{xy}$, the local parallel transport equation, Proposition 2.7.18, is a linear equation whose coefficient matrices are the connection matrices evaluated along $\dot{c}_{xy}$. Those matrices are bounded on the compact set under consideration, and the length of c_{xy} is comparable to $d_{\mathbf{h}}(x, y)$. The integral form of the equation and Grönwall's inequality in Lemma 1.8.19 therefore give

$$\|G(x, y) - I\| \le C_0 d_{\mathbf{h}}(x, y). \tag{9.73}$$

The same estimate holds for $G(x, y)^{-1} = G(y, x)$.
Uniform equivalence of the bundle norm and the component norm on $\overline{U}$ allows us to deduce from (9.73)

the two inequalities

$$|\mathbf{v}(x)-\mathbf{v}(y)|_{\mathbf{h_F};x,y} \le C_1\big(|v^\flat(x)-v^\flat(y)|+d_{\mathbf{h}}(x,y)|v^\flat(y)|\big), \tag{9.74}$$

$$|v^\flat(x)-v^\flat(y)| \le C_1\big(|\mathbf{v}(x)-\mathbf{v}(y)|_{\mathbf{h_F};x,y}+d_{\mathbf{h}}(x,y)|v^\flat(y)|\big), \tag{9.75}$$

where $v^\flat=(v^1,\dots,v^r)$. On a coordinate ball with compact closure there are constants $c_2, C_2>0$ such that

$$c_2\|\varphi(x)-\varphi(y)\| \le d_{\mathbf{h}}(x,y) \le C_2\|\varphi(x)-\varphi(y)\|, \qquad c_2\,dz \le d\lambda_{\mathbf{h}} \le C_2\,dz.$$

Divide the pth powers in (9.74) and (9.75) by $d_{\mathbf{h}}(x,y)^{d+\sigma p}$ and integrate. The error term is controlled by $\|\mathbf{v}\|^p_{L^p(U,\mathbf{F})}$, since, uniformly in y,

$$\int_{\varphi(U)} \frac{\|z-w\|^p}{\|z-w\|^{d+\sigma p}}\,dz \le C\int_0^{C_2\operatorname{diam}(U)} r^{p(1-\sigma)-1}\,dr<\infty.$$

Comparison of the L^p norms completes both inequalities in (9.72). □

Lemma 9.4.17. (Multiplication by smooth functions) Let $(N,\mathbf{h})$ be a compact connected Riemannian manifold without boundary, and let $\mathbf{F}\to N$ be a smooth vector bundle with a bundle metric and a compatible connection. Let $s>0$, $s\notin\mathbb{N}$, $1\le p<\infty$, and $\chi\in C^\infty(N)$. There is a constant $C_{\chi,s,p}>0$ such that

$$\|\chi\mathbf{v}\|_{W^{s,p}(N,\mathbf{F})} \le C_{\chi,s,p}\|\mathbf{v}\|_{W^{s,p}(N,\mathbf{F})}.$$

Proof. First suppose that $0<s<1$. Compatibility of parallel transport with scalar multiplication gives

$$\begin{aligned}|\chi(x)\mathbf{v}(x)-\chi(y)\mathbf{v}(y)|_{\mathbf{h_F};x,y} \le{}& |\chi(x)|\,|\mathbf{v}(x)-\mathbf{v}(y)|_{\mathbf{h_F};x,y}\\ &+|\chi(x)-\chi(y)|\,|\mathbf{v}(y)|_{\mathbf{h_F}}.\end{aligned}$$

Since N is compact, χ is bounded and Lipschitz. The second term is integrated using $\int_0^{\rho_N} r^{p(1-s)-1}\,dr<\infty$, exactly as in the preceding proof.

Now let $s=k+\sigma$, with $k\ge 1$. The iterated Leibniz rule expresses $\nabla^k_w(\chi\mathbf{v})$ as a finite sum of contractions of $\nabla^{k-j}\chi$ with $\nabla^j_w\mathbf{v}$. The term $j=k$ is controlled by the case of order σ applied to $\nabla^k_w\mathbf{v}$. If $j<k$, then $\nabla^j_w\mathbf{v}\in W^{1,p}(N,\mathbf{F}_j)$. Let us justify this assertion and, at the same time, the Leibniz rule just used. By Theorem 9.1.48, there is a sequence of smooth sections $(\mathbf{v}_\nu)$ converging to $\mathbf{v}$ in $W^{k,p}(N,\mathbf{F})$. For $j<k$, the identities

$$\nabla^j\mathbf{v}_\nu \longrightarrow \nabla^j_w\mathbf{v}, \qquad \nabla(\nabla^j\mathbf{v}_\nu)=\nabla^{j+1}\mathbf{v}_\nu \longrightarrow \nabla^{j+1}_w\mathbf{v}$$

hold in the corresponding spaces L^p. Lemma 9.1.12, applied to the bundle $\mathbf{F}_j$, shows that $\nabla^j_w\mathbf{v}$ belongs to $W^{1,p}(N,\mathbf{F}_j)$ and its first weak derivative is $\nabla^{j+1}_w\mathbf{v}$. The Leibniz formula for $\mathbf{v}_\nu$ passes to the limit in L^p because its coefficients are the smooth bounded derivatives of χ; this establishes the same formula for $\mathbf{v}$.

Now localize in a finite cover and apply a fixed cutoff equal to one on the support of each localization. The cut-off components, extended by zero, belong to $W^{1,p}(\mathbb{R}^d)$; Lemma 9.4.15 then applies on $\mathbb{R}^d$. The cross term from the extension is controlled by the positive distance of the support from the chart boundary. Lemma 9.4.16 allows us to return to the intrinsic seminorm. In each frame, contractions with $\nabla^{k-j}\chi$ are

finite sums of multiplications by smooth coefficients; the second part of the same Euclidean lemma controls them. The derivatives of χ are bounded and the family of charts is finite, so every term is controlled by $C\|\mathbf{v}\|_{W^{s,p}(N,\mathbf{F})}$. The integer-order part of the norm is controlled directly by the same Leibniz rule, the uniform bounds on the derivatives of χ, and the definition of $W^{k,p}(N,\mathbf{F})$. This completes the estimate of the integer-order norm and the fractional seminorm appearing in (9.71). □

Proposition 9.4.18. (Localized characterization of the intrinsic norm) Let $(N,\mathbf{h})$ be a compact connected Riemannian manifold without boundary of dimension $d \geq 1$, and let $\mathbf{F} \to N$ be a smooth vector bundle of rank r with a bundle metric and a compatible connection. Let $s > 0$, $s \notin \mathbb{N}$, and $1 \leq p < \infty$. Choose a finite cover $(U_i,\varphi_i)_{i=1}^N$ of N by regular coordinate balls as in Lemma 9.4.16, a partition of unity $(\psi_i)_{i=1}^N$ with $\operatorname{supp}\psi_i \Subset U_i$, and frames $(\mathbf{e}_{i,1},\dots,\mathbf{e}_{i,r})$ defined on neighborhoods of $\overline{U_i}$. Write

$$\psi_i\mathbf{v} = \sum_{a=1}^{r} v_i^a \mathbf{e}_{i,a} \qquad \text{in } U_i$$

and denote by $\widetilde{v}_i^a$ the extension of $v_i^a \circ \varphi_i^{-1}$ by zero to $\mathbb{R}^d$. There are constants $c,C > 0$, independent of $\mathbf{v}$, such that

$$c\|\mathbf{v}\|_{W^{s,p}(N,\mathbf{F})} \leq \sum_{i=1}^{N}\sum_{a=1}^{r} \|\widetilde{v}_i^a\|_{W^{s,p}(\mathbb{R}^d)} \leq C\|\mathbf{v}\|_{W^{s,p}(N,\mathbf{F})}. \tag{9.76}$$

In particular, the intrinsically defined space is independent of the charts, frames, and partition used to compute the localized norm.

Proof. First consider $0 < s < 1$. Lemma 9.4.17 controls $\psi_i\mathbf{v}$ by $\mathbf{v}$, and Lemma 9.4.16 controls its components inside $\varphi_i(U_i)$. Since $K_i := \varphi_i(\operatorname{supp}\psi_i) \Subset \varphi_i(U_i)$, extension by zero creates no singularity at the chart boundary. If $z \in K_i$ and $w \notin \varphi_i(U_i)$, then $\|z-w\| \geq \delta_i > 0$, and the cross part of the seminorm is bounded by

$$2\int_{K_i} |(v_i^a \circ \varphi_i^{-1})(z)|^p \left(\int_{\mathbb{R}^d\setminus\varphi_i(U_i)} \frac{dw}{\|z-w\|^{d+sp}}\right) dz \leq C\delta_i^{-sp}\|v_i^a \circ \varphi_i^{-1}\|^p_{L^p(\varphi_i(U_i))}.$$

The last inequality uses polar coordinates centered at z; the radial integral is $\displaystyle\int_{\delta_i}^{\infty} r^{-sp-1}\,dr = (sp)^{-1}\delta_i^{-sp}$. This proves the second inequality in (9.76).

For the reverse inequality, use $\mathbf{v} = \displaystyle\sum_{i=1}^{N} \psi_i\mathbf{v}$. The triangle inequality reduces the estimate to each section $\psi_i\mathbf{v}$. Pairs of points contained in U_i are controlled by (9.72). If one point belongs to the support of ψ_i and the other to $N \setminus U_i$, their distance is at least $\operatorname{dist}_{\mathbf{h}}(\operatorname{supp}\psi_i, N\setminus U_i) > 0$; this part is controlled by the L^p norm, as in Lemma 9.4.14. This proves the first inequality when $0 < s < 1$.

Let $s = k + \sigma$, with $k \geq 1$. In each chart, formula (9.39), applied to the localized section $\psi_i\mathbf{v}$, gives the triangular expression

$$(\nabla_w^k(\psi_i\mathbf{v}))^a_{i_1\dots i_k} = \partial_{i_k}\cdots\partial_{i_1} v_i^a + \sum_{b=1}^{r}\sum_{|\beta|\leq k-1} (B_{k,\beta})^a_{b\,i_1\dots i_k}\partial^\beta v_i^b. \tag{9.77}$$

The coefficients and all derivatives used are bounded on the fixed compact sets. Lemma 9.4.15 controls each derivative of order at most $k-1$ in $W^{\sigma,p}$ by the $W^{k,p}$ norm, and its second assertion permits multiplication by the coefficients $B_{k,\beta}$ after inserting a cutoff equal to one on the support under consideration. Consequently,

(9.77) and the formula obtained by solving for the partial derivatives of order k give, in both directions,

$$\sum_{a=1}^{r} \|\widetilde{v}_i^a\|_{W^{k+\sigma,p}(\mathbb{R}^d)} \le C\|\psi_i \mathbf{v}\|_{W^{k+\sigma,p}(N,\mathbf{F})}$$

and

$$\|\psi_i \mathbf{v}\|_{W^{k+\sigma,p}(N,\mathbf{F})} \le C \sum_{a=1}^{r} \|\widetilde{v}_i^a\|_{W^{k+\sigma,p}(\mathbb{R}^d)}.$$

Identity (9.39) was established directly for weak covariant derivatives; thus (9.77) holds almost everywhere for the given section. Its inverse form is obtained by algebraically solving for the leading partial derivatives, without approximation in the fractional norm. Comparison of the fractional part of the components of $\nabla_w^k(\psi_i \mathbf{v})$ with its intrinsic seminorm follows by applying Lemma 9.4.16 to the bundle $\mathbf{F}_k$ for pairs contained in U_i. The section $\nabla_w^k(\psi_i \mathbf{v})$ is supported in $\operatorname{supp}\psi_i \Subset U_i$; if only one of the two points is in U_i, the positive distance from the complement controls this part of the seminorm by the L^p norm, exactly as in the case $0 < s < 1$. This gives the full intrinsic seminorm. Summing over the finite family of charts yields (9.76) for every k. □

Corollary 9.4.19. (Functional structure and density) Let $(N, \mathbf{h})$ be a compact connected Riemannian manifold without boundary, and let $\mathbf{F} \to N$ be a smooth vector bundle with a bundle metric and a compatible connection. Let $s > 0$ and $1 \le p < \infty$. Then $W^{s,p}(N, \mathbf{F})$ is a Banach space. It is separable for $1 \le p < \infty$, reflexive for $1 < p < \infty$, and $\Gamma(\mathbf{F})$ is dense in it.

Proof. Proposition 9.4.18 identifies the norm with a finite norm of Euclidean localizations. If $(\mathbf{v}_j)$ is Cauchy, each localized component converges in the Banach space $W^{s,p}(\mathbb{R}^d)$ by Theorem 6.2.3. The change-of-frame relations and the identity $\sum_{i=1}^{N} \psi_i = 1$, valid for each $\mathbf{v}_j$, pass to the limit in L^p and reconstruct a section $\mathbf{v} \in L^p(N, \mathbf{F})$. The first inequality in (9.76) shows that $\mathbf{v}_j \to \mathbf{v}$ in $W^{s,p}(N, \mathbf{F})$.
For density, approximate each $\widetilde{v}_i^a$ in $W^{s,p}(\mathbb{R}^d)$ by functions in $C_c^\infty(\mathbb{R}^d)$ using the same theorem, and multiply the approximations by a fixed cutoff supported in $\varphi_i(U_i)$ and equal to one on K_i. Reconstruction in the frame $\mathbf{e}_{i,a}$ and summation over i, a give smooth sections; the localized estimate proves convergence. Separability follows from this same finite representation. For $1 < p < \infty$, the map

$$\mathbf{v} \longmapsto \left(\mathbf{v}, \frac{\mathbf{v}(x) - P^{\mathbf{F}}_{\gamma_{xy},1\to 0}\mathbf{v}(y)}{d_{\mathbf{h}}(x,y)^{\frac{d}{p}+\sigma}} \right)$$

realizes the case $0 < s < 1$ as a closed subspace of a product of reflexive L^p spaces. The image is closed because the norm induced by that product is equivalent to the norm already proved complete. For $s = k + \sigma$, consider in the same product the vector formed by $\mathbf{v}, \nabla_w \mathbf{v}, \dots, \nabla_w^k \mathbf{v}$ and the parallel difference quotient of $\nabla_w^k \mathbf{v}$. Closedness of weak covariant derivatives and the preceding completeness again show that the image is closed. A closed subspace of a finite product of reflexive L^p spaces is reflexive. □

Remark 9.4.20. (Dependence on the connection) The seminorm in (9.69) records the connection and is therefore geometrically natural. Nevertheless, two smooth compatible connections on the same compact bundle yield the same space with equivalent norms. Indeed, Proposition 9.4.18 compares both norms with

the same Euclidean localizations; the constants may differ because they depend on the local connection matrices.

We now apply this construction to $N = \partial M$ and $\mathbf{F} = \mathbf{E} \restriction_{\partial M}$, with the induced metric and the connection from Lemma 9.4.11. The resulting fractional regularity is therefore intrinsic to the geometric data of the boundary.

Theorem 9.4.21. (Exact intrinsic trace for vector bundles) Let $(M, \mathbf{g})$ be a compact Riemannian manifold with nonempty smooth boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. If $m \in \mathbb{N}$ and $1 < p < \infty$, restriction of smooth sections extends uniquely to a continuous surjective linear operator

$$\mathbf{Tr} : W^{m,p}(M, \mathbf{E}) \longrightarrow W^{m-\frac{1}{p},p}(\partial M, \mathbf{E} \restriction_{\partial M}). \tag{9.78}$$

There is a continuous linear operator

$$\mathbf{Ex} : W^{m-\frac{1}{p},p}(\partial M, \mathbf{E} \restriction_{\partial M}) \longrightarrow W^{m,p}(M, \mathbf{E}) \tag{9.79}$$

such that $\mathbf{TrEx} = I$. In particular, there are constants $C_1, C_2 > 0$ such that

$$\|\mathbf{Tru}\|_{W^{m-\frac{1}{p},p}(\partial M, \mathbf{E}\restriction_{\partial M})} \leq C_1 \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$$

and

$$\|\mathbf{Exv}\|_{W^{m,p}(M,\mathbf{E})} \leq C_2 \|\mathbf{v}\|_{W^{m-\frac{1}{p},p}(\partial M, \mathbf{E}\restriction_{\partial M})}.$$

Proof. If $\dim M = 1$, the boundary is a finite set and, under the preceding convention, the target space is the finite sum of the fibers of $\mathbf{E}$ over its points. On each interval component, the fundamental theorem of calculus and Hölder's inequality bound endpoint evaluation by the $W^{1,p}$ norm, and therefore by the $W^{m,p}$ norm. Choosing smooth sections supported in disjoint neighborhoods of the endpoints with prescribed values there gives a linear coretraction; its continuity is automatic because the target space is finite-dimensional. For the rest of the proof, assume that $\dim M \geq 2$.

Set $s := m - \frac{1}{p} = (m-1) + (1 - \frac{1}{p})$; since $1 < p < \infty$, its fractional part belongs to $(0,1)$. Choose the regular half-balls, frames, and finite partition $(\psi_\alpha)_{\alpha=1}^N$ from the proof of Theorem 9.4.12, refined so that the tangential charts of ∂M satisfy the hypotheses of Proposition 9.4.18.

First let $\mathbf{u} \in \Gamma(\mathbf{E})$. In a chart write

$$\psi_\alpha \mathbf{u} = \sum_{a=1}^{r} f_\alpha^a \mathbf{e}_{\alpha,a}, \qquad v_\alpha^a := f_\alpha^a \circ \varphi_\alpha^{-1} \quad \text{in } B_\alpha^+.$$

Each v_α^a vanishes on a neighborhood of the spherical part. Its extension by zero $\widetilde{v}_\alpha^a$ to $\mathbb{R}_+^n$ belongs to $W^{m,p}(\mathbb{R}_+^n)$ by Lemma 3.6.8, and

$$\|\widetilde{v}_\alpha^a\|_{W^{m,p}(\mathbb{R}_+^n)} = \|v_\alpha^a\|_{W^{m,p}(B_\alpha^+)}.$$

Apply the operator $\mathcal{E}_m$ from Corollary 3.6.4. The identification $W^{m,p}(\mathbb{R}^n) = H^{m,p}(\mathbb{R}^n)$ in Theorem 6.3.10,

followed by Theorem 6.7.2, gives

$$\begin{aligned}\|\operatorname{tr}(\mathcal{E}_m \widetilde{v}_\alpha^a)\|_{B_{p,p}^{m-\frac{1}{p}}(\mathbb{R}^{n-1})} &\le C\|\mathcal{E}_m \widetilde{v}_\alpha^a\|_{H^{m,p}(\mathbb{R}^n)} \\ &\le C'\|v_\alpha^a\|_{W^{m,p}(B_\alpha^+)}.\end{aligned}$$

Since $m-\frac{1}{p}$ is not an integer, Theorem 6.3.14 identifies $B_{p,p}^{m-\frac{1}{p}}(\mathbb{R}^{n-1})$ with $W^{m-\frac{1}{p},p}(\mathbb{R}^{n-1})$. On the flat ball, the preceding trace agrees with $(v_\alpha^a)\restriction_{B'_\alpha}$. Two extensions of $\widetilde{v}_\alpha^a$ agreeing on the half-space have the same trace: approximate in $W^{m,p}(\mathbb{R}^n_+)$ by functions smooth up to the boundary and use continuity of both trace operators; equivalently, apply the locality recorded in Lemma 3.7.2. Since $\widetilde{v}_\alpha^a$ vanishes on a relative neighborhood of the complement of B'_α in the boundary of the half-space, the same lemma shows that the trace vanishes there. Thus $\operatorname{tr}(\mathcal{E}_m \widetilde{v}_\alpha^a)$ is exactly the extension by zero of the trace on B'_α.

The localized characterization in Proposition 9.4.18 now gives, with $s = m - \frac{1}{p}$,

$$\begin{aligned}\|\mathbf{u}\restriction_{\partial M}\|_{W^{s,p}(\partial M,\mathbf{E}\restriction_{\partial M})} &\le C \sum_{\substack{1\le\alpha\le N\\ 1\le a\le r}} \|\operatorname{tr}(\mathcal{E}_m \widetilde{v}_\alpha^a)\|_{W^{s,p}(\mathbb{R}^{n-1})} \\ &\le C \sum_{\substack{1\le\alpha\le N\\ 1\le a\le r}} \|v_\alpha^a\|_{W^{m,p}(B_\alpha^+)} \\ &\le C \sum_{\alpha=1}^{N} \|\psi_\alpha \mathbf{u}\|_{W^{m,p}(U_\alpha,\mathbf{E})} \\ &\le C\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.\end{aligned}$$

The third line uses local equivalence between partial and covariant derivatives; the fourth uses continuity of multiplication by each ψ_α and finiteness of the atlas. The constants can be chosen simultaneously because only finitely many charts, frames, and cutoffs are involved.

Theorem 9.4.8 and Corollary 9.4.19 allow restriction to extend uniquely and continuously to all of $W^{m,p}(M,\mathbf{E})$. The continuous inclusion

$$W^{m-\frac{1}{p},p}(\partial M,\mathbf{E}\restriction_{\partial M}) \hookrightarrow W^{m-1,p}(\partial M,\mathbf{E}\restriction_{\partial M})$$

follows directly from the definition. By uniqueness of extension from smooth sections, the resulting operator agrees with that in Theorem 9.4.12.

We construct the coretraction. Retain the preceding cover and choose a partition of unity (ϑ_α) on ∂M with $\operatorname{supp}\vartheta_\alpha \Subset U_\alpha \cap \partial M$. For $\mathbf{v} \in W^{m-\frac{1}{p},p}(\partial M,\mathbf{E}\restriction_{\partial M})$ write

$$\vartheta_\alpha \mathbf{v} = \sum_{a=1}^{r} g_\alpha^a \mathbf{e}_{\alpha,a}$$

and extend $g_\alpha^a \circ (\varphi_\alpha\restriction_{\partial M})^{-1}$ by zero to $\mathbb{R}^{n-1}$; denote this extension by $\widetilde{g}_\alpha^a$. Proposition 9.4.18 controls the sum of the $W^{m-\frac{1}{p},p}(\mathbb{R}^{n-1})$ norms of these extensions by the norm of $\mathbf{v}$.

Apply to each component the coretraction ex from Theorem 6.7.2. The result belongs to $H^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$. Choose a cutoff $\chi_\alpha \in C_c^\infty(\mathbb{R}^n)$ whose support, intersected with the upper half-space, lies in the coordinate region, and whose restriction to the hyperplane equals one on a neighborhood of the supports

of all localized components. Multiply $\operatorname{ex}\widetilde{g}_\alpha^a$ by χ_α, transport it to the chart, reconstruct it with the frame $\mathbf{e}_{\alpha,a}$, and extend it by zero outside U_α. Denote the resulting global section by $\mathbf{V}_\alpha$. The support of the factor $\chi_\alpha \operatorname{ex}\widetilde{g}_\alpha^a$ is separated from the artificial part of the chart boundary; thus its transport to U_α and extension by zero have weak derivatives through order m without boundary terms. This follows from the same compactly supported test argument as in Lemma 3.6.8. The Euclidean multiplier bounds and local equivalence of norms give

$$\|\mathbf{V}_\alpha\|_{W^{m,p}(M,\mathbf{E})} \le C_\alpha \sum_{a=1}^{r} \|\widetilde{g}_\alpha^a\|_{W^{m-\frac{1}{p},p}(\mathbb{R}^{n-1})}.$$

Define

$$\mathbf{Exv} := \sum_{\alpha=1}^{N} \mathbf{V}_\alpha.$$

The sum is finite, so the localized estimate proves continuity. Since $\chi_\alpha = 1$ on the support of the boundary data and $\operatorname{tr}\operatorname{ex} = I$, we have

$$\mathbf{TrV}_\alpha = \vartheta_\alpha \mathbf{v}.$$

Summing over α gives $\mathbf{TrExv} = \sum_{\alpha=1}^{N} \vartheta_\alpha \mathbf{v} = \mathbf{v}$. This proves both surjectivity and the stated estimates. □

Theorem 9.4.22. (Higher-order traces) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $m \ge 1$ and $1 \le p < \infty$. For each $0 \le j \le m-1$, the smooth restriction operator

$$\mathbf{u} \longmapsto \gamma_0(\nabla^j \mathbf{u}) = (\nabla^j \mathbf{u}) \restriction_{\partial M}, \qquad \mathbf{u} \in \Gamma(\mathbf{E}),$$

extends uniquely to a continuous linear operator

$$\mathbf{Tr}_j \colon W^{m,p}(M,\mathbf{E}) \longrightarrow W^{m-j-1,p}\left(\partial M, \left(T^{(0,j)}(TM) \otimes \mathbf{E}\right) \restriction_{\partial M}\right).$$

Consequently, the operator

$$\mathcal{T}_m \colon W^{m,p}(M,\mathbf{E}) \longrightarrow \bigoplus_{j=0}^{m-1} W^{m-j-1,p}\left(\partial M, \left(T^{(0,j)}(TM) \otimes \mathbf{E}\right) \restriction_{\partial M}\right),$$

defined by $\mathcal{T}_m \mathbf{u} := \left(\mathbf{Tr}_0 \mathbf{u}, \mathbf{Tr}_1 \mathbf{u}, \dots, \mathbf{Tr}_{m-1} \mathbf{u}\right)$ is linear and continuous. If $1 < p < \infty$, we also have the sharper estimate

$$\mathbf{Tr}_j \colon W^{m,p}(M,\mathbf{E}) \longrightarrow W^{m-j-\frac{1}{p},p}\left(\partial M, \left(T^{(0,j)}(TM) \otimes \mathbf{E}\right) \restriction_{\partial M}\right)$$

for every $j \in \{0,\dots,m-1\}$.

Proof. Fix $0 \le j \le m-1$ and write $\mathbf{F}_j := T^{(0,j)}(TM) \otimes \mathbf{E}$. This bundle is equipped with the product metric and induced connection. Moreover, $\mathbf{F}_j \restriction_{\partial M} = \left(T^{(0,j)}(TM) \otimes \mathbf{E}\right) \restriction_{\partial M}$.
Let $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$. By Theorem 9.4.8, there is $(\mathbf{u}_\nu) \subseteq \Gamma(\mathbf{E})$ such that $\mathbf{u}_\nu \to \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. For smooth

sections, the recursive definition of covariant differentiation gives

$$\nabla^s(\nabla^j \mathbf{u}_\nu) = \nabla^{s+j}\mathbf{u}_\nu, \qquad 0 \le s \le m-j,$$

after the natural identification of cotangent factors. Consequently, $(\nabla^j \mathbf{u}_\nu)$ is Cauchy in $W^{m-j,p}(M,\mathbf{F}_j)$. Its limit in L^p is $\nabla^j_w \mathbf{u}$; closedness of weak covariant derivatives then shows that $\nabla^j_w \mathbf{u} \in W^{m-j,p}(M,\mathbf{F}_j)$ and its successive derivatives are those expected. Moreover,

$$\|\nabla^j_w \mathbf{u}\|_{W^{m-j,p}(M,\mathbf{F}_j)} \le \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

Since $m-j \ge 1$, we may apply Theorem 9.4.12 to the bundle $\mathbf{F}_j \to M$. This gives a continuous linear operator

$$\mathbf{Tr}^{\mathbf{F}_j} : W^{m-j,p}(M,\mathbf{F}_j) \longrightarrow W^{m-j-1,p}(\partial M, \mathbf{F}_j \restriction_{\partial M}).$$

For $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$, define $\mathbf{Tr}_j \mathbf{u} := \mathbf{Tr}^{\mathbf{F}_j}(\nabla^j_w \mathbf{u})$. Then

$$\|\mathbf{Tr}_j \mathbf{u}\|_{W^{m-j-1,p}(\partial M, \mathbf{F}_j\restriction_{\partial M})} \le C\|\nabla^j_w \mathbf{u}\|_{W^{m-j,p}(M,\mathbf{F}_j)} \le C\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

This proves that $\mathbf{Tr}_j$ is linear and continuous.

If $1 < p < \infty$, instead apply Theorem 9.4.21 to the bundle $\mathbf{F}_j$ at order $m-j$. Since $\nabla^j_w \mathbf{u} \in W^{m-j,p}(M,\mathbf{F}_j)$, we obtain

$$\|\mathbf{Tr}_j \mathbf{u}\|_{W^{m-j-\frac{1}{p},p}(\partial M, \mathbf{F}_j\restriction_{\partial M})} \le C\|\nabla^j_w \mathbf{u}\|_{W^{m-j,p}(M,\mathbf{F}_j)} \le C\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

After composition with the continuous inclusion into $W^{m-j-1,p}(\partial M, \mathbf{F}_j \restriction_{\partial M})$, the resulting operator and the one defined by the integer-order theorem agree on smooth sections and, by density, throughout $W^{m,p}(M,\mathbf{E})$. We now show that this operator extends smooth restriction. If $\mathbf{u} \in \Gamma(\mathbf{E})$, then $\nabla^j_w \mathbf{u} = \nabla^j \mathbf{u}$, and by the definition of the trace operator in Theorem 9.4.12, we have $\mathbf{Tr}^{\mathbf{F}_j}(\nabla^j \mathbf{u}) = (\nabla^j \mathbf{u}) \restriction_{\partial M} = \gamma_0(\nabla^j \mathbf{u})$. Thus $\mathbf{Tr}_j$ extends the smooth operator $\mathbf{u} \mapsto \gamma_0(\nabla^j \mathbf{u})$.

Uniqueness follows from density of smooth sections in $W^{m,p}(M,\mathbf{E})$. Finally, equip the direct sum $\bigoplus_{j=0}^{m-1} W^{m-j-1,p}(\partial M, \mathbf{F}_j \restriction_{\partial M})$ with any of the equivalent product norms. Since each $\mathbf{Tr}_j$ is linear and continuous, there is a constant $C_j > 0$ such that $\|\mathbf{Tr}_j \mathbf{u}\|_{W^{m-j-1,p}(\partial M, \mathbf{F}_j\restriction_{\partial M})} \le C_j \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$. Summing over $j \in \{0,\dots,m-1\}$ gives

$$\|\mathcal{T}_m \mathbf{u}\| \le \left(\sum_{j=0}^{m-1} C_j\right) \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

Therefore, $\mathcal{T}_m$ is linear and continuous. □

The integration-by-parts formula motivating this construction explains why $\mathcal{T}_m$ is the appropriate boundary operator. Green's formula for ∇^s contains terms of the form

$$\sum_{j=0}^{s-1} (-1)^{s-1-j} \int_{\partial M} \left\langle \nabla^j \mathbf{F}, \iota_{\boldsymbol{\nu}}\big((\mathrm{tr}_{\mathbf{g}} \circ \nabla)^{s-1-j} \mathbf{G}\big) \right\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} d\lambda_{\tilde{\mathbf{g}}}.$$

Thus the quantities arising naturally on the boundary are restrictions of $\nabla^j \mathbf{F}$ to ∂M, with $0 \le j \le s-1$. This is

why, to characterize $W_0^{m,p}(M,\mathbf{E})$ when $m \geq 2$, vanishing of the trace of $\mathbf{u}$ is insufficient: the traces of its covariant derivatives through order $m-1$ must vanish as well.

Theorem 9.4.23. (Characterization of $W_0^{m,p}(M,\mathbf{E})$ by traces) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $m \geq 1$ and $1 \leq p < \infty$. Then $W_0^{m,p}(M,\mathbf{E}) = \ker(\mathcal{T}_m)$. That is,

$$W_0^{m,p}(M,\mathbf{E}) = \left\{ \mathbf{u} \in W^{m,p}(M,\mathbf{E}) \middle| \mathbf{Tr}_j \mathbf{u} = 0 \text{ for every } 0 \leq j \leq m-1 \right\}.$$

Proof. We first prove that $W_0^{m,p}(M,\mathbf{E}) \subseteq \ker(\mathcal{T}_m)$. Let $\mathbf{u} \in W_0^{m,p}(M,\mathbf{E})$. By definition, there is a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}} \subseteq \Gamma_{c,\mathrm{Int}(M)}(\mathbf{E})$ such that $\mathbf{u}_k \to \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. Since each $\mathbf{u}_k$ has compact support contained in $\mathrm{Int}(M)$, there is a neighborhood of ∂M on which $\mathbf{u}_k$ vanishes. Thus, for every $0 \leq j \leq m-1$, $\nabla^j \mathbf{u}_k$ also vanishes on a neighborhood of ∂M. Consequently, $\mathbf{Tr}_j \mathbf{u}_k = 0$. Using continuity of $\mathbf{Tr}_j$ gives $\mathbf{Tr}_j \mathbf{u} = \lim_{k\to\infty} \mathbf{Tr}_j \mathbf{u}_k = 0$. Since this holds for every $0 \leq j \leq m-1$, it follows that $\mathbf{u} \in \ker(\mathcal{T}_m)$.

We now prove the reverse inclusion. Let $\mathbf{u} \in \ker(\mathcal{T}_m)$. Take a finite open cover $(U_\alpha)_{\alpha=1}^N$ of M consisting of regular coordinate balls contained in $\mathrm{Int}(M)$ and regular coordinate half-balls near ∂M. Let $(\psi_\alpha)_{\alpha=1}^N$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp}\psi_\alpha \Subset U_\alpha$. Since $\mathbf{u} = \sum_{\alpha=1}^N \psi_\alpha \mathbf{u}$, it suffices to prove that $\psi_\alpha \mathbf{u} \in W_0^{m,p}(M,\mathbf{E})$ for each $\alpha \in \{1,\dots,N\}$.

Fix $\alpha \in \{1,\dots,N\}$. First we prove that $\mathbf{Tr}_j(\psi_\alpha \mathbf{u}) = 0$ for every $0 \leq j \leq m-1$. Since $\mathbf{u} \in \ker(\mathcal{T}_m)$, we have $\mathbf{Tr}_t \mathbf{u} = 0$ for every $0 \leq t \leq m-1$. By Theorem 9.4.8, there is a sequence $(\mathbf{v}_k)_{k\in\mathbb{N}} \subseteq \Gamma(\mathbf{E})$ such that $\mathbf{v}_k \to \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. By continuity of the higher-order traces, given by Theorem 9.4.22, we obtain $\mathbf{Tr}_t \mathbf{v}_k \to \mathbf{Tr}_t \mathbf{u} = 0$ in $W^{m-t-1,p}\left(\partial M, \left(T^{(0,t)}(TM)\otimes\mathbf{E}\right)\restriction_{\partial M}\right)$ for every $0 \leq t \leq m-1$.

Now fix $j \in \{0,\dots,m-1\}$. For smooth sections, the iterated Leibniz rule for the product connection gives, up to the natural permutations of covariant indices,

$$\nabla^j(\psi_\alpha \mathbf{v}_k) = \sum_{t=0}^{j} \binom{j}{t} (\nabla^{j-t}\psi_\alpha) \otimes \nabla^t \mathbf{v}_k.$$

More precisely, this equality means that the right-hand side represents a finite sum of terms obtained from $(\nabla^{j-t}\psi_\alpha)\otimes\nabla^t\mathbf{v}_k$ by permutations of the j covariant indices. Restricting to ∂M gives

$$\mathbf{Tr}_j(\psi_\alpha \mathbf{v}_k) = \sum_{t=0}^{j} \binom{j}{t} \left((\nabla^{j-t}\psi_\alpha)\restriction_{\partial M}\right) \otimes \mathbf{Tr}_t \mathbf{v}_k,$$

again up to those index permutations. This notation suffices for the estimate we need, because index permutations are pointwise isometries, and tensor multiplication by the fixed smooth sections $(\nabla^{j-t}\psi_\alpha)\restriction_{\partial M}$ is continuous on the corresponding Sobolev spaces over the compact manifold ∂M. Since $\mathbf{Tr}_t \mathbf{v}_k \to 0$ for every $0 \leq t \leq j$, we conclude that $\mathbf{Tr}_j(\psi_\alpha \mathbf{v}_k) \to 0$ in $W^{m-j-1,p}\left(\partial M, \left(T^{(0,j)}(TM)\otimes\mathbf{E}\right)\restriction_{\partial M}\right)$.

On the other hand, the map $\mathbf{v} \longmapsto \mathbf{Tr}_j(\psi_\alpha \mathbf{v})$ is continuous from $W^{m,p}(M,\mathbf{E})$ to that same space: this follows from continuity of multiplication by ψ_α on $W^{m,p}(M,\mathbf{E})$, given by Lemma 9.4.9, and continuity of $\mathbf{Tr}_j$. Since $\mathbf{v}_k \to \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$, we have $\mathbf{Tr}_j(\psi_\alpha \mathbf{v}_k) \to \mathbf{Tr}_j(\psi_\alpha \mathbf{u})$. By uniqueness of the limit, $\mathbf{Tr}_j(\psi_\alpha \mathbf{u}) = 0$. Since j was arbitrary, we obtain $\mathbf{Tr}_j(\psi_\alpha \mathbf{u}) = 0$ for every $0 \leq j \leq m-1$.

Now consider the two types of charts. If U_α is a regular coordinate ball contained in $\mathrm{Int}(M)$, then $\psi_\alpha \mathbf{u}$ has

compact support contained in $\mathrm{Int}(M)$. Take a function $\chi_\alpha \in C_c^\infty(\mathrm{Int}(M))$ such that $\chi_\alpha \equiv 1$ on a neighborhood of $\mathrm{supp}(\psi_\alpha)$. By Theorem 9.4.8, there is a sequence $(\mathbf{z}_k)_{k\in\mathbb{N}} \subseteq \Gamma(\mathbf{E})$ such that $\mathbf{z}_k \to \psi_\alpha \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. By Lemma 9.4.9, we have $\chi_\alpha \mathbf{z}_k \to \chi_\alpha \psi_\alpha \mathbf{u} = \psi_\alpha \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. Moreover, each $\chi_\alpha \mathbf{z}_k$ is a smooth section with compact support contained in $\mathrm{Int}(M)$. Therefore, $\psi_\alpha \mathbf{u} \in W_0^{m,p}(M,\mathbf{E})$.

Now suppose that U_α is a regular coordinate half-ball. Write $B_\alpha^+ := \varphi_\alpha(U_\alpha \cap \mathrm{Int}(M))$ and $B'_\alpha := \varphi_\alpha(U_\alpha \cap \partial M)$. Take a smooth local frame $(\mathbf{e}_{1,\alpha},\dots,\mathbf{e}_{r,\alpha})$ of $\mathbf{E}$ on U_α, and write locally $\psi_\alpha \mathbf{u} = \sum_{a=1}^{r} w_\alpha^a \mathbf{e}_{a,\alpha}$. Set $q_\alpha^a := w_\alpha^a \circ \varphi_\alpha^{-1}$ on B_α^+, and denote by $\widetilde{q}_\alpha^a$ its extension by zero to $\mathbb{R}_+^n$. Since $\operatorname{supp}\psi_\alpha \Subset U_\alpha$, each q_α^a vanishes outside a compact subset of $B_\alpha^+ \cup B'_\alpha$ lying at positive distance from the spherical part of B_α^+. Thus we may use the preceding convention for these functions.

We want to prove that each q_α^a belongs to the closure of $C_c^\infty(B_\alpha^+)$ in $W^{m,p}(B_\alpha^+)$. By Corollary 3.7.3, it suffices to prove that $\gamma_0 D^\beta q_\alpha^a = 0$ on B'_α, in the sense of the preceding convention, for every $|\beta| \le m-1$ and every a. Explicitly, we must prove that $(\gamma_0 D^\beta \widetilde{q}_\alpha^a) \restriction_{B'_\alpha} = 0$.

We prove this by induction on ℓ. For $\ell = 0$, the equality $\mathbf{Tr}_0(\psi_\alpha \mathbf{u}) = 0$ means that the trace of $\psi_\alpha \mathbf{u}$ vanishes as a section of $\mathbf{E} \restriction_{\partial M}$. In the boundary chart and chosen frame, the trace operator was constructed precisely by taking components, extending them by zero to the half-space, and applying the half-space trace. Therefore, $(\gamma_0 \widetilde{q}_\alpha^a) \restriction_{B'_\alpha} = 0$ for every a.

Suppose that $(\gamma_0 D^\beta \widetilde{q}_\alpha^a) \restriction_{B'_\alpha} = 0$ has been proved for every $|\beta| \le \ell - 1$ and every a, where $1 \le \ell \le m-1$. Since $\mathbf{Tr}_\ell(\psi_\alpha \mathbf{u}) = 0$, the local components of the trace of $\nabla^\ell(\psi_\alpha \mathbf{u})$ vanish on B'_α. By Lemma 9.1.28, applied in the chart $(U_\alpha, \varphi_\alpha)$ and frame $(\mathbf{e}_{1,\alpha},\dots,\mathbf{e}_{r,\alpha})$, we have locally

$$(\nabla^\ell(\psi_\alpha \mathbf{u}))^a_{i_1\dots i_\ell} = \partial_{i_\ell}\cdots\partial_{i_1} w_\alpha^a + \sum_{b=1}^{r}\sum_{|\beta|\le \ell-1} (B_\beta)^a_{b\, i_1\dots i_\ell}\, \partial^\beta w_\alpha^b.$$

To pass this identity to traces, first fix the supports of the approximation. Choose $\chi_\alpha \in C_c^\infty(U_\alpha)$ equal to one near $\operatorname{supp}\psi_\alpha$ and take $\mathbf{z}_k \in \Gamma(\mathbf{E})$ with $\mathbf{z}_k \to \psi_\alpha \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$, by Theorem 9.4.8. Then $\mathbf{v}_k = \chi_\alpha \mathbf{z}_k$ converges to the same limit in that space by Lemma 9.4.9, and all its supports lie in the fixed compact set $\operatorname{supp}\chi_\alpha$. Denote its coordinate components by $q_{\alpha,k}^a$. Local equivalence of norms and Lemma 3.6.8 give

$$\widetilde{q}_{\alpha,k}^a \longrightarrow \widetilde{q}_\alpha^a \quad \text{in } W^{m,p}(\mathbb{R}_+^n).$$

Here extension by zero is from the half-ball to the half-space, across its spherical part; the fixed compact set is separated from that part, and the flat boundary is not crossed.

For $|\beta| \le \ell$, $D^\beta \widetilde{q}_{\alpha,k}^a \to D^\beta \widetilde{q}_\alpha^a$ in $W^{m-|\beta|,p}(\mathbb{R}_+^n)$, and by continuity of the trace,

$$\gamma_0 D^\beta \widetilde{q}_{\alpha,k}^a \longrightarrow \gamma_0 D^\beta \widetilde{q}_\alpha^a \quad \text{in } W^{m-|\beta|-1,p}(\mathbb{R}^{n-1}).$$

All these spaces embed continuously into $W^{m-\ell-1,p}(\mathbb{R}^{n-1})$. Multiply the coefficients B_β, expressed in the chart, by a cutoff equal to one near the fixed compact set; they then extend to smooth compactly supported coefficients on $\mathbb{R}^n$. Multiplication by them, before and after taking the trace, is continuous. Moreover, the components of $\nabla^\ell \mathbf{v}_k$ converge in $W^{m-\ell,p}(\mathbb{R}_+^n)$, since the covariant derivative of order ℓ is continuous from $W^{m,p}(M,\mathbf{E})$ to $W^{m-\ell,p}(M, T^{(0,\ell)}(TM)\otimes\mathbf{E})$, as proved in Theorem 9.4.22. Apply the local identity to $\mathbf{v}_k$ and take limits in $W^{m-\ell-1,p}(\mathbb{R}^{n-1})$, then restrict to B'_α. This gives exactly the required trace identity for $\psi_\alpha \mathbf{u}$, including when $p = 1$.

Now fix a multi-index γ with $|\gamma| = \ell$. Choose an ordered block $(i_1, \dots, i_\ell)$ in which the index q appears exactly γ_q times. Since weak partial derivatives commute, the leading term of the corresponding component is

$$\partial_{i_\ell} \cdots \partial_{i_1} q_\alpha^a = D^\gamma q_\alpha^a.$$

Taking the trace of that component on B'_α gives

$$0 = \gamma_0(D^\gamma \widetilde{q}_\alpha^a) + \sum_{b=1}^{r} \sum_{|\beta| \le \ell - 1} \gamma_0\left(\widetilde{B}_\beta \, D^\beta \widetilde{q}_\alpha^b\right) \quad \text{in } B'_\alpha,$$

where $\widetilde{B}_\beta$ denotes the smooth coefficient expressed in the chart. The left-hand side is zero because it is a component of $\mathbf{Tr}_\ell(\psi_\alpha \mathbf{u})$. For each term in the sum, compatibility of the trace with multiplication by a smooth function gives

$$\gamma_0\left(\widetilde{B}_\beta \, D^\beta \widetilde{q}_\alpha^b\right) = \widetilde{B}_\beta \restriction_{B'_\alpha} \gamma_0\left(D^\beta \widetilde{q}_\alpha^b\right) = 0,$$

since $|\beta| \le \ell - 1$ and the induction hypothesis applies. This compatibility uses the same sequence with fixed support: if $v_k \to v$ in $W^{m-|\beta|,p}(\mathbb{R}^n_+)$, both $\gamma_0(\widetilde{B}_\beta v_k)$ and $(\widetilde{B}_\beta|_{\mathbb{R}^{n-1}})\gamma_0 v_k$ converge in $W^{m-|\beta|-1,p}(\mathbb{R}^{n-1})$ by the multiplication and trace bounds. They agree for each smooth k; uniqueness of the limit proves the identity used. Consequently,

$$\gamma_0 \left(D^\gamma \widetilde{q}_\alpha^a\right) \restriction_{B'_\alpha} = 0$$

for every $|\gamma| = \ell$ and every a. This completes the induction.

Thus $\gamma_0 D^\beta q_\alpha^a = 0$ on B'_α, in the sense of the preceding convention, for every $|\beta| \le m - 1$ and every a. By Corollary 3.7.3, each q_α^a belongs to the closure of $C_c^\infty(B_\alpha^+)$ in $W^{m,p}(B_\alpha^+)$. Hence, for each a there is a sequence $(\varphi_k^a)_{k \in \mathbb{N}} \subseteq C_c^\infty(B_\alpha^+)$ such that $\varphi_k^a \to q_\alpha^a$ in $W^{m,p}(B_\alpha^+)$.

On $U_\alpha \cap \mathrm{Int}(M)$ define $\mathbf{s}_k := \sum_{a=1}^{r} (\varphi_k^a \circ \varphi_\alpha)\mathbf{e}_{a,\alpha}$. Since each φ_k^a has compact support contained in B_α^+, each $\mathbf{s}_k$ extends by zero to a smooth section with compact support contained in $\mathrm{Int}(M)$. By local equivalence of norms in the regular coordinate half-ball, we have $\mathbf{s}_k \to \psi_\alpha \mathbf{u}$ in $W^{m,p}(M, \mathbf{E})$. Therefore, $\psi_\alpha \mathbf{u} \in W_0^{m,p}(M, \mathbf{E})$.

We have proved that $\psi_\alpha \mathbf{u} \in W_0^{m,p}(M, \mathbf{E})$ for each $\alpha \in \{1, \dots, N\}$. Since $\mathbf{u} = \sum_{\alpha=1}^{N} \psi_\alpha \mathbf{u}$, we conclude that $\mathbf{u} \in W_0^{m,p}(M, \mathbf{E})$. This proves that $\ker(\mathcal{T}_m) \subseteq W_0^{m,p}(M, \mathbf{E})$ and completes the proof. □

9.4.3 Embeddings for sections on compact manifolds with boundary

The localization principle transfers to manifolds with boundary the embeddings established earlier for bundles over manifolds without boundary. In interior charts, the argument of Theorems 9.2.1 and 9.2.2 is repeated. In a boundary chart, the localized section is represented by Sobolev functions on a half-ball; extension by zero from Lemma 3.6.8 allows us to regard them as functions on the half-space, where the continuous and compact parts of Theorem 3.6.6 apply. A smooth partition of unity then combines the local estimates.

Theorem 9.4.24. (Sobolev embedding for bundles over manifolds with boundary) Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary, and let $\mathbf{E} \to M$ be a smooth real or complex vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ (Hermitian in the complex case)

and a compatible connection $\nabla^{\mathbf{E}}$. Let $j \geq 0$ and $m \geq 1$ be integers and $1 \leq p, q < \infty$ such that

$$j + m - \frac{n}{p} \geq j - \frac{n}{q}.$$

Then the inclusion

$$W^{j+m,p}(M, \mathbf{E}) \hookrightarrow W^{j,q}(M, \mathbf{E})$$

is continuous.

Proof. Choose a finite cover $(U_\alpha, \varphi_\alpha)_{\alpha=1}^N$ consisting of regular coordinate balls contained in $\mathrm{Int}(M)$ and regular coordinate half-balls centered on ∂M. Refining the cover, assume that $\overline{U_\alpha}$ lies in the domain of a smooth local frame $(\mathbf{e}_{1,\alpha}, \dots, \mathbf{e}_{r,\alpha})$ of $\mathbf{E}$. Take a smooth partition of unity $(\psi_\alpha)_{\alpha=1}^N$ subordinate to the cover, with $\operatorname{supp} \psi_\alpha \Subset U_\alpha$.

Let $\mathbf{u} \in W^{j+m,p}(M, \mathbf{E})$. By Lemma 9.4.9, each $\psi_\alpha \mathbf{u}$ belongs to $W^{j+m,p}(M, \mathbf{E})$ and there is $C_0 > 0$, independent of $\mathbf{u}$ and α, such that

$$\|\psi_\alpha \mathbf{u}\|_{W^{j+m,p}(M,\mathbf{E})} \leq C_0 \|\mathbf{u}\|_{W^{j+m,p}(M,\mathbf{E})}.$$

On U_α write

$$\psi_\alpha \mathbf{u} = \sum_{a=1}^{r} u_\alpha^a \mathbf{e}_{a,\alpha}$$

and set $v_\alpha^a := u_\alpha^a \circ \varphi_\alpha^{-1}$. By Lemmas 9.1.44 and 9.4.5, the norm defined by covariant derivatives is equivalent to the sum of the component norms; here we use compatibility of $\nabla^{\mathbf{E}}$ with $\mathbf{h}_{\mathbf{E}}$. In the complex case they apply to the real and imaginary parts. Thus there are constants $c_\alpha, C_\alpha > 0$ such that

$$c_\alpha \sum_{a=1}^{r} \|v_\alpha^a\|_{W^{j+m,p}(\Omega_\alpha)} \leq \|\psi_\alpha \mathbf{u}\|_{W^{j+m,p}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})}$$

and

$$\|\psi_\alpha \mathbf{u}\|_{W^{j,q}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})} \leq C_\alpha \sum_{a=1}^{r} \|v_\alpha^a\|_{W^{j,q}(\Omega_\alpha)},$$

where $\Omega_\alpha := \varphi_\alpha(U_\alpha)$ in an interior chart and $\Omega_\alpha := \varphi_\alpha(U_\alpha \cap \mathrm{Int}(M))$ in a boundary chart.

If U_α is interior, the support of v_α^a is contained in $\varphi_\alpha(\operatorname{supp} \psi_\alpha) \Subset \Omega_\alpha$. Thus $v_\alpha^a \in W_0^{j+m,p}(\Omega_\alpha)$, and we proceed exactly as in the proof of Theorem 9.2.1: apply Theorem 3.3.16 to each localized component. This gives a constant $K_\alpha > 0$ such that

$$\|v_\alpha^a\|_{W^{j,q}(\Omega_\alpha)} \leq K_\alpha \|v_\alpha^a\|_{W^{j+m,p}(\Omega_\alpha)}.$$

Suppose that U_α is a boundary chart. Then $\Omega_\alpha = B_\alpha^+$ is a half-ball. Since $\operatorname{supp} \psi_\alpha \Subset U_\alpha$, there is a compact set $K_\alpha \subseteq B_\alpha^+ \cup B_\alpha'$ lying at positive distance from the spherical part of B_α^+ such that $v_\alpha^a = 0$ almost everywhere on $B_\alpha^+ \setminus K_\alpha$ for every a. By Lemma 3.6.8, the extension by zero $\widetilde{v}_\alpha^a$ belongs to $W^{j+m,p}(\mathbb{R}_+^n)$ and

$$\|\widetilde{v}_\alpha^a\|_{W^{j+m,p}(\mathbb{R}_+^n)} = \|v_\alpha^a\|_{W^{j+m,p}(B_\alpha^+)}.$$

Moreover, all these extensions have support in a fixed compact subset of $\overline{\mathbb{R}_+^n}$ depending only on the chart

and ψ_α. Theorem 3.6.6 gives a constant $K_\alpha > 0$ such that

$$\|\widetilde{v}_\alpha^a\|_{W^{j,q}(\mathbb{R}^n_+)} \le K_\alpha \|\widetilde{v}_\alpha^a\|_{W^{j+m,p}(\mathbb{R}^n_+)}.$$

Restricting to B_α^+ gives

$$\|v_\alpha^a\|_{W^{j,q}(B_\alpha^+)} \le K_\alpha \|v_\alpha^a\|_{W^{j+m,p}(B_\alpha^+)}.$$

In both types of charts, local equivalence of norms gives

$$\|\psi_\alpha \mathbf{u}\|_{W^{j,q}(M,\mathbf{E})} \le \frac{C_\alpha K_\alpha}{c_\alpha} \|\psi_\alpha \mathbf{u}\|_{W^{j+m,p}(M,\mathbf{E})} \le C'_\alpha \|\mathbf{u}\|_{W^{j+m,p}(M,\mathbf{E})}.$$

Since the cover is finite and $\mathbf{u} = \sum_{\alpha=1}^{N} \psi_\alpha \mathbf{u}$, it follows that

$$\|\mathbf{u}\|_{W^{j,q}(M,\mathbf{E})} \le \sum_{\alpha=1}^{N} \|\psi_\alpha \mathbf{u}\|_{W^{j,q}(M,\mathbf{E})} \le C \|\mathbf{u}\|_{W^{j+m,p}(M,\mathbf{E})}.$$

□

Theorem 9.4.25. (Rellich–Kondrashov for bundles over manifolds with boundary) Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary, and let $\mathbf{E} \to M$ be a smooth real or complex vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ (Hermitian in the complex case) and a compatible connection $\nabla^\mathbf{E}$. Let $j \ge 0$ and $m \ge 1$ be integers and $1 \le p, q < \infty$ such that

$$j + m - \frac{n}{p} > j - \frac{n}{q}.$$

Then the inclusion

$$W^{j+m,p}(M,\mathbf{E}) \hookrightarrow W^{j,q}(M,\mathbf{E})$$

is compact.

Proof. Let $(\mathbf{u}_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{j+m,p}(M,\mathbf{E})$. Use the cover, partition of unity, and frames from the preceding proof. For each α and each a, define

$$v_{\alpha,k}^a := (\psi_\alpha \mathbf{u}_k)_\alpha^a \circ \varphi_\alpha^{-1}.$$

Continuity of multiplication by ψ_α and local equivalence of norms show that $(v_{\alpha,k}^a)_k$ is bounded in $W^{j+m,p}(\Omega_\alpha)$. These comparisons use compatibility of $\nabla^\mathbf{E}$ with $\mathbf{h}_\mathbf{E}$, as in Lemmas 9.1.44 and 9.4.5. If the bundle is complex, apply the Euclidean results to the real and imaginary parts of the components.

If U_α is an interior chart, proceed as in the proof of Theorem 9.2.2: Theorem 3.4.2 gives a subsequence converging in $W^{j,q}(\Omega_\alpha)$.

If U_α is a boundary chart, extend each $v_{\alpha,k}^a$ by zero to $\mathbb{R}^n_+$. Lemma 3.6.8 shows that the extensions form a bounded sequence in $W^{j+m,p}(\mathbb{R}^n_+)$ and their supports lie in the same compact subset of $\overline{\mathbb{R}^n_+}$, determined by $\operatorname{supp}\psi_\alpha$. The compact part of Theorem 3.6.6 gives a subsequence converging in $W^{j,q}(\mathbb{R}^n_+)$. Restricting to B_α^+ gives a subsequence converging in $W^{j,q}(B_\alpha^+)$.

Since the numbers of charts and components are finite, successive extractions give a single subsequence, again denoted by $(\mathbf{u}_k)$, for which all sequences $(v^a_{\alpha,k})_k$ are Cauchy in $W^{j,q}(\Omega_\alpha)$. Local equivalence of norms implies that $(\psi_\alpha \mathbf{u}_k)_k$ is Cauchy in $W^{j,q}(U_\alpha, \mathbf{E}\restriction_{U_\alpha})$ for each α. Therefore,

$$\|\mathbf{u}_k - \mathbf{u}_\ell\|_{W^{j,q}(M,\mathbf{E})} \le \sum_{\alpha=1}^{N} \|\psi_\alpha(\mathbf{u}_k - \mathbf{u}_\ell)\|_{W^{j,q}(M,\mathbf{E})} \longrightarrow 0$$

as $k, \ell \to \infty$. Since $W^{j,q}(M,\mathbf{E})$ is complete, the subsequence converges. This proves compactness. □

Corollary 9.4.26. (Embeddings and the critical exponent) Under the geometric hypotheses of the preceding theorems, let $j \ge 0$ and $m \ge 1$ be integers, and let $1 \le p < n/m$. Then

$$W^{j+m,p}(M,\mathbf{E}) \hookrightarrow W^{j,q}(M,\mathbf{E})$$

continuously for $1 \le q \le p^*_m := \dfrac{np}{n-mp}$, and compactly for $1 \le q < p^*_m$.

Proof. Since $mp < n$, the condition $j + m - \dfrac{n}{p} \ge j - \dfrac{n}{q}$ is equivalent to $q \le p^*_m$, and the strict inequality is equivalent to $q < p^*_m$. Apply Theorems 9.4.24 and 9.4.25, respectively. □

The scalar case follows by taking the trivial rank-one bundle with its canonical metric and connection.

Corollary 9.4.27. (Sobolev embedding on compact manifolds with boundary) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary. Let $j \ge 0$ and $m \ge 1$ be integers and $1 \le p, q < \infty$ such that $j + m - \dfrac{n}{p} \ge j - \dfrac{n}{q}$. Then the inclusion

$$W^{j+m,p}(M) \hookrightarrow W^{j,q}(M)$$

is continuous.

Proof. Apply Theorem 9.4.24 to the trivial bundle $M \times \mathbb{R} \to M$, equipped with the usual metric and the connection induced by the exterior differential. Under the canonical identification of its sections with functions, the Sobolev norms agree with the scalar norms. □

Corollary 9.4.28. (Rellich–Kondrashov on compact manifolds with boundary) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary. Let $j \ge 0$ and $m \ge 1$ be integers and $1 \le p, q < \infty$ such that $j + m - \frac{n}{p} > j - \frac{n}{q}$. Then the inclusion

$$W^{j+m,p}(M) \hookrightarrow W^{j,q}(M)$$

is compact.

Proof. This is the case of Theorem 9.4.25 corresponding to the trivial rank-one bundle. □

Corollary 9.4.29. (Compact embedding with a decrease of one order) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with possibly empty smooth boundary, and let $\mathbf{E}\to M$ be a smooth real or complex vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$ (Hermitian in the complex case) and a compatible connection $\nabla^{\mathbf{E}}$. For every integer $m\geq 1$ and every $1\leq p<\infty$, the inclusions

$$W^{m,p}(M,\mathbf{E})\hookrightarrow W^{m-1,p}(M,\mathbf{E}),\qquad W_0^{m,p}(M,\mathbf{E})\hookrightarrow W_0^{m-1,p}(M,\mathbf{E})$$

are compact. Here $W_0^{0,p}(M,\mathbf{E})=L^p(M,\mathbf{E})$.

Proof. In Theorems 9.2.2 and 9.4.25, take $j=m-1$, a gain of one derivative, and the same exponent p in the source and target. The inequality between indices reduces to $1>0$, giving the first inclusion for every $1\leq p<\infty$, including $p=1$.
For the second, if $\mathbf{u}\in W_0^{m,p}(M,\mathbf{E})$, choose $\mathbf{u}_\nu\in\Gamma_{c,\mathrm{Int}(M)}(\mathbf{E})$ with $\mathbf{u}_\nu\to\mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. The inequality $\|\mathbf{v}\|_{W^{m-1,p}(M,\mathbf{E})}\leq\|\mathbf{v}\|_{W^{m,p}(M,\mathbf{E})}$ implies that the same convergence holds in $W^{m-1,p}(M,\mathbf{E})$; by definition, $\mathbf{u}\in W_0^{m-1,p}(M,\mathbf{E})$. By the first inclusion, a bounded sequence in $W_0^{m,p}$ has a subsequence converging in $W^{m-1,p}$. All its terms belong to $W_0^{m-1,p}$, and this subspace is closed by definition, so the limit also belongs to it. The subspace norm is the induced norm, proving the asserted compactness. □

Remark 9.4.30. (Total orders and dependence of the constants) In terms of total orders, the two preceding theorems state that

$$W^{m,p}(M,\mathbf{E})\hookrightarrow W^{k,q}(M,\mathbf{E})\quad\text{is compact if}\quad m>k\geq 0,\quad 1\leq p,q<\infty,\quad m-\frac{n}{p}>k-\frac{n}{q}.$$

This formulation is obtained by substituting $j=k$ and replacing the gain of derivatives by $m-k$. The same assertion holds with subscript zero in both source and target: continuity of the inclusion carries smooth approximations supported in the interior into $W^{k,q}$, and closedness of $W_0^{k,q}$ retains the limit. If $(m-k)p<n$, the strict condition is equivalent to $q<np/(n-(m-k)p)$; at the critical exponent, only continuity has been asserted in the corresponding Sobolev theorems.
The continuity constants depend on m,k,p,q, M, $\mathbf{g}$, and the smooth data $\mathbf{E}$, $\mathbf{h_E}$, $\nabla^{\mathbf{E}}$. In the proofs they arise from a fixed finite cover, derivatives of its partition functions, and bounds on the coefficients of the metrics, their inverses, and the connections through the orders used. They do not depend on the section or the sequence index. For a varying family of geometric data, uniformity of these constants requires control of those bounds and the charts; compactness of each manifold separately does not provide a uniform constant for the entire family. No assertion in this paragraph includes infinite exponents; embeddings into continuous and Hölder sections are treated separately.

Corollary 9.4.31. (First-order embeddings for the Dirichlet space) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary, and let $\mathbf{E}\to M$ be a smooth real vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$ and a connection $\nabla^{\mathbf{E}}$ compatible with $\mathbf{h_E}$. Let $1\leq q<n$ and define $q^*:=\dfrac{nq}{n-q}$. Then

$$W_0^{1,q}(M,\mathbf{E})\hookrightarrow L^\ell(M,\mathbf{E})$$

continuously for every $1\leq\ell\leq q^*$, and compactly for every $1\leq\ell<q^*$.

Proof. The continuous inclusion follows from Theorem 9.4.24 with $j = 0$ and $m = 1$. The compact inclusion follows from Theorem 9.4.25. In both cases it suffices to restrict the inclusion operators to the closed subspace $W_0^{1,q}(M, \mathbf{E}) \subseteq W^{1,q}(M, \mathbf{E})$. □

To formulate Morrey's embedding up to the boundary, we need to extend the notation of Definition 9.2.6. We do not use the seminorm defined by minimizing geodesics, since that formulation assumes the manifold has no boundary. Instead, we use the Euclidean extension norms already fixed for $C^{k,\alpha}(\overline{\Omega})$.

Definition 9.4.32. (Hölder sections up to the boundary) Let $(M, \mathbf{g})$ be a compact Riemannian manifold with nonempty smooth boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of rank r, equipped with a bundle metric and a compatible connection. Choose a finite cover $(U_\lambda, \varphi_\lambda)_{\lambda=1}^N$ consisting of regular interior coordinate balls and regular boundary coordinate half-balls, a smooth partition of unity $(\psi_\lambda)_{\lambda=1}^N$ subordinate to it, with $\operatorname{supp}\psi_\lambda \Subset U_\lambda$, and local frames $(\mathbf{e}_{\lambda,1}, \dots, \mathbf{e}_{\lambda,r})$ defined on neighborhoods of $\overline{U_\lambda}$. Set

$$\Omega_\lambda := \begin{cases} \varphi_\lambda(U_\lambda), & U_\lambda \subseteq \operatorname{Int}(M), \\ \varphi_\lambda(U_\lambda \cap \operatorname{Int}(M)), & U_\lambda \cap \partial M \neq \varnothing. \end{cases}$$

If $\mathbf{u} \in \Gamma^k(\mathbf{E})$ and

$$\psi_\lambda \mathbf{u} = \sum_{a=1}^{r} u_\lambda^a \mathbf{e}_{\lambda,a} \qquad \text{in } U_\lambda,$$

define

$$\|\mathbf{u}\|_{\Gamma_{\mathcal{A}}^{k,\alpha}(\mathbf{E})} := \sum_{\lambda=1}^{N} \sum_{a=1}^{r} \|u_\lambda^a \circ \varphi_\lambda^{-1}\|_{C^{k,\alpha}(\overline{\Omega_\lambda})}, \tag{9.80}$$

where $k \in \mathbb{N}_0$, $0 < \alpha < 1$, and $\mathcal{A}$ represents the chosen set of local data. Denote by $\Gamma_{\mathcal{A}}^{k,\alpha}(\mathbf{E})$ the space of sections of class C^k up to the boundary for which (9.80) is finite.

Proposition 9.4.33. (Independence of the Hölder norm up to the boundary) Let $(M, \mathbf{g})$ be a compact Riemannian manifold with nonempty smooth boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. The space and topology in Definition 9.4.32 are independent of the chosen cover, partition of unity, charts, and frames. Moreover, the resulting space is Banach. Henceforth we denote it by $\Gamma^{k,\alpha}(\mathbf{E})$ and omit the subscript $\mathcal{A}$ from its norm.

Proof. Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be two choices of local data. Use a tilde for objects in the second system. For an index μ of $\widetilde{\mathcal{A}}$, the identity $\sum_{\lambda=1}^{N} \psi_\lambda = 1$ gives

$$\widetilde{\psi}_\mu \mathbf{u} = \sum_{\substack{1 \le \lambda \le N \\ U_\lambda \cap \widetilde{U}_\mu \neq \varnothing}} \widetilde{\psi}_\mu \psi_\lambda \mathbf{u}. \tag{9.81}$$

The sum is finite. On an overlap write

$$\mathbf{e}_{\lambda,a} = \sum_{b=1}^{r} (G_{\mu\lambda})^b{}_a \, \widetilde{\mathbf{e}}_{\mu,b}$$

and denote the coordinate change on its domain by

$$\Phi_{\lambda\mu} := \varphi_\lambda \circ \widetilde{\varphi}_\mu^{-1}$$

. If $f_\lambda^a := u_\lambda^a \circ \varphi_\lambda^{-1}$, the component b of the corresponding summand in (9.81), transported to the chart $\widetilde{\varphi}_\mu$, is

$$\left[(\widetilde{\psi}_\mu G_{\mu\lambda})^b_{\ a} \circ \widetilde{\varphi}_\mu^{-1}\right] \left(f_\lambda^a \circ \Phi_{\lambda\mu}\right). \tag{9.82}$$

We explain precisely why this expression defines a continuous operator between the extension norms. The support of the first factor lies in a compact subset of the domain of the coordinate change. In the chart $\widetilde{\varphi}_\mu$, choose a smooth cutoff equal to one on a neighborhood of that compact set, with support still contained in the same domain. Changes of charts on a manifold with boundary extend smoothly to Euclidean open sets containing those compact sets; for two boundary charts the extension crosses the flat face, while in an overlap with an interior chart the compact set lies away from that face. Take an extension $F_\lambda^a \in C^{k,\alpha}(\mathbb{R}^n)$ of f_λ^a. After inserting the preceding cutoff, (9.82) is the restriction of a product of a smooth compactly supported function and F_λ^a composed with a smooth diffeomorphism.
Denote by $T_{\mu\lambda}^{ba} f_\lambda^a$ the function on the right-hand side of (9.82). The Faà di Bruno formula in Theorem 1.1.19 controls derivatives of the composition through order k. For derivatives of order k, we also use

$$[fg]_{C^{0,\alpha}(K)} \le \|f\|_{C^0(K)} [g]_{C^{0,\alpha}(K)} + \|g\|_{C^0(K)} [f]_{C^{0,\alpha}(K)}$$

and the fact that a smooth map is Lipschitz on a compact set. We obtain a constant $C_{\lambda\mu} > 0$ such that

$$\|T_{\mu\lambda}^{ba} f_\lambda^a\|_{C^{k,\alpha}(\overline{\widetilde{\Omega}_\mu})} \le C_{\lambda\mu} \|f_\lambda^a\|_{C^{k,\alpha}(\overline{\Omega_\lambda})}.$$

The inequality is first obtained for an extension F_λ^a, and then the infimum over all extensions is taken in the norm definition.
Sum over the finitely many overlaps, components, and indices. The maximum of the preceding constants is finite, and (9.81) gives

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\widetilde{\mathcal{A}}}(\mathbf{E})} \le C \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{A}}(\mathbf{E})}.$$

Interchanging the two systems gives the reverse inequality.
We prove completeness. If $(\mathbf{u}_j)_{j\in\mathbb{N}}$ is Cauchy for one of these norms, each family of localized components converges in the Banach space $C^{k,\alpha}(\overline{\Omega_\lambda})$, whose completeness was established in Proposition 3.5.4. Reconstruct the limits in each frame, multiply them by cutoffs equal to one on the supports of ψ_λ, and extend them by zero outside U_λ. The finite sum of the resulting sections defines a section $\mathbf{u}$ of class C^k up to the boundary. Since $\mathbf{u}_j = \sum_{\lambda=1}^{N} \psi_\lambda \mathbf{u}_j$, the same transfer estimate shows that $\mathbf{u}_j \to \mathbf{u}$ in $\Gamma^{k,\alpha}(\mathbf{E})$. □

Theorem 9.4.34. (Sobolev–Morrey embedding for bundles over manifolds with boundary) Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank, equipped with a bundle metric and a compatible connection. Let $m \in \mathbb{N}$, $1 \le p < \infty$, $k \in \mathbb{N}_0$, and $0 < \alpha < 1$. If

$$k + \alpha \le m - \frac{n}{p}, \tag{9.83}$$

then each element of $W^{m,p}(M,\mathbf{E})$ has a unique representative in $\Gamma^{k,\alpha}(\mathbf{E})$, and there is $C>0$ such that

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}. \tag{9.84}$$

The restriction of this representative to ∂M agrees with **Tru**.

Proof. Use the local data from Definition 9.4.32. For $\mathbf{u}\in W^{m,p}(M,\mathbf{E})$ write

$$\psi_\lambda \mathbf{u} = \sum_{a=1}^{r} u_\lambda^a \mathbf{e}_{\lambda,a} \qquad \text{in } U_\lambda$$

and set $v_\lambda^a := u_\lambda^a \circ \varphi_\lambda^{-1}$. Continuity of multiplication by ψ_λ, local equivalence between ordinary and covariant derivatives, and finiteness of the family give a constant $C_0>0$, independent of $\mathbf{u}$, such that

$$\sum_{\lambda=1}^{N}\sum_{a=1}^{r} \|v_\lambda^a\|_{W^{m,p}(\Omega_\lambda)} \le C_0\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}. \tag{9.85}$$

If U_λ is an interior chart, the support of v_λ^a is compact in Ω_λ and therefore $v_\lambda^a \in W_0^{m,p}(\Omega_\lambda)$. Theorem 3.5.7 gives

$$\|v_\lambda^a\|_{C^{k,\alpha}(\overline{\Omega_\lambda})} \le C_\lambda \|v_\lambda^a\|_{W^{m,p}(\Omega_\lambda)}. \tag{9.86}$$

Suppose that U_λ is a boundary chart and write $\Omega_\lambda = B_\lambda^+$. There is a compact set

$$K_\lambda \subseteq B_\lambda^+ \cup B_\lambda'$$

containing the essential supports of all components v_λ^a and lying at positive distance from the spherical part of ∂B_λ^+. Lemma 3.6.8 permits extension of v_λ^a by zero across that spherical part, but not across the flat face. Denote this first extension by w_λ^a. Then

$$w_\lambda^a \in W^{m,p}(\mathbb{R}^n_+), \qquad \|w_\lambda^a\|_{W^{m,p}(\mathbb{R}^n_+)} = \|v_\lambda^a\|_{W^{m,p}(B_\lambda^+)}.$$

Now apply the extension operator

$$\mathcal{E}_m \colon W^{m,p}(\mathbb{R}^n_+) \longrightarrow W^{m,p}(\mathbb{R}^n)$$

from Corollary 3.6.4. Choose $R_\lambda>0$ and $\chi_\lambda \in C_c^\infty(B_{\mathrm{euc}}(0,2R_\lambda))$ such that $\chi_\lambda = 1$ on a neighborhood of K_λ, and define

$$T_\lambda v_\lambda^a := \chi_\lambda \mathcal{E}_m w_\lambda^a.$$

This function belongs to $W_0^{m,p}(B_{\mathrm{euc}}(0,2R_\lambda))$, its restriction to B_λ^+ agrees with v_λ^a, and

$$\|T_\lambda v_\lambda^a\|_{W^{m,p}(B_{\mathrm{euc}}(0,2R_\lambda))} \le C_\lambda' \|v_\lambda^a\|_{W^{m,p}(B_\lambda^+)}.$$

Theorem 3.5.7, applied to $T_\lambda v_\lambda^a$, and the definition of the extension norm give

$$\|v_\lambda^a\|_{C^{k,\alpha}(\overline{B_\lambda^+})} \le C_\lambda'' \|v_\lambda^a\|_{W^{m,p}(B_\lambda^+)}. \tag{9.87}$$

For each λ, the continuous representatives of $v_\lambda^1,\dots,v_\lambda^r$ reconstruct a localized section $\mathbf{s}_\lambda$ on U_λ. This sec-

tion agrees almost everywhere with $\psi_\lambda \mathbf{u}$ and, since its components vanish outside the fixed compact set determined by $\operatorname{supp}(\psi_\lambda)$, it extends by zero to a global section of class C^k up to the boundary. Define the global representative by

$$\widetilde{\mathbf{u}} := \sum_{\lambda=1}^{N} \mathbf{s}_\lambda.$$

The sum is finite and agrees almost everywhere with $\sum_{\lambda=1}^{N} \psi_\lambda \mathbf{u} = \mathbf{u}$. Definition 9.4.32, the local estimates, and (9.85) give

$$\|\widetilde{\mathbf{u}}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C \sum_{\lambda=1}^{N} \sum_{a=1}^{r} \|v_\lambda^a\|_{W^{m,p}(\Omega_\lambda)} \le C' \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

If two representatives continuous up to the boundary define the same class in $L^p(M,\mathbf{E})$, the relative set where they differ is open. Every nonempty relative coordinate ball or half-ball has positive Riemannian volume, so that set is empty. This proves uniqueness.

It remains to identify restriction to the boundary. By Theorem 9.4.8, choose $(\mathbf{u}_j)_{j\in\mathbb{N}} \subseteq \Gamma(\mathbf{E})$ such that $\mathbf{u}_j \to \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. Applying (9.84) to differences shows that $\mathbf{u}_j$ converges to the preceding representative in $\Gamma^{k,\alpha}(\mathbf{E})$ and, in particular, uniformly on ∂M. On the other hand, continuity in Theorem 9.4.12 implies

$$\mathbf{u}_j \restriction_{\partial M} = \mathbf{Tr}\mathbf{u}_j \longrightarrow \mathbf{Tr}\mathbf{u} \qquad \text{in } L^p(\partial M, \mathbf{E} \restriction_{\partial M}).$$

Uniform convergence gives the same limit in L^p; uniqueness of the limit identifies the restriction of the representative with the trace. □

Theorem 9.4.35. (Rellich–Kondrashov into Hölder sections up to the boundary) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $m \in \mathbb{N}$, $1 \le p < \infty$, $k \in \mathbb{N}_0$, and $0 < \alpha < 1$. If

$$k + \alpha < m - \frac{n}{p},$$

then the embedding

$$W^{m,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{k,\alpha}(\mathbf{E})$$

is compact.

Proof. Let $(\mathbf{u}_j)_{j\in\mathbb{N}}$ be a bounded sequence in $W^{m,p}(M,\mathbf{E})$, and use the localizations from the preceding proof. In an interior chart, each component sequence is bounded in $W_0^{m,p}(\Omega_\lambda)$; Theorem 3.5.10 gives a subsequence converging in $C^{k,\alpha}(\overline{\Omega_\lambda})$.

In a boundary chart, apply to each component the operator T_λ constructed in the preceding proof. The functions $T_\lambda v_{\lambda,j}^a$ form a bounded sequence in $W_0^{m,p}(B_{\mathrm{euc}}(0,2R_\lambda))$. The same Euclidean theorem gives a subsequence converging in $C^{k,\alpha}(\overline{B_{\mathrm{euc}}(0,2R_\lambda)})$; upon restriction, the original components converge in $C^{k,\alpha}(\overline{B_\lambda^+})$. There are only finitely many charts and components. A diagonal extraction produces a single subsequence for which all localized components are Cauchy in their Hölder spaces. By (9.80), the subsequence is Cauchy in $\Gamma^{k,\alpha}(\mathbf{E})$ and converges by completeness established in Proposition 9.4.33. □

Corollary 9.4.36. (Regularity exponents up to the boundary) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $m\in\mathbb{N}$ and $1\le p<\infty$, and let $\sigma := m-\frac{n}{p}>0$.

(a) If $\sigma=\ell+\theta$, with $\ell\in\mathbb{N}_0$ and $0<\theta<1$, then

$$W^{m,p}(M,\mathbf{E})\hookrightarrow \Gamma^{\ell,\theta}(\mathbf{E})$$

continuously, and the embedding into $\Gamma^{\ell,\beta}(\mathbf{E})$ is compact for every $0<\beta<\theta$.

(b) If $\sigma=s\in\mathbb{N}$, then, for every $0<\alpha<1$,

$$W^{m,p}(M,\mathbf{E})\hookrightarrow \Gamma^{s-1,\alpha}(\mathbf{E})$$

compactly and, in particular, continuously.

Proof. For $\sigma=\ell+\theta$, the Sobolev–Morrey theorem up to the boundary, Theorem 9.4.34, applied with $k=\ell$ and $\alpha=\theta$, gives the continuous embedding. If $0<\beta<\theta$, then

$$\ell+\beta<m-\frac{n}{p},$$

and Theorem 9.4.35 gives the compact embedding into $\Gamma^{\ell,\beta}(\mathbf{E})$.
If $\sigma=s\in\mathbb{N}$ and $0<\alpha<1$, then

$$(s-1)+\alpha<s=m-\frac{n}{p}.$$

Applying the Rellich–Kondrashov theorem up to the boundary once more with $k=s-1$ gives the compact, and hence continuous, embedding into $\Gamma^{s-1,\alpha}(\mathbf{E})$. □

9.5 The weak Kato inequality

Earlier we proved Kato's inequality for smooth sections. We now extend it to sections in $W^{m,p}(M,\mathbf{E})$ by Theorem 9.4.8. When $1<p<\infty$, reflexivity of the $L^p(M,\mathbf{F})$ spaces of sections of finite-rank bundles allows us to extract the weakly convergent subsequences needed. To include the case $p=1$, where $L^1(M,\mathbf{F})$ is not reflexive, we use the Dunford–Pettis theorem.

Recall that, if $(X,\mathcal{A},\mu)$ is a measure space, a family $\mathcal{F}\subseteq L^1(X,\mathcal{A},\mu)$ is uniformly integrable if $\sup_{f\in\mathcal{F}}\|f\|_{L^1(X)}<\infty$ and, for every $\varepsilon>0$, there is $\delta>0$ such that $\int_A |f|\,d\mu<\varepsilon$ for any $f\in\mathcal{F}$ and $A\in\mathcal{A}$ with $\mu(A)<\delta$.

Theorem 9.5.1. (Dunford–Pettis) Let $(X,\mathcal{A},\mu)$ be a finite measure space. A family $\mathcal{F}\subseteq L^1(X,\mathcal{A},\mu)$ has weakly compact closure in $L^1(X,\mathcal{A},\mu)$ if and only if $\mathcal{F}$ is uniformly integrable.

This is the classical form of the Dunford–Pettis criterion for a finite measure; see [22].

By Theorem B.7.7, the weak topology of $L^1(X,\mathcal{A},\mu)$ agrees with $\sigma(L^1(X,\mathcal{A},\mu),L^\infty(X,\mathcal{A},\mu))$. Thus the preceding statement is equivalent to the classical formulation using L^1–L^∞ duality.

Theorem 9.5.2. (Dunford–Pettis for vector bundles) Let $(M, \mathbf{g})$ be a compact Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$. For a family $\mathcal{F} \subseteq L^1(M, \mathbf{E})$, the following assertions are equivalent:

(a) The weak closure of $\mathcal{F}$ in $L^1(M, \mathbf{E})$ is compact.

(b) The family $|\mathcal{F}|_{\mathbf{h}_{\mathbf{E}}} := \{|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} \mid \mathbf{u} \in \mathcal{F}\} \subseteq L^1(M)$ is uniformly integrable.

Proof. Since M is compact, there is a finite open cover $(U_\alpha)_{\alpha=1}^N$ of M with a local trivialization of $\mathbf{E}$ defined over each U_α, together with a smooth partition of unity $(\chi_\alpha)_{\alpha=1}^N$ subordinate to this cover. On each U_α fix a smooth local orthonormal frame $(\mathbf{e}_{\alpha,1}, \dots, \mathbf{e}_{\alpha,r})$.
For $\alpha \in \{1, \dots, N\}$ and $a \in \{1, \dots, r\}$, define

$$T_{\alpha,a} \colon L^1(M, \mathbf{E}) \longrightarrow L^1(U_\alpha), \qquad T_{\alpha,a}(\mathbf{u}) := \langle \chi_\alpha \mathbf{u}, \mathbf{e}_{\alpha,a} \rangle_{\mathbf{h}_{\mathbf{E}}}.$$

Since

$$|T_{\alpha,a}(\mathbf{u})| \le \chi_\alpha |\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} \le |\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}},$$

we have

$$\|T_{\alpha,a}(\mathbf{u})\|_{L^1(U_\alpha)} \le \|\mathbf{u}\|_{L^1(M,\mathbf{E})}.$$

Thus $T_{\alpha,a}$ is linear and norm continuous. By Proposition B.11.12, it is also continuous for the weak topologies.
First suppose that the weak closure $C := \overline{\mathcal{F}}^{\sigma(L^1(M,\mathbf{E}),(L^1(M,\mathbf{E}))')}$ is compact. Weak continuity of $T_{\alpha,a}$ implies that $T_{\alpha,a}(C)$ is weakly compact in $L^1(U_\alpha)$ and contains $T_{\alpha,a}(\mathcal{F})$. By the Hahn–Banach theorem, $(L^1(U_\alpha))'$ separates points of $L^1(U_\alpha)$; by Proposition B.11.7, the weak topology of this space is Hausdorff. Consequently, $T_{\alpha,a}(C)$ is weakly closed. Thus

$$\overline{T_{\alpha,a}(\mathcal{F})}^{\sigma(L^1(U_\alpha),(L^1(U_\alpha))')} \subseteq T_{\alpha,a}(C),$$

and the weak closure of $T_{\alpha,a}(\mathcal{F})$ is compact. Theorem 9.5.1 shows that each family $T_{\alpha,a}(\mathcal{F})$ is uniformly integrable.
For every $\mathbf{u} \in \mathcal{F}$, since

$$\mathbf{u} = \sum_{\alpha=1}^N \chi_\alpha \mathbf{u}$$

and, on U_α,

$$\chi_\alpha \mathbf{u} = \sum_{a=1}^r T_{\alpha,a}(\mathbf{u}) \mathbf{e}_{\alpha,a},$$

we have

$$|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} \le \sum_{\alpha=1}^N |\chi_\alpha \mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} \le \sum_{\alpha=1}^N \sum_{a=1}^r |T_{\alpha,a}(\mathbf{u})|.$$

In particular,

$$\sup_{\mathbf{u}\in\mathcal{F}} \|\mathbf{u}\|_{L^1(M,\mathbf{E})} \le \sum_{\alpha=1}^N \sum_{a=1}^r \sup_{\mathbf{u}\in\mathcal{F}} \|T_{\alpha,a}(\mathbf{u})\|_{L^1(U_\alpha)} < \infty.$$

Given $\varepsilon > 0$, for each $(\alpha, a) \in \{1, \dots, N\} \times \{1, \dots, r\}$ there is $\delta_{\alpha,a} > 0$ such that

$$\int_A |T_{\alpha,a}(\mathbf{u})|\, d\lambda_{\mathbf{g}} < \frac{\varepsilon}{Nr}$$

for every $\mathbf{u} \in \mathcal{F}$ and every measurable set $A \subseteq U_\alpha$ with $\lambda_{\mathbf{g}}(A) < \delta_{\alpha,a}$. Taking

$$\delta := \min_{\substack{1 \le \alpha \le N \\ 1 \le a \le r}} \delta_{\alpha,a},$$

if $A \subseteq M$ is measurable and $\lambda_{\mathbf{g}}(A) < \delta$, then

$$\int_A |\mathbf{u}|_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \le \sum_{\alpha=1}^{N} \sum_{a=1}^{r} \int_{A \cap U_\alpha} |T_{\alpha,a}(\mathbf{u})|\, d\lambda_{\mathbf{g}} < \varepsilon$$

for every $\mathbf{u} \in \mathcal{F}$. Consequently, $|\mathcal{F}|_{\mathbf{h_E}}$ is uniformly integrable.
Conversely, suppose that $|\mathcal{F}|_{\mathbf{h_E}}$ is uniformly integrable. Since

$$|T_{\alpha,a}(\mathbf{u})| \le |\mathbf{u}|_{\mathbf{h_E}},$$

we have

$$\sup_{\mathbf{u} \in \mathcal{F}} \|T_{\alpha,a}(\mathbf{u})\|_{L^1(U_\alpha)} \le \sup_{\mathbf{u} \in \mathcal{F}} \|\mathbf{u}\|_{L^1(M,\mathbf{E})} < \infty.$$

Moreover, given $\varepsilon > 0$, there is $\delta > 0$ such that, if $A \subseteq U_\alpha$ is measurable and $\lambda_{\mathbf{g}}(A) < \delta$, then

$$\int_A |T_{\alpha,a}(\mathbf{u})|\, d\lambda_{\mathbf{g}} \le \int_A |\mathbf{u}|_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}} < \varepsilon$$

for every $\mathbf{u} \in \mathcal{F}$. Thus each family $T_{\alpha,a}(\mathcal{F})$ is uniformly integrable. Theorem 9.5.1 implies that its weak closure

$$K_{\alpha,a} := \overline{T_{\alpha,a}(\mathcal{F})}^{\sigma(L^1(U_\alpha),(L^1(U_\alpha))')}$$

is compact.
Consider the finite product

$$\mathbb{X} := \prod_{\alpha=1}^{N} \prod_{a=1}^{r} L^1(U_\alpha)$$

equipped, for example, with the norm

$$\|(f_{\alpha,a})_{\alpha,a}\|_{\mathbb{X}} := \sum_{\alpha=1}^{N} \sum_{a=1}^{r} \|f_{\alpha,a}\|_{L^1(U_\alpha)}.$$

Since the product has finitely many factors, the topology of this norm agrees with the product topology. By Proposition B.11.14, the weak topology of $\mathbb{X}$ agrees with the product of the weak topologies of the factors. Consequently,

$$K := \prod_{\alpha=1}^{N} \prod_{a=1}^{r} K_{\alpha,a}$$

is weakly compact in $\mathbb{X}$.

Define

$$T\colon L^1(M,\mathbf{E}) \longrightarrow \mathbb{X}, \qquad T(\mathbf{u}) := (T_{\alpha,a}(\mathbf{u}))_{\alpha,a},$$

and

$$S\colon \mathbb{X} \longrightarrow L^1(M,\mathbf{E}), \qquad S((f_{\alpha,a})_{\alpha,a}) := \sum_{\alpha=1}^{N}\sum_{a=1}^{r} \widetilde{f_{\alpha,a}\mathbf{e}_{\alpha,a}},$$

where the tilde denotes extension by zero outside U_α. The operator S is linear and satisfies

$$\|S((f_{\alpha,a})_{\alpha,a})\|_{L^1(M,\mathbf{E})} \le \sum_{\alpha=1}^{N}\sum_{a=1}^{r} \|f_{\alpha,a}\|_{L^1(U_\alpha)} = \|(f_{\alpha,a})_{\alpha,a}\|_{\mathbb{X}}.$$

Thus S is norm continuous and, by Proposition B.11.12, is continuous from the weak topology of $\mathbb{X}$ to the weak topology of $L^1(M,\mathbf{E})$. Since K is weakly compact, $S(K)$ is weakly compact in $L^1(M,\mathbf{E})$.
Finally, for every $\mathbf{u} \in L^1(M,\mathbf{E})$,

$$S(T(\mathbf{u})) = \sum_{\alpha=1}^{N}\sum_{a=1}^{r} \widetilde{\langle \chi_\alpha \mathbf{u}, \mathbf{e}_{\alpha,a}\rangle_{\mathbf{h}_\mathbf{E}} \mathbf{e}_{\alpha,a}} = \sum_{\alpha=1}^{N} \chi_\alpha \mathbf{u} = \mathbf{u}$$

almost everywhere. Since $T(\mathcal{F}) \subseteq K$, it follows that $\mathcal{F} = S(T(\mathcal{F})) \subseteq S(K)$. Again by Hahn–Banach and Proposition B.11.7, the weak topology of $L^1(M,\mathbf{E})$ is Hausdorff. Thus the compact set $S(K)$ is weakly closed. Consequently, the weak closure of $\mathcal{F}$ is contained in $S(K)$ and is a weakly closed subset of this compact set. Therefore, the weak closure of $\mathcal{F}$ is compact. □

We need the following consequence of weak lower semicontinuity of convex functionals.

Lemma 9.5.3. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{F} \to M$ be a vector bundle of finite rank equipped with a bundle metric $\mathbf{h}_\mathbf{F}$, let $1 \le p < \infty$, and let p' be the conjugate exponent of p. For every nonnegative function $\varphi \in L^{p'}(M)$, the functional

$$\Phi_\varphi\colon L^p(M,\mathbf{F}) \longrightarrow [0,\infty), \qquad \Phi_\varphi(\mathbf{Z}) := \int_M \varphi |\mathbf{Z}|_{\mathbf{h}_\mathbf{F}}\, d\lambda_\mathbf{g},$$

is convex, norm continuous, and lower semicontinuous with respect to $\sigma(L^p(M,\mathbf{F}),(L^p(M,\mathbf{F}))')$.

Proof. Convexity follows from the triangle inequality in the fibers. Moreover, for any $\mathbf{Z},\mathbf{W} \in L^p(M,\mathbf{F})$, Hölder's inequality (Proposition B.7.9) implies

$$|\Phi_\varphi(\mathbf{Z}) - \Phi_\varphi(\mathbf{W})| \le \int_M \varphi |\mathbf{Z}-\mathbf{W}|_{\mathbf{h}_\mathbf{F}}\, d\lambda_\mathbf{g} \le \|\varphi\|_{L^{p'}(M)} \|\mathbf{Z}-\mathbf{W}\|_{L^p(M,\mathbf{F})}.$$

Thus Φ_φ is norm continuous. Since it is also convex, Corollary B.11.17 implies that it is lower semicontinuous with respect to $\sigma(L^p(M,\mathbf{F}),(L^p(M,\mathbf{F}))')$. □

By Lemma 7.2.5, we know that, for a smooth section $\mathbf{u}$, $|\mathbf{u}|_{\mathbf{h}_\mathbf{E}}$ admits a weak derivative almost everywhere on M. Now that we have the concept of a *weak covariant derivative*, it is natural to ask whether this almost-everywhere defined classical derivative agrees with a weak derivative. The following proposition answers this question, which will be important in proving our weak version of Kato's inequality.

Lemma 9.5.4. (Classical and weak derivatives of the modulus of a smooth section) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{F}\to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_F}$ and a compatible connection $\nabla^{\mathbf{F}}$. Let $1\le p<\infty$ and $\mathbf{v}\in\Gamma(\mathbf{F})$. Then $|\mathbf{v}|_{\mathbf{h_F}}\in W^{1,p}_{\mathrm{loc}}(M)$ and $\nabla_w|\mathbf{v}|_{\mathbf{h_F}}=\nabla|\mathbf{v}|_{\mathbf{h_F}}$ almost everywhere on M, where the right-hand side denotes the classical covariant derivative defined almost everywhere. Moreover, $\left|\nabla_w|\mathbf{v}|_{\mathbf{h_F}}\right|_{\mathbf{g}}\le|\nabla^{\mathbf{F}}\mathbf{v}|_{\mathbf{g},\mathbf{h_F}}$ almost everywhere. If, in addition, $\mathbf{v}\in L^p(M,\mathbf{F})$ and $\nabla^{\mathbf{F}}\mathbf{v}\in L^p(M,T^*M\otimes\mathbf{F})$, then $|\mathbf{v}|_{\mathbf{h_F}}\in W^{1,p}(M)$. In particular, this last conclusion holds automatically when M is compact.

Proof. By Lemma 7.2.5, $\nabla|\mathbf{v}|_{\mathbf{h_F}}$ exists almost everywhere on M. Define the measurable 1-form G by

$$G(x):=\begin{cases}\nabla|\mathbf{v}|_{\mathbf{h_F}}(x), & |\mathbf{v}(x)|_{\mathbf{h_F}}>0,\\ 0, & |\mathbf{v}(x)|_{\mathbf{h_F}}=0.\end{cases}$$

On the set where $|\mathbf{v}|_{\mathbf{h_F}}>0$, the classical Kato inequality, Theorem 7.2.6, applied with $s=0$, gives $|G|_{\mathbf{g}}=\left|\nabla|\mathbf{v}|_{\mathbf{h_F}}\right|_{\mathbf{g}}\le|\nabla^{\mathbf{F}}\mathbf{v}|_{\mathbf{g},\mathbf{h_F}}$. On the set where $|\mathbf{v}|_{\mathbf{h_F}}=0$, this inequality also holds by the definition of G. Consequently, $|G|_{\mathbf{g}}\le|\nabla^{\mathbf{F}}\mathbf{v}|_{\mathbf{g},\mathbf{h_F}}$ almost everywhere on M.
For each $\varepsilon>0$, consider the smooth function

$$f_\varepsilon:=\sqrt{|\mathbf{v}|^2_{\mathbf{h_F}}+\varepsilon^2}-\varepsilon.$$

On the open set where $|\mathbf{v}|_{\mathbf{h_F}}>0$, the chain rule implies

$$\nabla f_\varepsilon=\frac{|\mathbf{v}|_{\mathbf{h_F}}}{\sqrt{|\mathbf{v}|^2_{\mathbf{h_F}}+\varepsilon^2}}\nabla|\mathbf{v}|_{\mathbf{h_F}}=\frac{|\mathbf{v}|_{\mathbf{h_F}}}{\sqrt{|\mathbf{v}|^2_{\mathbf{h_F}}+\varepsilon^2}}G.$$

If $|\mathbf{v}|_{\mathbf{h_F}}=0$ at a point, the smooth function $|\mathbf{v}|^2_{\mathbf{h_F}}$ attains its minimum there, and hence $\nabla|\mathbf{v}|^2_{\mathbf{h_F}}=0$. Consequently,

$$\nabla f_\varepsilon=\frac{\nabla|\mathbf{v}|^2_{\mathbf{h_F}}}{2\sqrt{|\mathbf{v}|^2_{\mathbf{h_F}}+\varepsilon^2}}=0$$

at that point. Thus throughout M we have

$$\nabla f_\varepsilon=\frac{|\mathbf{v}|_{\mathbf{h_F}}}{\sqrt{|\mathbf{v}|^2_{\mathbf{h_F}}+\varepsilon^2}}G.$$

Since

$$0\le\frac{|\mathbf{v}|_{\mathbf{h_F}}}{\sqrt{|\mathbf{v}|^2_{\mathbf{h_F}}+\varepsilon^2}}\le 1,$$

it follows that $|\nabla f_\varepsilon|_{\mathbf{g}}\le|G|_{\mathbf{g}}\le|\nabla^{\mathbf{F}}\mathbf{v}|_{\mathbf{g},\mathbf{h_F}}$.
Moreover, $\nabla f_\varepsilon\longrightarrow G$ as $\varepsilon\to 0^+$ pointwise on M, since on the set where $|\mathbf{v}|_{\mathbf{h_F}}>0$ the preceding quotient converges to 1, while on the set where $|\mathbf{v}|_{\mathbf{h_F}}=0$ both 1-forms vanish. Let $K\subseteq M$ be compact. Since $\mathbf{v}$ is smooth, $|\nabla^{\mathbf{F}}\mathbf{v}|_{\mathbf{g},\mathbf{h_F}}\in L^p(K)$ and, by the dominated convergence theorem, $\nabla f_\varepsilon\longrightarrow G$ in $L^p(K,T^*M)$.
On the other hand, $0\le f_\varepsilon\le|\mathbf{v}|_{\mathbf{h_F}}$ and $f_\varepsilon\longrightarrow|\mathbf{v}|_{\mathbf{h_F}}$ pointwise. Again by dominated convergence, $f_\varepsilon\longrightarrow|\mathbf{v}|_{\mathbf{h_F}}$ in

$L^p(K)$.
Now let $\eta \in \Gamma_{c,\mathrm{Int}(M)}(T^*M)$ and choose a compact set K containing the support of η. Since f_ε is smooth,

$$\int_M \langle \nabla f_\varepsilon, \eta \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M f_\varepsilon \nabla^* \eta \, d\lambda_{\mathbf{g}}.$$

The preceding strong convergences and Hölder's inequality allow us to pass to the limit and obtain

$$\int_M \langle G, \eta \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M |\mathbf{v}|_{\mathbf{h_F}} \nabla^* \eta \, d\lambda_{\mathbf{g}}.$$

By the definition of weak covariant derivatives and local Sobolev spaces,

$$|\mathbf{v}|_{\mathbf{h_F}} \in W^{1,p}_{\mathrm{loc}}(M) \qquad \text{and} \qquad \nabla_w |\mathbf{v}|_{\mathbf{h_F}} = G$$

almost everywhere.
It remains to identify G with the classical derivative. On the set where $|\mathbf{v}|_{\mathbf{h_F}} > 0$, this equality holds by the definition of G itself. If $|\mathbf{v}|_{\mathbf{h_F}} = 0$ at a point where the classical derivative exists, the nonnegative function $|\mathbf{v}|_{\mathbf{h_F}}$ attains its minimum there, so $\nabla |\mathbf{v}|_{\mathbf{h_F}} = 0 = G$.
Since the classical derivative exists almost everywhere, we conclude that $\nabla_w |\mathbf{v}|_{\mathbf{h_F}} = \nabla |\mathbf{v}|_{\mathbf{h_F}}$ almost everywhere on M. The estimate already obtained for G gives

$$\left| \nabla_w |\mathbf{v}|_{\mathbf{h_F}} \right|_{\mathbf{g}} \leq |\nabla^{\mathbf{F}} \mathbf{v}|_{\mathbf{g},\mathbf{h_F}}$$

almost everywhere.
Finally, if $\mathbf{v} \in L^p(M,\mathbf{F})$ and $\nabla^{\mathbf{F}} \mathbf{v} \in L^p(M, T^*M \otimes \mathbf{F})$, the preceding weak identity therefore implies that $|\mathbf{v}|_{\mathbf{h_F}} \in W^{1,p}(M)$. □

Theorem 9.5.5. (Weak Kato inequality) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $m \in \mathbb{N}$, $1 \leq p < \infty$, and $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$. Then, for every $s \in \{0,\dots,m-1\}$, $|\nabla_w^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \in W^{1,p}(M)$ and

$$\left| \nabla_w |\nabla_w^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \right|_{\mathbf{g}} \leq |\nabla_w^{s+1} \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$$

almost everywhere on M, where $\nabla_w^0 \mathbf{u} := \mathbf{u}$.

Proof. Fix $s \in \{0,\dots,m-1\}$. By Theorem 9.1.48 if $\partial M = \varnothing$, or Theorem 9.4.8 if $\partial M \neq \varnothing$, there is a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}} \subseteq \Gamma(\mathbf{E}) \cap W^{m,p}(M,\mathbf{E})$ such that $\mathbf{u}_k \to \mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$. The reverse triangle inequality in the fibers implies

$$\left| |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} - |\nabla_w^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \right| \leq |\nabla^s \mathbf{u}_k - \nabla_w^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}},$$

so that $|\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \to |\nabla_w^s \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ in $L^p(M)$.
By Lemma 9.5.4, each function $|\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}}$ admits a first weak covariant derivative agreeing almost everywhere with its classical derivative. Kato's inequality for smooth sections, Theorem 7.2.6, gives

$$\left| \nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \right|_{\mathbf{g}} \leq |\nabla^{s+1} \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}}$$

almost everywhere. Since $|\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \in L^p(M)$ and $|\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \in L^p(M)$, we conclude that $|\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \in W^{1,p}(M)$.

Moreover, again by the reverse triangle inequality,

$$\left| |\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} - |\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \right| \leq |\nabla^{s+1}\mathbf{u}_k - \nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}},$$

so that $|\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \to |\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ in $L^p(M)$. In particular, the sequence $\left(\nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}}\right)_{k\in\mathbb{N}}$ is bounded in $L^p(M, T^*M)$.

If $1 < p < \infty$, reflexivity of $L^p(M, T^*M)$ allows us to extract a subsequence, which we continue to denote in the same way, and find $\mathbf{Z} \in L^p(M, T^*M)$ such that

$$\nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \rightharpoonup \mathbf{Z} \quad \text{in } L^p(M, T^*M).$$

Now suppose that $p = 1$. Since $|\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \to |\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ in $L^1(M)$, the family

$$\left\{ |\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \;\middle|\; k \in \mathbb{N} \right\}$$

is uniformly integrable. Convergence in $L^1(M)$ implies that this family is bounded in $L^1(M)$. On the other hand, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\left\| |\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} - |\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \right\|_{L^1(M)} < \frac{\varepsilon}{2}$$

for every $k \geq N$. Since every function in $L^1(M)$ has an integral absolutely continuous with respect to $\lambda_{\mathbf{g}}$, and only the functions $|\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}, |\nabla^{s+1}\mathbf{u}_1|_{\mathbf{g},\mathbf{h_E}}, \ldots, |\nabla^{s+1}\mathbf{u}_{N-1}|_{\mathbf{g},\mathbf{h_E}}$ are involved, there is $\delta > 0$ such that $\lambda_{\mathbf{g}}(A) < \delta$ implies

$$\int_A |\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} < \frac{\varepsilon}{2}$$

and

$$\int_A |\nabla^{s+1}\mathbf{u}_j|_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} < \varepsilon$$

for every $j \in \{1, \ldots, N-1\}$. For $k \geq N$ we obtain

$$\int_A |\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} \leq \left\| |\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} - |\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}} \right\|_{L^1(M)} + \int_A |\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}} < \varepsilon.$$

Thus the preceding family is uniformly integrable. Since

$$\left| \nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \right|_{\mathbf{g}} \leq |\nabla^{s+1}\mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}},$$

the family

$$\left\{ \left| \nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \right|_{\mathbf{g}} \;\middle|\; k \in \mathbb{N} \right\}$$

is also uniformly integrable. Theorem 9.5.2, applied to the bundle T^*M, implies that the weak closure of $\{\nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \mid k \in \mathbb{N}\}$ in $L^1(M, T^*M)$ is compact. By the Eberlein–Šmulian theorem (Theorem B.11.36), there

are a subsequence and $\mathbf{Z} \in L^1(M, T^*M)$ such that

$$\nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \rightharpoonup \mathbf{Z} \quad \text{in } L^1(M, T^*M).$$

Thus in both cases, after extracting a subsequence if necessary, we may assume that

$$\nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \rightharpoonup \mathbf{Z} \quad \text{in } L^p(M, T^*M)$$

for some $\mathbf{Z} \in L^p(M, T^*M)$.
Let $\eta \in \Gamma_{c,\mathrm{Int}(M)}(T^*M)$. Since $\nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$ is the weak covariant derivative of $|\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$, we have

$$\int_M \left\langle \nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, \eta \right\rangle_\mathbf{g} d\lambda_\mathbf{g} = \int_M |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \nabla^* \eta \, d\lambda_\mathbf{g}.$$

Weak convergence of the left-hand side and strong convergence $|\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \to |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$ in $L^p(M)$ allow us to pass to the limit, since $\nabla^* \eta \in L^{p'}(M)$, with the interpretation $p' = \infty$ when $p = 1$, and by Hölder's inequality (Proposition B.7.9)

$$\left| \int_M \left(|\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} - |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right) \nabla^* \eta \, d\lambda_\mathbf{g} \right| \leq \left\| |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} - |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right\|_{L^p(M)} \|\nabla^* \eta\|_{L^{p'}(M)} \longrightarrow 0.$$

Consequently,

$$\int_M \langle \mathbf{Z}, \eta \rangle_\mathbf{g} \, d\lambda_\mathbf{g} = \int_M |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \nabla^* \eta \, d\lambda_\mathbf{g}.$$

By the definition of weak covariant derivatives, $|\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \in W^{1,p}(M)$ and $\mathbf{Z} = \nabla_w |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$.
Finally, let $\varphi \in C^\infty_c(\mathrm{Int}(M))$ be nonnegative. By Lemma 9.5.3,

$$\int_M \varphi \left| \nabla_w |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right|_\mathbf{g} d\lambda_\mathbf{g} \leq \liminf_{k\to\infty} \int_M \varphi \left| \nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right|_\mathbf{g} d\lambda_\mathbf{g}.$$

Kato's inequality for smooth sections and convergence $|\nabla^{s+1} \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \to |\nabla^{s+1}_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$ in $L^p(M)$ imply

$$\liminf_{k\to\infty} \int_M \varphi \left| \nabla_w |\nabla^s \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right|_\mathbf{g} d\lambda_\mathbf{g} \leq \lim_{k\to\infty} \int_M \varphi |\nabla^{s+1} \mathbf{u}_k|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} = \int_M \varphi |\nabla^{s+1}_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g},$$

where the last convergence follows from Hölder's inequality in Proposition B.7.9, since $\varphi \in L^{p'}(M)$. Thus

$$\int_M \varphi \left(\left| \nabla_w |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right|_\mathbf{g} - |\nabla^{s+1}_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right) d\lambda_\mathbf{g} \leq 0$$

for every nonnegative $\varphi \in C^\infty_c(\mathrm{Int}(M))$. Since the function in parentheses belongs to $L^1_{\mathrm{loc}}(\mathrm{Int}(M))$, we conclude that

$$\left| \nabla_w |\nabla^s_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \right|_\mathbf{g} \leq |\nabla^{s+1}_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

almost everywhere on $\mathrm{Int}(M)$. Since ∂M has measure zero, the inequality also holds almost everywhere on M. □

Remark 9.5.6. The restriction $p < \infty$ comes from the use of Theorem 9.4.8, since smooth sections are not, in general, dense in $W^{m,\infty}(M, \mathbf{E})$ in its norm.

Corollary 9.5.7. (Weak Kato inequality) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E}\to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $m\in\mathbb{N}$, $1\le p<\infty$, and $\mathbf{u}\in W^{m,p}(M,\mathbf{E})$. Then, for every $s\in\{0,\dots,m-1\}$, we have $|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\in W^{1,p}(M)$ and $\left|\nabla_w|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\right|_{\mathbf{g}}\le|\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ almost everywhere on M.

Proof. Apply a smooth exhaustion to the manifold $\mathrm{Int}(M)$, which has no boundary: choose increasing open sets $(\Omega_j)_{j\in\mathbb{N}}$ with smooth boundary, $\overline{\Omega_j}\Subset\mathrm{Int}(M)$ and $\mathrm{Int}(M)=\bigcup_{j\in\mathbb{N}}\Omega_j$. If M is already compact, the preceding theorem also gives the conclusion directly; the exhaustion allows all cases to be treated by the same interior argument. By Proposition 9.1.26, for every $j\in\mathbb{N}$ and every $r\le m$ we have $\mathbf{u}\restriction_{\Omega_j}\in W^{m,p}(\Omega_j,\mathbf{E}\restriction_{\Omega_j})$ and $\nabla_w^r(\mathbf{u}\restriction_{\Omega_j})=(\nabla_w^r\mathbf{u})\restriction_{\Omega_j}$ almost everywhere on Ω_j. Identifying Ω_j with the interior of the compact manifold with boundary $\overline{\Omega_j}$, Theorem 9.5.5 implies that $|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\restriction_{\Omega_j}\in W^{1,p}(\Omega_j)$ and there is $\mathbf{v}_j\in L^p(\Omega_j,T^*\Omega_j)$ such that $\mathbf{v}_j=\nabla_w|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ on Ω_j and $|\mathbf{v}_j|_{\mathbf{g}}\le|\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ almost everywhere on Ω_j.
If $j\le k$, then $|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\restriction_{\Omega_j}$ is the restriction of $|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\restriction_{\Omega_k}$ to Ω_j, and by Proposition 9.1.26 the weak derivative of this restriction is $\mathbf{v}_k\restriction_{\Omega_j}$, while by definition it is also $\mathbf{v}_j$. Uniqueness of the weak covariant derivative gives $\mathbf{v}_k\restriction_{\Omega_j}=\mathbf{v}_j$ almost everywhere on Ω_j. Thus there is a measurable section $\mathbf{v}$ of T^*M such that $\mathbf{v}\restriction_{\Omega_j}=\mathbf{v}_j$ almost everywhere on Ω_j for every $j\in\mathbb{N}$. Since $|\mathbf{v}|_{\mathbf{g}}\le|\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ almost everywhere on each Ω_j and $\mathrm{Int}(M)=\bigcup_{j\in\mathbb{N}}\Omega_j$, this inequality holds almost everywhere on M, since ∂M has zero n-dimensional measure. Extend $\mathbf{v}$ by any value, for example zero, on ∂M, giving $\int_M|\mathbf{v}|_{\mathbf{g}}^p\,d\lambda_{\mathbf{g}}\le\int_M|\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}^p\,d\lambda_{\mathbf{g}}<\infty$ and consequently $\mathbf{v}\in L^p(M,T^*M)$.
Finally, let $\mathbf{X}\in\Gamma_{c,\mathrm{Int}(M)}(T^*M)$. Since $\mathrm{supp}(\mathbf{X})$ is a compact subset of the interior and $(\Omega_j)_{j\in\mathbb{N}}$ is an increasing exhaustion of it, there is $j\in\mathbb{N}$ such that $\mathrm{supp}(\mathbf{X})\subseteq\Omega_j$. Since $\mathbf{v}\restriction_{\Omega_j}=\mathbf{v}_j$ and $\mathbf{v}_j$ is the weak derivative of $|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\restriction_{\Omega_j}$, we have

$$\int_M\langle\mathbf{v},\mathbf{X}\rangle_{\mathbf{g}}\,d\lambda_{\mathbf{g}}=\int_{\Omega_j}\langle\mathbf{v}_j,\mathbf{X}\rangle_{\mathbf{g}}\,d\lambda_{\mathbf{g}}=\int_{\Omega_j}|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\nabla^*\mathbf{X}\,d\lambda_{\mathbf{g}}=\int_M|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\nabla^*\mathbf{X}\,d\lambda_{\mathbf{g}}.$$

The first and last equalities follow because $\mathrm{supp}(\mathbf{X})\subseteq\Omega_j$ and, by locality of ∇^*, $\mathrm{supp}(\nabla^*\mathbf{X})\subseteq\mathrm{supp}(\mathbf{X})$. Thus $\mathbf{v}=\nabla_w|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$. Since $|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\in L^p(M)$ and $\mathbf{v}\in L^p(M,T^*M)$, we conclude that $|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\in W^{1,p}(M)$, and the inequality $\left|\nabla_w|\nabla_w^s\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}\right|_{\mathbf{g}}\le|\nabla_w^{s+1}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ holds almost everywhere on M. □

9.6 The Poincaré inequality in the compact case

The Poincaré inequality controls the L^p norm of a section, and hence its Sobolev norm, solely in terms of its first weak covariant derivative. More precisely, an estimate of the form $\|\mathbf{u}\|_{W^{1,p}(M,\mathbf{E})}\le C\|\nabla_w^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$ is equivalent, by the definition of the Sobolev norm, to estimating $\|\mathbf{u}\|_{L^p(M,\mathbf{E})}$ by $\|\nabla_w^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$. Such an estimate, however, cannot hold throughout $W^{1,p}(M,\mathbf{E})$ when the connection admits nonzero parallel sections.

The trivial bundle immediately exhibits this obstruction. If M is a compact Riemannian manifold and $\mathbf{E}=M\times\mathbb{R}$ is equipped with the trivial metric and connection, and $u\equiv c$ with $c\neq0$, then $u\in W^{1,p}(M)$ and $\nabla_w u=0$, while $\|u\|_{L^p(M)}>0$. In a general vector bundle, the same obstruction is given by the kernel of the weak covariant derivative, that is, the sections $\mathbf{u}$ satisfying $\nabla_w^{\mathbf{E}}\mathbf{u}=0$. Thus the Poincaré inequality does not control a section absolutely, but only modulo the space of parallel sections.

This observation allows a unified treatment of the different forms of the inequality. One possibility is to project each section onto the kernel of the connection and estimate the difference between the section and its parallel component. For the trivial bundle over a connected manifold, the kernel consists of constant functions and the corresponding projection is precisely the mean. Another possibility is to impose a homogeneous Dirichlet condition: if every connected component of the manifold has nonempty boundary, no nonzero parallel section can have zero trace. Both formulations remove the same obstruction by different mechanisms.

We begin by describing the space of parallel sections and a continuous projection onto it. For this we need a preliminary lemma about functions whose weak derivative is 0.

Lemma 9.6.1. (Functions with zero weak covariant derivative) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $1 \le p < \infty$, and let $u \in W^{1,p}(M)$. If

$$\nabla_w u = 0$$

almost everywhere on M, then u is constant almost everywhere on each connected component of M. In particular, if M is connected, there is $c \in \mathbb{R}$ such that $u = c$ almost everywhere on M.

Proof. If $\dim M = 0$, the assertion is immediate. Henceforth suppose that $\dim M = n \ge 1$ and fix a connected component M_α of M.

By Proposition 1.2.11, M_α admits a countable open cover by regular coordinate balls and regular coordinate half-balls; write this cover as

$$(U_i, \varphi_i)_{i \in I}.$$

For each $i \in I$, define

$$\Omega_i := \varphi_i\big(U_i \cap \operatorname{Int}(M)\big).$$

Then Ω_i is an open ball or an open half-ball in $\mathbb{R}^n$, as appropriate; in particular, Ω_i is connected.

We now examine the situation locally. Since $u \in W^{1,p}(M)$ and $\nabla_w u = 0$ almost everywhere on M, Proposition 9.1.26 implies that

$$u \restriction_{U_i} \in W^{1,p}(U_i) \quad \text{and} \quad \nabla_w(u \restriction_{U_i}) = (\nabla_w u) \restriction_{U_i} = 0 \quad \text{almost everywhere on } U_i.$$

If M has boundary, the preceding application is justified by the identification described in Section 9.4. Indeed, $u \restriction_{\operatorname{Int}(M)}$ belongs to $W^{1,p}(\operatorname{Int}(M))$ and its weak covariant derivative agrees almost everywhere with $(\nabla_w u) \restriction_{\operatorname{Int}(M)} = 0$. Apply the proposition to the manifold without boundary $\operatorname{Int}(M)$ and the open set $U_i \cap \operatorname{Int}(M) = \operatorname{Int}(U_i)$. Using that identification again, now on U_i, gives the preceding assertions: the test sections have compact support in $\operatorname{Int}(U_i)$ and $U_i \cap \partial M$ has zero Riemannian measure, so both membership in $W^{1,p}$ and the identity of weak derivatives are preserved.

Since we are dealing with scalar functions, the covariant derivative agrees with the differential. Thus, applying Lemmas 9.1.44 and 9.4.5 to the trivial rank-one bundle, we find that the local representation

$$v_i := u \circ \varphi_i^{-1}$$

belongs to $W^{1,p}(\Omega_i)$ and satisfies

$$D_j v_i = 0 \quad \text{almost everywhere on } \Omega_i, \qquad j = 1, \dots, n.$$

In other words, all first Euclidean weak derivatives of v_i vanish almost everywhere.
By Lemma 3.1.5, there is a constant $c_i \in \mathbb{R}$ such that

$$v_i = c_i \quad \text{almost everywhere on } \Omega_i.$$

Returning to the manifold, this means that

$$u = c_i \quad \text{almost everywhere on } U_i \cap \mathrm{Int}(M).$$

We claim that in fact also

$$u = c_i \quad \text{almost everywhere on } U_i.$$

Indeed, if U_i is a regular coordinate ball, then $U_i \subseteq \mathrm{Int}(M)$ and there is nothing to prove. If U_i is a regular coordinate half-ball, then

$$U_i \setminus \big(U_i \cap \mathrm{Int}(M)\big) = U_i \cap \partial M.$$

Under the chart φ_i, this set corresponds to the flat face of a half-ball, which has zero n-dimensional Lebesgue measure. Since Riemannian measure in coordinates is given by $\sqrt{\det(\mathbf{g})}\, d\lambda_n$, with a smooth strictly positive factor, it follows that $U_i \cap \partial M$ also has zero Riemannian measure. Therefore,

$$u = c_i \quad \text{almost everywhere on } U_i.$$

We now show that the local constants agree on overlaps. Let $i, j \in I$ be such that $U_i \cap U_j \neq \varnothing$. Since $U_i \cap U_j$ is a nonempty open subset of the manifold M_α, there is $x \in U_i \cap U_j$. If $x \in \mathrm{Int}(M)$, some open neighborhood of x in M lies in $\mathrm{Int}(M)$, and hence

$$(U_i \cap U_j) \cap \mathrm{Int}(M) \neq \varnothing.$$

If $x \in \partial M$, since $U_i \cap U_j$ is open in the manifold with boundary, its expression in half-ball coordinates contains a small half-ball; in particular, it also contains points of $\mathrm{Int}(M)$. In either case,

$$(U_i \cap U_j) \cap \mathrm{Int}(M)$$

is a nonempty open subset of $\mathrm{Int}(M)$ and therefore has positive Riemannian measure.
But on that set we have, almost everywhere,

$$u = c_i \qquad \text{and} \qquad u = c_j.$$

It follows that $c_i = c_j$. Thus the constants $\{c_i\}_{i \in I}$ are compatible on nonempty intersections.
Since M_α is connected, we conclude that all constants c_i agree with a single constant, denoted by c_α. Moreover, since the cover is countable, the union of the exceptional null sets where the local equalities may fail still has measure zero. Therefore,

$$u = c_\alpha \quad \text{almost everywhere on } M_\alpha.$$

Since the connected component M_α was arbitrary, we conclude that u is constant almost everywhere on

each connected component of M. In particular, if M is connected, there is $c \in \mathbb{R}$ such that

$$u = c \quad \text{almost everywhere on } M.$$

□

Proposition 9.6.2. (The space of parallel sections and its projection) Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension n with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. For fixed $1 \le p < \infty$, define $\mathcal{K}(\mathbf{E}) := \left\{ \mathbf{v} \in W^{1,p}(M, \mathbf{E}) \middle| \nabla^{\mathbf{E}}_w \mathbf{v} = 0 \right\}$. Then this definition is independent of the exponent p, the space $\mathcal{K}(\mathbf{E})$ is finite-dimensional, and there is a continuous linear projection $\mathsf{P}_{\mathbf{E}} \colon L^p(M, \mathbf{E}) \longrightarrow \mathcal{K}(\mathbf{E})$.
More precisely, if $(\sigma_1, \dots, \sigma_N)$ is an orthonormal basis of $\mathcal{K}(\mathbf{E})$ with respect to the inner product of $L^2(M, \mathbf{E})$, one may take $\mathsf{P}_{\mathbf{E}} \mathbf{u} := \sum_{j=1}^{N} \left(\int_M \langle \mathbf{u}, \sigma_j \rangle_{\mathbf{h_E}} \, d\lambda_{\mathbf{g}} \right) \sigma_j$, with the convention $\mathsf{P}_{\mathbf{E}} = 0$ when $\mathcal{K}(\mathbf{E}) = \{0\}$.

Proof. If $\mathbf{v} \in \mathcal{K}(\mathbf{E})$, the weak Kato inequality, Corollary 9.5.7, implies that $|\mathbf{v}|_{\mathbf{h_E}} \in W^{1,p}(M)$ and $\left| \nabla_w |\mathbf{v}|_{\mathbf{h_E}} \right|_{\mathbf{g}} \le |\nabla^{\mathbf{E}}_w \mathbf{v}|_{\mathbf{g}, \mathbf{h_E}} = 0$ almost everywhere. Lemma 9.6.1 shows that $|\mathbf{v}|_{\mathbf{h_E}}$ is constant almost everywhere on each connected component of M. Since M is compact and has finitely many connected components, it follows that $\mathbf{v} \in L^\infty(M, \mathbf{E})$. Consequently, $\mathbf{v} \in L^q(M, \mathbf{E})$ for every $1 \le q < \infty$ and, since $\nabla^{\mathbf{E}}_w \mathbf{v} = 0$ is independent of the integrability exponent, we have $\mathbf{v} \in W^{1,q}(M, \mathbf{E})$ and $\nabla^{\mathbf{E}}_w \mathbf{v} = 0$ for every q. This proves that the kernel under consideration is independent of p.
Consider the closed ball $B := \left\{ \mathbf{v} \in \mathcal{K}(\mathbf{E}) \middle| \|\mathbf{v}\|_{L^p(M,\mathbf{E})} \le 1 \right\}$. Since $\nabla^{\mathbf{E}}_w \mathbf{v} = 0$ for every $\mathbf{v} \in \mathcal{K}(\mathbf{E})$, the norms $L^p(M, \mathbf{E})$ and $W^{1,p}(M, \mathbf{E})$ agree on $\mathcal{K}(\mathbf{E})$. By Corollary 9.4.29, applied with $m = 1$, B is relatively compact in $L^p(M, \mathbf{E})$. Moreover, B is closed in $L^p(M, \mathbf{E})$: if $\mathbf{v}_k \in B$ and $\mathbf{v}_k \to \mathbf{v}$ in $L^p(M, \mathbf{E})$, then $\nabla^{\mathbf{E}}_w \mathbf{v}_k = 0 \to 0$ in $L^p(M, T^*M \otimes \mathbf{E})$ and, by Lemma 9.1.12, $\mathbf{v} \in W^{1,p}(M, \mathbf{E})$ and $\nabla^{\mathbf{E}}_w \mathbf{v} = 0$. Thus B is compact, and consequently $\mathcal{K}(\mathbf{E})$ is finite-dimensional.
Every element of $\mathcal{K}(\mathbf{E})$ belongs to $L^\infty(M, \mathbf{E})$ and, in particular, to $L^2(M, \mathbf{E})$, so a basis $(\sigma_1, \dots, \sigma_N)$ orthonormal with respect to the inner product of $L^2(M, \mathbf{E})$ may be chosen. For $\mathbf{u} \in L^p(M, \mathbf{E})$, the integrals in the definition of $\mathsf{P}_{\mathbf{E}} \mathbf{u}$ are well defined and, denoting the conjugate exponent of p by p', with $p' = \infty$ if $p = 1$, Hölder's inequality (Proposition B.7.9) gives $\|\mathsf{P}_{\mathbf{E}} \mathbf{u}\|_{L^p(M,\mathbf{E})} \le \sum_{j=1}^{N} \|\sigma_j\|_{L^p(M,\mathbf{E})} \|\sigma_j\|_{L^{p'}(M,\mathbf{E})} \|\mathbf{u}\|_{L^p(M,\mathbf{E})}$. Thus $\mathsf{P}_{\mathbf{E}}$ is linear and continuous. Finally, if $\mathbf{v} \in \mathcal{K}(\mathbf{E})$, its expansion in the orthonormal basis $(\sigma_1, \dots, \sigma_N)$ shows that $\mathsf{P}_{\mathbf{E}} \mathbf{v} = \mathbf{v}$, so $\mathsf{P}_{\mathbf{E}}$ is a projection onto $\mathcal{K}(\mathbf{E})$. □

The Poincaré inequality can now be formulated as an estimate for a section modulo its parallel component.

Theorem 9.6.3. (Poincaré inequality modulo the kernel of the connection) Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension n with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. If $1 \le p < \infty$, then there is a constant $C_P > 0$, depending only on M, $\mathbf{g}$, $\mathbf{E}$, $\mathbf{h_E}$, $\nabla^{\mathbf{E}}$, p, and the projection $\mathsf{P}_{\mathbf{E}}$, such that $\|\mathbf{u} - \mathsf{P}_{\mathbf{E}} \mathbf{u}\|_{L^p(M,\mathbf{E})} \le C_P \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M, T^*M \otimes \mathbf{E})}$ for every $\mathbf{u} \in W^{1,p}(M, \mathbf{E})$.
Consequently, $\|\mathbf{u}\|_{\nabla^{\mathbf{E}}, p} := \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M, T^*M \otimes \mathbf{E})}$ defines a norm on $\ker(\mathsf{P}_{\mathbf{E}}) \cap W^{1,p}(M, \mathbf{E})$ equivalent to the usual norm of $W^{1,p}(M, \mathbf{E})$.

Proof. Suppose, for a contradiction, that the inequality is false. Then, for each $k \in \mathbb{N}$, there is $\mathbf{u}_k \in W^{1,p}(M,\mathbf{E})$ such that $\|\mathbf{u}_k - \mathsf{P}_{\mathbf{E}}\mathbf{u}_k\|_{L^p(M,\mathbf{E})} > k\|\nabla^{\mathbf{E}}_w \mathbf{u}_k\|_{L^p(M,T^*M\otimes\mathbf{E})}$. Define $\mathbf{v}_k := \dfrac{\mathbf{u}_k - \mathsf{P}_{\mathbf{E}}\mathbf{u}_k}{\|\mathbf{u}_k - \mathsf{P}_{\mathbf{E}}\mathbf{u}_k\|_{L^p(M,\mathbf{E})}}$. Since $\mathsf{P}_{\mathbf{E}}^2 = \mathsf{P}_{\mathbf{E}}$ and $\nabla^{\mathbf{E}}_w(\mathsf{P}_{\mathbf{E}}\mathbf{u}_k) = 0$, we have $\mathsf{P}_{\mathbf{E}}\mathbf{v}_k = 0$, $\|\mathbf{v}_k\|_{L^p(M,\mathbf{E})} = 1$, and $\|\nabla^{\mathbf{E}}_w \mathbf{v}_k\|_{L^p(M,T^*M\otimes\mathbf{E})} < \dfrac{1}{k}$. In particular, $(\mathbf{v}_k)$ is bounded in $W^{1,p}(M,\mathbf{E})$.

By Corollary 9.4.29, applied with $m = 1$, after passing to a subsequence there is $\mathbf{v} \in L^p(M,\mathbf{E})$ such that $\mathbf{v}_k \to \mathbf{v}$ in $L^p(M,\mathbf{E})$. On the other hand, $\nabla^{\mathbf{E}}_w \mathbf{v}_k \to 0$ in $L^p(M, T^*M \otimes \mathbf{E})$ and, by Lemma 9.1.12, we have $\mathbf{v} \in W^{1,p}(M,\mathbf{E})$ and $\nabla^{\mathbf{E}}_w \mathbf{v} = 0$. Therefore, $\mathbf{v} \in \mathcal{K}(\mathbf{E})$.

Continuity of $\mathsf{P}_{\mathbf{E}}$ on $L^p(M,\mathbf{E})$ and the identity $\mathsf{P}_{\mathbf{E}}\mathbf{v}_k = 0$ imply $\mathsf{P}_{\mathbf{E}}\mathbf{v} = 0$. However, since $\mathbf{v} \in \mathcal{K}(\mathbf{E})$ and $\mathsf{P}_{\mathbf{E}}$ is the identity on $\mathcal{K}(\mathbf{E})$, we also have $\mathsf{P}_{\mathbf{E}}\mathbf{v} = \mathbf{v}$. Consequently, $\mathbf{v} = 0$, contradicting $\|\mathbf{v}\|_{L^p(M,\mathbf{E})} = \lim\limits_{k\to\infty} \|\mathbf{v}_k\|_{L^p(M,\mathbf{E})} = 1$. This proves the inequality.

If $\mathbf{u} \in \ker(\mathsf{P}_{\mathbf{E}}) \cap W^{1,p}(M,\mathbf{E})$, then $\mathbf{u} - \mathsf{P}_{\mathbf{E}}\mathbf{u} = \mathbf{u}$ and therefore $\|\mathbf{u}\|_{L^p(M,\mathbf{E})} \le C_P \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$. This gives $\|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})} \le \|\mathbf{u}\|_{W^{1,p}(M,\mathbf{E})} \le (1 + C_P^p)^{\frac{1}{p}} \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$, proving equivalence of norms. □

Corollary 9.6.4. (Poincaré inequality with the mean) Let $(M,\mathbf{g})$ be a compact connected Riemannian manifold of dimension n with or without boundary, and let $1 \le p < \infty$. Then there is a constant $C_P > 0$ such that $\|u - \overline{u}\|_{L^p(M)} \le C_P \|\nabla_w u\|_{L^p(M,T^*M)}$ for every $u \in W^{1,p}(M)$, where $\overline{u} := \dfrac{1}{\lambda_{\mathbf{g}}(M)} \displaystyle\int_M u \, d\lambda_{\mathbf{g}}$.

Proof. For the trivial bundle $M \times \mathbb{R}$ equipped with the trivial connection, the space $\mathcal{K}(M \times \mathbb{R})$ consists of constant functions and has dimension 1. Since the constant function 1 satisfies $\|1\|_{L^2(M)} = \lambda_{\mathbf{g}}(M)^{\frac{1}{2}}$, the function $\sigma = \lambda_{\mathbf{g}}(M)^{-\frac{1}{2}}$ has unit norm in $L^2(M)$, so $\{\sigma\}$ is an orthonormal basis of $\mathcal{K}(M \times \mathbb{R})$. Consequently, $\mathsf{P}_{M\times\mathbb{R}} u = \left(\displaystyle\int_M u \, \lambda_{\mathbf{g}}(M)^{-\frac{1}{2}} \, d\lambda_{\mathbf{g}} \right) \lambda_{\mathbf{g}}(M)^{-\frac{1}{2}} = \dfrac{1}{\lambda_{\mathbf{g}}(M)} \displaystyle\int_M u \, d\lambda_{\mathbf{g}} = \overline{u}$. The result follows from Theorem 9.6.3. □

Remark 9.6.5. If M is disconnected and $M_1, \dots, M_\ell$ are its connected components, the kernel of the weak derivative consists of functions constant on each component. Consequently, the projection onto this kernel is given by $\mathsf{P}_{M\times\mathbb{R}} u = \displaystyle\sum_{i=1}^{\ell} \left(\frac{1}{\lambda_{\mathbf{g}}(M_i)} \int_{M_i} u \, d\lambda_{\mathbf{g}} \right) \mathbf{1}_{M_i}$.

This expression corresponds to subtracting the mean on each connected component. In particular, the global mean gives the correct formulation only when M is connected.

There are geometric situations in which a vector bundle admits no nontrivial parallel sections. In such cases the kernel of the weak covariant derivative is trivial, and the Poincaré inequality takes a particularly simple form.

Corollary 9.6.6. (The case without parallel sections) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank with a bundle metric and a compatible connection. Let $1 \le p < \infty$. If $\mathcal{K}(\mathbf{E}) = \{0\}$, then there is a constant $C_P > 0$ such that $\|\mathbf{u}\|_{L^p(M,\mathbf{E})} \le C_P \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$ for every $\mathbf{u} \in W^{1,p}(M,\mathbf{E})$. In particular, $\|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$ defines a norm on $W^{1,p}(M,\mathbf{E})$ equivalent to the usual Sobolev norm.

Proof. If $\mathcal{K}(\mathbf{E}) = \{0\}$, then $\mathsf{P}_{\mathbf{E}} = 0$ and the result follows directly from Theorem 9.6.3. □

The preceding formulation removes the parallel component of a section by a projection. When the manifold

has boundary, another natural possibility is to impose a homogeneous Dirichlet condition. To prove the corresponding inequality, we need to relate the trace of a section to the trace of its bundle norm.

Lemma 9.6.7. (Compatibility of the trace with the bundle norm) Let $(M, \mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. If $1 \leq p < \infty$ and $\mathbf{u} \in W^{1,p}(M, \mathbf{E})$, then $|\mathbf{u}|_{\mathbf{h_E}} \in W^{1,p}(M)$ and $\operatorname{Tr}(|\mathbf{u}|_{\mathbf{h_E}}) = |\mathbf{Tr}(\mathbf{u})|_{\mathbf{h_E}}$ almost everywhere on ∂M.

Proof. By Corollary 9.5.7, we have $|\mathbf{u}|_{\mathbf{h_E}} \in W^{1,p}(M)$. By the Meyers–Serrin theorem for vector bundles on manifolds with boundary, Theorem 9.4.8, there is a sequence $(\mathbf{u}_k) \subseteq \Gamma(\mathbf{E}) \cap W^{1,p}(M, \mathbf{E})$ such that $\mathbf{u}_k \to \mathbf{u}$ in $W^{1,p}(M, \mathbf{E})$. The reverse triangle inequality implies $\big\| |\mathbf{u}_k|_{\mathbf{h_E}} - |\mathbf{u}|_{\mathbf{h_E}} \big\|_{L^p(M)} \leq \|\mathbf{u}_k - \mathbf{u}\|_{L^p(M,\mathbf{E})} \to 0$.
Applying Corollary 9.5.7 once more, now to each $\mathbf{u}_k$, gives $|\mathbf{u}_k|_{\mathbf{h_E}} \in W^{1,p}(M)$ and

$$\big|\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}}\big|_{\mathbf{g}} \leq |\nabla^{\mathbf{E}}_w \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}}$$

almost everywhere. Since $\nabla^{\mathbf{E}}_w \mathbf{u}_k \to \nabla^{\mathbf{E}}_w \mathbf{u}$ in $L^p(M, T^*M \otimes \mathbf{E})$, the sequence $\big(\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}}\big)$ is bounded in $L^p(M, T^*M)$.
If $1 < p < \infty$, reflexivity of $L^p(M, T^*M)$ gives a subsequence and an element $\mathbf{Z} \in L^p(M, T^*M)$ such that

$$\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}} \rightharpoonup \mathbf{Z} \qquad \text{in } L^p(M, T^*M).$$

If $p = 1$, convergence $|\nabla^{\mathbf{E}}_w \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \to |\nabla^{\mathbf{E}}_w \mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$ in $L^1(M)$ implies that the family

$$\Big\{ |\nabla^{\mathbf{E}}_w \mathbf{u}_k|_{\mathbf{g},\mathbf{h_E}} \;\Big|\; k \in \mathbb{N} \Big\}$$

is uniformly integrable. The weak Kato inequality then shows that

$$\Big\{ \big|\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}}\big|_{\mathbf{g}} \;\Big|\; k \in \mathbb{N} \Big\}$$

is also uniformly integrable. By Theorem 9.5.2, applied to the bundle T^*M, the weak closure of $\{\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}} \mid k \in \mathbb{N}\}$ in $L^1(M, T^*M)$ is compact. The Eberlein–Šmulian theorem (Theorem B.11.36) therefore allows us to extract a subsequence and find $\mathbf{Z} \in L^1(M, T^*M)$ such that

$$\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}} \rightharpoonup \mathbf{Z} \qquad \text{in } L^1(M, T^*M).$$

Thus in both cases, after passing to a subsequence, we may assume that $\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}} \rightharpoonup \mathbf{Z}$ in $L^p(M, T^*M)$ for some $\mathbf{Z} \in L^p(M, T^*M)$. For every $\eta \in \Gamma_{c,\mathrm{Int}(M)}(T^*M)$, since $\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}}$ is the weak covariant derivative of $|\mathbf{u}_k|_{\mathbf{h_E}}$, we have

$$\int_M \big\langle \nabla_w |\mathbf{u}_k|_{\mathbf{h_E}}, \eta \big\rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M |\mathbf{u}_k|_{\mathbf{h_E}} \nabla^* \eta \, d\lambda_{\mathbf{g}}.$$

Since $\eta \in L^{p'}(M, T^*M)$ and $\nabla^* \eta \in L^{p'}(M)$, with the convention $p' = \infty$ when $p = 1$, weak convergence of $\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}}$ allows passage to the limit on the left-hand side. On the right-hand side we use strong convergence

$|\mathbf{u}_k|_{\mathbf{h_E}} \to |\mathbf{u}|_{\mathbf{h_E}}$ in $L^p(M)$; indeed, by Hölder's inequality,

$$\left| \int_M (|\mathbf{u}_k|_{\mathbf{h_E}} - |\mathbf{u}|_{\mathbf{h_E}}) \nabla^* \eta \, d\lambda_{\mathbf{g}} \right| \leq \big\| |\mathbf{u}_k|_{\mathbf{h_E}} - |\mathbf{u}|_{\mathbf{h_E}} \big\|_{L^p(M)} \|\nabla^* \eta\|_{L^{p'}(M)} \longrightarrow 0.$$

Consequently,

$$\int_M \langle \mathbf{Z}, \eta \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M |\mathbf{u}|_{\mathbf{h_E}} \nabla^* \eta \, d\lambda_{\mathbf{g}},$$

so that $\mathbf{Z} = \nabla_w |\mathbf{u}|_{\mathbf{h_E}}$.
Consider the isometric embedding

$$J_1 : W^{1,p}(M) \longrightarrow L^p(M) \times L^p(M, T^*M), \qquad J_1(w) := (w, \nabla_w w),$$

where the product is equipped with the norm ℓ^p, whose topology agrees with the product topology. Strong convergence $|\mathbf{u}_k|_{\mathbf{h_E}} \to |\mathbf{u}|_{\mathbf{h_E}}$ in $L^p(M)$ also implies weak convergence in that space and, together with $\nabla_w |\mathbf{u}_k|_{\mathbf{h_E}} \rightharpoonup \nabla_w |\mathbf{u}|_{\mathbf{h_E}}$ in $L^p(M, T^*M)$, Proposition B.11.14 gives

$$J_1(|\mathbf{u}_k|_{\mathbf{h_E}}) \rightharpoonup J_1(|\mathbf{u}|_{\mathbf{h_E}})$$

in $L^p(M) \times L^p(M, T^*M)$. By Theorem 9.1.14, applied to the trivial bundle, the image of J_1 is closed. Proposition B.11.13 then shows that this convergence is also weak in the image of J_1. Since $J_1 \colon W^{1,p}(M) \longrightarrow \mathrm{Im}(J_1)$ is an isometric isomorphism, its inverse $J_1^{-1} \colon \mathrm{Im}(J_1) \longrightarrow W^{1,p}(M)$ is linear and continuous and, by Proposition B.11.12, is continuous for the weak topologies. Applying J_1^{-1} to the preceding convergence, we conclude that

$$|\mathbf{u}_k|_{\mathbf{h_E}} \rightharpoonup |\mathbf{u}|_{\mathbf{h_E}} \qquad \text{in } W^{1,p}(M).$$

By Theorem 9.4.12, applied respectively to the trivial bundle and to $\mathbf{E}$, the trace operators $\mathrm{Tr} \colon W^{1,p}(M) \longrightarrow L^p(\partial M)$ and $\mathbf{Tr} \colon W^{1,p}(M, \mathbf{E}) \longrightarrow L^p(\partial M, \mathbf{E}\restriction_{\partial M})$ are linear and continuous. By Proposition B.11.12, the operator Tr is continuous for the weak topologies, so $\mathrm{Tr}(|\mathbf{u}_k|_{\mathbf{h_E}}) \rightharpoonup \mathrm{Tr}(|\mathbf{u}|_{\mathbf{h_E}})$ in $L^p(\partial M)$. On the other hand, continuity of $\mathbf{Tr}$ and convergence $\mathbf{u}_k \to \mathbf{u}$ in $W^{1,p}(M, \mathbf{E})$ imply $\mathbf{Tr}(\mathbf{u}_k) \to \mathbf{Tr}(\mathbf{u})$ in $L^p(\partial M, \mathbf{E}\restriction_{\partial M})$. The reverse triangle inequality then gives

$$|\mathbf{Tr}(\mathbf{u}_k)|_{\mathbf{h_E}} \longrightarrow |\mathbf{Tr}(\mathbf{u})|_{\mathbf{h_E}} \qquad \text{in } L^p(\partial M).$$

To justify this identity for each smooth section $\mathbf{u}_k$, fix k and use the smooth functions

$$f_{k,\varepsilon} := \sqrt{|\mathbf{u}_k|^2_{\mathbf{h_E}} + \varepsilon^2} - \varepsilon, \qquad \varepsilon > 0.$$

The proof of Lemma 9.5.4, applied on the compact set M, proves that $f_{k,\varepsilon} \to |\mathbf{u}_k|_{\mathbf{h_E}}$ in $W^{1,p}(M)$ for every $1 \leq p < \infty$: the functions and their derivatives converge in L^p by dominated convergence. Continuity of the trace then gives $\mathrm{Tr}(f_{k,\varepsilon}) \to \mathrm{Tr}(|\mathbf{u}_k|_{\mathbf{h_E}})$ in $L^p(\partial M)$. Since $f_{k,\varepsilon}$ is smooth, its trace is its restriction, and the inequality

$$0 \leq |\mathbf{u}_k|_{\mathbf{h_E}} - f_{k,\varepsilon} \leq \varepsilon$$

proves that these restrictions converge uniformly on ∂M to $|\mathbf{u}_k \restriction_{\partial M}|_{\mathbf{h_E}}$. Since ∂M has finite measure, convergence also holds in $L^p(\partial M)$. Uniqueness of the limit gives $\mathrm{Tr}(|\mathbf{u}_k|_{\mathbf{h_E}}) = |\mathbf{Tr}(\mathbf{u}_k)|_{\mathbf{h_E}}$ almost everywhere on ∂M for every k. The preceding strong convergence also implies weak convergence, so the same sequence

$\operatorname{Tr}(|\mathbf{u}_k|_{\mathbf{h_E}}) = |\mathbf{Tr}(\mathbf{u}_k)|_{\mathbf{h_E}}$ converges weakly both to $\operatorname{Tr}(|\mathbf{u}|_{\mathbf{h_E}})$ and to $|\mathbf{Tr}(\mathbf{u})|_{\mathbf{h_E}}$. Uniqueness of the weak limit finally gives

$$\operatorname{Tr}(|\mathbf{u}|_{\mathbf{h_E}}) = |\mathbf{Tr}(\mathbf{u})|_{\mathbf{h_E}}$$

almost everywhere on ∂M. □

We can now prove the Poincaré inequality for sections satisfying a homogeneous Dirichlet condition.

Theorem 9.6.8. (Poincaré inequality in $W^{1,p}_0(M,\mathbf{E})$) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty smooth boundary, and suppose that every connected component of M has nonempty boundary. Let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. If $1 \le p < \infty$, then there is a constant $C_P > 0$, depending only on M, $\mathbf{g}$, $\mathbf{E}$, $\mathbf{h_E}$, $\nabla^{\mathbf{E}}$, and p, such that $\|\mathbf{u}\|_{L^p(M,\mathbf{E})} \le C_P \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$ for every $\mathbf{u} \in W^{1,p}_0(M,\mathbf{E})$. Consequently, $\|\mathbf{u}\|_{\nabla^{\mathbf{E}},p} := \left(\int_M |\nabla^{\mathbf{E}}_w \mathbf{u}|^p_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}\right)^{\frac{1}{p}}$ defines a norm on $W^{1,p}_0(M,\mathbf{E})$ equivalent to the usual norm of $W^{1,p}(M,\mathbf{E})$.

Proof. Suppose, for a contradiction, that the inequality is false. Then there is a sequence $(\mathbf{u}_k) \subseteq W^{1,p}_0(M,\mathbf{E})$ such that $\|\mathbf{u}_k\|_{L^p(M,\mathbf{E})} = 1$ for every $k \in \mathbb{N}$ and $\|\nabla^{\mathbf{E}}_w \mathbf{u}_k\|_{L^p(M,T^*M\otimes\mathbf{E})} \to 0$. In particular, $(\mathbf{u}_k)$ is bounded in $W^{1,p}(M,\mathbf{E})$. By Corollary 9.4.29, applied with $m = 1$, after passing to a subsequence there is $\mathbf{u} \in L^p(M,\mathbf{E})$ such that $\mathbf{u}_k \to \mathbf{u}$ in $L^p(M,\mathbf{E})$.

Since $\nabla^{\mathbf{E}}_w \mathbf{u}_k \to 0$ in $L^p(M, T^*M \otimes \mathbf{E})$ and the weak covariant derivative is a closed operator from $L^p(M,\mathbf{E})$ to $L^p(M, T^*M \otimes \mathbf{E})$, we have $\mathbf{u} \in W^{1,p}(M,\mathbf{E})$ and $\nabla^{\mathbf{E}}_w \mathbf{u} = 0$. Therefore, $\mathbf{u}_k \to \mathbf{u}$ in $W^{1,p}(M,\mathbf{E})$. This step uses only strong convergence and closedness of the weak derivative, Lemma 9.1.12, and is also valid for $p = 1$; it does not require reflexivity. Since $W^{1,p}_0(M,\mathbf{E})$ is a closed subspace of $W^{1,p}(M,\mathbf{E})$, it follows that $\mathbf{u} \in W^{1,p}_0(M,\mathbf{E})$ and, by Theorem 9.4.23 applied with $m = 1$, $\mathbf{Tr}(\mathbf{u}) = 0$.

Write $M = \bigsqcup_{i=1}^{\ell} M_i$, where $M_1, \dots, M_\ell$ are the connected components of M. By the weak Kato inequality, Corollary 9.5.7, we have $|\mathbf{u}|_{\mathbf{h_E}} \in W^{1,p}(M)$ and $\left|\nabla_w |\mathbf{u}|_{\mathbf{h_E}}\right|_{\mathbf{g}} \le |\nabla^{\mathbf{E}}_w \mathbf{u}|_{\mathbf{g},\mathbf{h_E}} = 0$ almost everywhere. Lemma 9.6.1 implies that, for each $i \in \{1,\dots,\ell\}$, there is a constant $c_i \ge 0$ such that $|\mathbf{u}|_{\mathbf{h_E}} = c_i$ almost everywhere on M_i.

By Lemma 9.6.7, $\operatorname{Tr}(|\mathbf{u}|_{\mathbf{h_E}}) = |\mathbf{Tr}(\mathbf{u})|_{\mathbf{h_E}} = 0$ almost everywhere on ∂M. Since $\partial M_i \neq \varnothing$ and the trace of the constant function c_i on ∂M_i is c_i, we conclude that $c_i = 0$ for each $i \in \{1,\dots,\ell\}$. Thus $|\mathbf{u}|_{\mathbf{h_E}} = 0$ almost everywhere on M and consequently $\mathbf{u} = 0$ almost everywhere.

However, convergence $\mathbf{u}_k \to \mathbf{u}$ in $L^p(M,\mathbf{E})$ implies $\|\mathbf{u}\|_{L^p(M,\mathbf{E})} = \lim_{k\to\infty} \|\mathbf{u}_k\|_{L^p(M,\mathbf{E})} = 1$, a contradiction. Thus there is $C_P > 0$ such that $\|\mathbf{u}\|_{L^p(M,\mathbf{E})} \le C_P \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$ for every $\mathbf{u} \in W^{1,p}_0(M,\mathbf{E})$.

Finally, for every $\mathbf{u} \in W^{1,p}_0(M,\mathbf{E})$ we have $\|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})} \le \|\mathbf{u}\|_{W^{1,p}(M,\mathbf{E})} \le (1 + C_P^p)^{\frac{1}{p}} \|\nabla^{\mathbf{E}}_w \mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$, so $\|\cdot\|_{\nabla^{\mathbf{E}},p}$ is a norm equivalent to the usual norm of $W^{1,p}(M,\mathbf{E})$. □

10

Distributions on vector bundles

Distribution theory on a manifold must respect the geometry of the space while extending the duality used in the Euclidean case. For scalar functions, a distribution can be viewed as a continuous functional on test functions. On a vector bundle, the natural formulation requires pairing test sections of the dual bundle and controlling their support and covariant derivatives.

We first develop the topology of spaces of compactly supported smooth sections. We then use it to define bundle-valued distributions, their restrictions, their support, and the distributional action of differential operators. The formal adjoint allows an operator to be transferred to the test section, just as integration by parts defines distributional derivatives on $\mathbb{R}^n$.

This framework will be needed later to introduce Sobolev spaces of real order and operators defined by functional calculus, since it allows their elements and equations to be interpreted without recourse to particular coordinates.

Duality as a defining principle

Schwartz's theory taught us that a singular object can be understood through its action on test functions. On a vector bundle, this idea requires care concerning which fibers are paired, which measure permits integration, and how an operator is transferred to the test factor. The result is a theory that preserves the local nature of distributions while respecting the global geometry of the bundle. This duality will underlie elliptic regularity and the kernel theorem appearing at the end of the chapter [131].

10.1 The space of test sections

In this section, we construct the spaces of test sections associated with a smooth vector bundle. The idea parallels the construction of $\mathcal{D}(\Omega)$ in the Euclidean case, but the C^m norms are now defined using covariant derivatives and bundle metrics. This formulation allows us to work intrinsically while retaining the functional properties needed to define bundle-valued distributions.

For a boundary chart, its image Ω is a relatively open subset of the closed half-space. The notation $C^\infty(\Omega)$ means smoothness up to the boundary, and $\mathcal{D}(\Omega)$ allows compact supports meeting it. Its dual is understood with this convention. All interiors and neighborhoods of compact subsets of a manifold are taken in its relative

topology. The extension by zero of a compactly supported function from a relatively open subset to the half-space is smooth; to extend it across the geometric boundary, we use Lemma B.12.19.

Definition 10.1.1. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E}\longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$, let ∇^{TM} be a connection on TM, and let $\nabla^{\mathbf{E}}$ be a connection on $\mathbf{E}$. Let $K\subseteq M$ be compact. For each $\varphi\in\Gamma^m(\mathbf{E})$, define

$$\|\varphi\|_{C^m(K)} := \max_{0\le j\le m}\sup_{x\in K}|\nabla^j\varphi(x)|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}},$$

where $\nabla^j\varphi$ denotes the covariant derivative of order j of φ, regarded as a section of the tensor bundle $T^{(0,j)}(TM)\otimes\mathbf{E}$ and induced by the connections ∇^{TM} and $\nabla^{\mathbf{E}}$ as in Definition 9.1.1. The norm $|\cdot|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}$ is the norm induced by $\mathbf{g}$ and $\mathbf{h}_{\mathbf{E}}$ on the corresponding tensor bundles; see Proposition 9.1.2.
Also define

$$\Gamma_K^m(\mathbf{E}) := \{\varphi\in\Gamma^m(\mathbf{E})\mid \operatorname{supp}(\varphi)\subseteq K\}.$$

When $m=\infty$, we write

$$\Gamma_K(\mathbf{E}) := \{\varphi\in\Gamma(\mathbf{E})\mid \operatorname{supp}(\varphi)\subseteq K\}.$$

For each $m\in\mathbb{N}_0$, the quantity $\|\cdot\|_{C^m(K)}$ defines a norm on $\Gamma_K^m(\mathbf{E})$. The triangle inequality and homogeneity are immediate. If $\|\varphi\|_{C^m(K)}=0$, then in particular $\varphi=0$ on K. Since $\operatorname{supp}(\varphi)\subseteq K$, it follows that $\varphi=0$ on $M\setminus K$, and hence $\varphi=0$ throughout M.

Although the preceding definition uses a Riemannian metric, a bundle metric, and auxiliary connections, the resulting topology on $\Gamma_K^m(\mathbf{E})$ is independent of these choices. This matters because the subsequent constructions should depend on the bundle and manifold, rather than on particular auxiliary data.

Theorem 10.1.2. Let M be a smooth manifold with or without boundary, let $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ be two Riemannian metrics on M, let ∇^{TM} and $\widetilde{\nabla}^{TM}$ be two connections on TM, let $\mathbf{E}\longrightarrow M$ be a smooth vector bundle of finite rank, let $\mathbf{h}_{\mathbf{E}}$ and $\widetilde{\mathbf{h}}_{\mathbf{E}}$ be two smooth bundle metrics on $\mathbf{E}$, and let $\nabla^{\mathbf{E}}$ and $\widetilde{\nabla}^{\mathbf{E}}$ be two connections on $\mathbf{E}$. Also, let $K\subseteq M$ be compact. Then the norms $\|\cdot\|_{C^m(K)}$ and $\|\cdot\|_{\widetilde{C^m}(K)}$ are equivalent on $\Gamma_K^m(\mathbf{E})$.

Proof. For $\mathbf{u}\in\Gamma_K^m(\mathbf{E})$, write

$$\|\mathbf{u}\|_{C^m(K)} = \max_{0\le j\le m}\sup_{x\in K}|\nabla^j\mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}},\qquad \|\mathbf{u}\|_{\widetilde{C^m}(K)} = \max_{0\le j\le m}\sup_{x\in K}|\widetilde{\nabla}^j\mathbf{u}(x)|_{\widetilde{\mathbf{g}},\widetilde{\mathbf{h}}_{\mathbf{E}}}.$$

Applying Lemma 9.2.16 with the integer m gives a constant $C_m>0$ such that, for every $x\in K$,

$$\sum_{j=0}^{m}|\widetilde{\nabla}^j\mathbf{u}(x)|^2_{\widetilde{\mathbf{g}},\widetilde{\mathbf{h}}_{\mathbf{E}}}\le C_m\sum_{j=0}^{m}|\nabla^j\mathbf{u}(x)|^2_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}.$$

Therefore

$$\max_{0\le j\le m}|\widetilde{\nabla}^j\mathbf{u}(x)|^2_{\widetilde{\mathbf{g}},\widetilde{\mathbf{h}}_{\mathbf{E}}}\le C_m(m+1)\max_{0\le j\le m}|\nabla^j\mathbf{u}(x)|^2_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}.$$

Taking the supremum over $x\in K$, we obtain

$$\|\mathbf{u}\|_{\widetilde{C^m}(K)}\le\sqrt{C_m(m+1)}\,\|\mathbf{u}\|_{C^m(K)}.$$

The reverse argument, interchanging the data with and without tildes, produces a constant $\widetilde{C} > 0$ such that

$$\|\mathbf{u}\|_{C^m(K)} \leq \widetilde{C}\,\|\mathbf{u}\|_{\widetilde{C}^m(K)}.$$

These two inequalities give the asserted comparison constants. □

To prove that $\Gamma_K^m(\mathbf{E})$ is complete, we adapt the Euclidean argument to uniform limits of sections and their covariant derivatives. This intrinsic formulation will later allow passage to the limit in covariant identities.

Lemma 10.1.3. (Uniform limit of continuous sections) Let M be a smooth manifold with or without boundary, and let $\mathbf{F} \longrightarrow M$ be a vector bundle of finite rank equipped with a continuous bundle metric $\mathbf{h_F}$. Let $(\mathbf{s}_n)_{n\in\mathbb{N}}$ be a sequence of continuous sections of $\mathbf{F}$, and suppose that there exists a section $\mathbf{s}\colon M \longrightarrow \mathbf{F}$ such that

$$\|\mathbf{s}_n - \mathbf{s}\|_{L^\infty(M,\mathbf{F})} \longrightarrow 0.$$

Then $\mathbf{s}$ is continuous.

Proof. Fix $p \in M$. Choose a local trivialization $\Psi\colon \mathbf{F}\restriction_U \longrightarrow U \times \mathbb{K}^r$ over an open neighborhood U of p, where r is the rank of $\mathbf{F}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Also choose an open neighborhood V of p such that $\overline{V} \subseteq U$ and $\overline{V}$ is compact.
For each $n \in \mathbb{N}$, write $\Psi(\mathbf{s}_n(x)) = (x, u_n(x))$ and $\Psi(\mathbf{s}(x)) = (x, u(x))$ for $x \in U$, where $u_n, u\colon U \longrightarrow \mathbb{K}^r$.
Since the bundle metric $\mathbf{h_F}$ is continuous and $\overline{V}$ is compact, the norms induced by $\mathbf{h_F}$ on the fibers of $\mathbf{F}\restriction_{\overline{V}}$ are uniformly equivalent to the Euclidean norm on $\mathbb{K}^r$. Thus there exists a constant $C > 0$ such that

$$|u_n(x) - u(x)| \leq C\,|\mathbf{s}_n(x) - \mathbf{s}(x)|_{\mathbf{h_F}}$$

for every $x \in \overline{V}$ and every $n \in \mathbb{N}$.
Consequently,

$$\|u_n - u\|_{C^0(\overline{V},\mathbb{K}^r)} \leq C\|\mathbf{s}_n - \mathbf{s}\|_{L^\infty(M,\mathbf{F})} \longrightarrow 0.$$

Thus $u_n \to u$ uniformly on $\overline{V}$. Since each u_n is continuous, u is also continuous on V. Therefore $\mathbf{s}$ is continuous on V.
Since $p \in M$ was arbitrary, we conclude that $\mathbf{s}$ is continuous throughout M. □

Proposition 10.1.4. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$, let ∇^{TM} be a connection on TM, let $\nabla^{\mathbf{E}}$ be a connection on $\mathbf{E}$, and let $K \subseteq M$ be compact. For each $m \in \mathbb{N}_0$, the space $\left(\Gamma_K^m(\mathbf{E}), \|\cdot\|_{C^m(K)}\right)$ is a Banach space.

Proof. We use the convention $\nabla^0\boldsymbol{\varphi} = \boldsymbol{\varphi}$. For each $r \in \{0,\dots,m\}$, denote by $\mathbf{F}_r := T^{(0,r)}(TM) \otimes \mathbf{E}$ the tensor bundle in which $\nabla^r\boldsymbol{\varphi}$ takes values. This bundle is equipped with the connection induced by ∇^{TM} and $\nabla^{\mathbf{E}}$, and the bundle metric induced by $\mathbf{g}$ and $\mathbf{h_E}$.
Let $(\boldsymbol{\varphi}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\Gamma_K^m(\mathbf{E})$. Fix $r \in \{0,\dots,m\}$.
If $x \in M \setminus K$, since $\operatorname{supp}(\boldsymbol{\varphi}_n) \subseteq K$, there is an open neighborhood of x on which $\boldsymbol{\varphi}_n$ vanishes. Thus $\nabla^r\boldsymbol{\varphi}_n(x) = 0$. Consequently, $\nabla^r\boldsymbol{\varphi}_n$ vanishes on $M \setminus K$ for every $n \in \mathbb{N}$.

Moreover, for any $n, p \in \mathbb{N}$,

$$\|\nabla^r \varphi_n - \nabla^r \varphi_p\|_{L^\infty(M, T^{(0,r)}(TM)\otimes \mathbf{E})} = \|\nabla^r \varphi_n - \nabla^r \varphi_p\|_{C^0(K, T^{(0,r)}(TM)\otimes \mathbf{E})} \leq \|\varphi_n - \varphi_p\|_{C^m(K)}.$$

Since $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy in $\Gamma_K^m(\mathbf{E})$, it follows that $(\nabla^r \varphi_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on M.
For each $x \in M$, the sequence $(\nabla^r \varphi_n(x))_{n\in\mathbb{N}}$ is Cauchy in the fiber $(\mathbf{F}_r)_x$, which is a finite-dimensional normed vector space and hence complete. Define

$$\mathbf{A}_r(x) := \lim_{n\to\infty} \nabla^r \varphi_n(x), \qquad x \in M.$$

This gives a section $\mathbf{A}_r \colon M \longrightarrow \mathbf{F}_r$.
Let us show that convergence is uniform. Let $\varepsilon > 0$. Since $(\nabla^r \varphi_n)_{n\in\mathbb{N}}$ is uniformly Cauchy, there exists $N \in \mathbb{N}$ such that, if $n, p \geq N$, then

$$\|\nabla^r \varphi_n - \nabla^r \varphi_p\|_{L^\infty(M, T^{(0,r)}(TM)\otimes \mathbf{E})} < \varepsilon.$$

Fix $n \geq N$ and let $p \to \infty$. For each $x \in M$,

$$\left|\nabla^r \varphi_n(x) - \mathbf{A}_r(x)\right|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \leq \varepsilon.$$

Taking the supremum over $x \in M$ gives

$$\|\nabla^r \varphi_n - \mathbf{A}_r\|_{L^\infty(M, T^{(0,r)}(TM)\otimes \mathbf{E})} \leq \varepsilon.$$

Therefore $\nabla^r \varphi_n \to \mathbf{A}_r$ uniformly on M.
Since each $\nabla^r \varphi_n$ is continuous, Lemma 10.1.3 implies that $\mathbf{A}_r$ is continuous. Moreover, since $\nabla^r \varphi_n = 0$ on $M \setminus K$ for every n, passage to the limit gives $\mathbf{A}_r = 0$ on $M \setminus K$.
Define $\varphi := \mathbf{A}_0$. Then φ is a continuous section of $\mathbf{E}$ and vanishes on $M \setminus K$. Since K is compact and hence closed, it follows that $\operatorname{supp}(\varphi) \subseteq K$.
It remains to prove that φ is of class C^m and that $\nabla^r \varphi = \mathbf{A}_r$ for every $r \in \{0, \ldots, m\}$.
Fix $r \in \{0, \ldots, m-1\}$. Consider the bundle $\mathbf{F}_r = T^{(0,r)}(TM)\otimes\mathbf{E}$ with the induced connection. Let $\gamma \colon [a, b] \longrightarrow M$ be a smooth curve, and let $t, t_0 \in [a, b]$.
Since $\varphi_n \in \Gamma_K^m(\mathbf{E})$ and $r \leq m-1$, the section $\nabla^r \varphi_n$ is of class C^1 as a section of $\mathbf{F}_r$. Apply Lemma 2.7.21 to the section $\nabla^r \varphi_n$ restricted to γ. We obtain

$$P^{\mathbf{F}_r}_{\gamma, t\to t_0}\big(\nabla^r \varphi_n(\gamma(t))\big) - \nabla^r \varphi_n(\gamma(t_0))$$

$$= \int_{t_0}^{t} P^{\mathbf{F}_r}_{\gamma, \tau\to t_0}\left(\nabla^{\mathbf{F}_r}_{\dot\gamma(\tau)}\big(\nabla^r \varphi_n\big)(\gamma(\tau))\right) d\tau.$$

By the definition of the higher-order covariant derivative, under the identification $T^*M\otimes\mathbf{F}_r \cong T^{(0,r+1)}(TM)\otimes \mathbf{E}$ we have $\nabla^{\mathbf{F}_r}(\nabla^r \varphi_n) = \nabla^{r+1}\varphi_n$. Therefore

$$\nabla^{\mathbf{F}_r}_{\dot\gamma(\tau)}\big(\nabla^r \varphi_n\big)(\gamma(\tau)) = \iota^{\mathrm{prim}}_{\dot\gamma(\tau)}\left(\nabla^{r+1}\varphi_n(\gamma(\tau))\right),$$

where ι_v^{prim} denotes contraction in the first covariant factor; that is,

$$\big(\iota_v^{\mathrm{prim}} A\big)(X_1,\dots,X_r) = A(v,X_1,\dots,X_r).$$

Thus

$$P^{\mathbf{F}_r}_{\gamma,t\to t_0}\big(\nabla^r \boldsymbol{\varphi}_n(\gamma(t))\big) - \nabla^r \boldsymbol{\varphi}_n(\gamma(t_0))$$
$$= \int_{t_0}^{t} P^{\mathbf{F}_r}_{\gamma,\tau\to t_0}\Big(\iota^{\mathrm{prim}}_{\dot\gamma(\tau)}\big(\nabla^{r+1}\boldsymbol{\varphi}_n(\gamma(\tau))\big)\Big)\,d\tau.$$

We let $n\to\infty$ in this identity. Since $\nabla^r\boldsymbol{\varphi}_n \to \mathbf{A}_r$ uniformly on M, the left-hand side converges to

$$P^{\mathbf{F}_r}_{\gamma,t\to t_0}\big(\mathbf{A}_r(\gamma(t))\big) - \mathbf{A}_r(\gamma(t_0)).$$

Let us now check that we can also pass to the limit in the integral. For fixed t and t_0, consider the compact interval between them. The function

$$\tau \longmapsto P^{\mathbf{F}_r}_{\gamma,\tau\to t_0}\circ \iota_{\dot\gamma(\tau)}$$

is continuous and takes values in the space of linear operators between finite-dimensional vector spaces. Thus there is a constant $C>0$ such that

$$\Big\| P^{\mathbf{F}_r}_{\gamma,\tau\to t_0}\circ \iota_{\dot\gamma(\tau)}\Big\| \le C$$

for every τ between t_0 and t.
Consequently,

$$\left|\int_{t_0}^{t} P^{\mathbf{F}_r}_{\gamma,\tau\to t_0}\Big(\iota_{\dot\gamma(\tau)}\big(\nabla^{r+1}\boldsymbol{\varphi}_n(\gamma(\tau)) - \mathbf{A}_{r+1}(\gamma(\tau))\big)\Big)\,d\tau\right|$$
$$\le C\,|t-t_0|\,\|\nabla^{r+1}\boldsymbol{\varphi}_n - \mathbf{A}_{r+1}\|_{L^\infty(M,T^{(0,r+1)}(TM)\otimes\mathbf{E})}.$$

The right-hand side converges to 0, since $\nabla^{r+1}\boldsymbol{\varphi}_n\to\mathbf{A}_{r+1}$ uniformly on M. Therefore

$$P^{\mathbf{F}_r}_{\gamma,t\to t_0}\big(\mathbf{A}_r(\gamma(t))\big) - \mathbf{A}_r(\gamma(t_0))$$
$$= \int_{t_0}^{t} P^{\mathbf{F}_r}_{\gamma,\tau\to t_0}\Big(\iota_{\dot\gamma(\tau)}\big(\mathbf{A}_{r+1}(\gamma(\tau))\big)\Big)\,d\tau.$$

This identity has exactly the form in Lemma 2.7.22, applied to the bundle $\mathbf{F}_r$, the section $\mathbf{A}_r$, and the tensor $\mathbf{A}_{r+1}$ viewed as an element of $\Gamma^0(T^*M\otimes\mathbf{F}_r)$. We conclude that $\mathbf{A}_r$ is of class C^1 and that $\nabla^{\mathbf{F}_r}\mathbf{A}_r = \mathbf{A}_{r+1}$.
Using this compatibility, we prove by induction on $r\in\{0,\dots,m\}$ that $\boldsymbol{\varphi}\in\Gamma^r(\mathbf{E})$ and $\nabla^r\boldsymbol{\varphi}=\mathbf{A}_r$.
For $r=0$, the assertion is immediate, since $\boldsymbol{\varphi}=\mathbf{A}_0$ and $\mathbf{A}_0$ is continuous.
Now suppose the assertion holds for some $r\in\{0,\dots,m-1\}$. Then $\boldsymbol{\varphi}\in\Gamma^r(\mathbf{E})$ and $\nabla^r\boldsymbol{\varphi}=\mathbf{A}_r$. Since we have just proved that $\mathbf{A}_r$ is of class C^1 and $\nabla^{\mathbf{F}_r}\mathbf{A}_r=\mathbf{A}_{r+1}$, the definition of higher-order covariant derivative gives $\boldsymbol{\varphi}\in\Gamma^{r+1}(\mathbf{E})$ and $\nabla^{r+1}\boldsymbol{\varphi}=\mathbf{A}_{r+1}$.
By induction, $\boldsymbol{\varphi}\in\Gamma^m(\mathbf{E})$ and $\nabla^r\boldsymbol{\varphi}=\mathbf{A}_r$ for every $r\in\{0,\dots,m\}$.
Since we also have $\operatorname{supp}(\boldsymbol{\varphi})\subseteq K$, we conclude that $\boldsymbol{\varphi}\in\Gamma^m_K(\mathbf{E})$.

Finally, for each $r \in \{0,\dots,m\}$, $\nabla^r \varphi_n \to \nabla^r \varphi$ uniformly on K. Therefore

$$\|\varphi_n - \varphi\|_{C^m(K)} = \max_{0\le r\le m} \|\nabla^r \varphi_n - \nabla^r \varphi\|_{C^0(K,T^{(0,r)}(TM)\otimes \mathbf{E})} \longrightarrow 0.$$

Thus every Cauchy sequence in $\Gamma_K^m(\mathbf{E})$ converges in $\Gamma_K^m(\mathbf{E})$. We conclude that $\left(\Gamma_K^m(\mathbf{E}), \|\cdot\|_{C^m(K)}\right)$ is a Banach space. □

Having constructed the Banach space $\Gamma_K^m(\mathbf{E})$ for each finite order, we pass to the space of smooth sections supported in K. This space is described by the countable family of norms C^m, rather than by a single norm.

Definition 10.1.5. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$, and let $K \subseteq M$ be compact. Equip $\Gamma_K(\mathbf{E})$ with the locally convex topology induced by the family of seminorms

$$\mathbf{u} \longmapsto \|\mathbf{u}\|_{C^m(K)}, \qquad m \in \mathbb{N}_0.$$

This is the topology of uniform convergence on K of sections and all their covariant derivatives.

Lemma 10.1.6. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h_E}$, and let $K \subseteq M$ be compact. Then the topology on $\Gamma_K(\mathbf{E})$ generated by the family of seminorms $(\|\cdot\|_{C^m(K)})_{m\in\mathbb{N}_0}$ has as a local base at 0 the family of sets

$$V_{K,m,\ell} := \left\{ \mathbf{u} \in \Gamma_K(\mathbf{E}) \;\middle|\; \|\mathbf{u}\|_{C^m(K)} < \frac{1}{\ell} \right\}, \qquad m \in \mathbb{N}_0, \quad \ell \in \mathbb{N}.$$

Consequently, a neighborhood base at $\mathbf{u} \in \Gamma_K(\mathbf{E})$ is given by

$$\mathbf{u} + V_{K,m,\ell} = \left\{ \mathbf{v} \in \Gamma_K(\mathbf{E}) \;\middle|\; \|\mathbf{u}-\mathbf{v}\|_{C^m(K)} < \frac{1}{\ell} \right\}.$$

Proof. By Definition B.10.18 of the topology induced by seminorms, a local base at 0 consists of sets of the form

$$U = \left\{ \mathbf{u} \in \Gamma_K(\mathbf{E}) \;\middle|\; \|\mathbf{u}\|_{C^{m_1}(K)} < \varepsilon_1, \dots, \|\mathbf{u}\|_{C^{m_q}(K)} < \varepsilon_q \right\}.$$

Let $m := \max\{m_1,\dots,m_q\}$. Then, since the norms C^m increase with the order,

$$\|\mathbf{u}\|_{C^{m_i}(K)} \le \|\mathbf{u}\|_{C^m(K)} \qquad \forall i \in \{1,\dots,q\}.$$

Choose $\ell \in \mathbb{N}$ such that $\frac{1}{\ell} < \min\{\varepsilon_1,\dots,\varepsilon_q\}$. If $\mathbf{u} \in V_{K,m,\ell}$, then

$$\|\mathbf{u}\|_{C^{m_i}(K)} \le \|\mathbf{u}\|_{C^m(K)} < \frac{1}{\ell} < \varepsilon_i$$

for every $i \in \{1,\dots,q\}$. Therefore $V_{K,m,\ell} \subseteq U$.
This proves that the family $V_{K,m,\ell}$ is a local base at 0. The general case follows by translation, since translations are homeomorphisms of a topological vector space. □

Proposition 10.1.7. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E}\to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$, and let $K\subseteq M$ be compact. For $\mathbf{u},\mathbf{v}\in\Gamma_K(\mathbf{E})$, define

$$d_K(\mathbf{u},\mathbf{v}) := \sum_{m=0}^{\infty} \frac{1}{2^m}\frac{\|\mathbf{u}-\mathbf{v}\|_{C^m(K)}}{1+\|\mathbf{u}-\mathbf{v}\|_{C^m(K)}}.$$

Then the following properties hold:

(a) d_K is a translation-invariant metric on $\Gamma_K(\mathbf{E})$.

(b) The topology on $\Gamma_K(\mathbf{E})$ generated by the family of seminorms $(\|\cdot\|_{C^m(K)})_{m\in\mathbb{N}_0}$ agrees with the topology induced by d_K.

(c) $(\Gamma_K(\mathbf{E}),d_K)$ is complete.

In particular, $\Gamma_K(\mathbf{E})$ is a Fréchet space.

Proof. Consider the countable family of seminorms

$$p_m(\mathbf{u}) := \|\mathbf{u}\|_{C^m(K)}, \qquad m\in\mathbb{N}_0,$$

defined on $\Gamma_K(\mathbf{E})$. In fact, each p_m is a norm on $\Gamma_K(\mathbf{E})$: if $p_m(\mathbf{u})=0$, then in particular $\mathbf{u}=0$ on K; since $\operatorname{supp}(\mathbf{u})\subseteq K$, it follows that $\mathbf{u}=0$ on $M\setminus K$, and hence $\mathbf{u}=0$ throughout M. Thus the family $(p_m)_{m\in\mathbb{N}_0}$ separates points.

Properties (a) and (b) are therefore a special case of Proposition B.10.24. The metric constructed there, applied to the family $(p_m)_{m\in\mathbb{N}_0}$, is precisely d_K; hence d_K is translation invariant and induces the locally convex topology generated by the norms $\|\cdot\|_{C^m(K)}$.

It remains to prove completeness. We use sequential completeness of projective intersections. First observe that, within the Hausdorff topological vector space $Y:=\Gamma_K^0(\mathbf{E})$, we have the algebraic equality

$$\Gamma_K(\mathbf{E}) = \bigcap_{m\in\mathbb{N}_0} \Gamma_K^m(\mathbf{E}).$$

If a section belongs to $\Gamma_K(\mathbf{E})$, it is smooth and therefore belongs to $\Gamma_K^m(\mathbf{E})$ for every $m\in\mathbb{N}_0$. Conversely, if a section belongs to $\Gamma_K^m(\mathbf{E})$ for every $m\in\mathbb{N}_0$, it is of class C^m for every m, hence smooth, and its support is contained in K.

For each $m\in\mathbb{N}_0$, the space $\Gamma_K^m(\mathbf{E})$ is Banach by Proposition 10.1.4. Moreover, the inclusion

$$\Gamma_K^m(\mathbf{E}) \hookrightarrow \Gamma_K^0(\mathbf{E})$$

is continuous, since

$$\|\mathbf{u}\|_{C^0(K)} \le \|\mathbf{u}\|_{C^m(K)}.$$

On the other hand, the topology on $\Gamma_K(\mathbf{E})$ generated by the norms $\|\cdot\|_{C^m(K)}$ agrees with the projective topology associated with the natural inclusions

$$\iota_m\colon \Gamma_K(\mathbf{E}) \longrightarrow \Gamma_K^m(\mathbf{E}), \qquad m\in\mathbb{N}_0.$$

This is because that topology is, by definition, the initial topology making all the inclusions ι_m continuous, or equivalently the topology generated by the seminorms $\mathbf{u} \mapsto \|\iota_m(\mathbf{u})\|_{C^m(K)} = \|\mathbf{u}\|_{C^m(K)}$; see Definition B.10.39 and Definition B.10.18.
Applying Proposition 4.2.4 and reindexing the family $\{\Gamma_K^m(\mathbf{E})\}_{m\in\mathbb{N}_0}$ by $n = m+1$, we conclude that $\Gamma_K(\mathbf{E})$ is sequentially complete in this projective topology. Since this same topology is induced by the translation-invariant metric d_K, a sequence is Cauchy with respect to d_K if and only if it is Cauchy in the corresponding vector topology. By sequential completeness, every Cauchy sequence for d_K converges in $\Gamma_K(\mathbf{E})$; since d_K induces the topology, this is convergence in the metric d_K. Therefore $(\Gamma_K(\mathbf{E}), d_K)$ is complete.
Finally, $\Gamma_K(\mathbf{E})$ is locally convex because its topology is generated by seminorms, Hausdorff because the family $(p_m)_{m\in\mathbb{N}_0}$ separates points, metrizable by d_K, and complete by what we have just proved. By the definition of a Fréchet space, or equivalently by Theorem B.10.38, we conclude that $\Gamma_K(\mathbf{E})$ is a Fréchet space. □

Proposition 10.1.8. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle equipped with a bundle metric $\mathbf{h}_\mathbf{E}$, and let $K, L \subseteq M$ be compact sets such that $K \subseteq L$. Then:

(a) for each $m \in \mathbb{N}_0$, the natural inclusion $\Gamma_K^m(\mathbf{E}) \hookrightarrow \Gamma_L^m(\mathbf{E})$ is an isometry onto its image, that is,

$$\|\mathbf{u}\|_{C^m(L)} = \|\mathbf{u}\|_{C^m(K)} \qquad \forall \mathbf{u} \in \Gamma_K^m(\mathbf{E});$$

(b) the topology on $\Gamma_K^m(\mathbf{E})$ agrees with the subspace topology induced by $\Gamma_L^m(\mathbf{E})$;

(c) the Fréchet topology on $\Gamma_K(\mathbf{E})$ agrees with the subspace topology induced by $\Gamma_L(\mathbf{E})$;

(d) the images of $\Gamma_K^m(\mathbf{E})$ in $\Gamma_L^m(\mathbf{E})$ and of $\Gamma_K(\mathbf{E})$ in $\Gamma_L(\mathbf{E})$ are closed.

Proof. Let $\mathbf{u} \in \Gamma_K^m(\mathbf{E})$. Since $\operatorname{supp}(\mathbf{u}) \subseteq K$, we have $\mathbf{u} = 0$ on the open subset $M \setminus K$. Thus, for each $j \in \{0, \dots, m\}$, we also have $\nabla^j \mathbf{u} = 0$ on $M \setminus K$. In particular, on $L \setminus K$ all covariant derivatives of $\mathbf{u}$ up to order m vanish. This gives

$$\sup_{x\in L} |\nabla^j \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}} = \sup_{x\in K} |\nabla^j \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \qquad 0 \le j \le m.$$

Taking the maximum over $0 \le j \le m$ yields $\|\mathbf{u}\|_{C^m(L)} = \|\mathbf{u}\|_{C^m(K)}$. This proves (a).
Part (b) follows immediately from (a), since the norm on $\Gamma_L^m(\mathbf{E})$ restricted to the subspace $\Gamma_K^m(\mathbf{E})$ agrees with the intrinsic norm of $\Gamma_K^m(\mathbf{E})$.
For (c), recall that the topology on $\Gamma_K(\mathbf{E})$ is induced by the family of norms $\|\cdot\|_{C^m(K)}$, with $m \in \mathbb{N}_0$, whereas the topology on $\Gamma_L(\mathbf{E})$ is induced by the family $\|\cdot\|_{C^m(L)}$, with $m \in \mathbb{N}_0$. By (a), restricting the latter norms to $\Gamma_K(\mathbf{E})$ gives exactly the family $\|\cdot\|_{C^m(K)}$. Therefore the topologies agree.
For (d), let $\mathbf{u}_j \in \Gamma_K^m(\mathbf{E})$ be a sequence converging to $\mathbf{u} \in \Gamma_L^m(\mathbf{E})$. In particular, $\mathbf{u}_j \to \mathbf{u}$ uniformly on L. Since $\mathbf{u}_j = 0$ on $L \setminus K$, we also have $\mathbf{u} = 0$ there; outside L the section $\mathbf{u}$ already vanishes, so $\operatorname{supp}\mathbf{u} \subseteq K$. This proves closedness of the image at the C^m level. If $\mathbf{u}_j \in \Gamma_K(\mathbf{E})$ converges to $\mathbf{u}$ in $\Gamma_L(\mathbf{E})$, the seminorm $C^0(L)$ gives the same argument. Thus both images are closed. □

Definition 10.1.9. Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth

vector bundle. Fix an increasing exhaustion of M by compact sets

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq M, \qquad K_n \subseteq \operatorname{Int}(K_{n+1}), \qquad M = \bigcup_{n=1}^{\infty} K_n.$$

(a) For each $m \in \mathbb{N}_0$, equip the space $\Gamma^m(\mathbf{E})$ with the locally convex topology generated by the family of seminorms

$$p_{n,j}(u) := \sup_{x \in K_n} |\nabla^j u(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, \qquad n \in \mathbb{N}, \quad 0 \le j \le m.$$

(b) Equip the space $\Gamma(\mathbf{E})$ with the locally convex topology generated by the family of seminorms

$$p_{n,m}(u) := \max_{0 \le j \le m} \sup_{x \in K_n} |\nabla^j u(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, \qquad n \in \mathbb{N}, \quad m \in \mathbb{N}_0.$$

(c) For each $m \in \mathbb{N}_0$, equip the space $\Gamma_c^m(\mathbf{E})$ with the final locally convex topology associated with the inclusions

$$\Gamma_{K_n}^m(\mathbf{E}) \hookrightarrow \Gamma_c^m(\mathbf{E}), \qquad n \in \mathbb{N}.$$

In other words,

$$\Gamma_c^m(\mathbf{E}) = \varinjlim_{n \in \mathbb{N}} \Gamma_{K_n}^m(\mathbf{E}).$$

By Proposition 10.1.8, the inclusions are embeddings with closed image. Since each $\Gamma_{K_n}^m(\mathbf{E})$ is a Banach space, $\Gamma_c^m(\mathbf{E})$ is a strict LB inductive limit and, in particular, a strict LF space.

(d) Equip the space $\Gamma_c(\mathbf{E})$ with the final locally convex topology associated with the inclusions

$$\Gamma_{K_n}(\mathbf{E}) \hookrightarrow \Gamma_c(\mathbf{E}), \qquad n \in \mathbb{N}.$$

That is,

$$\Gamma_c(\mathbf{E}) = \varinjlim_{n \in \mathbb{N}} \Gamma_{K_n}(\mathbf{E}).$$

By Proposition 10.1.8, the inclusions are embeddings with closed image. Since each $\Gamma_{K_n}(\mathbf{E})$ is a Fréchet space, $\Gamma_c(\mathbf{E})$ is a strict LF space.

Proposition 10.1.10. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank. The topologies defined in 10.1.9 are independent of the chosen metrics, connections, and increasing exhaustion by compact sets.

Proof. Independence of the metrics and auxiliary connections follows from Theorem 10.1.2. On each fixed compact set $K \subseteq M$, the norms $C^m(K)$ constructed using two different choices of metrics and connections are equivalent. Thus, for each fixed m, both choices generate the same topology on $\Gamma_K^m(\mathbf{E})$ and $\Gamma_K(\mathbf{E})$.
It remains to justify independence of the exhaustion. Let $(K_n)_{n\in\mathbb{N}}$ and $(L_q)_{q\in\mathbb{N}}$ be two increasing exhaustions of M by compact sets, with $K_n \subseteq \operatorname{Int}(K_{n+1})$, $L_q \subseteq \operatorname{Int}(L_{q+1})$, $M = \bigcup_{n=1}^{\infty} K_n$, and $M = \bigcup_{q=1}^{\infty} L_q$.
First show that each compact set in one exhaustion lies in a compact set of the other. Fix $n \in \mathbb{N}$. Since

$M = \bigcup_{q=1}^{\infty} L_q$ and $L_q \subseteq \mathrm{Int}(L_{q+1})$, the open subsets $\mathrm{Int}(L_q)$ cover M. In particular, they cover the compact set K_n. By compactness, there exist $q_1, \dots, q_s$ such that

$$K_n \subseteq \bigcup_{\alpha=1}^{s} \mathrm{Int}(L_{q_\alpha}).$$

If $q_0 := \max\{q_1, \dots, q_s\}$, then, since the family (L_q) is increasing,

$$K_n \subseteq L_{q_0}.$$

The same argument, interchanging K and L, shows that for each $q \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $L_q \subseteq K_{n_0}$. Now consider the topology on $\Gamma^m(\mathbf{E})$. For the exhaustion (K_n), it is generated by the seminorms

$$p^K_{n,j}(\mathbf{u}) := \sup_{x \in K_n} |\nabla^j \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, \qquad n \in \mathbb{N}, \quad 0 \le j \le m,$$

whereas for the exhaustion (L_q), it is generated by the seminorms

$$p^L_{q,j}(\mathbf{u}) := \sup_{x \in L_q} |\nabla^j \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}, \qquad q \in \mathbb{N}, \quad 0 \le j \le m.$$

If $K_n \subseteq L_{q_0}$, then $p^K_{n,j}(\mathbf{u}) \le p^L_{q_0,j}(\mathbf{u})$ for every $\mathbf{u} \in \Gamma^m(\mathbf{E})$. Thus every basic neighborhood defined using the exhaustion (K_n) contains a basic neighborhood defined using the exhaustion (L_q). The reverse argument, using the fact that each L_q lies in some K_{n_0}, gives the reverse inclusion between the topologies. Thus both exhaustions induce the same topology on $\Gamma^m(\mathbf{E})$. The same reasoning applies to $\Gamma(\mathbf{E})$, using the seminorms

$$p^K_{n,r}(\mathbf{u}) := \max_{0 \le j \le r} \sup_{x \in K_n} |\nabla^j \mathbf{u}(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}}$$

and their counterparts associated with (L_q).

We now prove independence of the exhaustion for the spaces with compact support. Denote by τ^m_K the final locally convex topology on $\Gamma^m_c(\mathbf{E})$ associated with the family $(\Gamma^m_{K_n}(\mathbf{E}))_{n\in\mathbb{N}}$, and by τ^m_L the final locally convex topology associated with the family $(\Gamma^m_{L_q}(\mathbf{E}))_{q\in\mathbb{N}}$.

We prove $\tau^m_K = \tau^m_L$ using the maximality property in Theorem 4.4.1. For each $n \in \mathbb{N}$, choose $q_0 \in \mathbb{N}$ such that $K_n \subseteq L_{q_0}$. Then the inclusion

$$\Gamma^m_{K_n}(\mathbf{E}) \hookrightarrow \Gamma^m_c(\mathbf{E})$$

viewed with the topology τ^m_L is continuous, since it factors as

$$\Gamma^m_{K_n}(\mathbf{E}) \hookrightarrow \Gamma^m_{L_{q_0}}(\mathbf{E}) \hookrightarrow (\Gamma^m_c(\mathbf{E}), \tau^m_L),$$

where the first arrow is continuous by Proposition 10.1.8 and the second by the definition of τ^m_L as a final locally convex topology.

Thus τ^m_L is a locally convex topology on $\Gamma^m_c(\mathbf{E})$ making all the inclusions $\Gamma^m_{K_n}(\mathbf{E}) \hookrightarrow \Gamma^m_c(\mathbf{E})$ continuous. Since τ^m_K is the finest topology with this property, we obtain $\tau^m_L \subseteq \tau^m_K$. Interchanging the exhaustions (K_n) and (L_q) gives $\tau^m_K \subseteq \tau^m_L$. Consequently, $\tau^m_K = \tau^m_L$.

The same argument proves independence of the topology on $\Gamma_c(\mathbf{E})$. If $K_n \subseteq L_{q_0}$, the inclusion

$$\Gamma_{K_n}(\mathbf{E}) \hookrightarrow \Gamma_{L_{q_0}}(\mathbf{E})$$

is continuous by Proposition 10.1.8; hence the final locally convex topology defined by the exhaustion (L_q) makes all the inclusions $\Gamma_{K_n}(\mathbf{E}) \hookrightarrow \Gamma_c(\mathbf{E})$ continuous. By maximality in Theorem 4.4.1, one topology contains the other. Interchanging the exhaustions gives the reverse containment, so the two agree. □

Definition 10.1.11. Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle. Define

$$\mathcal{D}(M,\mathbf{E}) := \Gamma_c(\mathbf{E}),$$

and equip this space with the strict locally convex inductive topology

$$\mathcal{D}(M,\mathbf{E}) = \varinjlim_{n\in\mathbb{N}} \Gamma_{K_n}(\mathbf{E}),$$

where $(K_n)_{n\in\mathbb{N}}$ is an increasing exhaustion of M by compact sets. By Proposition 10.1.10, this topology is independent of the chosen exhaustion. In particular, $\mathcal{D}(M,\mathbf{E})$ is an LF space.

The LF structure of $\mathcal{D}(M,\mathbf{E})$ gives a concrete description of sequential convergence: a sequence converges if and only if all its terms are supported in a common compact set and converge uniformly there together with all their covariant derivatives.

Corollary 10.1.12. Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle. Let $\varphi_n, \varphi \in \mathcal{D}(M,\mathbf{E})$, with $n \in \mathbb{N}$. Then $\varphi_n \to \varphi$ in $\mathcal{D}(M,\mathbf{E})$ if and only if there exists a compact set $K \subseteq M$ such that:

(a) $\operatorname{supp}(\varphi_n) \subseteq K$ for every $n \in \mathbb{N}$ and $\operatorname{supp}(\varphi) \subseteq K$;

(b) for every $k \in \mathbb{N}_0$,

$$\|\varphi_n - \varphi\|_{C^k(K)} \to 0 \qquad \text{when } n \to \infty.$$

Proof. By definition, $\mathcal{D}(M,\mathbf{E})$ is the LF space generated by the sequence of Fréchet spaces $\Gamma_{K_n}(\mathbf{E})$. By the corresponding part of Theorem 4.4.6, the sequence φ_n converges to φ in $\mathcal{D}(M,\mathbf{E})$ if and only if there exists an index $N \in \mathbb{N}$ such that

$$\{\varphi\} \cup \{\varphi_n \mid n \in \mathbb{N}\} \subseteq \Gamma_{K_N}(\mathbf{E})$$

and, moreover, $\varphi_n \to \varphi$ in the Fréchet topology of $\Gamma_{K_N}(\mathbf{E})$.
The first condition means that all supports lie in the compact set K_N. Taking $K = K_N$ gives (a). On the other hand, the topology on $\Gamma_{K_N}(\mathbf{E})$ is induced by the family of seminorms $\|\cdot\|_{C^k(K_N)}$, with $k \in \mathbb{N}_0$. By the characterization of convergence in seminorm-induced topologies, Proposition B.10.23, convergence $\varphi_n \to \varphi$ in $\Gamma_{K_N}(\mathbf{E})$ is equivalent to

$$\|\varphi_n - \varphi\|_{C^k(K_N)} \to 0 \qquad \text{for every } k \in \mathbb{N}_0.$$

This proves (b).
Conversely, suppose there exists a compact set $K \subseteq M$ satisfying (a) and (b). Since $(K_n)_{n\in\mathbb{N}}$ is an increasing

exhaustion with $K_n \subseteq \mathrm{Int}(K_{n+1})$, there exists $N \in \mathbb{N}$ such that $K \subseteq K_N$. Then $\varphi_n, \varphi \in \Gamma_{K_N}(\mathbf{E})$ for every n.
Fix $k \in \mathbb{N}_0$. Since the sections $\varphi_n - \varphi$ are supported in K, they vanish on $M \setminus K$. Thus all their covariant derivatives also vanish on $M \setminus K$. In particular,

$$\|\varphi_n - \varphi\|_{C^k(K_N)} = \|\varphi_n - \varphi\|_{C^k(K)}.$$

By hypothesis (b), the right-hand side tends to 0. Since this holds for every $k \in \mathbb{N}_0$, we obtain $\varphi_n \to \varphi$ in $\Gamma_{K_N}(\mathbf{E})$. Applying Theorem 4.4.6 once again, we conclude that $\varphi_n \to \varphi$ in $\mathcal{D}(M, \mathbf{E})$. □

10.2 Distributions on vector bundles

With these tools, we are ready to define distributions in the setting of vector bundles.

Definition 10.2.1. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional $\mathbb{K}$-vector space ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).
Denote by $\mathcal{D}'(M, \mathbf{E}, W)$ the space of all continuous $\mathbb{K}$-linear maps

$$\mathbf{F} \colon \mathcal{D}(M, \mathbf{E}^*) \longrightarrow W$$

.
Equip $\mathcal{D}'(M, \mathbf{E}, W)$ with the weak-$*$ topology, defined as the initial topology associated with the family of evaluation maps

$$\mathrm{ev}_\varphi \colon \mathcal{D}'(M, \mathbf{E}, W) \longrightarrow W, \qquad \mathrm{ev}_\varphi(\mathbf{F}) = \mathbf{F}(\varphi), \qquad \varphi \in \mathcal{D}(M, \mathbf{E}^*),$$

that is, the topology generated by sets of the form

$$\mathrm{ev}_\varphi^{-1}(U) = \{\mathbf{F} \in \mathcal{D}'(M, \mathbf{E}, W) \mid \mathbf{F}(\varphi) \in U\},$$

where $\varphi \in \mathcal{D}(M, \mathbf{E}^*)$ and $U \subseteq W$ is open.
Elements of $\mathcal{D}'(M, \mathbf{E}, W)$ are called **distributions** on $\mathbf{E}$ with values in W.
When $W = \mathbb{K}$, we simply write $\mathcal{D}'(M, \mathbf{E})$, and if in addition $\mathbf{E} = T^{(0,0)}(TM)$, we write $\mathcal{D}'(M)$.

Remark 10.2.2. The weak-$*$ topology on $\mathcal{D}'(M, \mathbf{E}, W)$ agrees with the topology of pointwise convergence on $\mathcal{D}(M, \mathbf{E}^*)$. Thus a sequence $(\mathbf{F}_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}'(M, \mathbf{E}, W)$ converges to $\mathbf{F} \in \mathcal{D}'(M, \mathbf{E}, W)$ if and only if

$$\mathbf{F}_n(\varphi) \to \mathbf{F}(\varphi) \qquad \forall \varphi \in \mathcal{D}(M, \mathbf{E}^*).$$

More generally, a net $(\mathbf{F}_\alpha)_\alpha$ converges to $\mathbf{F}$ if and only if

$$\mathbf{F}_\alpha(\varphi) \to \mathbf{F}(\varphi) \qquad \forall \varphi \in \mathcal{D}(M, \mathbf{E}^*).$$

Proposition 10.2.3. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, let W be a finite-dimensional $\mathbb{K}$-vector space, and let $N := \dim W$. Then there exists a natural

isomorphism of topological vector spaces

$$\mathcal{D}'(M,\mathbf{E},W)\cong\mathcal{D}'(M,\mathbf{E})^N,$$

where the right-hand side is equipped with the product topology.

Proof. Let $\{w_1,\dots,w_N\}$ be a basis of W and $\{w^1,\dots,w^N\}$ the induced dual basis. Given $\mathbf{F}\in\mathcal{D}'(M,\mathbf{E},W)$, define

$$\mathbf{F}_j := w^j\circ\mathbf{F}\in\mathcal{D}'(M,\mathbf{E}),\qquad j\in\{1,\dots,N\}.$$

Then, for every $\varphi\in\mathcal{D}(M,\mathbf{E}^*)$, we have

$$\mathbf{F}(\varphi)=\sum_{j=1}^{N}\mathbf{F}_j(\varphi)\,w_j.$$

This defines a linear isomorphism

$$\mathcal{D}'(M,\mathbf{E},W)\longrightarrow\mathcal{D}'(M,\mathbf{E})^N.$$

Moreover, by the definition of the weak-$*$ topology, convergence in $\mathcal{D}'(M,\mathbf{E},W)$ is equivalent to componentwise convergence, showing that this isomorphism is a homeomorphism. □

Remark 10.2.4. In particular, there is a natural identification

$$\mathcal{D}'(M,\mathbf{E},W)\cong\mathcal{D}'(M,\mathbf{E})\otimes W,$$

which, upon choosing a basis of W, reduces to the preceding identification with a finite product.

Continuity can be characterized as follows, analogously to the Euclidean case.

Theorem 10.2.5. Let M be a smooth manifold with or without boundary, $\mathbf{E}\longrightarrow M$ a smooth vector bundle, and W a finite-dimensional $\mathbb{K}$-vector space equipped with a norm $|\cdot|_W$. Let $\mathbf{F}\colon\mathcal{D}(M,\mathbf{E}^*)\longrightarrow W$ be a linear operator. Then the following properties are equivalent:

(a) $\mathbf{F}$ is continuous.

(b) If $\{\varphi_k\}_{k\in\mathbb{N}}$ converges to φ in $\mathcal{D}(M,\mathbf{E}^*)$, then

$$\lim_{k\to\infty}\mathbf{F}(\varphi_k)=\mathbf{F}(\varphi).$$

(c) For every compact set $K\subseteq M$, the restriction $\mathbf{F}\restriction_{\mathcal{D}_K(M,\mathbf{E}^*)}$ is continuous.

(d) For every compact set $K\subseteq M$, there exist $m\in\mathbb{N}_0$ and $c_K>0$ such that

$$|\mathbf{F}(\varphi)|_W\le c_K\|\varphi\|_{C^m(K)}\qquad\forall\varphi\in\mathcal{D}_K(M,\mathbf{E}^*).$$

Proof. (a)⇒(b). Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(M,\mathbf{E}^*)$ such that $\varphi_k\to\varphi$. Continuity of $\mathbf{F}$ directly gives

$$\mathbf{F}(\varphi_k)\longrightarrow\mathbf{F}(\varphi).$$

(b)⇒(c). Let $K \subseteq M$ be a compact set. Consider the canonical inclusion

$$\iota\colon \mathcal{D}_K(M,\mathbf{E}^*) \hookrightarrow \mathcal{D}(M,\mathbf{E}^*),$$

which is continuous by construction of the inductive topology.
Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}_K(M,\mathbf{E}^*)$ such that $\varphi_k \to \varphi$ in $\mathcal{D}_K(M,\mathbf{E}^*)$. Then $\varphi_k \to \varphi$ in $\mathcal{D}(M,\mathbf{E}^*)$, and by (b),

$$\mathbf{F}(\varphi_k) \longrightarrow \mathbf{F}(\varphi).$$

This shows that the restriction $\mathbf{F}\restriction_{\mathcal{D}_K(M,\mathbf{E}^*)}$ is sequentially continuous. Since $\mathcal{D}_K(M,\mathbf{E}^*)$ is a Fréchet space, it is in particular metrizable, and Proposition B.10.10 shows that sequential continuity implies continuity. Therefore $\mathbf{F}\restriction_{\mathcal{D}_K(M,\mathbf{E}^*)}$ is continuous.
(c)⇒(a). Let $\{K_n\}_{n\in\mathbb{N}}$ be an exhausting sequence of compact sets such that

$$\mathcal{D}(M,\mathbf{E}^*) = \varinjlim_{n\in\mathbb{N}} \mathcal{D}_{K_n}(M,\mathbf{E}^*).$$

By hypothesis, for every $n \in \mathbb{N}$ the restriction $\mathbf{F}\restriction_{\mathcal{D}_{K_n}(M,\mathbf{E}^*)}$ is continuous.
By the definition of the inductive topology (Theorem 4.4.1), a linear operator

$$\mathbf{F}\colon \mathcal{D}(M,\mathbf{E}^*) \longrightarrow W$$

is continuous if and only if all its restrictions to the spaces $\mathcal{D}_{K_n}(M,\mathbf{E}^*)$ are continuous. We conclude that $\mathbf{F}$ is continuous.

(c)⇒(d). Let $K \subseteq M$ be a compact set. Then $\mathcal{D}_K(M,\mathbf{E}^*)$ is a locally convex topological vector space whose topology is generated by the increasing family of seminorms

$$\|\varphi\|_{C^m(K)}, \qquad m \in \mathbb{N}_0.$$

By (c), the restriction $\mathbf{F}\restriction_{\mathcal{D}_K(M,\mathbf{E}^*)}$ is continuous. Applying Theorem B.10.25 gives $m \in \mathbb{N}_0$ and $c_K > 0$ such that

$$|\mathbf{F}(\varphi)|_W \leq c_K \|\varphi\|_{C^m(K)} \qquad \forall \varphi \in \mathcal{D}_K(M,\mathbf{E}^*).$$

(d)⇒(c). Let $K \subseteq M$ be compact. Suppose the estimate in (d) holds. If $\{\varphi_k\}_{k\in\mathbb{N}}$ converges to φ in $\mathcal{D}_K(M,\mathbf{E}^*)$, then in particular

$$\|\varphi_k - \varphi\|_{C^m(K)} \longrightarrow 0$$

for every $m \in \mathbb{N}_0$. Applying the estimate in (d), we obtain

$$|\mathbf{F}(\varphi_k) - \mathbf{F}(\varphi)|_W \leq c_K \|\varphi_k - \varphi\|_{C^m(K)} \longrightarrow 0.$$

This shows that $\mathbf{F}\restriction_{\mathcal{D}_K(M,\mathbf{E}^*)}$ is continuous.

□

With this in mind, we define the order of a distribution:

Definition 10.2.6. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, let W be a finite-dimensional vector space, and let $\mathbf{F}\colon \mathcal{D}(M,\mathbf{E}^*) \longrightarrow W$ be a distribution.
Let $K \subseteq M$ be a compact set. By part (d) of Theorem 10.2.5, there exist $m_K \in \mathbb{N}_0$ and $C_K > 0$ such that

$$|\mathbf{F}(\boldsymbol{\varphi})|_W \le C_K \|\boldsymbol{\varphi}\|_{C^{m_K}(K)} \qquad \forall \boldsymbol{\varphi} \in \mathcal{D}_K(M,\mathbf{E}^*).$$

The least integer m_K satisfying this inequality is called the *order* of $\mathbf{F}$ on the compact set K.
We say that $\mathbf{F}$ is a distribution of *order* m on M if m is the least nonnegative integer such that, for every compact set $K \subseteq M$, there exists a constant $C_K > 0$ with

$$|\mathbf{F}(\boldsymbol{\varphi})|_W \le C_K \|\boldsymbol{\varphi}\|_{C^{m}(K)} \qquad \forall \boldsymbol{\varphi} \in \mathcal{D}_K(M,\mathbf{E}^*).$$

If no such integer m exists, we say that $\mathbf{F}$ is a distribution of infinite order.

Lemma 10.2.7. (Approximation with controlled enlargement of the support) Let M be a smooth manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle. If $K \subseteq \operatorname{Int} L$ are compact, then for each $\mathbf{u} \in \Gamma^m_K(\mathbf{E})$ there exists a sequence $\mathbf{u}_j \in \Gamma_L(\mathbf{E})$ such that $\mathbf{u}_j \to \mathbf{u}$ in the $C^m(L)$ norm.

Proof. If $\dim M = 0$, every section is smooth, and it suffices to take $\mathbf{u}_j = \mathbf{u}$. Suppose that $\dim M = n \ge 1$. Choose an open subset V with $K \subseteq V \Subset \operatorname{Int} L$, a finite cover of $\overline{V}$ by trivializing charts relatively compact in $\operatorname{Int} L$, and functions $\rho_a \in C^\infty_c(M)$ subordinate to these charts, indexed by $a \in \{1,\dots,N\}$, whose sum is one on a neighborhood of K. Then $\mathbf{u} = \sum_{a=1}^{N} \rho_a \mathbf{u}$. In each chart, the support of the components of $\rho_a \mathbf{u}$ is a compact subset of its coordinate domain; in particular, it is separated from its artificial boundaries.
Fix a nonnegative mollifier $\eta \in C^\infty_c(B(0,1))$ with integral one, and set $\eta_\varepsilon(z) = \varepsilon^{-n}\eta(z/\varepsilon)$. In an interior chart, each component f extends by zero to $\mathbb{R}^n$ as a compactly supported function of class C^m. The convolution $\eta_\varepsilon * f$ converges to f uniformly together with derivatives of order at most m.
In a boundary chart, first extend f by zero within $\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n \ge 0\}$, only across the artificial boundaries of the chart. This extension remains of class C^m up to the boundary and has compact support in $\mathbb{H}^n$. Denote by f_0 its measurable extension by zero to the lower half-space, and define

$$f_\varepsilon(x) := (\eta_\varepsilon * f_0)(x + 2\varepsilon e_n), \qquad x \in \mathbb{H}^n.$$

The convolution is smooth on $\mathbb{R}^n$. Moreover, for $x_n \ge 0$ and $z \in \operatorname{supp}\eta$, the last coordinate of $x + 2\varepsilon e_n - \varepsilon z$ is at least ε. Thus, for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$, we may differentiate using only interior values of f, obtaining

$$D^\alpha f_\varepsilon(x) = \int_{\mathbb{R}^n} \eta(z) D^\alpha f(x + 2\varepsilon e_n - \varepsilon z)\, dz.$$

The derivatives $D^\alpha f$ are uniformly continuous on the half-space, since they are continuous up to the boundary and compactly supported. Since $|2\varepsilon e_n - \varepsilon z| \le 3\varepsilon$ in this integral and $\int \eta = 1$, it follows that

$$\max_{|\alpha| \le m} \sup_{x \in \mathbb{H}^n} |D^\alpha f_\varepsilon(x) - D^\alpha f(x)| \longrightarrow 0.$$

In either type of chart, the support of the approximant lies in the neighborhood of radius 3ε of the original

support, taken in the corresponding coordinate model. For each support, we can choose a slightly larger compact set still contained in the chart domain. Taking ε sufficiently small ensures that all approximants are supported in these fixed compact sets. They are then transferred to the bundle and extended by zero outside their charts; the resulting sections are smooth and supported in $\operatorname{Int} L$. Equivalence of coordinate and covariant norms on these compact sets, given by Theorem 10.1.2, turns convergence of the components into C^m convergence of the sections. Since there are only finitely many charts and components, choose a common sequence $\varepsilon_j \downarrow 0$ and sum the approximants. This gives $\mathbf{u}_j \in \Gamma_L(\mathbf{E})$ with $\mathbf{u}_j \to \mathbf{u}$ in $C^m(L)$. □

Remark 10.2.8. Let $\mathbf{F}\colon \mathcal{D}(M,\mathbf{E}^*) \to W$ be a distribution of global order m. For $\mathbf{u} \in \Gamma_K^m(\mathbf{E}^*)$, choose a compact set L with $K \subseteq \operatorname{Int} L$ and, by the preceding lemma, $\mathbf{u}_j \in \Gamma_L(\mathbf{E}^*)$ with $\mathbf{u}_j \to \mathbf{u}$ in $C^m(L)$. The order estimate on L shows that $(\mathbf{F}(\mathbf{u}_j))$ is Cauchy; define

$$\widetilde{\mathbf{F}}(\mathbf{u}) := \lim_j \mathbf{F}(\mathbf{u}_j).$$

The value is independent of L and the approximation. Indeed, for two choices L_1, L_2, take a compact set L_3 containing both in its interior. The differences of the two approximants converge to zero in $C^m(L_3)$, and the order estimate on L_3 forces the limits to agree. This gives a unique continuous linear extension

$$\widetilde{\mathbf{F}}\colon \Gamma_c^m(\mathbf{E}^*) \longrightarrow W$$

agreeing with $\mathbf{F}$ on $\mathcal{D}(M,\mathbf{E}^*)$. For each compact set K, its restriction to $\Gamma_K^m(\mathbf{E}^*)$ is continuous because, choosing L as above, $\|\mathbf{u}\|_{C^m(L)} = \|\mathbf{u}\|_{C^m(K)}$.

We give some examples of distributions.

Example 10.2.9. We present two fundamental examples of distributions. Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle. Fix an auxiliary bundle metric $\mathbf{h}_{\mathbf{E}}$ on $\mathbf{E}$, and denote by $\mathbf{h}_{\mathbf{E}}^*$ the dual metric on $\mathbf{E}^*$.

(a) The Dirac delta. Let $p \in M$. Define

$$\delta_p\colon \mathcal{D}(M,\mathbf{E}^*) \longrightarrow \mathbf{E}_p^*, \qquad \delta_p(\boldsymbol{\varphi}) := \boldsymbol{\varphi}(p).$$

Let $K \subseteq M$ be compact and $\boldsymbol{\varphi} \in \mathcal{D}_K(M,\mathbf{E}^*)$. If $p \notin K$, then $\delta_p(\boldsymbol{\varphi}) = 0$. If $p \in K$, we have

$$|\delta_p(\boldsymbol{\varphi})|_{\mathbf{h}_{\mathbf{E}}^*(p)} = |\boldsymbol{\varphi}(p)|_{\mathbf{h}_{\mathbf{E}}^*(p)} \le \sup_{x\in K} |\boldsymbol{\varphi}(x)|_{\mathbf{h}_{\mathbf{E}}^*(x)} \le \|\boldsymbol{\varphi}\|_{C^0(K,\mathbf{E}^*)}.$$

In either case, there exists $C_K = 1$ such that

$$|\delta_p(\boldsymbol{\varphi})|_{\mathbf{h}_{\mathbf{E}}^*(p)} \le C_K \|\boldsymbol{\varphi}\|_{C^0(K,\mathbf{E}^*)}.$$

By part (d) of Theorem 10.2.5, δ_p is continuous. Moreover, since the estimate holds with $m = 0$ for every compact set K, we conclude that δ_p is a distribution of order 0.

(b) Locally integrable sections. Let $\mathbf{f} \in L^1_{\mathrm{loc}}(M,\mathbf{E})$. Define

$$\mathbf{T}_{\mathbf{f}}(\boldsymbol{\varphi}) := \int_M \boldsymbol{\varphi}(\mathbf{f})\, d\lambda_{\mathbf{g}}, \qquad \boldsymbol{\varphi} \in \mathcal{D}(M,\mathbf{E}^*).$$

Let $K \subseteq M$ be compact and $\varphi \in \mathcal{D}_K(M, \mathbf{E}^*)$. Then

$$|\mathbf{T}_\mathbf{f}(\varphi)| \leq \int_K |\varphi(x)(\mathbf{f}(x))|\, d\lambda_\mathbf{g}(x) \leq \int_K |\varphi(x)|_{\mathbf{h}^*_\mathbf{E}(x)}\, |\mathbf{f}(x)|_{\mathbf{h}_\mathbf{E}(x)}\, d\lambda_\mathbf{g}(x).$$

Hence

$$|\mathbf{T}_\mathbf{f}(\varphi)| \leq \|\varphi\|_{C^0(K,\mathbf{E}^*)} \int_K |\mathbf{f}(x)|_{\mathbf{h}_\mathbf{E}(x)}\, d\lambda_\mathbf{g}(x).$$

Since $\mathbf{f} \in L^1_{\mathrm{loc}}(M, \mathbf{E})$, the integral is finite; denoting it by C_K, we obtain

$$|\mathbf{T}_\mathbf{f}(\varphi)| \leq C_K \|\varphi\|_{C^0(K,\mathbf{E}^*)}.$$

By part (d) of Theorem 10.2.5, $\mathbf{T}_\mathbf{f}$ is continuous and, since it depends only on the norm C^0, is a distribution of order 0.

Although every locally integrable section can be viewed as a distribution, we may ask when a distribution is induced by such a section. Because of their significance, these distributions have a special name, which we record in the following definition:

Definition 10.2.10. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle. A distribution $\mathbf{T}\colon \mathcal{D}(M, \mathbf{E}^*) \longrightarrow \mathbb{K}$ is called *regular* if there exists $\mathbf{f} \in L^1_{\mathrm{loc}}(M, \mathbf{E})$ such that

$$\mathbf{T}(\varphi) = \int_M \varphi(\mathbf{f})\, d\lambda_\mathbf{g} \qquad \forall \varphi \in \mathcal{D}(M, \mathbf{E}^*),$$

where $\varphi(\mathbf{f})(x) = \varphi(x)(\mathbf{f}(x))$.

Remark 10.2.11. The linear map $\mathbf{f} \mapsto \mathbf{T}_\mathbf{f}$ defines an operator from $L^1_{\mathrm{loc}}(M, \mathbf{E})$ to the space of distributions. We will later see that this operator is injective. Thus it is customary to identify the section $\mathbf{f}$ with the distribution it induces, writing simply $\mathbf{f}(\varphi)$ instead of $\mathbf{T}_\mathbf{f}(\varphi)$.

The appearance of $\mathbf{E}^*$ in the space of test sections is explained by the canonical pairing between a bundle and its dual:

Remark 10.2.12. The choice to define a distribution as a linear operator

$$\mathbf{T}\colon \mathcal{D}(M, \mathbf{E}^*) \longrightarrow W,$$

where W is a finite-dimensional normed vector space, is motivated by the existence of a canonical pairing between a vector bundle and its dual.
If $\mathbf{f} \in \Gamma(\mathbf{E})$ is a sufficiently regular section, then to each $\varphi \in \mathcal{D}(M, \mathbf{E}^*)$ we can naturally associate a scalar by fiberwise contraction

$$\varphi(\mathbf{f})(x) = \varphi(x)(\mathbf{f}(x)).$$

This allows us to define a linear operator

$$\mathbf{T}_\mathbf{f}\colon \mathcal{D}(M, \mathbf{E}^*) \longrightarrow \mathbb{K}, \qquad \mathbf{T}_\mathbf{f}(\varphi) = \int_M \varphi(x)(\mathbf{f}(x))\, d\lambda_\mathbf{g}(x),$$

generalizing the classical scalar case.
For example, if $\Omega \subseteq \mathbb{R}^n$ is open and $f \in L^1_{\mathrm{loc}}(\Omega)$, then the expression

$$\int_\Omega f(x)\,\varphi(x)\,dx, \qquad \varphi \in C_c^\infty(\Omega),$$

defines a scalar distribution. The preceding formulation is simply the intrinsic version of this construction in the setting of vector bundles.

Observe that this construction uses only the canonical pairing

$$\mathbf{E}_x^* \times \mathbf{E}_x \longrightarrow \mathbb{K},$$

which requires no additional structure. In particular, if we worked with sections of $\mathbf{E}$ instead of sections of $\mathbf{E}^*$, there would be no natural way to define an expression of the form

$$\int_M \psi(x)\,\mathbf{f}(x)\,d\lambda_{\mathbf{g}}(x),$$

for $\psi, \mathbf{f} \in \Gamma(\mathbf{E})$ without introducing a bundle metric identifying $\mathbf{E}$ with $\mathbf{E}^*$.
For this reason, the formulation in terms of $\mathbf{E}^*$ is intrinsic and independent of additional geometric choices.

The extension to the W-valued case arises naturally when considering finite families of distributions. If $W \cong \mathbb{K}^r$, a linear operator

$$\mathbf{T}\colon \mathcal{D}(M,\mathbf{E}^*) \longrightarrow W$$

can be written as

$$\mathbf{T}(\varphi) = \big(T_1(\varphi),\dots,T_r(\varphi)\big),$$

where each $T_i\colon \mathcal{D}(M,\mathbf{E}^*) \longrightarrow \mathbb{K}$ is a scalar distribution. In this sense, distributions with values in W introduce no new analytic difficulty, but allow finitely many scalar distributions to be treated simultaneously.

This formulation will be fundamental later when considering distributions on product manifolds and describing linear operators by distributional kernels.

10.2.1 Partial differential operators and distributions

We have seen that distributions are defined as linear functionals on the space of test sections $\mathcal{D}(M,\mathbf{E}^*)$. In particular, to extend the action of differential operators to distributions, we must understand how these operators interact with this space.

Let

$$P\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{F})$$

be a differential operator. If $\mathbf{T}$ is a distribution, we would like to define a new distribution $P\mathbf{T}$ representing the action of P on $\mathbf{T}$. By analogy with the classical case, it is natural to expect this action to be determined by an identity of the form

$$(P\mathbf{T})(\varphi) = \mathbf{T}(\text{expression depending on } P \text{ and } \varphi).$$

Since $\mathbf{T}$ is defined only on $\mathcal{D}(M,\mathbf{E}^*)$, the preceding expression suggests that to each $\varphi \in \mathcal{D}(M,\mathbf{F}^*)$ we should

associate a section of $\mathcal{D}(M, \mathbf{E}^*)$. That is, we need to construct an operator

$$P' \colon \mathcal{D}(M, \mathbf{F}^*) \longrightarrow \mathcal{D}(M, \mathbf{E}^*),$$

that transfers the action of P to test sections.

Unlike the pointwise case, transferring differential operators requires the integral structure of the manifold in order to integrate by parts. This suggests that the operator P' should be characterized by an integral identity involving the natural evaluation between bundles and their duals.

Definition 10.2.13. (Formal transpose) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be complex vector bundles. If $P \in \mathbf{PDO}(\mathbf{E}, \mathbf{F})$, its *formal transpose* is an operator

$$P' \colon \Gamma(\mathbf{F}^*) \longrightarrow \Gamma(\mathbf{E}^*)$$

satisfying

$$\int_M \psi(P\mathbf{u})\, d\lambda_{\mathbf{g}} = \int_M (P'\psi)(\mathbf{u})\, d\lambda_{\mathbf{g}} \tag{10.1}$$

for every $\mathbf{u} \in \Gamma_c(\mathbf{E})$ and every $\psi \in \Gamma_c(\mathbf{F}^*)$. The pairing in this identity is complex-bilinear. In the real case, the same definition applies, with all conjugations below omitted.

Proposition 10.2.14. (Intrinsic construction of the formal transpose) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be complex vector bundles. Let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$. The formal transpose exists, is unique, and belongs to $\mathbf{PDO}^{(m)}(\mathbf{F}^*, \mathbf{E}^*)$. For any auxiliary Hermitian metrics $\mathbf{h}_{\mathbf{E}}, \mathbf{h}_{\mathbf{F}}$, linear in the first variable, we have

$$P' = \mathcal{R}_{\mathbf{E}} \circ \overline{P_h^*} \circ \mathcal{R}_{\mathbf{F}}^{-1}. \tag{10.2}$$

The domains and codomains of the composition are displayed in the diagram

$$\mathbf{F}^* \xrightarrow{\mathcal{R}_{\mathbf{F}}^{-1}} \overline{\mathbf{F}} \xrightarrow{\overline{P_h^*}} \overline{\mathbf{E}} \xrightarrow{\mathcal{R}_{\mathbf{E}}} \mathbf{E}^*.$$

In particular, the right-hand side of (10.2) is independent of the auxiliary bundle metrics.

Proof. The Riesz isomorphisms $\mathcal{R}_{\mathbf{F}}^{-1} \colon \mathbf{F}^* \to \overline{\mathbf{F}}$ and $\mathcal{R}_{\mathbf{E}} \colon \overline{\mathbf{E}} \to \mathbf{E}^*$ are complex-linear and of order zero. By the theory of operator conjugation, $\overline{P_h^*} \colon \Gamma(\overline{\mathbf{F}}) \to \Gamma(\overline{\mathbf{E}})$ is complex-linear and has order at most m. By composition, the right-hand side of (10.2) is a complex-linear operator of order at most m.
Let $\mathbf{u} \in \Gamma_c(\mathbf{E})$ and $\psi \in \Gamma_c(\mathbf{F}^*)$, and write

$$\mathcal{R}_{\mathbf{F}}^{-1}\psi = \overline{\mathbf{v}}, \qquad \mathbf{v} \in \Gamma_c(\mathbf{F}).$$

By the definition $\mathcal{R}_{\mathbf{F}}(\overline{\mathbf{v}})(\mathbf{w}) = \mathbf{h}_{\mathbf{F}}(\mathbf{w}, \mathbf{v})$ and the Hermitian adjoint identity,

$$\begin{aligned} \int_M \psi(P\mathbf{u})\, d\lambda_{\mathbf{g}} &= \int_M \mathbf{h}_{\mathbf{F}}(P\mathbf{u}, \mathbf{v})\, d\lambda_{\mathbf{g}} \\ &= \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u}, P_h^*\mathbf{v})\, d\lambda_{\mathbf{g}} \end{aligned}$$

$$= \int_M \mathcal{R}_{\mathbf{E}}\left(\overline{P_h^*\mathbf{v}}\right)(\mathbf{u})\, d\lambda_{\mathbf{g}}$$
$$= \int_M \left(\mathcal{R}_{\mathbf{E}} \circ \overline{P_h^*} \circ \mathcal{R}_{\mathbf{F}}^{-1}\right)(\psi)(\mathbf{u})\, d\lambda_{\mathbf{g}}.$$

This proves (10.1).

For uniqueness, let $S_1, S_2 \in \mathbf{PDO}(\mathbf{F}^*, \mathbf{E}^*)$ be two operators satisfying this identity. Fix $\psi \in \Gamma_c(\mathbf{F}^*)$ and set $\eta = (S_1 - S_2)\psi \in \Gamma_c(\mathbf{E}^*)$. If $\mathcal{R}_{\mathbf{E}}^{-1}\eta = \overline{\mathbf{w}}$, then $\mathbf{w} \in \Gamma_c(\mathbf{E})$ and, taking $\mathbf{u} = \mathbf{w}$, we obtain

$$0 = \int_M \eta(\mathbf{w})\, d\lambda_{\mathbf{g}} = \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{w}, \mathbf{w})\, d\lambda_{\mathbf{g}}.$$

Therefore $\mathbf{w} = 0$ and $S_1\psi = S_2\psi$. For a section ψ without compact support, choose a cutoff equal to one near a given point and use locality of S_1, S_2. Thus $S_1 = S_2$ on all sections. Since any choice of auxiliary metrics produces an operator satisfying the same integral identity, uniqueness proves independence of these metrics. □

Corollary 10.2.15. (Local formula for the formal transpose) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, let $\mathbf{E}, \mathbf{F} \to M$ be complex vector bundles, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$. In a chart $\varphi\colon U \to \Omega \subseteq \mathbb{R}^n$, with coordinate functions $(x^1, \dots, x^n)$, let $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ and $(\mathbf{f}_1, \dots, \mathbf{f}_s)$ be local frames of $\mathbf{E}$ and $\mathbf{F}$, and let $(\mathbf{e}^1, \dots, \mathbf{e}^r)$ and $(\mathbf{f}^1, \dots, \mathbf{f}^s)$ be their dual frames. Set

$$D_\varphi^\alpha a := \left[D^\alpha (a \circ \varphi^{-1})\right] \circ \varphi.$$

If

$$P\left(\sum_{i=1}^r u^i \mathbf{e}_i\right) = \sum_{j=1}^s \left(\sum_{i=1}^r \sum_{|\alpha| \le m} a_{\alpha i}^j D_\varphi^\alpha u^i\right) \mathbf{f}_j,$$

then, for $\psi = \sum_{j=1}^s \psi_j \mathbf{f}^j$,

$$P'\psi = \frac{1}{\sqrt{\det(\mathbf{g})}} \sum_{i=1}^r \left[\sum_{j=1}^s \sum_{|\alpha| \le m} (-1)^{|\alpha|} D_\varphi^\alpha \left(\sqrt{\det(\mathbf{g})}\, a_{\alpha i}^j \psi_j\right)\right] \mathbf{e}^i, \tag{10.3}$$

where $g_{ab} = \mathbf{g}(\partial_a, \partial_b)$.

Proof. Let $\mathbf{u} = \sum_{i=1}^r u^i \mathbf{e}_i \in \Gamma_c(\mathbf{E}|_U)$ and let $\psi = \sum_{j=1}^s \psi_j \mathbf{f}^j \in \Gamma(\mathbf{F}^*|_U)$. By the local formulas for P and the Riemannian measure,

$$\int_U \psi(P\mathbf{u})\, d\lambda_{\mathbf{g}} = \sum_{i=1}^r \sum_{j=1}^s \sum_{|\alpha| \le m} \int_\Omega \left(a_{\alpha i}^j \circ \varphi^{-1}\right)\left(\psi_j \circ \varphi^{-1}\right) D^\alpha \left(u^i \circ \varphi^{-1}\right)\left(\sqrt{\det(\mathbf{g})} \circ \varphi^{-1}\right) d\lambda_n.$$

The function $u^i \circ \varphi^{-1}$ has compact support in Ω. Integrating by parts $|\alpha|$ times, without boundary terms,

the summand corresponding to i, j, α becomes

$$(-1)^{|\alpha|}\int_{\Omega}\big(u^{i}\circ\varphi^{-1}\big)D^{\alpha}\Big[\Big(\sqrt{\det(\mathbf{g})}\,a^{j}_{\alpha i}\psi_{j}\Big)\circ\varphi^{-1}\Big]\,d\lambda_{n}.$$

Summing all terms and inserting $\big(\sqrt{\det(\mathbf{g})}\big)^{-1}\sqrt{\det(\mathbf{g})}$ gives

$$\int_{U}\boldsymbol{\psi}(P\mathbf{u})\,d\lambda_{\mathbf{g}}=\int_{U}(P'\boldsymbol{\psi})(\mathbf{u})\,d\lambda_{\mathbf{g}},$$

where $P'\boldsymbol{\psi}$ is precisely the right-hand side of (10.3). Uniqueness of the formal transpose identifies this local expression with the global operator already constructed. □

Proposition 10.2.16. (Rules for the formal transpose) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E},\mathbf{F},\mathbf{G}\to M$ be complex vector bundles. Let $P\in\mathbf{PDO}(\mathbf{E},\mathbf{F})$, $Q\in\mathbf{PDO}(\mathbf{F},\mathbf{G})$, and $R\in\mathbf{PDO}(\mathbf{E},\mathbf{F})$. Then:

(a) for every $\lambda\in\mathbb{C}$,
$$(P+\lambda R)'=P'+\lambda R';$$

(b)
$$(Q\circ P)'=P'\circ Q';$$

(c) if $U\subseteq M$ is open and P_{U} denotes the local restriction of P, then
$$(P_{U})'=(P')_{U};$$

(d) if P has order zero, then
$$(P'\boldsymbol{\psi})(\mathbf{u})=\boldsymbol{\psi}(P\mathbf{u})$$
pointwise; in particular, $(M_{f})'=M_{f}$;

(e) if
$$\kappa_{\mathbf{E}}\colon\overline{\mathbf{E}^{*}}\longrightarrow(\overline{\mathbf{E}})^{*},\qquad\kappa_{\mathbf{E}}(\overline{\boldsymbol{\eta}})(\overline{v}):=\overline{\boldsymbol{\eta}(v)},$$
and
$$\kappa_{\mathbf{F}}\colon\overline{\mathbf{F}^{*}}\longrightarrow(\overline{\mathbf{F}})^{*},\qquad\kappa_{\mathbf{F}}(\overline{\psi})(\overline{w}):=\overline{\psi(w)},$$
then
$$(\overline{P})'=\kappa_{\mathbf{E}}\circ\overline{P'}\circ\kappa_{\mathbf{F}}^{-1}.\tag{10.4}$$

Proof. The first identity follows from bilinearity of (10.1); unlike the Hermitian adjoint, no $\overline{\lambda}$ appears here. For $\mathbf{u}\in\Gamma_{c}(\mathbf{E})$ and $\boldsymbol{\theta}\in\Gamma_{c}(\mathbf{G}^{*})$,

$$\int_{M}\boldsymbol{\theta}(QP\mathbf{u})\,d\lambda_{\mathbf{g}}=\int_{M}(Q'\boldsymbol{\theta})(P\mathbf{u})\,d\lambda_{\mathbf{g}}=\int_{M}(P'Q'\boldsymbol{\theta})(\mathbf{u})\,d\lambda_{\mathbf{g}},$$

which proves the second identity by uniqueness. The same integral identity, applied entirely within U, proves the third. The fourth is the definition of the fiberwise algebraic transpose.

The map $\kappa_{\mathbf{E}}$ is a complex-linear isomorphism. For $\mathbf{u} \in \Gamma_c(\mathbf{E})$ and $\eta \in \Gamma_c(\mathbf{F}^*)$, we have

$$\kappa_{\mathbf{F}}(\overline{\eta})(\overline{P}\,\overline{\mathbf{u}}) = \overline{\eta(P\mathbf{u})}, \qquad \kappa_{\mathbf{E}}(\overline{P'\eta})(\overline{\mathbf{u}}) = \overline{(P'\eta)(\mathbf{u})}.$$

Conjugating the integral identity for P' yields the integral identity for the right-hand side of (10.4); uniqueness of the formal transpose finishes the proof. □

The preceding construction extends the action of differential operators to distributions. If $P\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{F})$ is a differential operator and $P'\colon \Gamma(\mathbf{F}^*) \longrightarrow \Gamma(\mathbf{E}^*)$ denotes its formal transpose with respect to the bilinear evaluation pairing, then P' sends test sections of $\mathbf{F}^*$ to test sections of $\mathbf{E}^*$. This allows the action of P on a distribution to be defined by duality.

Lemma 10.2.17. (C^k estimate for differential operators) Let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles of ranks r and s, respectively, over a smooth manifold M with or without boundary, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$. Then, for every compact set $K \subseteq M$ and every $k \in \mathbb{N}_0$, there exists a constant $C_{K,k} > 0$ such that

$$\|P\mathbf{u}\|_{C^k(K,\mathbf{F})} \leq C_{K,k}\|\mathbf{u}\|_{C^{k+m}(K,\mathbf{E})}, \qquad \mathbf{u} \in \Gamma_K^{k+m}(\mathbf{E}).$$

In particular, P induces a continuous map

$$P\colon \mathcal{D}_K(M,\mathbf{E}) \longrightarrow \mathcal{D}_K(M,\mathbf{F}).$$

Proof. Let $K \subseteq M$ be compact. Take a finite family of smooth charts (U_i, φ_i), with $i \in \{1,\dots,N\}$, such that each U_i is a regular coordinate ball, local trivializations

$$\boldsymbol{\tau}_{\mathbf{E},i}\colon \mathbf{E} \restriction_{U_i} \longrightarrow U_i \times \mathbb{K}^r, \qquad \boldsymbol{\tau}_{\mathbf{F},i}\colon \mathbf{F} \restriction_{U_i} \longrightarrow U_i \times \mathbb{K}^s,$$

exist, and $K \subseteq \bigcup_{i=1}^{N} U_i$ holds.

Consider a partition of unity $\{\psi_i\}_{i=1}^N$ subordinate to the cover $\{U_i\}_{i=1}^N$ of K, with sum one on a neighborhood of K, and define

$$K_i := K \cap \operatorname{supp}(\psi_i).$$

Since $\operatorname{supp}(\psi_i)$ is closed in M and contained in U_i, each K_i is a compact subset of U_i. Omit the indices for which $K_i = \varnothing$ and renumber the rest. If $\mathbf{u} \in \Gamma_K^{k+m}(\mathbf{E})$, then $\operatorname{supp}(\mathbf{u}) \subseteq K$ and

$$\mathbf{u} = \sum_{i=1}^{N} \psi_i \mathbf{u}.$$

Moreover, since $\operatorname{supp}(\psi_i) \subseteq U_i$, we have

$$\operatorname{supp}(\psi_i\mathbf{u}) \subseteq \operatorname{supp}(\mathbf{u}) \cap \operatorname{supp}(\psi_i) \subseteq K \cap \operatorname{supp}(\psi_i) = K_i.$$

Fix $i \in \{1,\dots,N\}$ and write $\mathbf{v}_i := \psi_i\mathbf{u}$. Define its local representation in the trivialization $\boldsymbol{\tau}_{\mathbf{E},i}$ by

$$\widetilde{v_i} := \pi_2 \circ \boldsymbol{\tau}_{\mathbf{E},i} \circ (\mathbf{v}_i \restriction_{U_i})\colon U_i \longrightarrow \mathbb{K}^r,$$

and similarly define the local representation of $P\mathbf{v}_i$ in the trivialization $\boldsymbol{\tau}_{\mathbf{F},i}$ by

$$\widetilde{P\mathbf{v}_i} := \pi_2 \circ \boldsymbol{\tau}_{\mathbf{F},i} \circ ((P\mathbf{v}_i) \restriction_{U_i}) \colon U_i \longrightarrow \mathbb{K}^s.$$

By Theorem 8.1.34, there exist smooth functions

$$a_{\alpha,i} \colon U_i \longrightarrow \operatorname{Hom}(\mathbb{K}^r, \mathbb{K}^s), \qquad |\alpha| \le m,$$

such that

$$\widetilde{P\mathbf{v}_i} = \sum_{|\alpha| \le m} a_{\alpha,i}\, D^\alpha \widetilde{v}_i.$$

Let β be a multi-index with $|\beta| \le k$. We apply the Leibniz rule to the bilinear map

$$\operatorname{Hom}(\mathbb{K}^r, \mathbb{K}^s) \times \mathbb{K}^r \longrightarrow \mathbb{K}^s, \qquad (A, w) \longmapsto A(w).$$

To make the notation precise, if $A \colon U_i \longrightarrow \operatorname{Hom}(\mathbb{K}^r, \mathbb{K}^s)$ and we write $A(x) = (A^b_a(x))$, with $1 \le a \le r$ and $1 \le b \le s$, then we define $D^\gamma A$ componentwise, that is,

$$D^\gamma A(x) := \big(D^\gamma A^b_a(x)\big).$$

With this convention, $D^\gamma A(x)$ remains an element of $\operatorname{Hom}(\mathbb{K}^r, \mathbb{K}^s)$.
Then, for each $|\beta| \le k$, we have

$$D^\beta(\widetilde{P\mathbf{v}_i}) = \sum_{|\alpha| \le m} \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \big(D^\gamma a_{\alpha,i}\big)\big(D^{\alpha+\beta-\gamma} \widetilde{v}_i\big).$$

This identity follows by applying the Leibniz rule componentwise to the preceding matrix expression.
Since K_i is compact and the coefficients $a_{\alpha,i}$ are smooth, the functions $D^\gamma a_{\alpha,i}$ are bounded on K_i for every $|\gamma| \le k$. Moreover, if $|\alpha| \le m$, $\gamma \le \beta$, and $|\beta| \le k$, then $|\alpha + \beta - \gamma| \le m + k$. Thus there exists a constant $A_i > 0$ such that

$$\max_{|\beta| \le k} \|D^\beta(\widetilde{P\mathbf{v}_i})\|_{C^0(K_i, \mathbb{K}^s)} \le A_i \max_{|\eta| \le k+m} \|D^\eta \widetilde{v}_i\|_{C^0(K_i, \mathbb{K}^r)}.$$

Now relate this estimate to the C^j norms. On U_i, write $\varphi_i = (x_i^1, \dots, x_i^n)$ and denote the coordinate vector fields by $\partial_{x_i^1}, \dots, \partial_{x_i^n}$. Define a connection $\widetilde{\nabla}^{TM,i}$ on TU_i by declaring these coordinate vector fields parallel, that is,

$$\widetilde{\nabla}^{TM,i}_{\partial_{x_i^\ell}} \partial_{x_i^a} = 0, \qquad a, \ell \in \{1, \dots, n\}.$$

Likewise, if $\mathbf{e}_1^i, \dots, \mathbf{e}_r^i$ is the local frame of $\mathbf{E} \restriction_{U_i}$ induced by the trivialization $\boldsymbol{\tau}_{\mathbf{E},i}$, define a connection $\widetilde{\nabla}^{\mathbf{E},i}$ on $\mathbf{E} \restriction_{U_i}$ by declaring this frame parallel:

$$\widetilde{\nabla}^{\mathbf{E},i}_{\partial_{x_i^\ell}} \mathbf{e}_a^i = 0, \qquad a \in \{1, \dots, r\},\ \ell \in \{1, \dots, n\}.$$

Also take the Riemannian metric $\widetilde{\mathbf{g}}_i$ making the coordinate frame orthonormal and the bundle metric $\widetilde{\mathbf{h}}_{\mathbf{E},i}$ making the frame $\mathbf{e}_1^i, \dots, \mathbf{e}_r^i$ orthonormal.

With these choices, if $\mathbf{v}_i = \sum_{a=1}^{r} v_i^a \mathbf{e}_a^i$ on U_i, then $\widetilde{v}_i = (v_i^1, \dots, v_i^r)$ and the components of the iterated covariant derivatives of $\mathbf{v}_i$, computed using the connections $\widetilde{\nabla}^{TM,i}$ and $\widetilde{\nabla}^{\mathbf{E},i}$, are precisely the partial derivatives of the components of $\widetilde{v}_i$. Consequently, for each $j \le k+m$ there exists a constant $B_{i,j} > 0$ such that

$$\max_{|\eta|\le j} \|D^\eta \widetilde{v}_i\|_{C^0(K_i,\mathbb{K}^r)} \le B_{i,j} \|\mathbf{v}_i\|_{\widetilde{C^j}(K_i,\mathbf{E})}.$$

Here $\|\cdot\|_{\widetilde{C^j}(K_i,\mathbf{E})}$ denotes the C^j norm on sections of $\mathbf{E}$ constructed using the choices $\widetilde{\mathbf{g}}_i$, $\widetilde{\nabla}^{TM,i}$, $\widetilde{\mathbf{h}}_{\mathbf{E},i}$, and $\widetilde{\nabla}^{\mathbf{E},i}$. By Theorem 10.1.2, applied to the compact set K_i and the bundle $\mathbf{E}$, the norms $\|\cdot\|_{\widetilde{C^{k+m}}(K_i,\mathbf{E})}$ and $\|\cdot\|_{C^{k+m}(K_i,\mathbf{E})}$ are equivalent. In particular, there exists a constant $C_i > 0$ such that

$$\|\mathbf{v}_i\|_{\widetilde{C^{k+m}}(K_i,\mathbf{E})} \le C_i \|\mathbf{v}_i\|_{C^{k+m}(K_i,\mathbf{E})}.$$

Since $\mathbf{v}_i = \psi_i \mathbf{u}$, the Leibniz rule for covariant derivatives implies the existence of a constant $L_i > 0$ such that

$$\|\mathbf{v}_i\|_{C^{k+m}(K_i,\mathbf{E})} = \|\psi_i \mathbf{u}\|_{C^{k+m}(K_i,\mathbf{E})} \le L_i \|\mathbf{u}\|_{C^{k+m}(K,\mathbf{E})}.$$

Apply the same reasoning to the bundle $\mathbf{F}$. If $\mathbf{f}_1^i, \dots, \mathbf{f}_s^i$ is the local frame of $\mathbf{F}\restriction_{U_i}$ induced by $\boldsymbol{\tau}_{\mathbf{F},i}$, take the connection $\widetilde{\nabla}^{\mathbf{F},i}$ determined by

$$\widetilde{\nabla}^{\mathbf{F},i}_{\partial_{x_i^\ell}} \mathbf{f}_b^i = 0, \qquad b \in \{1,\dots,s\},\ \ell \in \{1,\dots,n\},$$

and the bundle metric $\widetilde{h}_{\mathbf{F},i}$ making that frame orthonormal. Then, using Theorem 10.1.2 again, there exists a constant $D_i > 0$ such that

$$\|P\mathbf{v}_i\|_{C^k(K_i,\mathbf{F})} \le D_i \max_{|\beta|\le k} \|D^\beta(\widetilde{P\mathbf{v}_i})\|_{C^0(K_i,\mathbb{K}^s)}.$$

Combining the preceding estimates gives

$$\|P\mathbf{v}_i\|_{C^k(K_i,\mathbf{F})} \le D_i A_i B_{i,k+m} C_i L_i \|\mathbf{u}\|_{C^{k+m}(K,\mathbf{E})}.$$

Thus, for each $i \in \{1,\dots,N\}$ there exists a constant $C_i' > 0$ such that

$$\|P(\psi_i \mathbf{u})\|_{C^k(K_i,\mathbf{F})} \le C_i' \|\mathbf{u}\|_{C^{k+m}(K,\mathbf{E})}.$$

Since $\mathbf{u} = \sum_{i=1}^{N} \psi_i \mathbf{u}$ and P is linear, we have $P\mathbf{u} = \sum_{i=1}^{N} P(\psi_i \mathbf{u})$. Moreover, locality of P implies $\operatorname{supp}(P(\psi_i \mathbf{u})) \subseteq \operatorname{supp}(\psi_i \mathbf{u}) \subseteq K_i$. Therefore $\|P(\psi_i \mathbf{u})\|_{C^k(K,\mathbf{F})} = \|P(\psi_i \mathbf{u})\|_{C^k(K_i,\mathbf{F})}$. Using the triangle inequality for the norm $C^k(K,\mathbf{F})$, we obtain

$$\|P\mathbf{u}\|_{C^k(K,\mathbf{F})} \le \sum_{i=1}^{N} \|P(\psi_i \mathbf{u})\|_{C^k(K,\mathbf{F})} = \sum_{i=1}^{N} \|P(\psi_i \mathbf{u})\|_{C^k(K_i,\mathbf{F})}.$$

The preceding estimates give

$$\|P\mathbf{u}\|_{C^k(K,\mathbf{F})} \le \left(\sum_{i=1}^{N} C_i' \right) \|\mathbf{u}\|_{C^{k+m}(K,\mathbf{E})}.$$

Setting $C_{K,k} := \sum_{i=1}^{N} C_i'$ yields the desired estimate.
Locality of P ensures that $\operatorname{supp}(P\mathbf{u}) \subseteq \operatorname{supp}(\mathbf{u})$, so the map is well defined, and the preceding argument also shows that it is continuous. □

Proposition 10.2.18. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $\mathbf{E},\mathbf{F} \longrightarrow M$ be smooth vector bundles, let W be a finite-dimensional $\mathbb{K}$-vector space, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$. Then P induces a continuous linear operator

$$P\colon \mathcal{D}'(M,\mathbf{E},W) \longrightarrow \mathcal{D}'(M,\mathbf{F},W)$$

given by

$$(P\mathbf{T})(\boldsymbol{\varphi}) := \mathbf{T}(P'\boldsymbol{\varphi}), \qquad \boldsymbol{\varphi} \in \mathcal{D}(M,\mathbf{F}^*),$$

where $P'\colon \Gamma(\mathbf{F}^*) \longrightarrow \Gamma(\mathbf{E}^*)$ is the bilinear formal transpose. Continuity is understood with respect to the weak-$*$ topologies on both distribution spaces.

Proof. First check that the definition makes sense. By Proposition 10.2.14, we have $P' \in \mathbf{PDO}^{(m)}(\mathbf{F}^*,\mathbf{E}^*)$. Thus, if $\boldsymbol{\varphi} \in \mathcal{D}(M,\mathbf{F}^*)$, then $P'\boldsymbol{\varphi}$ is a smooth section of $\mathbf{E}^*$.
Moreover, since differential operators are local, we have

$$\operatorname{supp}(P'\boldsymbol{\varphi}) \subseteq \operatorname{supp}(\boldsymbol{\varphi}).$$

In particular, if $\boldsymbol{\varphi}$ has compact support, then $P'\boldsymbol{\varphi}$ also has compact support. Thus $P'\boldsymbol{\varphi} \in \mathcal{D}(M,\mathbf{E}^*)$, and consequently the expression $\mathbf{T}(P'\boldsymbol{\varphi})$ is well defined.
Define $(P\mathbf{T})(\boldsymbol{\varphi}) := \mathbf{T}(P'\boldsymbol{\varphi})$ for $\boldsymbol{\varphi} \in \mathcal{D}(M,\mathbf{F}^*)$. Linearity of $P\mathbf{T}$ follows from linearity of $\mathbf{T}$ and P'.
Now prove that $P\mathbf{T} \in \mathcal{D}'(M,\mathbf{F},W)$. Let $K \subseteq M$ be compact. If $\boldsymbol{\varphi} \in \mathcal{D}_K(M,\mathbf{F}^*)$, then locality of P' gives $P'\boldsymbol{\varphi} \in \mathcal{D}_K(M,\mathbf{E}^*)$. Since $\mathbf{T} \in \mathcal{D}'(M,\mathbf{E},W)$, Theorem 10.2.5 provides $\ell \in \mathbb{N}_0$ and $c_K > 0$ such that

$$|\mathbf{T}(\boldsymbol{\psi})|_W \le c_K \|\boldsymbol{\psi}\|_{C^\ell(K,\mathbf{E}^*)} \qquad \forall \boldsymbol{\psi} \in \mathcal{D}_K(M,\mathbf{E}^*).$$

Applying this estimate to $\boldsymbol{\psi} = P'\boldsymbol{\varphi}$ gives

$$|(P\mathbf{T})(\boldsymbol{\varphi})|_W = |\mathbf{T}(P'\boldsymbol{\varphi})|_W \le c_K \|P'\boldsymbol{\varphi}\|_{C^\ell(K,\mathbf{E}^*)}.$$

Since P' is a differential operator of order m, Lemma 10.2.17 provides a constant $C_{K,\ell} > 0$ such that

$$\|P'\boldsymbol{\varphi}\|_{C^\ell(K,\mathbf{E}^*)} \le C_{K,\ell} \|\boldsymbol{\varphi}\|_{C^{\ell+m}(K,\mathbf{F}^*)}.$$

Therefore

$$|(P\mathbf{T})(\boldsymbol{\varphi})|_W \le c_K C_{K,\ell} \|\boldsymbol{\varphi}\|_{C^{\ell+m}(K,\mathbf{F}^*)} \qquad \forall \boldsymbol{\varphi} \in \mathcal{D}_K(M,\mathbf{F}^*).$$

Applying Theorem 10.2.5 again, now to the bundle $\mathbf{F}$, we conclude that $P\mathbf{T} \in \mathcal{D}'(M,\mathbf{F},W)$.
Next check linearity of the operator induced by P. If $\lambda,\mu \in \mathbb{K}$ and $\mathbf{T}_1,\mathbf{T}_2 \in \mathcal{D}'(M,\mathbf{E},W)$, then, for every $\boldsymbol{\varphi} \in \mathcal{D}(M,\mathbf{F}^*)$,

$$P(\lambda\mathbf{T}_1 + \mu\mathbf{T}_2)(\boldsymbol{\varphi}) = (\lambda\mathbf{T}_1 + \mu\mathbf{T}_2)(P'\boldsymbol{\varphi}) = \lambda\mathbf{T}_1(P'\boldsymbol{\varphi}) + \mu\mathbf{T}_2(P'\boldsymbol{\varphi}).$$

Therefore $P(\lambda \mathbf{T}_1 + \mu \mathbf{T}_2) = \lambda P\mathbf{T}_1 + \mu P\mathbf{T}_2$.
Finally, prove continuity of the induced operator in the weak-$*$ topologies. Let $(\mathbf{T}_\alpha)_\alpha$ be a net in $\mathcal{D}'(M, \mathbf{E}, W)$ such that $\mathbf{T}_\alpha \to \mathbf{T}$ in $\mathcal{D}'(M, \mathbf{E}, W)$. By Remark 10.2.2, this means that

$$\mathbf{T}_\alpha(\psi) \longrightarrow \mathbf{T}(\psi) \qquad \forall \psi \in \mathcal{D}(M, \mathbf{E}^*).$$

Now let $\varphi \in \mathcal{D}(M, \mathbf{F}^*)$. As already shown, $P'\varphi \in \mathcal{D}(M, \mathbf{E}^*)$. Therefore

$$(P\mathbf{T}_\alpha)(\varphi) = \mathbf{T}_\alpha(P'\varphi) \longrightarrow \mathbf{T}(P'\varphi) = (P\mathbf{T})(\varphi).$$

Since this holds for every $\varphi \in \mathcal{D}(M, \mathbf{F}^*)$, Remark 10.2.2 again gives $P\mathbf{T}_\alpha \to P\mathbf{T}$ in $\mathcal{D}'(M, \mathbf{F}, W)$. Thus the operator induced by P is continuous. □

Remark 10.2.19. If $\mathbf{u} \in \Gamma(\mathbf{E})$ is a smooth section, then $P\mathbf{u} \in \Gamma(\mathbf{F})$, and the regular distribution associated with $P\mathbf{u}$ agrees with the distribution obtained by applying P to the regular distribution associated with $\mathbf{u}$. If $\mathbf{T}_\mathbf{u}$ is given by

$$\mathbf{T}_\mathbf{u}(\psi) := \int_M \psi(\mathbf{u})\, d\lambda_\mathbf{g}, \qquad \psi \in \mathcal{D}(M, \mathbf{E}^*),$$

then, for every $\varphi \in \mathcal{D}(M, \mathbf{F}^*)$,

$$(P\mathbf{T}_\mathbf{u})(\varphi) = \mathbf{T}_\mathbf{u}(P'\varphi) = \int_M (P'\varphi)(\mathbf{u})\, d\lambda_\mathbf{g}.$$

By the definition of the bilinear formal transpose,

$$\int_M (P'\varphi)(\mathbf{u})\, d\lambda_\mathbf{g} = \int_M \varphi(P\mathbf{u})\, d\lambda_\mathbf{g}.$$

Therefore $P\mathbf{T}_\mathbf{u} = \mathbf{T}_{P\mathbf{u}}$.
In particular, if P has order 0, then P corresponds to a smooth section of $\operatorname{Hom}(\mathbf{E}, \mathbf{F})$ and P' corresponds to the pointwise transpose $\mathbf{F}^* \to \mathbf{E}^*$. In the special case where $\mathbf{E} = \mathbf{F}$ and P is multiplication by a function $f \in C^\infty(M)$, we obtain

$$(fT)(\varphi) = T(f\varphi).$$

The preceding proposition shows that the action of a differential operator on distributions is determined by the action of its bilinear formal transpose on test sections. This construction suffices to define distributional derivatives and, more generally, to transfer differential operators to the distributional setting. We now give some examples illustrating what has been developed so far.

Example 10.2.20. (Differential operators and their distributional extensions on bundles) Let $(M, \mathbf{g})$ be a closed Riemannian manifold, let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth vector bundles, and let W be a finite-dimensional $\mathbb{K}$-vector space.
If $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$, the induced operator on distributions is defined by

$$(P\mathbf{T})(\varphi) := \mathbf{T}(P'\varphi), \qquad \mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W),\ \varphi \in \mathcal{D}(M, \mathbf{F}^*),$$

where $P' \colon \Gamma(\mathbf{F}^*) \longrightarrow \Gamma(\mathbf{E}^*)$ is the bilinear formal transpose. This definition is valid for arbitrary distributions with values in W.

To make the integral formulas concrete, we also consider the special case $W = \mathbb{K}$. If $\mathbf{u} \in \Gamma(\mathbf{E})$, denote by $\mathbf{T}_{\mathbf{u}}$ the regular distribution induced by $\mathbf{u}$, given by

$$\mathbf{T}_{\mathbf{u}}(\psi) := \int_M \psi(\mathbf{u})\, d\lambda_{\mathbf{g}}, \qquad \psi \in \mathcal{D}(M, \mathbf{E}^*).$$

(a) Let $\nabla^{\mathbf{E}} \colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(T^*M \otimes \mathbf{E})$ be a connection. The induced operator is

$$\nabla^{\mathbf{E}} \colon \mathcal{D}'(M, \mathbf{E}, W) \longrightarrow \mathcal{D}'(M, T^*M \otimes \mathbf{E}, W),$$

and is given by

$$(\nabla^{\mathbf{E}}\mathbf{T})(\varphi) = \mathbf{T}\big((\nabla^{\mathbf{E}})'\varphi\big), \qquad \varphi \in \mathcal{D}(M, TM \otimes \mathbf{E}^*).$$

In the regular case, that is, when $W = \mathbb{K}$ and $\mathbf{T} = \mathbf{T}_{\mathbf{u}}$, we have

$$(\nabla^{\mathbf{E}}\mathbf{T}_{\mathbf{u}})(\varphi) = \int_M \varphi(\nabla^{\mathbf{E}}\mathbf{u})\, d\lambda_{\mathbf{g}}.$$

If $\mathbf{E}$ is also equipped with a real bundle metric in the real case or a Hermitian metric in the complex case, the Riesz isomorphism of the bundle $\mathbf{H} := T^*M \otimes \mathbf{E}$ uniquely determines $\varphi^\sharp \in \Gamma_c(\mathbf{H})$ by

$$\overline{\varphi^\sharp} = \mathcal{R}_{\mathbf{H}}^{-1}\varphi, \qquad \varphi(w) = \mathbf{h}_{\mathbf{H}}(w, \varphi^\sharp).$$

In the complex case, $\varphi \mapsto \varphi^\sharp$ is antilinear; the map $\mathcal{R}_{\mathbf{H}}^{-1} : \mathbf{H}^* \to \overline{\mathbf{H}}$ is complex-linear. Then

$$(\nabla^{\mathbf{E}}\mathbf{T}_{\mathbf{u}})(\varphi) = \int_M \mathbf{h}_{\mathbf{H}}(\nabla^{\mathbf{E}}\mathbf{u}, \varphi^\sharp)\, d\lambda_{\mathbf{g}}.$$

By the definition of the formal Hermitian adjoint,

$$\int_M \mathbf{h}_{\mathbf{H}}(\nabla^{\mathbf{E}}\mathbf{u}, \varphi^\sharp)\, d\lambda_{\mathbf{g}} = \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u}, (\nabla^{\mathbf{E}})^*_h \varphi^\sharp)\, d\lambda_{\mathbf{g}}.$$

Thus, in the presence of bundle metrics, the same identity can be read as an integration-by-parts formula.

(b) For the covariant derivative of order k,

$$\nabla^k \colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(T^{(0,k)}(TM) \otimes \mathbf{E}),$$

the induced operator is given by

$$(\nabla^k\mathbf{T})(\varphi) = \mathbf{T}\big((\nabla^k)'\varphi\big), \qquad \varphi \in \mathcal{D}(M, T^{(k,0)}(TM) \otimes \mathbf{E}^*).$$

If $W = \mathbb{K}$ and $\mathbf{T} = \mathbf{T}_{\mathbf{u}}$, then

$$(\nabla^k\mathbf{T}_{\mathbf{u}})(\varphi) = \int_M \varphi(\nabla^k\mathbf{u})\, d\lambda_{\mathbf{g}}.$$

In the presence of a real or Hermitian bundle metric, set $\mathbf{H}_k := T^{(0,k)}(TM) \otimes \mathbf{E}$ and define $\varphi^\sharp$ by

$\overline{\varphi^\sharp} = \mathcal{R}_{\mathbf{H}_k}^{-1}\varphi$. We obtain

$$(\nabla^k \mathbf{T}_\mathbf{u})(\varphi) = \int_M \mathbf{h}_{\mathbf{H}_k}(\nabla^k \mathbf{u}, \varphi^\sharp)\, d\lambda_\mathbf{g} = \int_M \mathbf{h}_\mathbf{E}(\mathbf{u}, (\nabla^k)_h^* \varphi^\sharp)\, d\lambda_\mathbf{g}.$$

(c) Let $\operatorname{div}\colon \mathfrak{X}(M) \longrightarrow C^\infty(M)$ be the divergence operator associated with $\mathbf{g}$. Its distributional extension is given by

$$(\operatorname{div}\mathbf{T})(\varphi) = \mathbf{T}\big((\operatorname{div})'\varphi\big), \qquad \varphi \in \mathcal{D}(M).$$

In this case, $(\operatorname{div})'\varphi = -d\varphi$. Thus, if $W = \mathbb{K}$ and $\mathbf{T} = \mathbf{T}_\mathbf{X}$ for a smooth vector field $\mathbf{X} \in \mathfrak{X}(M)$, then

$$(\operatorname{div}\mathbf{T}_\mathbf{X})(\varphi) = \int_M \varphi \operatorname{div}\mathbf{X}\, d\lambda_\mathbf{g}.$$

Using the Riemannian metric,

$$\int_M \varphi \operatorname{div}\mathbf{X}\, d\lambda_\mathbf{g} = -\int_M d\varphi(\mathbf{X})\, d\lambda_\mathbf{g} = -\int_M \langle \operatorname{grad}\varphi, \mathbf{X}\rangle_\mathbf{g}\, d\lambda_\mathbf{g}.$$

Thus the bilinear formal transpose of divergence is written as $-d$, while the metric identifies it with the negative gradient.

(d) Suppose that $\mathbf{E}$ is equipped with a bundle metric $\mathbf{h}_\mathbf{E}$, real or Hermitian according to the scalar field, and that $\nabla^\mathbf{E}$ is compatible with it. On ∇^2, use the tensor connection induced by $\nabla^\mathbf{E}$ and the Levi–Civita connection of $\mathbf{g}$. Let $\Delta_B\colon \Gamma(\mathbf{E}) \longrightarrow \Gamma(\mathbf{E})$ be the Bochner Laplacian associated with $\mathbf{g}$ and $\nabla^\mathbf{E}$, defined with the positive sign convention by $\Delta_B = (\nabla^\mathbf{E})_h^* \nabla^\mathbf{E} = -\operatorname{tr}_\mathbf{g}(\nabla^2)$. Its distributional extension is given by

$$(\Delta_B \mathbf{T})(\varphi) = \mathbf{T}\big((\Delta_B)'\varphi\big), \qquad \varphi \in \mathcal{D}(M, \mathbf{E}^*).$$

If $W = \mathbb{K}$ and $\mathbf{T} = \mathbf{T}_\mathbf{u}$, then

$$(\Delta_B \mathbf{T}_\mathbf{u})(\varphi) = \int_M \varphi(\Delta_B \mathbf{u})\, d\lambda_\mathbf{g}.$$

With this bundle metric $\mathbf{h}_\mathbf{E}$, define the test section $\varphi^\sharp \in \Gamma_c(\mathbf{E})$ by $\overline{\varphi^\sharp} := \mathcal{R}_\mathbf{E}^{-1}\varphi$. Then

$$(\Delta_B \mathbf{T}_\mathbf{u})(\varphi) = \int_M \mathbf{h}_\mathbf{E}(\Delta_B \mathbf{u}, \varphi^\sharp)\, d\lambda_\mathbf{g} = \int_M \mathbf{h}_\mathbf{E}(\mathbf{u}, (\Delta_B)_h^* \varphi^\sharp)\, d\lambda_\mathbf{g}.$$

Since $\Delta_B = (\nabla^\mathbf{E})_h^* \nabla^\mathbf{E}$, it is formally self-adjoint and satisfies $(\Delta_B)_h^* = \Delta_B$.

(e) *A Lorentzian digression, outside the Riemannian convention used throughout the rest of the chapter.* If $(M, \mathbf{g})$ is a closed Lorentzian manifold, the d'Alembert operator

$$\Box\colon C^\infty(M) \longrightarrow C^\infty(M), \qquad \Box := \operatorname{tr}_\mathbf{g}(\nabla^2),$$

also induces an operator on distributions

$$\Box\colon \mathcal{D}'(M, W) \longrightarrow \mathcal{D}'(M, W),$$

defined by

$$(\Box\mathbf{T})(\varphi) = \mathbf{T}(\Box'\varphi), \qquad \varphi \in \mathcal{D}(M).$$

In the regular scalar case, that is, if $W = \mathbb{K}$ and $\mathbf{T} = \mathbf{T_u}$, we obtain

$$(\Box\mathbf{T_u})(\varphi) = \int_M \varphi\, \Box\mathbf{u}\, d\lambda_{\mathbf{g}}.$$

In this case, $\lambda_{\mathbf{g}}$ denotes the positive Borel measure associated with the Lorentzian density: in coordinates, $d\lambda_{\mathbf{g}} = |\det(\mathbf{g})|^{1/2}\, dx$. No assertion is made that $\mathbf{g}$ defines a Riemannian distance, and none of the positive estimates specific to Riemannian geometry is used here.

(f) Finally, if $P \in \mathbf{PDO}^{(0)}(\mathbf{E},\mathbf{F})$ corresponds to a smooth section of $\operatorname{Hom}(\mathbf{E},\mathbf{F})$, then

$$(P\mathbf{T})(\varphi) = \mathbf{T}(P'\varphi), \qquad \varphi \in \mathcal{D}(M,\mathbf{F}^*).$$

If $W = \mathbb{K}$ and $\mathbf{T} = \mathbf{T_u}$, then

$$(P\mathbf{T_u})(\varphi) = \int_M \varphi(P\mathbf{u})\, d\lambda_{\mathbf{g}}.$$

In the presence of bundle metrics $\mathbf{h_E}$ and $\mathbf{h_F}$, define $\varphi^\sharp \in \Gamma_c(\mathbf{F})$ by $\overline{\varphi^\sharp} := \mathcal{R}_{\mathbf{F}}^{-1}\varphi$. This can then be written as

$$(P\mathbf{T_u})(\varphi) = \int_M \mathbf{h_F}(P\mathbf{u}, \varphi^\sharp)\, d\lambda_{\mathbf{g}} = \int_M \mathbf{h_E}(\mathbf{u}, P_h^*\varphi^\sharp)\, d\lambda_{\mathbf{g}}.$$

In general, the action on distributions is determined by the bilinear formal transpose P'. For regular distributions with fixed bundle metrics, the preceding formulas can be rewritten using the formal Hermitian adjoint P_h^*, recovering the usual integration-by-parts identities.

Example 10.2.21. (An example in which the space W plays a role) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, take the trivial bundle $\mathbf{E} = M \times \mathbb{K}$, and let $W = \mathbb{K}^2$. Fix a smooth function $u \in C^\infty(M)$ and a point $p \in M$. Define

$$T\colon \mathcal{D}(M) \longrightarrow \mathbb{K}^2$$

by

$$T(\varphi) := \left(\int_M \varphi u\, d\lambda_{\mathbf{g}},\ \varphi(p) \right).$$

Then $T \in \mathcal{D}'(M,\mathbb{K}^2)$. The first component is a regular distribution, while the second is the Dirac delta at p. If $P \in \mathbf{PDO}^{(m)}(M\times\mathbb{K}, M\times\mathbb{K})$, the distributional extension of P is given by

$$(PT)(\varphi) = T(P'\varphi).$$

Therefore

$$(PT)(\varphi) = \left(\int_M (P'\varphi) u\, d\lambda_{\mathbf{g}},\ (P'\varphi)(p) \right).$$

In particular, if $P = \Delta_{\mathbf{g}}$, which is formally self-adjoint, then

$$(\Delta_{\mathbf{g}} T)(\varphi) = \left(\int_M (\Delta_{\mathbf{g}}\varphi) u\, d\lambda_{\mathbf{g}},\ (\Delta_{\mathbf{g}}\varphi)(p) \right).$$

The first component represents the action of the Laplacian on the regular part, while the second represents its action on the Dirac delta. In this sense, taking $W = \mathbb{K}^2$ combines two different types of sources in a single distributional object: an extended source given by u and a point source concentrated at p.

In the preceding examples, we mainly considered the scalar case $W = \mathbb{K}$. We now extend these constructions to distributions with values in a finite-dimensional vector space.

Definition 10.2.22. (Regular distribution with values in W) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional $\mathbb{K}$-vector space. A distribution

$$\mathbf{T} \colon \mathcal{D}(M, \mathbf{E}^*) \longrightarrow W$$

is called *regular* if there exists $\mathbf{f} \in L^1_{\mathrm{loc}}(M, \mathbf{E} \otimes W)$ such that

$$\mathbf{T}(\boldsymbol{\varphi}) = \int_M (\boldsymbol{\varphi} \otimes \mathrm{id}_W)(\mathbf{f})\, d\lambda_{\mathbf{g}}, \qquad \forall \boldsymbol{\varphi} \in \mathcal{D}(M, \mathbf{E}^*).$$

Remark 10.2.23. When $W = \mathbb{K}$, this definition agrees with Definition 10.2.10.

Definition 10.2.24. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, let W be a finite-dimensional $\mathbb{K}$-vector space, and let $\mathbf{u} \in L^1_{\mathrm{loc}}(M, \mathbf{E} \otimes W)$. Define the regular distribution associated with $\mathbf{u}$ by

$$\mathbf{T}_{\mathbf{u}}(\boldsymbol{\varphi}) := \int_M (\boldsymbol{\varphi} \otimes \mathrm{id}_W)(\mathbf{u})\, d\lambda_{\mathbf{g}}, \qquad \boldsymbol{\varphi} \in \mathcal{D}(M, \mathbf{E}^*).$$

Remark 10.2.25. When $\mathbf{u} \in \Gamma(\mathbf{E} \otimes W)$ is a smooth section, the preceding expression defines a smooth regular distribution. In particular, this construction extends the classical association of smooth sections with regular distributions.

10.2.2 Local properties

A fundamental property of distributions is their local character. In particular, a distribution can be restricted to an open subset, reflecting the fact that its action is completely determined by the behavior of test functions on small neighborhoods. This construction is compatible with the vector bundle structure and will be used repeatedly below.

Definition 10.2.26. (Restriction of distributions on bundles) Let M be a smooth manifold with or without boundary, $\mathbf{E} \longrightarrow M$ a smooth vector bundle, and $U \subseteq M$ an open subset. Let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$. Define the restriction of $\mathbf{T}$ to U as the distribution

$$\mathbf{T} \restriction_U \in \mathcal{D}'(U, \mathbf{E}, W)$$

given by

$$\mathbf{T} \restriction_U (\boldsymbol{\varphi}) := \mathbf{T}(\widetilde{\boldsymbol{\varphi}}), \qquad \boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{E}^*),$$

where $\widetilde{\boldsymbol{\varphi}} \in \mathcal{D}(M, \mathbf{E}^*)$ is the extension of $\boldsymbol{\varphi}$ by zero outside U.

Lemma 10.2.27. (Well-definedness of restriction) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle, let W be a finite-dimensional $\mathbb{K}$-vector space, and let $U \subseteq M$ be open. If $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$, the formula

$$\mathbf{T} \restriction_U (\varphi) := \mathbf{T}(\widetilde{\varphi}), \qquad \varphi \in \mathcal{D}(U, \mathbf{E}^*),$$

where $\widetilde{\varphi}$ denotes the extension of φ by zero outside U, defines an element of $\mathcal{D}'(U, \mathbf{E}, W)$. If $V \subseteq U$ is open, then

$$(\mathbf{T} \restriction_U) \restriction_V = \mathbf{T} \restriction_V .$$

Proof. Denote extension by zero by

$$e_U : \mathcal{D}(U, \mathbf{E}^* \restriction_U) \longrightarrow \mathcal{D}(M, \mathbf{E}^*)$$

. This map is linear. If $K \Subset U$, every $\varphi \in \Gamma_K(\mathbf{E}^* \restriction_U)$ vanishes on a neighborhood of the boundary of U, so $e_U \varphi$ is smooth and supported in K. For each $m \in \mathbb{N}_0$, its covariant derivatives agree with those of φ on U and vanish outside U; in particular,

$$\|e_U \varphi\|_{C^m(K;M)} = \|\varphi\|_{C^m(K;U)}.$$

Thus the restriction of e_U to every fixed-support stage is continuous. The final property of the *LF* topology, given by Theorem 4.4.1, implies that e_U is continuous on the full test spaces.
Therefore $\mathbf{T} \restriction_U = \mathbf{T} \circ e_U$ is linear and continuous, that is, it belongs to $\mathcal{D}'(U, \mathbf{E}, W)$. Finally, if $V \subseteq U$, extending by zero first from V to U and then from U to M gives exactly the extension by zero from V to M. Composing with $\mathbf{T}$ yields the restriction identity. □

Restriction makes the local nature of distributions precise: the open subsets on which a distribution vanishes determine its support.

Definition 10.2.28. (Support of a distribution) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, let W be a finite-dimensional vector space, and let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$. Define the support of $\mathbf{T}$ as the set

$$\operatorname{supp}(\mathbf{T}) := M \setminus \bigcup \{U \subseteq M \text{ open} \mid \mathbf{T} \restriction_U = 0\}.$$

The following proposition gives an equivalent definition:

Proposition 10.2.29. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \longrightarrow M$ be a smooth vector bundle, and let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$. Then

$$\operatorname{supp}(\mathbf{T}) = \{x \in M \mid \text{for every open neighborhood } U \text{ of } x, \text{ there exists } \varphi \in \mathcal{D}(M, \mathbf{E}^*) \text{ with } \operatorname{supp}(\varphi) \subseteq U \text{ and } \mathbf{T}(\varphi) \neq 0\}.$$

Proof. Define

$$A := \bigcup \{U \subseteq M \text{ open} \mid \mathbf{T} \restriction_U = 0\}.$$

Observe that A is open, so its complement $M \setminus A$ is closed. We prove that $M \setminus A$ agrees with the set described

in the second definition.
Let $x \in M \setminus A$. Consider an open neighborhood U of x. We cannot have $\mathbf{T} \restriction_U = 0$, since this would give $U \subseteq A$ and, in particular, $x \in A$, a contradiction. Thus $\mathbf{T} \restriction_U \neq 0$. This means there exists a test section $\boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{E}^*)$ such that

$$\mathbf{T} \restriction_U (\boldsymbol{\varphi}) \neq 0.$$

If $\widetilde{\boldsymbol{\varphi}} \in \mathcal{D}(M, \mathbf{E}^*)$ denotes the extension of $\boldsymbol{\varphi}$ by zero outside U, then $\operatorname{supp}(\widetilde{\boldsymbol{\varphi}}) \subseteq U$ and

$$\mathbf{T}(\widetilde{\boldsymbol{\varphi}}) = \mathbf{T} \restriction_U (\boldsymbol{\varphi}) \neq 0.$$

We conclude that every open neighborhood of x contains the support of a test section on which $\mathbf{T}$ does not vanish, so x belongs to the second set.
Conversely, suppose x belongs to the second set. If $x \in A$, there is an open subset $U \subseteq M$ such that $x \in U$ and $\mathbf{T} \restriction_U = 0$. By hypothesis, however, there exists a test section $\boldsymbol{\varphi} \in \mathcal{D}(M, \mathbf{E}^*)$ with $\operatorname{supp}(\boldsymbol{\varphi}) \subseteq U$ such that

$$\mathbf{T}(\boldsymbol{\varphi}) \neq 0.$$

Since $\operatorname{supp}(\boldsymbol{\varphi}) \subseteq U$, the restriction $\boldsymbol{\varphi} \restriction_U$ belongs to $\mathcal{D}(U, \mathbf{E}^*)$, and its extension by zero to M agrees with $\boldsymbol{\varphi}$. Therefore

$$\mathbf{T}(\boldsymbol{\varphi}) = \mathbf{T} \restriction_U (\boldsymbol{\varphi} \restriction_U) = 0,$$

contradicting the choice of $\boldsymbol{\varphi}$. Consequently, $x \notin A$.
We have proved that the two sets agree, and hence that the definitions are equivalent. □

Support makes the local character of distributions precise. In particular, a distribution does not "detect" test sections supported in regions where it vanishes. This is expressed by the following fundamental result.

Proposition 10.2.30. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional vector space. Let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$ and let $\boldsymbol{\varphi} \in \mathcal{D}(M, \mathbf{E}^*)$. If $\operatorname{supp}(\boldsymbol{\varphi}) \cap \operatorname{supp}(\mathbf{T}) = \varnothing$, then $\mathbf{T}(\boldsymbol{\varphi}) = 0$.

Proof. Let $K := \operatorname{supp}(\boldsymbol{\varphi})$, which is compact. For each $x \in K$, since $x \notin \operatorname{supp}(\mathbf{T})$, there is an open neighborhood U_x of x such that $\mathbf{T} \restriction_{U_x} = 0$. The family $\{U_x\}_{x \in K}$ is an open cover of K, so there exist open subsets $U_1, \dots, U_k$ still covering K.
Let $\{\chi_j\}_{j \in \{1,\dots,k\}}$ be a partition of unity subordinate to $\{U_j\}_{j \in \{1,\dots,k\}}$. For each $j \in \{1, \dots, k\}$, define $\boldsymbol{\varphi}_j := \chi_j \boldsymbol{\varphi}$. Then $\boldsymbol{\varphi} = \sum_{j=1}^{k} \boldsymbol{\varphi}_j$ and $\operatorname{supp}(\boldsymbol{\varphi}_j) \subseteq U_j$ for every $j \in \{1, \dots, k\}$.
Since $\mathbf{T} \restriction_{U_j} = 0$ and $\operatorname{supp}(\boldsymbol{\varphi}_j) \subseteq U_j$, it follows that $\mathbf{T}(\boldsymbol{\varphi}_j) = 0$ for every $j \in \{1, \dots, k\}$. Therefore $\mathbf{T}(\boldsymbol{\varphi}) = \sum_{j=1}^{k} \mathbf{T}(\boldsymbol{\varphi}_j) = 0$.

□

Remark 10.2.31. The condition $\operatorname{supp}(\varphi) \cap \operatorname{supp}(T) = \varnothing$ cannot be replaced by the condition that φ vanish on $\operatorname{supp}(T)$. For example, if $M = \mathbb{R}$, $\mathbf{E} = M \times \mathbb{K}$, and $T \in \mathcal{D}'(\mathbb{R}, M \times \mathbb{K}, \mathbb{K})$ is given by $T(\varphi) = \varphi'(0)$, then $\operatorname{supp}(T) = \{0\}$, but we may have $\varphi(0) = 0$ while $T(\varphi) \neq 0$.

The preceding property naturally extends the action of a distribution to certain sections without compact support.

Proposition 10.2.32. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional vector space. Let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$ and let $\boldsymbol{\varphi} \in \Gamma(M, \mathbf{E}^*)$. If $\operatorname{supp}(\mathbf{T}) \cap \operatorname{supp}(\boldsymbol{\varphi})$ is compact, the quantity $\mathbf{T}(\boldsymbol{\varphi})$ can be defined consistently.

Proof. Let $K := \operatorname{supp}(\mathbf{T}) \cap \operatorname{supp}(\boldsymbol{\varphi})$, which is compact by hypothesis. Since M is a smooth manifold, there exists a relatively compact open subset $V \subseteq M$ such that $K \subseteq V$. For each $x \in K$, there is a relatively compact chart V_x containing x; compactness of K gives a finite subcover $V_{x_1}, \ldots, V_{x_N}$, and taking $V := \bigcup_{j=1}^{N} V_{x_j}$ gives an open subset with $\overline{V}$ compact and $K \subseteq V$.
Since $K \subseteq V$ and K is compact, choose an open subset $U \subseteq M$ such that

$$K \subseteq U \subseteq \overline{U} \subseteq V.$$

Now take a function $\sigma \in \mathcal{D}(M, \mathbb{K})$ such that $\sigma \equiv 1$ on $\overline{U}$ and $\operatorname{supp}(\sigma) \subseteq V$. Define

$$\mathbf{T}(\boldsymbol{\varphi}) := \mathbf{T}(\sigma \boldsymbol{\varphi}).$$

Observe that $\sigma\boldsymbol{\varphi} \in \mathcal{D}(M, \mathbf{E}^*)$, so the preceding expression makes sense.
To see that the definition is independent of the choice of σ, let σ and σ' be two functions with the preceding property. Then $\sigma - \sigma'$ vanishes on a neighborhood of K, and in particular

$$\operatorname{supp}\big((\sigma - \sigma')\boldsymbol{\varphi}\big) \cap \operatorname{supp}(\mathbf{T}) = \varnothing.$$

By Proposition 10.2.30, it follows that

$$\mathbf{T}\big((\sigma - \sigma')\boldsymbol{\varphi}\big) = 0.$$

Therefore

$$\mathbf{T}(\sigma\boldsymbol{\varphi}) - \mathbf{T}(\sigma'\boldsymbol{\varphi}) = \mathbf{T}\big((\sigma - \sigma')\boldsymbol{\varphi}\big) = 0,$$

proving that the definition is independent of the choice of σ. □

Remark 10.2.33. The preceding proposition shows that the action of a distribution extends to smooth sections without compact support, provided the intersection with the support of the distribution is compact. This reinforces the idea that the behavior of a distribution is completely determined by its support.

The localization used later to define function spaces requires coordinate representations of distributions. The following construction also fixes the volume factor; omitting it would change the components of a regular distribution.

Proposition 10.2.34. (Local representation of a distribution) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a vector bundle of rank r, let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E})$, and let

$$\varphi \colon U \longrightarrow \Omega \subseteq \mathbb{R}^n, \qquad \mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_r)$$

be a chart and a local frame of $\mathbf{E}$, with dual frame $\mathbf{e}^* = (\mathbf{e}^1, \dots, \mathbf{e}^r)$. If

$$J_\varphi(x) := \sqrt{\det\big(\mathbf{g}(\varphi^{-1}(x))\big)}, \qquad x \in \Omega,$$

define the coordinate component of $\mathbf{T}$ by

$$\big\langle T^a_{\varphi,\mathbf{e}}, \psi \big\rangle := (\mathbf{T} \restriction_U)\left(\left(\frac{\psi}{J_\varphi} \circ \varphi\right)\mathbf{e}^a\right), \qquad \psi \in \mathcal{D}(\Omega), \tag{10.5}$$

for $a \in \{1, \dots, r\}$. Then $T^a_{\varphi,\mathbf{e}} \in \mathcal{D}'(\Omega)$. If $\mathbf{u} = \sum_{a=1}^{r} u^a \mathbf{e}_a \in L^1_{\mathrm{loc}}(U, \mathbf{E})$ and $\mathbf{T} = \mathbf{T}_\mathbf{u}$, we have

$$T^a_{\varphi,\mathbf{e}} = T_{u^a \circ \varphi^{-1}} \quad \text{on } \Omega,$$

where the right-hand side is the Euclidean distribution defined using Lebesgue measure.
In the scalar case, we omit the frame and write $(\mathbf{T} \restriction_U) \circ \varphi^{-1}$ for this distribution. If $\kappa = \varphi^{-1} \colon \Omega \longrightarrow U$ is the corresponding parametrization, we also write $(\mathbf{T} \restriction_U) \circ \kappa$; both expressions denote the same object.
If $h \in C^\infty_c(U)$, each component of $h\mathbf{T}$ has compact support contained in Ω and admits a canonical extension by zero to $\mathbb{R}^n$. Specifically, if $K := \operatorname{supp}((h\mathbf{T})^a_{\varphi,\mathbf{e}}) \Subset \Omega$ and $\chi \in C^\infty_c(\Omega)$ satisfies $\chi = 1$ on a neighborhood of K, this extension is given by

$$\big\langle \widetilde{(h\mathbf{T})^a_{\varphi,\mathbf{e}}}, \psi \big\rangle := \big\langle (h\mathbf{T})^a_{\varphi,\mathbf{e}}, \chi(\psi \restriction_\Omega) \big\rangle, \qquad \psi \in \mathcal{D}(\mathbb{R}^n), \tag{10.6}$$

and is independent of χ.

Proof. The map

$$\psi \longmapsto \left(\frac{\psi}{J_\varphi} \circ \varphi\right)\mathbf{e}^a$$

is linear and continuous from $\mathcal{D}(\Omega)$ to $\mathcal{D}(U, \mathbf{E}^* \restriction_U)$: on each compact set, the Leibniz and chain rules control all its derivatives. Continuity of $\mathbf{T} \restriction_U$ proves that (10.5) defines a Euclidean distribution. For a regular distribution, use the change-of-variables formula

$$\int_U \eta \, d\lambda_\mathbf{g} = \int_\Omega (\eta \circ \varphi^{-1})(x) J_\varphi(x) \, d\lambda_n(x),$$

valid for every integrable function η supported in U. Thus

$$\int_U \left(\frac{\psi}{J_\varphi} \circ \varphi\right) u^a \, d\lambda_\mathbf{g} = \int_\Omega \psi(x) u^a(\varphi^{-1}(x)) \, dx,$$

proving the second assertion.
The identity $\operatorname{supp}(h\mathbf{T}) \subseteq \operatorname{supp}(h) \Subset U$ follows directly from the definition of the product of a distribution with a smooth function and Proposition 10.2.30. Thus the set K in the last assertion is compact in Ω. If χ_1 and χ_2 equal one near K, then $(\chi_1 - \chi_2)(\psi \restriction_\Omega)$ has support separated from K; Proposition 10.2.30 shows that the two formulas give the same value. Continuity of (10.6) is immediate on each space of test functions supported in a fixed compact set. □

Support describes where a distribution "lives." We can refine this idea by distinguishing points where the distribution is smooth from those where it has singularities.

Definition 10.2.35. (Singular support) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, let W be a finite-dimensional vector space, and let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$. Define the singular support of $\mathbf{T}$ as the set

$$\operatorname{sing\,supp}(\mathbf{T}) := \{x \in M \mid \text{for every neighborhood } U \text{ of } x,\ \mathbf{T} \restriction_U \neq \mathbf{T}_{\mathbf{u}} \text{ for every } \mathbf{u} \in \Gamma(U, \mathbf{E} \otimes W)\}.$$

Equivalently, $x \notin \operatorname{sing\,supp}(\mathbf{T})$ if and only if there exist an open neighborhood U of x and a smooth section $\mathbf{u} \in \Gamma(U, \mathbf{E} \otimes W)$ such that $\mathbf{T} \restriction_U = \mathbf{T}_{\mathbf{u}}$.

Proposition 10.2.36. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, let W be a finite-dimensional vector space, and let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$. Then $\operatorname{sing\,supp}(\mathbf{T})$ is a closed subset of M, and

$$\operatorname{sing\,supp}(\mathbf{T}) \subseteq \operatorname{supp}(\mathbf{T}).$$

Proof. Let $x \notin \operatorname{sing\,supp}(\mathbf{T})$. Then there exist an open neighborhood U of x and a smooth section $\mathbf{u} \in \Gamma(U, \mathbf{E} \otimes W)$ such that $\mathbf{T} \restriction_U = \mathbf{T}_{\mathbf{u}}$. In particular, for every $y \in U$, $\mathbf{T} \restriction_U$ agrees with a distribution induced by a smooth section on a neighborhood of y, so $y \notin \operatorname{sing\,supp}(\mathbf{T})$. This shows that $U \subseteq M \setminus \operatorname{sing\,supp}(\mathbf{T})$, and hence $M \setminus \operatorname{sing\,supp}(\mathbf{T})$ is open. We conclude that $\operatorname{sing\,supp}(\mathbf{T})$ is closed.
For the second assertion, let $x \notin \operatorname{supp}(\mathbf{T})$. There is then an open neighborhood U of x such that $\mathbf{T} \restriction_U = 0$. Since $0 = \mathbf{T}_0$ with $0 \in \Gamma(U, \mathbf{E} \otimes W)$, $\mathbf{T} \restriction_U$ agrees with a distribution induced by a smooth section. Thus $x \notin \operatorname{sing\,supp}(\mathbf{T})$, implying $\operatorname{sing\,supp}(\mathbf{T}) \subseteq \operatorname{supp}(\mathbf{T})$. □

Example 10.2.37. Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let $p \in M$. Suppose that $\mathbf{E}_p^* \neq \{0\}$. Fix an auxiliary bundle metric $\mathbf{h}_{\mathbf{E}}$ on $\mathbf{E}$ and its dual metric $\mathbf{h}_{\mathbf{E}}^*$ on $\mathbf{E}^*$. Consider

$$\delta_p \colon \mathcal{D}(M, \mathbf{E}^*) \longrightarrow \mathbf{E}_p^*, \qquad \delta_p(\boldsymbol{\varphi}) := \boldsymbol{\varphi}(p),$$

the Dirac delta distribution at p. Recall that δ_p is a distribution with values in $\mathbf{E}_p^*$, that is,

$$\delta_p \in \mathcal{D}'(M, \mathbf{E}, \mathbf{E}_p^*).$$

We claim that

$$\operatorname{supp}(\delta_p) = \{p\}.$$

If $\dim M \geq 1$, we also have $\operatorname{sing\,supp}(\delta_p) = \{p\}$; if $\dim M = 0$, its singular support is empty.
Let $U \subseteq M$ be an open subset such that $p \notin U$. Then, for every test section $\boldsymbol{\varphi} \in \mathcal{D}(M, \mathbf{E}^*)$ with $\operatorname{supp}(\boldsymbol{\varphi}) \subseteq U$, we have $\boldsymbol{\varphi}(p) = 0$, and hence $\delta_p(\boldsymbol{\varphi}) = 0$. This shows that $\delta_p \restriction_U = 0$, giving

$$\operatorname{supp}(\delta_p) \subseteq \{p\}.$$

On the other hand, if U is any open subset containing p, there is a test section $\boldsymbol{\varphi} \in \mathcal{D}(M, \mathbf{E}^*)$ with $\operatorname{supp}(\boldsymbol{\varphi}) \subseteq U$ such that $\boldsymbol{\varphi}(p) \neq 0$. It suffices to take a nonzero element of $\mathbf{E}_p^*$ and use Lemma 1.7.28. Thus $\delta_p(\boldsymbol{\varphi}) = \boldsymbol{\varphi}(p) \neq$

0, implying $p \in \operatorname{supp}(\delta_p)$. We conclude that

$$\operatorname{supp}(\delta_p) = \{p\}.$$

If $\dim M = 0$, the manifold is discrete, and Riemannian measure is counting measure. Let $\mathbf{v}$ be the section of $\mathbf{E} \otimes \mathbf{E}_p^*$ that vanishes outside p and whose value at p is the canonical tensor $\sum_{a=1}^{r} e_a \otimes e^a$, where $\{e_a\}_{a=1}^r$ is any basis of $\mathbf{E}_p$ and $\{e^a\}_{a=1}^r$ its dual basis. This tensor is independent of the basis, and the section is smooth because every point is open. For every test section,

$$\mathbf{T}_{\mathbf{v}}(\boldsymbol{\varphi}) = \sum_{a=1}^{r} \boldsymbol{\varphi}(p)(e_a)e^a = \boldsymbol{\varphi}(p) = \delta_p(\boldsymbol{\varphi}).$$

Hence δ_p is regular and has empty singular support.
Now suppose that $d := \dim M \geq 1$. Let us show that $p \in \operatorname{sing\,supp}(\delta_p)$. Suppose, for a contradiction, that there exist an open subset U with $p \in U$ and a smooth section $\mathbf{u} \in \Gamma(U, \mathbf{E} \otimes \mathbf{E}_p^*)$ such that

$$\delta_p \restriction_U = \mathbf{T}_{\mathbf{u}}.$$

Then, for every test section $\boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{E}^*)$, we would have

$$\boldsymbol{\varphi}(p) = \delta_p(\boldsymbol{\varphi}) = \mathbf{T}_{\mathbf{u}}(\boldsymbol{\varphi}) = \int_U (\boldsymbol{\varphi} \otimes \mathrm{id}_{\mathbf{E}_p^*})(\mathbf{u})\, d\lambda_{\mathbf{g}}.$$

Consider an open subset $V \subseteq M$ such that $p \in V$, $\overline{V}$ is compact, and $\overline{V} \subseteq U$. We can choose it so that $\mathbf{E}^*$ admits a local trivialization over V and a chart (V, φ) exists. By Lemma 1.7.28, choose $\boldsymbol{\eta} \in \Gamma_c(V, \mathbf{E}^*)$ such that $\boldsymbol{\eta}(p) \neq 0$, and extend it by zero outside V.
Let $H = \mathbb{R}^d$ if the chart is interior and $H = \mathbb{H}^d$ if it is a boundary chart. Since $\varphi(V)$ is open in H, choose $r_n \downarrow 0$ so that $\overline{B_{\mathrm{euc}}(\varphi(p), r_1)} \cap H \subseteq \varphi(V)$. Restricting Euclidean cutoff functions centered at $\varphi(p)$ to H and transferring them by the chart gives $\rho_n \in C_c^\infty(V)$ with

$$0 \leq \rho_n \leq 1, \qquad \rho_n(p) = 1, \qquad \operatorname{supp}(\rho_n) \subseteq \varphi^{-1}\big(B_{\mathrm{euc}}(\varphi(p), r_n) \cap H\big).$$

The weight $\sqrt{\det(\mathbf{g})} \circ \varphi^{-1}$ is bounded on the compact coordinate set of radius r_1. Consequently, there exists $C_0 > 0$ independent of n such that

$$\lambda_{\mathbf{g}}(\operatorname{supp}\rho_n) \leq C_0 \lambda_d\big(B_{\mathrm{euc}}(\varphi(p), r_n)\big) \longrightarrow 0,$$

since $d \geq 1$. All these supports lie in $V \subseteq U$.
Now define

$$\boldsymbol{\varphi}_n := \rho_n \boldsymbol{\eta}.$$

Then $\boldsymbol{\varphi}_n \in \mathcal{D}(U, \mathbf{E}^*)$ and

$$\boldsymbol{\varphi}_n(p) = \boldsymbol{\eta}(p) \neq 0$$

for every n.
On the other hand, the section $\mathbf{s} := (\boldsymbol{\eta} \otimes \mathrm{id}_{\mathbf{E}_p^*})(\mathbf{u})$ is smooth on V and supported in the compact set $\operatorname{supp}\boldsymbol{\eta} \Subset$

V. It is therefore bounded on that compact set and zero outside it. Thus there exists a constant $C > 0$ such that $|\mathbf{s}(q)|_{\mathbf{h}^*_{\mathbf{E}}(p)} \le C$ for every $q \in V$.
Since $\boldsymbol{\varphi}_n = \rho_n \boldsymbol{\eta}$ and $\operatorname{supp}(\rho_n) \subseteq V$, we have

$$\int_U (\boldsymbol{\varphi}_n \otimes \mathrm{id}_{\mathbf{E}^*_p})(\mathbf{u})\, d\lambda_{\mathbf{g}} = \int_V \rho_n \mathbf{s}\, d\lambda_{\mathbf{g}}.$$

By the triangle inequality for integrals,

$$\left| \int_V \rho_n \mathbf{s}\, d\lambda_{\mathbf{g}} \right|_{\mathbf{h}^*_{\mathbf{E}}(p)} \le \int_V |\rho_n| |\mathbf{s}|_{\mathbf{h}^*_{\mathbf{E}}(p)}\, d\lambda_{\mathbf{g}}.$$

Since $0 \le \rho_n \le 1$ and $|\mathbf{s}|_{\mathbf{h}^*_{\mathbf{E}}(p)} \le C$ on $\operatorname{supp}(\rho_n)$, we obtain

$$\left| \int_U (\boldsymbol{\varphi}_n \otimes \mathrm{id}_{\mathbf{E}^*_p})(\mathbf{u})\, d\lambda_{\mathbf{g}} \right|_{\mathbf{h}^*_{\mathbf{E}}(p)} \le C\, \lambda_{\mathbf{g}}(\operatorname{supp}(\rho_n)).$$

Recall that $\lambda_{\mathbf{g}}(\operatorname{supp}(\rho_n)) \longrightarrow 0$; therefore

$$\int_U (\boldsymbol{\varphi}_n \otimes \mathrm{id}_{\mathbf{E}^*_p})(\mathbf{u})\, d\lambda_{\mathbf{g}} \longrightarrow 0.$$

This is a contradiction, since these integrals equal $\boldsymbol{\varphi}_n(p) = \boldsymbol{\eta}(p) \neq 0$. Thus no such smooth section $\mathbf{u}$ exists, and it follows that $p \in \operatorname{sing\,supp}(\delta_p)$.

We now examine further properties of support, including its relation to restrictions and partial differential operators.

Proposition 10.2.38. (Compatibility of restriction with support) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional vector space. Let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$ and let $U \subseteq M$ be open. Then

$$\operatorname{supp}(\mathbf{T} \restriction_U) = \operatorname{supp}(\mathbf{T}) \cap U.$$

Proof. First prove that $\operatorname{supp}(\mathbf{T} \restriction_U) \subseteq \operatorname{supp}(\mathbf{T}) \cap U$. Let $x \in U$ be such that $x \notin \operatorname{supp}(\mathbf{T})$. There is then an open neighborhood $V \subseteq M$ of x such that $\mathbf{T} \restriction_V = 0$. Since $U \cap V$ is an open neighborhood of x in U, we have

$$(\mathbf{T} \restriction_U) \restriction_{U \cap V} = \mathbf{T} \restriction_{U \cap V} = 0.$$

Therefore $x \notin \operatorname{supp}(\mathbf{T} \restriction_U)$.
Conversely, let $x \in \operatorname{supp}(\mathbf{T}) \cap U$. Consider an open neighborhood $V \subseteq U$ of x. Since U is open in M, we may also view V as an open subset of M. Since $x \in \operatorname{supp}(\mathbf{T})$, we cannot have $\mathbf{T} \restriction_V = 0$. Thus there is a test section $\boldsymbol{\varphi} \in \mathcal{D}(V, \mathbf{E}^*)$ such that $\mathbf{T} \restriction_V (\boldsymbol{\varphi}) \neq 0$. But $\mathbf{T} \restriction_V = (\mathbf{T} \restriction_U) \restriction_V$, so $(\mathbf{T} \restriction_U) \restriction_V \neq 0$. Since this holds for every open neighborhood V of x in U, it follows that $x \in \operatorname{supp}(\mathbf{T} \restriction_U)$.
We conclude that $\operatorname{supp}(\mathbf{T} \restriction_U) = \operatorname{supp}(\mathbf{T}) \cap U$. □

Proposition 10.2.39. (Compatibility of restriction with differential operators) Let M be a smooth manifold without boundary, let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles, and let W be a finite-dimensional vector space. Let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$, let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$, and let $U \subseteq M$ be open. If P_U is the restriction of P to U, then

$$(P\mathbf{T})|_U = P_U(\mathbf{T}|_U) \qquad \text{on } \mathcal{D}'(U, \mathbf{F}|_U, W). \tag{10.7}$$

Proof. Compatibility of the formal transpose with restrictions gives $(P_U)' = (P')_U$. Let $\boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{F}^*|_U)$ and denote its extension by zero by $\widetilde{\boldsymbol{\varphi}}$; it is smooth because $\operatorname{supp} \boldsymbol{\varphi} \Subset U$. Locality of P' implies that $P'\widetilde{\boldsymbol{\varphi}}$ is the extension by zero of $(P_U)'\boldsymbol{\varphi}$. Therefore

$$\begin{aligned}((P\mathbf{T})|_U)(\boldsymbol{\varphi}) &= (P\mathbf{T})(\widetilde{\boldsymbol{\varphi}}) = \mathbf{T}(P'\widetilde{\boldsymbol{\varphi}}) \\ &= (\mathbf{T}|_U)((P_U)'\boldsymbol{\varphi}) = (P_U(\mathbf{T}|_U))(\boldsymbol{\varphi}),\end{aligned}$$

which is (10.7). □

Proposition 10.2.40. (Locality of differential operators on distributions) Let M be a smooth manifold without boundary, let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles, and let W be a finite-dimensional vector space. Let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$ and let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$. Then

$$\operatorname{supp}(P\mathbf{T}) \subseteq \operatorname{supp}(\mathbf{T}).$$

Proof. Let $x \notin \operatorname{supp}(\mathbf{T})$. There is then an open neighborhood U of x such that $\mathbf{T} \restriction_U = 0$.
Let $\boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{F}^*)$ and let $\widetilde{\boldsymbol{\varphi}} \in \mathcal{D}(M, \mathbf{F}^*)$ be its extension by zero outside U. By Proposition 10.2.39, we have

$$(P\mathbf{T}) \restriction_U (\boldsymbol{\varphi}) = \mathbf{T} \restriction_U \left((P'\widetilde{\boldsymbol{\varphi}}) \restriction_U\right).$$

Since $\mathbf{T} \restriction_U = 0$, we obtain

$$(P\mathbf{T}) \restriction_U (\boldsymbol{\varphi}) = 0.$$

This holds for every $\boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{F}^*)$, so $(P\mathbf{T}) \restriction_U = 0$. Consequently, $x \notin \operatorname{supp}(P\mathbf{T})$.
Therefore

$$\operatorname{supp}(P\mathbf{T}) \subseteq \operatorname{supp}(\mathbf{T}).$$

□

Lemma 10.2.41. (Restriction of regular distributions) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional vector space. Let $\mathbf{u} \in \Gamma(\mathbf{E} \otimes W)$ and let $U \subseteq M$ be open. Then

$$(\mathbf{T}_{\mathbf{u}}) \restriction_U = \mathbf{T}_{\mathbf{u} \restriction_U}$$

as elements of $\mathcal{D}'(U, \mathbf{E} \restriction_U, W)$.

Proof. Let $\boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{E}^* \restriction_U)$. By the definition of restriction of a distribution,

$$(\mathbf{T}_{\mathbf{u}}) \restriction_U (\boldsymbol{\varphi}) = \mathbf{T}_{\mathbf{u}}(\widetilde{\boldsymbol{\varphi}}),$$

where $\widetilde{\varphi} \in \mathcal{D}(M, \mathbf{E}^*)$ denotes the extension of φ by zero outside U.
By the definition of $\mathbf{T}_{\mathbf{u}}$, we have

$$\mathbf{T}_{\mathbf{u}}(\widetilde{\varphi}) = \int_M (\widetilde{\varphi} \otimes \mathrm{id}_W)(\mathbf{u})\, d\lambda_{\mathbf{g}}.$$

Since $\widetilde{\varphi}$ vanishes outside U, this integral reduces to

$$\int_U (\varphi \otimes \mathrm{id}_W)(\mathbf{u} \restriction_U)\, d\lambda_{\mathbf{g}}.$$

But the last expression is precisely

$$\mathbf{T}_{\mathbf{u}\restriction_U}(\varphi).$$

Therefore $(\mathbf{T}_{\mathbf{u}}) \restriction_U (\varphi) = \mathbf{T}_{\mathbf{u}\restriction_U}(\varphi)$ for every $\varphi \in \mathcal{D}(U, \mathbf{E}^* \restriction_U)$, proving the equality. □

Proposition 10.2.42. (Smooth regular distributions and support) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional vector space. Let $\mathbf{u} \in \Gamma(\mathbf{E} \otimes W)$ and let $\mathbf{T}_{\mathbf{u}} \in \mathcal{D}'(M, \mathbf{E}, W)$ be the smooth regular distribution induced by $\mathbf{u}$. Then

$$\operatorname{supp}(\mathbf{T}_{\mathbf{u}}) = \operatorname{supp}(\mathbf{u}).$$

Proof. Let $x \notin \operatorname{supp}(\mathbf{u})$. There is then an open neighborhood U of x such that $\mathbf{u} \restriction_U = 0$. If $\varphi \in \mathcal{D}(U, \mathbf{E}^*)$, we have

$$(\mathbf{T}_{\mathbf{u}} \restriction_U)(\varphi) = \mathbf{T}_{\mathbf{u}\restriction_U}(\varphi) = \int_U (\varphi \otimes \mathrm{id}_W)(\mathbf{u} \restriction_U)\, d\lambda_{\mathbf{g}} = 0.$$

It follows that $\mathbf{T}_{\mathbf{u}} \restriction_U = 0$, and hence $x \notin \operatorname{supp}(\mathbf{T}_{\mathbf{u}})$. This proves that $\operatorname{supp}(\mathbf{T}_{\mathbf{u}}) \subseteq \operatorname{supp}(\mathbf{u})$.
Conversely, let $x \in \operatorname{supp}(\mathbf{u})$. We show that $x \in \operatorname{supp}(\mathbf{T}_{\mathbf{u}})$. Let U be an arbitrary open neighborhood of x. Since $x \in \operatorname{supp}(\mathbf{u})$, there exists $y \in U$ such that $\mathbf{u}(y) \neq 0$.
Since $\mathbf{u}(y) \in \mathbf{E}_y \otimes W$ is nonzero, there exist $\ell_0 \in \mathbf{E}_y^*$ and $\omega \in W^*$ such that

$$\omega\big((\ell_0 \otimes \mathrm{id}_W)(\mathbf{u}(y))\big) \neq 0.$$

Applying Lemma 1.7.28 to the bundle $\mathbf{E}^* \to M$, choose a smooth section $\boldsymbol{\ell} \in \Gamma(\mathbf{E}^*)$ with $\operatorname{supp}(\boldsymbol{\ell}) \subseteq U$ and $\boldsymbol{\ell}(y) = \ell_0$.
Multiplying $\boldsymbol{\ell}$ by a nonzero scalar if necessary, we may assume that

$$\operatorname{Re}\, \omega\big((\boldsymbol{\ell}(y) \otimes \mathrm{id}_W)(\mathbf{u}(y))\big) > 0,$$

where, if $\mathbb{K} = \mathbb{R}$, Re is understood as the identity. By continuity, there is a relatively compact open subset $V \subseteq U$, with $y \in V$, such that

$$\operatorname{Re}\, \omega\big((\boldsymbol{\ell}(z) \otimes \mathrm{id}_W)(\mathbf{u}(z))\big) > 0 \qquad \forall z \in V.$$

Now let $\rho \in \mathcal{D}(U, \mathbb{K})$ be a cutoff function such that $\rho \geq 0$, $\rho \not\equiv 0$, and $\operatorname{supp}(\rho) \subseteq V$. Define

$$\varphi := \rho\, \boldsymbol{\ell} \restriction_U \in \mathcal{D}(U, \mathbf{E}^*).$$

Then

$$\omega\big(\mathbf{T}_{\mathbf{u}} \restriction_U (\boldsymbol{\varphi})\big) = \omega\left(\int_U (\boldsymbol{\varphi} \otimes \mathrm{id}_W)(\mathbf{u})\, d\lambda_{\mathbf{g}}\right).$$

Since W is finite dimensional, we may move ω inside the integral, obtaining

$$\omega\big(\mathbf{T}_{\mathbf{u}} \restriction_U (\boldsymbol{\varphi})\big) = \int_U \omega\big((\boldsymbol{\varphi} \otimes \mathrm{id}_W)(\mathbf{u})\big)\, d\lambda_{\mathbf{g}}.$$

Using the definition of $\boldsymbol{\varphi}$ and the fact that $\operatorname{supp}(\rho) \subseteq V$, this reduces to

$$\omega\big(\mathbf{T}_{\mathbf{u}} \restriction_U (\boldsymbol{\varphi})\big) = \int_V \rho\, \omega\big((\boldsymbol{\ell} \otimes \mathrm{id}_W)(\mathbf{u})\big)\, d\lambda_{\mathbf{g}}.$$

Taking real parts,

$$\operatorname{Re}\, \omega\big(\mathbf{T}_{\mathbf{u}} \restriction_U (\boldsymbol{\varphi})\big) = \int_V \rho\, \operatorname{Re}\, \omega\big((\boldsymbol{\ell} \otimes \mathrm{id}_W)(\mathbf{u})\big)\, d\lambda_{\mathbf{g}}.$$

The integrand is nonnegative and does not vanish identically, since $\rho \geq 0$, $\rho \not\equiv 0$, and

$$\operatorname{Re}\, \omega\big((\boldsymbol{\ell}(z) \otimes \mathrm{id}_W)(\mathbf{u}(z))\big) > 0 \qquad \forall z \in V.$$

Consequently,

$$\operatorname{Re}\, \omega\big(\mathbf{T}_{\mathbf{u}} \restriction_U (\boldsymbol{\varphi})\big) > 0,$$

and in particular $\mathbf{T}_{\mathbf{u}} \restriction_U (\boldsymbol{\varphi}) \neq 0$. Therefore $\mathbf{T}_{\mathbf{u}} \restriction_U \neq 0$.
Since U was an arbitrary neighborhood of x, we conclude that $x \in \operatorname{supp}(\mathbf{T}_{\mathbf{u}})$. Consequently,

$$\operatorname{supp}(\mathbf{T}_{\mathbf{u}}) = \operatorname{supp}(\mathbf{u}).$$

□

Proposition 10.2.43. (Compatibility of restriction with singular support) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle, and let W be a finite-dimensional vector space. Let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$ and let $U \subseteq M$ be open. Then

$$\operatorname{sing\,supp}(\mathbf{T} \restriction_U) = \operatorname{sing\,supp}(\mathbf{T}) \cap U.$$

Proof. Let $x \in U$ be such that $x \notin \operatorname{sing\,supp}(\mathbf{T})$. There then exist an open neighborhood $V \subseteq M$ of x and a smooth section $\mathbf{u} \in \Gamma(V, \mathbf{E} \otimes W)$ such that

$$\mathbf{T} \restriction_V = \mathbf{T}_{\mathbf{u}}.$$

Restricting to $U \cap V$ gives

$$(\mathbf{T} \restriction_U) \restriction_{U\cap V} = \mathbf{T} \restriction_{U\cap V} = (\mathbf{T}_{\mathbf{u}}) \restriction_{U\cap V} = \mathbf{T}_{\mathbf{u}\restriction_{U\cap V}}.$$

Therefore $x \notin \operatorname{sing\,supp}(\mathbf{T} \restriction_U)$. This proves that

$$\operatorname{sing\,supp}(\mathbf{T} \restriction_U) \subseteq \operatorname{sing\,supp}(\mathbf{T}) \cap U.$$

Conversely, let $x \in U$ be such that $x \notin \operatorname{sing\,supp}(\mathbf{T} \restriction_U)$. There then exist an open neighborhood $V \subseteq U$ of x

and a smooth section $\mathbf{u} \in \Gamma(V, \mathbf{E} \otimes W)$ such that

$$(\mathbf{T} \restriction_U) \restriction_V = \mathbf{T}_{\mathbf{u}}.$$

Since V is also open in M, we have $\mathbf{T} \restriction_V = (\mathbf{T} \restriction_U) \restriction_V = \mathbf{T}_{\mathbf{u}}$. Therefore $x \notin \operatorname{sing\,supp}(\mathbf{T})$. This proves the reverse containment.
We conclude that

$$\operatorname{sing\,supp}(\mathbf{T} \restriction_U) = \operatorname{sing\,supp}(\mathbf{T}) \cap U.$$

□

Proposition 10.2.44. (Locality of differential operators and singular support) Let M be a smooth manifold without boundary, let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles, and let W be a finite-dimensional vector space. Let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$ and let $\mathbf{T} \in \mathcal{D}'(M, \mathbf{E}, W)$. Then

$$\operatorname{sing\,supp}(P\mathbf{T}) \subseteq \operatorname{sing\,supp}(\mathbf{T}).$$

Proof. Let $x \notin \operatorname{sing\,supp}(\mathbf{T})$. There then exist an open neighborhood U of x and a smooth section $\mathbf{u} \in \Gamma(U, \mathbf{E} \otimes W)$ such that

$$\mathbf{T} \restriction_U = \mathbf{T}_{\mathbf{u}}.$$

We show that $(P\mathbf{T}) \restriction_U$ is also a smooth regular distribution.
Let $\boldsymbol{\varphi} \in \mathcal{D}(U, \mathbf{F}^*)$ and let $\widetilde{\boldsymbol{\varphi}} \in \mathcal{D}(M, \mathbf{F}^*)$ be its extension by zero outside U. By Proposition 10.2.39, we have

$$(P\mathbf{T}) \restriction_U (\boldsymbol{\varphi}) = \mathbf{T} \restriction_U \big((P'\widetilde{\boldsymbol{\varphi}}) \restriction_U\big).$$

Using $\mathbf{T} \restriction_U = \mathbf{T}_{\mathbf{u}}$, we obtain

$$(P\mathbf{T}) \restriction_U (\boldsymbol{\varphi}) = \mathbf{T}_{\mathbf{u}}\big((P'\widetilde{\boldsymbol{\varphi}}) \restriction_U\big) = \int_U \big((P'\widetilde{\boldsymbol{\varphi}}) \restriction_U \otimes \mathrm{id}_W\big)(\mathbf{u})\, d\lambda_{\mathbf{g}}.$$

Now observe that $\widetilde{\boldsymbol{\varphi}}$ agrees with $\boldsymbol{\varphi}$ on U, so $\widetilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi} = 0$ on U. Since P' is a differential operator, it is local, and therefore

$$P'(\widetilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) = 0 \qquad \text{on } U.$$

Consequently,

$$(P'\widetilde{\boldsymbol{\varphi}}) \restriction_U = P'\boldsymbol{\varphi},$$

where $P'\boldsymbol{\varphi}$ is computed in U.
Substitute this into the preceding integral. To justify the resulting identity, work locally: write

$$\mathbf{u} = \sum_{a=1}^{N} \mathbf{u}_a \otimes w_a,$$

where $\{w_1, \dots, w_N\}$ is a basis of W and each $\mathbf{u}_a$ is a smooth local section of $\mathbf{E}$. Then

$$\big((P'\widetilde{\boldsymbol{\varphi}}) \restriction_U \otimes \mathrm{id}_W\big)(\mathbf{u}) = \sum_{a=1}^{N} (P'\boldsymbol{\varphi})(\mathbf{u}_a)\, w_a.$$

Applying the definition of the bilinear formal transpose (on U), we have

$$\int_U (P'\varphi)(\mathbf{u}_a)\,d\lambda_{\mathbf{g}} = \int_U \varphi(P\mathbf{u}_a)\,d\lambda_{\mathbf{g}} \qquad \text{for every } a \in \{1,\dots,N\}.$$

Therefore

$$\int_U \big((P'\widetilde{\varphi}) \restriction_U \otimes \mathrm{id}_W\big)(\mathbf{u})\,d\lambda_{\mathbf{g}} = \sum_{a=1}^{N} \left(\int_U \varphi(P\mathbf{u}_a)\,d\lambda_{\mathbf{g}}\right) w_a.$$

On the other hand, since P acts only on the component in $\mathbf{E}$, we have

$$(P\otimes \mathrm{id}_W)\mathbf{u} = \sum_{a=1}^{N} P\mathbf{u}_a \otimes w_a \in \Gamma(U, \mathbf{F}\otimes W).$$

It follows that

$$\sum_{a=1}^{N} \left(\int_U \varphi(P\mathbf{u}_a)\,d\lambda_{\mathbf{g}}\right) w_a = \int_U \big(\varphi\otimes \mathrm{id}_W\big)\big((P\otimes \mathrm{id}_W)\mathbf{u}\big)\,d\lambda_{\mathbf{g}}.$$

Consequently,

$$(P\mathbf{T}) \restriction_U (\varphi) = \int_U \big(\varphi\otimes \mathrm{id}_W\big)\big((P\otimes \mathrm{id}_W)\mathbf{u}\big)\,d\lambda_{\mathbf{g}} = \mathbf{T}_{(P\otimes \mathrm{id}_W)\mathbf{u}}(\varphi).$$

Since this holds for every $\varphi \in \mathcal{D}(U, \mathbf{F}^*)$, we conclude that

$$(P\mathbf{T}) \restriction_U = \mathbf{T}_{(P\otimes \mathrm{id}_W)\mathbf{u}}.$$

In particular, $(P\mathbf{T}) \restriction_U$ is a smooth regular distribution on a neighborhood of x, implying

$$x \notin \operatorname{sing\,supp}(P\mathbf{T}).$$

Since x was arbitrary, we obtain

$$\operatorname{sing\,supp}(P\mathbf{T}) \subseteq \operatorname{sing\,supp}(\mathbf{T}).$$

□

10.2.3 External products and parameter-dependent sections

To study operators from a more global viewpoint, it is useful to consider sections depending on two variables. In the Euclidean case, these arise when operators are written using functions of two variables, usually called kernels. For manifolds and vector bundles, the corresponding expression must be formulated intrinsically.

Remark 10.2.45. (Products with boundary and corners) If both factors have boundary, $M\times N$ is a manifold with corners. Its product charts take values in relatively open subsets of $\mathbb{R}^{m+n-r}\times[0,\infty)^r$, with $r \in \{0,1,2\}$. A section is smooth up to the faces if all ordinary derivatives of its interior components extend continuously to them. Lemma B.12.19, applied locally after multiplication by a cutoff function, shows that this condition is equivalent to admitting local smooth extensions to Euclidean space. The chain rule ensures independence of the product charts.

Bundles over the product, their compactly supported sections, and their distributions will be understood in this sense. On each compact set, covariant derivative norms are equivalent to ordinary derivative norms in

finitely many product charts: this follows by applying the triangular expressions for covariant derivatives and boundedness of their coefficients, which are smooth up to the faces. The fixed-support spaces are Fréchet. Indeed, a uniform limit of all derivatives in these charts preserves the differentiation identities by the fundamental theorem of calculus along interior segments; continuity extends these identities to the faces, and the support remains in the fixed compact set. Inclusions of supports are closed and preserve their seminorms, so an exhaustion by compact sets defines the same strict inductive limit LF as in the preceding section. Its arguments for completeness and stagewise continuity apply unchanged to these products. In particular, test functions may be nonzero on any of the faces.

If $\mathbf{E} \longrightarrow M$ and $\mathbf{F} \longrightarrow N$ are smooth vector bundles, a section depending on two variables and taking values in $\mathbf{E}_x \otimes \mathbf{F}_y^*$ for each $(x, y) \in M \times N$ must be understood as a section of a vector bundle over $M \times N$.

Pullback by the canonical projections of the product gives vector bundles over $M \times N$ from bundles defined over each factor. This allows us to introduce the external product.

Definition 10.2.46. (External product of vector bundles) Let M and N be smooth manifolds with or without boundary, and let $\mathbf{E} \longrightarrow M$ and $\mathbf{F} \longrightarrow N$ be smooth vector bundles. Denote the canonical projections by $\pi_M : M \times N \longrightarrow M$ and $\pi_N : M \times N \longrightarrow N$.
Define the *external product* of $\mathbf{E}$ and $\mathbf{F}$ as the smooth vector bundle over $M \times N$ given by

$$\mathbf{E} \boxtimes \mathbf{F} := \pi_M^* \mathbf{E} \otimes \pi_N^* \mathbf{F}.$$

In particular, for each $(x, y) \in M \times N$, its fiber is $(\mathbf{E} \boxtimes \mathbf{F})_{(x,y)} = \mathbf{E}_x \otimes \mathbf{F}_y$.

Remark 10.2.47. There is a canonical isomorphism

$$(\mathbf{E} \boxtimes \mathbf{F})^* \cong \mathbf{E}^* \boxtimes \mathbf{F}^*.$$

Fiberwise, we have $(\mathbf{E}_x \otimes \mathbf{F}_y)^* \cong \mathbf{E}_x^* \otimes \mathbf{F}_y^*$, since the vector spaces involved are finite dimensional. These fiberwise isomorphisms are compatible with local trivializations and therefore define a smooth bundle isomorphism.

The external product allows an expression of the form $\varphi(x, y) \in \mathbf{E}_x \otimes \mathbf{F}_y^*$ to be interpreted intrinsically as a section of $\mathbf{E} \boxtimes \mathbf{F}^*$ over $M \times N$. Before applying distributions in one variable, we specify an elementary isomorphism that will be used repeatedly.

Remark 10.2.48. (Convention) If V is a finite-dimensional $\mathbb{K}$-vector space and N is a smooth manifold with or without boundary, we identify V with the trivial vector bundle $N \times V \to N$. Consequently, if $\mathbf{F} \to N$ is a vector bundle, we write simply $V \otimes \mathbf{F}$ for the tensor product of the trivial bundle $N \times V$ with $\mathbf{F}$. With this convention,

$$(V \otimes \mathbf{F})_y \cong V \otimes \mathbf{F}_y, \qquad y \in N.$$

Proposition 10.2.49. Let N be a smooth manifold with or without boundary, let $\mathbf{F} \longrightarrow N$ be a smooth vector bundle, let V be a finite-dimensional $\mathbb{K}$-vector space, and let $K \subseteq N$ be compact. Then, for every

$k \in \mathbb{N}_0$, the linear map

$$\Theta_K : V \otimes_\pi \Gamma_K^k(\mathbf{F}) \longrightarrow \Gamma_K^k(V \otimes \mathbf{F}), \qquad \Theta_K(v \otimes \mathbf{s})(y) := v \otimes \mathbf{s}(y),$$

extended by linearity, is a linear isomorphism.
Moreover, if we fix a norm on V, a Riemannian metric $\mathbf{g}$ on N, and a bundle metric and connection on $\mathbf{F}$, then Θ_K is a topological isomorphism from $V \otimes_\pi \Gamma_K^k(\mathbf{F})$, equipped with the projective tensor norm induced by $\|\cdot\|_V$ and $\|\cdot\|_{C^k(K,\mathbf{F})}$, onto $\Gamma_K^k(V \otimes \mathbf{F})$ with the norm $\|\cdot\|_{C^k(K,V\otimes\mathbf{F})}$.

Proof. First prove that Θ_K is well defined. If $v \in V$ and $\mathbf{s} \in \Gamma_K^k(\mathbf{F})$, then, for each $y \in N$, $v \otimes \mathbf{s}(y) \in (V \otimes \mathbf{F})_y$, so the map $y \longmapsto v \otimes \mathbf{s}(y)$ defines a section of class C^k of the bundle $V \otimes \mathbf{F}$. Moreover, $\operatorname{supp}(\Theta_K(v \otimes \mathbf{s})) \subseteq \operatorname{supp}(\mathbf{s}) \subseteq K$. Therefore $\Theta_K(v \otimes \mathbf{s}) \in \Gamma_K^k(V \otimes \mathbf{F})$. Extending by linearity gives a linear map

$$\Theta_K : V \otimes \Gamma_K^k(\mathbf{F}) \longrightarrow \Gamma_K^k(V \otimes \mathbf{F}).$$

Let us check surjectivity. Take a basis $\{e_1, \dots, e_r\}$ of V, and let $\{\varepsilon^1, \dots, \varepsilon^r\}$ be the dual basis of V^*. If $\sigma \in \Gamma_K^k(V \otimes \mathbf{F})$, define $\sigma_i(y) := (\varepsilon^i \otimes \mathrm{id}_{\mathbf{F}_y})(\sigma(y))$ for $i \in \{1, \dots, r\}$.
Since $\varepsilon^i \otimes \mathrm{id}_{\mathbf{F}}$ defines a smooth bundle homomorphism $V \otimes \mathbf{F} \longrightarrow \mathbf{F}$, it follows that σ_i is of class C^k. Moreover, if $y \notin K$, then $\sigma(y) = 0$ and hence $\sigma_i(y) = 0$. Thus $\sigma_i \in \Gamma_K^k(\mathbf{F})$.
For each $y \in N$, using the basis $\{e_i\}$ gives $\sigma(y) = \sum_{i=1}^r e_i \otimes \sigma_i(y)$. Therefore $\sigma = \Theta_K\left(\sum_{i=1}^r e_i \otimes \sigma_i\right)$. This proves that Θ_K is surjective.
Now prove injectivity. Suppose that

$$\Theta_K\left(\sum_{i=1}^r e_i \otimes \mathbf{s}_i\right) = 0, \qquad \mathbf{s}_i \in \Gamma_K^k(\mathbf{F}).$$

Then, for every $y \in N$, $\sum_{i=1}^r e_i \otimes \mathbf{s}_i(y) = 0$ in $V \otimes \mathbf{F}_y$. Composing with $\varepsilon^j \otimes \mathrm{id}_{\mathbf{F}_y}$ gives $\mathbf{s}_j(y) = 0$ for every $y \in N$.
Since this holds for every $j \in \{1, \dots, r\}$, it follows that $\mathbf{s}_j = 0$ for every j. Therefore $\sum_{i=1}^r e_i \otimes \mathbf{s}_i = 0$ in $V \otimes \Gamma_K^k(\mathbf{F})$, and Θ_K is injective.
Finally, prove the topological assertion. The connection on $\mathbf{F}$ and the trivial connection on $N \times V$ induce a connection on $V \otimes \mathbf{F}$. If $\sigma = \sum_{i=1}^r e_i \otimes \mathbf{s}_i$, then, for $0 \le j \le k$, $\nabla^j \sigma = \sum_{i=1}^r e_i \otimes \nabla^j \mathbf{s}_i$. Since V is finite dimensional, there exist constants $A, B > 0$, depending on the basis and fixed metrics but independent of the sections, such that for every $y \in K$ and $0 \le j \le k$,

$$A \max_{1\le i\le r} |\nabla^j \mathbf{s}_i(y)|_{\mathbf{g},\mathbf{h}_\mathbf{F}} \le \left|\sum_{i=1}^r e_i \otimes \nabla^j \mathbf{s}_i(y)\right|_{\mathbf{g},\mathbf{h}_{V\otimes\mathbf{F}}} \le B \max_{1\le i\le r} |\nabla^j \mathbf{s}_i(y)|_{\mathbf{g},\mathbf{h}_\mathbf{F}}.$$

Taking the supremum over $y \in K$ and the maximum over $0 \le j \le k$ gives constants $C_1, C_2 > 0$ such that

$$C_1 \max_{1\le i\le r} \|\mathbf{s}_i\|_{C^k(K,\mathbf{F})} \le \left\| \Theta_K \left(\sum_{i=1}^{r} e_i \otimes \mathbf{s}_i \right) \right\|_{C^k(K, V\otimes\mathbf{F})} \le C_2 \max_{1\le i\le r} \|\mathbf{s}_i\|_{C^k(K,\mathbf{F})}.$$

For the fixed basis (e_i), the quantity $\max_{1\le i\le r} \|\mathbf{s}_i\|_{C^k(K,\mathbf{F})}$ is equivalent to the projective norm of $\sum_{i=1}^{r} e_i \otimes \mathbf{s}_i$. These inequalities show that Θ_K and Θ_K^{-1} are continuous. □

Corollary 10.2.50. Let N be a smooth manifold with or without boundary, let $\mathbf{F} \longrightarrow N$ be a smooth vector bundle, and let V be a finite-dimensional $\mathbb{K}$-vector space. Then, for every $k \in \mathbb{N}_0$, the linear map

$$\Theta : V \otimes \mathcal{D}^k(N,\mathbf{F}) \longrightarrow \mathcal{D}^k(N, V\otimes\mathbf{F}), \qquad \Theta(v\otimes\mathbf{s})(y) := v \otimes \mathbf{s}(y),$$

extended by linearity, is a continuous linear isomorphism with continuous inverse.

Proof. Recall that $\mathcal{D}^k(N,\mathbf{F}) = \Gamma_c^k(\mathbf{F})$ and that this space is equipped with the inductive topology

$$\mathcal{D}^k(N,\mathbf{F}) = \varinjlim_{\ell\in\mathbb{N}} \Gamma^k_{K_\ell}(\mathbf{F}),$$

where $(K_\ell)_{\ell\in\mathbb{N}}$ is an increasing exhaustion of N by compact sets. Likewise,

$$\mathcal{D}^k(N, V\otimes\mathbf{F}) = \varinjlim_{\ell\in\mathbb{N}} \Gamma^k_{K_\ell}(V\otimes\mathbf{F}).$$

For each ℓ, Proposition 10.2.49 provides a continuous linear isomorphism with continuous inverse

$$\Theta_{K_\ell} : V \otimes \Gamma^k_{K_\ell}(\mathbf{F}) \longrightarrow \Gamma^k_{K_\ell}(V\otimes\mathbf{F}).$$

Moreover, if $K_\ell \subseteq K_{\ell'}$, the corresponding maps are compatible with the natural inclusions, since all are defined by the same formula $v\otimes\mathbf{s} \mapsto (y \mapsto v\otimes\mathbf{s}(y))$.
Thus the isomorphisms Θ_{K_ℓ} induce a linear map

$$\Theta : V \otimes \mathcal{D}^k(N,\mathbf{F}) \longrightarrow \mathcal{D}^k(N, V\otimes\mathbf{F}).$$

Let us prove that this map is bijective. If $\mathbf{s} \in \mathcal{D}^k(N,\mathbf{F})$, there exists ℓ such that $\mathbf{s} \in \Gamma^k_{K_\ell}(\mathbf{F})$; by the preceding proposition, $\Theta(v\otimes\mathbf{s}) \in \Gamma^k_{K_\ell}(V\otimes\mathbf{F}) \subseteq \mathcal{D}^k(N, V\otimes\mathbf{F})$. This shows that Θ is well defined.
Now let $\sigma \in \mathcal{D}^k(N, V\otimes\mathbf{F})$. There exists ℓ such that $\sigma \in \Gamma^k_{K_\ell}(V\otimes\mathbf{F})$. By Proposition 10.2.49, there exist $\mathbf{s}_1, \dots, \mathbf{s}_r \in \Gamma^k_{K_\ell}(\mathbf{F})$ such that

$$\sigma = \sum_{i=1}^{r} e_i \otimes \mathbf{s}_i.$$

Since $\mathbf{s}_i \in \mathcal{D}^k(N,\mathbf{F})$, we obtain

$$\sigma = \Theta\left(\sum_{i=1}^{r} e_i \otimes \mathbf{s}_i \right).$$

This proves surjectivity. Injectivity follows in the same way by choosing a compact set containing the supports of all sections involved and applying injectivity of Θ_K on that compact set.
Finally, continuity of Θ and its inverse follows from the definition of the inductive topology, since it suffices to check continuity on each space $\Gamma^k_{K_\ell}$, where it was already proved in Proposition 10.2.49. This proves that Θ is a topological isomorphism. □

We apply this corollary with $V = \mathbf{E}_x$ and the bundle $\mathbf{F}^*$ over N. Thus, for each $x \in M$, there is a canonical isomorphism

$$\mathbf{E}_x \otimes \mathcal{D}^k(N, \mathbf{F}^*) \cong \mathcal{D}^k(N, \mathbf{E}_x \otimes \mathbf{F}^*).$$

This justifies the action of a distribution in the variable y on sections of $\mathbf{E}_x \otimes \mathbf{F}^*$.

Lemma 10.2.51. Let $(M, \mathbf{g})$ and $(N, \widetilde{\mathbf{g}})$ be Riemannian manifolds with or without boundary, equipped with their respective Riemann–Lebesgue measures $\lambda_{\mathbf{g}}$ and $\lambda_{\widetilde{\mathbf{g}}}$. Let $\mathbf{E} \longrightarrow M$ and $\mathbf{F} \longrightarrow N$ be smooth vector bundles, let $K \subseteq N$ be compact, and let $\varphi \in \Gamma^k(M \times N, \mathbf{E} \boxtimes \mathbf{F}^*)$ be a section such that $\operatorname{supp}(\varphi) \subseteq M \times K$.
Let $m < k$ and let $\mathbf{T} \in \mathcal{D}'(N, \mathbf{F}, \mathbb{K})$ be a distribution of order m. Then the map $\mathbf{f}\colon M \longrightarrow \mathbf{E}$ given by $\mathbf{f}(x) := \mathbf{T}(\varphi(x, \cdot))$ defines a section in $\Gamma^{k-m}(M, \mathbf{E})$ supported in the projection of $\operatorname{supp}(\varphi)$ onto the first factor, that is,

$$\operatorname{supp}(\mathbf{f}) \subseteq \{x \in M \mid \exists y \in K \text{ such that } (x, y) \in \operatorname{supp}(\varphi)\}.$$

Moreover, if P is a linear differential operator of order $\leq k - m$ acting on sections of $\mathbf{E}$, then

$$P\mathbf{f} = \mathbf{T}(P_x \varphi(x, \cdot)),$$

where P_x indicates that P acts in the variable x.
In particular, if φ is smooth and compactly supported and $\mathbf{T} \in \mathcal{D}'(N, \mathbf{F}, \mathbb{K})$ is an arbitrary distribution, then $\mathbf{f}$ is a compactly supported smooth section of $\mathbf{E}$.

Proof. First explain the meaning of $\mathbf{T}(\varphi(x, \cdot))$. Fix $x \in M$. Since $\varphi(x, y) \in \mathbf{E}_x \otimes \mathbf{F}^*_y$, the section $\varphi(x, \cdot)$ belongs to $\mathcal{D}^k(N, \mathbf{E}_x \otimes \mathbf{F}^*)$. Moreover, since $\operatorname{supp}(\varphi) \subseteq M \times K$, we have $\operatorname{supp}(\varphi(x, \cdot)) \subseteq K$.
By Corollary 10.2.50, applied to $V = \mathbf{E}_x$ and the bundle $\mathbf{F}^*$, there is a canonical isomorphism

$$\mathbf{E}_x \otimes \mathcal{D}^k(N, \mathbf{F}^*) \cong \mathcal{D}^k(N, \mathbf{E}_x \otimes \mathbf{F}^*).$$

Thus we may regard $\varphi(x, \cdot)$ as an element of $\mathbf{E}_x \otimes \mathcal{D}^k(N, \mathbf{F}^*)$. Writing

$$\varphi(x, \cdot) = \sum_{i=1}^{r} v_i \otimes \varphi_i(x, \cdot),$$

with $v_i \in \mathbf{E}_x$ and $\varphi_i(x, \cdot) \in \mathcal{D}^k(N, \mathbf{F}^*)$, define

$$\mathbf{T}(\varphi(x, \cdot)) := (\mathrm{id}_{\mathbf{E}_x} \otimes \mathbf{T})(\varphi(x, \cdot)) = \sum_{i=1}^{r} v_i \, \mathbf{T}(\varphi_i(x, \cdot)).$$

This definition is independent of the chosen decomposition because it comes from the linear map $\mathrm{id}_{\mathbf{E}_x} \otimes \mathbf{T}$.
For simplicity, we write $\mathbf{T}(\varphi(x, \cdot))$ instead of $(\mathrm{id}_{\mathbf{E}_x} \otimes \mathbf{T})(\varphi(x, \cdot))$.

Now prove that $\mathbf{f}$ is of class C^{k-m}. Since the issue is local in M, take a coordinate chart and a local trivialization of $\mathbf{E}$. Identifying the chart with an open subset $U \subseteq \mathbb{R}^n$, we may assume locally that $\mathbf{E} \restriction_U \cong U \times \mathbb{K}^r$. Independence of the C^m norms from the metric and connection on compact sets allows us to work locally with the trivial connection and Euclidean norm in the factor $\mathbb{K}^r$. Also fix an auxiliary bundle metric $\mathbf{h}_{\mathbf{F}}$ and a connection on $\mathbf{F}$; on $\mathbb{K}^r \otimes \mathbf{F}^*$, use the product metric h_{prod} induced by the Euclidean norm and $\mathbf{h}_{\mathbf{F}}^*$. Varying the chart and trivialization gives the global assertion.

Fix $x_0 \in U$ and take a compact convex neighborhood $U' \subseteq U$ of x_0. Since $\boldsymbol{\varphi}$ is of class C^k and has support in the variable y contained in K, for every multi-index α with $|\alpha| \le k-m$ we have

$$D_x^\alpha \boldsymbol{\varphi}(x,\cdot) \in \mathcal{D}^m(N, \mathbb{K}^r \otimes \mathbf{F}^*)$$

for every $x \in U'$.

Use Taylor's formula with integral remainder in the variable x, centered at x_0, through order $k-m$. For $x \in U'$ and $y \in N$, we have

$$\boldsymbol{\varphi}(x,y) = \sum_{|\alpha| \le k-m} \frac{1}{\alpha!} D_x^\alpha \boldsymbol{\varphi}(x_0,y)(x-x_0)^\alpha$$

$$+ \sum_{|\alpha| = k-m} \frac{k-m}{\alpha!} \int_0^1 (1-t)^{k-m-1} \left[D_x^\alpha \boldsymbol{\varphi}((1-t)x_0 + tx, y) - D_x^\alpha \boldsymbol{\varphi}(x_0,y) \right] dt\, (x-x_0)^\alpha.$$

For $|\alpha| = k-m$, define

$$R_\alpha(x,\cdot) := \int_0^1 (1-t)^{k-m-1} \left[D_x^\alpha \boldsymbol{\varphi}((1-t)x_0 + tx, \cdot) - D_x^\alpha \boldsymbol{\varphi}(x_0,\cdot) \right] dt.$$

Then

$$\boldsymbol{\varphi}(x,\cdot) = \sum_{|\alpha| \le k-m} \frac{1}{\alpha!} D_x^\alpha \boldsymbol{\varphi}(x_0,\cdot)(x-x_0)^\alpha + \sum_{|\alpha| = k-m} \frac{k-m}{\alpha!} R_\alpha(x,\cdot)(x-x_0)^\alpha.$$

Since $D_x^\alpha \boldsymbol{\varphi}(x_0,\cdot) \in \mathcal{D}^m(N, \mathbb{K}^r \otimes \mathbf{F}^*)$ for $|\alpha| \le k-m$, and since $\mathbf{T}$ has order m on K, we may apply $\mathrm{id}_{\mathbb{K}^r} \otimes \mathbf{T}$ to each term. We obtain

$$\mathbf{T}(\boldsymbol{\varphi}(x,\cdot)) = \sum_{|\alpha| \le k-m} \frac{1}{\alpha!} \mathbf{T}\big(D_x^\alpha \boldsymbol{\varphi}(x_0,\cdot)\big)(x-x_0)^\alpha + \sum_{|\alpha| = k-m} \frac{k-m}{\alpha!} \mathbf{T}(R_\alpha(x,\cdot))(x-x_0)^\alpha.$$

Let us prove that $\mathbf{T}(R_\alpha(x,\cdot)) \to 0$ as $x \to x_0$. Here ∇_y^j denotes the iterated connection acting only on the factor N of the bundle $\mathbb{K}^r \otimes \mathbf{F}^*$. Since $D_x^\alpha \boldsymbol{\varphi}$ is of class C^m in the variable y and continuous in x, the sections $\nabla_y^j D_x^\alpha \boldsymbol{\varphi}(x,y)$, with $0 \le j \le m$, are continuous on $U' \times K$. Since $U' \times K$ is compact, they are uniformly continuous. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|x - x_0\|_{\mathbb{R}^n} < \delta$, then for every $t \in [0,1]$, every $y \in K$, and every $0 \le j \le m$,

$$\left| \nabla_y^j D_x^\alpha \boldsymbol{\varphi}((1-t)x_0 + tx, y) - \nabla_y^j D_x^\alpha \boldsymbol{\varphi}(x_0,y) \right|_{\widetilde{g}, h_{\mathrm{prod}}} < \varepsilon.$$

It follows that

$$\|R_\alpha(x,\cdot)\|_{C^m(K, \mathbb{K}^r \otimes \mathbf{F}^*)} \le \varepsilon \int_0^1 (1-t)^{k-m-1}\, dt = \frac{\varepsilon}{k-m}.$$

Consequently, $\|R_\alpha(x,\cdot)\|_{C^m(K, \mathbb{K}^r \otimes \mathbf{F}^*)} \to 0$ as $x \to x_0$.

Since $\mathbf{T}$ has order m, there exists a constant $C_K > 0$ such that

$$|\mathbf{T}(\psi)| \le C_K \|\psi\|_{C^m(K,\mathbf{F}^*)}$$

for every test section ψ of $\mathbf{F}^*$ supported in K. Using this estimate componentwise in $\mathbb{K}^r \otimes \mathbf{F}^*$ and increasing C_K if necessary gives

$$|\mathbf{T}(R_\alpha(x,\cdot))|_{\mathbb{K}^r} \le C_K \|R_\alpha(x,\cdot)\|_{C^m(K,\mathbb{K}^r\otimes\mathbf{F}^*)} \longrightarrow 0.$$

Thus the preceding expression for $\mathbf{T}(\varphi(x,\cdot))$ shows that $\mathbf{f}$ admits a Taylor expansion at x_0 of order $k-m$ with coefficients $\mathbf{T}(D_x^\alpha \varphi(x_0,\cdot))$. Therefore

$$D_x^\alpha \mathbf{f}(x_0) = \mathbf{T}(D_x^\alpha \varphi(x_0,\cdot)), \qquad |\alpha| \le k-m.$$

It remains to check continuity of these derivatives. Let $|\alpha| \le k-m$. For $x, x_0 \in U'$, using again that $\mathbf{T}$ has order m on K, we have

$$\begin{aligned} |D_x^\alpha \mathbf{f}(x) - D_x^\alpha \mathbf{f}(x_0)|_{\mathbb{K}^r} &= \left|\mathbf{T}(D_x^\alpha \varphi(x,\cdot) - D_x^\alpha \varphi(x_0,\cdot))\right|_{\mathbb{K}^r} \\ &\le C_K \|D_x^\alpha \varphi(x,\cdot) - D_x^\alpha \varphi(x_0,\cdot)\|_{C^m(K,\mathbb{K}^r\otimes\mathbf{F}^*)}. \end{aligned}$$

The last term tends to 0 as $x \to x_0$, by uniform continuity of the derivatives on $U' \times K$. Thus $D_x^\alpha \mathbf{f}$ is continuous. Since x_0 was arbitrary, we conclude that $\mathbf{f} \in \Gamma^{k-m}(M,\mathbf{E})$.

Now prove the support assertion. Since $\operatorname{supp}(\varphi)$ is closed in $M\times N$ and contained in $M\times K$, with K compact, its projection onto M is closed. If x does not belong to this projection, there is a neighborhood U_x of x such that $(U_x \times K) \cap \operatorname{supp}(\varphi) = \varnothing$. Therefore $\varphi(x,\cdot) = 0$ and consequently $\mathbf{f}(x) = 0$. This proves the support inclusion.

Now let P be a linear differential operator of order $\le k-m$ acting on sections of $\mathbf{E}$. In local coordinates, P is a finite combination of derivatives in the variable x of order at most $k-m$, with smooth coefficients and bundle homomorphisms. Since we have already proved that

$$D_x^\alpha \mathbf{f}(x) = \mathbf{T}(D_x^\alpha \varphi(x,\cdot)) \qquad |\alpha| \le k-m,$$

combining these identities linearly with the coefficients of P gives

$$(P\mathbf{f})(x) = \mathbf{T}(P_x \varphi(x,\cdot)).$$

By locality, this identity is independent of the chart and trivialization.

Finally, suppose φ is smooth and compactly supported and $\mathbf{T} \in \mathcal{D}'(N,\mathbf{F},\mathbb{K})$ is an arbitrary distribution. Since $\operatorname{supp}(\varphi)$ is compact in $M\times N$, there exists a compact set $K \subseteq N$ such that $\operatorname{supp}(\varphi) \subseteq M \times K$. Moreover, the projection of $\operatorname{supp}(\varphi)$ onto M is compact. By the definition of a distribution, for this compact set K there is an integer m_K such that $\mathbf{T}$ has order $\le m_K$ on K. Since φ is smooth, the result already proved applies with any $k > m_K$. Thus $\mathbf{f}$ is of class C^ℓ for every $\ell \in \mathbb{N}_0$, and is therefore smooth. The compact support assertion follows from the support inclusion already proved and compactness of the projection of a compact set under $M \times N \to M$. □

The preceding result allows a distribution to be evaluated in one variable of a section defined on a product. The resulting section in the remaining variable loses at most as many derivatives as the order of the distribution.

10.2.4 Tensor products of test spaces

The topological description of test sections on $M \times N$ is obtained first with fixed supports, then by passage to the inductive limit. The first stage involves projective products of nuclear Fréchet spaces. The second involves the inductive product, which linearizes separately continuous bilinear maps on LF spaces. Definitions and abstract results concerning these topologies are given in Section B.12.

Lemma 10.2.52. (Localization in models with faces) Let $U \subseteq G_{m,r} := \mathbb{R}^{m-r} \times [0,\infty)^r$ and $V \subseteq G_{n,s}$ be relatively open subsets. For every compact set $C \subseteq U$, the space $\mathcal{D}_C(U)$ of functions smooth up to the faces and supported in C is nuclear Fréchet. If

$$C_0 \subseteq \mathrm{Int}_{G_{m,r}}(C_1) \subseteq U, \qquad D_0 \subseteq \mathrm{Int}_{G_{n,s}}(D_1) \subseteq V$$

are compact, there exists a continuous linear operator

$$\mathscr{R} : \mathcal{D}_{C_0 \times D_0}(U \times V) \longrightarrow \mathcal{D}_{C_1}(U) \widehat{\otimes}_\pi \mathcal{D}_{D_1}(V)$$

whose composition with multiplication of functions is the identity.

Proof. For $f \in \mathcal{D}_C(U)$, its extension by zero to the complement of U in $G_{m,r}$ is smooth: the support C lies in the relatively open subset U. This operation crosses only the boundary of the open subset within the model. Next apply the operator $E_{m,r}$ from Lemma B.12.19. We obtain a continuous map

$$J_C : \mathcal{D}_C(U) \longrightarrow \mathcal{D}_{\widehat{C}}(\mathbb{R}^m), \qquad J_C f = E_{m,r}\widetilde{f},$$

for a fixed compact set $\widehat{C}$. Restriction to U is the inverse of J_C on its image and is continuous for every derivative seminorm. This image is closed. Indeed, if $J_C f_j \to h$ in $\mathcal{D}_{\widehat{C}}(\mathbb{R}^m)$, the restrictions converge with all derivatives to $f = h|_U$, which is supported in C; continuity of extension gives $J_C f = h$. Thus $\mathcal{D}_C(U)$ is isomorphic to a closed subspace of a nuclear Fréchet space, and the first assertion follows from Corollary B.12.20 and Proposition B.12.15.

To construct $\mathscr{R}$, first extend $\Phi \in \mathcal{D}_{C_0 \times D_0}(U \times V)$ by zero to $G_{m,r} \times G_{n,s}$, then extend in each group of variables:

$$\widehat{\Phi} := E^x_{m,r} E^y_{n,s} \widetilde{\Phi}.$$

The extension depends continuously on Φ in every seminorm and is supported in $A_0 \times B_0$, for fixed compact sets $A_0 \subseteq \mathbb{R}^m$ and $B_0 \subseteq \mathbb{R}^n$. This follows by applying support control in each group while treating the other as a parameter. Choose compact sets A_1, B_1 containing A_0, B_0 in their respective interiors. Lemma B.12.22 provides a continuous operator

$$R_{\mathrm{eucl}} : \mathcal{D}_{A_0 \times B_0}(\mathbb{R}^{m+n}) \longrightarrow \mathcal{D}_{A_1}(\mathbb{R}^m) \widehat{\otimes}_\pi \mathcal{D}_{B_1}(\mathbb{R}^n)$$

recovering each function upon application of the external product. Its construction by Fourier series on tori applies to $\widehat{\Phi}$, which is smooth across all faces.

Take $\chi \in \mathcal{D}_{C_1}(U)$ and $\eta \in \mathcal{D}_{D_1}(V)$ equal to one near C_0 and D_0. These functions are obtained by restricting Euclidean cutoffs: compactness allows each smaller compact set to be covered by finitely many balls whose

intersections with the model lie in the relative interior of the larger compact set. The maps

$$P(f) := \chi(f|_U), \qquad Q(g) := \eta(g|_V)$$

are continuous into $\mathcal{D}_{C_1}(U)$ and $\mathcal{D}_{D_1}(V)$ by the Leibniz rule. Define

$$\mathscr{R}\Phi := (P\widehat{\otimes}Q)R_{\mathrm{eucl}}\widehat{\Phi}.$$

All operations are continuous and, by the restriction identity for the extensions, the product of their factors is

$$\chi(x)\eta(y)\widehat{\Phi}|_{U\times V} = \chi(x)\eta(y)\Phi = \Phi.$$

This proves the second assertion. □

Proposition 10.2.53. (Nuclearity of sections with fixed support) Let M be a smooth manifold with or without boundary, let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank, and let $K \subseteq M$ be compact. Then $\Gamma_K(\mathbf{E})$ is a nuclear Fréchet space.

Proof. By Proposition 10.1.7, only nuclearity remains to be proved. Choose coordinate and trivializing open subsets $U_1,\dots,U_s$ covering K, with U_j relatively compact, and functions $\chi_j \in C_c^\infty(U_j)$ such that $\sum_{j=1}^{s}\chi_j = 1$ on a neighborhood of K. Fix a chart $\kappa_j\colon U_j \longrightarrow \kappa_j(U_j)$ whose image is a relatively open subset of $\mathbb{R}^{\dim M}$ or of the closed half-space, and a local frame $(\mathbf{e}_{j,1},\dots,\mathbf{e}_{j,r})$ of $\mathbf{E}$ on U_j, where $r = \operatorname{rank}(\mathbf{E})$.
For $\mathbf{u} \in \Gamma_K(\mathbf{E})$, write

$$(\chi_j\mathbf{u})|_{U_j} = \sum_{a=1}^{r} u_j^a \mathbf{e}_{j,a}$$

and set $K_j := \kappa_j(K \cap \operatorname{supp}\chi_j)$. The map

$$I\colon \Gamma_K(\mathbf{E}) \longrightarrow \prod_{j=1}^{s} \mathcal{D}_{K_j}(\kappa_j(U_j))^r, \qquad I(\mathbf{u}) := \left(u_j^a \circ \kappa_j^{-1}\right)_{j,a},$$

is linear and continuous. Define a map in the opposite direction by

$$S\left((v_j^a)_{j,a}\right) := \sum_{j=1}^{s}\left(\sum_{a=1}^{r}\widetilde{(v_j^a \circ \kappa_j)}\mathbf{e}_{j,a}\right)^M.$$

The tilde with superscript M denotes extension by zero from U_j to M. Each section in the sum is supported in the compact set $\kappa_j^{-1}(K_j) \Subset U_j$, so this extension is smooth. Moreover, $\kappa_j^{-1}(K_j) \subseteq K$, so S takes values in $\Gamma_K(\mathbf{E})$. Local derivative expressions show that S is continuous: each C^k seminorm of a summand is bounded by a finite sum of C^k seminorms of its component functions.
Since $\sum_{j=1}^{s}\chi_j = 1$ on a neighborhood of K and $\mathbf{u}$ vanishes outside K, we have $S(I(\mathbf{u})) = \mathbf{u}$ for every $\mathbf{u} \in \Gamma_K(\mathbf{E})$.
Consequently, I is a topological embedding, and $I \circ S$ is a continuous projection onto its image. This image is therefore a complemented subspace of the finite product.

Each space $\mathcal{D}_{K_j}(\kappa_j(U_j))$ is nuclear by Lemma 10.2.52. Proposition B.12.15 shows that the finite product and its complemented subspace $I(\Gamma_K(\mathbf{E}))$ are nuclear. Since I is a topological isomorphism onto its image, $\Gamma_K(\mathbf{E})$ is nuclear. The same argument with quadrant charts proves the assertion for the products with corners in Remark 10.2.45. □

Theorem 10.2.54. (Localized tensor product representation of sections) Let M and N be smooth manifolds with or without boundary, and let $\mathbf{E} \to M$ and $\mathbf{F} \to N$ be smooth vector bundles of finite rank. Let the following sets be compact:

$$K_0 \subseteq \operatorname{Int}(K_1) \subseteq M, \qquad L_0 \subseteq \operatorname{Int}(L_1) \subseteq N.$$

The external product induces a topological embedding

$$T_{K_1,L_1} : \Gamma_{K_1}(\mathbf{E}) \widehat{\otimes}_\pi \Gamma_{L_1}(\mathbf{F}) \longrightarrow \Gamma_{K_1\times L_1}(\mathbf{E} \boxtimes \mathbf{F}).$$

Its image contains $\Gamma_{K_0\times L_0}(\mathbf{E} \boxtimes \mathbf{F})$, and the restriction of $T^{-1}_{K_1,L_1}$ to this subspace is continuous.

Proof. Local derivative estimates for $u \boxtimes v$ show that the bilinear map

$$\Gamma_{K_1}(\mathbf{E}) \times \Gamma_{L_1}(\mathbf{F}) \longrightarrow \Gamma_{K_1\times L_1}(\mathbf{E} \boxtimes \mathbf{F})$$

is continuous. The universal property of the projective product and completeness of the target give the continuous operator T_{K_1,L_1}.

We first construct a continuous preimage for sections supported in $K_0 \times L_0$. Choose coordinate and trivializing open subsets

$$U_1, \dots, U_A \Subset \operatorname{Int}(K_1), \qquad V_1, \dots, V_B \Subset \operatorname{Int}(L_1),$$

covering K_0 and L_0. Let $(\mathbf{e}_{i,1}, \dots, \mathbf{e}_{i,r})$ and $(\mathbf{f}_{j,1}, \dots, \mathbf{f}_{j,s})$ be the local frames. Take cutoff functions $\chi_i \in C^\infty_c(U_i)$ and $\eta_j \in C^\infty_c(V_j)$ such that

$$\sum_{i=1}^{A} \chi_i = 1 \quad \text{on a neighborhood of } K_0, \qquad \sum_{j=1}^{B} \eta_j = 1 \quad \text{on a neighborhood of } L_0.$$

Denote the charts by $\kappa_i \colon U_i \longrightarrow \kappa_i(U_i)$ and $\lambda_j \colon V_j \longrightarrow \lambda_j(V_j)$. Choose compact sets $\widetilde{C}_i \Subset \kappa_i(U_i)$ and $\widetilde{D}_j \Subset \lambda_j(V_j)$ whose interiors contain, respectively,

$$C_i := \kappa_i(K_0 \cap \operatorname{supp}(\chi_i)), \qquad D_j := \lambda_j(L_0 \cap \operatorname{supp}(\eta_j)).$$

Let $\mathbf{\Phi} \in \Gamma_{K_0\times L_0}(\mathbf{E} \boxtimes \mathbf{F})$. On $U_i \times V_j$, write

$$(\chi_i \boxtimes \eta_j)\mathbf{\Phi} = \sum_{a=1}^{r} \sum_{b=1}^{s} \big(\Phi^{ab}_{ij} \circ (\kappa_i \times \lambda_j)\big) \mathbf{e}_{i,a} \boxtimes \mathbf{f}_{j,b},$$

where

$$\Phi^{ab}_{ij} \in \mathcal{D}_{C_i\times D_j}\big(\kappa_i(U_i) \times \lambda_j(V_j)\big).$$

Extraction of each coefficient is continuous in the fixed-support topologies. Lemma 10.2.52 gives a contin-

uous operator

$$\mathscr{R}_{ij} : \mathcal{D}_{C_i \times D_j}\big(\kappa_i(U_i) \times \lambda_j(V_j)\big) \longrightarrow \mathcal{D}_{\widetilde{C}_i}(\kappa_i(U_i)) \widehat{\otimes}_\pi \mathcal{D}_{\widetilde{D}_j}(\lambda_j(V_j))$$

whose composition with multiplication of functions is the identity.
For $h \in \mathcal{D}_{\widetilde{C}_i}(\kappa_i(U_i))$, the section $(h \circ \kappa_i)\mathbf{e}_{i,a}$ is supported in $\kappa_i^{-1}(\widetilde{C}_i) \Subset U_i$. Denote by

$$\widetilde{(h \circ \kappa_i)\mathbf{e}_{i,a}}^{M}$$

its extension by zero from U_i to M. If $k \in \mathcal{D}_{\widetilde{D}_j}(\lambda_j(V_j))$, write

$$\widetilde{(k \circ \lambda_j)\mathbf{f}_{j,b}}^{N}$$

for the extension by zero from V_j to N. Define

$$P_{i,a}(h) := \widetilde{(h \circ \kappa_i)\mathbf{e}_{i,a}}^{M}, \qquad Q_{j,b}(k) := \widetilde{(k \circ \lambda_j)\mathbf{f}_{j,b}}^{N}.$$

These extensions are smooth, and the maps $P_{i,a}$ and $Q_{j,b}$ are continuous. Moreover, they take values in $\Gamma_{K_1}(\mathbf{E})$ and $\Gamma_{L_1}(\mathbf{F})$ because their supports lie in $\operatorname{Int}(K_1)$ and $\operatorname{Int}(L_1)$. By the universal property of the projective product, $P_{i,a} \widehat{\otimes} Q_{j,b}$ is continuous. Set

$$R^{K_1,L_1}_{K_0,L_0}\mathbf{\Phi} := \sum_{i=1}^{A}\sum_{j=1}^{B}\sum_{a=1}^{r}\sum_{b=1}^{s}(P_{i,a}\widehat{\otimes}Q_{j,b})\mathscr{R}_{ij}(\Phi_{ij}^{ab}).$$

The sum is finite, so $R^{K_1,L_1}_{K_0,L_0}$ is continuous. Applying T_{K_1,L_1}, each term recovers the corresponding coefficient of $(\chi_i \boxtimes \eta_j)\mathbf{\Phi}$. The partition-of-unity identities give

$$T_{K_1,L_1} R^{K_1,L_1}_{K_0,L_0}\mathbf{\Phi} = \sum_{i=1}^{A}\sum_{j=1}^{B}(\chi_i \boxtimes \eta_j)\mathbf{\Phi} = \mathbf{\Phi}.$$

It remains to prove that T_{K_1,L_1} is a topological embedding. Cover K_1 and L_1 by finitely many trivializing charts $(U_\alpha, \kappa_\alpha)$ and (V_β, λ_β). Choose cutoff functions $\zeta_\alpha \in C_c^\infty(U_\alpha)$ and $\theta_\beta \in C_c^\infty(V_\beta)$ whose sums equal one on neighborhoods of K_1 and L_1. Taking the components of $\zeta_\alpha u$ and $\theta_\beta v$ in the local frames, first extend them by zero within their coordinate models and then, by Lemma B.12.19, to the full Euclidean space. In interior charts, the latter operator is the identity. We obtain topological embeddings

$$I_{\mathbf{E}} : \Gamma_{K_1}(\mathbf{E}) \hookrightarrow \prod_{c=1}^{C}\mathcal{D}_{A_c}(\mathbb{R}^{\dim M}), \qquad I_{\mathbf{F}} : \Gamma_{L_1}(\mathbf{F}) \hookrightarrow \prod_{d=1}^{D}\mathcal{D}_{B_d}(\mathbb{R}^{\dim N}).$$

The index c ranges over pairs consisting of a chart of the first factor and a frame component, and d ranges over the analogous pairs for the second factor. Let the compact supports before extension be

$$A_c^0 = \kappa_\alpha(K_1 \cap \operatorname{supp}(\zeta_\alpha)), \qquad B_d^0 = \lambda_\beta(L_1 \cap \operatorname{supp}(\theta_\beta)),$$

repeated once for each component. Choose A_c and B_d containing the fixed supports of their smooth ex-

tensions. For example, if u_α^a is the component of $\zeta_\alpha u$ in U_α, the corresponding coordinate of $I_{\mathbf{E}}u$ is $E_\alpha\big(\widetilde{u_\alpha^a \circ \kappa_\alpha^{-1}}\big)$, where the tilde denotes extension by zero from $\kappa_\alpha(U_\alpha)$ to its model $G_{\dim M, r_\alpha}$, and $E_\alpha = E_{\dim M, r_\alpha}$, with $r_\alpha \in \{0,1\}$. The extension with a tilde is smooth because $\operatorname{supp}(\zeta_\alpha) \Subset U_\alpha$, and E_α smoothly extends its jets across the geometric boundary. The same convention, with operators E_β, is used for $I_{\mathbf{F}}$.

That $I_{\mathbf{E}}$ and $I_{\mathbf{F}}$ are embeddings is checked with the C^k seminorms: components are bounded by seminorms of the section, while the cutoff identity bounds the section seminorms by a finite sum of seminorms of its components. The same construction, using $\zeta_\alpha \boxtimes \theta_\beta$ and the extension $E_\alpha^x E_\beta^y$ of each localized component, gives an embedding

$$I_{\mathbf{E}\boxtimes\mathbf{F}} : \Gamma_{K_1\times L_1}(\mathbf{E}\boxtimes\mathbf{F}) \hookrightarrow \prod_{c,d} \mathcal{D}_{A_c\times B_d}(\mathbb{R}^{\dim M+\dim N}).$$

Support control in each group of variables allows us to use precisely the compact sets $A_c \times B_d$. The product identity in the extension lemma ensures, for elementary sections, that extending a component of $u \boxtimes v$ is the same as multiplying the extended components of u and v. By continuity, the same identity holds in the completed tensor products.

By Proposition 10.2.53, the two domain spaces are nuclear. The finite products in the codomain are nuclear as well. Theorem B.12.14 identifies the topologies π and ε, and Proposition B.12.18 shows that $I_{\mathbf{E}}\widehat{\otimes}I_{\mathbf{F}}$ is a topological embedding. After applying Proposition B.12.3, the external product becomes, in each coordinate (c,d), the scalar map

$$\mathcal{D}_{A_c}(\mathbb{R}^{\dim M})\widehat{\otimes}_\pi \mathcal{D}_{B_d}(\mathbb{R}^{\dim N}) \longrightarrow \mathcal{D}_{A_c\times B_d}(\mathbb{R}^{\dim M+\dim N}),$$

which is an embedding by Lemma B.12.22.

Let (z_γ) be a net in $\Gamma_{K_1}(\mathbf{E})\widehat{\otimes}_\pi\Gamma_{L_1}(\mathbf{F})$, and suppose that $T_{K_1,L_1}z_\gamma \to 0$. Commutativity of the localization maps implies that the image of $(I_{\mathbf{E}}\widehat{\otimes}I_{\mathbf{F}})z_\gamma$ under all the preceding scalar embeddings converges to zero. Since their finite product is an embedding, $(I_{\mathbf{E}}\widehat{\otimes}I_{\mathbf{F}})z_\gamma \to 0$. The map $I_{\mathbf{E}}\widehat{\otimes}I_{\mathbf{F}}$ is also an embedding, so $z_\gamma \to 0$. Applying the same argument to a constant net proves injectivity of T_{K_1,L_1}. Thus its inverse on the image is continuous. By injectivity, the operator constructed at the outset is the restriction of this inverse to $\Gamma_{K_0\times L_0}(\mathbf{E}\boxtimes\mathbf{F})$. □

Theorem 10.2.55. (Tensor product representation of test spaces) Let M and N be smooth manifolds with or without boundary, and let $\mathbf{E} \to M$ and $\mathbf{F} \to N$ be smooth vector bundles of finite rank. The external product induces a canonical topological isomorphism

$$\mathcal{D}(M,\mathbf{E})\widehat{\otimes}_\iota\mathcal{D}(N,\mathbf{F}) \cong \mathcal{D}(M\times N, \mathbf{E}\boxtimes\mathbf{F}).$$

In particular, finite linear combinations of sections $\mathbf{u}\boxtimes\mathbf{v}$ are dense in $\mathcal{D}(M\times N,\mathbf{E}\boxtimes\mathbf{F})$.

Proof. Choose exhaustions by compact sets $(K_j)_{j\in\mathbb{N}}$ and $(L_j)_{j\in\mathbb{N}}$ such that $K_j \subseteq \operatorname{Int}(K_{j+1})$ and $L_j \subseteq \operatorname{Int}(L_{j+1})$. Then $(K_j\times L_j)_{j\in\mathbb{N}}$ is an exhaustion of $M\times N$ with the same property.

First identify the topology on the algebraic tensor product. The ι topology on $\mathcal{D}(M,\mathbf{E})\otimes\mathcal{D}(N,\mathbf{F})$ agrees with the final locally convex topology for the maps

$$\Gamma_{K_p}(\mathbf{E})\otimes_\pi\Gamma_{L_q}(\mathbf{F}) \longrightarrow \mathcal{D}(M,\mathbf{E})\otimes\mathcal{D}(N,\mathbf{F}), \qquad p,q\in\mathbb{N}.$$

To check this, let $b\colon \mathcal{D}(M,\mathbf{E})\times\mathcal{D}(N,\mathbf{F}) \longrightarrow Z$ be a bilinear map. If b is separately continuous, its restriction

to $\Gamma_{K_p}(\mathbf{E}) \times \Gamma_{L_q}(\mathbf{F})$ is separately continuous as well. These two spaces are Fréchet, so Proposition B.12.10 makes the restriction continuous. Conversely, suppose all restrictions are continuous. If $\mathbf{u} \in \mathcal{D}(M,\mathbf{E})$, there exists p such that $\mathbf{u} \in \Gamma_{K_p}(\mathbf{E})$. For each q, the map $\mathbf{v} \mapsto b(\mathbf{u},\mathbf{v})$ is continuous on $\Gamma_{L_q}(\mathbf{F})$. The final property of the LF topology on $\mathcal{D}(N,\mathbf{F})$ implies continuity on the entire space. The same argument with the factors interchanged proves continuity in the first variable. The universal property of the inductive product, Proposition B.12.8, gives equality of the topologies.

The external product is separately continuous. Indeed, if $\mathbf{v} \in \Gamma_{L_q}(\mathbf{F})$ is fixed, for each p the local estimates C^k show that $\mathbf{u} \longmapsto \mathbf{u} \boxtimes \mathbf{v}$ is continuous from $\Gamma_{K_p}(\mathbf{E})$ to $\Gamma_{K_p \times L_q}(\mathbf{E} \boxtimes \mathbf{F})$. The final property of the LF topology gives continuity on $\mathcal{D}(M,\mathbf{E})$. The other factor is treated in the same way. Thus the external product linearizes to a continuous operator on $\mathcal{D}(M,\mathbf{E}) \otimes_\iota \mathcal{D}(N,\mathbf{F})$. Since $\mathcal{D}(M \times N, \mathbf{E} \boxtimes \mathbf{F})$ is complete by Theorem 4.4.6, this operator extends uniquely to

$$T : \mathcal{D}(M,\mathbf{E}) \widehat{\otimes}_\iota \mathcal{D}(N,\mathbf{F}) \longrightarrow \mathcal{D}(M \times N, \mathbf{E} \boxtimes \mathbf{F}).$$

We construct the inverse of T. For $\mathbf{\Phi} \in \Gamma_{K_j \times L_j}(\mathbf{E} \boxtimes \mathbf{F})$, Theorem 10.2.54 gives a unique element

$$R_j \mathbf{\Phi} \in \Gamma_{K_{j+1}}(\mathbf{E}) \widehat{\otimes}_\pi \Gamma_{L_{j+1}}(\mathbf{F})$$

whose image under the external product is $\mathbf{\Phi}$. The map R_j is continuous. Inclusion of the two fixed-support stages into the test spaces induces a continuous operator from this completed projective product to $\mathcal{D}(M,\mathbf{E}) \widehat{\otimes}_\iota \mathcal{D}(N,\mathbf{F})$.

We must check independence of the stage. Suppose that $j < q$ and $\mathbf{\Phi} \in \Gamma_{K_j \times L_j}(\mathbf{E} \boxtimes \mathbf{F})$. Include $R_j \mathbf{\Phi}$ and $R_q \mathbf{\Phi}$ in

$$\Gamma_{K_{q+1}}(\mathbf{E}) \widehat{\otimes}_\pi \Gamma_{L_{q+1}}(\mathbf{F}).$$

The inclusions used here are topological embeddings. Indeed, fixed-support spaces are nuclear by Proposition 10.2.53, so $\pi = \varepsilon$, and the assertion follows from Proposition B.12.18. Both preimages have external-product image $\mathbf{\Phi}$. Injectivity in Theorem 10.2.54, applied at stage $K_{q+1} \times L_{q+1}$, shows that they agree.

We may therefore define

$$R : \mathcal{D}(M \times N, \mathbf{E} \boxtimes \mathbf{F}) \longrightarrow \mathcal{D}(M,\mathbf{E}) \widehat{\otimes}_\iota \mathcal{D}(N,\mathbf{F}), \qquad R\mathbf{\Phi} := R_j \mathbf{\Phi}$$

when $\operatorname{supp} \mathbf{\Phi} \subseteq K_j \times L_j$. Each restriction of R to $\Gamma_{K_j \times L_j}(\mathbf{E} \boxtimes \mathbf{F})$ is continuous. The LF topology on the domain is final with respect to these stages, so R is continuous. By construction, $T \circ R$ is the identity.

Now let $\mathbf{u} \in \mathcal{D}(M,\mathbf{E})$ and $\mathbf{v} \in \mathcal{D}(N,\mathbf{F})$. Choose j so that $\operatorname{supp} \mathbf{u} \subseteq K_j$ and $\operatorname{supp} \mathbf{v} \subseteq L_j$. The elements $R(\mathbf{u} \boxtimes \mathbf{v})$ and $\mathbf{u} \otimes \mathbf{v}$, viewed at the corresponding stage, have the same image under the external product. Localized injectivity gives

$$R(\mathbf{u} \boxtimes \mathbf{v}) = \mathbf{u} \otimes \mathbf{v}.$$

Thus $R \circ T$ is the identity on the algebraic tensor product. This subspace is dense in its completion, and continuity of $R \circ T$ extends the identity to all of $\mathcal{D}(M,\mathbf{E}) \widehat{\otimes}_\iota \mathcal{D}(N,\mathbf{F})$. Therefore T and R are inverse topological isomorphisms. The image of the algebraic tensor product under T is the space of finite linear combinations of sections $\mathbf{u} \boxtimes \mathbf{v}$, so this space is dense in the test space of the product. $\square$

Corollary 10.2.56. (Tensor decomposition with support control) Let M and N be smooth manifolds with or without boundary, let $\mathbf{E} \to M$ and $\mathbf{F} \to N$ be smooth vector bundles of finite rank, and let $K_0 \subseteq \operatorname{Int}(K_1) \subseteq M$ and $L_0 \subseteq \operatorname{Int}(L_1) \subseteq N$ be compact. Every section

$$\mathbf{\Phi} \in \Gamma_{K_0 \times L_0}(\mathbf{E}^* \boxtimes \mathbf{F}^*)$$

has a unique preimage

$$u_{\mathbf{\Phi}} \in \Gamma_{K_1}(\mathbf{E}^*) \widehat{\otimes}_\pi \Gamma_{L_1}(\mathbf{F}^*)$$

under the external-product map. Moreover, $\mathbf{\Phi} \mapsto u_{\mathbf{\Phi}}$ is continuous.

Proof. Apply Theorem 10.2.54 to the bundles $\mathbf{E}^*$ and $\mathbf{F}^*$. The external-product map at stage $K_1 \times L_1$ is injective and its image contains $\Gamma_{K_0 \times L_0}(\mathbf{E}^* \boxtimes \mathbf{F}^*)$. Thus each $\mathbf{\Phi}$ has a unique preimage $u_{\mathbf{\Phi}}$. Continuity of $\mathbf{\Phi} \mapsto u_{\mathbf{\Phi}}$ is precisely continuity of the inverse restricted to this subspace. □

10.2.5 The Schwartz kernel theorem

A continuous operator can be studied as a distribution on the product of its source and target manifolds. In the presence of bundles, the kernel takes values in the appropriate tensor product; the following formulation makes this identification explicit.

Theorem 10.2.57. (Schwartz kernel theorem for vector bundles) Let M and N be smooth manifolds with or without boundary, let $\mathbf{E} \to M$ and $\mathbf{F} \to N$ be smooth vector bundles, and let W be a finite-dimensional $\mathbb{K}$-vector space.
Let

$$A \colon \mathcal{D}(N, \mathbf{F}^*) \longrightarrow \mathcal{D}'(M, \mathbf{E}, W)$$

be a linear map, where the target is equipped with the weak-$*$ topology, or equivalently the pointwise convergence topology of Definition 10.2.1. The following conditions are equivalent:

(a) A is continuous;

(b) the bilinear map

$$B_A \colon \mathcal{D}(M, \mathbf{E}^*) \times \mathcal{D}(N, \mathbf{F}^*) \longrightarrow W, \qquad B_A(\varphi, \psi) := (A\psi)(\varphi),$$

is separately continuous;

(c) for any compact sets $K \subseteq M$ and $L \subseteq N$, the restriction of B_A to $\Gamma_K(\mathbf{E}^*) \times \Gamma_L(\mathbf{F}^*)$ is continuous.

When these conditions hold, there exists a unique distribution

$$\mathbf{K}_A \in \mathcal{D}'(M \times N, \mathbf{E} \boxtimes \mathbf{F}, W)$$

such that

$$\mathbf{K}_A(\varphi \boxtimes \psi) = (A\psi)(\varphi)$$

for every $\varphi \in \mathcal{D}(M, \mathbf{E}^*)$ and every $\psi \in \mathcal{D}(N, \mathbf{F}^*)$.

Conversely, every distribution $\mathbf{K} \in \mathcal{D}'(M \times N, \mathbf{E} \boxtimes \mathbf{F}, W)$ defines a linear map

$$A_{\mathbf{K}} \colon \mathcal{D}(N, \mathbf{F}^*) \longrightarrow \mathcal{D}'(M, \mathbf{E}, W)$$

by

$$(A_{\mathbf{K}}\psi)(\varphi) := \mathbf{K}(\varphi \boxtimes \psi),$$

and this map is continuous in the weak-$*$ topology. Moreover, the correspondences $A \mapsto \mathbf{K}_A$ and $\mathbf{K} \mapsto A_{\mathbf{K}}$ are linear and mutually inverse. In other words, there is a canonical linear bijection

$$\mathcal{D}'(M \times N, \mathbf{E} \boxtimes \mathbf{F}, W) \cong \mathcal{L}\big(\mathcal{D}(N, \mathbf{F}^*), \mathcal{D}'(M, \mathbf{E}, W)_\sigma\big),$$

where the subscript σ indicates the weak-$*$ topology.

Proof. If A is continuous in the topology of pointwise convergence, then, for each $\varphi \in \mathcal{D}(M, \mathbf{E}^*)$, the map

$$\psi \longmapsto (A\psi)(\varphi) = \mathrm{ev}_\varphi(A\psi)$$

is continuous. For each ψ, the map $\varphi \mapsto (A\psi)(\varphi)$ is continuous because $A\psi$ is a distribution. This proves $(a) \Rightarrow (b)$.

Suppose (b). The restriction of B_A to $\Gamma_K(\mathbf{E}^*) \times \Gamma_L(\mathbf{F}^*)$ is separately continuous. Both factors are Fréchet spaces by Proposition 10.1.7, and Proposition B.12.10 shows that the restriction is continuous. Therefore $(b) \Rightarrow (c)$.

Finally, suppose (c) and fix $\varphi \in \mathcal{D}(M, \mathbf{E}^*)$. There exists a compact set $K \subseteq M$ such that $\varphi \in \Gamma_K(\mathbf{E}^*)$. For every compact set $L \subseteq N$, condition (c) implies that $\psi \mapsto B_A(\varphi, \psi)$ is continuous on $\Gamma_L(\mathbf{F}^*)$. Since the topology on $\mathcal{D}(N, \mathbf{F}^*)$ is final with respect to these fixed-support spaces, the same map is continuous on $\mathcal{D}(N, \mathbf{F}^*)$. Thus $\mathrm{ev}_\varphi \circ A$ is continuous for every φ. The weak-$*$ topology on $\mathcal{D}'(M, \mathbf{E}, W)$ is the initial topology for evaluations, so Proposition B.10.40 proves continuity of A. This gives $(c) \Rightarrow (a)$.

Now suppose these conditions hold. By (b) and the universal property of the inductive product, Proposition B.12.8, there exists a unique continuous operator

$$\widetilde{B}_A : \mathcal{D}(M, \mathbf{E}^*) \otimes_\iota \mathcal{D}(N, \mathbf{F}^*) \longrightarrow W$$

such that $\widetilde{B}_A(\varphi \otimes \psi) = B_A(\varphi, \psi)$. The space W is complete because it is finite dimensional. The universal property of completion extends $\widetilde{B}_A$ uniquely to a continuous operator

$$\widehat{B}_A : \mathcal{D}(M, \mathbf{E}^*) \widehat{\otimes}_\iota \mathcal{D}(N, \mathbf{F}^*) \longrightarrow W.$$

Let

$$T : \mathcal{D}(M, \mathbf{E}^*) \widehat{\otimes}_\iota \mathcal{D}(N, \mathbf{F}^*) \longrightarrow \mathcal{D}(M \times N, \mathbf{E}^* \boxtimes \mathbf{F}^*)$$

be the isomorphism in Theorem 10.2.55. Define

$$\mathbf{K}_A := \widehat{B}_A \circ T^{-1}.$$

This is a continuous linear operator on $\mathcal{D}(M \times N, \mathbf{E}^* \boxtimes \mathbf{F}^*)$. Through the isomorphism $(\mathbf{E} \boxtimes \mathbf{F})^* \cong \mathbf{E}^* \boxtimes \mathbf{F}^*$ in

Remark 10.2.47, it is identified with an element of $\mathcal{D}'(M\times N,\mathbf{E}\boxtimes\mathbf{F},W)$. Moreover,

$$\mathbf{K}_A(\varphi\boxtimes\psi)=\widehat{B}_A(\varphi\otimes\psi)=B_A(\varphi,\psi)=(A\psi)(\varphi).$$

For uniqueness, let $\mathbf{S}$ be another distribution satisfying the same identity. Then $\mathbf{S}-\mathbf{K}_A$ vanishes on every section $\varphi\boxtimes\psi$ and, by linearity, on all finite linear combinations of them. These combinations are dense by Theorem 10.2.55. Since $\mathbf{S}-\mathbf{K}_A$ is continuous, it vanishes on all of $\mathcal{D}(M\times N,\mathbf{E}^*\boxtimes\mathbf{F}^*)$. Therefore $\mathbf{S}=\mathbf{K}_A$.
Conversely, let $\mathbf{K}\in\mathcal{D}'(M\times N,\mathbf{E}\boxtimes\mathbf{F},W)$. For $\psi\in\mathcal{D}(N,\mathbf{F}^*)$, define

$$(A_{\mathbf{K}}\psi)(\varphi):=\mathbf{K}(\varphi\boxtimes\psi).$$

For fixed ψ, the map $\varphi\mapsto\varphi\boxtimes\psi$ is continuous, as proved in constructing the isomorphism of Theorem 10.2.55. Its composition with $\mathbf{K}$ is therefore a continuous operator from $\mathcal{D}(M,\mathbf{E}^*)$ to W. This proves that $A_{\mathbf{K}}\psi\in\mathcal{D}'(M,\mathbf{E},W)$. Linearity of $A_{\mathbf{K}}$ follows from linearity of $\mathbf{K}$.
To prove continuity of $A_{\mathbf{K}}$ into the weak-$*$ dual, fix $\varphi\in\mathcal{D}(M,\mathbf{E}^*)$. The map

$$\mathrm{ev}_\varphi\circ A_{\mathbf{K}}:\psi\longmapsto\mathbf{K}(\varphi\boxtimes\psi)$$

is continuous because $\psi\mapsto\varphi\boxtimes\psi$ is continuous. Every evaluation is continuous. By the universal property of the initial topology, $A_{\mathbf{K}}$ is continuous in the weak-$*$ topology.
Finally,

$$(A_{\mathbf{K}_A}\psi)(\varphi)=\mathbf{K}_A(\varphi\boxtimes\psi)=(A\psi)(\varphi),$$

so $A_{\mathbf{K}_A}=A$. Starting with $\mathbf{K}$, the distributions $\mathbf{K}_{A_{\mathbf{K}}}$ and $\mathbf{K}$ agree on every elementary product. Density proved in Theorem 10.2.55 and continuity of both distributions imply that they agree on the entire test space. Therefore $\mathbf{K}_{A_{\mathbf{K}}}=\mathbf{K}$. The definitions also show that the two correspondences are linear. □

Remark 10.2.58. (Variants of the kernel theorem) The scalar Euclidean formulation can be found in [149, Theorem 51.7]. A proof by regularization and a formulation on manifolds appear in [64, Theorem 3.1]. Taylor proves the compact-manifold version using Sobolev estimates and Hilbert–Schmidt operators in [142, Chapter 4, §6]. For the geometric formulation with vector bundles over closed manifolds, see also [90, Chapter 13, §3.5]. The result here holds for noncompact manifolds and different bundles on the two factors. Control through the stages $K_j\times L_j$ replaces a global compactness hypothesis.

11

Principal symbol and elliptic operators

Chapters 8 and 10 construct differential operators between sections, their formal adjoints, and their distributional extensions. Building on this foundation, this chapter isolates the highest-order part, formulates ellipticity intrinsically, and derives the regularity estimates that will be used to study the heat semigroup and fractional Sobolev spaces.

The principal part of an operator contains less information than the full operator, but it governs several of its fundamental analytic properties. On a compact manifold, invertibility of the principal symbol allows us to recover all derivatives of a section from the operator applied to that section and a zeroth-order term. Elliptic regularity and the equivalence of the Sobolev norm and the graph norm follow from this observation. The same symbol will later be the geometric object that enters index theory.

Throughout this chapter, M will be a smooth manifold of dimension n, without boundary unless otherwise specified, and $\mathbf{E}, \mathbf{F}, \mathbf{G} \to M$ will be smooth real or complex vector bundles of finite rank. Whenever L^2 or Sobolev norms are involved, we fix a Riemannian metric $\mathbf{g}$ on M, bundle metrics $\mathbf{h}_{\mathbf{E}}$, $\mathbf{h}_{\mathbf{F}}$, and $\mathbf{h}_{\mathbf{G}}$ on the corresponding bundles, Hermitian in the complex case, and compatible connections $\nabla^{\mathbf{E}}$, $\nabla^{\mathbf{F}}$, and $\nabla^{\mathbf{G}}$. On tensor factors constructed from TM and T^*M, we use the Levi–Civita connection of $\mathbf{g}$.

Ellipticity: a local condition with global consequences

Elliptic theory took its modern form with the recognition that the highest-order terms control regularity, whereas lower-order terms can be treated as perturbations. The principal symbol encodes this dominant part on the cotangent bundle. Its invertibility away from the zero section is a pointwise condition, yet it leads to global estimates, finite-dimensional kernels, and stability of the index. This contrast between a microlocal hypothesis and global consequences will recur throughout the remainder of the book.

11.1 The principal symbol

Recall that $\mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$ denotes the space of partial differential operators of order at most m introduced in Definition 8.1.3. For $f \in C^\infty(M)$, we write $\mathrm{ad}(f)(P) = [P, f]$, where $[P, f](u) = P(fu) - fP(u)$. Proposition 8.1.25 shows that the pointwise evaluation of an iterated commutator of length m depends only on the differentials of the functions involved.

Definition 11.1.1. (Principal symbol) Let M be a smooth manifold without boundary, let $\mathbf{E},\mathbf{F}\to M$ be smooth vector bundles of finite rank, and let $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$, with $m\geq 1$. For $x\in M$, $\xi_1,\dots,\xi_m\in T_x^*M$, and $v\in\mathbf{E}_x$, choose functions $f_1,\dots,f_m\in C^\infty(M)$ such that $df_j(x)=\xi_j$ and define

$$\widetilde{\sigma}_m(P)_x(\xi_1,\dots,\xi_m)v:=\frac{1}{m!}\operatorname{ad}(f_1)\cdots\operatorname{ad}(f_m)(P)|_x(v).$$

The **polarized form of the principal symbol** is the map $\widetilde{\sigma}_m(P)_x\colon (T_x^*M)^m\longrightarrow\operatorname{Hom}(\mathbf{E}_x,\mathbf{F}_x)$, and the **principal symbol** of P is the map

$$\sigma_m(P)\colon T^*M\longrightarrow\operatorname{Hom}(\mathbf{E},\mathbf{F}),\qquad \sigma_m(P)(x,\xi):=\widetilde{\sigma}_m(P)_x(\xi,\dots,\xi).$$

If $m=0$, the operator P is $C^\infty(M)$-linear and corresponds to a bundle homomorphism; in this case we set $\sigma_0(P)(x):=P_x$.

The normalization by $m!$ ensures that, in coordinates, the symbol is exactly the homogeneous polynomial formed from the coefficients of order m.

Proposition 11.1.2. (Well-definedness and elementary properties) Let M be a smooth manifold without boundary, let $\mathbf{E},\mathbf{F}\to M$ be smooth vector bundles of finite rank, and let $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$. The form $\widetilde{\sigma}_m(P)_x$ is well defined, symmetric, and multilinear in the covariables. Moreover, $\sigma_m(P)$ is smooth and satisfies

$$\sigma_m(P)(x,t\xi)=t^m\sigma_m(P)(x,\xi)$$

for all $x\in M$, $\xi\in T_x^*M$, and $t\in\mathbb{R}$.

Proof. Independence of the chosen functions is precisely the content of Proposition 8.1.25. We prove linearity in the first covariable; the other arguments are treated in the same way. Let $\xi,\eta,\xi_2,\dots,\xi_m\in T_x^*M$ and $a,b\in\mathbb{R}$. Choose smooth functions $f,g,f_2,\dots,f_m$ such that $df(x)=\xi$, $dg(x)=\eta$, and $df_j(x)=\xi_j$. Since $d(af+bg)(x)=a\xi+b\eta$, the cited proposition allows us to use $af+bg$ to represent this covariable. Linearity of $f\mapsto\operatorname{ad}(f)$ then gives

$$\widetilde{\sigma}_m(P)_x(a\xi+b\eta,\xi_2,\dots,\xi_m)=a\widetilde{\sigma}_m(P)_x(\xi,\xi_2,\dots,\xi_m)+b\widetilde{\sigma}_m(P)_x(\eta,\xi_2,\dots,\xi_m).$$

Lemma 8.1.21 allows us to interchange two consecutive commutators; hence the form is symmetric, and the linearity just proved in the first entry implies linearity in every entry.
It remains to check smoothness. In a chart (U,φ), with $\varphi=(x^1,\dots,x^n)$, and local frames $(\mathbf{e}_1,\dots,\mathbf{e}_r)$ of $\mathbf{E}$ and $(\mathbf{f}_1,\dots,\mathbf{f}_s)$ of $\mathbf{F}$, Corollary 8.1.35 allows us to write

$$P\left(\sum_{a=1}^{r}u^a\mathbf{e}_a\right)=\sum_{b=1}^{s}\left(\sum_{|\alpha|\leq m}\sum_{a=1}^{r}(A_\alpha)_{ba}D^\alpha u^a\right)\mathbf{f}_b,$$

where $A_\alpha\colon U\longrightarrow M_{s\times r}(\mathbb{K})$ is smooth. For $x\in U$ and $\xi=\sum_{i=1}^{n}\xi_i\,\mathbf{d}x^i|_x$, consider near x the linear coordinate

function $\ell_\xi := \sum_{i=1}^{n} \xi_i x^i$. Multiplying it by a smooth function equal to one on a neighborhood of x produces a function in $C^\infty(M)$ with the same differential at x. Proposition 8.1.25 therefore allows us to use ℓ_ξ in the m entries defining $\sigma_m(P)(x,\xi)$.

Since $[\partial_i, \ell_\xi] = \xi_i \operatorname{Id}$ and $[AB, \ell_\xi] = A[B,\ell_\xi] + [A,\ell_\xi]B$, each commutator replaces one of the $|\alpha|$ differential factors of D^α with the corresponding component of ξ. If $|\alpha| < m$, the result vanishes after m commutators. If $|\alpha| = m$, a term survives only if each of the m factors is replaced exactly once, and the commutators can select these factors in $m!$ orders. Consequently,

$$\operatorname{ad}(\ell_\xi)^m(D^\alpha) = \begin{cases} m!\,\xi^\alpha \operatorname{Id}, & |\alpha| = m, \\ 0, & |\alpha| < m. \end{cases}$$

The matrices A_α act by multiplication and commute with scalar functions. Dividing by the normalization $m!$ in the definition yields

$$\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha.$$

The coefficients A_α are smooth, and the right-hand side is a homogeneous polynomial of degree m in ξ; therefore $\sigma_m(P)$ is smooth and $\sigma_m(P)(x,t\xi) = t^m \sigma_m(P)(x,\xi)$. □

Proposition 11.1.3. (Local expression) Let M be a smooth manifold without boundary, let $\mathbf{E},\mathbf{F} \to M$ be smooth vector bundles of finite rank, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$. In a chart and local frames in which $P = \sum_{|\alpha|\le m} A_\alpha D^\alpha$, if $\xi = \sum_{j=1}^{n} \xi_j dx^j|_x$, then

$$\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha, \qquad \xi^\alpha := \prod_{j=1}^{n} \xi_j^{\alpha_j}.$$

Consequently, P has order at most $m-1$ if and only if $\sigma_m(P) = 0$.

Proof. This formula is the diagonal computation carried out in the proof of Proposition 11.1.2. If P has order at most $m-1$, no coefficients A_α with $|\alpha| = m$ occur, and the symbol vanishes. Conversely, if $\sigma_m(P) = 0$, every entry of the polynomial matrix $\sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha$ vanishes for all $\xi \in \mathbb{R}^n$. Uniqueness of polynomial coefficients implies $A_\alpha(x) = 0$ for every $|\alpha| = m$ and every x in the chart. Corollary 8.1.35 then shows that $P \in \mathbf{PDO}^{(m-1)}(\mathbf{E},\mathbf{F})$. □

Remark 11.1.4. The preceding definition does not include powers of i. With this convention, the symbol of $-\partial_j^2$ is $-\xi_j^2$. Applying the Fourier transform instead introduces the multiplier $(i\xi)^\alpha$. We will explicitly account for this distinction in the proof of the elliptic estimates.

11.1.1 Elementary symbol calculus

The principal symbol records the highest-order part of an operator. Before using it to formulate ellipticity, we need to check how it behaves under composition and passage to adjoints.

Proposition 11.1.5. (Symbol of a composition) Let M be a smooth manifold without boundary, and let $\mathbf{E},\mathbf{F},\mathbf{G} \to M$ be smooth vector bundles of finite rank. Let $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ and $Q \in \mathbf{PDO}^{(\ell)}(\mathbf{F},\mathbf{G})$. Then $Q \circ P \in \mathbf{PDO}^{(m+\ell)}(\mathbf{E},\mathbf{G})$ and

$$\sigma_{m+\ell}(Q \circ P)(x,\xi) = \sigma_{\ell}(Q)(x,\xi) \circ \sigma_{m}(P)(x,\xi).$$

Proof. Membership in $\mathbf{PDO}^{(m+\ell)}(\mathbf{E},\mathbf{G})$ was proved in Proposition 8.1.19. In a chart and local frames, write

$$P = \sum_{|\alpha| \le m} A_\alpha D^\alpha, \qquad Q = \sum_{|\beta| \le \ell} B_\beta D^\beta.$$

Upon expanding $Q(Pu)$, terms of order $m+\ell$ arise only when all the derivatives D^β act on $D^\alpha u$ and none acts on A_α. Thus the highest-order part is

$$\sum_{|\beta|=\ell} \sum_{|\alpha|=m} B_\beta A_\alpha D^{\alpha+\beta} u.$$

Evaluating the symbol at ξ gives

$$\sum_{|\beta|=\ell} \sum_{|\alpha|=m} B_\beta(x) A_\alpha(x) \xi^{\alpha+\beta} = \left(\sum_{|\beta|=\ell} B_\beta(x) \xi^\beta \right) \left(\sum_{|\alpha|=m} A_\alpha(x) \xi^\alpha \right),$$

which is the asserted identity. □

The formal Hermitian adjoint was already constructed in Theorem 8.2.3. We need only record the effect of that construction on the principal part.

Proposition 11.1.6. (Symbol of the formal Hermitian adjoint) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E},\mathbf{F} \to M$ be smooth vector bundles of finite rank equipped with real or Hermitian bundle metrics $\mathbf{h}_{\mathbf{E}}$ and $\mathbf{h}_{\mathbf{F}}$. If $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ and P_h^* is its formal Hermitian adjoint, then

$$\sigma_m(P_h^*)(x,\xi) = (-1)^m \sigma_m(P)(x,\xi)^*,$$

where the adjoint on the right is taken between the fibers $(\mathbf{E}_x,\mathbf{h}_{\mathbf{E}})$ and $(\mathbf{F}_x,\mathbf{h}_{\mathbf{F}})$.

Proof. In orthonormal or unitary local frames, write

$$P\left(\sum_{i=1}^{r} u^i \mathbf{e}_i \right) = \sum_{j=1}^{s} \left(\sum_{i=1}^{r} \sum_{|\alpha| \le m} a_{\alpha i}^{j} D_\varphi^\alpha u^i \right) \mathbf{f}_j.$$

Theorem 8.2.3 shows that the component along $\mathbf{e}_i$ of $P_h^*(\sum_{j=1}^{s} v^j \mathbf{f}_j)$ is

$$\frac{1}{\sqrt{\det(\mathbf{g})}} \sum_{j=1}^{s} \sum_{|\alpha| \le m} (-1)^{|\alpha|} D_\varphi^\alpha \Big(\sqrt{\det(\mathbf{g})}\, \overline{a_{\alpha i}^{j}}\, v^j \Big).$$

Terms of order m arise only when all $|\alpha|$ derivatives act on v^j. If

$$A_\alpha = (a^j_{\alpha i})\colon U \longrightarrow M_{s\times r}(\mathbb{K}),$$

their coefficients form, pointwise, the matrix $(-1)^m A^*_\alpha$ for $|\alpha| = m$. Proposition 11.1.3 gives

$$\sigma_m(P^*_h)(x,\xi) = (-1)^m \sum_{|\alpha|=m} A_\alpha(x)^* \xi^\alpha = (-1)^m \sigma_m(P)(x,\xi)^*.$$

□

Corollary 11.1.7. (Symbol under conjugation) Let M be a smooth manifold without boundary, let $\mathbf{E},\mathbf{F} \to M$ be smooth complex vector bundles of finite rank, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$. Then

$$\sigma_m(\overline{P})(x,\xi) = \overline{\sigma_m(P)(x,\xi)}\colon \overline{\mathbf{E}_x} \longrightarrow \overline{\mathbf{F}_x}.$$

In particular, P is elliptic if and only if $\overline{P}$ is elliptic.

Proof. In conjugate frames, the principal coefficients of $\overline{P}$ are $\overline{A_\alpha}$ by Proposition 8.1.36. Proposition 11.1.3 gives the formula; conjugation preserves invertibility. □

Corollary 11.1.8. (Symbol of the formal transpose) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, let $\mathbf{E},\mathbf{F} \to M$ be smooth vector bundles of finite rank, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$. If P' is its bilinear formal transpose, then

$$\sigma_m(P')(x,\xi) = (-1)^m \sigma_m(P)(x,\xi)^T\colon \mathbf{F}^*_x \longrightarrow \mathbf{E}^*_x, \tag{11.1}$$

where T is the algebraic transpose, without conjugation. Thus P is elliptic if and only if P' is elliptic.

Proof. Corollary 10.2.15 shows that the coefficient of D^α of order m is $(-1)^m A^T_\alpha$; every derivative falling on the coordinate weight of the measure or on A_α has lower order. The local formula for the symbol yields (11.1). The transpose of a finite-dimensional map is invertible exactly when the map itself is invertible. □

Example 11.1.9. (The connection) Let M be a smooth manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with connection $\nabla^{\mathbf{E}}$. By Proposition 8.1.12, $\nabla^{\mathbf{E}}$ is a first-order differential operator from $\mathbf{E}$ to $T^*M \otimes \mathbf{E}$. The Leibniz rule gives

$$[\nabla^{\mathbf{E}}, f]\mathbf{u} = df \otimes \mathbf{u},$$

and consequently

$$\sigma_1(\nabla^{\mathbf{E}})(x,\xi)v = \xi \otimes v.$$

This symbol is injective for $\xi \neq 0$, although it is not surjective when $n > 1$.

Example 11.1.10. (The Bochner Laplacian) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with a real or Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection

$\nabla^{\mathbf{E}}$. The Bochner Laplacian $\Delta_B = (\nabla^{\mathbf{E}})_h^* \nabla^{\mathbf{E}}$, defined in Definition 8.2.10, satisfies

$$\sigma_2(\Delta_B)(x,\xi) = -\|\xi\|_{\mathbf{g}}^2 \operatorname{Id}_{\mathbf{E}_x}.$$

This identity follows from Propositions 11.1.5 and 11.1.6, since

$$\sigma_1((\nabla^{\mathbf{E}})_h^*)(x,\xi)(\eta \otimes v) = -\langle \xi, \eta \rangle_{\mathbf{g}} v.$$

11.2 Elliptic operators

Ellipticity expresses the invertibility of the principal symbol in every nonzero cotangent direction. This condition allows derivatives of the solution to be recovered from the equation and leads to regularity estimates.

Definition 11.2.1. (Ellipticity) Let M be a smooth manifold without boundary, let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles of finite rank, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$, with $m \geq 1$. We say that P is **elliptic** if, for every $x \in M$ and every $\xi \in T_x^* M \setminus \{0\}$, the map

$$\sigma_m(P)(x,\xi) \colon \mathbf{E}_x \longrightarrow \mathbf{F}_x$$

is a linear isomorphism.

The definition requires $\mathbf{E}$ and $\mathbf{F}$ to have the same rank. When the symbol is only injective or only surjective, one speaks of an overdetermined or underdetermined elliptic operator, respectively. These variants will be useful in index theory, but are not needed for the estimates in this chapter.

Proposition 11.2.2. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, let $\mathbf{E}, \mathbf{F}, \mathbf{G} \to M$ be smooth vector bundles of finite rank equipped with bundle metrics, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$.

(a) P is elliptic if and only if P_h^* is elliptic;

(b) if P is elliptic, then $P_h^* P$ and $P P_h^*$ are elliptic of order $2m$;

(c) if $Q \in \mathbf{PDO}^{(\ell)}(\mathbf{F}, \mathbf{G})$ and both P and Q are elliptic, then $Q \circ P$ is elliptic.

Proof. Part (a) follows from Proposition 11.1.6, because a linear map between finite-dimensional spaces is invertible if and only if its adjoint is invertible. For (b), Propositions 11.1.5 and 11.1.6 give

$$\sigma_{2m}(P_h^* P)(x,\xi) = (-1)^m \sigma_m(P)(x,\xi)^* \sigma_m(P)(x,\xi),$$

which is invertible for $\xi \neq 0$. The argument for $P P_h^*$ is identical. Part (c) follows because a composition of isomorphisms is an isomorphism. □

Example 11.2.3. The Bochner Laplacian is elliptic by Example 11.1.10. The Weitzenböck identity, Theorem 8.2.33, shows that the bundle-valued Hodge Laplacian differs from the Bochner Laplacian by a zeroth-order endomorphism. They therefore have the same principal symbol and are both elliptic.

Proposition 11.2.4. (Differential symbol of a Dirac-type operator) Let $(\mathbf{S}, c, \nabla^{\mathbf{S}})$ be a Clifford module with connection, and let $D = c \circ \nabla^{\mathbf{S}}$ be the operator in Definition 8.3.1. Its differential principal symbol is

$$\sigma_1(D)(x,\xi) = c(\xi)\colon \mathbf{S}_x \longrightarrow \mathbf{S}_x. \tag{11.2}$$

In particular, D is elliptic and, for $\xi \neq 0$,

$$\sigma_1(D)(x,\xi)^{-1} = -\frac{c(\xi)}{|\xi|_{\mathbf{g}}^2}.$$

If the module is graded, the block $D^+\colon \Gamma(\mathbf{S}^+) \to \Gamma(\mathbf{S}^-)$ is elliptic and its symbol is the restriction $c(\xi)\colon \mathbf{S}^+_x \to \mathbf{S}^-_x$.

Proof. By (8.26), for any function f with $df_x = \xi$,

$$\sigma_1(D)(x,\xi) = [D, f]_x = c(df_x) = c(\xi).$$

The Clifford relation $c(\xi)^2 = -|\xi|_{\mathbf{g}}^2 I$ gives the stated inverse. In the graded case, $c(\xi)$ is odd and its square satisfies the same identity on each summand, so both restrictions are isomorphisms for $\xi \neq 0$. □

11.3 Elliptic estimates on vector bundles

We first prove the estimate for systems with constant coefficients, then freeze the coefficients at a point, and finally globalize the result by means of a partition of unity. The only ingredient from Fourier analysis is Plancherel's identity.

11.3.1 An interpolation inequality

Local estimates involve lower-order terms that must be absorbed. The following inequality quantifies this absorption in terms of a small portion of the higher-order norm and a lower-order norm.

Lemma 11.3.1. (Interpolation between integer orders) Let $m \in \mathbb{N}$ and $0 \leq j < m$. For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|u\|_{H^j(\mathbb{R}^n,\mathbb{K}^r)} \leq \varepsilon \|u\|_{H^m(\mathbb{R}^n,\mathbb{K}^r)} + C_\varepsilon \|u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)}$$

for every $u \in H^m(\mathbb{R}^n, \mathbb{K}^r)$.

Proof. We use the Fourier transform with unitary normalization, write $\langle \xi \rangle := (1 + \|\xi\|^2)^{\frac{1}{2}}$, and set

$$\|u\|_{s,\mathbf{F}} := \|\langle \xi \rangle^s \widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)}.$$

For each $s \in \mathbb{N}_0$, the Leibniz formula for $(1 + \|\xi\|^2)^s$ and Plancherel's identity give constants $a_s, b_s > 0$, depending only on n, r, s, and the chosen normalization of the Sobolev norm, such that

$$a_s \|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} \leq \|u\|_{s,\mathbf{F}} \leq b_s \|u\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)}.$$

Let $R \geq 1$ and set $\mathbf{E}_R := \{\xi \in \mathbb{R}^n \mid \langle \xi \rangle \leq R\}$. On $\mathbf{E}_R$ we have $\langle \xi \rangle^j \leq R^j$, whereas on $\mathbf{E}_R^c$ we have $\langle \xi \rangle^{j-m} \leq R^{j-m}$. By the triangle inequality in $L^2(\mathbb{R}^n, \mathbb{K}^r)$, applied componentwise, and Plancherel's identity,

$$\begin{aligned} \|u\|_{j,\mathbf{F}} &= \|\langle \xi \rangle^j \widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} \\ &\leq \|\mathbf{1}_{\mathbf{E}_R} \langle \xi \rangle^j \widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} + \|\mathbf{1}_{\mathbf{E}_R^c} \langle \xi \rangle^j \widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} \\ &\leq R^j \|\widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} + R^{j-m} \|\langle \xi \rangle^m \widehat{u}\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} \\ &= R^j \|u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} + R^{j-m} \|u\|_{m,\mathbf{F}}. \end{aligned}$$

Let $\delta := \frac{a_j \varepsilon}{b_m}$ and choose $R_\varepsilon := \max\left\{1, \delta^{-\frac{1}{m-j}}\right\}$. Then $R_\varepsilon^{j-m} \leq \delta$ and, using the two preceding inequalities,

$$\|u\|_{H^j(\mathbb{R}^n,\mathbb{K}^r)} \leq a_j^{-1} \|u\|_{j,\mathbf{F}} \leq \varepsilon \|u\|_{H^m(\mathbb{R}^n,\mathbb{K}^r)} + a_j^{-1} R_\varepsilon^j \|u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)}.$$

Thus it suffices to take $C_\varepsilon := a_j^{-1} R_\varepsilon^j$. □

11.3.2 Systems with constant coefficients

Freezing the coefficients first reduces the problem to a translation-invariant system. In this model, the Fourier transform converts ellipticity into an algebraic estimate for the symbol.

Lemma 11.3.2. (Estimate for a constant-coefficient elliptic system) Let $n, r, m \in \mathbb{N}$, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $A_\alpha \in \mathrm{End}(\mathbb{K}^r)$ for $|\alpha| \leq m$. Consider the operator

$$P_0(D) \colon C_c^\infty(\mathbb{R}^n, \mathbb{K}^r) \longrightarrow C_c^\infty(\mathbb{R}^n, \mathbb{K}^r), \qquad P_0(D)u = \sum_{|\alpha| \leq m} A_\alpha D^\alpha u,$$

with constant coefficients, and suppose that

$$p_m(\xi) := \sum_{|\alpha| = m} A_\alpha \xi^\alpha$$

is invertible for $\xi \neq 0$. There exists $C > 0$ such that

$$\|u\|_{H^m(\mathbb{R}^n,\mathbb{K}^r)} \leq C\left(\|P_0(D)u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} + \|u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)}\right)$$

for every $u \in C_c^\infty(\mathbb{R}^n, \mathbb{K}^r)$.

Proof. If $\mathbb{K} = \mathbb{R}$, we complexify the system to use the Fourier transform; the resulting estimate is then restricted to real-valued functions. The function $(\xi, v) \mapsto \|p_m(\xi)v\|_{\mathbb{C}^r}$ is continuous on the compact set $S^{n-1} \times S^{2r-1}$ and does not vanish. Thus there exists $c > 0$ such that

$$\|p_m(\xi)v\|_{\mathbb{C}^r} \geq c\|\xi\|^m \|v\|_{\mathbb{C}^r}$$

for all $\xi \in \mathbb{R}^n$ and $v \in \mathbb{C}^r$.

We must distinguish the preceding principal symbol from the Fourier multiplier. Set

$$p^{\#}(\xi) := \sum_{|\alpha|\le m} A_\alpha (i\xi)^\alpha, \qquad p_m^{\#}(\xi) := \sum_{|\alpha|=m} A_\alpha (i\xi)^\alpha = i^m p_m(\xi).$$

Since $|i^m| = 1$, the preceding estimate gives $\|p_m^{\#}(\xi)v\|_{\mathbb{C}^r} \ge c\|\xi\|^m \|v\|_{\mathbb{C}^r}$. If

$$r^{\#}(\xi) := p^{\#}(\xi) - p_m^{\#}(\xi),$$

then there exists $C_0 > 0$ such that $\|r^{\#}(\xi)v\|_{\mathbb{C}^r} \le C_0(1+\|\xi\|)^{m-1}\|v\|_{\mathbb{C}^r}$. Consequently,

$$\|p^{\#}(\xi)v\|_{\mathbb{C}^r} \ge c\|\xi\|^m\|v\|_{\mathbb{C}^r} - C_0(1+\|\xi\|)^{m-1}\|v\|_{\mathbb{C}^r}.$$

Choose $R > 1$ so that $C_0(1+\|\xi\|)^{m-1} \le \dfrac{c}{2}\|\xi\|^m$ for $\|\xi\| \ge R$. In this region,

$$(1+\|\xi\|^2)^{\frac{m}{2}}\|v\|_{\mathbb{C}^r} \le C_1\|p^{\#}(\xi)v\|_{\mathbb{C}^r}.$$

For $\|\xi\| < R$, we have $(1+\|\xi\|^2)^{\frac{m}{2}}\|v\|_{\mathbb{C}^r} \le C_R\|v\|_{\mathbb{C}^r}$. Therefore

$$(1+\|\xi\|^2)^{\frac{m}{2}}\|\widehat{u}(\xi)\|_{\mathbb{C}^r} \le C\big(\|p^{\#}(\xi)\widehat{u}(\xi)\|_{\mathbb{C}^r} + \|\widehat{u}(\xi)\|_{\mathbb{C}^r}\big).$$

We square, integrate, and apply Plancherel's identity. Since

$$\widehat{P_0(D)u}(\xi) = p^{\#}(\xi)\widehat{u}(\xi),$$

the asserted estimate follows. □

Lemma 11.3.3. (Constant-coefficient estimate at an arbitrary Sobolev order) Let $n, r, m \in \mathbb{N}$, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $A_\alpha \in \mathrm{End}(\mathbb{K}^r)$ for $|\alpha| = m$. Let the operator

$$P_m(D)\colon \mathcal{S}(\mathbb{R}^n, \mathbb{K}^r) \longrightarrow \mathcal{S}(\mathbb{R}^n, \mathbb{K}^r), \qquad P_m(D)v = \sum_{|\alpha|=m} A_\alpha D^\alpha v,$$

be a homogeneous system with constant coefficients, and suppose that its symbol

$$p_m(\xi) := \sum_{|\alpha|=m} A_\alpha \xi^\alpha$$

is invertible for $\xi \neq 0$. For every $s \in \mathbb{R}$, there exists $C_s > 0$ such that

$$\|v\|_{H^{s+m}(\mathbb{R}^n,\mathbb{K}^r)} \le C_s\big(\|P_m(D)v\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} + \|v\|_{H^{s+m-1}(\mathbb{R}^n,\mathbb{K}^r)}\big)$$

for every $v \in \mathcal{S}(\mathbb{R}^n, \mathbb{K}^r)$.

Proof. As in the proof of Lemma 11.3.2, ellipticity and compactness of the sphere give $c > 0$ such that

$$\|p_m^{\#}(\xi)z\|_{\mathbb{C}^r} \ge c\|\xi\|^m\|z\|_{\mathbb{C}^r}, \qquad p_m^{\#}(\xi) := \sum_{|\alpha|=m} A_\alpha (i\xi)^\alpha,$$

for all $\xi \in \mathbb{R}^n$ and $z \in \mathbb{C}^r$. Choose $R \geq 1$. If $\|\xi\| \geq R$, the preceding inequality implies

$$\langle \xi \rangle^{s+m} \|z\|_{\mathbb{C}^r} \leq C_{s,R} \langle \xi \rangle^s \|p_m^{\#}(\xi) z\|_{\mathbb{C}^r}.$$

If $\|\xi\| < R$, then

$$\langle \xi \rangle^{s+m} \|z\|_{\mathbb{C}^r} \leq C_{s,R} \langle \xi \rangle^{s+m-1} \|z\|_{\mathbb{C}^r}.$$

Combining the two regions gives, for every ξ and every z,

$$\langle \xi \rangle^{s+m} \|z\|_{\mathbb{C}^r} \leq C_s \left(\langle \xi \rangle^s \|p_m^{\#}(\xi) z\|_{\mathbb{C}^r} + \langle \xi \rangle^{s+m-1} \|z\|_{\mathbb{C}^r} \right).$$

We apply this inequality to $z = \widehat{v}(\xi)$, square, and integrate. Since $\widehat{P_m(D)v} = p_m^{\#} \widehat{v}$, Plancherel's identity completes the proof. □

11.3.3 Freezing coefficients

On a small ball, a variable-coefficient operator is compared with the constant-coefficient operator obtained by fixing at the center only the coefficients of its principal part. This part determines ellipticity. The principal error retains order m, but its coefficients become small; the remaining terms need not be small because they have order at most $m-1$.

Lemma 11.3.4. (Local estimate for elliptic systems) Let $n, r, m \in \mathbb{N}$, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $U \subseteq \mathbb{R}^n$ be open, and let $A_\alpha \in C^\infty(U; \mathrm{End}(\mathbb{K}^r))$. Let

$$P \colon C_c^\infty(U, \mathbb{K}^r) \longrightarrow C_c^\infty(U, \mathbb{K}^r), \qquad Pu = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u,$$

be an elliptic system with smooth coefficients. For every $x_0 \in U$, there exist an open subset V, with $x_0 \in V \Subset U$, and a constant $C > 0$ such that

$$\|u\|_{H^m(\mathbb{R}^n, \mathbb{K}^r)} \leq C \left(\|Pu\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} + \|u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} \right)$$

for every $u \in C_c^\infty(V, \mathbb{K}^r)$, extended by zero outside V.

Proof. We freeze only the principal coefficients at x_0 and write

$$P = P_{x_0}^{(m)} + R_1 + R_0,$$

where

$$P_{x_0}^{(m)} := \sum_{|\alpha| = m} A_\alpha(x_0) D^\alpha, \qquad R_1 := \sum_{|\alpha| = m} \big(A_\alpha(x) - A_\alpha(x_0) \big) D^\alpha,$$

and R_0 collects the terms of order at most $m-1$. For every $W \Subset U$, the operators $P_{x_0}^{(m)}$, R_1, and R_0 send $C_c^\infty(W, \mathbb{K}^r)$ into $C_c^\infty(U, \mathbb{K}^r)$. Lemma 11.3.2 applied to $P_{x_0}^{(m)}$ gives

$$\|u\|_{H^m(\mathbb{R}^n, \mathbb{K}^r)} \leq C_0 \left(\|P_{x_0}^{(m)} u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} + \|u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} \right).$$

Choose $r_0 > 0$ so that $\overline{B(x_0, r_0)} \subseteq U$. For $|\alpha| = m$, set

$$C_\alpha(r_0) := \sup_{z \in \overline{B(x_0,r_0)}} \sum_{j=1}^{n} \left\| \frac{\partial A_\alpha}{\partial x^j}(z) \right\|, \qquad C_*(r_0) := \sum_{|\alpha|=m} C_\alpha(r_0).$$

Hadamard's lemma 1.4.10, applied to the matrix entries of A_α, gives, for $x \in B(x_0, r_0)$,

$$A_\alpha(x) - A_\alpha(x_0) = \sum_{j=1}^{n} (x^j - x_0^j) \int_0^1 \frac{\partial A_\alpha}{\partial x^j}\big(x_0 + t(x - x_0)\big)\, dt. \tag{11.3}$$

Since the segment lies in the ball, for $0 < r \le r_0$ we obtain the quantitative bound

$$\sup_{x \in B(x_0,r)} \|A_\alpha(x) - A_\alpha(x_0)\| \le r C_\alpha(r_0). \tag{11.4}$$

Choose $r \in (0, r_0)$ small enough that

$$r C_*(r_0) \le \frac{1}{4 C_0 C_1}$$

, and take $V = B(x_0, r)$. If $C_*(r_0) = 0$, any $r < r_0$ satisfies this condition. Then

$$\sup_{x \in V} \sum_{|\alpha|=m} \|A_\alpha(x) - A_\alpha(x_0)\| \le \frac{1}{4 C_0 C_1},$$

where C_1 satisfies $\sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} \le C_1 \|u\|_{H^m(\mathbb{R}^n, \mathbb{K}^r)}$. Thus

$$\|R_1 u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} \le \frac{1}{4 C_0} \|u\|_{H^m(\mathbb{R}^n, \mathbb{K}^r)}.$$

Since R_0 has order at most $m - 1$, there exists $C_2 > 0$ such that $\|R_0 u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} \le C_2 \|u\|_{H^{m-1}(\mathbb{R}^n, \mathbb{K}^r)}$. Applying Lemma 11.3.1 with $\varepsilon = (4 C_0 C_2)^{-1}$ yields

$$\|R_0 u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} \le \frac{1}{4 C_0} \|u\|_{H^m(\mathbb{R}^n, \mathbb{K}^r)} + C_3 \|u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)}.$$

Since $P^{(m)}_{x_0} u = Pu - R_1 u - R_0 u$,

$$\|u\|_{H^m(\mathbb{R}^n, \mathbb{K}^r)} \le C_0 \|Pu\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)} + \frac{1}{2} \|u\|_{H^m(\mathbb{R}^n, \mathbb{K}^r)} + C_4 \|u\|_{L^2(\mathbb{R}^n, \mathbb{K}^r)}.$$

Moving the middle term to the left-hand side completes the proof. □

Proposition 11.3.5. (Local estimate at an arbitrary Sobolev order) Let $n, r, m \in \mathbb{N}$, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $U \subseteq \mathbb{R}^n$ be open, and let $A_\alpha \in C^\infty(U; \mathrm{End}(\mathbb{K}^r))$. Let

$$P \colon C_c^\infty(U, \mathbb{K}^r) \longrightarrow C_c^\infty(U, \mathbb{K}^r), \qquad Pv = \sum_{|\alpha| \le m} A_\alpha(x) D^\alpha v,$$

be an elliptic system with smooth coefficients. Given $x_0 \in U$ and $s \in \mathbb{R}$, there exist an open subset V, with

$x_0 \in V \Subset U$, and a constant $C_s > 0$ such that

$$\|v\|_{H^{s+m}(\mathbb{R}^n,\mathbb{K}^r)} \le C_s\big(\|Pv\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} + \|v\|_{H^{s+m-1}(\mathbb{R}^n,\mathbb{K}^r)}\big)$$

for every $v \in C_c^\infty(V,\mathbb{K}^r)$, extended by zero outside V.

Proof. We freeze only the principal coefficients at x_0 and set

$$P_{x_0}^{(m)} := \sum_{|\alpha|=m} A_\alpha(x_0) D^\alpha.$$

Lemma 11.3.3 provides a constant $C_0 > 0$ such that

$$\|v\|_{H^{s+m}(\mathbb{R}^n,\mathbb{K}^r)} \le C_0\big(\|P_{x_0}^{(m)} v\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} + \|v\|_{H^{s+m-1}(\mathbb{R}^n,\mathbb{K}^r)}\big).$$

Write

$$P = P_{x_0}^{(m)} + R_1 + R_0,$$

where

$$R_1 := \sum_{|\alpha|=m} \big(A_\alpha(x) - A_\alpha(x_0)\big) D^\alpha$$

and R_0 collects the terms of order at most $m-1$. For every $W \Subset U$, the operators $P_{x_0}^{(m)}$, R_1, and R_0 send $C_c^\infty(W,\mathbb{K}^r)$ into $C_c^\infty(U,\mathbb{K}^r)$.

We will apply the refined multiplication estimate, Lemma 6.1.8, after extending the coefficients to all of $\mathbb{R}^n$. Let us make this localization precise. Choose $r_0 > 0$ with $\overline{B(x_0,r_0)} \subseteq U$, and define $C_\alpha(r_0)$ and $C_*(r_0)$ as in the preceding proof. Identities (11.3) and (11.4) show that, for $0 < r < r_0$,

$$\sum_{|\alpha|=m} \sup_{x \in B(x_0,r)} \|A_\alpha(x) - A_\alpha(x_0)\| \le r C_*(r_0). \tag{11.5}$$

Let $C_1 > 0$ be a constant bounding the product of the matrix constant in Lemma 6.1.8 and the norms of the operators $D^\alpha : H^{s+m}(\mathbb{R}^n,\mathbb{K}^r) \longrightarrow H^s(\mathbb{R}^n,\mathbb{K}^r)$, $|\alpha| = m$. This constant depends on n, s, m and the ranks, but not on r, θ, or the coefficients. Choose $r \in (0, r_0)$ so that $C_0 C_1 r C_*(r_0) \le \frac{1}{2}$, and take $V = B(x_0, r/2)$. Choose $\theta \in C_c^\infty(B(x_0,r))$ with $0 \le \theta \le 1$ and $\theta = 1$ on a neighborhood of $\overline{V}$. If $v \in C_c^\infty(V,\mathbb{K}^r)$, we may replace each coefficient of R_1 by

$$B_\alpha(x) := \theta(x)\big(A_\alpha(x) - A_\alpha(x_0)\big),$$

interpreted as zero outside $B(x_0,r)$. This coefficient is smooth and agrees with $A_\alpha - A_\alpha(x_0)$ on a neighborhood of the support of v. Moreover, since $0 \le \theta \le 1$ and its support is contained in $B(x_0,r)$, estimate (11.5) gives

$$\sum_{|\alpha|=m} \|B_\alpha\|_{L^\infty(\mathbb{R}^n)} \le r C_*(r_0).$$

Lemma 6.1.8, applied componentwise, and continuity of $D^\alpha : H^{s+m}(\mathbb{R}^n,\mathbb{K}^r) \longrightarrow H^s(\mathbb{R}^n,\mathbb{K}^r)$ give

$$\|R_1 v\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} \le C_1 r C_*(r_0) \|v\|_{H^{s+m}(\mathbb{R}^n,\mathbb{K}^r)} + C_2 \|v\|_{H^{s+m-1}(\mathbb{R}^n,\mathbb{K}^r)}.$$

Here the second term collects the remainders that lose one derivative; their constants may depend on derivatives of the coefficients and on θ, but not on v.
To handle R_0, we extend its coefficients in the same way using θ and then by zero to $\mathbb{R}^n$. Since each summand has the form $B_\alpha D^\alpha v$ with $|\alpha| \le m-1$, Lemma 6.1.8, together with Proposition 6.1.6, gives

$$\|R_0 v\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} \le C_3 \|v\|_{H^{s+m-1}(\mathbb{R}^n,\mathbb{K}^r)}.$$

The identity $P_{x_0}^{(m)} v = Pv - R_1 v - R_0 v$ implies

$$\|v\|_{H^{s+m}(\mathbb{R}^n,\mathbb{K}^r)} \le C_0 \|Pv\|_{H^s(\mathbb{R}^n,\mathbb{K}^r)} + C_0 C_1 r C_*(r_0) \|v\|_{H^{s+m}(\mathbb{R}^n,\mathbb{K}^r)} + C_4 \|v\|_{H^{s+m-1}(\mathbb{R}^n,\mathbb{K}^r)}.$$

Our choice of radius satisfies $C_0 C_1 r C_*(r_0) \le \frac{1}{2}$; if $C_*(r_0) = 0$, any radius for which the preceding localizations are defined suffices. This condition explains quantitatively why the term R_1, although it retains order m, can be absorbed into the left-hand side. The coefficients of R_0 have not been made small: the loss of one order is what allows them to remain in the last term. This gives the asserted estimate. □

Proposition 11.3.6. (Higher-order local estimate) Under the hypotheses of Lemma 11.3.4, for each $k \in \mathbb{N}_0$ the open subset V can be shrunk, if necessary, so that there exists $C_k > 0$ such that

$$\|u\|_{H^{k+m}(\mathbb{R}^n,\mathbb{K}^r)} \le C_k \left(\|Pu\|_{H^k(\mathbb{R}^n,\mathbb{K}^r)} + \|u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} \right)$$

for every $u \in C_c^\infty(V,\mathbb{K}^r)$, extended by zero outside V.

Proof. Apply Proposition 11.3.5 with $s = k$. After shrinking V if necessary, we obtain

$$\|u\|_{H^{k+m}(\mathbb{R}^n,\mathbb{K}^r)} \le C_1 \left(\|Pu\|_{H^k(\mathbb{R}^n,\mathbb{K}^r)} + \|u\|_{H^{k+m-1}(\mathbb{R}^n,\mathbb{K}^r)} \right).$$

Lemma 11.3.1, applied with orders $k+m-1$ and $k+m$, provides, for every $\varepsilon > 0$, a constant $C_\varepsilon > 0$ such that

$$\|u\|_{H^{k+m-1}(\mathbb{R}^n,\mathbb{K}^r)} \le \varepsilon \|u\|_{H^{k+m}(\mathbb{R}^n,\mathbb{K}^r)} + C_\varepsilon \|u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)}.$$

Choose ε so that $C_1 \varepsilon < 1$ and absorb the corresponding term into the left-hand side. We obtain

$$\|u\|_{H^{k+m}(\mathbb{R}^n,\mathbb{K}^r)} \le C_k \left(\|Pu\|_{H^k(\mathbb{R}^n,\mathbb{K}^r)} + \|u\|_{L^2(\mathbb{R}^n,\mathbb{K}^r)} \right),$$

as desired. □

11.3.4 Globalization on a closed manifold

The local estimates give a global estimate by means of a finite partition of unity. Commutators with cutoff functions have lower order, so the interpolation inequality controls the terms introduced by patching.

Theorem 11.3.7. (Global elliptic estimate) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, let $\mathbf{E},\mathbf{F} \to M$ be smooth vector bundles with bundle metrics $\mathbf{h_E},\mathbf{h_F}$, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ be elliptic. For every $k \in \mathbb{N}_0$,

there exists $C_k > 0$ such that

$$\|\mathbf{u}\|_{H^{k+m}(M,\mathbf{E})} \leq C_k\big(\|P\mathbf{u}\|_{H^k(M,\mathbf{F})} + \|\mathbf{u}\|_{L^2(M,\mathbf{E})}\big)$$

for every $\mathbf{u} \in \Gamma(\mathbf{E})$. Moreover, P extends uniquely to a continuous linear operator

$$P\colon H^{k+m}(M,\mathbf{E}) \longrightarrow H^k(M,\mathbf{F}).$$

Proof. Continuity of $P\colon H^{k+m}(M,\mathbf{E}) \longrightarrow H^k(M,\mathbf{F})$ follows from the local expression in Corollary 8.1.35. Each derivative of order at most k of $P\mathbf{u}$ is a finite sum of smooth coefficients multiplied by derivatives of the components of $\mathbf{u}$ of order at most $k+m$. Lemma 9.1.44, applied with $p = 2$, locally identifies the norm defined using covariant derivatives with the sum of the Sobolev norms of the components. This allows us to sum the estimates over a finite cover.

If $m = 0$, the operator P is a smooth bundle homomorphism, and ellipticity means that each P_x is invertible. In local frames, the inverse matrix formula shows that P^{-1} is smooth. The same Leibniz estimate, applied to P^{-1} on a finite cover, gives $\|\mathbf{u}\|_{H^k(M,\mathbf{E})} \leq C_k\|P\mathbf{u}\|_{H^k(M,\mathbf{F})}$. This also proves the reverse inequality at order zero.

Now suppose that $m \geq 1$. For the reverse inequality, choose a finite cover $(U_a)_{a=1}^N$ by charts over which $\mathbf{E}$ and $\mathbf{F}$ admit smooth frames, and a subordinate partition of unity $(\chi_a)_{a=1}^N$. Refine the cover so that $\operatorname{supp}(\chi_a)$ lies in an open subset on which Proposition 11.3.6 applies. In coordinates and components,

$$\|\chi_a\mathbf{u}\|_{H^{k+m}(M,\mathbf{E})} \leq C_a\big(\|P(\chi_a\mathbf{u})\|_{H^k(M,\mathbf{F})} + \|\chi_a\mathbf{u}\|_{L^2(M,\mathbf{E})}\big).$$

Since

$$P(\chi_a\mathbf{u}) = \chi_a Pu + [P,\chi_a]\mathbf{u}$$

and $[P,\chi_a]$ has order at most $m-1$,

$$\|[P,\chi_a]\mathbf{u}\|_{H^k(M,\mathbf{F})} \leq C_a'\|\mathbf{u}\|_{H^{k+m-1}(M,\mathbf{E})}.$$

Summing over a and using Lemma 9.1.44 once again,

$$\|\mathbf{u}\|_{H^{k+m}(M,\mathbf{E})} \leq C\big(\|P\mathbf{u}\|_{H^k(M,\mathbf{F})} + \|\mathbf{u}\|_{H^{k+m-1}(M,\mathbf{E})} + \|\mathbf{u}\|_{L^2(M,\mathbf{E})}\big).$$

The global version of Lemma 11.3.1, obtained by applying it in the finitely many charts, gives

$$\|\mathbf{u}\|_{H^{k+m-1}(M,\mathbf{E})} \leq \varepsilon\|\mathbf{u}\|_{H^{k+m}(M,\mathbf{E})} + C_\varepsilon\|\mathbf{u}\|_{L^2(M,\mathbf{E})}.$$

Choose ε so that the first term can be absorbed into the left-hand side. This proves the estimate for smooth sections. Continuity of P and density of $\Gamma(\mathbf{E})$ in $H^{k+m}(M,\mathbf{E})$ yield the indicated extension and extend the inequality to all of $H^{k+m}(M,\mathbf{E})$. □

Corollary 11.3.8. (Equivalence with the graph norm) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, let $\mathbf{E},\mathbf{F} \to M$ be smooth vector bundles with bundle metrics $\mathbf{h_E},\mathbf{h_F}$, and let $P \in \mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ be elliptic. For

each $k \in \mathbb{N}_0$, there exist constants $c_P, C_P > 0$, depending on P, k, and the fixed geometric data, such that

$$c_P \|\mathbf{u}\|_{H^{k+m}(M,\mathbf{E})} \le \|\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|P\mathbf{u}\|_{H^k(M,\mathbf{F})} \le C_P \|\mathbf{u}\|_{H^{k+m}(M,\mathbf{E})}$$

for every $\mathbf{u} \in H^{k+m}(M,\mathbf{E})$.

Proof. Theorem 11.3.7 gives one inequality. The other follows from continuity of $P\colon H^{k+m}(M,\mathbf{E}) \longrightarrow H^k(M,\mathbf{F})$ and the continuous inclusion $H^{k+m}(M,\mathbf{E}) \hookrightarrow L^2(M,\mathbf{E})$. □

In the language of operators on bundles, the preceding estimate agrees with the fundamental elliptic inequality in [21, Exercise 9.12(b)]. An alternative construction of elliptic regularity, based on microlocal tools that have not yet been defined, will be developed in Chapter 15.

11.4 Elliptic regularity

The operator P acts on distributions through the construction in Chapter 10; see Proposition 10.2.18. Thus the equation $P\mathbf{u} = \mathbf{f}$ makes sense for $\mathbf{u} \in \mathcal{D}'(M,\mathbf{E})$ under the definition of distributional derivative already established.

Lemma 11.4.1. (Local gain of one derivative) Let $\Omega \subseteq \mathbb{R}^n$ be open, let

$$P = \sum_{|\alpha| \le m} A_\alpha(x) D^\alpha$$

be an elliptic system with smooth coefficients, and let $t \in \mathbb{Z}$. If

$$v \in H^t_{\mathrm{loc}}(\Omega, \mathbb{K}^r) \qquad \text{and} \qquad Pv \in H^{t-m+1}_{\mathrm{loc}}(\Omega, \mathbb{K}^r),$$

then

$$v \in H^{t+1}_{\mathrm{loc}}(\Omega, \mathbb{K}^r).$$

Proof. Fix $x_0 \in \Omega$. Apply Proposition 11.3.5 with the index

$$s := t - m + 1.$$

We obtain an open subset V_0, with $x_0 \in V_0 \Subset \Omega$, and a constant $C > 0$ such that

$$\|z\|_{H^{t+1}(\mathbb{R}^n,\mathbb{K}^r)} \le C\big(\|Pz\|_{H^{t-m+1}(\mathbb{R}^n,\mathbb{K}^r)} + \|z\|_{H^t(\mathbb{R}^n,\mathbb{K}^r)}\big)$$

for every $z \in C^\infty_c(V_0, \mathbb{K}^r)$.
Choose open subsets W, V such that

$$x_0 \in W \Subset V \Subset V_0$$

and a function $\chi \in C^\infty_c(V)$ equal to one on a neighborhood of $\overline{W}$. Define

$$w := \chi v,$$

extended by zero outside Ω. Since $v \in H^t_{\mathrm{loc}}$, Definition 6.1.7 gives $w \in H^t(\mathbb{R}^n, \mathbb{K}^r)$ and $\operatorname{supp}(w) \subseteq \operatorname{supp}(\chi) \Subset V$.

In the distributional sense,

$$Pw = \chi Pv + [P, \chi]v.$$

Let us justify the regularity of both summands. The hypothesis on Pv gives $\chi Pv \in H^{t-m+1}(\mathbb{R}^n, \mathbb{K}^r)$. On the other hand, $[P, \chi]$ is a differential system of order at most $m-1$. Choose $\eta \in C^\infty_c(V_0)$ such that $\eta = 1$ on a neighborhood of $\operatorname{supp}(\chi)$. Locality of differential operators allows us to write

$$[P, \chi]v = [P, \chi](\eta v).$$

Since $\eta v \in H^t(\mathbb{R}^n, \mathbb{K}^r)$, continuity of differentiation and multiplication by smooth coefficients, Proposition 6.1.6 and Lemma 6.1.8, gives

$$[P, \chi]v \in H^{t-m+1}(\mathbb{R}^n, \mathbb{K}^r).$$

Therefore

$$Pw \in H^{t-m+1}(\mathbb{R}^n, \mathbb{K}^r).$$

Let J_ε be the mollifier in Proposition 6.1.11 and set $w_\varepsilon := J_\varepsilon w$. Since $\operatorname{supp}(w) \Subset V$, there exists $\varepsilon_0 > 0$ such that

$$\operatorname{supp}(w_\varepsilon) \subseteq V_0 \qquad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Moreover, $w_\varepsilon \in C^\infty_c(V_0, \mathbb{K}^r)$, so the preceding elliptic estimate applies:

$$\|w_\varepsilon\|_{H^{t+1}(\mathbb{R}^n, \mathbb{K}^r)} \leq C\big(\|Pw_\varepsilon\|_{H^{t-m+1}(\mathbb{R}^n, \mathbb{K}^r)} + \|w_\varepsilon\|_{H^t(\mathbb{R}^n, \mathbb{K}^r)}\big).$$

To control the first term, we extend the coefficients of P outside V_0. Choose $\vartheta \in C^\infty_c(\Omega)$ equal to one on a neighborhood of $\overline{V_0}$ and replace A_α with the extension by zero of ϑA_α to $\mathbb{R}^n$. This replacement does not change Pw, Pw_ε, or the commutator on their supports. Then

$$Pw_\varepsilon = P(J_\varepsilon w) = J_\varepsilon(Pw) + [P, J_\varepsilon]w.$$

Proposition 6.1.11 and Corollary 6.1.13 give, uniformly for $0 < \varepsilon < \varepsilon_0$,

$$\|Pw_\varepsilon\|_{H^{t-m+1}(\mathbb{R}^n, \mathbb{K}^r)} \leq \|Pw\|_{H^{t-m+1}(\mathbb{R}^n, \mathbb{K}^r)} + C_1\|w\|_{H^t(\mathbb{R}^n, \mathbb{K}^r)}.$$

Also, $\|w_\varepsilon\|_{H^t(\mathbb{R}^n, \mathbb{K}^r)} \leq \|w\|_{H^t(\mathbb{R}^n, \mathbb{K}^r)}$. Consequently,

$$\|w_\varepsilon\|_{H^{t+1}(\mathbb{R}^n, \mathbb{K}^r)} \leq C_2\big(\|Pw\|_{H^{t-m+1}(\mathbb{R}^n, \mathbb{K}^r)} + \|w\|_{H^t(\mathbb{R}^n, \mathbb{K}^r)}\big),$$

and the right-hand side is independent of ε.

The space $H^{t+1}(\mathbb{R}^n, \mathbb{K}^r)$ is a Hilbert space. Thus there exist a sequence $\varepsilon_j \to 0$ and an element $z \in H^{t+1}(\mathbb{R}^n, \mathbb{K}^r)$ such that $w_{\varepsilon_j} \rightharpoonup z$ weakly in $H^{t+1}(\mathbb{R}^n, \mathbb{K}^r)$. The continuous inclusion $H^{t+1}(\mathbb{R}^n, \mathbb{K}^r) \hookrightarrow H^t(\mathbb{R}^n, \mathbb{K}^r)$ implies that the same sequence converges weakly to z in $H^t(\mathbb{R}^n, \mathbb{K}^r)$. However, Proposition 6.1.11 gives $w_{\varepsilon_j} \to w$ strongly in $H^t(\mathbb{R}^n, \mathbb{K}^r)$. Uniqueness of the weak limit shows that $z = w$. Consequently, $w \in H^{t+1}(\mathbb{R}^n, \mathbb{K}^r)$.

Since $\chi = 1$ on a neighborhood of $\overline{W}$, we have $v = w$ on W. Thus $v \in H^{t+1}_{\mathrm{loc}}(W, \mathbb{K}^r)$ on a neighborhood of x_0.

Since x_0 was arbitrary, the localization criterion in Lemma 6.1.9 yields $v \in H^{t+1}_{\mathrm{loc}}(\Omega, \mathbb{K}^r)$. □

Theorem 11.4.2. (Local elliptic regularity) Let M be a smooth manifold without boundary, let $\mathbf{E}, \mathbf{F} \to M$ be smooth vector bundles of finite rank, let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$ be elliptic, and let $k \in \mathbb{N}_0$. If $\mathbf{u} \in \mathcal{D}'(M, \mathbf{E})$ and $P\mathbf{u} \in H^k_{\mathrm{loc}}(M, \mathbf{F})$, then $\mathbf{u} \in H^{k+m}_{\mathrm{loc}}(M, \mathbf{E})$. In particular, if $P\mathbf{u}$ is smooth, then $\mathbf{u}$ is smooth.

Proof. If $m = 0$, ellipticity gives a smooth inverse homomorphism P^{-1}. Multiplication of distributions by its smooth coefficients is compatible with composition, so $\mathbf{u} = P^{-1}(P\mathbf{u})$. The Leibniz rule in each chart proves the local continuity of P^{-1} on H^k and its preservation of smoothness. Both conclusions follow directly in this case.

Suppose that $m \geq 1$ and fix $x_0 \in M$. Choose a regular coordinate ball (U, φ) around x_0, with $\Omega := \varphi(U) \subseteq \mathbb{R}^n$, and smooth frames

$$\mathbf{e}_1, \dots, \mathbf{e}_r \qquad \text{and} \qquad \mathbf{f}_1, \dots, \mathbf{f}_r$$

for $\mathbf{E}|_U$ and $\mathbf{F}|_U$, respectively. The ranks agree because the principal symbol of an elliptic operator is an isomorphism between fibers. Shrinking U if necessary, we may assume that U is relatively compact and that x_0 corresponds to the point $y_0 := \varphi(x_0)$.

Let $q \in C^\infty(\Omega)$ be the positive coordinate weight of the Riemannian measure, characterized by the change-of-variables formula

$$\int_U \eta \, d\lambda_{\mathbf{g}} = \int_\Omega (\eta \circ \varphi^{-1})(y) q(y) \, d\lambda_n(y)$$

for every integrable function η supported in U. Let $\mathbf{e}^1, \dots, \mathbf{e}^r$ and $\mathbf{f}^1, \dots, \mathbf{f}^r$ be the dual frames of $\mathbf{E}^*|_U$ and $\mathbf{F}^*|_U$, respectively. The local representation of $\mathbf{u}$ is the vector-valued distribution

$$\widehat{u} = (u^1, \dots, u^r) \in \mathcal{D}'(\Omega, \mathbb{K}^r)$$

defined by

$$\langle u^a, \psi \rangle := \mathbf{u}\left(((q^{-1}\psi) \circ \varphi) \mathbf{e}^a \right), \qquad \psi \in C^\infty_c(\Omega),$$

where the test section is extended by zero outside U. Define $\widehat{Pu} = ((Pu)^1, \dots, (Pu)^r)$ by

$$\langle (Pu)^b, \psi \rangle := P\mathbf{u}\left(((q^{-1}\psi) \circ \varphi) \mathbf{f}^b \right), \qquad \psi \in C^\infty_c(\Omega), \quad 1 \leq b \leq r,$$

again extending the test section by zero. This normalization by q^{-1} makes the components correspond to Lebesgue measure on Ω.

In the chosen frames, the operator has the form

$$P_U = \sum_{|\alpha| \leq m} A_\alpha(y) D^\alpha.$$

The distributional extension of P is compatible with this local representation. Let us verify this assertion. If $\Psi = (\Psi_1, \dots, \Psi_r) \in C^\infty_c(\Omega, (\mathbb{K}^r)^*)$, identify $q^{-1}\Psi$ with the section of $\mathbf{F}^*|_U$ given by $\sum_{b=1}^r ((q^{-1}\Psi_b) \circ \varphi) \mathbf{f}^b$. Let P^{t}_U be the bilinear formal transpose of the system with respect to Lebesgue measure; its matrices are transposed,

not conjugated. Corollary 10.2.15 gives

$$P'\big(q^{-1}\Psi\big) = q^{-1}P_U^{\mathrm{t}}\Psi.$$

By the definition of the distributional action,

$$\langle\widehat{Pu},\Psi\rangle = (P\mathbf{u})(q^{-1}\Psi) = \mathbf{u}\big(P'(q^{-1}\Psi)\big) = \langle\widehat{u},P_U^{\mathrm{t}}\Psi\rangle = \langle P_U\widehat{u},\Psi\rangle.$$

Thus

$$\widehat{Pu} = P_U\widehat{u} \qquad \text{on } \mathcal{D}'(\Omega,\mathbb{K}^r).$$

By Proposition 11.1.3, the principal symbol of the system is

$$p_m(y,\xi) = \sum_{|\alpha|=m} A_\alpha(y)\xi^\alpha.$$

Ellipticity of P means that $p_m(y,\xi)$ is invertible for every $y\in\Omega$ and every $\xi\neq 0$. Thus P_U is an elliptic system with smooth coefficients.
The hypothesis $P\mathbf{u}\in H^k_{\mathrm{loc}}(M,\mathbf{F})$ and Lemma 9.1.44 imply that the components of $P\mathbf{u}$ in the frame $\mathbf{f}_1,\dots,\mathbf{f}_r$ belong to $H^k_{\mathrm{loc}}(\Omega,\mathbb{K}^r)$. The preceding normalization merely multiplies test functions by the positive smooth function q^{-1}, so the local representation satisfies

$$P_U\widehat{u} = \widehat{Pu} \in H^k_{\mathrm{loc}}(\Omega,\mathbb{K}^r).$$

We now need a starting point for the iteration. Choose $\chi_0\in C_c^\infty(\Omega)$ such that $\chi_0 = 1$ on a neighborhood of y_0. By Proposition 6.1.10, there exists $N\in\mathbb{N}$ such that

$$\chi_0\widehat{u}\in H^{-N}(\mathbb{R}^n,\mathbb{K}^r).$$

The localization criterion, Lemma 6.1.9, shows that

$$\widehat{u}\in H^{-N}_{\mathrm{loc}}(\Omega_0,\mathbb{K}^r)$$

on some open neighborhood $\Omega_0\Subset\Omega$ of y_0 on which $\chi_0 = 1$.
Apply Lemma 11.4.1 repeatedly. In the first step, take $t = -N$. Since

$$-N-m+1\le k,$$

the continuous inclusion in Proposition 6.1.6 gives

$$P_U\widehat{u}\in H^k_{\mathrm{loc}}(\Omega_0)\hookrightarrow H^{-N-m+1}_{\mathrm{loc}}(\Omega_0).$$

The derivative-gain lemma yields

$$\widehat{u}\in H^{-N+1}_{\mathrm{loc}}(\Omega_0).$$

Suppose inductively that, for some integer t with

$$-N \le t < k+m,$$

we have obtained $\widehat{u} \in H^t_{\mathrm{loc}}(\Omega_0)$. Since $t \le k+m-1$, we have $t-m+1 \le k$, and again

$$P_U \widehat{u} \in H^k_{\mathrm{loc}}(\Omega_0) \hookrightarrow H^{t-m+1}_{\mathrm{loc}}(\Omega_0).$$

Lemma 11.4.1 implies

$$\widehat{u} \in H^{t+1}_{\mathrm{loc}}(\Omega_0).$$

After finitely many steps, we reach

$$\widehat{u} \in H^{k+m}_{\mathrm{loc}}(\Omega_0, \mathbb{K}^r).$$

Lemma 9.1.44, applied in the frame $\mathbf{e}_1, \dots, \mathbf{e}_r$, identifies the Sobolev regularity of a section with that of its local components. Thus

$$\mathbf{u} \in H^{k+m}_{\mathrm{loc}}(M, \mathbf{E})$$

on a neighborhood of x_0. Since x_0 was arbitrary, the first assertion is proved.
Finally, suppose that $P\mathbf{u}$ is smooth. Then $P\mathbf{u} \in H^k_{\mathrm{loc}}(M, \mathbf{F})$ for every $k \in \mathbb{N}_0$, and the first part gives

$$\mathbf{u} \in H^{k+m}_{\mathrm{loc}}(M, \mathbf{E}) \qquad \text{for every } k \in \mathbb{N}_0.$$

Fix $\ell \in \mathbb{N}_0$ and choose k so that $k+m > \ell + \dfrac{n}{2}$. In a chart and a local frame, the Euclidean Sobolev embedding gives

$$H^{k+m}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{K}^r) \hookrightarrow C^{\ell}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{K}^r).$$

Thus $\mathbf{u} \in C^{\ell}(\mathbf{E})$. Since ℓ was arbitrary, we conclude that $\mathbf{u}$ is smooth. □

Remark 11.4.3. The geometric part of the proof consists of using a chart and local frames to identify the symbol of the system with the intrinsic symbol of the operator. Once this identification is made, the analytic argument is the same for functions and sections: the matrices A_α couple the components, but uniform invertibility of the symbol on the covector sphere gives the same Fourier estimate.

11.4.1 Minimal and maximal realizations on a closed manifold

Let $P \in \mathbf{PDO}^{(m)}(\mathbf{E}, \mathbf{F})$ and suppose that M is closed. Define the initial realization

$$P_0 \colon \Gamma(\mathbf{E}) \subseteq L^2(M, \mathbf{E}) \longrightarrow L^2(M, \mathbf{F}), \qquad P_0\mathbf{u} := P\mathbf{u}.$$

Since $\Gamma(\mathbf{E})$ is dense, P_0 is densely defined. Its closure, when it exists, is called the minimal realization and is denoted by $P_{\min}$. The maximal realization is defined through the distributional extension of P:

$$\mathcal{D}(P_{\max}) := \{\mathbf{u} \in L^2(M, \mathbf{E}) \mid P\mathbf{u} \in L^2(M, \mathbf{F}) \text{ in the distributional sense}\}, \qquad P_{\max}\mathbf{u} := P\mathbf{u}.$$

Theorem 11.4.4. (Domain of an elliptic operator on a closed manifold) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, let $\mathbf{E},\mathbf{F}\to M$ be smooth vector bundles equipped with bundle metrics, and let $P\in\mathbf{PDO}^{(m)}(\mathbf{E},\mathbf{F})$ be elliptic. Then P_0 is closable,

$$P_{\min}=P_{\max},\qquad \mathcal{D}(P_{\min})=\mathcal{D}(P_{\max})=H^m(M,\mathbf{E}),$$

and there exist constants $c_P,C_P>0$, depending on P and the fixed geometric data, such that

$$c_P\|\mathbf{u}\|_{H^m(M,\mathbf{E})}\le\|\mathbf{u}\|_{L^2(M,\mathbf{E})}+\|P_{\max}\mathbf{u}\|_{L^2(M,\mathbf{F})}\le C_P\|\mathbf{u}\|_{H^m(M,\mathbf{E})}$$

for every $\mathbf{u}\in H^m(M,\mathbf{E})$. In particular, every closed realization of P lying between the minimal and maximal realizations agrees with them.

Proof. Let $\mathbf{u}\in\mathcal{D}(P_{\max})$. Then $\mathbf{u}\in L^2(M,\mathbf{E})$ and $P\mathbf{u}\in L^2(M,\mathbf{F})$ distributionally. Theorem 11.4.2 with $k=0$ gives $\mathbf{u}\in H^m_{\mathrm{loc}}(M,\mathbf{E})$. Since M is compact, a finite cover by charts turns this local regularity into $\mathbf{u}\in H^m(M,\mathbf{E})$. Thus

$$\mathcal{D}(P_{\max})\subseteq H^m(M,\mathbf{E}).$$

The reverse inclusion follows from the continuity $P\colon H^m(M,\mathbf{E})\longrightarrow L^2(M,\mathbf{F})$ in Theorem 11.3.7. Therefore $\mathcal{D}(P_{\max})=H^m(M,\mathbf{E})$. Corollary 11.3.8 shows that this domain is complete in the graph norm; hence $P_{\max}$ is closed. Since $P_0\subseteq P_{\max}$, the initial operator is closable.

Now take $\mathbf{u}\in H^m(M,\mathbf{E})$. By density, there exists a sequence $(\mathbf{u}_j)\subseteq\Gamma(\mathbf{E})$ such that $\mathbf{u}_j\to\mathbf{u}$ in $H^m(M,\mathbf{E})$. Continuity of P gives $P\mathbf{u}_j\to P\mathbf{u}$ in $L^2(M,\mathbf{F})$. Thus $(\mathbf{u}_j,P\mathbf{u}_j)$ converges to $(\mathbf{u},P\mathbf{u})$ in $L^2(M,\mathbf{E})\times L^2(M,\mathbf{F})$, which shows that $\mathbf{u}\in\mathcal{D}(P_{\min})$ and $P_{\min}\mathbf{u}=P\mathbf{u}$. We obtain $H^m(M,\mathbf{E})\subseteq\mathcal{D}(P_{\min})$. Since we always have $P_{\min}\subseteq P_{\max}$, the two realizations agree.

Equivalence of the graph norm and the $H^m(M,\mathbf{E})$ norm is Corollary 11.3.8 with $k=0$. If $\widetilde{P}$ is any closed realization of P satisfying

$$P_{\min}\subseteq\widetilde{P}\subseteq P_{\max},$$

then the equalities already proved imply $P_{\min}=\widetilde{P}=P_{\max}$. This proves the last assertion. □

Corollary 11.4.5. (Closed realization of the connection Laplacian) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, and let $(\mathbf{E},\mathbf{h}_{\mathbf{E}},\nabla^{\mathbf{E}})$ be a real vector bundle with a bundle metric or a Hermitian bundle with a compatible connection. Consider

$$L_{\mathbf{E},0}:=(\nabla^{\mathbf{E}})^*_h\nabla^{\mathbf{E}}\colon\Gamma(\mathbf{E})\subset L^2(M,\mathbf{E})\longrightarrow L^2(M,\mathbf{E}).$$

Then

$$\overline{L_{\mathbf{E},0}}=L_{\mathbf{E},\min}=L_{\mathbf{E},\max}=L^*_{\mathbf{E},0},\qquad \mathcal{D}(\overline{L_{\mathbf{E},0}})=H^2(M,\mathbf{E}).\tag{11.6}$$

This realization is self-adjoint, nonnegative, and agrees with the Friedrichs realization. In particular, $L_{\mathbf{E},0}$ is essentially self-adjoint.

Proof. The differential operator $L_{\mathbf{E},0}$ is elliptic of order two and formally self-adjoint. Theorem 11.4.4 gives $L_{\mathbf{E},\min}=L_{\mathbf{E},\max}$ with domain $H^2(M,\mathbf{E})$. By the definition of the adjoint of a densely defined operator and

the formal adjoint identity,

$$\mathbf{v} \in \mathcal{D}(L^*_{\mathbf{E},0}) \iff L_{\mathbf{E},0}\mathbf{v} \in L^2(M,\mathbf{E}) \text{ distributionally}, \qquad L^*_{\mathbf{E},0}\mathbf{v} = L_{\mathbf{E},0}\mathbf{v};$$

hence $L^*_{\mathbf{E},0} = L_{\mathbf{E},\max}$. Since the minimal realization is the closure of the initial operator, we obtain (11.6), and equality with its adjoint proves self-adjointness.
For $\mathbf{u} \in \Gamma(\mathbf{E})$,

$$\langle L_{\mathbf{E},0}\mathbf{u}, \mathbf{u}\rangle_{L^2(M,\mathbf{E})} = \|\nabla^{\mathbf{E}}\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})} \geq 0.$$

Taking the closure preserves nonnegativity. The Friedrichs realization is a self-adjoint extension of $L_{\mathbf{E},0}$ contained in $L^*_{\mathbf{E},0}$; since the minimal and maximal realizations agree, there is no distinct intermediate closed extension, and the Friedrichs realization is the one in (11.6). The equality $\overline{L_{\mathbf{E},0}} = L^*_{\mathbf{E},0}$ is precisely essential self-adjointness of the initial operator. □

11.5 Powers of the Bochner Laplacian

The equivalence in Theorem 9.1.42 was obtained by integration by parts and commutation identities for covariant derivatives. Elliptic theory allows us to interpret it as an equivalence of graph norms.

Corollary 11.5.1. (Powers of the Laplacian and Sobolev norms) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, and let $\mathbf{E} \to M$ be a smooth vector bundle with a real or Hermitian bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $L_{\mathbf{E}} = (\nabla^{\mathbf{E}})^*_h\nabla^{\mathbf{E}}$ be the realization in (11.6). For every $q \in \mathbb{N}$,

$$\mathcal{D}(L^q_{\mathbf{E}}) = H^{2q}(M,\mathbf{E}),$$

and there exist constants $c_q, C_q > 0$ such that

$$c_q\|\mathbf{u}\|_{H^{2q}(M,\mathbf{E})} \leq \|\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|L^q_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})} \leq C_q\|\mathbf{u}\|_{H^{2q}(M,\mathbf{E})}.$$

Moreover,

$$\bigcap_{q=1}^{\infty} \mathcal{D}(L^q_{\mathbf{E}}) = \Gamma(\mathbf{E}).$$

Proof. Proposition 11.1.5 and Example 11.1.10 show that the differential operator $((\nabla^{\mathbf{E}})^*_h\nabla^{\mathbf{E}})^q$ is elliptic of order $2q$. Theorem 11.4.4 gives its maximal domain $H^{2q}(M,\mathbf{E})$ and constants $c_q, C_q > 0$ such that

$$c_q\|\mathbf{u}\|_{H^{2q}(M,\mathbf{E})} \leq \|\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|L^q_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})} \leq C_q\|\mathbf{u}\|_{H^{2q}(M,\mathbf{E})}.$$

The operator $L_{\mathbf{E}}$ is nonnegative and self-adjoint. By the spectral theorem B.4.6, the power $L^q_{\mathbf{E}}$ is closed. On $\Gamma(\mathbf{E})$, it agrees with the differential power $L^q_{\mathbf{E},0}$ and therefore extends that initial operator. Moreover, if $\mathbf{u} \in \mathcal{D}(L^q_{\mathbf{E}})$, then $L^q_{\mathbf{E},0}\mathbf{u} = L^q_{\mathbf{E}}\mathbf{u} \in L^2(M,\mathbf{E})$ in the distributional sense; thus $L^q_{\mathbf{E}}$ is contained in the maximal realization of $L^q_{\mathbf{E},0}$. Theorem 11.4.4, applied to the elliptic operator $L^q_{\mathbf{E},0}$, states that its minimal and maximal realizations agree. Consequently,

$$(L^q_{\mathbf{E},0})_{\min} \subseteq L^q_{\mathbf{E}} \subseteq (L^q_{\mathbf{E},0})_{\max} = (L^q_{\mathbf{E},0})_{\min},$$

and the three realizations are equal. The domain identity and both estimates follow. The direct inequality can also be read from Theorem 9.1.42, where the $H^{2q}(M,\mathbf{E})$ norm is expressed in terms of powers of the Laplacian and intermediate covariant derivatives.
If $\mathbf{u}$ belongs to all the domains, then $\mathbf{u} \in H^{2q}(M,\mathbf{E})$ for every q. The local Sobolev embeddings imply $\mathbf{u} \in C^k(M,\mathbf{E})$ for each k, so $\mathbf{u}$ is smooth. The reverse inclusion is immediate. □

Remark 11.5.2. On a complete, noncompact manifold without boundary, local elliptic regularity remains valid, but a global estimate with a single constant requires uniform control of the geometry and the coefficients. Analogous estimates hold under bounded geometry hypotheses; we will return to this point in the corresponding chapter.

The principal symbol constructed here is also the symbol that enters index theory. If $\pi\colon T^*M \longrightarrow M$ is the projection, an elliptic operator determines an isomorphism over $T^*M \setminus 0_M$

$$\pi^*\mathbf{E} \longrightarrow \pi^*\mathbf{F}, \qquad (x,\xi,v) \longmapsto (x,\xi,\boldsymbol{\sigma}_m(P)(x,\xi)v).$$

The restriction of this isomorphism to the cosphere bundle contains the topological information that will be used later. This covariant formulation agrees with the one developed in [21, Definition 6.36 and Chapter 9].

12

Fractional Sobolev spaces on vector bundles and the heat kernel: an intrinsic approach

The integer-order Sobolev spaces studied in the preceding chapters are constructed using the covariant derivatives $\nabla^j \mathbf{u}$. When the order is no longer an integer, no finite family of derivatives by itself gives an intrinsic definition. One alternative is to choose a nonnegative elliptic operator, construct the semigroup it generates, and define its negative powers through a Laplace integral. This is the viewpoint of Bessel potentials.

The semigroup arises in solving the evolution problem

$$\partial_t \mathbf{u} + L_{\mathbf{E}} \mathbf{u} = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0.$$

Formally, its solution is $\mathbf{u}(t) = e^{-tL_{\mathbf{E}}} \mathbf{u}_0$. Integrating these operators with respect to t with the appropriate weight produces negative powers of $I + L_{\mathbf{E}}$ and, with them, a scale of fractional regularity. Heat thus links a parabolic PDE to the functional calculus of an elliptic operator.

The development proceeds on two levels. In $L^2(M, \mathbf{E})$, we have the Hilbert space structure, closed sesquilinear forms, and spectral calculus for self-adjoint operators. Once the heat semigroup has been constructed on this space, the Kato–Simon inequality allows us to compare it with the scalar semigroup and extend it consistently to $L^p(M, \mathbf{E})$. Bessel potentials are then defined by Bochner integrals. For scalar functions on complete manifolds, this program was developed by Strichartz; see [137, Theorems 3.5 and 4.2]. Here we formulate it directly for sections and prove the covariant steps involved in the construction.

Throughout this chapter, $(M, \mathbf{g})$ will be a connected, complete Riemannian manifold without boundary. The bundle $\mathbf{E} \to M$ will be a smooth real or complex vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$, Hermitian in the complex case, and $\nabla^{\mathbf{E}}$ will be a connection compatible with $\mathbf{h}_{\mathbf{E}}$. In the real case, spectral theory is applied to the complexification of $\mathbf{E}$; the Laplacian, semigroup, and Bessel potentials of real order preserve real sections. We write $L^p(M, \mathbf{E})$ for the spaces defined with respect to the Riemann–Lebesgue measure $\lambda_{\mathbf{g}}$. On Sobolev classes, $\nabla^j \mathbf{u}$ denotes the weak covariant derivative $\nabla^j_w \mathbf{u}$.

Heat as a tool for regularity

Since Fourier, the heat equation has served to separate and damp frequencies. On a manifold, the semigroup generated by a Laplacian retains this role, even though global translations are no longer available. Spectral theory constructs heat from the operator; its estimates reveal the geometry of the manifold; and integrals of the semigroup produce Bessel potentials and fractional orders. The same object thus connects evolution, regularization, and functional analysis.

12.1 The connection Laplacian on $L^2(M,\mathbf{E})$

It is useful to establish at the outset the relation between the two signs used in this book. For functions,

$$\Delta_{\mathbf{g}} = \operatorname{div}_{\mathbf{g}} \operatorname{grad}_{\mathbf{g}} = \operatorname{tr}_{\mathbf{g}}(\nabla^2)$$

has Fourier symbol $-|\xi|^2_{\mathbf{g}}$, whereas $d^*d = -\Delta_{\mathbf{g}}$ is nonnegative. For a compatible connection, we similarly define the covariant trace

$$\Delta_{\mathbf{E}}\mathbf{u} := \operatorname{tr}_{\mathbf{g}}(\nabla^2\mathbf{u}), \qquad (\nabla^{\mathbf{E}})^*_h\nabla^{\mathbf{E}} = -\Delta_{\mathbf{E}}$$

on compactly supported smooth sections; the trace contracts the first two derivative indices of $\nabla^2\mathbf{u}$. The operator $L_{\mathbf{E}}$ constructed below is the nonnegative realization of $(\nabla^{\mathbf{E}})^*_h\nabla^{\mathbf{E}}$; thus the equation $\partial_t\mathbf{u} + L_{\mathbf{E}}\mathbf{u} = 0$ is equivalent, in the smooth setting, to $\partial_t\mathbf{u} = \Delta_{\mathbf{E}}\mathbf{u}$.

12.1.1 The realization determined by the energy form

The Bochner Laplacian was defined in Definition 8.2.10. In this chapter, we use the notation

$$L_{\mathbf{E},0} := \Delta_B \restriction_{\Gamma_c(\mathbf{E})} = (\nabla^{\mathbf{E}})^*_h\nabla^{\mathbf{E}} = -\operatorname{tr}_{\mathbf{g}}(\nabla^2), \qquad \mathcal{D}(L_{\mathbf{E},0}) = \Gamma_c(\mathbf{E}),$$

for its initial restriction to compactly supported smooth sections. Its distributional action requires no new definition: it is obtained by applying Proposition 10.2.18 to the differential operator $L_{\mathbf{E},0}$. In particular, if $\mathbf{u} \in \mathcal{D}'(M,\mathbf{E})$, the expression $L_{\mathbf{E},0}\mathbf{u}$ has the meaning established in the chapter on distributions on vector bundles.

The local formula for the Bochner Laplacian, proved in the chapter on partial differential operators, gives

$$L_{\mathbf{E},0}\mathbf{u} = -\sum_{i=1}^{n}\left(\nabla^{\mathbf{E}}_{\mathbf{e}_i}\nabla^{\mathbf{E}}_{\mathbf{e}_i}\mathbf{u} - \nabla^{\mathbf{E}}_{\nabla^{TM}_{\mathbf{e}_i}\mathbf{e}_i}\mathbf{u}\right)$$

in any local orthonormal frame $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ of TM. Example 11.1.10 shows that its principal symbol is $-|\xi|^2_{\mathbf{g}}\operatorname{Id}_{\mathbf{E}_x}$; it is therefore elliptic.

Notation 12.1.1. (Connection Laplacian) In what follows, we call the preceding operator $L_{\mathbf{E},0}$ the **initial connection Laplacian**. The letter $L_{\mathbf{E}}$ is reserved for its self-adjoint realization on $L^2(M,\mathbf{E})$.

Corollary 12.1.2. (The closed case) If, in addition, M is closed, the elliptic results already proved give

$$\overline{L_{\mathbf{E},0}} = L_{\mathbf{E},\min} = L_{\mathbf{E},\max} = L^*_{\mathbf{E},0}, \qquad \mathcal{D}(\overline{L_{\mathbf{E},0}}) = H^2(M,\mathbf{E}).$$

The common realization is self-adjoint, nonnegative, and agrees with the Friedrichs realization. In particular, the initial operator $L_{\mathbf{E},0}$ is essentially self-adjoint.

Proof. This is Corollary 11.4.5, obtained from elliptic regularity, equality of the minimal and maximal realizations, and formal adjunction. □

Proposition 12.1.3. Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. The operator $L_{\mathbf{E},0}$ is densely defined, symmetric, and nonnegative. For any $\mathbf{u},\mathbf{v}\in\Gamma_c(\mathbf{E})$, we have

$$\langle L_{\mathbf{E},0}\mathbf{u},\mathbf{v}\rangle_{L^2(M,\mathbf{E})}=\int_M\langle\nabla^{\mathbf{E}}\mathbf{u},\nabla^{\mathbf{E}}\mathbf{v}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}=\langle\mathbf{u},L_{\mathbf{E},0}\mathbf{v}\rangle_{L^2(M,\mathbf{E})}.$$

In particular,

$$\langle L_{\mathbf{E},0}\mathbf{u},\mathbf{u}\rangle_{L^2(M,\mathbf{E})}=\|\nabla^{\mathbf{E}}\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})}.$$

Proof. Density of $\Gamma_c(\mathbf{E})$ in $L^2(M,\mathbf{E})$ was proved in the chapter on Sobolev spaces on bundles. The first equality is the integration-by-parts formula characterizing $(\nabla^{\mathbf{E}})^*_h$; the second follows by interchanging $\mathbf{u}$ and $\mathbf{v}$ and using the Hermitian symmetry of the inner product. Taking $\mathbf{v}=\mathbf{u}$ gives the last identity and hence nonnegativity. □

Define on $\Gamma_c(\mathbf{E})$ the form

$$\mathfrak{q}_{\mathbf{E}}(\mathbf{u},\mathbf{v}):=\int_M\langle\nabla^{\mathbf{E}}\mathbf{u},\nabla^{\mathbf{E}}\mathbf{v}\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}.$$

Its closure has domain $W^{1,2}_0(M,\mathbf{E})$. To identify this space with $W^{1,2}(M,\mathbf{E})$, we need cutoff functions whose gradients tend uniformly to zero. Before constructing them, we give a detailed proof of the existence of a smooth exhaustion function with bounded gradient.

Lemma 12.1.4. (Smoothing Lipschitz functions) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary. Let $f\colon M\longrightarrow\mathbb{R}$ be an L-Lipschitz function with respect to the Riemannian distance, and let $\varepsilon>0$. There exists $h\in C^\infty(M)$ such that

$$|h-f|<\varepsilon\qquad\text{and}\qquad|dh|_{\mathbf{g}}\le L+\varepsilon$$

throughout M.

Proof. If $L=0$, connectedness of M implies that f is constant, and it suffices to take $h=f$. Suppose that $L>0$ and fix

$$\theta:=\min\left\{\frac{1}{4},\frac{\varepsilon}{16L}\right\}.$$

For each $p\in M$, Proposition 2.8.10, applied with this value of θ, provides a regular normal coordinate ball U_p and a normal chart $\kappa_p\colon U_p\longrightarrow B_{\mathrm{euc}}(0,r_p)$ such that, for every $z\in B_{\mathrm{euc}}(0,r_p)$ and $w\in\mathbb{R}^n$,

$$(1-\theta)\|w\|\le\left|d(\kappa_p^{-1})_z w\right|_{\mathbf{g}}\le(1+\theta)\|w\|.$$

These balls cover M. Consider the family $\mathcal{B}$ of regular coordinate balls V for which $\overline{V} \subseteq U_p$ for some $p \in M$. This family is a basis: if $q \in O$ and O is open, choose p with $q \in U_p$ and, inside the Euclidean open subset $\kappa_p(O \cap U_p)$, two concentric balls centered at $\kappa_p(q)$ whose radii satisfy $0 < s < s'$ and whose closed ball of radius s' is contained in $\kappa_p(O \cap U_p)$. The preimage of the ball of radius s is a regular coordinate ball V such that $q \in V$ and $\overline{V} \subseteq O \cap U_p$, in accordance with Definition 1.2.9. Theorem 1.2.8 allows us to choose from $\mathcal{B}$ a countable, locally finite refinement $(V_j)_{j\in\mathbb{N}}$ of the cover $(U_p)_{p\in M}$. For each j, fix a normal ball U_j from the original cover such that $\overline{V}_j \subseteq U_j$, and denote its normal chart by $\kappa_j \colon U_j \longrightarrow B_{\mathrm{euc}}(0, r_j)$. Let $(\varphi_j)_{j\in\mathbb{N}}$ be a smooth partition of unity subordinate to $(V_j)_{j\in\mathbb{N}}$, whose existence is guaranteed by Theorem 1.3.5. Since $\operatorname{supp}(\varphi_j)$ is closed in M and contained in the compact set $\overline{V}_j$, the numbers

$$C_j := \|d\varphi_j\|_{L^\infty(M,T^*M)}, \qquad \delta_j := \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{2^{j+4}(1+C_j)}\right\}$$

are well defined, with $C_j \geq 0$ and $\delta_j > 0$.

Fix $j \in \mathbb{N}$ and write $F_j := f \circ \kappa_j^{-1}$. If $z, z' \in B_{\mathrm{euc}}(0, r_j)$, the segment joining them remains in this Euclidean ball. The curve $\gamma(t) := \kappa_j^{-1}((1-t)z + tz')$ lies in U_j and satisfies

$$d_{\mathbf{g}}\big(\kappa_j^{-1}(z), \kappa_j^{-1}(z')\big) \leq L_{\mathbf{g}}(\gamma) \leq (1+\theta)\|z - z'\|.$$

Since f is L-Lipschitz with respect to $d_{\mathbf{g}}$, it follows that

$$|F_j(z) - F_j(z')| \leq L(1+\theta)\|z - z'\|.$$

Thus F_j is L_j-Lipschitz, where $L_j := L(1+\theta)$. McShane's extension theorem, Theorem B.1.6, gives an extension $\widetilde{F}_j \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ with the same constant L_j. Take the mollifier ρ from Theorem B.9.1 and set

$$\tau_j := \min\left\{1, \frac{\delta_j}{2L_j m_1(\rho)}\right\}, \qquad f_j := \big(\widetilde{F}_j * \rho_{\tau_j}\big) \circ \kappa_j \quad \text{in } U_j.$$

The estimates in that theorem imply $|f_j - f| \leq \dfrac{\delta_j}{2}$ on U_j and $\|D(\widetilde{F}_j * \rho_{\tau_j})\| \leq L_j$. If $q \in U_j$, $z = \kappa_j(q)$, and $v \in T_qM$ satisfies $|v|_{\mathbf{g}} = 1$, the first metric inequality gives $\|d\kappa_j(v)\| \leq \dfrac{1}{1-\theta}$. Therefore

$$|df_j|_{\mathbf{g}} \leq \sup_{\substack{v\in T_qM \\ |v|_{\mathbf{g}}=1}} \Big|D(\widetilde{F}_j * \rho_{\tau_j})_z\big(d\kappa_j(v)\big)\Big| \leq \frac{L_j}{1-\theta} = L\frac{1+\theta}{1-\theta}.$$

Moreover,

$$L\frac{1+\theta}{1-\theta} - L = \frac{2L\theta}{1-\theta} \leq \frac{\varepsilon}{6}.$$

Indeed, if $\theta = \dfrac{\varepsilon}{16L}$, use $\theta \leq \dfrac{1}{4}$; if $\theta = \dfrac{1}{4}$, then $\varepsilon \geq 4L$ and the same bound follows.

Each product $\varphi_j f_j$, defined to be zero outside U_j, is smooth because $\operatorname{supp}(\varphi_j) \subseteq V_j \Subset U_j$. Define $h := \displaystyle\sum_{j=1}^{\infty} \varphi_j f_j$.

The sum is locally finite, so $h \in C^\infty(M)$. Since $\sum_{j=1}^{\infty} \varphi_j = 1$ and $\varphi_j \geq 0$,

$$|h - f| \leq \sum_{j=1}^{\infty} \varphi_j |f_j - f| \leq \frac{1}{2} \sum_{j=1}^{\infty} \varphi_j \delta_j \leq \frac{\varepsilon}{8} < \varepsilon.$$

The identity $\sum_{j=1}^{\infty} d\varphi_j = 0$ allows us to write, without differentiating the Lipschitz function f,

$$dh = \sum_{j=1}^{\infty} \varphi_j \, df_j + \sum_{j=1}^{\infty} (f_j - f) \, d\varphi_j.$$

By the preceding bounds and local finiteness,

$$|dh|_{\mathbf{g}} \leq L + \frac{\varepsilon}{6} + \frac{1}{2} \sum_{j=1}^{\infty} \delta_j C_j \leq L + \frac{\varepsilon}{6} + \frac{\varepsilon}{2} \sum_{j=1}^{\infty} 2^{-j-4} < L + \varepsilon.$$

This proves both assertions. □

Remark 12.1.5. The preceding proof does not require the balls U_j to be strongly geodesically convex. The convexity used is that of $\kappa_j(U_j) = B_{\mathrm{euc}}(0, r_j)$: the Euclidean segment determines a curve contained in U_j, and its length gives an upper bound for the Riemannian distance. Thus the local control in Proposition 2.8.10 is exactly the geometric result needed.

Proposition 12.1.6. (Exhaustion function with bounded gradient) Let $(M, \mathbf{g})$ be a connected, complete Riemannian manifold without boundary. There exists a strictly positive proper function $\rho \in C^\infty(M)$ such that

$$|d\rho|_{\mathbf{g}} \leq 2$$

throughout M.

Proof. Fix $o \in M$ and consider $r(x) := d_{\mathbf{g}}(o, x)$. The triangle inequality shows that r is 1-Lipschitz. Since M is complete, the Hopf–Rinow theorem, Theorem 2.9.9, implies that the closed balls $\overline{B}_{\mathbf{g}}(o, R)$ are compact; hence r is proper.

Apply Lemma 12.1.4 to r with $\varepsilon = 1$. We obtain $h \in C^\infty(M)$ such that $|h - r| < 1$ and $|dh|_{\mathbf{g}} \leq 2$. In particular, $|h(o)| < 1$ because $r(o) = 0$. Set $\rho := h - h(o) + 2$. This function is strictly positive because

$$\rho(x) \geq r(x) - 1 - h(o) + 2 \geq 1 - h(o) > 0,$$

, and it is proper: if $\rho(x) \leq R$, then $r(x) \leq R + h(o) - 1$, so the sublevel set $\{\rho \leq R\}$ is contained in a closed ball and is closed; by Theorem 2.9.9, it is compact. Finally, $d\rho = dh$, so $|d\rho|_{\mathbf{g}} \leq 2$. □

Lemma 12.1.7. (Cutoff functions on complete manifolds) Let $(M, \mathbf{g})$ be a connected, complete Riemannian manifold without boundary. There exists a sequence $(\chi_j)_{j \in \mathbb{N}} \subseteq C_c^\infty(M)$ such that $0 \leq \chi_j \leq 1$, $\chi_j \to 1$

uniformly on every compact set, and

$$\|d\chi_j\|_{L^\infty(M,T^*M)} \longrightarrow 0.$$

Proof. Let ρ be the function in Proposition 12.1.6. Choose a nonincreasing function $\eta \in C^\infty(\mathbb{R})$ such that $0 \le \eta \le 1$, $\eta(s) = 1$ for $s \le 1$, and $\eta(s) = 0$ for $s \ge 2$, and define

$$\chi_j(x) := \eta\left(\frac{\rho(x)}{j}\right).$$

The support of χ_j is contained in the sublevel set $\{\rho \le 2j\}$, which is compact because ρ is proper. Since $\rho > 0$ and η is nonincreasing, the sequence (χ_j) is increasing. If $K \subseteq M$ is compact and $R_K := \max_{x\in K} \rho(x)$, then $\chi_j = 1$ on K for every integer $j \ge \max\{1, \lceil R_K \rceil\}$. In particular, χ_j converges pointwise to 1. Finally,

$$|d\chi_j|_{\mathbf{g}} = \frac{1}{j}\left|\eta'\left(\frac{\rho}{j}\right)\right| |d\rho|_{\mathbf{g}} \le \frac{2\|\eta'\|_{L^\infty(\mathbb{R})}}{j},$$

which proves the asserted convergence. □

Proposition 12.1.8. Let $(M, \mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Then $W_0^{1,2}(M, \mathbf{E}) = W^{1,2}(M, \mathbf{E})$.

Proof. Let $\mathbf{u} \in W^{1,2}(M, \mathbf{E})$, and let (χ_j) be the functions in the preceding lemma. The weak Leibniz rule gives

$$\nabla^{\mathbf{E}}(\chi_j \mathbf{u}) = d\chi_j \otimes \mathbf{u} + \chi_j \nabla^{\mathbf{E}} \mathbf{u}.$$

Since $0 \le \chi_j \le 1$ and $\chi_j \to 1$ pointwise, we have

$$|(1-\chi_j)\mathbf{u}|^2_{\mathbf{h}_{\mathbf{E}}} \le |\mathbf{u}|^2_{\mathbf{h}_{\mathbf{E}}}, \qquad |(1-\chi_j)\nabla^{\mathbf{E}}\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} \le |\nabla^{\mathbf{E}}\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}.$$

Both dominating functions are integrable. Lebesgue's dominated convergence theorem B.6.14 implies

$$(1-\chi_j)\mathbf{u} \longrightarrow 0 \quad \text{in } L^2(M, \mathbf{E})$$

and

$$(1-\chi_j)\nabla^{\mathbf{E}}\mathbf{u} \longrightarrow 0 \quad \text{in } L^2(M, T^*M \otimes \mathbf{E}).$$

Moreover,

$$\|d\chi_j \otimes \mathbf{u}\|_{L^2(M,T^*M\otimes\mathbf{E})} \le \|d\chi_j\|_{L^\infty(M,T^*M)} \|\mathbf{u}\|_{L^2(M,\mathbf{E})} \longrightarrow 0.$$

Therefore $\chi_j \mathbf{u} \to \mathbf{u}$ in $W^{1,2}(M, \mathbf{E})$. For each j, the Meyers–Serrin theorem for bundles, Theorem 9.1.48, approximates $\chi_j \mathbf{u}$ in $W^{1,2}(M, \mathbf{E})$ by smooth sections $\mathbf{u}_{j,k}$. Choose $\zeta_j \in C_c^\infty(M)$ equal to 1 on a neighborhood of $\operatorname{supp}(\chi_j \mathbf{u})$. Since multiplication by ζ_j is continuous on $W^{1,2}(M, \mathbf{E})$, we have $\zeta_j \mathbf{u}_{j,k} \to \chi_j \mathbf{u}$, and now $\zeta_j \mathbf{u}_{j,k} \in \Gamma_c(\mathbf{E})$. Thus $\mathbf{u} \in W_0^{1,2}(M, \mathbf{E})$. □

The closure of $\mathfrak{q}_{\mathbf{E}}$ is therefore a densely defined, symmetric, nonnegative, closed sesquilinear form with domain $W^{1,2}(M, \mathbf{E})$.

Theorem 12.1.9. (Friedrichs realization of the connection Laplacian) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. There exists a unique nonnegative self-adjoint operator

$$L_{\mathbf{E}}\colon \mathcal{D}(L_{\mathbf{E}})\subseteq L^2(M,\mathbf{E})\longrightarrow L^2(M,\mathbf{E})$$

such that $\mathcal{D}(L_{\mathbf{E}})\subseteq W^{1,2}(M,\mathbf{E})$ and

$$\mathfrak{q}_{\mathbf{E}}(\mathbf{u},\mathbf{v})=\langle L_{\mathbf{E}}\mathbf{u},\mathbf{v}\rangle_{L^2(M,\mathbf{E})},\qquad \mathbf{u}\in\mathcal{D}(L_{\mathbf{E}}),\quad \mathbf{v}\in W^{1,2}(M,\mathbf{E}).$$

Its domain is

$$\mathcal{D}(L_{\mathbf{E}})=\left\{\mathbf{u}\in W^{1,2}(M,\mathbf{E})\;\middle|\;\begin{array}{l}\text{there exists } \mathbf{f}\in L^2(M,\mathbf{E}) \text{ such that}\\ \mathfrak{q}_{\mathbf{E}}(\mathbf{u},\mathbf{v})=\langle \mathbf{f},\mathbf{v}\rangle_{L^2(M,\mathbf{E})} \text{ for every } \mathbf{v}\in W^{1,2}(M,\mathbf{E})\end{array}\right\},$$

and, in this case, $L_{\mathbf{E}}\mathbf{u}=\mathbf{f}$. Moreover, $L_{\mathbf{E}}$ extends $L_{\mathbf{E},0}$.

Proof. Existence, uniqueness, and the characterization of the domain follow from Theorem B.4.2. If $\mathbf{u}\in\Gamma_c(\mathbf{E})$, Proposition 12.1.3 gives

$$\mathfrak{q}_{\mathbf{E}}(\mathbf{u},\mathbf{v})=\langle L_{\mathbf{E},0}\mathbf{u},\mathbf{v}\rangle_{L^2(M,\mathbf{E})}$$

for $\mathbf{v}\in\Gamma_c(\mathbf{E})$. Both sides depend continuously on $\mathbf{v}$ in the $W^{1,2}(M,\mathbf{E})$ norm, and $\Gamma_c(\mathbf{E})$ is dense in that space by Proposition 12.1.8. The identity extends to all of $\mathbf{v}\in W^{1,2}(M,\mathbf{E})$, which proves that $\mathbf{u}\in\mathcal{D}(L_{\mathbf{E}})$ and $L_{\mathbf{E}}\mathbf{u}=L_{\mathbf{E},0}\mathbf{u}$. □

Theorem 12.1.10. (Essential self-adjointness) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. The operator $L_{\mathbf{E},0}$ is essentially self-adjoint on $L^2(M,\mathbf{E})$, and its closure agrees with $L_{\mathbf{E}}$.

Proof. Since $L_{\mathbf{E},0}$ is symmetric and nonnegative, Proposition B.4.5 reduces the proof to verifying that $\ker(L_{\mathbf{E},0}^{*}+I)=\{0\}$. Let $\mathbf{u}$ belong to this kernel. Then

$$L_{\mathbf{E},0}\mathbf{u}=-\mathbf{u}$$

in the distributional sense. Theorem 11.4.2, applied to the elliptic operator $L_{\mathbf{E},0}+I$, implies that $\mathbf{u}$ is smooth. Take the cutoff functions (χ_j) from Lemma 12.1.7. The Leibniz identities

$$\nabla^{\mathbf{E}}(\chi_j\mathbf{u})=d\chi_j\otimes\mathbf{u}+\chi_j\nabla^{\mathbf{E}}\mathbf{u}$$

and

$$\nabla^{\mathbf{E}}(\chi_j^2\mathbf{u})=2\chi_j d\chi_j\otimes\mathbf{u}+\chi_j^2\nabla^{\mathbf{E}}\mathbf{u}$$

give, after expanding and taking real parts,

$$|\nabla^{\mathbf{E}}(\chi_j\mathbf{u})|^2_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}=\operatorname{Re}\langle\nabla^{\mathbf{E}}\mathbf{u},\nabla^{\mathbf{E}}(\chi_j^2\mathbf{u})\rangle_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}+|d\chi_j\otimes\mathbf{u}|^2_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}.$$

The section $\chi_j^2\mathbf{u}$ is smooth and compactly supported. Integrating and using the definition of the distributional adjoint,

$$\|\nabla^{\mathbf{E}}(\chi_j\mathbf{u})\|^2_{L^2(M,T^*M\otimes\mathbf{E})} = \mathrm{Re}\langle L^*_{\mathbf{E},0}\mathbf{u}, \chi_j^2\mathbf{u}\rangle_{L^2(M,\mathbf{E})} + \|d\chi_j\otimes\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})}.$$

Since $L^*_{\mathbf{E},0}\mathbf{u} = -\mathbf{u}$,

$$0 \le \|\nabla^{\mathbf{E}}(\chi_j\mathbf{u})\|^2_{L^2(M,T^*M\otimes\mathbf{E})} = -\|\chi_j\mathbf{u}\|^2_{L^2(M,\mathbf{E})} + \|d\chi_j\otimes\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})}.$$

Therefore

$$\|\chi_j\mathbf{u}\|^2_{L^2(M,\mathbf{E})} \le \|d\chi_j\|^2_{L^\infty(M,T^*M)}\|\mathbf{u}\|^2_{L^2(M,\mathbf{E})} \longrightarrow 0.$$

On the other hand, $|\chi_j\mathbf{u}-\mathbf{u}|^2_{\mathbf{h}_\mathbf{E}} = |1-\chi_j|^2|\mathbf{u}|^2_{\mathbf{h}_\mathbf{E}} \le |\mathbf{u}|^2_{\mathbf{h}_\mathbf{E}}$, the dominating function belongs to $L^1(M)$, and $\chi_j\mathbf{u}\to\mathbf{u}$ pointwise. Lebesgue's dominated convergence theorem B.6.14 then gives $\|\chi_j\mathbf{u}-\mathbf{u}\|_{L^2(M,\mathbf{E})}\to 0$. Since at the same time $\|\chi_j\mathbf{u}\|_{L^2(M,\mathbf{E})}\to 0$, it follows that $\mathbf{u}=0$. □

12.1.2 The spectral semigroup and the scalar semigroup

For $t\ge 0$, define

$$P_t^{\mathbf{E}} := e^{-tL_{\mathbf{E}}}$$

by spectral calculus. On the trivial bundle of rank one, we write $L = \Delta_B = -\Delta$ for the positive scalar realization and $P_t = e^{-tL}$.

Theorem 12.1.11. (Covariant heat semigroup on L^2) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. The family $(P_t^{\mathbf{E}})_{t\ge0}$ satisfies:

(a) $P_0^{\mathbf{E}} = I$ and $P_{t+s}^{\mathbf{E}} = P_t^{\mathbf{E}}P_s^{\mathbf{E}}$ for $s,t\ge 0$;

(b) each $P_t^{\mathbf{E}}$ is self-adjoint and $\|P_t^{\mathbf{E}}\|_{\mathcal{L}(L^2(M,\mathbf{E}))}\le 1$;

(c) $P_t^{\mathbf{E}}\mathbf{u}\to\mathbf{u}$ in $L^2(M,\mathbf{E})$ as $t\to 0^+$;

(d) if $t>0$, then $P_t^{\mathbf{E}}\mathbf{u}\in\mathcal{D}(L_{\mathbf{E}}^m)$ for every $m\in\mathbb{N}$;

(e) for $\mathbf{u}\in L^2(M,\mathbf{E})$, the map $t\mapsto P_t^{\mathbf{E}}\mathbf{u}$ is differentiable in $(0,\infty)$ and

$$\frac{d}{dt}P_t^{\mathbf{E}}\mathbf{u} = -L_{\mathbf{E}}P_t^{\mathbf{E}}\mathbf{u};$$

if $\mathbf{u}\in\mathcal{D}(L_{\mathbf{E}})$, the right derivative at $t=0$ exists and equals $-L_{\mathbf{E}}\mathbf{u}$.

Proof. This is Proposition B.4.9 applied to $A = L_{\mathbf{E}}$. For regularization, it suffices to observe that, if $t>0$, the function $\lambda\mapsto\lambda^m e^{-t\lambda}$ is bounded on $[0,\infty)$. □

Before comparing the scalar and covariant semigroups, we need positivity of the scalar resolvent.

Lemma 12.1.12. (Positivity and uniform bound for the scalar resolvent) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary. Let $\lambda>0$ and let $R_\lambda = (\lambda I + L)^{-1}$. Then:

(a) if $f\in L^2(M)$ is real and $f\ge 0$, then $R_\lambda f\ge 0$ almost everywhere;

(b) if $f \in L^2(M)\cap L^\infty(M)$, then

$$\|R_\lambda f\|_{L^\infty(M)} \le \frac{1}{\lambda}\|f\|_{L^\infty(M)}.$$

Proof. Let $u = R_\lambda f$. The variational characterization of the resolvent, Corollary B.4.3, gives

$$\int_M \langle du, d\varphi\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} + \lambda \int_M u\varphi\, d\lambda_{\mathbf{g}} = \int_M f\varphi\, d\lambda_{\mathbf{g}}$$

for every $\varphi \in W^{1,2}(M)$.

First suppose that $f \ge 0$ and take $u_- := \max\{-u, 0\}$. Theorem 9.1.49, applied to the negative part, gives $u_- \in W^{1,2}(M)$ and

$$du_- = -\mathbf{1}_{\{u<0\}}\, du$$

almost everywhere. Choosing $\varphi = u_-$ yields

$$-\|du_-\|^2_{L^2(M,T^*M)} - \lambda\|u_-\|^2_{L^2(M)} = \int_M f u_-\, d\lambda_{\mathbf{g}} \ge 0.$$

The left-hand side is nonpositive; therefore $u_- = 0$.

For the uniform bound, we may first assume that f is real. The resolvent preserves real-valued functions because its defining form has real coefficients. Let $c = \frac{\|f\|_{L^\infty(M)}}{\lambda}$. If $c = 0$, the first part applied to u and $-u$ gives $u = 0$. Thus suppose that $c > 0$ and set $z = (u-c)_+$. The function z belongs to $W^{1,2}(M)$; moreover, the set $\{u > c\}$ has finite measure because $u \in L^2(M)$, and the Cauchy–Schwarz inequality in Theorem B.3.2 gives $z \in L^1(M)$. On $\{u > c\}$, we have $u = z + c$, whereas $dz = du$. Taking $\varphi = z$,

$$\|dz\|^2_{L^2(M,T^*M)} + \lambda\|z\|^2_{L^2(M)} + \lambda c\int_M z\, d\lambda_{\mathbf{g}} = \int_M fz\, d\lambda_{\mathbf{g}} \le \lambda c\int_M z\, d\lambda_{\mathbf{g}}.$$

Thus $z = 0$ and $u \le c$. Applying the same argument to $-u$ gives $|u| \le c$. Now let f be complex-valued, and write again $u = R_\lambda f$. The resolvent preserves real-valued functions and is positive. For each $\theta \in \mathbb{R}$,

$$\mathrm{Re}(e^{-i\theta}u) = R_\lambda\big(\mathrm{Re}(e^{-i\theta}f)\big) \le R_\lambda|f|.$$

The identity $|z| = \sup_{\theta\in 2\pi\mathbb{Q}} \mathrm{Re}(e^{-i\theta}z)$, valid for every $z \in \mathbb{C}$, and countability of $2\pi\mathbb{Q}$ allow us to take the supremum outside a single null set. Thus $|R_\lambda f| \le R_\lambda|f|$ almost everywhere. Applying the real-valued bound already proved to $|f|$ yields $|u| \le \frac{1}{\lambda}\|f\|_{L^\infty(M)}$. □

Corollary 12.1.13. (Sub-Markov property of the scalar semigroup) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary. The semigroup $(P_t)_{t\ge 0}$ preserves positivity. Moreover, for $f \in L^2(M)\cap L^\infty(M)$,

$$\|P_t f\|_{L^\infty(M)} \le \|f\|_{L^\infty(M)}.$$

Proof. The exponential formula in Proposition B.4.16, applied to the generator $-L$, gives

$$P_t f = \lim_{n\to\infty}\left(I + \frac{t}{n}L\right)^{-n} f$$

in $L^2(M)$. Each factor is

$$\left(I + \frac{t}{n}L\right)^{-1} = \frac{n}{t}\left(\frac{n}{t}I + L\right)^{-1}$$

and, by the preceding lemma, preserves positivity and is a contraction on $L^\infty(M)$. The iterates have the same properties. From convergence in $L^2(M)$, extract a subsequence converging almost everywhere; passage to the limit preserves positivity and the pointwise bound. □

12.1.3 The Kato–Simon inequality

The weak Kato inequality, Corollary 9.5.7, asserts that if $\mathbf{u} \in W^{1,2}(M,\mathbf{E})$, then $|\mathbf{u}|_{\mathbf{h_E}} \in W^{1,2}(M)$ and

$$|d|\mathbf{u}|_{\mathbf{h_E}}|_{\mathbf{g}} \le |\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}$$

almost everywhere. To obtain semigroup domination, we first prove a resolvent inequality.

Lemma 12.1.14. (Resolvent domination) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. For $\lambda > 0$ and $\mathbf{f} \in L^2(M,\mathbf{E})$,

$$\left|(\lambda I + L_{\mathbf{E}})^{-1}\mathbf{f}\right|_{\mathbf{h_E}} \le (\lambda I + L)^{-1}|\mathbf{f}|_{\mathbf{h_E}}$$

almost everywhere.

Proof. Set

$$\mathbf{u} := (\lambda I + L_{\mathbf{E}})^{-1}\mathbf{f}$$

and, for $\varepsilon > 0$,

$$a_\varepsilon := \left(|\mathbf{u}|^2_{\mathbf{h_E}} + \varepsilon^2\right)^{\frac{1}{2}}.$$

The function a_ε belongs to $W^{1,2}_{\mathrm{loc}}(M)$, and

$$da_\varepsilon(\mathbf{X}) = \frac{\mathrm{Re}\langle\nabla^{\mathbf{E}}_{\mathbf{X}}\mathbf{u},\mathbf{u}\rangle_{\mathbf{h_E}}}{a_\varepsilon}$$

for almost every point and every vector field $\mathbf{X}$. In particular,

$$|da_\varepsilon|_{\mathbf{g}} \le |\nabla^{\mathbf{E}}\mathbf{u}|_{\mathbf{g},\mathbf{h_E}}.$$

Let $0 \le \varphi \in C^\infty_c(M)$. The section

$$\mathbf{v}_\varepsilon := \frac{\varphi}{a_\varepsilon}\mathbf{u}$$

belongs to $W^{1,2}(M,\mathbf{E})$ and has compact support. The variational resolvent equation is the one in Corollary B.4.3 applied to the form $\mathfrak{q}_{\mathbf{E}}$:

$$\mathfrak{q}_{\mathbf{E}}(\mathbf{u},\mathbf{v}_\varepsilon) + \lambda\langle\mathbf{u},\mathbf{v}_\varepsilon\rangle_{L^2(M,\mathbf{E})} = \langle\mathbf{f},\mathbf{v}_\varepsilon\rangle_{L^2(M,\mathbf{E})}.$$

The Leibniz rule gives

$$\nabla^{\mathbf{E}} \mathbf{v}_\varepsilon = \frac{d\varphi}{a_\varepsilon} \otimes \mathbf{u} + \frac{\varphi}{a_\varepsilon} \nabla^{\mathbf{E}} \mathbf{u} - \frac{\varphi}{a_\varepsilon^2} da_\varepsilon \otimes \mathbf{u}.$$

Contracting with $\nabla^{\mathbf{E}} \mathbf{u}$ and taking real parts, the first term produces $\langle da_\varepsilon, d\varphi \rangle_{\mathbf{g}}$. For the other two terms, we obtain

$$\operatorname{Re} \left\langle \nabla^{\mathbf{E}} \mathbf{u}, \frac{\varphi}{a_\varepsilon} \nabla^{\mathbf{E}} \mathbf{u} - \frac{\varphi}{a_\varepsilon^2} da_\varepsilon \otimes \mathbf{u} \right\rangle_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} = \frac{\varphi}{a_\varepsilon} \left(|\nabla^{\mathbf{E}} \mathbf{u}|^2_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} - |da_\varepsilon|^2_{\mathbf{g}} \right) \geq 0.$$

Therefore

$$\operatorname{Re} \mathfrak{q}_{\mathbf{E}}(\mathbf{u}, \mathbf{v}_\varepsilon) \geq \int_M \langle da_\varepsilon, d\varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}.$$

Moreover,

$$\operatorname{Re} \langle \mathbf{u}, \mathbf{v}_\varepsilon \rangle_{\mathbf{h}_{\mathbf{E}}} = \varphi \frac{|\mathbf{u}|^2_{\mathbf{h}_{\mathbf{E}}}}{a_\varepsilon} = \varphi \left(a_\varepsilon - \frac{\varepsilon^2}{a_\varepsilon} \right),$$

and

$$\operatorname{Re} \langle \mathbf{f}, \mathbf{v}_\varepsilon \rangle_{\mathbf{h}_{\mathbf{E}}} \leq \varphi |\mathbf{f}|_{\mathbf{h}_{\mathbf{E}}} \frac{|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}}}{a_\varepsilon} \leq \varphi |\mathbf{f}|_{\mathbf{h}_{\mathbf{E}}}.$$

Substituting these estimates into the variational equation,

$$\int_M \langle da_\varepsilon, d\varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} + \lambda \int_M a_\varepsilon \varphi \, d\lambda_{\mathbf{g}} \leq \int_M |\mathbf{f}|_{\mathbf{h}_{\mathbf{E}}} \varphi \, d\lambda_{\mathbf{g}} + \lambda \varepsilon \int_M \varphi \, d\lambda_{\mathbf{g}},$$

since $\frac{\varepsilon^2}{a_\varepsilon} \leq \varepsilon$.

Now let $\varepsilon \to 0^+$. We have $a_\varepsilon \to |\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}}$ pointwise and $|a_\varepsilon - |\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}}| \leq \varepsilon$, so the zeroth-order term converges. Likewise,

$$da_\varepsilon = \frac{\operatorname{Re} \langle \nabla^{\mathbf{E}} \mathbf{u}, \mathbf{u} \rangle_{\mathbf{h}_{\mathbf{E}}}}{a_\varepsilon} \longrightarrow d|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}}$$

almost everywhere; on the set where $\mathbf{u} = 0$, the weak differential of $|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}}$ vanishes almost everywhere. Since $|da_\varepsilon|_{\mathbf{g}} \leq |\nabla^{\mathbf{E}} \mathbf{u}|_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}$ and $d\varphi$ is bounded and compactly supported, Lebesgue's dominated convergence theorem B.6.14 gives

$$\int_M \langle d|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}}, d\varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} + \lambda \int_M |\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} \varphi \, d\lambda_{\mathbf{g}} \leq \int_M |\mathbf{f}|_{\mathbf{h}_{\mathbf{E}}} \varphi \, d\lambda_{\mathbf{g}}.$$

Let

$$w := (\lambda I + L)^{-1} |\mathbf{f}|_{\mathbf{h}_{\mathbf{E}}}.$$

Subtracting the weak equation for w, we wish to use the test function

$$z := (|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} - w)_+.$$

Theorem 9.1.49, applied to $|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} - w$ and the function $t \mapsto t_+$, shows that $z \in W^{1,2}(M)$ and that

$$dz = \mathbf{1}_{\{|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} > w\}} (d|\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}} - dw)$$

almost everywhere. By Proposition 12.1.8, $W^{1,2}(M) = W^{1,2}_0(M)$. To use z as a test function, we combine smooth approximation with nonnegative truncations. Choose $\varphi_j \in C^\infty_c(M)$ with $\varphi_j \to z$ in $W^{1,2}(M)$ and, after passing to a subsequence, assume that $\varphi_j \to z$ almost everywhere. For $\delta > 0$, let $\vartheta_\delta \in C^\infty(\mathbb{R})$ be a

convex function such that

$$\vartheta_\delta(t) = 0 \quad \text{if } t \le 0, \qquad 0 \le \vartheta_\delta'(t) \le 1, \qquad |\vartheta_\delta(t) - t_+| \le \delta.$$

These functions are obtained by integrating a smooth nondecreasing approximation to the indicator function of $(0, +\infty)$. Choose $\delta_j \to 0^+$ so that

$$\delta_j \, \lambda_{\mathbf{g}}(\operatorname{supp} \varphi_j)^{\frac{1}{2}} \longrightarrow 0$$

and define $z_j := \vartheta_{\delta_j}(\varphi_j)$. Then $z_j \in C_c^\infty(M)$, $z_j \ge 0$, and

$$\|z_j - z\|_{L^2(M)} \le \|\varphi_j - z\|_{L^2(M)} + \delta_j \lambda_{\mathbf{g}}(\operatorname{supp} \varphi_j)^{\frac{1}{2}} \longrightarrow 0.$$

The chain rule gives $dz_j = \vartheta_{\delta_j}'(\varphi_j) d\varphi_j$. Therefore

$$|dz_j - dz|_{\mathbf{g}} \le |d\varphi_j - dz|_{\mathbf{g}} + |\vartheta_{\delta_j}'(\varphi_j) - \mathbf{1}_{\{z>0\}}| |dz|_{\mathbf{g}}.$$

The first term converges to zero in $L^2(M, T^*M)$. In the second, the factor multiplying $|dz|_{\mathbf{g}}$ converges to zero almost everywhere: this is immediate on $\{z > 0\}$, whereas $dz = 0$ almost everywhere on the level set $\{z = 0\}$. Since this factor is bounded by 1, Lebesgue's dominated convergence theorem B.6.14 shows that $dz_j \to dz$ in $L^2(M, T^*M)$. Thus $z_j \to z$ in $W^{1,2}(M)$.

The preceding weak inequality is continuous with respect to this norm, since $d|\mathbf{u}|_{\mathbf{h_E}} - dw \in L^2(M, T^*M)$ and $\lambda(|\mathbf{u}|_{\mathbf{h_E}} - w) \in L^2(M)$. We may therefore apply it to z_j and pass to the limit. On the set $\{z > 0\}$, we have $|\mathbf{u}|_{\mathbf{h_E}} - w = z$, and obtain

$$\int_M |dz|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} + \lambda \int_M z^2 \, d\lambda_{\mathbf{g}} \le 0.$$

Both terms are nonnegative, so $z = 0$ almost everywhere. This is equivalent to $|\mathbf{u}|_{\mathbf{h_E}} \le w$. □

Theorem 12.1.15. (Kato–Simon inequality) Let $(M, \mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. For every $t \ge 0$ and every $\mathbf{u} \in L^2(M, \mathbf{E})$,

$$|P_t^{\mathbf{E}} \mathbf{u}|_{\mathbf{h_E}} \le P_t(|\mathbf{u}|_{\mathbf{h_E}})$$

almost everywhere.

Proof. For $\tau > 0$, the preceding lemma with $\lambda = \frac{1}{\tau}$ is equivalent to

$$\left| (I + \tau L_{\mathbf{E}})^{-1} v \right|_{\mathbf{h_E}} \le (I + \tau L)^{-1} |v|_{\mathbf{h_E}}.$$

Fix $\tau > 0$ and write $R_{\mathbf{E}} = (I + \tau L_{\mathbf{E}})^{-1}$ and $R = (I + \tau L)^{-1}$. The preceding inequality is the case $k = 1$ of $|R_{\mathbf{E}}^k \mathbf{u}| \le R^k |\mathbf{u}|$. If it holds for $k \in \mathbb{N}$, apply it first to $R_{\mathbf{E}}^k \mathbf{u}$ and then use the fact that R preserves positivity:

$$|R_{\mathbf{E}}^{k+1} \mathbf{u}| \le R\big(|R_{\mathbf{E}}^k \mathbf{u}|\big) \le R\big(R^k |\mathbf{u}|\big) = R^{k+1} |\mathbf{u}|.$$

This proves the inequality for each $k \in \mathbb{N}$. Given $t > 0$, take $\tau = t/N$ and $k = N$, with $N \in \mathbb{N}$, to obtain

$$\left| \left(I + \frac{t}{N} L_{\mathbf{E}} \right)^{-N} \mathbf{u} \right|_{\mathbf{h_E}} \leq \left(I + \frac{t}{N} L \right)^{-N} |\mathbf{u}|_{\mathbf{h_E}}.$$

Proposition B.4.16 shows that the left-hand side converges in $L^2(M, \mathbf{E})$ to $P_t^{\mathbf{E}} \mathbf{u}$ and the right-hand side in $L^2(M)$ to $P_t |\mathbf{u}|_{\mathbf{h_E}}$. Choose a common subsequence converging almost everywhere and pass to the limit in the pointwise inequality. The resulting inequality is the assertion for $t > 0$; at $t = 0$, both sides agree. □

12.2 The covariant heat kernel

The kernel $\mathbf{K}_t^{\mathbf{E}}$ is a section of $\mathbf{E} \boxtimes \mathbf{E}^* \to M_x \times M_y$, equipped with the product connection. In every differential calculation, we distinguish the variables by writing

$$\nabla_x^r \nabla_y^s \mathbf{K}_t^{\mathbf{E}}.$$

After fixing charts or frames, we write its components as

$$\left(\nabla_x^r \nabla_y^s \mathbf{K}_t^{\mathbf{E}} \right)_{i_1 \cdots i_r \, ; \, j_1 \cdots j_s},$$

where the semicolon separates the indices of x from those of y. Each group follows the recursive convention in Chapter 9; in particular, these derivatives will not be mixed with undeclared indexed compositions. We denote by

$$\Delta_{B,\mathbf{E}}^{(x)} \quad \text{and} \quad \Delta_{B,\mathbf{E}^*}^{(y)}$$

the positive Bochner Laplacians acting, respectively, on the factor $\mathbf{E}_x$ and the dual factor $\mathbf{E}_y^*$ of the product connection.

12.2.1 Regularization and continuity of evaluations

The essential step in constructing the kernel is that $P_t^{\mathbf{E}}$ transforms sections in $L^2(M, \mathbf{E})$ into smooth sections when $t > 0$. This property follows directly from spectral calculus and the elliptic regularity developed in the preceding chapter.

Lemma 12.2.1. (Semigroup regularization) Let $(M, \mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $t > 0$, $k \in \mathbb{N}_0$, and let $K \subseteq M$ be compact. The operator $P_t^{\mathbf{E}}$ induces a continuous linear map

$$P_t^{\mathbf{E}} \colon L^2(M, \mathbf{E}) \longrightarrow C^k(K, \mathbf{E}).$$

In particular, $P_t^{\mathbf{E}} \mathbf{u}$ has a smooth representative for every $\mathbf{u} \in L^2(M, \mathbf{E})$.

Proof. Set $\mathbf{v} = P_t^{\mathbf{E}} \mathbf{u}$. Spectral calculus gives $\mathbf{v} \in \mathcal{D}(L_{\mathbf{E}}^m)$ for every $m \in \mathbb{N}$ and

$$L_{\mathbf{E},0}^j \mathbf{v} = L_{\mathbf{E}}^j \mathbf{v} \in L^2(M, \mathbf{E}), \qquad j \in \mathbb{N}_0,$$

where the first equality is distributional. Let us see how local regularity follows. If $\mathbf{v}, L_{\mathbf{E},0}\mathbf{v} \in L^2$, Theorem 11.4.2 gives $\mathbf{v} \in H^2_{\mathrm{loc}}$. Suppose that membership of the powers up to order m in L^2 already implies regularity H^{2m}_{loc}. If powers up to order $m+1$ also exist, apply this assertion to $L_{\mathbf{E},0}\mathbf{v}$ to obtain $L_{\mathbf{E},0}\mathbf{v} \in H^{2m}_{\mathrm{loc}}$. The same elliptic regularity theorem, now with right-hand side in H^{2m}_{loc}, gives $\mathbf{v} \in H^{2m+2}_{\mathrm{loc}}$. This proves the assertion for every $m \in \mathbb{N}$. The local embeddings in Chapter 9 provide a smooth representative.

Continuity can be expressed by a bound that also controls time. Choose relatively compact neighborhoods of K, each with closure contained in the next. Apply the local elliptic estimate to the operator $L^m_{\mathbf{E},0}$, of order $2m$, and then the Sobolev embedding with $2m > k + n/2$. This yields

$$\|\mathbf{v}\|_{C^k(K,\mathbf{E})} \le C_{K,k,m}\big(\|L^m_{\mathbf{E}}\mathbf{v}\|_{L^2(M,\mathbf{E})} + \|\mathbf{v}\|_{L^2(M,\mathbf{E})}\big).$$

The constant uses only the charts, frames, and coefficients on the chosen compact neighborhoods. By spectral calculus,

$$\|L^m_{\mathbf{E}} P^{\mathbf{E}}_t\|_{\mathcal{L}(L^2)} \le \sup_{\lambda\in[0,\infty)} \lambda^m e^{-t\lambda} = \left(\frac{m}{et}\right)^m.$$

Together with contractivity of $P^{\mathbf{E}}_t$, this gives

$$\|P^{\mathbf{E}}_t \mathbf{u}\|_{C^k(K,\mathbf{E})} \le C_{K,k,m}\left(1 + \left(\frac{m}{et}\right)^m\right)\|\mathbf{u}\|_{L^2(M,\mathbf{E})}. \tag{12.1}$$

In particular, the bound is uniform for $t \ge t_0 > 0$. Here the constant may depend on the compact set K; completeness alone does not impose uniformity with respect to its location in M. □

For $t > 0$ and $x \in M$, denote the regularized evaluation operator by

$$R_{t,x} \colon L^2(M,\mathbf{E}) \longrightarrow \mathbf{E}_x, \qquad R_{t,x}\mathbf{u} := P^{\mathbf{E}}_t \mathbf{u}(x),$$

. The preceding lemma shows that $R_{t,x}$ is continuous.

Theorem 12.2.2. (Covariant heat kernel) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. There exists a unique smooth map

$$\mathbf{K}^{\mathbf{E}} \colon (0,\infty)\times M \times M \longrightarrow \mathbf{E} \boxtimes \mathbf{E}^*, \qquad (t,x,y) \longmapsto \mathbf{K}^{\mathbf{E}}_t(x,y) \in \mathrm{Hom}(\mathbf{E}_y, \mathbf{E}_x),$$

such that, for every $t > 0$, $\mathbf{u} \in L^2(M,\mathbf{E})$, and $x \in M$,

$$P^{\mathbf{E}}_t \mathbf{u}(x) = \int_M \mathbf{K}^{\mathbf{E}}_t(x,y)\mathbf{u}(y)\, d\lambda_{\mathbf{g}}(y).$$

The integral converges absolutely in the finite-dimensional fiber $\mathbf{E}_x$. Moreover:

(a) $\mathbf{K}^{\mathbf{E}}_t(y,x) = \mathbf{K}^{\mathbf{E}}_t(x,y)^*$;

(b) for $s,t > 0$,

$$\mathbf{K}^{\mathbf{E}}_{t+s}(x,y) = \int_M \mathbf{K}^{\mathbf{E}}_t(x,z)\mathbf{K}^{\mathbf{E}}_s(z,y)\, d\lambda_{\mathbf{g}}(z);$$

(c) the kernel satisfies, smoothly in both variables,

$$\frac{\partial}{\partial t}\mathbf{K}_t^{\mathbf{E}} = -\Delta_{B,\mathbf{E}}^{(x)}\mathbf{K}_t^{\mathbf{E}} = -\Delta_{B,\mathbf{E}^*}^{(y)}\mathbf{K}_t^{\mathbf{E}}.$$

In particular, for $y \in M$ and $\eta \in \mathbf{E}_y$, the first equality follows by evaluating the kernel on η;

(d) if $p_t(x,y)$ is the kernel obtained by the same construction for the trivial bundle of rank one, then

$$\|\mathbf{K}_t^{\mathbf{E}}(x,y)\|_{\mathcal{L}((\mathbf{E}_y,\mathbf{h}_{\mathbf{E}}(y)),(\mathbf{E}_x,\mathbf{h}_{\mathbf{E}}(x)))} \le p_t(x,y).$$

Proof. Fix $t > 0$ and set $s = \frac{t}{2}$. Since $R_{s,x}$ is continuous, it has an adjoint

$$R_{s,x}^*\colon \mathbf{E}_x \longrightarrow L^2(M,\mathbf{E}).$$

Define

$$\mathbf{K}_t^{\mathbf{E}}(x,y) := R_{s,x}R_{s,y}^*.$$

This formula already shows that $\mathbf{K}_t^{\mathbf{E}}(x,y)$ is a linear map from $\mathbf{E}_y$ to $\mathbf{E}_x$.
Smoothness. Let $U, V \subseteq M$ be open subsets over which smooth orthonormal frames $(\mathbf{e}_a)_{a=1}^r$ and $(\mathbf{f}_b)_{b=1}^r$ of $\mathbf{E}$ exist. For $s > 0$ and $y \in V$, define

$$\mathbf{\Psi}_b(s,y) := R_{s,y}^*\mathbf{f}_b(y) \in L^2(M,\mathbf{E}).$$

Fix $\mathbf{u} \in L^2(M,\mathbf{E})$. By the definition of the adjoint,

$$\langle \mathbf{u}, \mathbf{\Psi}_b(s,y)\rangle_{L^2(M,\mathbf{E})} = \langle P_s^{\mathbf{E}}\mathbf{u}(y), \mathbf{f}_b(y)\rangle_{\mathbf{h}_{\mathbf{E}}}. \tag{12.2}$$

Let us check the joint regularity that we will use. Let $(s_0,y_0) \in (0,\infty)\times V$ and choose a compact interval $J \Subset (0,\infty)$ containing s_0 in its interior. For $a \in \mathbb{N}_0$, we have, in $L^2(M,\mathbf{E})$,

$$\partial_s^a P_s^{\mathbf{E}}\mathbf{u} = (-L_{\mathbf{E}})^a P_s^{\mathbf{E}}\mathbf{u}, \qquad s \in J.$$

The function $s \mapsto (-L_{\mathbf{E}})^a P_s^{\mathbf{E}}\mathbf{u}$ is continuous in $L^2(M,\mathbf{E})$. Let $\delta > 0$ be such that $2\delta < \inf_{s\in J} s$. For $s \in J$, write

$$(-L_{\mathbf{E}})^a P_s^{\mathbf{E}}\mathbf{u} = P_\delta^{\mathbf{E}}\big((-L_{\mathbf{E}})^a P_{s-\delta}^{\mathbf{E}}\mathbf{u}\big).$$

The map $s \mapsto (-L_{\mathbf{E}})^a P_{s-\delta}^{\mathbf{E}}\mathbf{u}$ is continuous in $L^2(M,\mathbf{E})$, whereas, by Lemma 12.2.1, $P_\delta^{\mathbf{E}}\colon L^2(M,\mathbf{E}) \longrightarrow C^q(K,\mathbf{E})$ is continuous for every $q \in \mathbb{N}_0$ and every compact set $K \Subset V$. Thus

$$s \longmapsto (-L_{\mathbf{E}})^a P_s^{\mathbf{E}}\mathbf{u}$$

is continuous on J with values in each of these spaces $C^q(K,\mathbf{E})$. In a trivialization over V, for each multi-index β, the derivative

$$\partial_s^a\partial_y^\beta\langle P_s^{\mathbf{E}}\mathbf{u}(y), \mathbf{f}_b(y)\rangle_{\mathbf{h}_{\mathbf{E}}}$$

is, by the Leibniz rule, a finite sum of pairings between coordinate derivatives of $(-L_{\mathbf{E}})^a P_s^{\mathbf{E}}\mathbf{u}$ and derivatives of the frame $\mathbf{f}_b$. Each summand is continuous in (s,y). Consequently, the right-hand side of (12.2) is jointly

smooth in (s,y).

We have proved that $(s,y)\mapsto \mathbf{\Psi}_b(s,y)$ is weakly smooth with values in the Hilbert space $L^2(M,\mathbf{E})$. Theorem B.3.26, applied in charts on $(0,\infty)\times V$, shows that this map is smooth in the norm of $L^2(M,\mathbf{E})$.

If $s=\frac{t}{2}$, then

$$\left\langle \mathbf{K}_t^{\mathbf{E}}(x,y)\mathbf{f}_b(y),\mathbf{e}_a(x)\right\rangle_{\mathbf{h_E}} = \left\langle \mathbf{\Psi}_b(s,y),\mathbf{\Psi}_a(s,x)\right\rangle_{L^2(M,\mathbf{E})}.$$

The inner product is a continuous sesquilinear map; hence the last expression is smooth in (t,x,y). Since this holds for every component in local frames, $\mathbf{K}^{\mathbf{E}}$ is a smooth section of $\mathbf{E}\boxtimes\mathbf{E}^*$.

Integral representation. Fix $x\in M$ and $\xi\in\mathbf{E}_x$. For $\mathbf{u}\in L^2(M,\mathbf{E})$, the section

$$y\longmapsto \mathbf{K}_t^{\mathbf{E}}(x,y)^*\xi = R_{s,y}R_{s,x}^*\xi = P_s^{\mathbf{E}}(R_{s,x}^*\xi)(y)$$

belongs to $L^2(M,\mathbf{E})$. The Cauchy–Schwarz inequality in Theorem B.3.2 shows that its scalar product with $\mathbf{u}$ is integrable. Using the definition of the adjoint and self-adjointness of $P_s^{\mathbf{E}}$,

$$\begin{aligned}\int_M \left\langle \mathbf{K}_t^{\mathbf{E}}(x,y)\mathbf{u}(y),\xi\right\rangle_{\mathbf{h_E}} d\lambda_{\mathbf{g}}(y) &= \left\langle \mathbf{u},P_s^{\mathbf{E}}R_{s,x}^*\xi\right\rangle_{L^2(M,\mathbf{E})}\\ &= \left\langle P_s^{\mathbf{E}}\mathbf{u},R_{s,x}^*\xi\right\rangle_{L^2(M,\mathbf{E})}\\ &= \left\langle R_{s,x}P_s^{\mathbf{E}}\mathbf{u},\xi\right\rangle_{\mathbf{h_E}}\\ &= \left\langle P_t^{\mathbf{E}}\mathbf{u}(x),\xi\right\rangle_{\mathbf{h_E}}.\end{aligned}$$

Convergence is also absolute in the fiber norm. If $(\xi_a)_{a=1}^{r_{\mathbf{E}}}$ is an orthonormal basis of $\mathbf{E}_x$, then

$$\int_M |\mathbf{K}_t^{\mathbf{E}}(x,y)\mathbf{u}(y)|\,d\lambda_{\mathbf{g}}(y)\le \sum_{a=1}^{r_{\mathbf{E}}}\|\mathbf{K}_t^{\mathbf{E}}(x,\cdot)^*\xi_a\|_{L^2(M,\mathbf{E})}\|\mathbf{u}\|_{L^2(M,\mathbf{E})}<\infty.$$

Since $\xi\in\mathbf{E}_x$ was arbitrary, the integral formula follows.

If another smooth kernel satisfies the same formula, their difference, paired with local frames, integrates to zero against every compactly supported smooth section. The fundamental lemma of distribution theory implies that each component vanishes; hence the kernel is unique.

Symmetry and the semigroup property. The definition gives

$$\mathbf{K}_t^{\mathbf{E}}(y,x)^* = (R_{s,y}R_{s,x}^*)^* = R_{s,x}R_{s,y}^* = \mathbf{K}_t^{\mathbf{E}}(x,y).$$

For the semigroup property, fix $\eta\in\mathbf{E}_y$. The section $z\mapsto \mathbf{K}_s^{\mathbf{E}}(z,y)\eta$ belongs to $L^2(M,\mathbf{E})$ because it can be written as $P_{\frac{s}{2}}^{\mathbf{E}}R_{\frac{s}{2},y}^*\eta$. Applying the integral formula and $P_t^{\mathbf{E}}P_s^{\mathbf{E}}=P_{t+s}^{\mathbf{E}}$,

$$\int_M \mathbf{K}_t^{\mathbf{E}}(x,z)\mathbf{K}_s^{\mathbf{E}}(z,y)\eta\,d\lambda_{\mathbf{g}}(z) = P_t^{\mathbf{E}}\big(\mathbf{K}_s^{\mathbf{E}}(\cdot,y)\eta\big)(x) = \mathbf{K}_{t+s}^{\mathbf{E}}(x,y)\eta.$$

The heat equation. Fix $t>0$, $y\in M$, and $\eta\in\mathbf{E}_y$. Choose $t_0\in(0,t)$ and set

$$\mathbf{w} := \mathbf{K}_{t_0}^{\mathbf{E}}(\cdot,y)\eta\in L^2(M,\mathbf{E}).$$

The semigroup property gives

$$\mathbf{K}^{\mathbf{E}}_{\tau}(\cdot,y)\eta = P^{\mathbf{E}}_{\tau-t_0}\mathbf{w}, \qquad \tau > t_0.$$

Since $\mathbf{w} = P^{\mathbf{E}}_{\frac{t_0}{2}} R^{*}_{\frac{t_0}{2},y}\eta$, spectral calculus shows that $\mathbf{w} \in \mathcal{D}(L^r_{\mathbf{E}})$ for every $r \in \mathbb{N}$. Differentiability of the semigroup gives, for $h \to 0^+$,

$$\frac{P^{\mathbf{E}}_{\tau-t_0+h}\mathbf{w} - P^{\mathbf{E}}_{\tau-t_0}\mathbf{w}}{h} \longrightarrow -L_{\mathbf{E}}P^{\mathbf{E}}_{\tau-t_0}\mathbf{w}$$

in $L^2(M,\mathbf{E})$. Since smoothness of the kernel in the time variable has already been proved, this one-sided derivative identifies the ordinary derivative. Convergence also holds in the graph norm of $L^r_{\mathbf{E}}$. Indeed, powers of $L_{\mathbf{E}}$ commute with the semigroup and

$$L^r_{\mathbf{E}}\left(\frac{P^{\mathbf{E}}_h - I}{h}P^{\mathbf{E}}_{\tau-t_0}\mathbf{w} + L_{\mathbf{E}}P^{\mathbf{E}}_{\tau-t_0}\mathbf{w}\right) = \left(\frac{P^{\mathbf{E}}_h - I}{h} + L_{\mathbf{E}}\right)L^r_{\mathbf{E}}P^{\mathbf{E}}_{\tau-t_0}\mathbf{w} \longrightarrow 0$$

in $L^2(M,\mathbf{E})$.

Let $K \subseteq M$ be compact and choose r with $2r > k + \frac{n}{2}$. Local elliptic estimates for the expression $(\Delta^{(x)}_{B,\mathbf{E}})^r$ turn convergence in the graph norm into convergence in $H^{2r}_{\mathrm{loc}}(M,\mathbf{E})$ and, in particular, on a neighborhood of K. The local Sobolev embedding for bundles then turns this into convergence in $C^k(K,\mathbf{E})$. Thus the time derivative exists as a smooth section and

$$\frac{\partial}{\partial t}\mathbf{K}^{\mathbf{E}}_t(\cdot,y)\eta = -\Delta^{(x)}_{B,\mathbf{E}}\mathbf{K}^{\mathbf{E}}_t(\cdot,y)\eta.$$

Since k and K were arbitrary, the equality holds on $\Gamma(\mathbf{E})$. Applying this equality to the dual bundle and using $\mathbf{K}^{\mathbf{E}}_t(y,x) = \mathbf{K}^{\mathbf{E}}_t(x,y)^*$ gives, upon interchanging x and y,

$$\partial_t\mathbf{K}^{\mathbf{E}}_t = -\Delta^{(y)}_{B,\mathbf{E}^*}\mathbf{K}^{\mathbf{E}}_t.$$

Pointwise domination of the kernel. Fix $y \in M$ and $\eta \in \mathbf{E}_y$. Let $r_{\mathbf{E}} = \operatorname{rank}\mathbf{E}$. Choose an open neighborhood V of y with a smooth orthonormal frame $(\mathbf{e}_a)_{a=1}^{r_{\mathbf{E}}}$ and write $\eta = \sum_{a=1}^{r_{\mathbf{E}}}\eta_a\mathbf{e}_a(y)$. Define the local section of constant norm, with $\eta_a \in \mathbb{K}$ for $a \in \{1,\dots,r_{\mathbf{E}}\}$,

$$\widetilde{\eta}(z) := \sum_{a=1}^{r_{\mathbf{E}}}\eta_a\mathbf{e}_a(z).$$

We construct the required approximate identity explicitly. Take a chart $\kappa\colon V \longrightarrow \Omega \subseteq \mathbb{R}^n$ with $\kappa(y) = 0$ and choose $\rho > 0$ so that $\overline{B}_{\mathrm{euc}}(0,\rho) \subseteq \Omega$. Let $a \in C^\infty(\Omega)$ be the positive coordinate weight of the Riemannian measure, determined by

$$\int_V \varphi\, d\lambda_{\mathbf{g}} = \int_\Omega (\varphi\circ\kappa^{-1})(z)a(z)\,d\lambda_n(z)$$

for every integrable function φ supported in V. Denote by ω the mollifier in Theorem B.9.1; for $j \in \mathbb{N}$, define

$$\varepsilon_j := \frac{\rho}{j+1}, \qquad c_j := \left(\int_\Omega \varepsilon_j^{-n}\omega\left(\frac{z}{\varepsilon_j}\right)a(z)\,dz\right)^{-1},$$

where the integrand vanishes outside $\overline{B}_{\mathrm{euc}}(0,\varepsilon_j) \subseteq \Omega$. The number c_j is well defined and positive. Set

$$\rho_j(q) := \begin{cases} c_j \varepsilon_j^{-n} \omega\left(\dfrac{\kappa(q)}{\varepsilon_j}\right), & q \in V, \\ 0, & q \notin V. \end{cases}$$

Since the support of the first expression is compact and contained in V, its extension by zero is smooth. Moreover, $\rho_j \geq 0$, $\displaystyle\int_M \rho_j \, d\lambda_{\mathbf{g}} = 1$, and $\operatorname{supp}(\rho_j) \subseteq \kappa^{-1}(\overline{B}_{\mathrm{euc}}(0,\varepsilon_j))$. If W is a neighborhood of y, choose $0 < r_W \leq \rho$ such that $\kappa^{-1}(B_{\mathrm{euc}}(0,r_W)) \subseteq W$; then $\operatorname{supp}(\rho_j) \subseteq W$ for every j satisfying $j+1 > \dfrac{\rho}{r_W}$. Set $\mathbf{u}_j = \rho_j \widetilde{\eta}$. Then

$$|\mathbf{u}_j|_{\mathbf{h}_{\mathbf{E}}} = |\eta|_{\mathbf{h}_{\mathbf{E}}} \rho_j.$$

The Kato–Simon inequality initially gives, for almost every $x \in M$,

$$|P_t^{\mathbf{E}} \mathbf{u}_j(x)|_{\mathbf{h}_{\mathbf{E}}} \leq |\eta|_{\mathbf{h}_{\mathbf{E}}} P_t \rho_j(x).$$

For $t > 0$, both functions in this inequality are continuous, since the scalar and covariant semigroups are regularizing. If the inequality failed at a point, it would fail on a nonempty open subset and hence on a set of positive measure. It therefore holds for every $x \in M$. By the integral formula,

$$P_t^{\mathbf{E}} \mathbf{u}_j(x) = \int_M \mathbf{K}_t^{\mathbf{E}}(x,z) \widetilde{\eta}(z) \rho_j(z) \, d\lambda_{\mathbf{g}}(z).$$

The function $z \mapsto \mathbf{K}_t^{\mathbf{E}}(x,z)\widetilde{\eta}(z)$ is continuous. Given $\delta > 0$, there exists a neighborhood W of y on which its difference from $\mathbf{K}_t^{\mathbf{E}}(x,y)\eta$ has norm less than δ throughout W. The explicit choice above gives an index beyond which $\operatorname{supp}(\rho_j) \subseteq W$, and then

$$\left| P_t^{\mathbf{E}} \mathbf{u}_j(x) - \mathbf{K}_t^{\mathbf{E}}(x,y)\eta \right|_{\mathbf{h}_{\mathbf{E}}} \leq \int_M \delta \rho_j \, d\lambda_{\mathbf{g}} = \delta.$$

Therefore $P_t^{\mathbf{E}} \mathbf{u}_j(x) \to \mathbf{K}_t^{\mathbf{E}}(x,y)\eta$. The same argument applied to the scalar kernel gives $P_t \rho_j(x) \to p_t(x,y)$. Passing to the limit,

$$|\mathbf{K}_t^{\mathbf{E}}(x,y)\eta|_{\mathbf{h}_{\mathbf{E}}} \leq p_t(x,y) |\eta|_{\mathbf{h}_{\mathbf{E}}}.$$

Taking the supremum over unit vectors in $\mathbf{E}_y$ gives the operator bound. □

Remark 12.2.3. (Scope of the kernel construction) The preceding construction gives a smooth kernel for every $t > 0$ and the domination estimate $\|\mathbf{K}_t^{\mathbf{E}}(x,y)\| \leq p_t(x,y)$. Estimate (12.1) describes regularization on compact sets and is uniform when time stays bounded away from zero. Asymptotic expansions as $t \to 0^+$ additionally require the local analysis in Chapter 15. For the scalar construction of these expansions, see [34, Chapter VI, §§3–4].

The following scalar properties will be needed to extend the semigroup to the entire L^p scale.

Proposition 12.2.4. (Properties of the scalar kernel) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary. The scalar kernel $p_t(x,y)$ is smooth, symmetric, and nonnegative. It satisfies

$$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)\,d\lambda_{\mathbf{g}}(z)$$

and

$$\int_M p_t(x,y)\,d\lambda_{\mathbf{g}}(y) \le 1.$$

Proof. Smoothness, symmetry, and the composition identity are the trivial-bundle case of the preceding theorem. Suppose that $p_t(x_0,y_0) < 0$. By continuity, there exist a neighborhood W of y_0 and a constant $c > 0$ such that $p_t(x_0,y) \le -c$ for every $y \in W$. Choose a nonzero function $f \in C_c^\infty(W)$ with $f \ge 0$. Then

$$P_t f(x_0) = \int_W p_t(x_0,y)f(y)\,d\lambda_{\mathbf{g}}(y) \le -c\int_W f\,d\lambda_{\mathbf{g}} < 0,$$

contradicting positivity of the scalar semigroup proved in Corollary 12.1.13. Therefore $p_t \ge 0$.
Take the functions (χ_j) from Lemma 12.1.7. We chose this sequence so that $0 \le \chi_j \le 1$, it is increasing, and it converges pointwise to 1. Contractivity on $L^\infty(M)$ gives $0 \le P_t\chi_j(x) \le 1$. Since $p_t(x,y) \ge 0$, the sequence $p_t(x,\cdot)\chi_j$ is also increasing and converges pointwise to $p_t(x,\cdot)$. The monotone convergence theorem B.6.11 gives

$$\int_M p_t(x,y)\,d\lambda_{\mathbf{g}}(y) = \lim_{j\to\infty}\int_M p_t(x,y)\chi_j(y)\,d\lambda_{\mathbf{g}}(y) = \lim_{j\to\infty} P_t\chi_j(x) \le 1.$$

□

12.3 Extension of the semigroup to $L^p(M,\mathbf{E})$

12.3.1 The scalar semigroup on the L^p scale

Proposition 12.3.1. (Scalar semigroup on L^p) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary. For each $1 \le p \le \infty$ and $t \ge 0$, the operator P_t extends to a positive contraction P_t^p on $L^p(M)$. The extensions are consistent. If $1 \le p < \infty$, $(P_t^p)_{t\ge 0}$ is a C_0-semigroup.

Proof. The bound on L^∞ was proved in Corollary 12.1.13. For $f \in L^1(M) \cap L^2(M)$, self-adjointness and the bound on L^∞ give

$$\|P_t f\|_{L^1(M)} = \sup_{\substack{\varphi\in L^2(M)\cap L^\infty(M)\\ \|\varphi\|_{L^\infty(M)}\le 1}} \left|\int_M (P_t f)\varphi\,d\lambda_{\mathbf{g}}\right| = \sup_{\substack{\varphi\in L^2(M)\cap L^\infty(M)\\ \|\varphi\|_{L^\infty(M)}\le 1}} \left|\int_M f(P_t\varphi)\,d\lambda_{\mathbf{g}}\right| \le \|f\|_{L^1(M)}.$$

In the first line, the norm is initially understood to take values in $[0,\infty]$; the last bound proves a posteriori that $P_t f \in L^1$. Let us justify the supremum formula without assuming this conclusion. If $h \in L^2(M)$, choose an increasing exhaustion $(K_j)_{j\in\mathbb{N}}$ by compact sets; each K_j has finite measure. In the real case, set $\varphi_j =$

$\mathbf{1}_{K_j}\operatorname{sgn}(h)$ and, in the complex case,

$$\varphi_j(x) := \begin{cases} \mathbf{1}_{K_j}(x)\,\dfrac{\overline{h(x)}}{|h(x)|}, & h(x) \neq 0, \\ 0, & h(x) = 0. \end{cases}$$

Then $\varphi_j \in L^2(M) \cap L^\infty(M)$, $\|\varphi_j\|_{L^\infty(M)} \leq 1$, and

$$\left| \int_M h\varphi_j \, d\lambda_{\mathbf{g}} \right| = \int_{K_j} |h| \, d\lambda_{\mathbf{g}} \longrightarrow \int_M |h| \, d\lambda_{\mathbf{g}}$$

by monotone convergence, even if the value is infinite. The reverse inequality holds for every φ by the definition of the extended integral. The Riesz–Thorin theorem 5.7.7 gives contractivity on $L^p(M)$ for $1 < p < \infty$. Density of $L^2(M) \cap L^p(M)$ in $L^p(M)$ when $p < \infty$ gives the unique extension. For $p = \infty$, the integral formula with $p_t \geq 0$ and mass at most one defines the extension on $L^\infty(M)$.

The integral formula also represents the extensions for finite exponents. Indeed, if $f \in L^p(M)$ and $1 \leq p < \infty$, Jensen's inequality for the subprobability measure $p_t(x,y)\,d\lambda_{\mathbf{g}}(y)$ gives

$$\left(\int_M p_t(x,y)|f(y)| \, d\lambda_{\mathbf{g}}(y) \right)^p \leq \int_M p_t(x,y)|f(y)|^p \, d\lambda_{\mathbf{g}}(y).$$

For $p = 1$, this inequality is an equality. Tonelli's theorem, symmetry of the kernel, and its mass bound show that the integral of the right-hand side with respect to x does not exceed $\|f\|^p_{L^p(M)}$. Thus the kernel integral converges absolutely for almost every x and defines a contraction on $L^p(M)$. It agrees with P_t on the intersection with L^2, and density identifies this contraction with P^p_t.

For consistency, let $p, q < \infty$ and $f \in L^p(M) \cap L^q(M)$. Using the preceding exhaustion (K_j), set

$$f_j = \mathbf{1}_{K_j}\mathbf{1}_{\{|f| \leq j\}} f.$$

Each f_j is bounded and has support of finite measure; hence it belongs simultaneously to $L^2(M)$, $L^p(M)$, and $L^q(M)$. Dominated convergence applied to $|f_j - f|^p$ and $|f_j - f|^q$ gives convergence in both spaces. By contractivity, $P^p_t f_j \to P^p_t f$ in $L^p(M)$ and $P^q_t f_j \to P^q_t f$ in $L^q(M)$. The terms of the two sequences agree for every j because they arise from the same operator on $L^2(M)$. Pairing with a compactly supported smooth test function, Hölder's inequality identifies their distributional limits; thus $P^p_t f = P^q_t f$ almost everywhere. If one exponent is infinite, the integral formula already proved for each exponent directly gives the same function.

For $p < \infty$, now take $f_j \to f$ in $L^p(M)$ with $f_j \in L^2(M) \cap L^p(M)$. Consistency allows us to apply the semigroup identity on $L^2(M)$ to each f_j, and

$$\|P^p_t P^p_s f - P^p_{t+s} f\|_{L^p(M)} \leq \|P^p_t P^p_s (f - f_j)\|_{L^p(M)} + \|P^p_{t+s}(f_j - f)\|_{L^p(M)} \leq 2\|f - f_j\|_{L^p(M)} \longrightarrow 0.$$

For $p = \infty$ and $s, t > 0$, the double integral arising in the composition of kernels converges absolutely, since

$$\int_M \int_M p_t(x,z) p_s(z,y) |f(y)| \, d\lambda_{\mathbf{g}}(y) \, d\lambda_{\mathbf{g}}(z) \leq \|f\|_{L^\infty(M)}.$$

Fubini's theorem and the Chapman–Kolmogorov identity for the kernel give $P^\infty_t P^\infty_s f = P^\infty_{t+s} f$. When one

time is zero, the equality follows from $P_0^p = I$.

Let us prove strong continuity for $p < \infty$. If $f \in C_c^\infty(M)$, then $Lf \in L^p(M) \cap L^2(M)$. In $L^2(M)$, spectral differentiability gives

$$P_t f - f = -\int_0^t P_s(Lf)\,ds.$$

Let us justify strong measurability of the integrand before using the strong continuity we are proving. Set $g = Lf$ and use the cutoffs χ_j from Lemma 12.1.7. For $s > 0$, the map $s \mapsto \chi_j P_s g$ is continuous with values in $L^p(M)$: heat regularization gives uniform continuity on the compact support of χ_j on each compact time interval in $(0,\infty)$. For every $s > 0$, dominated convergence in M gives $\chi_j P_s g \to P_s^p g$ in $L^p(M)$. Thus $s \mapsto P_s^p g$ is the pointwise limit of strongly measurable maps; assigning its value at $s = 0$ does not change this property. Moreover, its norm does not exceed $\|g\|_{L^p(M)}$. The Bochner integral exists in $L^p(M)$. Pairing the integrals in L^2 and L^p with compactly supported smooth test functions, consistency gives the same result; their distributional realizations agree. Therefore the same identity holds in $L^p(M)$:

$$P_t^p f - f = -\int_0^t P_s^p(Lf)\,ds.$$

Thus

$$\|P_t^p f - f\|_{L^p(M)} \le \int_0^t \|P_s^p(Lf)\|_{L^p(M)}\,ds \le t\|Lf\|_{L^p(M)} \longrightarrow 0$$

as $t \to 0^+$. For $f \in L^p(M)$, take $g_j \in C_c^\infty(M)$ with $g_j \to f$ in $L^p(M)$. Then

$$\|P_t^p f - f\|_{L^p(M)} \le 2\|f - g_j\|_{L^p(M)} + t\|Lg_j\|_{L^p(M)}.$$

First fix j to control the first term, and then let $t \to 0^+$. This proves strong continuity at zero for each f. The semigroup property and contractivity transfer it to any time: for $h > 0$, both the right and left differences at a positive time are bounded by $\|P_h^p f - f\|_{L^p(M)}$. □

12.3.2 The covariant semigroup on $L^p(M,\mathbf{E})$

Theorem 12.3.2. (Covariant semigroup on L^p) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $1 \le p \le \infty$. For each $t \ge 0$, $P_t^{\mathbf{E}}$ admits a contractive extension

$$P_t^{\mathbf{E},p} \colon L^p(M,\mathbf{E}) \longrightarrow L^p(M,\mathbf{E}).$$

The extensions are consistent and satisfy:

(a) $P_{t+s}^{\mathbf{E},p} = P_t^{\mathbf{E},p} P_s^{\mathbf{E},p}$;

(b) if $1 \le p < \infty$, $(P_t^{\mathbf{E},p})_{t\ge 0}$ is a C_0-semigroup;

(c) if $t > 0$ and $\mathbf{u} \in L^p(M,\mathbf{E})$,

$$P_t^{\mathbf{E},p}\mathbf{u}(x) = \int_M \mathbf{K}_t^{\mathbf{E}}(x,y)\mathbf{u}(y)\,d\lambda_{\mathbf{g}}(y)$$

for almost every x;

(d) for $t \geq 0$,

$$|P_t^{\mathbf{E},p}\mathbf{u}|_{\mathbf{h_E}} \leq P_t^p(|\mathbf{u}|_{\mathbf{h_E}})$$

almost everywhere.

For $p = \infty$, the preceding conclusions do not include strong continuity on $t = 0$.

Proof. If $\mathbf{u} \in L^2(M,\mathbf{E}) \cap L^p(M,\mathbf{E})$, the Kato–Simon inequality and scalar contractivity give

$$\|P_t^{\mathbf{E}}\mathbf{u}\|_{L^p(M,\mathbf{E})} \leq \|P_t^p(|\mathbf{u}|_{\mathbf{h_E}})\|_{L^p(M)} \leq \|\mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

For $p < \infty$, the subspace $L^2(M,\mathbf{E}) \cap L^p(M,\mathbf{E})$ is dense in $L^p(M,\mathbf{E})$, and Theorem B.2.36 provides the unique extension. The pointwise inequality passes to the limit: if $\mathbf{u}_j \to \mathbf{u}$ in $L^p(M,\mathbf{E})$, then, after passing to a subsequence, $P_t^{\mathbf{E},p}\mathbf{u}_j \to P_t^{\mathbf{E},p}\mathbf{u}$ and $P_t^p(|\mathbf{u}_j|_{\mathbf{h_E}}) \to P_t^p(|\mathbf{u}|_{\mathbf{h_E}})$ almost everywhere.
For $p = \infty$ and $t > 0$, define

$$P_t^{\mathbf{E},\infty}\mathbf{u}(x) := \int_M \mathbf{K}_t^{\mathbf{E}}(x,y)\mathbf{u}(y)\, d\lambda_{\mathbf{g}}(y),$$

and set $P_0^{\mathbf{E},\infty} := I$. The kernel bound and Proposition 12.2.4 give

$$|P_t^{\mathbf{E},\infty}\mathbf{u}(x)|_{\mathbf{h_E}} \leq \|\mathbf{u}\|_{L^\infty(M,\mathbf{E})} \int_M p_t(x,y)\, d\lambda_{\mathbf{g}}(y) \leq \|\mathbf{u}\|_{L^\infty(M,\mathbf{E})}.$$

The same estimate proves absolute convergence of the integral and pointwise domination.
For $p < \infty$ and $\mathbf{u} \in L^p(M,\mathbf{E})$, kernel domination and the Jensen–Tonelli argument in the scalar proposition show that

$$\int_M \|\mathbf{K}_t^{\mathbf{E}}(x,y)\mathbf{u}(y)\|_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}}(y) \leq P_t^p(|\mathbf{u}|_{\mathbf{h_E}})(x) < \infty$$

for almost every x. This integral defines a contractive operator on $L^p(M,\mathbf{E})$ agreeing with $P_t^{\mathbf{E}}$ on $L^2(M,\mathbf{E}) \cap L^p(M,\mathbf{E})$. The approximation is obtained by taking

$$\mathbf{u}_j = \mathbf{1}_{K_j}\mathbf{1}_{\{|\mathbf{u}|_{\mathbf{h_E}} \leq j\}}\mathbf{u}.$$

Then $\mathbf{u}_j \in L^2(M,\mathbf{E}) \cap L^p(M,\mathbf{E})$ and $\|\mathbf{u}_j - \mathbf{u}\|_{L^p(M,\mathbf{E})} \to 0$. Both the integral operator and $P_t^{\mathbf{E},p}$ send this difference to a section whose norm does not exceed $\|\mathbf{u}_j - \mathbf{u}\|_{L^p(M,\mathbf{E})}$. Their limits agree, proving the integral formula for every $\mathbf{u}$.
Since all exponents are represented by the same kernel, the extensions are consistent, including the infinite exponent. For $s, t > 0$, the norm of the integrand in the composition of operators is dominated by

$$p_t(x,z)p_s(z,y)|\mathbf{u}(y)|_{\mathbf{h_E}}.$$

The double integral of this function is $P_{t+s}^p(|\mathbf{u}|_{\mathbf{h_E}})(x)$, by Tonelli's theorem and the scalar Chapman–Kolmogorov identity. It is finite for almost every x when $p < \infty$, and for every x when $p = \infty$. Fubini's theorem in the finite-dimensional fiber $\mathbf{E}_x$ therefore allows the integrals to be interchanged. The composition identity for the covariant kernel gives $P_t^{\mathbf{E},p}P_s^{\mathbf{E},p}\mathbf{u} = P_{t+s}^{\mathbf{E},p}\mathbf{u}$ almost everywhere. Zero times are included by setting $P_0^{\mathbf{E},p} = I$.

Finally, if $\mathbf{u}\in\Gamma_c(\mathbf{E})$, then $L_{\mathbf{E},0}\mathbf{u}\in L^p(M,\mathbf{E})\cap L^2(M,\mathbf{E})$. The identity

$$P_t^{\mathbf{E}}\mathbf{u}-\mathbf{u}=-\int_0^t P_s^{\mathbf{E}}(L_{\mathbf{E},0}\mathbf{u})\,ds$$

is obtained first in $L^2(M,\mathbf{E})$. Strong measurability in $L^p(M,\mathbf{E})$ is checked as in Proposition 12.3.1: for each compact cutoff χ_j, covariant regularization makes $s\mapsto \chi_j P_s^{\mathbf{E}}(L_{\mathbf{E},0}\mathbf{u})$ continuous when $s>0$, and these maps converge pointwise in L^p to the integrand. Contractivity bounds its norm by $\|L_{\mathbf{E},0}\mathbf{u}\|_{L^p(M,\mathbf{E})}$, so it is Bochner integrable. Consistency, checked against test sections, identifies this integral with the one obtained in L^2. Therefore

$$P_t^{\mathbf{E},p}\mathbf{u}-\mathbf{u}=-\int_0^t P_s^{\mathbf{E},p}(L_{\mathbf{E},0}\mathbf{u})\,ds.$$

Thus

$$\|P_t^{\mathbf{E},p}\mathbf{u}-\mathbf{u}\|_{L^p(M,\mathbf{E})}\le t\|L_{\mathbf{E},0}\mathbf{u}\|_{L^p(M,\mathbf{E})}\longrightarrow 0$$

as $t\to 0^+$. For an arbitrary section $\mathbf{u}\in L^p(M,\mathbf{E})$, choose $\mathbf{v}_j\in\Gamma_c(\mathbf{E})$ with $\mathbf{v}_j\to\mathbf{u}$ in $L^p(M,\mathbf{E})$ and use

$$\|P_t^{\mathbf{E},p}\mathbf{u}-\mathbf{u}\|_{L^p(M,\mathbf{E})}\le 2\|\mathbf{u}-\mathbf{v}_j\|_{L^p(M,\mathbf{E})}+t\|L_{\mathbf{E},0}\mathbf{v}_j\|_{L^p(M,\mathbf{E})}.$$

First fix j and then let $t\to 0^+$; finally, let $j\to\infty$. The semigroup property transfers strong continuity to all times, just as in the scalar case. □

Definition 12.3.3. (Generator on L^p) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $1\le p<\infty$. Define

$$\mathcal{D}(L_{\mathbf{E},p}):=\left\{\mathbf{u}\in L^p(M,\mathbf{E})\,\middle|\,\lim_{t\to 0^+}\frac{\mathbf{u}-P_t^{\mathbf{E},p}\mathbf{u}}{t}\text{ exists in }L^p(M,\mathbf{E})\right\}$$

and

$$L_{\mathbf{E},p}\mathbf{u}:=\lim_{t\to 0^+}\frac{\mathbf{u}-P_t^{\mathbf{E},p}\mathbf{u}}{t}.$$

Then $-L_{\mathbf{E},p}$ is the generator of $(P_t^{\mathbf{E},p})_{t\ge 0}$.

Proposition 12.3.4. Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. For $1\le p<\infty$, $L_{\mathbf{E},p}$ is closed, densely defined, and maximally accretive. Moreover,

$$\Gamma_c(\mathbf{E})\subseteq\mathcal{D}(L_{\mathbf{E},p}),\qquad L_{\mathbf{E},p}\mathbf{u}=L_{\mathbf{E},0}\mathbf{u}\quad\text{for }\mathbf{u}\in\Gamma_c(\mathbf{E}),$$

and $L_{\mathbf{E},2}=L_{\mathbf{E}}$.

Proof. Closedness and density of the domain follow from Proposition B.4.12; maximal accretivity is the

Lumer–Phillips theorem, Theorem B.4.15, applied to the generator $-L_{\mathbf{E},p}$. If $\mathbf{u} \in \Gamma_c(\mathbf{E})$, then

$$\frac{\mathbf{u} - P_t^{\mathbf{E},p}\mathbf{u}}{t} = \frac{1}{t}\int_0^t P_s^{\mathbf{E},p}(L_{\mathbf{E},0}\mathbf{u})\,ds.$$

Strong continuity of the semigroup implies that the right-hand side converges to $L_{\mathbf{E},0}\mathbf{u}$ in $L^p(M,\mathbf{E})$ as $t \to 0^+$. For $p = 2$, the semigroups defined by extension and by spectral calculus agree; uniqueness of the generator gives $L_{\mathbf{E},2} = L_{\mathbf{E}}$. □

12.4 Bessel potentials and spaces of real order

12.4.1 Bessel potentials

The functional calculus of the Laplacian transfers the Euclidean multiplier $(1 + \|\xi\|^2)^{-\frac{s}{2}}$ to the manifold. The resulting Bessel potentials encode a gain of regularity without choosing coordinates.

Definition 12.4.1. (Bessel potential) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $s > 0$ and $1 \le p < \infty$. Define

$$J_{s,p}^{\mathbf{E}}\mathbf{u} := \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}P_t^{\mathbf{E},p}\mathbf{u}\,dt, \qquad \mathbf{u} \in L^p(M,\mathbf{E}),$$

where the integral is a Bochner integral in $L^p(M,\mathbf{E})$.

Proposition 12.4.2. (Properties of Bessel potentials) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $r, s > 0$ and $1 \le p, q < \infty$.

(a) $J_{s,p}^{\mathbf{E}} \in \mathcal{L}(L^p(M,\mathbf{E}))$ and $\|J_{s,p}^{\mathbf{E}}\|_{\mathcal{L}(L^p(M,\mathbf{E}))} \le 1$;

(b) $J_{s,p}^{\mathbf{E}} J_{r,p}^{\mathbf{E}} = J_{s+r,p}^{\mathbf{E}}$;

(c) $J_{2,p}^{\mathbf{E}} = (I + L_{\mathbf{E},p})^{-1}$ and

$$J_{2m,p}^{\mathbf{E}} = (I + L_{\mathbf{E},p})^{-m}$$

for $m \in \mathbb{N}$;

(d) $J_{s,p}^{\mathbf{E}}$ is injective and its range is dense in $L^p(M,\mathbf{E})$;

(e) if $\mathbf{u} \in L^p(M,\mathbf{E}) \cap L^q(M,\mathbf{E})$, then $J_{s,p}^{\mathbf{E}}\mathbf{u} = J_{s,q}^{\mathbf{E}}\mathbf{u}$ almost everywhere.

Proof. Contractivity of the semigroup gives

$$\int_0^\infty \left\| \frac{t^{\frac{s}{2}-1}e^{-t}}{\Gamma(\frac{s}{2})} P_t^{\mathbf{E},p}\mathbf{u} \right\|_{L^p(M,\mathbf{E})} dt \le \frac{\|\mathbf{u}\|_{L^p(M,\mathbf{E})}}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}\,dt = \|\mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

This proves convergence of the Bochner integral and $\|J_{s,p}^{\mathbf{E}}\|_{\mathcal{L}(L^p(M,\mathbf{E}))} \le 1$.

For the semigroup law, the norm of the double integrand is bounded by

$$\frac{t^{\frac{s}{2}-1}\tau^{\frac{r}{2}-1}e^{-(t+\tau)}}{\Gamma(\frac{s}{2})\Gamma(\frac{r}{2})}\|\mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

The integral of this function over $(0,\infty)^2$ is $\|\mathbf{u}\|_{L^p(M,\mathbf{E})}$. Thus Fubini's theorem for Bochner integrals B.6.26 applies and

$$J^{\mathbf{E}}_{s,p}J^{\mathbf{E}}_{r,p}\mathbf{u} = \frac{1}{\Gamma(\frac{s}{2})\Gamma(\frac{r}{2})}\int_0^\infty\int_0^\infty t^{\frac{s}{2}-1}\tau^{\frac{r}{2}-1}e^{-(t+\tau)}P^{\mathbf{E},p}_{t+\tau}\mathbf{u}\,d\tau\,dt.$$

Make the change of variables

$$a = t+\tau, \qquad \theta = \frac{t}{t+\tau}.$$

Its Jacobian is a, and the beta–gamma identity in Lemma B.4.18 gives

$$\int_0^1 \theta^{\frac{s}{2}-1}(1-\theta)^{\frac{r}{2}-1}\,d\theta = \frac{\Gamma(\frac{s}{2})\Gamma(\frac{r}{2})}{\Gamma\left(\frac{s+r}{2}\right)}.$$

We obtain $J^{\mathbf{E}}_{s+r,p}\mathbf{u}$.
The resolvent formula, Proposition B.4.14, applied to the generator $-L_{\mathbf{E},p}$ with $\lambda = 1$, gives $J^{\mathbf{E}}_{2,p} = (I+L_{\mathbf{E},p})^{-1}$. The semigroup law gives the formula for $J^{\mathbf{E}}_{2m,p}$.
If $J^{\mathbf{E}}_{s,p}\mathbf{u} = 0$, choose $m\in\mathbb{N}$ with $2m > s$. Then

$$J^{\mathbf{E}}_{2m,p}\mathbf{u} = J^{\mathbf{E}}_{2m-s,p}J^{\mathbf{E}}_{s,p}\mathbf{u} = 0.$$

The operator $(I+L_{\mathbf{E},p})^{-m}$ is injective, so $\mathbf{u}=0$. For density, observe that

$$J^{\mathbf{E}}_{2m,p} = J^{\mathbf{E}}_{s,p}J^{\mathbf{E}}_{2m-s,p}.$$

Therefore

$$\operatorname{Ran}(J^{\mathbf{E}}_{2m,p}) \subseteq \operatorname{Ran}(J^{\mathbf{E}}_{s,p}).$$

Moreover,

$$\operatorname{Ran}(J^{\mathbf{E}}_{2m,p}) = \operatorname{Ran}\big((I+L_{\mathbf{E},p})^{-m}\big) = \mathcal{D}\big((I+L_{\mathbf{E},p})^{m}\big),$$

where the last space is viewed with its inclusion into $L^p(M,\mathbf{E})$. The domain of a power of the generator is dense by Proposition B.4.12. Consequently, $\operatorname{Ran}(J^{\mathbf{E}}_{s,p})$ is dense in $L^p(M,\mathbf{E})$.
Finally, consistency of the semigroups implies that, for $\mathbf{u}\in L^p(M,\mathbf{E})\cap L^q(M,\mathbf{E})$, the truncated integrals

$$J^{\mathbf{E},[a,b]}_{s}\mathbf{u} := \frac{1}{\Gamma(\frac{s}{2})}\int_a^b t^{\frac{s}{2}-1}e^{-t}P^{\mathbf{E}}_t\mathbf{u}\,dt$$

are the same sections for all $0<a<b<\infty$. As $a\to 0^+$ and $b\to+\infty$, these sections converge to $J^{\mathbf{E}}_{s,p}\mathbf{u}$ in $L^p(M,\mathbf{E})$ and to $J^{\mathbf{E}}_{s,q}\mathbf{u}$ in $L^q(M,\mathbf{E})$. From any sequence of intervals exhausting $(0,+\infty)$, one can extract a subsequence converging almost everywhere in both spaces: first extract one for convergence in $L^p(M,\mathbf{E})$, and then a further subsequence for convergence in $L^q(M,\mathbf{E})$. The two pointwise limits agree, and hence $J^{\mathbf{E}}_{s,p}\mathbf{u} = J^{\mathbf{E}}_{s,q}\mathbf{u}$ almost everywhere. □

12.4.2 The Bessel kernel

The spectral representation of the potential can be converted into an integral representation. Its kernel separates the singularity near the diagonal from the decay at large distances and makes the regularization properties visible.

Definition 12.4.3. (Covariant Bessel kernel) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. For $s>0$, define

$$G_s^{\mathbf{E}}(x,y):=\frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}\mathbf{K}_t^{\mathbf{E}}(x,y)\,dt$$

whenever the integral converges. Also define

$$g_s(x,y):=\frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}p_t(x,y)\,dt,$$

initially allowing the value $+\infty$.

Proposition 12.4.4. Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. For almost every $(x,y)\in M\times M$, $g_s(x,y)<\infty$, the integral defining $G_s^{\mathbf{E}}(x,y)$ converges and

$$\|G_s^{\mathbf{E}}(x,y)\|_{\mathcal{L}((\mathbf{E}_y,\mathbf{h}_{\mathbf{E}}(y)),(\mathbf{E}_x,\mathbf{h}_{\mathbf{E}}(x)))}\le g_s(x,y).$$

Moreover,

$$\int_M g_s(x,y)\,d\lambda_{\mathbf{g}}(y)\le 1$$

for every x, and, if $\mathbf{u}\in L^p(M,\mathbf{E})$ with $1\le p<\infty$,

$$J_{s,p}^{\mathbf{E}}\mathbf{u}(x)=\int_M G_s^{\mathbf{E}}(x,y)\mathbf{u}(y)\,d\lambda_{\mathbf{g}}(y)$$

for almost every x.

Proof. The function $(t,x,y)\mapsto p_t(x,y)$ is nonnegative and measurable. Thus g_s is measurable as the integral of a nonnegative function. For each $x\in M$, Tonelli's theorem B.6.17 and the mass bound for p_t give

$$\begin{aligned}\int_M g_s(x,y)\,d\lambda_{\mathbf{g}}(y)&=\frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}\left(\int_M p_t(x,y)\,d\lambda_{\mathbf{g}}(y)\right)dt\\&\le\frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}\,dt=1.\end{aligned}$$

Since $p_t(x,y)=p_t(y,x)$, also

$$g_s(x,y)=g_s(y,x). \tag{12.3}$$

In particular, $g_s(x,\cdot)$ is finite almost everywhere for each x. Since the set

$$N_s:=\{(x,y)\in M\times M\mid g_s(x,y)=+\infty\}$$

is measurable, Tonelli's theorem applied to $\mathbf{1}_{N_s}$ shows that $(\lambda_{\mathbf{g}} \otimes \lambda_{\mathbf{g}})(N_s) = 0$.
If $(x, y) \notin N_s$, the bound

$$\|\mathbf{K}_t^{\mathbf{E}}(x, y)\|_{\mathcal{L}((\mathbf{E}_y, \mathbf{h}_{\mathbf{E}}(y)),(\mathbf{E}_x, \mathbf{h}_{\mathbf{E}}(x)))} \le p_t(x, y)$$

proves that the function with values in the finite-dimensional space $\operatorname{Hom}(\mathbf{E}_y, \mathbf{E}_x)$ appearing in the definition of $G_s^{\mathbf{E}}(x, y)$ is absolutely integrable. Thus the integral converges and

$$\|G_s^{\mathbf{E}}(x, y)\|_{\mathcal{L}((\mathbf{E}_y, \mathbf{h}_{\mathbf{E}}(y)),(\mathbf{E}_x, \mathbf{h}_{\mathbf{E}}(x)))} \le g_s(x, y).$$

On N_s, we henceforth assign the value $G_s^{\mathbf{E}}(x, y) = 0$. This choice gives a measurable kernel. Indeed, on the product of two trivializing open subsets, the components of

$$\frac{1}{\Gamma(\frac{s}{2})} \int_{\frac{1}{\nu}}^{\nu} t^{\frac{s}{2}-1} e^{-t} \mathbf{K}_t^{\mathbf{E}}(x, y)\, dt, \qquad \nu \in \mathbb{N},$$

are measurable, in fact smooth, functions. Outside N_s, they converge componentwise to those of $G_s^{\mathbf{E}}$, while N_s is measurable. Thus the limit defined on $M \times M \setminus N_s$ and extended by zero to N_s is measurable in every pair of local frames. This proves the first assertions.
Before treating the covariant kernel, we record boundedness of the scalar integral operator. For a measurable function $f \ge 0$, set

$$(\mathcal{T}_s f)(x) := \int_M g_s(x, y) f(y)\, d\lambda_{\mathbf{g}}(y),$$

initially allowing the value $+\infty$. If $1 \le p < \infty$, Jensen's inequality for the subprobability measure $g_s(x, y)\, d\lambda_{\mathbf{g}}(y)$ gives

$$|\mathcal{T}_s f(x)|^p \le \int_M g_s(x, y) |f(y)|^p\, d\lambda_{\mathbf{g}}(y).$$

For $p = 1$, the same inequality is immediate. Integrate in x and use Tonelli's theorem, the symmetry (12.3), and the mass bound already proved. We obtain

$$\|\mathcal{T}_s f\|_{L^p(M)} \le \|f\|_{L^p(M)}. \tag{12.4}$$

Decomposition into positive and negative parts gives the same conclusion for arbitrary scalar functions. In particular, the integral defining $\mathcal{T}_s f$ is finite for almost every x when $f \in L^p(M)$. For $f \in C_c^\infty(M)$, Fubini's theorem and the semigroup representation show that $\mathcal{T}_s f = J_{s,p} f$. Density and (12.4) extend this identity to all of $L^p(M)$.
First let $\mathbf{u} \in \Gamma_c(\mathbf{E})$. For almost every x,

$$\begin{aligned} &\frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty \int_M t^{\frac{s}{2}-1} e^{-t} |\mathbf{K}_t^{\mathbf{E}}(x, y)\mathbf{u}(y)|_{\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}}(y)\, dt \\ &\qquad \le \int_M g_s(x, y) |\mathbf{u}(y)|_{\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}}(y). \end{aligned}$$

Tonelli's theorem and the representation of the scalar semigroup by p_t show, for almost every x, that

$$\begin{aligned}\int_M g_s(x,y)|\mathbf{u}(y)|_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}(y) &= \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}P_t(|\mathbf{u}|_{\mathbf{h_E}})(x)\,dt\\ &= \big(\mathcal{T}_s(|\mathbf{u}|_{\mathbf{h_E}})\big)(x) = J_{s,p}(|\mathbf{u}|_{\mathbf{h_E}})(x).\end{aligned}$$

Here $J_{s,p}$ denotes the scalar Bessel potential. The last expression belongs to $L^p(M)$ and is therefore finite for almost every x. Fubini's theorem for Bochner integrals B.6.26 then allows us to interchange the integrals and gives

$$\begin{aligned}J^{\mathbf{E}}_{s,p}\mathbf{u}(x) &= \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}\left(\int_M \mathbf{K}^{\mathbf{E}}_t(x,y)\mathbf{u}(y)\,d\lambda_{\mathbf{g}}(y)\right)dt\\ &= \int_M G^{\mathbf{E}}_s(x,y)\mathbf{u}(y)\,d\lambda_{\mathbf{g}}(y)\end{aligned}$$

for almost every x.
Whenever the integral converges absolutely, define

$$\mathcal{G}^{\mathbf{E}}_s\mathbf{u}(x) := \int_M G^{\mathbf{E}}_s(x,y)\mathbf{u}(y)\,d\lambda_{\mathbf{g}}(y).$$

The preceding bound gives

$$|\mathcal{G}^{\mathbf{E}}_s\mathbf{u}(x)|_{\mathbf{h_E}} \le \int_M g_s(x,y)|\mathbf{u}(y)|_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}(y) = J_{s,p}(|\mathbf{u}|_{\mathbf{h_E}})(x) \tag{12.5}$$

for almost every x. The right-hand side is finite almost everywhere, and its L^p norm does not exceed $\|\mathbf{u}\|_{L^p(M,\mathbf{E})}$. Thus $\mathcal{G}^{\mathbf{E}}_s$ defines a contractive operator on the subspace $\Gamma_c(\mathbf{E})$.
Now take $\mathbf{u}\in L^p(M,\mathbf{E})$ and choose $\mathbf{u}_j\in\Gamma_c(\mathbf{E})$ with $\mathbf{u}_j\to\mathbf{u}$ in $L^p(M,\mathbf{E})$. Inequality (12.5), applied to $\mathbf{u}_j-\mathbf{u}_k$, shows that $(\mathcal{G}^{\mathbf{E}}_s\mathbf{u}_j)_j$ is Cauchy in $L^p(M,\mathbf{E})$. Therefore $\mathcal{G}^{\mathbf{E}}_s$ extends uniquely to a contractive operator on $L^p(M,\mathbf{E})$. Since $\mathcal{G}^{\mathbf{E}}_s\mathbf{u}_j = J^{\mathbf{E}}_{s,p}\mathbf{u}_j$ and both operators are continuous, we obtain

$$\mathcal{G}^{\mathbf{E}}_s\mathbf{u} = J^{\mathbf{E}}_{s,p}\mathbf{u}$$

in $L^p(M,\mathbf{E})$.
Finally, the preceding scalar identity, now applied to $|\mathbf{u}|_{\mathbf{h_E}}$, shows that

$$\big(\mathcal{T}_s(|\mathbf{u}|_{\mathbf{h_E}})\big)(x) = \int_M g_s(x,y)|\mathbf{u}(y)|_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}(y) < \infty$$

for almost every x. At these points, the integral

$$\int_M G^{\mathbf{E}}_s(x,y)\mathbf{u}(y)\,d\lambda_{\mathbf{g}}(y)$$

converges absolutely. Since $\mathcal{T}_s$ is contractive on $L^p(M)$, $\mathcal{T}_s(|\mathbf{u}_j-\mathbf{u}|_{\mathbf{h_E}})\to 0$ in $L^p(M)$. Choose a subsequence, without relabeling, for which

$$\mathcal{T}_s(|\mathbf{u}_j-\mathbf{u}|_{\mathbf{h_E}})(x)\longrightarrow 0 \quad\text{and}\quad \mathcal{G}^{\mathbf{E}}_s\mathbf{u}_j(x)\longrightarrow\mathcal{G}^{\mathbf{E}}_s\mathbf{u}(x)$$

for almost every x. For each j, let X_j be a set of full measure on which the integral representation already proved for $\mathbf{u}_j$ holds. Intersecting the sets X_j, the set on which the preceding scalar integrals are finite, and the two sets of pointwise convergence gives a single set $X \subseteq M$ of full measure on which all these properties hold simultaneously. If $x \in X$,

$$\begin{aligned}&\left|\mathcal{G}_s^{\mathbf{E}}\mathbf{u}_j(x) - \int_M G_s^{\mathbf{E}}(x,y)\mathbf{u}(y)\,d\lambda_{\mathbf{g}}(y)\right|_{\mathbf{h}_{\mathbf{E}}}\\ &\qquad \le \int_M g_s(x,y)|\mathbf{u}_j(y)-\mathbf{u}(y)|_{\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}(y) = \mathcal{T}_s(|\mathbf{u}_j-\mathbf{u}|_{\mathbf{h}_{\mathbf{E}}})(x),\end{aligned}$$

so the left-hand side tends to zero. By uniqueness of the limit, the extension $\mathcal{G}_s^{\mathbf{E}}\mathbf{u}$ agrees almost everywhere with the kernel integral. Since $\mathcal{G}_s^{\mathbf{E}} = J_{s,p}^{\mathbf{E}}$ in $L^p(M,\mathbf{E})$, the asserted representation follows. □

12.4.3 Definition of the fractional scale

With the potential and its kernel available, we can define real orders of regularity intrinsically. The definition is arranged to agree with integer orders and to be stable under spectral calculus.

Definition 12.4.5. (Bessel Sobolev spaces of positive order) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $s > 0$ and $1 \le p < \infty$. Define

$$H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E}) := J^{\mathbf{E}}_{s,p}(L^p(M,\mathbf{E})).$$

If $\mathbf{u} = J^{\mathbf{E}}_{s,p}\mathbf{f}$, set

$$\|\mathbf{u}\|_{H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})} := \|\mathbf{f}\|_{L^p(M,\mathbf{E})}.$$

Injectivity of $J^{\mathbf{E}}_{s,p}$ ensures that this norm is well defined. For $s = 0$, set $H^{0,p}_{L_{\mathbf{E}}}(M,\mathbf{E}) := L^p(M,\mathbf{E})$.

Definition 12.4.6. (Spaces of negative order) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $s < 0$ and $1 \le p < \infty$. Define $H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})$ as the completion of $L^p(M,\mathbf{E})$ with respect to the norm

$$\|\mathbf{u}\|_{H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})} := \|J^{\mathbf{E}}_{-s,p}\mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

Notation 12.4.7. (Scalar Bessel scale) On the trivial bundle of rank one, equipped with the trivial connection, the connection Laplacian is the positive realization $L = \Delta_B = -\Delta$ introduced in the section on the scalar semigroup. For every $s \in \mathbb{R}$ and $1 \le p < \infty$, we write

$$H^{s,p}_{L}(M) := H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})$$

in this case. Thus $H^{s,p}_{L}(M)$ always denotes the intrinsic scale constructed through the heat semigroup and the potentials $(I+L)^{-\frac{s}{2}}$, while the notation without a subscript is reserved for the spaces defined by localization when these are introduced.

Proposition 12.4.8. (Conjugation of the Bessel scale) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth complex vector bundle of finite rank with a Hermitian metric and a compatible connection. The conjugate connection $\nabla^{\overline{\mathbf{E}}}$ is compatible with $\mathbf{h}_{\overline{\mathbf{E}}}$, and its Laplacian satisfies

$$L_{\overline{\mathbf{E}},0}(\overline{\mathbf{u}}) = \overline{L_{\mathbf{E},0}\mathbf{u}}, \qquad \mathbf{u}\in\Gamma_c(\mathbf{E}).$$

For every $1\le p<\infty$, $t\ge 0$, and $r>0$,

$$P_t^{\overline{\mathbf{E}},p}(\overline{\mathbf{u}}) = \overline{P_t^{\mathbf{E},p}\mathbf{u}},$$
$$J_{r,p}^{\overline{\mathbf{E}}}(\overline{\mathbf{u}}) = \overline{J_{r,p}^{\mathbf{E}}\mathbf{u}}.$$

Consequently, for each $s\in\mathbb{R}$, the map

$$\mathfrak{C}_{s,p,\mathbf{E}}\colon \overline{H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})}\longrightarrow H^{s,p}_{L_{\overline{\mathbf{E}}}}(M,\overline{\mathbf{E}}), \qquad \mathfrak{C}_{s,p,\mathbf{E}}(\overline{\mathbf{u}})=\overline{\mathbf{u}}, \tag{12.6}$$

is a complex-linear isometric isomorphism.

Proof. Compatibility of $\nabla^{\overline{\mathbf{E}}}$ and parallelism of $\mathcal{R}_{\mathbf{E}}$ were proved in Proposition 2.5.7. Conjugating the local formula $L_{\mathbf{E},0}\mathbf{u}=-\operatorname{tr}_{\mathbf{g}}(\nabla^2\mathbf{u})$ gives the first identity. Conjugation of sections is an antilinear isometry between the L^2 spaces and between the domains of the energy forms; uniqueness of the associated self-adjoint realization gives $L_{\overline{\mathbf{E}}}\mathfrak{C}=\mathfrak{C}L_{\mathbf{E}}$. Spectral calculus, followed by uniqueness of the consistent extensions to L^p, gives the semigroup identity. Integrating this identity in the Laplace formula for $J_{r,p}$ gives the potential identity. For $s>0$, the norm of the image of $J^{\mathbf{E}}_{s,p}\mathbf{u}$ is the norm of $\overline{\mathbf{u}}$ in $L^p(M,\overline{\mathbf{E}})$, equal to that of $\mathbf{u}$. For $s<0$, the same conclusion follows from the norm defining the completion. Surjectivity follows by conjugating again. Finally, the scalar structure of the conjugate space shows that (12.6) is complex-linear. □

Theorem 12.4.9. (Properties of the Bessel scale) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $1\le p<\infty$.

(a) $H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})$ is a Banach space for every $s\in\mathbb{R}$;

(b) if $s_1>s_0$, the inclusion

$$H^{s_1,p}_{L_{\mathbf{E}}}(M,\mathbf{E})\hookrightarrow H^{s_0,p}_{L_{\mathbf{E}}}(M,\mathbf{E})$$

is continuous and has dense image;

(c) for $r>0$ and $s\in\mathbb{R}$, $J^{\mathbf{E}}_{r,p}$ extends to an isometric isomorphism

$$J^{\mathbf{E}}_{r,p}\colon H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})\longrightarrow H^{s+r,p}_{L_{\mathbf{E}}}(M,\mathbf{E});$$

(d) if $s>0$, $J^{\mathbf{E}}_{s,p}(\Gamma_c(\mathbf{E}))$ is dense in $H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})$;

(e) if $1<p<\infty$, p' is the conjugate exponent, and $s\in\mathbb{R}$, integral duality, initially defined on $L^p(M,\mathbf{E})\times L^{p'}(M,\overline{\mathbf{E}})$, extends uniquely to a continuous complex-bilinear pairing

$$\mathfrak{b}_{s,p;\mathbf{E}}\colon H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})\times H^{-s,p'}_{L_{\overline{\mathbf{E}}}}(M,\overline{\mathbf{E}})\longrightarrow\mathbb{C}$$

and induces the complex-linear isometric isomorphism

$$H^{-s,p'}_{L_{\overline{\mathbf{E}}}}(M,\overline{\mathbf{E}}) \longrightarrow \big(H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})\big)', \qquad \mathbf{w} \longmapsto \big[\mathbf{u} \longmapsto \mathfrak{b}_{s,p;\mathbf{E}}(\mathbf{u},\mathbf{w})\big]. \tag{12.7}$$

Equivalently, using (12.6), we obtain

$$\overline{H^{-s,p'}_{L_{\mathbf{E}}}(M,\mathbf{E})} \longrightarrow \big(H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})\big)', \qquad \overline{\mathbf{v}} \longmapsto \left[\mathbf{u} \longmapsto \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},\mathbf{v})\, d\lambda_{\mathbf{g}}\right].$$

In the last formula, the integral has its literal meaning when $\mathbf{u} \in L^p(M,\mathbf{E})$ and $\mathbf{v} \in L^{p'}(M,\mathbf{E})$; for elements of general orders, it denotes, by definition, the unique continuous extension $\mathfrak{b}_{s,p;\mathbf{E}}(\mathbf{u}, \mathfrak{C}_{-s,p',\mathbf{E}}(\overline{\mathbf{v}}))$. This convention does not assign pointwise values to a distribution of negative order.

Proof. Parts (a)–(c) are Theorem B.4.19 applied to the semigroup $(P^{\mathbf{E},p}_t)_{t\geq 0}$. For (d), if $\mathbf{u} = J^{\mathbf{E}}_{s,p}\mathbf{f}$, choose $\mathbf{f}_j \in \Gamma_c(\mathbf{E})$ with $\mathbf{f}_j \to \mathbf{f}$ in $L^p(M,\mathbf{E})$. Then

$$\|J^{\mathbf{E}}_{s,p}\mathbf{f}_j - \mathbf{u}\|_{H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})} = \|\mathbf{f}_j - \mathbf{f}\|_{L^p(M,\mathbf{E})} \longrightarrow 0.$$

For duality, self-adjointness on L^2 and consistency of the extensions imply

$$\int_M \mathbf{h}_{\mathbf{E}}(P^{\mathbf{E},p}_t\mathbf{u},\mathbf{v})\, d\lambda_{\mathbf{g}} = \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},P^{\mathbf{E},p'}_t\mathbf{v})\, d\lambda_{\mathbf{g}}$$

for $\mathbf{u}$ and $\mathbf{v}$ in dense intersections. The integral defining $J^{\mathbf{E}}_{r,p}$ is dominated by

$$\frac{t^{\frac{r}{2}-1}e^{-t}}{\Gamma(\frac{r}{2})}\|\mathbf{u}\|_{L^p(M,\mathbf{E})}\|\mathbf{v}\|_{L^{p'}(M,\mathbf{E})},$$

which is integrable on $(0,\infty)$. Fubini's theorem for Bochner integrals B.6.26 gives

$$\int_M \mathbf{h}_{\mathbf{E}}(J^{\mathbf{E}}_{r,p}\mathbf{u},\mathbf{v})\, d\lambda_{\mathbf{g}} = \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},J^{\mathbf{E}}_{r,p'}\mathbf{v})\, d\lambda_{\mathbf{g}}. \tag{12.8}$$

Proposition B.4.23, applied to the bilinear pairing $L^p(M,\mathbf{E})\times L^{p'}(M,\overline{\mathbf{E}})$ induced by $\mathcal{R}_{\mathbf{E}}$, uses (12.8) and proves that (12.7) is isometric and surjective. It is complex-linear in $\mathbf{w}$ because $\mathcal{R}_{\mathbf{E}}\colon \overline{\mathbf{E}} \to \mathbf{E}^*$ is complex-linear. Composition with the linear isomorphism $\mathfrak{C}_{-s,p',\mathbf{E}}$ gives the equivalent formulation from the conjugate space. In the real case, $\overline{\mathbf{E}} = \mathbf{E}$ and we recover the usual real duality. □

12.4.4 Spectral characterization in L^2

Theorem 12.4.10. (Spectral characterization) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. Let $s > 0$. Then

$$J^{\mathbf{E}}_{s,2} = (I + L_{\mathbf{E}})^{-\frac{s}{2}}$$

and

$$H^{s,2}_{L_{\mathbf{E}}}(M,\mathbf{E}) = \mathcal{D}\big((I + L_{\mathbf{E}})^{\frac{s}{2}}\big).$$

For $\mathbf{u}$ in this space,

$$\|\mathbf{u}\|_{H^{s,2}_{L_{\mathbf{E}}}(M,\mathbf{E})} = \|(I+L_{\mathbf{E}})^{\frac{s}{2}}\mathbf{u}\|_{L^2(M,\mathbf{E})}.$$

In particular,

$$H^{1,2}_{L_{\mathbf{E}}}(M,\mathbf{E}) = W^{1,2}(M,\mathbf{E})$$

and

$$\|\mathbf{u}\|^2_{H^{1,2}_{L_{\mathbf{E}}}(M,\mathbf{E})} = \|\mathbf{u}\|^2_{L^2(M,\mathbf{E})} + \|\nabla^{\mathbf{E}}\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})}.$$

Proof. For $\lambda \geq 0$,

$$(1+\lambda)^{-\frac{s}{2}} = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}e^{-t\lambda}\,dt.$$

To justify the passage to spectral calculus precisely, fix $\mathbf{u} \in L^2(M,\mathbf{E})$ and, for $0 < \varepsilon < R < \infty$, set

$$m_{\varepsilon,R}(\lambda) := \frac{1}{\Gamma(\frac{s}{2})}\int_\varepsilon^R t^{\frac{s}{2}-1}e^{-t}e^{-t\lambda}\,dt.$$

On the compact interval $[\varepsilon, R]$, the map $t \mapsto e^{-tL_{\mathbf{E}}}$ is continuous in operator norm: indeed, the spectral calculus in Theorem B.4.6 and the scalar fundamental theorem of calculus give

$$\|e^{-tL_{\mathbf{E}}} - e^{-\tau L_{\mathbf{E}}}\|_{\mathcal{L}(L^2(M,\mathbf{E}))} \leq |t-\tau| \sup_{\lambda\in[0,\infty)} \lambda e^{-\varepsilon\lambda} \leq \frac{|t-\tau|}{e\varepsilon}.$$

Approximating the integral by Riemann sums and using the rules of the same functional calculus yields

$$\frac{1}{\Gamma(\frac{s}{2})}\int_\varepsilon^R t^{\frac{s}{2}-1}e^{-t}e^{-tL_{\mathbf{E}}}\mathbf{u}\,dt = m_{\varepsilon,R}(L_{\mathbf{E}})\mathbf{u}.$$

As $\varepsilon \to 0^+$ and $R \to \infty$, the left-hand side converges to $J^{\mathbf{E}}_{s,2}\mathbf{u}$ by the Bochner dominated convergence theorem B.6.25. On the other hand, $m_{\varepsilon,R}(\lambda) \to (1+\lambda)^{-\frac{s}{2}}$ and $0 \leq m_{\varepsilon,R}(\lambda) \leq 1$. The norm formula in Theorem B.4.6, followed by Lebesgue's dominated convergence theorem B.6.14, implies

$$m_{\varepsilon,R}(L_{\mathbf{E}})\mathbf{u} \longrightarrow (I+L_{\mathbf{E}})^{-\frac{s}{2}}\mathbf{u} \quad \text{in } L^2(M,\mathbf{E}).$$

Consequently, $J^{\mathbf{E}}_{s,2}\mathbf{u} = (I+L_{\mathbf{E}})^{-\frac{s}{2}}\mathbf{u}$. Corollary B.4.8 identifies the range of $(I+L_{\mathbf{E}})^{-\frac{s}{2}}$ with the domain of $(I+L_{\mathbf{E}})^{\frac{s}{2}}$ and shows that these operators are inverses on the respective spaces.
For $s = 1$, Corollary B.4.7 gives

$$\mathcal{D}\big((I+L_{\mathbf{E}})^{\frac{1}{2}}\big) = W^{1,2}(M,\mathbf{E})$$

and

$$\|(I+L_{\mathbf{E}})^{\frac{1}{2}}\mathbf{u}\|^2_{L^2(M,\mathbf{E})} = \|\mathbf{u}\|^2_{L^2(M,\mathbf{E})} + \mathfrak{q}_{\mathbf{E}}(\mathbf{u},\mathbf{u}).$$

The definition of $\mathfrak{q}_{\mathbf{E}}$ completes the proof. □

Proposition 12.4.11. (Spectral, dual, and interpolation compatibility in L^2) Let $(M,\mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric $\mathbf{h}_{\mathbf{E}}$, Hermitian in the complex case, and a compatible connection. First suppose that

$\mathbf{E}$ is complex, and set $B_{\mathbf{E}} := I + L_{\mathbf{E}}$. Let $s_0 < s_1$, $0 < \theta < 1$, and $s_\theta := (1-\theta)s_0 + \theta s_1$. Then

$$[H^{s_0,2}_{L_{\mathbf{E}}}(M,\mathbf{E}), H^{s_1,2}_{L_{\mathbf{E}}}(M,\mathbf{E})]_\theta = H^{s_\theta,2}_{L_{\mathbf{E}}}(M,\mathbf{E}) \tag{12.9}$$

as topological spaces, with the norms determined by spectral calculus. This identification is the identity on the intersection of the two endpoints and commutes with Bessel potentials and the conjugation (12.6). Moreover, if $s > 0$, $\mathbf{w} \in H^{-s,2}_{L_{\overline{\mathbf{E}}}}(M,\overline{\mathbf{E}})$,

$$z := J^{\overline{\mathbf{E}}}_{s,2}\mathbf{w} \in L^2(M,\overline{\mathbf{E}})$$

, and $\mathbf{u} \in H^{s,2}_{L_{\mathbf{E}}}(M,\mathbf{E})$, denote by $\langle \mathbf{u},\mathbf{w}\rangle_{s,2;\mathbf{E}}$ the pairing in Theorem 12.4.9. It is given by the spectral formula

$$\langle \mathbf{u},\mathbf{w}\rangle_{s,2;\mathbf{E}} = \int_M \mathcal{R}_{\mathbf{E}}(z)\big(B^{\frac{s}{2}}_{\mathbf{E}}\mathbf{u}\big)\, d\lambda_{\mathbf{g}}. \tag{12.10}$$

In particular,

$$H^{-s,2}_{L_{\overline{\mathbf{E}}}}(M,\overline{\mathbf{E}}) \xrightarrow{\cong} \big(\mathcal{D}(B^{\frac{s}{2}}_{\mathbf{E}}), \|B^{\frac{s}{2}}_{\mathbf{E}}\cdot\|_{L^2(M,\mathbf{E})}\big)'$$

is precisely the identification in (12.7); passage to the spectral characterization introduces no second duality convention. When interpolating two orders, the pairing at order s_θ is the unique continuous extension of the same pairings on the common spectral core. Thus duality, conjugation, and interpolation give compatible identifications.

If $\mathbf{E}$ is real, the same assertions are obtained in the complexification and restricted to the invariant real subspace; in the dual formula, $\overline{\mathbf{E}}$ and $\mathcal{R}_{\mathbf{E}}$ are omitted and the real bundle inner product is used.

Proof. First prove (12.9) when $0 \le s_0 < s_1$. Write

$$a_j := \frac{s_j}{2}, \qquad a(z) := (1-z)a_0 + za_1, \qquad a := a(\theta), \qquad \delta := a_1 - a_0.$$

By the preceding theorem, $H^{s_j,2}_{L_{\mathbf{E}}}(M,\mathbf{E}) = \mathcal{D}(B^{a_j}_{\mathbf{E}})$, with norm $\|B^{a_j}_{\mathbf{E}}\cdot\|_{L^2(M,\mathbf{E})}$. Spectral calculus gives

$$\|B^{it}_{\mathbf{E}}\mathbf{v}\|_{L^2(M,\mathbf{E})} = \|\mathbf{v}\|_{L^2(M,\mathbf{E})}, \qquad t \in \mathbb{R}. \tag{12.11}$$

Let $\mathbf{u} \in \mathcal{D}(B^a_{\mathbf{E}})$. For $\varepsilon > 0$, the family

$$F_\varepsilon(z) := e^{\varepsilon(z-\theta)^2} B^{a-a(z)}_{\mathbf{E}}\mathbf{u}$$

is admissible for the complex method. Indeed, this is first verified after spectrally truncating $\mathbf{u}$ to the interval $[1,N]$; the spectral norm formula and dominated convergence then allow us to let $N \to \infty$, uniformly on closed substrips and in the two boundary norms. For $j \in \{0,1\}$ and $t \in \mathbb{R}$,

$$B^{a_j}_{\mathbf{E}} F_\varepsilon(j+it) = e^{\varepsilon(j+it-\theta)^2} B^{a-it\delta}_{\mathbf{E}}\mathbf{u}.$$

Identity (12.11) gives

$$\|F_\varepsilon(j+it)\|_{H^{s_j,2}_{L_{\mathbf{E}}}(M,\mathbf{E})} \le e^{\varepsilon(j-\theta)^2} e^{-\varepsilon t^2} \|B^a_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})}.$$

Since $F_\varepsilon(\theta) = \mathbf{u}$, we let $\varepsilon \to 0^+$ and obtain the embedding of $H^{s_\theta,2}_{L_{\mathbf{E}}}(M,\mathbf{E})$ into the left-hand side of (12.9).

For the reverse embedding, let F be an admissible family with $\mathbf{u} = F(\theta)$, and let

$$\mathscr{C}_{\mathbf{E}} := \bigcup_{N\in\mathbb{N}} \mathbf{1}_{[1,N]}(B_{\mathbf{E}})L^2(M,\mathbf{E}).$$

This subspace is a core for every nonnegative real power of $B_{\mathbf{E}}$. For $\mathbf{v} \in \mathscr{C}_{\mathbf{E}}$, the function

$$\Phi_v(z) := \int_M \mathbf{h}_{\mathbf{E}}\big(F(z), B_{\mathbf{E}}^{a(\overline{z})}\mathbf{v}\big)\, d\lambda_{\mathbf{g}}$$

is holomorphic: the second entry is antiholomorphic, and $\mathbf{h}_{\mathbf{E}}$ is antilinear in that entry. On the boundary $z = j + it$, self-adjointness and (12.11) imply

$$|\Phi_v(j+it)| \le \|F(j+it)\|_{H^{s_j,2}_{L_{\mathbf{E}}}(M,\mathbf{E})} \|B_{\mathbf{E}}^{-a_j} B_{\mathbf{E}}^{a_j - it\delta}\mathbf{v}\|_{L^2(M,\mathbf{E})} = \|F(j+it)\|_{H^{s_j,2}_{L_{\mathbf{E}}}(M,\mathbf{E})} \|\mathbf{v}\|_{L^2(M,\mathbf{E})}.$$

The three-lines lemma 5.7.3 gives

$$\left| \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u}, B_{\mathbf{E}}^a \mathbf{v})\, d\lambda_{\mathbf{g}} \right| \le \|F\|_{\mathscr{F}} \|\mathbf{v}\|_{L^2(M,\mathbf{E})}.$$

Since $\mathscr{C}_{\mathbf{E}}$ is a core for $B_{\mathbf{E}}^a$, conjugating the preceding inequality gives a bounded linear functional

$$\mathbf{v} \longmapsto \int_M \mathbf{h}_{\mathbf{E}}(B_{\mathbf{E}}^a \mathbf{v}, \mathbf{u})\, d\lambda_{\mathbf{g}}$$

on $\mathscr{C}_{\mathbf{E}}$ with respect to the norm of $L^2(M,\mathbf{E})$. This extends to all of $L^2(M,\mathbf{E})$, and the Riesz representation theorem B.3.12 provides $\mathbf{g} \in L^2(M,\mathbf{E})$ such that

$$\int_M \mathbf{h}_{\mathbf{E}}(B_{\mathbf{E}}^a \mathbf{v}, \mathbf{u})\, d\lambda_{\mathbf{g}} = \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{v}, \mathbf{g})\, d\lambda_{\mathbf{g}}, \qquad \mathbf{v} \in \mathscr{C}_{\mathbf{E}}.$$

Since $\mathscr{C}_{\mathbf{E}}$ is a core for $B_{\mathbf{E}}^a$, the identity extends to all of $v \in \mathcal{D}(B_{\mathbf{E}}^a)$. By the definition of the adjoint and self-adjointness of $B_{\mathbf{E}}^a$, this proves that $\mathbf{u} \in \mathcal{D}(B_{\mathbf{E}}^a)$, $B_{\mathbf{E}}^a \mathbf{u} = \mathbf{g}$, and $\|B_{\mathbf{E}}^a \mathbf{u}\|_{L^2(M,\mathbf{E})} \le \|F\|_{\mathscr{F}}$. Taking the infimum over F gives the reverse embedding.

For arbitrary orders $s_0 < s_1$, choose $r > \max\{0, -s_0\}$. By Theorem 12.4.9, the same potential $J_{r,2}^{\mathbf{E}}$ is a simultaneous isometric isomorphism

$$H^{s_j,2}_{L_{\mathbf{E}}}(M,\mathbf{E}) \longrightarrow H^{s_j+r,2}_{L_{\mathbf{E}}}(M,\mathbf{E}), \qquad j \in \{0,1\},$$

and also at the intermediate order. The complex interpolation theorem for operators 5.7.5, applied to $J_{r,2}^{\mathbf{E}}$ and its inverse on these compatible pairs, reduces the general case to the one already proved.

Now prove (12.10). Set $\mathbf{f} := B_{\mathbf{E}}^{\frac{s}{2}}\mathbf{u}$, so that $\mathbf{u} = J_{s,2}^{\mathbf{E}}\mathbf{f}$. The bilinear symmetry of the potentials, expressed between $\mathbf{E}$ and $\overline{\mathbf{E}}$, gives

$$\langle J_{s,2}^{\mathbf{E}}\mathbf{f}, \mathbf{w}\rangle_{s,2;\mathbf{E}} = \int_M \mathcal{R}_{\mathbf{E}}(J_{s,2}^{\overline{\mathbf{E}}}\mathbf{w})(\mathbf{f})\, d\lambda_{\mathbf{g}},$$

which is exactly the asserted formula. The operators $J_{r,2}$, spectral powers of real exponent, and conjugation commute because their scalar multipliers are real. Finally, spectral truncations form a dense subspace at every order; on this subspace, all the preceding identifications are the same algebraic formulas. Uniqueness

of continuous extensions proves the final compatibility assertion. □

Remark 12.4.12. (Complex interpolation outside L^2) The Hilbert space proof uses the fact that $(I + L_{\mathbf{E}})^{it}$ is unitary on L^2 for $t \in \mathbb{R}$. For $1 < p < \infty$, the argument requires a bound for these powers on $L^p(M, \mathbf{E})$; heat contractivity alone does not provide this bound for an abstract semigroup. Lemma 13.3.26 will prove it for the covariant Laplacian on a complete manifold, and Theorem 13.3.27 will establish the corresponding interpolation result. Comparison with localized norms additionally requires bounded geometry of the base and bundle; it is established in Theorem 14.5.6.

Corollary 12.4.13. Let $(M, \mathbf{g})$ be a connected, complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle of finite rank with a bundle metric and a compatible connection. If $m \in \mathbb{N}$ and $1 \le p < \infty$, then

$$H^{2m,p}_{L_{\mathbf{E}}}(M, \mathbf{E}) = \mathcal{D}\big((I + L_{\mathbf{E},p})^m\big)$$

and

$$\|\mathbf{u}\|_{H^{2m,p}_{L_{\mathbf{E}}}(M,\mathbf{E})} = \|(I + L_{\mathbf{E},p})^m \mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

Proof. Proposition 12.4.2 gives

$$J^{\mathbf{E}}_{2m,p} = (I + L_{\mathbf{E},p})^{-m}.$$

The range of this iterated resolvent is the domain of $(I+L_{\mathbf{E},p})^m$, and the range norm agrees with the indicated norm. □

Corollary 12.4.14. (Comparison on a closed manifold) If M is closed, then, for every $m \in \mathbb{N}_0$,

$$H^{m,2}_{L_{\mathbf{E}}}(M, \mathbf{E}) = H^m(M, \mathbf{E})$$

and there exist constants $c_m, C_m > 0$, depending only on m and the fixed data $(M, g, \mathbf{E}, \mathbf{h}_{\mathbf{E}}, \nabla^{\mathbf{E}})$, such that

$$c_m \|\mathbf{u}\|_{H^m(M,\mathbf{E})} \le \|\mathbf{u}\|_{H^{m,2}_{L_{\mathbf{E}}}(M,\mathbf{E})} \le C_m \|\mathbf{u}\|_{H^m(M,\mathbf{E})}, \qquad \mathbf{u} \in H^m(M, \mathbf{E}).$$

Proof. For $m = 0$, the two norms agree. Let $m = 2q$ with $q \ge 1$. Corollary 11.5.1 gives

$$\mathcal{D}(L^q_{\mathbf{E}}) = H^{2q}(M, \mathbf{E})$$

and constants $a_q, A_q > 0$ for which

$$a_q \|\mathbf{u}\|_{H^{2q}(M,\mathbf{E})} \le \|\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|L^q_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})} \le A_q \|\mathbf{u}\|_{H^{2q}(M,\mathbf{E})}.$$

For $\lambda \ge 0$,

$$1 + \lambda^{2q} \le (1 + \lambda)^{2q} \le 2^{2q-1}(1 + \lambda^{2q}).$$

Spectral calculus, together with

$$\big(\|\mathbf{u}\|^2_{L^2(M,\mathbf{E})} + \|L^q_{\mathbf{E}}\mathbf{u}\|^2_{L^2(M,\mathbf{E})}\big)^{\frac{1}{2}} \le \|\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|L^q_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})} \le \sqrt{2}\,\big(\|\mathbf{u}\|^2_{L^2(M,\mathbf{E})} + \|L^q_{\mathbf{E}}\mathbf{u}\|^2_{L^2(M,\mathbf{E})}\big)^{\frac{1}{2}},$$

proves the asserted inequality with $c_{2q} = \frac{a_q}{\sqrt{2}}$ and $C_{2q} = 2^{q-\frac{1}{2}} A_q$.
For $m = 2q+1$, spectral calculus gives

$$\mathcal{D}\big((I+L_{\mathbf{E}})^{q+\frac{1}{2}}\big) = \Big\{ \mathbf{u} \in \mathcal{D}\big((I+L_{\mathbf{E}})^{q}\big) \,\Big|\, (I+L_{\mathbf{E}})^{q}\mathbf{u} \in \mathcal{D}\big((I+L_{\mathbf{E}})^{\frac{1}{2}}\big) \Big\}.$$

Theorem 12.4.10 identifies the last domain with $W^{1,2}(M,\mathbf{E})$. Set

$$\mathcal{N}_q(\mathbf{u})^2 := \sum_{j=0}^{q} \|L_{\mathbf{E}}^{j}\mathbf{u}\|^2_{L^2(M,\mathbf{E})} + \|\nabla^{\mathbf{E}} L_{\mathbf{E}}^{q}\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})}.$$

The identity $\|\nabla^{\mathbf{E}} L_{\mathbf{E}}^{j}\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})} = \langle L_{\mathbf{E}}^{j+1}\mathbf{u}, L_{\mathbf{E}}^{j}\mathbf{u}\rangle_{L^2(M,\mathbf{E})}$ and Cauchy–Schwarz give

$$\|\nabla^{\mathbf{E}} L_{\mathbf{E}}^{j}\mathbf{u}\|^2_{L^2(M,T^*M\otimes\mathbf{E})} \le \frac{1}{2}\big(\|L_{\mathbf{E}}^{j+1}\mathbf{u}\|^2_{L^2(M,\mathbf{E})} + \|L_{\mathbf{E}}^{j}\mathbf{u}\|^2_{L^2(M,\mathbf{E})}\big), \qquad 0 \le j < q.$$

If $\|\mathbf{u}\|_{\Delta_B,2q+1}$ denotes the norm in Theorem 9.1.42, these inequalities imply

$$\mathcal{N}_q(\mathbf{u})^2 \le \|\mathbf{u}\|^2_{\Delta_B,2q+1} \le (q+1)\mathcal{N}_q(\mathbf{u})^2.$$

The cited theorem provides $\alpha_q, b_q > 0$, depending only on q and the fixed geometric data, such that

$$\alpha_q \|\mathbf{u}\|^2_{H^{2q+1}(M,\mathbf{E})} \le \|\mathbf{u}\|^2_{\Delta_B,2q+1} \le b_q \|\mathbf{u}\|^2_{H^{2q+1}(M,\mathbf{E})}.$$

On the other hand, for $\lambda \ge 0$,

$$\frac{1}{q+2}\left(\sum_{j=0}^{q} \lambda^{2j} + \lambda^{2q+1}\right) \le (1+\lambda)^{2q+1} \le 2^{2q}\left(\sum_{j=0}^{q} \lambda^{2j} + \lambda^{2q+1}\right).$$

Spectral calculus turns this polynomial comparison into

$$\frac{1}{\sqrt{q+2}}\mathcal{N}_q(\mathbf{u}) \le \|(I+L_{\mathbf{E}})^{q+\frac{1}{2}}\mathbf{u}\|_{L^2(M,\mathbf{E})} \le 2^q \mathcal{N}_q(\mathbf{u}).$$

Consequently, we may take

$$c_{2q+1} := \sqrt{\frac{\alpha_q}{(q+1)(q+2)}}, \qquad C_{2q+1} := 2^q\sqrt{b_q}.$$

Moreover, $\Gamma(\mathbf{E})$ is dense in both norms: it is dense in $H^{2q+1}(M,\mathbf{E})$ by definition and, since M is compact, it is also dense in $H^{2q+1,2}_{L_{\mathbf{E}}}(M,\mathbf{E})$. Indeed, spectral calculus gives

$$J^{\mathbf{E}}_{2q+1,2}(\Gamma(\mathbf{E})) \subseteq \bigcap_{k\in\mathbb{N}} \mathcal{D}(L_{\mathbf{E}}^{k}) = \Gamma(\mathbf{E}),$$

while part (d) of Theorem 12.4.9 states that the subset on the left-hand side is dense in $H^{2q+1,2}_{L_{\mathbf{E}}}(M,\mathbf{E})$. Thus the inequalities extend to the completions and simultaneously prove equality of the two spaces. □

Remark 12.4.15. (Comparison beyond the compact case) The equality $H^{1,2}_{L^2_{\mathbf{E}}}(M,\mathbf{E}) = W^{1,2}(M,\mathbf{E})$ holds on every complete manifold. For higher orders and for $p \neq 2$, identifying the Bessel scale with the spaces defined by covariant derivatives requires global elliptic estimates. The chapter on elliptic operators provides these estimates on closed manifolds, while the chapter on bounded geometry explains the uniform hypotheses needed in the noncompact case. Completeness alone guarantees the existence of the Bessel scale, but not global equivalence with a norm formed from finitely many covariant derivatives.

Remark 12.4.16. (Choice of operator) The construction can be carried out with other self-adjoint elliptic operators bounded from below, provided that their semigroups admit consistent extensions to $L^p(M,\mathbf{E})$. In this chapter, we chose the connection Laplacian because it is determined by $(g,\mathbf{h}_{\mathbf{E}},\nabla^{\mathbf{E}})$, its quadratic form is exactly the energy of the connection, and Kato's inequality gives a direct comparison with the scalar semigroup.

13

Sobolev spaces on manifolds of bounded geometry

An estimate obtained in a chart may be correct and yet provide no global information: its constant may grow as the center of the chart tends to infinity. On a compact manifold, choosing a finite cover suffices. On a noncompact manifold, we need another property: the ability to repeat the same local argument with radii and constants independent of the point.

Bounded geometry makes this possible. The injectivity radius provides normal charts of a common size; bounds on the curvature and its derivatives control the metric in these charts. From these data we will construct covers of uniformly bounded multiplicity and partitions of unity whose derivatives remain bounded. These are the conditions that allow us to sum the local estimates without losing control of the norm.

We will then compare spaces defined by localization with those defined using covariant derivatives or powers of the Laplacian. In this comparison, we must distinguish integer regularity from fractional regularity and, within the latter, the different orders of summation defining Besov and Triebel–Lizorkin spaces. We will retain these distinctions in interpolation and duality arguments as well.

The manifolds in this chapter are assumed to have no boundary. The case with boundary is developed in Chapter 14, where normal charts near the boundary are replaced by Fermi charts, and Euclidean norms by restriction norms on the half-space.

Uniformity in noncompact spaces

On a compact manifold, a finite cover allows the distortions of the charts and the metric to be absorbed into a single constant. This advantage disappears in the noncompact case. Bounded geometry replaces finiteness by uniformity: charts of controlled size, bounded multiplicity, and estimates independent of the center. Eichhorn's work and the framework of uniformly regular manifolds developed by Amann provide two closely related ways of turning this geometric uniformity into a stable theory of function spaces [53, 6].

13.1 Riemannian manifolds of bounded geometry

In the analytic study of a Riemannian manifold, it is essential to have local coordinate systems in which the geometry is controlled and comparable to its Euclidean counterpart. One of the most effective tools for this purpose is normal coordinates, constructed using the Riemannian exponential map at a point. These coordinates give a particularly simple representation of the metric near their center and facilitate the transfer of classical techniques from $\mathbb{R}^n$ to manifolds.

However, this procedure has limitations: at some distance, the exponential map may cease to be a diffeomorphism onto its image because conjugate points occur or distinct geodesics issuing from the same point intersect. This leads to the need to quantify the largest radius within which normal coordinates behave appropriately.

This parameter is the *injectivity radius* of the manifold, which plays a central role in geometry and analysis. A positive lower bound for this radius allows us to use normal charts of a common size. To control their metric coefficients and derivatives as well, we will need the curvature bounds introduced below. Both conditions enter into comparisons of Sobolev, Besov, and Triebel–Lizorkin norms.

Definition 13.1.1. (Injectivity radius) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and let $p \in M$. The *injectivity radius at* p, denoted by $\operatorname{inj}(p)$, is the supremum of the numbers $r > 0$ for which the Riemannian exponential map

$$\exp_p : B_{\mathbf{g}_p}(0, r) \subset T_pM \longrightarrow M$$

is a diffeomorphism onto its image.
The *injectivity radius of* M is defined by

$$\operatorname{inj}(M) := \inf_{p \in M} \operatorname{inj}(p).$$

Remark 13.1.2. Let M be a Riemannian manifold without boundary. If $\operatorname{inj}(M) > 0$, then every connected component of M is complete. Indeed, choose $0 < r < \operatorname{inj}(M)$. Every unit-speed geodesic defined on a maximal interval $[0, b)$, with $b < \infty$, can be extended from any time $t_0 < b$ for at least a time r, since $\exp_{\gamma(t_0)}$ is defined on the ball of radius r. Taking $b - r < t_0 < b$ gives an extension beyond b, a contradiction. Thus, each component is geodesically complete; the Hopf–Rinow Theorem 2.9.9 implies its metric completeness.

Since $\operatorname{inj}(M) \geq 0$, there are only two possibilities: $\operatorname{inj}(M) = 0$ or $\operatorname{inj}(M) > 0$. The first behavior can occur, for example, on manifolds containing geodesic loops of arbitrarily small length, as illustrated in the following figure.

Remark 13.1.3. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and suppose that its injectivity radius satisfies $\operatorname{inj}(M) > 0$. Then, for each $p \in M$ and every $0 < r < \operatorname{inj}(M)$, the exponential map

$$\exp_p : B_{\mathbf{g}_p}(0, r) \subseteq T_pM \longrightarrow B_{\mathbf{g}}(p, r) \subseteq M$$

is a diffeomorphism.
In particular, every point $q \in B_{\mathbf{g}}(p, r)$ can be joined to p by a unique minimizing geodesic of length less than r, given by

$$\gamma(t) = \exp_p(tv), \qquad t \in [0, 1],$$

where $v \in T_pM$ satisfies $\|v\|_{\mathbf{g}_p} < r$ and $\exp_p(v) = q$.

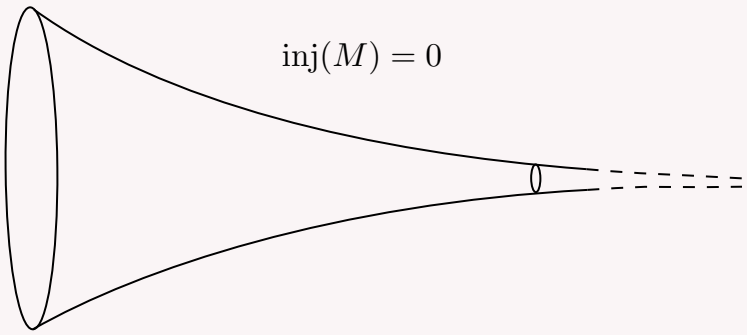


Figure 13.1. Schematic representation of a manifold with geodesic loops of arbitrarily small length, a phenomenon that can give rise to zero injectivity radius.

To quantify how closely the Riemannian metric approximates the Euclidean metric in normal coordinates, we will study the Taylor expansion of its coefficients. The following proposition makes this comparison explicit.

Proposition 13.1.4. Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, and let $p \in \mathrm{Int}(M)$. If (U, φ) is a normal coordinate chart centered at p, with $\varphi(U)$ convex and $\varphi(p) = 0$, then, for every $q \in U$, writing $x = (x^1, \dots, x^n) = \varphi(q)$, we have the following Taylor expansion:

$$(g_{ij} \circ \varphi^{-1})(x) = \delta_{ij} - \frac{1}{3} R_{iklj}(p) x^k x^l + R_{2,0,g_{ij}\circ\varphi^{-1}}(x),$$

where, for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|R_{2,0,g_{ij}\circ\varphi^{-1}}(x)| \leq \varepsilon \|x\|^2 \quad \text{if } \|x\| < \delta_\varepsilon.$$

Proof. Since $p \in \mathrm{Int}(M)$, we may consider the manifold without boundary $\mathrm{Int}(M)$ with the restricted metric. By Proposition 2.8.9 applied to this manifold, in normal coordinates centered at p we have

$$g_{ij}(p) = \delta_{ij}, \qquad \Gamma_{ij}^k(p) = 0.$$

By Corollary 2.4.16 (metric compatibility),

$$\partial_k g_{ij} = g_{mj}\Gamma_{ik}^m + g_{im}\Gamma_{jk}^m. \tag{13.1}$$

Evaluating at p and using $g_{mj}(p) = \delta_{mj}$, $g_{im}(p) = \delta_{im}$, and the vanishing of the Christoffel symbols at p, we obtain

$$(\partial_k g_{ij})(p) = 0.$$

Thus, the first-order derivatives vanish at p, and the corresponding linear term does not appear in the Taylor expansion.

Now, for arbitrary $v = v^i \partial_i|_p \in T_pM$, Proposition 2.8.9 gives $\gamma_v(t) = (tv^1, \dots, tv^n)$, so the geodesic equation

$$\frac{d^2x^k}{dt^2}(t) + \frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t)\Gamma_{ij}^k(x(t)) = 0$$

with $x(t) = \gamma_v(t)$ becomes

$$\Gamma_{ij}^k(tv)v^iv^j = 0.$$

Differentiating with respect to t gives:

$$0 = \frac{d}{dt}(\Gamma_{ij}^k(tv)v^iv^j) = \left(\frac{d}{dt}(\Gamma_{ij}^k(tv))\right)v^iv^j = \partial_l(\Gamma_{ij}^k)(tv)\frac{d(tv^l)}{dt}v^iv^j = \partial_l(\Gamma_{ij}^k)(tv)v^iv^jv^l.$$

Evaluating at $t = 0$ gives

$$\partial_l\Gamma_{ij}^k(p)v^iv^jv^l = 0.$$

This is a homogeneous polynomial of degree 3 that vanishes identically (since v was arbitrary). The coefficient of each term $v^iv^jv^l$ is the sum over 6 permutations of these indices (i, j and l). Since $\partial_l\Gamma_{ij}^k = \partial_l\Gamma_{ji}^k$, we obtain

$$\partial_l\Gamma_{ij}^k(p) + \partial_i\Gamma_{jl}^k(p) + \partial_j\Gamma_{li}^k(p) = 0. \tag{13.2}$$

We now differentiate (13.1) once more:

$$\partial_l\partial_k g_{ij} = \partial_l\left(g_{mj}\Gamma_{ik}^m + g_{im}\Gamma_{jk}^m\right) = (\partial_l g_{mj})\Gamma_{ik}^m + g_{mj}(\partial_l\Gamma_{ik}^m) + (\partial_l g_{im})\Gamma_{jk}^m + g_{im}(\partial_l\Gamma_{jk}^m).$$

Evaluating at p and again using the vanishing of the Christoffel symbols and the partial derivatives of order 1 at p, only terms containing derivatives of the Christoffel symbols remain. Consequently,

$$(\partial_l\partial_k g_{ij})(p) = g_{mj}(p)(\partial_l\Gamma_{ik}^m)(p) + g_{im}(p)(\partial_l\Gamma_{jk}^m)(p).$$

Finally, since the metric coefficients evaluated at p are Kronecker deltas, we are left with:

$$(\partial_l\partial_k g_{ij})(p) = (\partial_l\Gamma_{ik}^j)(p) + (\partial_l\Gamma_{jk}^i)(p). \tag{13.3}$$

On the other hand, Proposition 2.10.3 gives

$$R_{ijk}{}^l = \partial_i\Gamma_{jk}^l - \partial_j\Gamma_{ik}^l + \Gamma_{jk}^m\Gamma_{im}^l - \Gamma_{ik}^m\Gamma_{jm}^l,$$

so evaluating at p and using the vanishing of the Christoffel symbols at p, we obtain

$$R_{ikl}{}^j(p) = \partial_i\Gamma_{kl}^j(p) - \partial_k\Gamma_{il}^j(p), \qquad R_{ilk}{}^j(p) = \partial_i\Gamma_{lk}^j(p) - \partial_l\Gamma_{ik}^j(p).$$

Adding these terms gives

$$R_{ikl}{}^j(p) + R_{ilk}{}^j(p) = 2\partial_i\Gamma_{kl}^j(p) - \partial_l\Gamma_{ik}^j(p) - \partial_k\Gamma_{il}^j(p)$$

and using (13.2) with k and j interchanged, we obtain

$$\partial_l\Gamma_{ik}^j(p) + \partial_i\Gamma_{kl}^j(p) + \partial_k\Gamma_{li}^j(p) = 0,$$

so that

$$R_{ikl}{}^j + R_{ilk}{}^j = 3\partial_i\Gamma_{kl}^j(p).$$

Since $g_{ij} = \delta_{ij}$ at p, the preceding identity can be written in terms of the Riemann curvature tensor as follows:

$$R_{iklj}(p) + R_{ilkj}(p) = R_{ikl}{}^{j}(p) + R_{ilk}{}^{j}(p) = 3\partial_i \Gamma^j_{kl}(p).$$

Interchanging i and l gives

$$R_{likj}(p) + R_{lkij}(p) = 3\partial_l \Gamma^j_{ik}(p)$$

and interchanging i and l, and then i and j, gives

$$R_{ljki}(p) + R_{lkji}(p) = 3\partial_l \Gamma^i_{jk}(p).$$

Using identity (13.3), we obtain

$$3\partial_l \partial_k g_{ij}(p) = R_{likj}(p) + R_{lkij}(p) + R_{ljki}(p) + R_{lkji}(p).$$

By Corollary 2.10.7, $R_{lkij}(p) = -R_{lkji}(p)$ and $R_{ljki}(p) = R_{kilj}(p)$, so

$$\partial_l \partial_k g_{ij}(p) = \frac{1}{3}\big(R_{likj}(p) + R_{kilj}(p)\big) = -\frac{1}{3}(R_{iklj}(p) + R_{ilkj}(p)),$$

where the last equality uses $R_{likj}(p) = -R_{ilkj}(p)$ and $R_{kilj}(p) = -R_{iklj}(p)$.
By (Taylor's) Theorem 1.4.9,

$$\begin{aligned}(g_{ij} \circ \varphi^{-1})(x) &= \delta_{ij} + \partial_k g_{ij}(p)x^k + \frac{1}{2}\partial_k \partial_l g_{ij}(p)x^k x^l + R_{2,0,g_{ij}\circ\varphi^{-1}}(x) \\ &= \delta_{ij} - \frac{1}{6}\big(R_{iklj}(p) + R_{ilkj}(p)\big)x^k x^l + R_{2,0,g_{ij}\circ\varphi^{-1}}(x) \\ &= \delta_{ij} - \frac{1}{3}R_{iklj}(p)x^k x^l + R_{2,0,g_{ij}\circ\varphi^{-1}}(x).\end{aligned}$$

In the last equality, the dummy indices k and l are interchanged in the second summand. □

Thus, curvature determines the first deviation of the metric from the Euclidean metric in normal coordinates. Its covariant derivatives enter into higher-order expansions; see [106]. However, an expansion at each center does not by itself provide a uniform radius or bound for its remainders. To obtain this control, we will work directly with the Jacobi equation. The hypotheses that make this possible are as follows.

Definition 13.1.5. (Manifolds of bounded geometry) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary. We say that M has bounded geometry if the following two conditions hold:

(a) $\operatorname{inj}(M) > 0$.

(b) For each $k \in \mathbb{N}_0$, there exists $C_k > 0$ such that

$$\sup_{x \in M} |\nabla^k \mathbf{R}(x)|_{\mathbf{g}} \leq C_k.$$

Here $\mathbf{R} \in \Gamma(T^{(1,3)}(TM))$ is the curvature tensor of Definition 2.10.1, $\nabla^k \mathbf{R}$ is its kth derivative for the tensor connection induced by the Levi–Civita connection, and $|\cdot|_{\mathbf{g}}$ is the pointwise norm induced by $\mathbf{g}$ and its dual metric on all tensor factors.

Remark 13.1.6. (Euclidean open subsets and manifold boundary) An open subset of $\mathbb{R}^n$ equipped with the restricted Euclidean metric need not have bounded geometry. For example, let $M = B_{\mathrm{euc}}(0,1)$, regarded as a manifold without boundary. Its curvature is zero, but, for $x \in M$, the exponential map is $\exp_x(v) = x + v$ only as long as the corresponding segment remains in the ball. Consequently,

$$\mathrm{inj}(x) = 1 - |x|, \qquad \mathrm{inj}(M) = 0.$$

The radial geodesic moving toward the sphere reaches it in finite time and cannot be extended within M. Completeness does not appear as a third hypothesis in the preceding definition: it follows from the positive injectivity radius by Remark 13.1.2, and this is precisely the condition that fails here.
The sphere is the boundary of this open set as a Euclidean subset; it is not a boundary included in the manifold M. In contrast, $\overline{B}_{\mathrm{euc}}(0,1)$, with its smooth structure as a compact manifold with boundary and the same Euclidean metric, belongs to the theory with boundary of Chapter 14. These are two distinct geometric objects. The metric matters as well: replacing it by a complete metric can change the conclusion concerning completeness. Thus, the Sobolev and Rellich theorems for arbitrary Euclidean open subsets, particularly the zero-extension arguments for $W_0^{m,p}$, do not follow simply by applying the global bounded-geometry theorems.

Since throughout this chapter we will work with ***normal coordinates*** (also called ***geodesic coordinates***), as well as with coordinates in general, we introduce some terminology that will allow us to refer to them more clearly.

Definition 13.1.7. (Geodesic atlas) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary.
We will denote by

$$\mathcal{A}^{\mathrm{geo}} = (U_i^{\mathrm{geo}}, \varphi_i^{\mathrm{geo}})_{i \in J}$$

an atlas consisting of normal coordinate charts (see Proposition 2.8.9).
More precisely, for each $i \in J$, there exist a point $p_i \in M$, a radius $r_i > 0$, and a linear isomorphism

$$B_{p_i} : \mathbb{R}^n \longrightarrow T_{p_i}M,$$

taking the standard basis of $\mathbb{R}^n$ to an orthonormal basis of $T_{p_i}M$, such that

$$U_i^{\mathrm{geo}} = B_{\mathbf{g}}(p_i, r_i), \qquad \varphi_i^{\mathrm{geo}} = B_{p_i}^{-1} \circ \exp_{p_i}^{-1}.$$

We say that $\mathcal{A}^{\mathrm{geo}}$ is a *uniform geodesic atlas* if there exists $r > 0$ such that $r < \mathrm{inj}(M)$ and

$$U_i^{\mathrm{geo}} = B_{\mathbf{g}}(p_i, r)$$

for every $i \in J$.

The condition that the covariant derivatives of the curvature are bounded can be expressed in other ways which, together with $\mathrm{inj}(M) > 0$, yield useful equivalent characterizations of manifolds of bounded geometry.

Theorem 13.1.8. (Equivalent characterizations of bounded geometry) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary such that $\operatorname{inj}(M) > 0$. The following conditions are equivalent:

(a) For each $k \in \mathbb{N}_0$, there exists $C_k > 0$ such that

$$|\nabla^k \mathbf{R}|_{\mathbf{g}} \leq C_k.$$

(b) There exists r_0, with $0 < r_0 < \operatorname{inj}(M)$, such that the full family of normal charts of radius r_0 satisfies the following condition: for each $k \in \mathbb{N}_0$, there exists $C_{k,r_0} > 0$ such that, in every chart, the components of $\mathbf{g}$ and of its inverse satisfy

$$|D^\alpha g_{ij}| \leq C_{k,r_0}, \qquad |D^\alpha g^{ij}| \leq C_{k,r_0}, \qquad |\alpha| \leq k.$$

By restriction, the same estimates hold for every radius $0 < r \leq r_0$.

Whenever either condition holds, transitions between normal charts of any fixed radius $0 < r \leq r_0$ have uniformly bounded jets of every order: for each $k \in \mathbb{N}_0$, there exists $A_{k,r} > 0$ such that, if the charts i and j overlap,

$$\left|D^\alpha (\varphi_j^{\mathrm{geo}} \circ (\varphi_i^{\mathrm{geo}})^{-1})(x)\right| \leq A_{k,r}, \qquad |\alpha| \leq k,$$

throughout the domain of the transition. The same assertion holds with i and j interchanged.

Proof. Fix a common radius

$$0 < \rho < \min\{\operatorname{inj}(M), 1\}.$$

When necessary, we will decrease it once; all normal charts of radius ρ will still cover M. For each $p \in M$, choose a linear isometry $B_p \colon \mathbb{R}^n \longrightarrow T_pM$ and write

$$\Psi_p(v) := \exp_p(B_p v).$$

We denote by $g_{ij}^{(p)}$ and $g_{(p)}^{ij}$ the components of $\Psi_p^* \mathbf{g}$ and of its inverse metric, respectively, in the standard basis of $\mathbb{R}^n$.

First assume (1). For $v \in B(0, \rho)$ and $i \in \{1, \dots, n\}$, the variation

$$\alpha_i(s, t) := \exp_p\big(tB_p(v + se_i)\big)$$

produces the Jacobi field $\mathbf{J}_i(t) = \partial_s \alpha_i(0, t)$ along $\gamma_v(t) = \Psi_p(tv)$. Its initial data and equation are

$$\mathbf{J}_i(0) = 0, \qquad \frac{D\mathbf{J}_i}{dt}(0) = B_p e_i, \qquad \frac{D^2\mathbf{J}_i}{dt^2} + \mathbf{R}(\mathbf{J}_i, \dot\gamma_v)\dot\gamma_v = 0.$$

Moreover, $\mathbf{J}_i(1) = d\Psi_p(v)e_i$. Parallel transport the basis $(B_p e_a)_{a=1}^n$ along γ_v. In this frame, the equation is a linear system

$$y_i''(t) + \mathcal{R}_{p,v}(t) y_i(t) = 0, \qquad y_i(0) = 0, \qquad y_i'(0) = e_i,$$

and $\|\mathcal{R}_{p,v}(t)\| \leq C_0 \|v\|^2$. Writing it as a first-order system and applying Lemma 1.8.19 gives a bound for

$\|y_i\| + \|y_i'\|$ independent of p and v. The integral form of the equation also gives

$$\|y_i(t) - te_i\| \le C_0\|v\|^2 \int_0^t (t-\tau)\|y_i(\tau)\|\,d\tau.$$

Decrease ρ, depending only on C_0 and n, so that the right-hand side at $t=1$ is less than $\frac{1}{2\sqrt{n}}$ for every i. If $\mathbf{Y}(v)$ is the matrix whose columns are $y_i(1)$, then

$$\|\mathbf{Y}(v) - I\|_{\mathrm{op}} \le \|\mathbf{Y}(v) - I\|_{\mathrm{HS}} < \frac{1}{2}.$$

By the Neumann series, $\mathbf{Y}(v)$ is invertible and $\|\mathbf{Y}(v)^{-1}\|_{\mathrm{op}} \le 2$, uniformly in p and v. Since the parallel frame is orthonormal,

$$g_{ij}^{(p)}(v) = \langle \mathbf{J}_i(1), \mathbf{J}_j(1)\rangle_{\mathbf{g}},$$

which proves the uniform ellipticity of the coefficient matrix $(g_{ij}^{(p)})$.
We now justify the derivatives of arbitrary order. Fix $m \in \mathbb{N}_0$ and vectors $w_0, \dots, w_m \in \{e_1, \dots, e_n\}$ and, for sufficiently small $\boldsymbol{s} = (s_0, \dots, s_m)$, consider the multiparameter variation

$$\alpha(\boldsymbol{s}, t) := \exp_p\left(tB_p\left(v + \sum_{a=0}^{m} s_a w_a \right)\right), \qquad 0 \le t \le 1.$$

Let $\boldsymbol{\mathcal{E}} := \alpha^* TM$, and denote by $\mathbb{D}$ the connection on $\boldsymbol{\mathcal{E}}$ induced by the Levi–Civita connection. Set

$$D_t := \mathbb{D}_{\partial_t}, \qquad D_a := \mathbb{D}_{\partial_{s_a}}, \qquad \mathbf{T} := d\alpha(\partial_t), \qquad \mathbf{A}_a := d\alpha(\partial_{s_a}).$$

For each $\boldsymbol{s}$, the curve $t \mapsto \alpha(\boldsymbol{s}, t)$ is geodesic; moreover, since the Levi–Civita connection is torsion-free,

$$D_t \mathbf{T} = 0, \qquad D_a \mathbf{T} = D_t \mathbf{A}_a.$$

With the convention $\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}$, the commutators on the pullback bundle are precisely

$$(D_a D_t - D_t D_a)\mathbf{Z} = \mathbf{R}(\mathbf{A}_a, \mathbf{T})\mathbf{Z}, \qquad (D_a D_b - D_b D_a)\mathbf{Z} = \mathbf{R}(\mathbf{A}_a, \mathbf{A}_b)\mathbf{Z}.$$

In particular, differentiating $D_t \mathbf{T} = 0$ in direction s_a gives, with this sign,

$$D_t^2 \mathbf{A}_a + \mathbf{R}(\mathbf{A}_a, \mathbf{T})\mathbf{T} = 0.$$

We will carry out the induction without permuting covariant derivatives. For an ordered word $I = (a_1, \dots, a_k)$, write

$$|I| = k, \qquad D_I := D_{a_k} \cdots D_{a_1}, \qquad I^- := (a_1, \dots, a_{k-1}),$$

and set $D_\varnothing = \mathrm{id}$. Introduce the operator

$$\mathcal{L}\mathbf{Z} := D_t^2 \mathbf{Z} + \mathbf{R}(\mathbf{Z}, \mathbf{T})\mathbf{T}, \qquad \mathcal{L} \colon \Gamma(\boldsymbol{\mathcal{E}}) \longrightarrow \Gamma(\boldsymbol{\mathcal{E}}).$$

A direct calculation using the preceding commutators gives, for every $\mathbf{Z} \in \Gamma(\boldsymbol{\mathcal{E}})$,

$$\mathcal{L}(D_a\mathbf{Z}) = D_a(\mathcal{L}\mathbf{Z}) + \mathcal{C}(\mathbf{A}_a, \mathbf{Z}),$$

where $\mathcal{C}(\mathbf{A}, \mathbf{Z}) \in \Gamma(\boldsymbol{\mathcal{E}})$ is

$$\begin{aligned}\mathcal{C}(\mathbf{A}, \mathbf{Z}) = & -(\nabla_{\mathbf{T}}\mathbf{R})(\mathbf{A}, \mathbf{T})\mathbf{Z} - \mathbf{R}(D_t\mathbf{A}, \mathbf{T})\mathbf{Z} - 2\mathbf{R}(\mathbf{A}, \mathbf{T})D_t\mathbf{Z} \\ & -(\nabla_{\mathbf{A}}\mathbf{R})(\mathbf{Z}, \mathbf{T})\mathbf{T} - \mathbf{R}(\mathbf{Z}, D_t\mathbf{A})\mathbf{T} - \mathbf{R}(\mathbf{Z}, \mathbf{T})D_t\mathbf{A}.\end{aligned}$$

Indeed,

$$D_t^2 D_a\mathbf{Z} = D_a D_t^2\mathbf{Z} - (\nabla_{\mathbf{T}}\mathbf{R})(\mathbf{A}_a, \mathbf{T})\mathbf{Z} - \mathbf{R}(D_t\mathbf{A}_a, \mathbf{T})\mathbf{Z} - 2\mathbf{R}(\mathbf{A}_a, \mathbf{T})D_t\mathbf{Z},$$

and the derivative of $\mathbf{R}(\mathbf{Z}, \mathbf{T})\mathbf{T}$, using $D_a\mathbf{T} = D_t\mathbf{A}_a$, contributes precisely the remaining three terms. This verifies both the signs and the tensor spaces: each summand of $\mathcal{C}(\mathbf{A}, \mathbf{Z})$ is a section of $\boldsymbol{\mathcal{E}}$.
For a distinguished letter $b \in \{0, \dots, m\}$, define

$$\mathbf{Z}_{I;b} := D_I\mathbf{A}_b \in \Gamma(\boldsymbol{\mathcal{E}}), \qquad \widetilde{\mathbf{Q}}_{\varnothing;b} := 0,$$

and, if $I = (I^-, a)$, define recursively

$$\widetilde{\mathbf{Q}}_{I;b} := D_a\widetilde{\mathbf{Q}}_{I^-;b} + \mathcal{C}(\mathbf{A}_a, \mathbf{Z}_{I^-;b}) \in \Gamma(\boldsymbol{\mathcal{E}}).$$

We prove the recurrence by induction on $|I|$. For $I = \varnothing$, the Jacobi equation gives

$$\mathcal{L}\mathbf{Z}_{\varnothing;b} = \mathcal{L}\mathbf{A}_b = 0 = \widetilde{\mathbf{Q}}_{\varnothing;b}.$$

Now suppose that $\mathcal{L}\mathbf{Z}_{I^-;b} = \widetilde{\mathbf{Q}}_{I^-;b}$ and write $I = (I^-, a)$. Since $\mathbf{Z}_{I;b} = D_a\mathbf{Z}_{I^-;b}$, the preceding commutation identity gives, term by term,

$$\begin{aligned}\mathcal{L}\mathbf{Z}_{I;b} &= \mathcal{L}(D_a\mathbf{Z}_{I^-;b}) \\ &= D_a(\mathcal{L}\mathbf{Z}_{I^-;b}) + \mathcal{C}(\mathbf{A}_a, \mathbf{Z}_{I^-;b}) \\ &= D_a\widetilde{\mathbf{Q}}_{I^-;b} + \mathcal{C}(\mathbf{A}_a, \mathbf{Z}_{I^-;b}) \\ &= \widetilde{\mathbf{Q}}_{I;b}.\end{aligned}$$

Thus, induction on $|I|$ proves the identity

$$\mathcal{L}\mathbf{Z}_{I;b} = \widetilde{\mathbf{Q}}_{I;b}.$$

This recurrence enumerates all summands. To make this explicit, consider formal expressions obtained by contracting, using $\mathbf{g}$, factors $\nabla^r\mathbf{R}$, $\mathbf{T}$, $\mathbf{Z}_{J;c}$, and $D_t\mathbf{Z}_{J;c}$, leaving a single free vector index. Define a formal derivation $\mathfrak{d}_a$ by the Leibniz rule on products and contractions and by the rules

$$\begin{aligned}\mathfrak{d}_a\mathbf{T} &= D_t\mathbf{A}_a, & \mathfrak{d}_a\mathbf{Z}_{J;c} &= \mathbf{Z}_{(J,a);c}, \\ \mathfrak{d}_a(D_t\mathbf{Z}_{J;c}) &= D_t\mathbf{Z}_{(J,a);c} + \mathbf{R}(\mathbf{A}_a, \mathbf{T})\mathbf{Z}_{J;c}, & \mathfrak{d}_a(\nabla^r\mathbf{R}) &= \nabla_{\mathbf{A}_a}(\nabla^r\mathbf{R}) = (\nabla^{r+1}\mathbf{R})(\mathbf{A}_a, \cdot).\end{aligned}$$

In the last rule, $\cdot$ represents, in their original order, the r differentiation arguments already present; the

intrinsic indices of $\mathbf{R}$ remain within the tensor value. The new argument is placed first. The remaining tensor arguments are also differentiated, one at a time, by the Leibniz rule. Differentiating the contractions produces no terms because $\nabla\mathbf{g} = 0$. Let $\mathfrak{C}_{a,J;b}$ be the six summands, with their coefficients, of

$$\begin{aligned}&-(\nabla_{\mathbf{T}}\mathbf{R})(\mathbf{A}_a,\mathbf{T})\mathbf{Z}_{J;b} - \mathbf{R}(D_t\mathbf{A}_a,\mathbf{T})\mathbf{Z}_{J;b} - 2\mathbf{R}(\mathbf{A}_a,\mathbf{T})D_t\mathbf{Z}_{J;b}\\ &\quad-(\nabla_{\mathbf{A}_a}\mathbf{R})(\mathbf{Z}_{J;b},\mathbf{T})\mathbf{T} - \mathbf{R}(\mathbf{Z}_{J;b},D_t\mathbf{A}_a)\mathbf{T} - \mathbf{R}(\mathbf{Z}_{J;b},\mathbf{T})D_t\mathbf{A}_a.\end{aligned}$$

We now define finite multisets of expressions by

$$\mathfrak{A}_{\varnothing;b} := \varnothing, \qquad \mathfrak{A}_{(J,a);b} := \mathfrak{d}_a\mathfrak{A}_{J;b} \sqcup \mathfrak{C}_{a,J;b},$$

where $\mathfrak{d}_a\mathfrak{A}_{J;b}$ contains all summands produced by the preceding rules, retaining multiplicities and coefficients. Let us verify the resulting equality. For $I = \varnothing$, both sides are zero because $\widetilde{\mathbf{Q}}_{\varnothing;b} = 0$ and $\mathfrak{A}_{\varnothing;b} = \varnothing$. Suppose that $\widetilde{\mathbf{Q}}_{J;b} = \sum_{\mathfrak{a}\in\mathfrak{A}_{J;b}} \mathfrak{a}$ and let $I = (J,a)$. Covariant differentiation D_a acts on each product and contraction exactly according to the rules defining $\mathfrak{d}_a$. Hence,

$$\widetilde{\mathbf{Q}}_{I;b} = D_a\widetilde{\mathbf{Q}}_{J;b} + \mathcal{C}(\mathbf{A}_a,\mathbf{Z}_{J;b}) = \sum_{\mathfrak{a}\in\mathfrak{A}_{J;b}} \mathfrak{d}_a\mathfrak{a} + \sum_{\mathfrak{c}\in\mathfrak{C}_{a,J;b}} \mathfrak{c} = \sum_{\mathfrak{a}\in\mathfrak{A}_{(J,a);b}} \mathfrak{a}.$$

This proves by induction on the length of I that

$$\widetilde{\mathbf{Q}}_{I;b} = \sum_{\mathfrak{a}\in\mathfrak{A}_{I;b}} \mathfrak{a}. \tag{13.4}$$

Thus, $\mathfrak{A}_{(J,a);b}$ is determined by $\mathfrak{A}_{J;b}$ through the preceding union, which preserves the order of the letters and indexes every summand.

The indexing also has an exact grading. Assign weight zero to $\mathbf{T}$ and weight $|J|+1$ to every occurrence of $\mathbf{Z}_{J;c}$ or $D_t\mathbf{Z}_{J;c}$; in particular, $\mathbf{A}_c = \mathbf{Z}_{\varnothing;c}$ has weight one. If $|I| = k \geq 1$, every summand of (13.4) satisfies

$$\sum_{\text{occurrences of } \mathbf{Z}_{J;c} \text{ or } D_t\mathbf{Z}_{J;c}} (|J|+1) = k+1. \tag{13.5}$$

It contains one or more factors $\nabla^r\mathbf{R}$, with $0 \leq r \leq k$, and each field factor contains at most one derivative D_t. Moreover, every field appearing in the source has weight at most k, strictly less than the weight $k+1$ of the unknown $\mathbf{Z}_{I;b}$. We verify the four assertions simultaneously. If $|I| = 1$, write $I = (a)$. The six terms of $\mathfrak{C}_{a,\varnothing;b}$ contain the fields $\mathbf{A}_a$ and $\mathbf{A}_b$, or one of their derivatives D_t; their total weight is $1+1 = 2 = |I|+1$. They contain a factor $\mathbf{R}$ or $\nabla\mathbf{R}$, contain no second time derivatives, and the fields appearing in them have weight one.

Now assume the assertions hold for a word J of length $k-1$, and let $I = (J,a)$. In the terms arising from $\mathfrak{d}_a\mathfrak{A}_{J;b}$, the Leibniz rule makes $\mathfrak{d}_a$ act on one factor at a time. If it acts on $\mathbf{Z}_{L;c}$ or $D_t\mathbf{Z}_{L;c}$, it replaces that factor by one whose weight is exactly one greater; the additional curvature term in the rule for $D_t\mathbf{Z}_{L;c}$ also introduces $\mathbf{A}_a$, again of weight one. If it acts on $\mathbf{T}$, it introduces $D_t\mathbf{A}_a$, of weight one; and if it acts on $\nabla^r\mathbf{R}$, it raises the order of the curvature derivative from r to $r+1$ and introduces the argument $\mathbf{A}_a$. In every case, the total weight increases by exactly one, from k to $k+1$, no $D_t^2\mathbf{Z}$ appears, and the curvature order remains at most k. Finally, the six new terms of $\mathfrak{C}_{a,J;b}$ have weight $1+(|J|+1) = k+1$, curvature order at most one,

and only first time derivatives. Since the field of greatest weight among them is $\mathbf{Z}_{J;b}$, of weight k, the entire source contains only fields of weight less than $k+1$. This completes the induction.

Restricting to $\boldsymbol{s}=0$, set

$$\mathbf{Y}_{I;b} := \mathbf{Z}_{I;b}|_{\boldsymbol{s}=0}, \qquad \mathbf{Q}_{I;b} := \widetilde{\mathbf{Q}}_{I;b}|_{\boldsymbol{s}=0} \in \Gamma(\gamma_v^* TM).$$

Therefore,

$$\frac{D^2\mathbf{Y}_{I;b}}{dt^2} + \mathbf{R}(\mathbf{Y}_{I;b}, \dot\gamma_v)\dot\gamma_v = \mathbf{Q}_{I;b}.$$

All initial data are explicit. Since $\alpha(\boldsymbol{s},0)=p$,

$$\mathbf{A}_b(\boldsymbol{s},0)=0, \qquad D_t\mathbf{A}_b(\boldsymbol{s},0)=B_p w_b;$$

and this, together with the same commutators, implies

$$\begin{aligned} &\mathbf{Y}_{\varnothing;b}(0)=0, \quad \frac{D\mathbf{Y}_{\varnothing;b}}{dt}(0)=B_p w_b,\\ &\mathbf{Y}_{I;b}(0)=0, \quad \frac{D\mathbf{Y}_{I;b}}{dt}(0)=0, \qquad |I|\ge 1. \end{aligned}$$

Indeed, at $t=0$, all purely parametric derivatives of $\mathbf{A}_b(\boldsymbol{s},0)=0$ vanish, and every term arising when D_t is commuted with a nonempty word contains one of these fields; moreover, $B_p w_b$ does not depend on $\boldsymbol{s}$.

Restricting (13.4) preserves the grading (13.5): it replaces $\mathbf{T}$ by $\dot\gamma_v$, $\mathbf{Z}_{J;c}$ by $\mathbf{Y}_{J;c}$, and $D_t\mathbf{Z}_{J;c}$ by $\frac{D\mathbf{Y}_{J;c}}{dt}$. Since $|\dot\gamma_v|_{\mathbf{g}} = |B_p v|_{\mathbf{g}_p} = \|v\|_{\mathbb{R}^n} \le \rho$, the hypothesis on $\nabla^r\mathbf{R}$ allows the bound to be expressed using L^∞ norms over the interval. For $k\in\mathbb{N}$, let $\mathcal{W}_{m,k} := \bigcup_{\nu=0}^{k-1}\{0,\dots,m\}^\nu$ be the set of words of length less than k, including the empty word. Define

$$\mathcal{M}_k := \sum_{J\in\mathcal{W}_{m,k}}\sum_{c=0}^{m}\left(\|\mathbf{Y}_{J;c}\|_{L^\infty([0,1])} + \left\|\frac{D\mathbf{Y}_{J;c}}{dt}\right\|_{L^\infty([0,1])}\right),$$

with the pointwise norms induced by $\mathbf{g}$. The index set of the sum is finite. By the grading already proved, each field factor in $\mathbf{Q}_{I;b}$ has weight at most k when $|I|=k$. Therefore, there exist $C_k>0$ and $N_k\in\mathbb{N}$, independent of p, v, and the chosen coordinate directions, such that

$$\|\mathbf{Q}_{I;b}\|_{L^\infty([0,1])} \le C_k(1+\mathcal{M}_k)^{N_k}, \qquad |I|=k.$$

Here C_k and N_k depend only on n, k, ρ, the bounds $\|\nabla^r \mathrm{Rm}_{\mathbf{g}}\|_\infty$ for $0\le r\le k$, and the finite number of contractions produced by the recurrence $\widetilde{\mathbf{Q}}_{(I^-,a);b} = D_a\widetilde{\mathbf{Q}}_{I^-;b} + \mathcal{C}(\mathbf{A}_a, \mathbf{Z}_{I^-;b})$; in particular, they do not depend on the center p.

To establish the bounds, fix v and use along γ_v the orthonormal frame obtained by parallel transport of $(B_p e_r)_{r=1}^n$. In this frame, the last equation has the form

$$y''_{I;b}(t) + \mathcal{R}_{p,v}(t)y_{I;b}(t) = q_{I;b}(t), \qquad \|\mathcal{R}_{p,v}(t)\|_{\mathrm{op},\mathbb{R}^n} \le C_0\rho^2.$$

It is equivalent to the first-order system

$$\frac{d}{dt}\begin{pmatrix} y_{I;b} \\ y'_{I;b}\end{pmatrix} = \begin{pmatrix} 0 & I \\ -\mathcal{R}_{p,v}(t) & 0\end{pmatrix}\begin{pmatrix} y_{I;b} \\ y'_{I;b}\end{pmatrix} + \begin{pmatrix} 0 \\ q_{I;b}\end{pmatrix}.$$

We proceed by induction on the weight. For weight one, $I = \varnothing$ and the equation is the homogeneous Jacobi equation with initial data $y_{\varnothing;b}(0) = 0$ and $y'_{\varnothing;b}(0) = w_b$; the estimate obtained at the beginning of the proof provides a constant B_1 such that $\mathcal{M}_1 \le B_1$.

Suppose that, for some $k \ge 1$, all fields of weight at most k satisfy the corresponding bounds, and hence $\mathcal{M}_k \le B_k$. If $|I| = k$, the preceding estimate gives

$$\|q_{I;b}\|_{L^\infty([0,1])} \le C_k(1 + B_k)^{N_k} =: Q_k.$$

Let $z_{I;b} = (y_{I;b}, y'_{I;b})$. The matrix of the first-order system has norm at most $L := 1 + C_0\rho^2$. Since $z_{I;b}(0) = 0$ for $k \ge 1$, the integral inequality associated with the system is

$$|z_{I;b}(t)| \le L \int_0^t |z_{I;b}(\tau)|\, d\tau + tQ_k.$$

Grönwall's Lemma 1.8.19 implies

$$\|y_{I;b}\|_{L^\infty([0,1])} + \|y'_{I;b}\|_{L^\infty([0,1])} \le 2Q_k e^L.$$

There are only finitely many words I and letters b, their number depending on k and m. Adding these bounds to those forming $\mathcal{M}_k$ gives a constant B_{k+1} with $\mathcal{M}_{k+1} \le B_{k+1}$. This constant depends only on the geometric bounds listed above and on $B_1, \dots, B_k$, never on p, v, or the chosen letters. This completes the induction on the weight.

It remains to pass from these covariant derivatives to ordinary derivatives of the matrices. To this end, over the full variation, let $(\mathbf{E}_r(s,t))_{r=1}^n$ be the frame determined by

$$D_t\mathbf{E}_r = 0, \qquad \mathbf{E}_r(s,0) = B_p e_r.$$

It is smooth in s and orthonormal for every s. If $\mathbf{W}_{I;r} := D_I\mathbf{E}_r$, then

$$D_t\mathbf{W}_{\varnothing;r} = 0, \qquad \mathbf{W}_{\varnothing;r}(0) = B_p e_r,$$

and, defining $\mathbf{S}_{\varnothing;r} := 0$ and, for $I = (I^-, a)$,

$$\mathbf{S}_{I;r} := D_a\mathbf{S}_{I^-;r} - \mathbf{R}(\mathbf{A}_a, \mathbf{T})\mathbf{W}_{I^-;r},$$

the commutator fixed above gives precisely

$$D_t\mathbf{W}_{I;r} = \mathbf{S}_{I;r}, \qquad \mathbf{W}_{I;r}(0) = 0 \quad (|I| \ge 1).$$

After expanding $D_a\mathbf{S}_{I^-;r}$, its right-hand side contains only curvature, the variational fields already bounded, and $\mathbf{W}_{J;r}$ with $|J| < |I|$. For $I = \varnothing$, $\mathbf{W}_{\varnothing;r} = \mathbf{E}_r$ is parallel and unit length, so $|\mathbf{W}_{\varnothing;r}|_{\mathbf{g}} = 1$. If $I = (a)$, then

$$\mathbf{S}_{(a);r} = -\mathbf{R}(\mathbf{A}_a, \mathbf{T})\mathbf{E}_r, \qquad \mathbf{W}_{(a);r}(t) = \int_0^t \mathbf{S}_{(a);r}(\tau)\, d\tau,$$

where the integral is taken after identifying the fibers by parallel transport; the bounds on $\mathbf{R}$, $\mathbf{A}_a$, and $\mathbf{T}$ bound both sides uniformly. Now suppose all $\mathbf{W}_{J;r}$ with $|J| < k$ are bounded, and let $|I| = k$. The full Leibniz

expansion of $\mathbf{S}_{I;r}$ contains only these lower-order fields, curvature derivatives of order at most $k-1$, and the variational fields already bounded. Therefore, there exists C_k such that $\|\mathbf{S}_{I;r}\|_{L^\infty([0,1])} \le C_k$. Since $\mathbf{W}_{I;r}(0) = 0$,

$$\mathbf{W}_{I;r}(t) = \int_0^t \mathbf{S}_{I;r}(\tau)\, d\tau, \qquad \|\mathbf{W}_{I;r}\|_{L^\infty([0,1])} \le C_k.$$

This completes the induction.

Now take $w_0 = e_j$. If

$$y_j^r(v,t) := \langle \mathbf{J}_j(t), \mathbf{E}_r(v,t)\rangle_{\mathbf{g}},$$

the variations with $w_1,\dots,w_k$ compute its ordinary derivatives in v. More precisely, for an ordered word $I = (a_1,\dots,a_k)$ and a subset $S \subseteq \{1,\dots,k\}$, denote by I_S the subword retaining the original order. Metric compatibility of the connection and the Leibniz rule give the exact triangular identity

$$\partial_I y_j^r(v,t) = \sum_{S\subseteq\{1,\dots,k\}} \left\langle \mathbf{Y}_{I_S;0}(t), \mathbf{W}_{I_{S^c};r}(t)\right\rangle_{\mathbf{g}},$$

where $\partial_I = \partial_{s_{a_k}} \cdots \partial_{s_{a_1}}|_{s=0}$. The term corresponding to $S = \{1,\dots,k\}$ is $\langle \mathbf{Y}_{I;0}, \mathbf{E}_r\rangle_{\mathbf{g}}$, and all the other terms contain fewer derivatives on $\mathbf{A}_0$. The bounds obtained for $\mathbf{Y}_{I;0}$ and $\mathbf{W}_{I;r}$ therefore imply uniform bounds for all ordinary derivatives of $y_j^r(v,t)$.

At $t = 1$, the matrix $\mathbf{Y}(v) = (y_j^r(v,1))_{rj}$ represents $d\Psi_p(v)$ in the parallel frame, and

$$g_{ij}^{(p)}(v) = \langle \mathbf{J}_i(1), \mathbf{J}_j(1)\rangle_{\mathbf{g}} = \sum_{r=1}^n y_i^r(v,1) y_j^r(v,1).$$

The scalar Leibniz rule thus gives, for every multi-index α,

$$\sup_{p\in M} \sup_{v\in B(0,\rho)} |D^\alpha g_{ij}^{(p)}(v)| < \infty.$$

The uniform ellipticity already established gives a uniform bound for $g_{(p)}^{ij}$. For $|\alpha| \ge 1$, differentiate the identity $\sum_{\ell=1}^n g_{(p)}^{i\ell} g_{\ell j}^{(p)} = \delta_j^i$, isolate the term containing $\beta = \alpha$, and multiply by the inverse metric. Componentwise, this gives

$$D^\alpha g_{(p)}^{ik} = - \sum_{\substack{\beta\in\mathbb{N}_0^n \\ \beta\le\alpha,\ \beta\ne\alpha}} \binom{\alpha}{\beta} \sum_{\ell,j=1}^n (D^\beta g_{(p)}^{i\ell})(D^{\alpha-\beta} g_{\ell j}^{(p)}) g_{(p)}^{jk}.$$

Induction on $|\alpha|$ is now direct. For $|\alpha| = 0$, we already have the uniform bound for $g_{(p)}^{ij}$. Suppose all derivatives $D^\beta g_{(p)}^{ij}$ with $|\beta| < k$ are bounded, and let $|\alpha| = k$. In every summand of the preceding formula, $|\beta| < k$ and $1 \le |\alpha - \beta| \le k$; by the induction hypothesis, the bounds already proved for $D^{\alpha-\beta} g_{\ell j}^{(p)}$, and the bound for $g_{(p)}^{jk}$, every product is uniformly bounded. The sum contains finitely many terms, their number depending only on α. Thus, $D^\alpha g_{(p)}^{ik}$ is also uniformly bounded, and induction proves the assertion for all derivatives of the components $g_{(p)}^{ij}$.

We have proved (2) with $r_0 := \rho$. For $0 < r \le r_0$, the bounds in charts of radius r follow by restricting the charts of radius r_0.

We now prove the additional assertion about coordinate transitions. Fix $0 < r \le r_0$. On an overlap, set $\Phi = \varphi_j^{\mathrm{geo}} \circ (\varphi_i^{\mathrm{geo}})^{-1}$. The tensor identity

$$g_{ab}^{(i)} = \sum_{\ell,m=1}^{n} (g_{\ell m}^{(j)} \circ \Phi)(\partial_a \Phi^\ell)(\partial_b \Phi^m)$$

and uniform ellipticity bound $D\Phi$: indeed, evaluating the equality on a vector ξ, the lower bound for $g^{(j)}$ and the upper bound for $g^{(i)}$ give $c|D\Phi\,\xi|^2 \le C|\xi|^2$. Interchanging i and j bounds the differential of the inverse transition. The Christoffel symbols and all their derivatives are uniformly bounded because their components are universal expressions in $g^{ab}_{(i)}$ and the derivatives of $g^{(i)}_{ab}$. The transformation law for the connection reads

$$\partial_a \partial_b \Phi^k = \Gamma^c_{i,ab}\, \partial_c \Phi^k - \Gamma^k_{j,\ell m}(\Phi) \partial_a \Phi^\ell \partial_b \Phi^m.$$

This equality bounds $D^2\Phi$. For the inductive step, suppose that, for some $m \ge 2$, all derivatives of Φ of order at most m are uniformly bounded, and let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m-1$. Apply D^α to the preceding identity. The Leibniz rule expresses the derivative of the first summand in terms of derivatives of Γ_i of order at most $m-1$ and derivatives of Φ of order at most m.

For the second summand, first apply the Leibniz rule to its three factors. In the composite factor, set $A := \Gamma^k_{j,\ell m}$. The Faà di Bruno formula of Theorem 1.1.19, written in the same partial-derivative notation used here, states that, for $1 \le |\mu| \le m-1$,

$$D^\mu (A \circ \Phi)(x) = \sum_{\substack{\beta \in \mathbb{N}_0^n,\ 1 \le |\beta| \le |\mu| \\ \gamma = (\gamma_1, \dots, \gamma_{|\beta|}),\quad |\gamma_\nu| > 0 \\ \gamma_1 + \cdots + \gamma_{|\beta|} = \mu \\ l = (l_1, \dots, l_{|\beta|}),\quad l_\nu \in \{1, \dots, n\} \\ \#\{\nu \mid l_\nu = a\} = \beta_a,\quad 1 \le a \le n}} c_{\mu,\beta,\gamma,l}\, (D^\beta A)(\Phi(x)) \prod_{\nu=1}^{|\beta|} D^{\gamma_\nu} \Phi^{l_\nu}(x). \tag{13.6}$$

In each term, $|\beta| \le |\mu| \le m-1$, $\#\{\nu \mid l_\nu = a\} = \beta_a$ for $1 \le a \le n$, and $1 \le |\gamma_\nu| \le |\mu| \le m-1$. Thus, the derivatives of A are bounded by the metric bounds, and the derivatives of Φ appearing in (13.6) are bounded by the induction hypothesis. The other two factors are first derivatives of Φ; after applying the Leibniz rule, they involve derivatives of order at most m. Every term on the right-hand side is therefore bounded by a constant independent of the charts. The left-hand side is $D^\alpha \partial_a \partial_b \Phi^k$, of order $m+1$, which completes the induction: every multi-index of length $m+1 \ge 2$ can be written as $\alpha + e_a + e_b$, allowing $a = b$. Interchanging i and j gives the same estimate for the inverse transition.

It remains to prove that (2) implies (1). Fix $x \in M$ and use the normal chart of radius r_0 centered at x. The formula

$$\Gamma^\ell_{ij} = \frac{1}{2} \sum_{a=1}^{n} g^{\ell a} \left(\partial_i g_{ja} + \partial_j g_{ia} - \partial_a g_{ij} \right)$$

gives, for each multi-index α, the exact identity

$$\partial^\alpha \Gamma^\ell_{ij} = \frac{1}{2} \sum_{a=1}^{n} \sum_{\substack{\beta \in \mathbb{N}_0^n \\ \beta \le \alpha}} \binom{\alpha}{\beta} (\partial^\beta g^{\ell a}) \left(\partial^{\alpha-\beta+e_i} g_{ja} + \partial^{\alpha-\beta+e_j} g_{ia} - \partial^{\alpha-\beta+e_a} g_{ij} \right).$$

The sum over a and β is finite, and condition (2) bounds each of its factors uniformly.

The curvature components satisfy

$$R^{\ell}{}_{ijk} = \partial_i \Gamma^{\ell}_{jk} - \partial_j \Gamma^{\ell}_{ik} + \sum_{m=1}^{n} \left(\Gamma^{m}_{jk}\Gamma^{\ell}_{im} - \Gamma^{m}_{ik}\Gamma^{\ell}_{jm} \right).$$

This formula and the Leibniz rule give uniform bounds for all partial derivatives of R. Finally, the coordinate formula for the covariant derivative of a tensor consists of its partial derivative plus a finite sum of contractions with Christoffel symbols. We proceed by induction on q. For $q = 0$, the components of R and all their partial derivatives are uniformly bounded by the preceding calculation. Suppose that each component of $\nabla^q R$ is a finite universal sum of products of partial derivatives of R and partial derivatives of the Christoffel symbols, all of order at most q. Write the curvature tensor as a tensor of type $(0,4)$, lowering its contravariant index using $\mathbf{g}$. Lemma 9.1.28 gives, for $I = (i_1, \dots, i_{q+4}) \in \{1, \dots, n\}^{q+4}$ and $a \in \{1, \dots, n\}$,

$$(\nabla_a \nabla^q R)_{i_1 \cdots i_{q+4}} = \partial_a (\nabla^q R)_{i_1 \cdots i_{q+4}} - \sum_{\nu=1}^{q+4} \sum_{c=1}^{n} \Gamma^{c}_{a i_\nu} (\nabla^q R)_{i_1 \cdots i_{\nu-1} c i_{\nu+1} \cdots i_{q+4}}.$$

When differentiating the products in the induction hypothesis, the Leibniz rule increases by one the order of exactly one factor; the connection terms add a factor Γ without increasing the other orders. Thus, each component of $\nabla^{q+1} R$ has the same form with orders at most $q+1$. All these components are uniformly bounded, completing the induction. At the center of the normal chart, $g_{ij}(x) = \delta_{ij}$; consequently, the tensor norm of $\nabla^q \mathbf{R}(x)$ is bounded by a constant independent of x. Since x was arbitrary, this gives (1) and concludes the equivalence. □

To justify the calculation of the injectivity radius for the nonpositive-curvature models appearing below, we record the required global result.

Theorem 13.1.9. (Cartan–Hadamard) Let $(M, \mathbf{g})$ be a complete, connected Riemannian manifold without boundary whose sectional curvature is nonpositive. For every $p \in M$, the exponential map

$$\exp_p \colon T_pM \longrightarrow M$$

is a covering map. If, in addition, M is simply connected, then $\exp_p$ is a global diffeomorphism and $\operatorname{inj}(p) = +\infty$.

Proof. Let γ be a geodesic starting at p, and let $\mathbf{J}$ be a Jacobi field perpendicular to $\dot\gamma$, with $\mathbf{J}(0) = 0$. The Jacobi equation and the curvature hypothesis give

$$\frac{1}{2}\frac{d^2}{dt^2}\|\mathbf{J}(t)\|_{\mathbf{g}}^2 = \left\| \frac{D\mathbf{J}}{dt}(t) \right\|_{\mathbf{g}}^2 - \left\langle \mathbf{R}(\mathbf{J}, \dot\gamma)\dot\gamma, \mathbf{J} \right\rangle_{\mathbf{g}} \geq 0.$$

If $\mathbf{J}$ vanished again, the nonnegative convex function $\|\mathbf{J}\|_{\mathbf{g}}^2$ would vanish between the two endpoints; uniqueness for the Jacobi equation would imply $\mathbf{J} \equiv 0$. Thus, there are no conjugate points and $d\exp_p(v)$ is invertible for every $v \in T_pM$; hence, $\exp_p$ is a local diffeomorphism.

To make the differential estimate precise, decompose the initial datum of a Jacobi field with $\mathbf{J}(0) = 0$ into its components parallel and perpendicular to $\dot\gamma(0)$. If P_t denotes parallel transport along γ, the parallel component is exactly $\mathbf{J}_{\parallel}(t) = tP_t(D_t\mathbf{J}(0)_{\parallel})$. For the perpendicular component, at points where $f(t) := \|\mathbf{J}(t)\|_{\mathbf{g}}$

does not vanish, the Jacobi equation gives

$$f''(t) = \frac{\|D_t\mathbf{J}\|_{\mathbf{g}}^2 f(t)^2 - \langle D_t\mathbf{J},\mathbf{J}\rangle_{\mathbf{g}}^2 - \langle \mathbf{R}(\mathbf{J},\dot\gamma)\dot\gamma,\mathbf{J}\rangle_{\mathbf{g}} f(t)^2}{f(t)^3} \geq 0.$$

Here we used Cauchy–Schwarz and the nonpositive sectional curvature. Since $f(0) = 0$ and $f'(0^+) = \|D_t\mathbf{J}(0)\|_{\mathbf{g}}$, convexity implies $f(t) \geq t\|D_t\mathbf{J}(0)\|_{\mathbf{g}}$. The parallel and perpendicular components remain orthogonal. Evaluating at $t = 1$ gives

$$\|d\exp_p(v)w\|_{\mathbf{g}} \geq \|w\|_{\mathbf{g}_p}, \qquad v, w \in T_pM.$$

Consequently, the metric $\exp_p^*\mathbf{g}$ dominates the Euclidean metric of T_pM and is complete: a Cauchy sequence for $\exp_p^*\mathbf{g}$ is Cauchy in T_pM. If v is its limit, continuity and positivity of $\exp_p^*\mathbf{g}$ provide a neighborhood V of v and constants $c_v, C_v > 0$ such that

$$c_v\|w\|_{\mathbf{g}_p}^2 \leq (\exp_p^*\mathbf{g})_z(w,w) \leq C_v\|w\|_{\mathbf{g}_p}^2, \qquad z \in V, \quad w \in T_pM.$$

The sequence then converges for the distance of $\exp_p^*\mathbf{g}$ as well.

The exponential map is surjective by Hopf–Rinow (Theorem 2.9.9). We record the passage from local isometry to covering map. Let $y \in M$, and choose $\varepsilon > 0$ so that $\exp_y$ is a diffeomorphism on the ball of radius ε. For each $x \in \exp_p^{-1}(y)$, completeness of $(T_pM, \exp_p^*\mathbf{g})$ allows us to define

$$s_x(\exp_y v) := \exp_x^{\exp_p^*\mathbf{g}}\big((d\exp_p)_x^{-1}v\big), \qquad |v|_{\mathbf{g}_y} < \varepsilon.$$

A local isometry maps geodesics to geodesics and preserves their initial data; geodesic uniqueness proves $\exp_p \circ s_x = \mathrm{id}_{B_{\mathbf{g}}(y,\varepsilon)}$. The images of s_x are disjoint: if two contained the same point, the lifts of the reversed radial geodesic, with the same initial datum determined by $d\exp_p$, would coincide and have the same endpoint over y. Finally, if $q \in \exp_p^{-1}(B_{\mathbf{g}}(y,\varepsilon))$, lift from q the radial geodesic ending at y; completeness extends the lift to its endpoint $x \in \exp_p^{-1}(y)$, and the reversed geodesic shows that $q = s_x(\exp_p(q))$. Therefore,

$$\exp_p^{-1}(B_{\mathbf{g}}(y,\varepsilon)) = \coprod_{x \in \exp_p^{-1}(y)} s_x(B_{\mathbf{g}}(y,\varepsilon)),$$

and each summand projects diffeomorphically onto the ball. This proves that $\exp_p$ is a covering map. Since the finite-dimensional vector space T_pM is simply connected, it is the universal covering space; if M is also simply connected, the covering has only one sheet. The last assertion follows from Definition 13.1.1. □

Example 13.1.10. We present some fundamental examples of smooth Riemannian manifolds without boundary and with bounded geometry, explicitly verifying the conditions of Definition 13.1.5.

(a) Euclidean space $(\mathbb{R}^n, \overline{\mathbf{g}})$ has bounded geometry.

Work in the standard coordinates of $\mathbb{R}^n$. In these coordinates, $\overline{g}_{ij} = \delta_{ij}$, so $\partial_k\overline{g}_{ij} = 0$ for all $i, j, k \in \{1, \dots, n\}$.

By Corollary 2.4.11, the Christoffel symbols of the Levi–Civita connection satisfy $\Gamma_{ij}^k = 0$ for all $i, j, k \in \{1, \dots, n\}$.

Substituting into Proposition 2.10.3 gives $R_{ijk}{}^{l} = 0$ for all $i,j,k,l \in \{1,\dots,n\}$, and consequently $\mathbf{R} = 0$. Therefore, $\nabla^m \mathbf{R} = 0$, $\forall m \in \mathbb{N}_0$.

We now study the exponential map. Let $p \in \mathbb{R}^n$ and $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$. In standard coordinates, the geodesic equation is $\dfrac{d^2x^k}{dt^2}(t) = 0$, $k \in \{1,\dots,n\}$, since $\Gamma^k_{ij} = 0$.

Integrating twice shows that every solution has the form $x(t) = a + tb$, with $a,b \in \mathbb{R}^n$. Imposing the initial conditions $x(0) = p$, $\dfrac{dx}{dt}(0) = v$, gives $a = p$ and $b = v$, so the geodesic with initial data (p,v) is given by $\gamma_v(t) = p + tv$.

By definition of the exponential map, $\exp_p(v) = \gamma_v(1)$, and therefore $\exp_p(v) = p + v$.

In particular, the exponential map $\exp_p \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a global diffeomorphism, with inverse given by $q \mapsto q - p$. Consequently, $\operatorname{inj}(p) = +\infty$ for every $p \in \mathbb{R}^n$, and therefore $\operatorname{inj}(\mathbb{R}^n) = +\infty$.

We conclude that $(\mathbb{R}^n, \overline{\mathbf{g}})$ has bounded geometry.

(b) Consider the upper half-plane

$$\mathbb{H}^2 := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$$

equipped with the Riemannian metric

$$\mathbf{g} = \frac{(\mathbf{d}x)^2 + (\mathbf{d}y)^2}{y^2}.$$

Work in the global coordinates induced by the inclusion $\mathbb{H}^2 \subseteq \mathbb{R}^2$, namely $(x^1,x^2) = (x,y)$. Denote the coordinate fields by $\partial_i := \partial/\partial x^i$.

The coefficients of the metric and its inverse are given by $g_{ij} = \dfrac{1}{y^2}\delta_{ij}$ and $g^{ij} = y^2\delta_{ij}$. In particular, $g_{11} = g_{22} = \dfrac{1}{y^2}$ and $g_{12} = g_{21} = 0$.

Since g_{ij} depends only on the variable $y = x^2$, $\partial_1 g_{ij} = 0$ for all $i,j \in \{1,2\}$, whereas $\partial_2 g_{11} = \partial_2 g_{22} = -\dfrac{2}{y^3}$ and $\partial_2 g_{12} = \partial_2 g_{21} = 0$.

Using Corollary 2.4.11, we obtain

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right).$$

First compute the symbols with upper index 1. Since $g^{11} = y^2$ and $g^{12} = 0$, we have

$$\Gamma^1_{11} = \frac{1}{2}g^{1\ell}\left(\partial_1 g_{1\ell} + \partial_1 g_{1\ell} - \partial_\ell g_{11}\right) = 0.$$

Moreover,

$$\Gamma^1_{12} = \frac{1}{2}g^{1\ell}\left(\partial_1 g_{2\ell} + \partial_2 g_{1\ell} - \partial_\ell g_{12}\right) = \frac{1}{2}g^{11}\partial_2 g_{11} = \frac{1}{2}y^2\left(-\frac{2}{y^3}\right) = -\frac{1}{y}.$$

By symmetry of the Levi–Civita connection, $\Gamma^1_{21} = \Gamma^1_{12} = -\dfrac{1}{y}$. Finally,

$$\Gamma^1_{22} = \frac{1}{2}g^{1\ell}\left(\partial_2 g_{2\ell} + \partial_2 g_{2\ell} - \partial_\ell g_{22}\right) = 0.$$

Now compute the symbols with upper index 2. Since $g^{22} = y^2$ and $g^{21} = 0$, we obtain

$$\Gamma^2_{11} = \frac{1}{2}g^{2\ell}\left(\partial_1 g_{1\ell} + \partial_1 g_{1\ell} - \partial_\ell g_{11}\right) = -\frac{1}{2}g^{22}\partial_2 g_{11} = -\frac{1}{2}y^2\left(-\frac{2}{y^3}\right) = \frac{1}{y}.$$

Similarly,

$$\Gamma^2_{12} = \frac{1}{2}g^{2\ell}\left(\partial_1 g_{2\ell} + \partial_2 g_{1\ell} - \partial_\ell g_{12}\right) = 0,$$

and, by symmetry, $\Gamma^2_{21} = 0$. Lastly,

$$\Gamma^2_{22} = \frac{1}{2}g^{2\ell}\left(\partial_2 g_{2\ell} + \partial_2 g_{2\ell} - \partial_\ell g_{22}\right) = \frac{1}{2}g^{22}\partial_2 g_{22} = \frac{1}{2}y^2\left(-\frac{2}{y^3}\right) = -\frac{1}{y}.$$

Thus,

$$\Gamma^1_{11} = 0, \qquad \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y}, \qquad \Gamma^1_{22} = 0,$$

$$\Gamma^2_{11} = \frac{1}{y}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = 0, \qquad \Gamma^2_{22} = -\frac{1}{y}.$$

We now compute the curvature tensor using Proposition 2.10.3:

$$R_{ijk}{}^{\ell} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}.$$

Consider the component $R_{122}{}^1$. We have

$$\partial_1 \Gamma^1_{22} = 0, \qquad \partial_2 \Gamma^1_{12} = \partial_2\left(-\frac{1}{y}\right) = \frac{1}{y^2}.$$

Moreover,

$$\Gamma^m_{22}\Gamma^1_{1m} = \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} = 0 + \left(-\frac{1}{y}\right)\left(-\frac{1}{y}\right) = \frac{1}{y^2},$$

and

$$\Gamma^m_{12}\Gamma^1_{2m} = \Gamma^1_{12}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^1_{22} = \left(-\frac{1}{y}\right)\left(-\frac{1}{y}\right) + 0 = \frac{1}{y^2}.$$

Therefore,

$$R_{122}{}^1 = 0 - \frac{1}{y^2} + \frac{1}{y^2} - \frac{1}{y^2} = -\frac{1}{y^2}.$$

Lowering indices gives

$$R_{1221} = g_{1\ell}R_{122}{}^\ell = g_{11}R_{122}{}^1 = \frac{1}{y^2}\left(-\frac{1}{y^2}\right) = -\frac{1}{y^4}.$$

We now compute the sectional curvature of the plane spanned by ∂_1 and ∂_2. By definition,

$$K(\partial_1, \partial_2) = \frac{\langle \mathbf{R}(\partial_1, \partial_2)\partial_2, \partial_1\rangle}{\langle \partial_1, \partial_1\rangle\langle \partial_2, \partial_2\rangle - \langle \partial_1, \partial_2\rangle^2}.$$

The denominator is

$$\langle \partial_1, \partial_1\rangle\langle \partial_2, \partial_2\rangle - \langle \partial_1, \partial_2\rangle^2 = g_{11}g_{22} - (g_{12})^2 = \frac{1}{y^4},$$

whereas the numerator is

$$\langle \mathbf{R}(\partial_1,\partial_2)\partial_2,\partial_1\rangle = R_{1221} = -\frac{1}{y^4}.$$

Consequently,

$$K(\partial_1,\partial_2) = \frac{-\frac{1}{y^4}}{\frac{1}{y^4}} = -1.$$

We have proved that the sectional curvature is constant and equal to -1. By Theorem 2.10.12, it follows that

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = -\big(\mathbf{g}(\mathbf{Y},\mathbf{Z})\mathbf{X} - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{Y}\big).$$

We now verify that $\nabla\mathbf{R} = 0$. By definition of the covariant derivative of a tensor of type $(1,3)$,

$$(\nabla_{\mathbf{W}}\mathbf{R})(\mathbf{X},\mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{W}}\big(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\big) - \mathbf{R}(\nabla_{\mathbf{W}}\mathbf{X},\mathbf{Y})\mathbf{Z} - \mathbf{R}(\mathbf{X},\nabla_{\mathbf{W}}\mathbf{Y})\mathbf{Z} - \mathbf{R}(\mathbf{X},\mathbf{Y})\nabla_{\mathbf{W}}\mathbf{Z}.$$

Substituting the preceding expression for $\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}$ gives

$$\nabla_{\mathbf{W}}\big(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\big) = -\nabla_{\mathbf{W}}\big(\mathbf{g}(\mathbf{Y},\mathbf{Z})\mathbf{X} - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{Y}\big).$$

Applying the Leibniz rule for the connection gives

$$\nabla_{\mathbf{W}}\big(\mathbf{g}(\mathbf{Y},\mathbf{Z})\mathbf{X}\big) = \mathbf{W}\big(\mathbf{g}(\mathbf{Y},\mathbf{Z})\big)\mathbf{X} + \mathbf{g}(\mathbf{Y},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{X},$$

$$\nabla_{\mathbf{W}}\big(\mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{Y}\big) = \mathbf{W}\big(\mathbf{g}(\mathbf{X},\mathbf{Z})\big)\mathbf{Y} + \mathbf{g}(\mathbf{X},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{Y}.$$

Therefore,

$$\nabla_{\mathbf{W}}\big(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\big) = -\Big(\mathbf{W}(\mathbf{g}(\mathbf{Y},\mathbf{Z}))\mathbf{X} + \mathbf{g}(\mathbf{Y},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{X} - \mathbf{W}(\mathbf{g}(\mathbf{X},\mathbf{Z}))\mathbf{Y} - \mathbf{g}(\mathbf{X},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{Y}\Big).$$

Now using compatibility of the Levi–Civita connection with the metric,

$$\mathbf{W}(\mathbf{g}(\mathbf{Y},\mathbf{Z})) = \mathbf{g}(\nabla_{\mathbf{W}}\mathbf{Y},\mathbf{Z}) + \mathbf{g}(\mathbf{Y},\nabla_{\mathbf{W}}\mathbf{Z}), \qquad \mathbf{W}(\mathbf{g}(\mathbf{X},\mathbf{Z})) = \mathbf{g}(\nabla_{\mathbf{W}}\mathbf{X},\mathbf{Z}) + \mathbf{g}(\mathbf{X},\nabla_{\mathbf{W}}\mathbf{Z}),$$

we obtain

$$\begin{aligned}\nabla_{\mathbf{W}}\big(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\big) = -\Big(&\mathbf{g}(\nabla_{\mathbf{W}}\mathbf{Y},\mathbf{Z})\mathbf{X} + \mathbf{g}(\mathbf{Y},\nabla_{\mathbf{W}}\mathbf{Z})\mathbf{X} + \mathbf{g}(\mathbf{Y},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{X}\\ &-\mathbf{g}(\nabla_{\mathbf{W}}\mathbf{X},\mathbf{Z})\mathbf{Y} - \mathbf{g}(\mathbf{X},\nabla_{\mathbf{W}}\mathbf{Z})\mathbf{Y} - \mathbf{g}(\mathbf{X},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{Y}\Big).\end{aligned}$$

On the other hand,

$$\mathbf{R}(\nabla_{\mathbf{W}}\mathbf{X},\mathbf{Y})\mathbf{Z} = -\big(\mathbf{g}(\mathbf{Y},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{X} - \mathbf{g}(\nabla_{\mathbf{W}}\mathbf{X},\mathbf{Z})\mathbf{Y}\big),$$

$$\mathbf{R}(\mathbf{X},\nabla_{\mathbf{W}}\mathbf{Y})\mathbf{Z} = -\big(\mathbf{g}(\nabla_{\mathbf{W}}\mathbf{Y},\mathbf{Z})\mathbf{X} - \mathbf{g}(\mathbf{X},\mathbf{Z})\nabla_{\mathbf{W}}\mathbf{Y}\big),$$

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\nabla_{\mathbf{W}}\mathbf{Z} = -\big(\mathbf{g}(\mathbf{Y},\nabla_{\mathbf{W}}\mathbf{Z})\mathbf{X} - \mathbf{g}(\mathbf{X},\nabla_{\mathbf{W}}\mathbf{Z})\mathbf{Y}\big).$$

Substituting these expressions into the definition of $(\nabla_{\mathbf{W}}\mathbf{R})(\mathbf{X},\mathbf{Y})\mathbf{Z}$ shows that each colored term in $\nabla_{\mathbf{W}}\big(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\big)$ cancels exactly with the term of the same color arising from one of the other three

summands. Therefore,

$$(\nabla_{\mathbf{W}}\mathbf{R})(\mathbf{X},\mathbf{Y})\mathbf{Z} = 0$$

for all vector fields $\mathbf{W},\mathbf{X},\mathbf{Y},\mathbf{Z}$. Consequently,

$$\nabla^k\mathbf{R} = 0$$

for every $k \geq 1$.

We also prove the global hypotheses we will use. If $c(t) = (x(t), y(t))$ is a piecewise C^1 curve in $\mathbb{H}^2$, then

$$L_{\mathbf{g}}(c) = \int \frac{\sqrt{|x'(t)|^2 + |y'(t)|^2}}{y(t)}\,dt \geq \int \frac{|y'(t)|}{y(t)}\,dt \geq \bigl|\log y(b) - \log y(a)\bigr|.$$

Let $q_j = (x_j, y_j)$ be a Cauchy sequence for the Riemannian distance. The preceding inequality shows that $(\log y_j)$ is Cauchy; in particular, for large j, $a \leq y_j \leq b$ for some constants $0 < a < b < \infty$. Given two large indices, choose a curve between q_j and q_k whose length is less than $2d_{\mathbf{g}}(q_j, q_k)$. If this length is less than 1, the inequality already proved, applied to each subarc, shows that the coordinate y along the entire curve is at most eb. Consequently,

$$\|x_j - x_k\| \leq eb\, L_{\mathbf{g}}(c) \leq 2eb\, d_{\mathbf{g}}(q_j, q_k).$$

Thus, (x_j) is also Cauchy and q_j converges in the upper half-plane. Let $q = (x, y)$ be its Euclidean limit. For large j, the straight segment from q_j to q lies in a region where the second coordinate is at least $\frac{a}{2}$. In this region, for every vector v, $|v|_{\mathbf{g}} \leq 2a^{-1}|v|$; consequently, $d_{\mathbf{g}}(q_j, q) \leq 2a^{-1}|q_j - q| \to 0$. This proves completeness. The upper half-plane is convex and hence simply connected.

The Cartan–Hadamard Theorem 13.1.9 now applies: for every $p \in \mathbb{H}^2$, the exponential map

$$\exp_p \colon T_p\mathbb{H}^2 \longrightarrow \mathbb{H}^2$$

is a global diffeomorphism. In particular,

$$\operatorname{inj}(p) = +\infty$$

for every $p \in \mathbb{H}^2$, and therefore

$$\operatorname{inj}(\mathbb{H}^2) = +\infty.$$

We conclude that $(\mathbb{H}^2, \mathbf{g})$ has bounded geometry.

(c) The unit sphere $(S^n, \mathbf{g}_{\mathrm{can}})$ has bounded geometry. Its sectional curvature is constant and equal to 1, so Theorem 2.10.12 gives

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \mathbf{g}(\mathbf{Y},\mathbf{Z})\mathbf{X} - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{Y}.$$

Compatibility $\nabla\mathbf{g} = 0$ implies, by the same Leibniz rule used in the hyperbolic example, that $\nabla\mathbf{R} = 0$ and hence $\nabla^k\mathbf{R} = 0$ for every $k \geq 1$.

It remains to compute the injectivity radius, since Cartan–Hadamard does not apply here. For $p \in S^n$

and $v \in T_pS^n \setminus \{0\}$, the geodesic with initial data (p, v) is

$$\gamma_v(t) = \cos\big(t|v|_{\mathbf{g}_{\mathrm{can}},p}\big)p + \sin\big(t|v|_{\mathbf{g}_{\mathrm{can}},p}\big)\frac{v}{|v|_{\mathbf{g}_{\mathrm{can}},p}};$$

this formula is verified by differentiating twice and observing that $\gamma_v''(t) = -|v|^2_{\mathbf{g}_{\mathrm{can}},p}\gamma_v(t)$, whose acceleration is normal to the sphere. Consequently,

$$\exp_p(v) = \cos\big(|v|_{\mathbf{g}_{\mathrm{can}},p}\big)p + \sin\big(|v|_{\mathbf{g}_{\mathrm{can}},p}\big)\frac{v}{|v|_{\mathbf{g}_{\mathrm{can}},p}}.$$

If $|v|_{\mathbf{g}_{\mathrm{can}},p} < \pi$, the Euclidean inner product with p determines $|v|_{\mathbf{g}_{\mathrm{can}},p} = \arccos\langle p, \exp_p(v)\rangle_{\mathbb{R}^{n+1}}$, and the orthogonal component then determines $\frac{v}{|v|_{\mathbf{g}_{\mathrm{can}},p}}$. Thus, $\exp_p$ is injective on $B_{\mathbf{g}_{\mathrm{can}},p}(0,\pi)$. We also verify that it is a local diffeomorphism on this ball. If $v = re$, with $0 < r < \pi$ and $|e| = 1$, write $w = be + w_\perp$, with $w_\perp \perp e$. Differentiating the explicit formula gives

$$d\exp_p(v)w = b(-\sin r\, p + \cos r\, e) + \frac{\sin r}{r}w_\perp, \qquad |d\exp_p(v)w|^2 = b^2 + \left(\frac{\sin r}{r}\right)^2 |w_\perp|^2.$$

The two vector summands are orthogonal, and $\sin r/r > 0$. The differential is therefore invertible; at $v = 0$ it is the identity. Injectivity and the inverse function theorem show that $\exp_p$ is a diffeomorphism onto its image on the indicated ball. In contrast, all vectors of $\mathbf{g}_{\mathrm{can},p}$ norm equal to π are mapped to the antipodal point $-p$. We conclude that

$$\mathrm{inj}(p) = \pi, \qquad \mathrm{inj}(S^n) = \pi.$$

In particular, the sphere also satisfies both conditions of Definition 13.1.5.

In general, a Riemannian manifold of constant sectional curvature has a curvature tensor expressed algebraically in terms of the metric, and hence $\nabla\mathbf{R} = 0$. The models $\mathbb{R}^n$, S^n, and $\mathbb{H}^n$ thus have bounded curvature derivatives of all orders. Their injectivity radii are $+\infty$, π, and $+\infty$, respectively, for the normalizations used here.

Compact Riemannian manifolds have bounded geometry, providing many examples of such manifolds.

Theorem 13.1.11. Let $(M, \mathbf{g})$ be a compact Riemannian manifold without boundary. Then $(M, \mathbf{g})$ has bounded geometry.

Proof. The connected components of a manifold are open and form a cover of M. Compactness gives a finite subcover, so there are only finitely many components. Write

$$M = \bigsqcup_{j=1}^{N} M_j,$$

where each M_j is a connected component of M. Since connected components are closed in every topological space, each M_j is closed in M and hence compact.
Consequently, each Riemannian manifold $(M_j, \mathbf{g}\restriction_{M_j})$ is compact and therefore complete. For each $j \in$

$\{1,\dots,N\}$, Proposition 2.13.15 implies that the function

$$\mathrm{inj}\colon M_j \longrightarrow (0,\infty]$$

is continuous. (Note that $\mathrm{inj}_M(p) = \mathrm{inj}_{M_j}(p)\ \forall p \in M_j$, since M_j is a connected component of M and the geodesics of a manifold lie in a single connected component; this is why we simply denote it by $\mathrm{inj}(p)$.) Moreover, since $\mathrm{inj}(p) > 0$ for every $p \in M_j$, there exists $p_j \in M_j$ such that

$$\mathrm{inj}(p_j) = \min_{p\in M_j} \mathrm{inj}(p) > 0.$$

Define

$$c := \min_{j\in\{1,\dots,N\}} \mathrm{inj}(p_j) > 0.$$

Then $\mathrm{inj}(p) \geq c$ for every $p \in M$, and consequently $\mathrm{inj}(M) > 0$. On the other hand, for each $k \in \mathbb{N}_0$, the tensor $\nabla^k R$ is smooth on M, so the function

$$p \longmapsto |\nabla^k R(p)|_{\mathbf{g}}$$

is continuous. Since M is compact, for each $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that $|\nabla^k R|_{\mathbf{g}} \leq C_k$. Therefore, $(M,\mathbf{g})$ has bounded geometry. □

A localized norm depends on three choices: the charts, their domains, and the cutoff functions. It is useful to collect them into a single object so that we can compare two constructions without losing track of any of these choices.

Definition 13.1.12. (Trivialization) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary. We say that $\mathcal{T} = (U_i,\varphi_i,h_i)_{i\in J}$ is a *trivialization* of M if $(U_i,\varphi_i)_{i\in J}$ is an atlas of M and $(h_i)_{i\in J}$ is a partition of unity subordinate to the open cover $(U_i)_{i\in J}$.

For an infinite cover, local finiteness alone does not allow estimates to be summed with a common constant. We need a uniform bound for the number of charts that may be involved in each overlap.

Definition 13.1.13. If X is a topological space and $(U_i)_{i\in J}$ is an open cover of X, we say that the cover is uniformly locally finite if there exists $L > 0$ such that, for each $i \in J$,

$$\#\{j \in J \mid U_j \cap U_i \neq \varnothing\} \leq L.$$

We say that a trivialization is *uniformly locally finite* if its underlying cover has this property.

Definition 13.1.14. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry. If $\mathcal{T} = (U_i,\varphi_i,h_i)_{i\in I}$ is a uniformly locally finite trivialization of M, we say that $\mathcal{T}$ is *admissible* if the following conditions hold:

(B1) Let $r_0 > 0$ be a radius for which condition (2) of Theorem 13.1.8 holds. There exists a uniform geodesic atlas

$$\mathcal{A} = (U_j^{\mathrm{geo}},\varphi_j^{\mathrm{geo}})_{j\in J}$$

of common radius $0 < r \leq \frac{r_0}{10}$, with centers p_j, for which the balls $B_{\mathbf{g}}(p_j,\frac{r}{4})$ are pairwise disjoint and

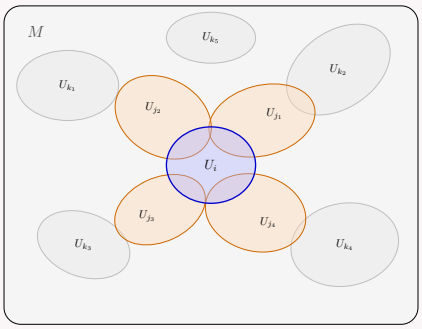


Figure 13.2. A uniformly locally finite cover. For any open set U_i in the cover, shown in blue, the number of open sets in the cover meeting it, shown in orange, is bounded by a constant L independent of i.

the balls $B_{\mathbf{g}}(p_j, \frac{r}{2})$ cover M. Moreover, there exists $N \in \mathbb{N}$ such that

$$\sup_{i \in I} \#\{j \in J \mid U_i \cap U_j^{\mathrm{geo}} \neq \varnothing\} \leq N$$

and

$$\sup_{j \in J} \#\{i \in I \mid U_i \cap U_j^{\mathrm{geo}} \neq \varnothing\} \leq N.$$

Finally, for each $k \in \mathbb{N}_0$, there exists $C_k > 0$ with the following property: if $U_i \cap U_j^{\mathrm{geo}} \neq \varnothing$ and $|\alpha| \leq k$, then

$$\sup_{x \in \varphi_j^{\mathrm{geo}}(U_i \cap U_j^{\mathrm{geo}})} \left| D^\alpha \left(\varphi_i \circ (\varphi_j^{\mathrm{geo}})^{-1} \right)(x) \right| \leq C_k,$$

and

$$\sup_{y \in \varphi_i(U_i \cap U_j^{\mathrm{geo}})} \left| D^\alpha \left(\varphi_j^{\mathrm{geo}} \circ \varphi_i^{-1} \right)(y) \right| \leq C_k.$$

(B2) For each $k \in \mathbb{N}_0$, there exists $c_k > 0$ such that, for every $i \in I$ and every multi-index $|\alpha| \leq k$,

$$\sup_{x \in \varphi_i(U_i)} \left| D^\alpha (h_i \circ \varphi_i^{-1})(x) \right| \leq c_k.$$

(B3) For each $i \in I$, there exists a function $\chi_i \in C_c^\infty(U_i)$ such that

$$0 \leq \chi_i \leq 1, \qquad \chi_i = 1 \quad \text{on a neighborhood of } \operatorname{supp}(h_i),$$

and, for each $k \in \mathbb{N}_0$, there exists $d_k > 0$, independent of i, for which

$$\sup_{x \in \varphi_i(U_i)} \left| D^\alpha (\chi_i \circ \varphi_i^{-1})(x) \right| \leq d_k, \qquad |\alpha| \leq k.$$

The functions χ_i in item (B3) will be called *auxiliary cutoff functions* of the trivialization. Their presence allows a localized expression to be extended by zero, denoted by a tilde, before a coordinate change is applied. The condition is not redundant: bounds on the derivatives of h_i do not by themselves provide a uniform separation

between $\operatorname{supp}(h_i)$ and the boundary of U_i.

Lemma 13.1.15. (Uniform margin for cutoff functions) The auxiliary functions in (B3) can be chosen so that there exists $\delta > 0$, independent of $i \in I$, with

$$\operatorname{dist}_{\mathbb{R}^n}\big(\varphi_i(\operatorname{supp}\chi_i), \mathbb{R}^n \setminus \varphi_i(U_i)\big) \geq \delta.$$

The new functions still equal one on a neighborhood of $\operatorname{supp} h_i$ and retain uniform bounds on all their derivatives.

Proof. Temporarily denote the functions in (B3) by χ_i^0. Their coordinate representations, extended by zero, are smooth on $\mathbb{R}^n$. The bound on their first derivatives gives a constant $D > 0$ such that

$$\big\|\widetilde{D\chi_i^0 \circ \varphi_i^{-1}}\big\|_\infty \leq D, \qquad i \in I.$$

Choose $\vartheta \in C^\infty(\mathbb{R})$ with $0 \leq \vartheta \leq 1$, $\vartheta = 0$ on $(-\infty, 1/4]$, and $\vartheta = 1$ on $[1/2, \infty)$, and set $\chi_i = \vartheta \circ \chi_i^0$. If $x \in \varphi_i(\operatorname{supp}\chi_i)$, then $(\chi_i^0 \circ \varphi_i^{-1})(x) \geq 1/4$. For $y \in \mathbb{R}^n \setminus \varphi_i(U_i)$, the extension is zero. The mean value theorem on the Euclidean segment from x to y implies $1/4 \leq D|x - y|$. Taking the infimum over these y gives the margin $\delta = (4D)^{-1}$. The Faà di Bruno formula of Theorem 1.1.19, applied to the fixed function ϑ, provides the higher-order bounds. Since $\chi_i^0 = 1$ near $\operatorname{supp} h_i$, the new function also equals one there. □

From now on, we choose the auxiliary functions with this margin. Thus, the constants in a zero extension or a localized coordinate change incorporate a common positive distance from the artificial boundary of the charts.

Definition 13.1.16. (Quadratic localization system) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry. A *quadratic localization system* is a family

$$\mathcal{Q} = (U_i, \varphi_i, h_i)_{i \in I}$$

such that $(U_i, \varphi_i)_{i \in I}$ is a uniformly locally finite atlas, $h_i \in C_c^\infty(U_i)$, $0 \leq h_i \leq 1$, and

$$\sum_{i \in I} h_i^2 = 1 \qquad \text{on } M.$$

We say that $\mathcal{Q}$ is *admissible* if the atlas satisfies condition (B1), the functions h_i satisfy the estimates in (B2), with h_i in place of the partition functions, and there is a family of auxiliary cutoff functions as in (B3), equal to one on a neighborhood of $\operatorname{supp}(h_i)$.

Remark 13.1.17. If $\mathcal{Q} = (U_i, \varphi_i, h_i)_{i \in I}$ is an admissible quadratic localization system and $\rho_i := h_i^2$, then $\sum_{i \in I} \rho_i = 1$ and

$$\mathcal{T}_{\mathcal{Q}} := (U_i, \varphi_i, \rho_i)_{i \in I}$$

is an admissible trivialization. Indeed, the Leibniz rule converts the uniform derivative bounds for h_i into the corresponding bounds for h_i^2, and the same auxiliary cutoff functions serve both families. This observation distinguishes the quadratic system used in the analysis and synthesis formulas from the genuine partition of unity.

Remark 13.1.18. (Meaning and use of quadratic localization) A quadratic partition is not a second definition of the function spaces. Its usefulness lies in the fact that the same factor h_i enters both the analysis and the synthesis operators. Analysis sends u to the family

$$\left(\widetilde{(h_i u)\circ\varphi_i^{-1}}\right)_{i\in I}.$$

For synthesis, each component is first multiplied by a cutoff function whose support is compact in the chart image, reconstructed on U_i, extended by zero to M, and multiplied by h_i. The precise distributional definition will be given in Proposition 13.3.11. The identity $\sum_{i\in I} h_i^2 = 1$ makes the composition of the two operators exactly the identity before any comparison of norms. This is Amann's convention: the cutoff system appears in [6, Lemma 3.6], the two operators in formulas (4.5)–(4.6), and the retraction–coretraction property in Theorem 4.2 of that reference.

The local definitions of the spaces given below use, instead, an ordinary partition (ρ_i), as in [72, 130]. Lemma 13.3.10 proves that, when $\rho_i = h_i^2$, both localizations satisfy the explicit inequalities with the positive constants c_H, C_H, c_F, C_F of that lemma, independent of the section. It is therefore useful to distinguish the two families in the notation, even though they describe the same space.

Lemma 13.1.19. (Uniform quadratic normalization) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let $\mathcal{T} = (U_i,\varphi_i,\psi_i)_{i\in I}$ be an admissible trivialization such that $0\le\psi_i\le 1$ and $\operatorname{supp}\psi_i \Subset U_i$ for every i. Define

$$S(x) := \left(\sum_{j\in I}\psi_j(x)^2\right)^{\frac12}, \qquad h_i(x) := \frac{\psi_i(x)}{S(x)}.$$

Then $\mathcal{Q} = (U_i,\varphi_i,h_i)_{i\in I}$ is an admissible quadratic localization system.

Proof. The sum defining S is locally finite. If L is a uniform bound for the number of members of the cover that can contain a given point, the equality $\sum_{j\in I}\psi_j = 1$ and the Cauchy–Schwarz inequality give

$$1 = \left(\sum_{j\in I}\psi_j(x)\right)^2 \le L\sum_{j\in I}\psi_j(x)^2.$$

Since the functions are nonnegative, we also have $\sum_{j\in I}\psi_j(x)^2 \le (\sum_{j\in I}\psi_j(x))^2 = 1$. Therefore,

$$L^{-\frac12} \le S(x) \le 1 \qquad \text{for every } x\in M.$$

In particular, S is smooth, $0\le h_i\le 1$, $\operatorname{supp} h_i \subseteq \operatorname{supp}\psi_i \Subset U_i$, and

$$\sum_{i\in I} h_i^2 = S^{-2}\sum_{i\in I}\psi_i^2 = 1.$$

It remains to verify uniformity of the derivatives. Fix a chart (U_i,φ_i). Only indices j for which $U_i\cap U_j\neq$

$\varnothing$ contribute, and there are at most L of them. In chart i, the representation of ψ_j is understood to be extended by zero outside $\varphi_i(U_i \cap U_j)$. This extension is smooth: points where ψ_j may be nonzero lie inside $\operatorname{supp}(\psi_j) \Subset U_j$, whereas outside that support the function vanishes on a neighborhood. On the overlap,

$$\psi_j \circ \varphi_i^{-1} = (\psi_j \circ \varphi_j^{-1}) \circ (\varphi_j \circ \varphi_i^{-1}).$$

Bounds for the coordinate transitions follow directly from (B1): near each point of the overlap, insert a geodesic chart from the atlas appearing in that condition and write

$$\varphi_j \circ \varphi_i^{-1} = \big(\varphi_j \circ (\varphi_\nu^{\mathrm{geo}})^{-1}\big) \circ \big(\varphi_\nu^{\mathrm{geo}} \circ \varphi_i^{-1}\big).$$

Condition (B2), these bounds, and the Faà di Bruno formula of Theorem 1.1.19 bound all derivatives of these functions uniformly. The Leibniz rule and the bound L then bound all derivatives of $\sum_{j \in I_i} (\psi_j \circ \varphi_i^{-1})^2$. This last function takes values in the compact interval $[L^{-1}, 1]$. Applying Faà di Bruno once more to $t \mapsto t^{-\frac{1}{2}}$, whose derivatives are bounded on that interval, gives uniform bounds for the derivatives of $S^{-1} \circ \varphi_i^{-1}$. Finally, applying the Leibniz rule to

$$h_i \circ \varphi_i^{-1} = (\psi_i \circ \varphi_i^{-1})(S^{-1} \circ \varphi_i^{-1})$$

gives the estimates in (B2) for h_i. This proves admissibility: the auxiliary cutoff functions of the original trivialization still equal one on a neighborhood of $\operatorname{supp}(h_i) \subseteq \operatorname{supp}(\psi_i)$ and retain the estimates in (B3). □

Remark 13.1.20. The coordinate-transition estimates in (B1) do not depend on the particular normal atlas. Indeed, if $0 < r' \le r_0$, extend the charts of radii r and r' to radius $\max\{r, r'\} \le r_0$ when necessary, apply the additional assertion of Theorem 13.1.8, and restrict again.

The combinatorial condition is also independent of the atlas when the second atlas, of radius $0 < r' \le \frac{r_0}{10}$, has disjoint balls $B_{\mathbf{g}}(p'_j, \frac{r'}{4})$ and the balls $B_{\mathbf{g}}(p'_j, \frac{r'}{2})$ cover M. To verify this, if a chart of radius r meets a chart of radius r', the disjoint ball of radius $\frac{r'}{4}$ associated with the second lies in a ball whose radius does not exceed $r + r' + \frac{r'}{4} < r_0$. The uniform bounds on the metric and its inverse in normal charts, given by Theorem 13.1.8, place the eigenvalues of its matrix between two positive constants independent of the center. Integrating the square root of its determinant gives, for each fixed radius less than r_0, uniform upper and lower bounds for the volume of the balls. Comparing the large ball with the disjoint balls of radius $\frac{r'}{4}$ bounds their number uniformly. Interchanging r and r' gives the other overlap bound. Combining this fact with the two overlap bounds in (B1) also transfers the cross-overlap bounds between $\mathcal{T}$ and the second atlas.

Moreover, (B1) controls transitions between two admissible trivializations. Let $\mathcal{T} = (U_i, \varphi_i)_{i \in I}$ and $\mathcal{T}^\sharp = (U_j^\sharp, \varphi_j^\sharp)_{j \in J}$. By the preceding two paragraphs, we may fix a common uniform geodesic atlas $(U_\nu^{\mathrm{geo}}, \varphi_\nu^{\mathrm{geo}})_{\nu \in K}$ relative to which both satisfy the bounds in (B1). If $p \in U_i \cap U_j^\sharp$, choose ν with $p \in U_\nu^{\mathrm{geo}}$. On a neighborhood of p contained in the triple overlap, we have exactly

$$\varphi_j^\sharp \circ \varphi_i^{-1} = \big(\varphi_j^\sharp \circ (\varphi_\nu^{\mathrm{geo}})^{-1}\big) \circ \big(\varphi_\nu^{\mathrm{geo}} \circ \varphi_i^{-1}\big).$$

The Faà di Bruno formula of Theorem 1.1.19 expresses every derivative of order k of the left-hand side at p as a finite sum of products of derivatives of the two factors of orders at most k. The bounds in (B1) are

independent of i, j, ν; since p is arbitrary, for each $k \in \mathbb{N}_0$ there exists C_k, independent of i, j, such that

$$\left\| D^\alpha \left(\varphi_j^\sharp \circ \varphi_i^{-1} \right) \right\|_{L^\infty(\varphi_i(U_i \cap U_j^\sharp))} \leq C_k, \qquad |\alpha| \leq k.$$

Interchanging the two trivializations gives the same estimate for the inverse transition. The case $\mathcal{T}^\sharp = \mathcal{T}$ provides, in particular, uniform control of the internal transitions of an admissible trivialization.

Normal geodesic coordinates can be used to define a trivialization of any manifold of bounded geometry that is also *uniformly locally finite*; we will call it a ***geodesic trivialization***.

Lemma 13.1.21. (Uniform control of geodesic ball volumes) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry. If $r_0 > 0$ is a radius for which condition (2) of Theorem 13.1.8 holds, then, for each $0 < r \leq r_0$, there exist constants $v_-(r), v_+(r) > 0$ such that

$$v_-(r) \leq \lambda_{\mathbf{g}}(B_{\mathbf{g}}(p, r)) \leq v_+(r), \qquad \forall p \in M.$$

Proof. Fix $0 < r \leq r_0$ and let $p \in M$. The Riemannian exponential map at p induces a normal chart on $B_{\mathbf{g}}(p, r)$. Fix a linear isomorphism

$$B_p : \mathbb{R}^n \longrightarrow T_p M$$

taking the standard basis of $\mathbb{R}^n$ to an orthonormal basis of $T_p M$, and consider the chart

$$\varphi_p := B_p^{-1} \circ \exp_p^{-1} : B_{\mathbf{g}}(p, r) \longrightarrow B_{\mathrm{euc}}(0, r).$$

Denote by $g_{ij}^{(p)}$ and $g_{(p)}^{ij}$ the components of the metric and its inverse in these coordinates. Since $(M, \mathbf{g})$ has bounded geometry, Theorem 13.1.8 implies that there exists a constant $C > 0$, independent of p, such that

$$|g_{ij}^{(p)}(q)| \leq C \qquad \text{and} \qquad |g_{(p)}^{ij}(q)| \leq C, \qquad \forall q \in B_{\mathbf{g}}(p, r),$$

for all $i, j \in \{1, \ldots, n\}$.
Fix $q \in B_{\mathbf{g}}(p, r)$ and let $\lambda_1(q), \ldots, \lambda_n(q)$ be the eigenvalues of the coefficient matrix $(g_{ij}^{(p)}(q))_{i,j=1}^n$. This matrix is symmetric and positive definite. The bound on its entries implies that its Frobenius norm is at most nC; therefore,

$$\lambda_\ell(q) \leq nC, \qquad \ell = 1, \ldots, n.$$

The inverse matrix has coefficients $(g_{(p)}^{ij}(q))$ and its Frobenius norm is also at most nC. Its eigenvalues are $\lambda_\ell(q)^{-1}$, whence

$$\lambda_\ell(q)^{-1} \leq nC \quad \Longrightarrow \quad \lambda_\ell(q) \geq (nC)^{-1}.$$

Multiplying the n inequalities gives directly

$$(nC)^{-n} \leq \det\left(\mathbf{g}^{(p)}(q)\right) \leq (nC)^n, \qquad q \in B_{\mathbf{g}}(p, r),$$

uniformly in p. Consequently,

$$(nC)^{-\frac{n}{2}} \leq \sqrt{\det\left(\mathbf{g}^{(p)}(q)\right)} \leq (nC)^{\frac{n}{2}}.$$

Using the local expression for the Riemannian measure, we obtain

$$\lambda_{\mathbf{g}}(B_{\mathbf{g}}(p,r)) = \int_{B_{\mathrm{euc}}(0,r)} \sqrt{\det\big(\mathbf{g}^{(p)}(\varphi_p^{-1}(x))\big)}\, dx.$$

By the preceding bounds,

$$(nC)^{-\frac{n}{2}}\, \lambda_n(B_{\mathrm{euc}}(0,r)) \le \lambda_{\mathbf{g}}(B_{\mathbf{g}}(p,r)) \le (nC)^{\frac{n}{2}}\, \lambda_n(B_{\mathrm{euc}}(0,r)).$$

It therefore suffices to define

$$v_-(r) := (nC)^{-\frac{n}{2}}\, \lambda_n(B_{\mathrm{euc}}(0,r)) \qquad \text{and} \qquad v_+(r) := (nC)^{\frac{n}{2}}\, \lambda_n(B_{\mathrm{euc}}(0,r)).$$

This concludes the proof. □

Theorem 13.1.22. (Geodesic trivialization) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry. Then there exist $r > 0$ and an admissible trivialization

$$\mathcal{T}^{\mathrm{geo}} = \big(B_{\mathbf{g}}(p_i, 2r), \varphi_i, h_i^{\mathrm{geo}}\big)_{i\in J}$$

of M such that:

(a) For each $i \in J$,

$$\varphi_i = B_{p_i}^{-1} \circ \exp_{p_i}^{-1} : B_{\mathbf{g}}(p_i, 2r) \longrightarrow B_{\mathrm{euc}}(0, 2r) \subseteq \mathbb{R}^n$$

is a normal coordinate chart, where $B_{p_i} : \mathbb{R}^n \longrightarrow T_{p_i}M$ is an isomorphism.

(b) The balls $B_{\mathbf{g}}(p_i, \frac{r}{2})$ are pairwise disjoint.

(c) $M = \bigcup_{i\in J} B_{\mathbf{g}}(p_i, r)$, and the cover $(B_{\mathbf{g}}(p_i, 2r))_{i\in J}$ is uniformly locally finite.

(d) $(h_i^{\mathrm{geo}})_{i\in J}$ is a partition of unity subordinate to $(B_{\mathbf{g}}(p_i, 2r))_{i\in J}$.

(e) For every $k \in \mathbb{N}_0$, there exists $C_k > 0$ such that

$$\left| D^\alpha (h_j^{\mathrm{geo}} \circ \varphi_i^{-1}) \right| \le C_k, \qquad |\alpha| \le k,$$

whenever $B_{\mathbf{g}}(p_i, 2r) \cap B_{\mathbf{g}}(p_j, 2r) \neq \varnothing$.

(f) There exist functions $\chi_i^{\mathrm{geo}} \in C_c^\infty(B_{\mathbf{g}}(p_i, 2r))$, equal to one on a neighborhood of $\mathrm{supp}(h_i^{\mathrm{geo}})$, whose representations in the normal charts have derivatives of every order uniformly bounded in i.

Proof. Since $(M,\mathbf{g})$ has bounded geometry, $\mathrm{inj}(M) > 0$. Let $r_0 > 0$ be a radius for which condition (2) of Theorem 13.1.8 holds. Choose

$$0 < r < \min\left\{ \frac{\mathrm{inj}(M)}{5}, \frac{r_0}{20} \right\}.$$

Consider

$$\mathcal{F} = \{A \subseteq M \mid d(p,q) \ge r \text{ for all } p,q \in A, \text{ with } p \neq q\}.$$

We can order $\mathcal{F}$ by set inclusion $\subseteq$, obtaining a partial order. If $\mathcal{C} \subseteq \mathcal{F}$ is a chain, then $A_{\mathcal{C}} = \bigcup_{A \in \mathcal{C}} A$ belongs to $\mathcal{F}$, since, given $p, q \in A_{\mathcal{C}}$, total ordering of $(\mathcal{C}, \subseteq)$ ensures that both belong to the same member of the chain. Thus, $A_{\mathcal{C}}$ is an upper bound for $\mathcal{C}$, so Zorn's Lemma, Theorem A.0.1, guarantees the existence of a maximal set $A = \{p_i\}_{i \in J}$. Maximality preserves the separation $d(p_i, p_j) \geq r$ whenever $i \neq j$, which implies that the balls $B_{\mathbf{g}}(p_i, \frac{r}{2})$ are disjoint.

We claim that $M = \bigcup_{i \in J} B_{\mathbf{g}}(p_i, r)$. Otherwise, there would exist $x \in M$ with $d(x, p_i) \geq r$ for every $i \in J$. In particular, this gives $x \notin A$, so

$$A \subsetneq A \cup \{x\} \in \mathcal{F},$$

contradicting maximality of A.

For each $i \in J$, since $2r < \operatorname{inj}(M)$, the Riemannian exponential map at p_i induces a normal chart on $B_{\mathbf{g}}(p_i, 2r)$. Define

$$\varphi_i := B_{p_i}^{-1} \circ \exp_{p_i}^{-1} \colon B_{\mathbf{g}}(p_i, 2r) \longrightarrow B_{\mathrm{euc}}(0, 2r),$$

where $B_{p_i} \colon \mathbb{R}^n \longrightarrow T_{p_i} M$ is an isomorphism taking the standard basis to an orthonormal basis. This proves (a).

We now prove that the cover $(B_{\mathbf{g}}(p_i, 2r))_{i \in J}$ is uniformly locally finite. For fixed $i \in J$, define

$$J_i := \{j \in J \mid B_{\mathbf{g}}(p_i, 2r) \cap B_{\mathbf{g}}(p_j, 2r) \neq \varnothing\}.$$

If $j \in J_i$, then $B_{\mathbf{g}}(p_i, 2r) \cap B_{\mathbf{g}}(p_j, 2r) \neq \varnothing$, whence

$$d(p_i, p_j) < 4r, \qquad \text{and therefore} \qquad p_j \in B_{\mathbf{g}}(p_i, 4r).$$

The balls $B_{\mathbf{g}}(p_j, \frac{r}{2})$ are pairwise disjoint, and moreover,

$$B_{\mathbf{g}}(p_j, \tfrac{r}{2}) \subseteq B_{\mathbf{g}}(p_i, \tfrac{9r}{2}), \qquad \forall j \in J_i.$$

By Lemma 13.1.21, there exist constants $v_-, v_+ > 0$ such that

$$\lambda_{\mathbf{g}}\big(B_{\mathbf{g}}(p, \tfrac{r}{2})\big) \geq v_-, \qquad \lambda_{\mathbf{g}}\big(B_{\mathbf{g}}(p, \tfrac{9r}{2})\big) \leq v_+, \qquad \forall p \in M.$$

Summing volumes over $j \in J_i$ gives

$$\sum_{j \in J_i} \lambda_{\mathbf{g}}\big(B_{\mathbf{g}}(p_j, \tfrac{r}{2})\big) \leq \lambda_{\mathbf{g}}\big(B_{\mathbf{g}}(p_i, \tfrac{9r}{2})\big),$$

and therefore

$$\#J_i \, v_- \leq v_+, \qquad \text{whence} \qquad \#J_i \leq \frac{v_+}{v_-}.$$

This implies that the cover $(B_{\mathbf{g}}(p_i, 2r))_{i \in J}$ is uniformly locally finite, proving (c).

We now construct a partition of unity. Use the flat function from the proof of Theorem 1.3.5, which we

write again here:

$$\vartheta(t) := \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

All its derivatives at the origin vanish. Define explicitly

$$\psi(x) := \frac{\vartheta\left((\frac{4r}{3})^2 - \|x\|^2\right)}{\vartheta\left((\frac{4r}{3})^2 - \|x\|^2\right) + \vartheta(\|x\|^2 - r^2)}.$$

The denominator does not vanish, and flatness of ϑ at the origin shows that $\psi \in C_c^\infty(\mathbb{R}^n)$. Moreover,

$$0 \le \psi \le 1, \qquad \psi \equiv 1 \text{ on } B_{\text{euc}}(0, r), \qquad \operatorname{supp}(\psi) \subseteq B_{\text{euc}}(0, \frac{3r}{2}).$$

For each $i \in J$, define

$$\psi_i(x) := \begin{cases} \psi(\varphi_i(x)), & x \in B_{\mathbf{g}}(p_i, 2r), \\ 0, & \text{otherwise.} \end{cases}$$

Then ψ_i is smooth and

$$\operatorname{supp}(\psi_i) \subseteq B_{\mathbf{g}}(p_i, \frac{3r}{2}) \subset B_{\mathbf{g}}(p_i, 2r),$$

so its extension by zero is smooth.
Define

$$\Psi(x) := \sum_{i \in J} \psi_i(x).$$

Since the cover $(B_{\mathbf{g}}(p_i, 2r))_{i \in J}$ is uniformly locally finite, for every $x \in M$ there exists a neighborhood U of x such that the set

$$\{i \in J \mid U \cap B_{\mathbf{g}}(p_i, 2r) \neq \varnothing\}$$

is finite. In particular, for every $y \in U$, the sum defining $\Psi(y)$ has finitely many nonzero summands, with a bound independent of $y \in U$. It follows that Ψ is a smooth function on M.
Moreover, since

$$M = \bigcup_{i \in J} B_{\mathbf{g}}(p_i, r)$$

and $\psi_i \equiv 1$ on $B_{\mathbf{g}}(p_i, r)$, for every $x \in M$ there exists $i \in J$ such that $\psi_i(x) = 1$. Consequently,

$$\Psi(x) \ge 1, \qquad \forall x \in M.$$

In particular, Ψ does not vanish on M, so the functions

$$h_i^{\text{geo}}(x) := \frac{\psi_i(x)}{\Psi(x)}$$

are well defined and smooth. Moreover, since $0 \le \psi_i \le 1$ and $\Psi \ge 1$,

$$0 \le h_i^{\text{geo}} \le 1, \qquad \forall i \in J.$$

Thus, $(h_i^{\text{geo}})_{i \in J}$ is a partition of unity subordinate to $(B_{\mathbf{g}}(p_i, 2r))_{i \in J}$, proving (d).

We prove the uniform derivative bounds. For fixed $i, j \in J$ such that $B_{\mathbf{g}}(p_i, 2r) \cap B_{\mathbf{g}}(p_j, 2r) \neq \varnothing$, in normal coordinates we have

$$h_j^{\mathrm{geo}} \circ \varphi_i^{-1} = \frac{\psi_j \circ \varphi_i^{-1}}{\sum_{k \in J_i} \psi_k \circ \varphi_i^{-1}},$$

where J_i is finite and has uniformly bounded cardinality.
First, we show that the functions $\psi_k \circ \varphi_i^{-1}$ have uniformly bounded derivatives of every order. Fix $m \in \mathbb{N}_0$. Since ψ is a fixed cutoff function on $\mathbb{R}^n$ with compact support contained in $B_{\mathrm{euc}}(0, \frac{3r}{2})$, there exists a constant $A_m > 0$ such that

$$|D^\alpha \psi(x)| \leq A_m, \qquad \forall x \in \mathbb{R}^n, \quad |\alpha| \leq m.$$

On the other hand, for each $k \in J_i$,

$$\psi_k \circ \varphi_i^{-1} = \psi \circ \varphi_k \circ \varphi_i^{-1}.$$

The equality is used on $\varphi_i(B_{\mathbf{g}}(p_i, 2r) \cap B_{\mathbf{g}}(p_k, 2r))$. Outside this overlap, $\psi_k \circ \varphi_i^{-1} = 0$. Indeed, $\operatorname{supp} \psi \subset B_{\mathrm{euc}}(0, \frac{3r}{2})$, so ψ_k vanishes on a neighborhood of the boundary of $B_{\mathbf{g}}(p_k, 2r)$. Thus, the coordinate expression is smooth throughout $\varphi_i(B_{\mathbf{g}}(p_i, 2r))$, and its derivatives vanish outside the overlap. Since the geodesic coordinate transitions $\varphi_k \circ \varphi_i^{-1}$ have uniformly bounded derivatives through order m by Theorem 13.1.8, the Faà di Bruno formula of Theorem 1.1.19 implies that there exists a constant $C'_m > 0$, independent of i and k, such that

$$|D^\alpha(\psi_k \circ \varphi_i^{-1})| \leq C'_m, \qquad |\alpha| \leq m.$$

Moreover, since the sum in the denominator involves a uniformly bounded number of terms, for each $m \in \mathbb{N}_0$ there exists a constant $C''_m > 0$ such that

$$\left| D^\alpha \left(\sum_{k \in J_i} \psi_k \circ \varphi_i^{-1} \right) \right| \leq C''_m, \qquad |\alpha| \leq m.$$

On the other hand, for every $x \in B_{\mathrm{euc}}(0, 2r)$,

$$\sum_{k \in J_i} \psi_k \circ \varphi_i^{-1}(x) = \Psi\left(\varphi_i^{-1}(x)\right) \geq 1.$$

Thus, the denominator is uniformly bounded below by 1. Set

$$F_i := \sum_{k \in J_i} \psi_k \circ \varphi_i^{-1}.$$

The lower bound gives $\|F_i^{-1}\|_{L^\infty(B_{\mathrm{euc}}(0,2r))} \leq 1$. For $|\alpha| \geq 1$, applying the Leibniz rule to $F_i^{-1} F_i = 1$ and isolating the term $D^\alpha(F_i^{-1})F_i$ gives the recurrence

$$D^\alpha(F_i^{-1}) = -F_i^{-1} \sum_{\substack{\beta \in \mathbb{N}_0^n \\ \beta \leq \alpha,\ \beta \neq \alpha}} \binom{\alpha}{\beta} D^\beta(F_i^{-1}) D^{\alpha-\beta} F_i.$$

For $|\alpha| = 1$, the right-hand side contains only F_i^{-1} and first derivatives of F_i, which are already bounded. If the derivatives of F_i^{-1} of order less than $|\alpha|$ are uniformly bounded, every term in the sum is bounded by

the induction hypothesis and the bounds on F_i. Thus, order by order, we obtain uniform bounds for all derivatives of F_i^{-1}. Writing

$$h_j^{\mathrm{geo}} \circ \varphi_i^{-1} = (\psi_j \circ \varphi_i^{-1}) \cdot F_i^{-1},$$

the multi-index Leibniz rule and the bounds just proved imply that, for each $m \in \mathbb{N}_0$, there exists a constant $C_m > 0$ such that

$$|D^\alpha (h_j^{\mathrm{geo}} \circ \varphi_i^{-1})| \le C_m, \qquad |\alpha| \le m.$$

This proves (e). The underlying atlas has common radius $2r$; the balls of radius $\frac{2r}{4} = \frac{r}{2}$ are disjoint, and those of radius $\frac{2r}{2} = r$ cover M. Moreover, $2r < \frac{r_0}{10}$ by the choice of radius. The overlap bounds already proved and the additional assertion of Theorem 13.1.8 verify (B1), while (e) verifies (B2).
To construct the auxiliary cutoff functions, use the same flat function and set

$$\chi(x) := \frac{\vartheta\big((\frac{15r}{8})^2 - \|x\|^2\big)}{\vartheta\big((\frac{15r}{8})^2 - \|x\|^2\big) + \vartheta\big(\|x\|^2 - (\frac{7r}{4})^2\big)}.$$

Then $\chi \in C_c^\infty(B_{\mathrm{euc}}(0,2r))$, $0 \le \chi \le 1$, and $\chi = 1$ on $B_{\mathrm{euc}}(0, \frac{7r}{4})$. Define

$$\chi_i^{\mathrm{geo}} := \begin{cases} \chi \circ \varphi_i, & \text{on } B_{\mathbf{g}}(p_i, 2r), \\ 0, & \text{outside } B_{\mathbf{g}}(p_i, 2r). \end{cases}$$

The construction of h_i^{geo} shows that its support is contained in $B_{\mathbf{g}}(p_i, \frac{3r}{2})$; therefore, $\chi_i^{\mathrm{geo}} = 1$ on a neighborhood of that support. In its own chart, its representation is the fixed function χ, so the required bounds are independent of i. This verifies (B3) and the last item of the statement. Therefore, $\mathcal{T}^{\mathrm{geo}}$ is admissible. □

We next present a useful lemma:

Lemma 13.1.23. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary, and let $\mathcal{T} = (U_i, \varphi_i, h_i)_{i \in I}$ be a trivialization of M. Suppose that there exists a constant $C > 0$ such that, for every $i \in I$, the components of the metric and its inverse in the chart (U_i, φ_i) satisfy

$$|g_{ab}^{(i)}(p)| \le C \qquad \text{and} \qquad |g_{(i)}^{ab}(p)| \le C$$

for every $p \in U_i$ and all $a, b \in \{1, \dots, n\}$.
Then there exist positive constants c_0, C_0, independent of i and p, such that

$$c_0 \|d(\varphi_i)_p(v)\|^2 \le |v|_{\mathbf{g}}^2 \le C_0 \|d(\varphi_i)_p(v)\|^2, \qquad \forall p \in U_i,\ \forall v \in T_pM.$$

Proof. Fix $i \in I$ and $p \in U_i$. Let $v \in T_pM$ and define

$$w := d(\varphi_i)_p(v) \in \mathbb{R}^n.$$

Since (U_i, φ_i) is a chart, the vector v can be written uniquely as

$$v = w^a \partial_a \Big|_p,$$

so, by definition of the metric coefficients,

$$|v|_{\mathbf{g}}^2 = \mathbf{g}_p(v,v) = \sum_{a,b=1}^{n} g_{ab}^{(i)}(p) w^a w^b. \tag{13.7}$$

Let $\lambda_1(p),\dots,\lambda_n(p)$ be the eigenvalues of the symmetric positive definite matrix $(g_{ab}^{(i)}(p))_{a,b=1}^n$. The bound on its entries gives

$$\left(\sum_{a,b=1}^{n} |g_{ab}^{(i)}(p)|^2 \right)^{1/2} \leq nC,$$

and Theorem C.1.10 implies $\lambda_\ell(p) \leq nC$. The inverse matrix has entries $g_{(i)}^{ab}(p)$, so the same argument applied to it gives

$$\lambda_\ell(p)^{-1} \leq nC \quad \Longrightarrow \quad \lambda_\ell(p) \geq \frac{1}{nC}.$$

The spectral theorem provides an orthonormal basis $(\varepsilon_\ell)_{\ell=1}^n$ in which, if $w = \sum_{\ell=1}^{n} w_\ell \varepsilon_\ell$, the last member of (13.7) is $\sum_{\ell=1}^{n} \lambda_\ell(p) w_\ell^2$. Therefore,

$$\frac{1}{nC} \|d(\varphi_i)_p(v)\|^2 \leq |v|_{\mathbf{g}}^2 \leq nC \, \|d(\varphi_i)_p(v)\|^2.$$

The constants $c_0 = (nC)^{-1}$ and $C_0 = nC$ are independent of i and p, as required. □

Remark 13.1.24. (Meaning of uniform comparison in coordinates) If $\overline{\mathbf{g}}$ is the Euclidean metric and $\boldsymbol{\delta}_i := \varphi_i^* \overline{\mathbf{g}}$, then

$$(\boldsymbol{\delta}_i)_p(v,v) = \|d(\varphi_i)_p(v)\|^2.$$

Thus, the conclusion of the preceding lemma can be abbreviated as $c_0 \boldsymbol{\delta}_i \leq \mathbf{g} \leq C_0 \boldsymbol{\delta}_i$ on the domain of each chart, in the sense of Definition 2.1.5. The lemma's comparison between intrinsic and coordinate quantities consists precisely of evaluating these two metrics on the same vector $v \in T_pM$. Proposition 7.1.9 then provides uniform bounds for covectors, tensors of any fixed type, and volume elements.

Here c_0 and C_0 are independent of the index i because the hypothesis controls the coefficients in all charts and their inverses by a single constant. Applying the comparison separately on each compact set would generally give constants c_i, C_i; an infinite family of charts does not allow them to be replaced by common constants without also proving $\inf_{i\in I} c_i > 0$ and $\sup_{i\in I} C_i < \infty$.

13.2 Uniform regularity as a coordinate formulation of bounded geometry

Bounded geometry is the intrinsic notion adopted in this book. Amann's uniform regularity expresses the same control through an atlas with normalized domains; its usefulness lies in making visible the constants associated with the coefficients, cutoff functions, and localization operators. It does not introduce a uniformity on a set of

configurations; the notion based on neighborhoods of the diagonal is developed in Section 27.2. The spaces BC^j and BUC^j, including the convention on a half-box, were defined in Definition 6.4.1.

Set

$$Q^m := (-1,1)^m, \qquad Q^m_+ := [0,1)\times(-1,1)^{m-1}.$$

If $0<\rho<1$, write $\rho Q^m = (-\rho,\rho)^m$ and $\rho Q^m_+ = [0,\rho)\times(-\rho,\rho)^{m-1}$.

Definition 13.2.1. (Uniformly regular atlas) Let M be a separable smooth manifold of dimension m, with or without boundary. An atlas $\mathfrak{K} = (U_\kappa,\kappa)_{\kappa\in K}$ is *normalized* if

$$\kappa\colon U_\kappa \longrightarrow Q_\kappa, \qquad Q_\kappa \in \{Q^m, Q^m_+\},$$

, with Q^m_+ used precisely for charts meeting the boundary. It is *shrinkable* if there exists a number $\rho\in(0,1)$, common to all charts, such that $\{\kappa^{-1}(\rho Q_\kappa)\mid \kappa\in K\}$ covers M. Its multiplicity is

$$\operatorname{mult}(\mathfrak{K}) := \sup_{p\in M} \#\{\kappa\in K\mid p\in U_\kappa\}.$$

The atlas is *uniformly regular* if it is normalized, shrinkable, has finite multiplicity, and, for each $j\in\mathbb{N}_0$, there exists $C_{\mathrm{tr}}(j)>0$, independent of the charts, such that

$$\|\widetilde{\kappa}\circ\kappa^{-1}\|_{BUC^j(\kappa(U_\kappa\cap U_{\widetilde{\kappa}}),\mathbb{R}^m)} \le C_{\mathrm{tr}}(j)$$

if $U_\kappa\cap U_{\widetilde{\kappa}}\neq\varnothing$. Equivalently, every transition belongs to $BUC^\infty(\kappa(U_\kappa\cap U_{\widetilde{\kappa}}),\mathbb{R}^m)$ and hence to BC^∞ on that domain, with bounds uniform over the charts for each finite-order seminorm.

Definition 13.2.2. (Uniformly regular structure) Let M be a smooth manifold with or without boundary. Two uniformly regular atlases of M are equivalent if there exists $N\in\mathbb{N}$ bounding, in both directions, the number of charts of one atlas meeting any chart of the other, and if, for each $j\in\mathbb{N}_0$, the mixed transitions and their inverses have BUC^j norm bounded by a constant independent of the charts. An equivalence class is a *uniformly regular structure*.

Definition 13.2.3. (Uniformly regular Riemannian manifold) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathfrak{K}$ be a uniformly regular atlas. For each chart, define, without ambiguity concerning pushforwards,

$$\mathbf{g}_\kappa := (\kappa^{-1})^*\mathbf{g}.$$

The metric is *uniformly regular relative to* $\mathfrak{K}$ if there exist $c_{\mathbf{g}}, C_{\mathbf{g}}>0$, independent of κ, such that

$$c_{\mathbf{g}}|\xi|^2 \le \mathbf{g}_\kappa(x)(\xi,\xi) \le C_{\mathbf{g}}|\xi|^2 \tag{13.8}$$

for every $x\in Q_\kappa$ and $\xi\in\mathbb{R}^m$, and if, for each $j\in\mathbb{N}_0$, there exists $G_j>0$, independent of κ, such that

$$\|(g_\kappa)_{ab}\|_{BUC^j(Q_\kappa)} \le G_j, \qquad 1\le a,b\le m. \tag{13.9}$$

A manifold equipped with the corresponding class is called a *uniformly regular Riemannian manifold*.

Bounds on the inverse metric are not an additional hypothesis. Indeed, (13.8) implies $\|\mathbf{g}_\kappa^{-1}(x)\|_{\mathrm{op}} \le c_{\mathbf{g}}^{-1}$. Differentiating $\mathbf{g}_\kappa^{-1}\mathbf{g}_\kappa = I$ gives

$$D(\mathbf{g}_\kappa^{-1})[v] = -\mathbf{g}_\kappa^{-1} D\mathbf{g}_\kappa[v]\mathbf{g}_\kappa^{-1}.$$

For a multi-index $\alpha \neq 0$, applying the Leibniz rule to $\mathbf{g}_\kappa^{-1}\mathbf{g}_\kappa = I$ gives

$$0 = \sum_{\substack{\beta \in \mathbb{N}_0^m \\ \beta \le \alpha}} \binom{\alpha}{\beta} (D^\beta \mathbf{g}_\kappa^{-1})(D^{\alpha-\beta}\mathbf{g}_\kappa).$$

Separating the term $\beta = \alpha$ and multiplying on the right by $\mathbf{g}_\kappa^{-1}$ gives the exact recurrence

$$D^\alpha \mathbf{g}_\kappa^{-1} = - \sum_{\substack{\beta \in \mathbb{N}_0^m \\ \beta \le \alpha,\ \beta \neq \alpha}} \binom{\alpha}{\beta} (D^\beta \mathbf{g}_\kappa^{-1})(D^{\alpha-\beta}\mathbf{g}_\kappa)\mathbf{g}_\kappa^{-1}. \tag{13.10}$$

Order zero is bounded by $c_{\mathbf{g}}^{-1}$. Suppose all derivatives of the inverse of order less than j are bounded, and let $|\alpha| = j$. In each summand of (13.10), we have $|\beta| < j$ and $1 \le |\alpha - \beta| \le j$; the induction hypothesis, the bound G_j for the metric, and the order-zero bound for its inverse control the product. The sum is finite, and its coefficients depend only on α. This proves the bound of order j and completes the induction. Thus, for each j, there exists a constant

$$G_j' = P_j(c_{\mathbf{g}}^{-1}, G_1, \dots, G_j) > 0,$$

where P_j is a universal polynomial with nonnegative coefficients, such that

$$\|(\mathbf{g}_\kappa^{-1})^{ab}\|_{BUC^j(Q_\kappa)} \le G_j' \tag{13.11}$$

for all charts and indices. The same transformation rule, the Faà di Bruno formula of Theorem 1.1.19, and the bounds on mixed transitions show that the definition is independent of the representative atlas.

Theorem 13.2.4. (From bounded geometry to uniform regularity) Every Riemannian manifold without boundary and with bounded geometry admits a uniformly regular structure. The atlas constants can be chosen to depend on m, a common normal radius, and the curvature bounds in Definition 13.1.5, never on the chart.

Proof. Let $i_0 := \mathrm{inj}(M) > 0$, and let $r_0 > 0$ be the radius in Theorem 13.1.8. Choose $a > 0$ so that

$$4a\sqrt{m} < \min\{i_0, r_0\}, \qquad \delta := \frac{a}{4}, \qquad \rho := \frac{1}{2}.$$

By the maximality argument of Theorem 13.1.22, there exists a family $(p_i)_{i\in I}$ that is δ-separated and whose balls of radius δ cover M. Choose isometries $B_i : \mathbb{R}^m \longrightarrow T_{p_i}M$ and define

$$\kappa_i^{-1}(x) := \exp_{p_i}(B_i(ax)), \qquad x \in Q^m.$$

The charts are well defined because $aQ^m \subset B(0, a\sqrt{m})$. If $p \in B_{\mathbf{g}}(p_i, \delta)$, then $p = \exp_{p_i}(B_i v)$ with $|v| < \frac{a}{4}$ and $\frac{v}{a} \in \rho Q^m$. Thus, the shrunken charts cover M.
If $U_i \cap U_j \neq \varnothing$, then $d_{\mathbf{g}}(p_i, p_j) < 2a\sqrt{m}$. The balls $B_{\mathbf{g}}(p_j, \frac{\delta}{2})$ are pairwise disjoint and, for every index meeting

U_i, lie in $B_{\mathbf{g}}(p_i, 2a\sqrt{m}+\frac{\delta}{2})$. With the constants from Lemma 13.1.21, we obtain

$$\operatorname{mult}(\mathfrak{K}) \le N_0 := \left\lceil \frac{v_+(2a\sqrt{m}+\frac{\delta}{2})}{v_-(\frac{\delta}{2})} \right\rceil. \tag{13.12}$$

Let $\varphi_i = B_i^{-1} \circ \exp_{p_i}^{-1}$ be the unscaled normal chart. If $U_i \cap U_j \neq \varnothing$, then $d_{\mathbf{g}}(p_i, p_j) < 2a\sqrt{m}$ and, for every $x \in Q^m$,

$$d_{\mathbf{g}}\big(p_j, \exp_{p_i}(B_i(ax))\big) < 3a\sqrt{m}.$$

The choice $4a\sqrt{m} < \min\{i_0, r_0\}$ therefore allows the coordinate transition to be extended to all of Q^m by

$$T_{ji}^{\mathrm{ext}}(x) := a^{-1} B_j^{-1} \exp_{p_j}^{-1}\big(\exp_{p_i}(B_i(ax))\big).$$

Theorem 13.1.8 provides constants $A_\ell > 0$, independent of i, j, for these exponential comparisons at the preceding radii. Since, on the actual domain of the transition,

$$\kappa_j \circ \kappa_i^{-1}(x) = a^{-1}(\varphi_j \circ \varphi_i^{-1})(ax),$$

we have, for $\ell \ge 1$,

$$\sup_{x \in \kappa_i(U_i \cap U_j)} \|D^\ell(\kappa_j \circ \kappa_i^{-1})(x)\|_{\mathrm{op}} \le a^{\ell-1} A_\ell, \tag{13.13}$$

and the order-zero bound for the actual transition is $\sqrt{m}$ because its image lies in Q^m. Since Q^m is convex, the estimate of order $\ell+1$ for T_{ji}^{ext} and the mean value theorem give

$$\|D^\ell T_{ji}^{\mathrm{ext}}(x) - D^\ell T_{ji}^{\mathrm{ext}}(y)\|_{\mathrm{op}} \le a^\ell A_{\ell+1} |x-y|, \qquad x, y \in Q^m.$$

Restricting to each overlap domain gives the required uniform continuity; the constants depend only on $a, m, A_1, \dots, A_{\ell+1}$.

Denote by C_ℓ the bounds for the metric coefficients and their inverses in normal charts of radius $2a\sqrt{m}$. The transformation rule under the rescaling $x \mapsto ax$ is, componentwise,

$$(g_{\kappa_i})_{bc}(x) = a^2 (g_{\varphi_i})_{bc}(ax).$$

The bounds on the entries of $(g_{\varphi_i})_{bc}$ and $(g_{\varphi_i})^{bc}$ imply, by the spectral argument of Lemma 13.1.23, that the eigenvalues of $(g_{\varphi_i})_{bc}$ lie in $[(mC_0)^{-1}, mC_0]$. Consequently,

$$c_{\mathbf{g}} := \frac{a^2}{mC_0}, \qquad C_{\mathbf{g}} := a^2 m C_0$$

satisfy (13.8). Moreover,

$$\|D^\ell (\mathbf{g}_{\kappa_i})_{ab}\|_{L^\infty(Q^m)} \le a^{\ell+2} C_\ell, \qquad 0 \le \ell \le j.$$

Summing these bounds provides $G_j := \sum_{\ell=0}^{j} a^{\ell+2} C_\ell$ in (13.9); the bound of order $j+1$ gives uniform continuity of the derivative of order j. This verifies all axioms for the uniformly regular atlas and metric. □

Theorem 13.2.5. (Full comparison in the absence of boundary) A Riemannian manifold without boundary is uniformly regular if and only if it has bounded geometry in the sense of Definition 13.1.5.

Proof. The implication from bounded geometry is the preceding theorem. For the converse, let $\mathfrak{K}$ be a uniformly regular atlas with shrinking factor $\rho \in (0,1)$ and the metric bounds in (13.8). Work in the normalized charts $Q^m = (-1,1)^m$; there are no boundary charts here. The formula

$$\Gamma^a_{bc} = \frac{1}{2}\sum_{d=1}^{m} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

and (13.10) provide constants $A_0, A_1 \geq 0$ such that

$$\|\Gamma(x)\|_{\mathrm{op}} \leq A_0, \qquad \|D\Gamma(x)\|_{\mathrm{op}} \leq A_1, \qquad x \in Q^m,$$

uniformly over the charts. Regard $\Gamma(x)$ as a bilinear form with values in $\mathbb{R}^m$ and use Euclidean norms. Set $\delta = 1 - \rho > 0$. Choose $a > 0$ independent of the chart and small enough that

$$2a < \frac{\delta}{2}, \qquad 4A_0 a \leq \frac{1}{2}, \qquad b := 4A_1 a^2 + 4A_0 a \leq \frac{1}{4}.$$

Let $p \in M$. The shrinking property allows us to choose a chart κ with $x = \kappa(p) \in \rho Q^m$. For $v \in \mathbb{R}^m$, $|v| < a$, solve the coordinate geodesic equation

$$z''(t) = -\Gamma(z(t))(z'(t), z'(t)), \qquad z(0) = x, \quad z'(0) = v.$$

As long as $0 \leq t \leq 1$ and $|z'(t)| \leq 2a$, its integral form gives

$$|z'(t)| \leq a + 4A_0 a^2 \leq \frac{3}{2}a, \qquad |z(t) - x| \leq 2a < \frac{\delta}{2}.$$

By continuity, the speed cannot reach $2a$ before $t = 1$. The position remains at distance at least $\delta/2$ from the boundary of the cube, and the velocity remains bounded. Thus, the solution stays in a compact subset of the domain of the first-order equation for (z, z'), so Theorem B.13.4 extends it to all of $[0,1]$. In particular, the exponential map is defined on these initial vectors.

Fix $w \in \mathbb{R}^m$ and denote by $Y(t) = D_v z(t)[w]$ the variation with respect to the initial velocity. Smooth dependence of ODE solutions on their data, Theorem B.13.5, and differentiation of the equation give

$$\begin{aligned} Y''(t) = &-D\Gamma(z(t))[Y(t)](z'(t), z'(t)) - 2\Gamma(z(t))(Y'(t), z'(t)), \\ Y(0) = 0, &\qquad Y'(0) = w. \end{aligned}$$

Let $S = \max_{0 \leq t \leq 1} |Y'(t)|$. Since $|Y(t)| \leq S$, the integrated equation gives

$$|Y'(t) - w| \leq (4A_1 a^2 + 4A_0 a)S = bS.$$

Thus, $S \leq |w| + bS$, whence

$$S \leq \frac{|w|}{1-b}, \qquad |Y(1) - w| \leq \frac{b}{1-b}|w| \leq \frac{1}{3}|w|.$$

If $F_x(v) = z(1)$, we have obtained $\|DF_x(v) - I\|_{\mathrm{op}} \le 1/3$ on $B(0,a)$. This domain is convex; integrating the derivative along the segment between v and w gives

$$|F_x(v) - F_x(w) - (v-w)| \le \frac{1}{3}|v-w|, \qquad |F_x(v) - F_x(w)| \ge \frac{2}{3}|v-w|.$$

Thus, F_x is injective and its derivative is invertible. The inverse function theorem makes it a diffeomorphism onto its image. Since F_x is the coordinate expression of $\exp_p$ and

$$|d\kappa_p(\mathbf{v})| \le c_{\mathbf{g}}^{-1/2}|\mathbf{v}|_{\mathbf{g}_p}, \qquad \mathbf{v} \in T_pM,$$

the exponential map is a diffeomorphism onto its image on the tangent ball of radius $a\sqrt{c_{\mathbf{g}}}$. This bound is independent of p and proves

$$\mathrm{inj}(M) \ge a\sqrt{c_{\mathbf{g}}} > 0.$$

Completeness of the components now follows from Remark 13.1.2.
The curvature bounds remain. In each chart,

$$R_{ijb}{}^{a} = \partial_i \Gamma^a_{jb} - \partial_j \Gamma^a_{ib} + \sum_{d=1}^{m} (\Gamma^d_{jb}\Gamma^a_{id} - \Gamma^d_{ib}\Gamma^a_{jd}).$$

Every coordinate derivative of order k of these components is a finite sum of products of derivatives of Γ of orders at most $k+1$. These are uniformly bounded by the metric bounds through order $k+2$ and the recurrence for its inverse. To pass to covariant derivatives, use the tensor formula: add to the coordinate derivative a term ΓT for each contravariant index and subtract one for each covariant index. At order zero, we have just bounded R. If the components of $\nabla^j R$ have been expressed as sums of products of coordinate derivatives of R and Γ, applying this formula and the Leibniz rule produces an expression of the same type for $\nabla^{j+1}R$, with orders increased by at most one. Induction provides uniform bounds for all these components. Finally, (13.8) and the inverse-metric bounds convert component bounds into the tensor bounds $\sup_{p\in M} |\nabla^k \mathbf{R}(p)|_{\mathbf{g}} < \infty$ for every $k \in \mathbb{N}_0$. Both bounded-geometry conditions hold. □

The definitions and this equivalence correspond to [6, Definitions 3.1–3.3 and Theorem 3.5].

Proposition 13.2.6. (Amann's localization system) Let M be a smooth manifold with or without boundary, and let $\mathfrak{K}$ be a uniformly regular atlas of M with shrinking factor ρ. There exist functions $\pi_\kappa \in C_c^\infty(U_\kappa, [0,1])$ and a function $\chi \in C_c^\infty(Q^m, [0,1])$ such that

$$\sum_{\kappa \in K} \pi_\kappa^2 = 1, \qquad \chi = 1 \quad \text{en } \mathrm{supp}(\pi_\kappa \circ \kappa^{-1}),$$

and, for each $j \in \mathbb{N}_0$, there exists $D_j > 0$, independent of κ, with

$$\|\pi_\kappa \circ \kappa^{-1}\|_{BUC^j(Q_\kappa)} \le D_j.$$

If $\mathfrak{K}$ is the atlas constructed in Theorem 13.2.4, or a uniformly regular atlas equivalent to it, this is also an admissible quadratic localization system in the sense of Definition 13.1.16.

Proof. Choose $\rho < \theta < \theta' < 1$. For $0 < a < b < 1$, let

$$\eta_{a,b}(t) := \frac{\vartheta(b^2 - t^2)}{\vartheta(b^2 - t^2) + \vartheta(t^2 - a^2)}, \qquad t \in \mathbb{R},$$

where ϑ is the flat function used in the proof of Theorem 1.3.5. If $b_0 = (\rho + \theta)/2$, define

$$\psi(x_1, \dots, x_m) := \prod_{a=1}^{m} \eta_{\rho,b_0}(x_a).$$

Then $\psi \in C_c^\infty(Q^m, [0,1])$, it equals one on ρQ^m, and its support lies in $[-b_0, b_0]^m \subset \theta Q^m$. On a half-box, use its restriction. Let $\psi_\kappa = \psi \circ \kappa$ on U_κ, extended by zero, and set

$$S(p) := \left(\sum_{\kappa \in K} \psi_\kappa(p)^2 \right)^{\frac{1}{2}}, \qquad \pi_\kappa := \frac{\psi_\kappa}{S}.$$

The family of supports entering this sum is not merely locally finite: the number of these supports meeting any given chart has a uniform bound. We prove both assertions at once. Set

$$N := \operatorname{mult}(\mathfrak{K}), \qquad \varepsilon := \frac{1 - \theta}{2 \max\{1, C_{\mathrm{tr}}(1)\}} > 0.$$

Fix a chart κ_0 and consider the indices κ for which $\operatorname{supp} \psi_\kappa \cap U_{\kappa_0} \neq \varnothing$. For each one, choose $q_\kappa \in \operatorname{supp} \psi_\kappa \cap U_{\kappa_0}$. Then $\kappa(q_\kappa) \in \operatorname{supp} \psi \subset \theta Q_\kappa$, whose distance from the nongeometric faces of Q_κ is at least $1 - \theta$. The bound BUC^1 for $\kappa \circ \kappa_0^{-1}$ and the first-exit-time argument, applied to each segment starting at $\kappa_0(q_\kappa)$, show that

$$B\big(\kappa_0(q_\kappa), \varepsilon\big) \cap Q_{\kappa_0} \subseteq \kappa_0(U_\kappa \cap U_{\kappa_0}). \tag{13.14}$$

On a half-box, a path contained in Q_{κ_0} cannot leave through the geometric face: chart transitions preserve the boundary. Thus, only distance from the nongeometric faces enters the argument.
All sets on the left-hand side of (13.14) lie in the bounded set

$$W := \{x \in \mathbb{R}^m \mid \operatorname{dist}(x, Q_{\kappa_0}) < \varepsilon\}.$$

Since Q_{κ_0} is a box or a half-box, there exists $c_m > 0$, depending only on the dimension, such that

$$\lambda_m\big(B(z, \varepsilon) \cap Q_{\kappa_0}\big) \geq c_m \varepsilon^m, \qquad z \in Q_{\kappa_0}.$$

On the other hand, each point of U_{κ_0} belongs to at most N charts. Summing the indicator functions of the preceding balls and integrating over W gives

$$\#\{\kappa \mid \operatorname{supp} \psi_\kappa \cap U_{\kappa_0} \neq \varnothing\} c_m \varepsilon^m \leq N \lambda_m(W).$$

The measure of W has a bound depending only on m and ε, not on κ_0. Thus, there exists an integer N_1, independent of the chart, bounding the cardinality of the preceding index set. This proves, in particular, local finiteness of the family of supports.
Since the shrunken charts cover, $1 \leq S \leq \sqrt{N}$. In chart κ, at most N_1 functions of the form $\psi \circ \widetilde{\kappa} \circ \kappa^{-1}$

contribute. The Faà di Bruno formula of Theorem 1.1.19, the transition bounds, and the Leibniz rule bound all their BUC^j norms by constants independent of the indices. If $F_\kappa := \sum_{\widetilde{\kappa}\in K} \psi_{\widetilde{\kappa}}^2$ is expressed in chart κ, then $1 \le F_\kappa \le N$ and $S^{-1} = F_\kappa^{-1/2}$. For $|\alpha| \ge 1$, the same formula is a finite sum of terms of the form

$$c_{\alpha,q,\beta} F_\kappa^{-\frac{1}{2}-q} \prod_{\nu=1}^{q} D^{\beta_\nu} F_\kappa, \qquad \beta_\nu \neq 0, \quad \beta_1 + \cdots + \beta_q = \alpha,$$

with $1 \le q \le |\alpha|$. The lower bound for F_κ and its derivative bounds explicitly control every product, uniformly over the charts. Applying the Leibniz rule to $\pi_\kappa = \psi_\kappa S^{-1}$ gives the required bounds. Moreover, $\sum_{\kappa\in K} \pi_\kappa^2 = S^{-2} \sum_{\kappa\in K} \psi_\kappa^2 = 1$.
If $b_1 = (\theta + \theta')/2$, take

$$\chi(x_1,\dots,x_m) := \prod_{a=1}^{m} \eta_{\theta,b_1}(x_a).$$

Then $\chi \in C_c^\infty(Q^m,[0,1])$, it equals one on θQ^m, and its support lies in $\theta' Q^m$. Therefore, $\chi = 1$ on all coordinate supports. When $\mathfrak{K}$ is the rescaled normal atlas of the preceding theorem, the transition bounds already established verify (B1); the bounds just obtained for the cutoffs verify the remaining conditions of Definition 13.1.16. For an equivalent atlas, the bounds on mixed transitions transfer these conditions through the same Faà di Bruno formula and Leibniz rule. □

The preceding system gives the following localization operators. Taking $h_i = \pi_{\kappa_i}$ and $\chi_i := \chi \circ \kappa_i$, the elementary operators on smooth functions are

$$\mathcal{S}_0 u = \left(\widetilde{(h_i u) \circ \varphi_i^{-1}} \right)_{i\in I}, \qquad (\mathcal{R}_0 v)(p) = \sum_{\{i\in I \mid p\in U_i\}} h_i(p)\chi_i(p) v_i(\varphi_i(p)).$$

The second sum is locally finite. Moreover, $\chi_i = 1$ on $\operatorname{supp} h_i$ and $\sum_{i\in I} h_i^2 = 1$; hence, $\mathcal{R}_0\mathcal{S}_0 u = u$. These formulas are used here only on smooth functions. Distributional reconstruction and the Sobolev norm bounds are established after these spaces are defined, using the admissibility conditions verified for this system.

The differential part of (B1) admits the following metric criterion. The overlap hypothesis is kept separate because it does not follow from estimates on the metric coefficients.

Lemma 13.2.7. (Metric criterion for admissibility) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary with $\operatorname{inj}(M) > 0$, and let $\mathcal{T} = (U_i,\varphi_i,h_i)_{i\in I}$ be a uniformly locally finite trivialization. Assume also that the chart images are uniformly normalized: there exists $R > 0$ such that $\varphi_i(U_i) \subseteq B_{\mathrm{euc}}(0,R)$ for every $i \in I$. Suppose that its cover satisfies the two cross-overlap bounds in (B1) relative to a uniform geodesic atlas with the separation and covering properties specified there, chosen with sufficiently small radius. Let $g_{ab}^{(i)}$ and $g_{(i)}^{ab}$ be the coefficients of $\mathbf{g}$ and its inverse in coordinates φ_i. Then $(M,\mathbf{g})$ has bounded geometry and $\mathcal{T}$ satisfies (B1) if and only if, for each $k \in \mathbb{N}_0$, there exists $C_k > 0$ such that, for $|\alpha| \le k$,

$$|D^\alpha g_{ab}^{(i)}| \le C_k \qquad \text{and} \qquad |D^\alpha g_{(i)}^{ab}| \le C_k$$

in every chart (U_i,φ_i).

Proof. ($\Leftarrow$) Suppose that, for every $k \in \mathbb{N}_0$, there exists $C_k > 0$ such that, for every chart (U_i, φ_i) of the trivialization $\mathcal{T}$,

$$|D^\alpha g^{(i)}_{ab}| \le C_k \qquad \text{and} \qquad |D^\alpha g^{ab}_{(i)}| \le C_k, \qquad \forall |\alpha| \le k.$$

First, we prove that $(M, \mathbf{g})$ has bounded geometry. Since $\operatorname{inj}(M) > 0$ by hypothesis, it suffices to show that, for each $m \in \mathbb{N}_0$, there exists a constant $C'_m > 0$ such that

$$|\nabla^m R|_{\mathbf{g}} \le C'_m$$

throughout M.
Fix a chart (U_i, φ_i) of $\mathcal{T}$ and write $\varphi_i = (x^1, \dots, x^n)$.
To simplify notation, denote simply by

$$g_{ab}, \qquad g^{ab}, \qquad \Gamma^c_{ab}, \qquad R_{abc}{}^d$$

the components of the metric, inverse metric, Christoffel symbols, and curvature tensor in these coordinates, which of course depend on i.
By Proposition 2.4.11, the local expression for the Christoffel symbols of the Levi–Civita connection is:

$$\Gamma^c_{ab} = \frac{1}{2} g^{c\ell} \left(\partial_a g_{b\ell} + \partial_b g_{a\ell} - \partial_\ell g_{ab} \right).$$

Applying the Leibniz rule and the hypothesis on the partial derivatives of g_{ab} and g^{ab}, we conclude that, for each $k \in \mathbb{N}_0$, there exists a constant $C^{(1)}_k > 0$, independent of the chart, such that

$$|D^\alpha \Gamma^c_{ab}| \le C^{(1)}_k, \qquad |\alpha| \le k.$$

Now using the local expression for the curvature tensor (Proposition 2.10.3),

$$R_{abc}{}^\ell = \partial_a \Gamma^\ell_{bc} - \partial_b \Gamma^\ell_{ac} + \Gamma^m_{bc} \Gamma^\ell_{am} - \Gamma^m_{ac} \Gamma^\ell_{bm},$$

the Leibniz rule again gives, for each $k \in \mathbb{N}_0$, a constant $C^{(2)}_k > 0$, independent of the chart, such that

$$|D^\alpha R_{abc}{}^\ell| \le C^{(2)}_k, \qquad |\alpha| \le k.$$

Finally, by Lemma 9.1.28, the coordinate components of $\nabla^m R$ are finite linear combinations of partial derivatives of the components of R of order at most m, with coefficients polynomial in the Christoffel symbols and their partial derivatives of order at most $m-1$. Since all these quantities are uniformly bounded, for each $m \in \mathbb{N}_0$ there exists a constant $C''_m > 0$ such that the components of $\nabla^m R$ are uniformly bounded in every chart of $\mathcal{T}$.
Moreover, the uniform bounds on the components of the metric and its inverse imply that, for the fixed tensor type of $\nabla^m R$, there exists $C_{m,\mathbf{g}} > 0$, independent of the chart, such that the pointwise norm induced by $\mathbf{g}$ does not exceed $C_{m,\mathbf{g}}$ times the Euclidean norm of its components. Consequently, for each $m \in \mathbb{N}_0$, there exists a constant $C'_m > 0$ such that

$$|\nabla^m R|_{\mathbf{g}} \le C'_m$$

throughout M. This proves that $(M,\mathbf{g})$ has bounded geometry.

We now prove that $\mathcal{T}$ satisfies (B1). Let $r_0>0$ be a radius for which condition (2) of Theorem 13.1.8 holds, and let

$$\mathcal{A}^{\mathrm{geo}}=(U_j^{\mathrm{geo}},\varphi_j^{\mathrm{geo}})_{j\in J}$$

be the geodesic atlas in the hypothesis. Here "sufficiently small radius" means that its common radius satisfies $0<r\le\frac{r_0}{10}$. Fix $i\in I$ and $j\in J$ such that

$$U_i\cap U_j^{\mathrm{geo}}\neq\varnothing.$$

Write

$$\varphi_i=(x^1,\dots,x^n),\qquad \varphi_j^{\mathrm{geo}}=(y^1,\dots,y^n).$$

Denote by

$$\widetilde{g}_{ab},\qquad \widetilde{g}^{ab},\qquad \widetilde{\Gamma}^c_{ab}$$

the components of the metric and its inverse and the Christoffel symbols in geodesic coordinates y.

Since we have already proved that $(M,\mathbf{g})$ has bounded geometry, Theorem 13.1.8 implies that, in the uniform geodesic atlas, the components $\widetilde{g}_{ab}$, $\widetilde{g}^{ab}$, and all their partial derivatives are uniformly bounded. In particular, for each $k\in\mathbb{N}_0$, there exists $A_k>0$ such that

$$|D^\alpha\widetilde{g}_{ab}|\le A_k\qquad\text{and}\qquad |D^\alpha\widetilde{g}^{ab}|\le A_k,\qquad\forall|\alpha|\le k.$$

Order zero is controlled by normalization: the image of $\varphi_i\circ(\varphi_j^{\mathrm{geo}})^{-1}$ lies in $B_{\mathrm{euc}}(0,R)$ and that of its inverse in $B_{\mathrm{euc}}(0,r)$. Metric bounds do not replace this hypothesis, since translating the origin of a chart leaves its metric coefficients unchanged. We now begin with first derivatives of the coordinate transition

$$\varphi_i\circ(\varphi_j^{\mathrm{geo}})^{-1}\qquad\text{and}\qquad \varphi_j^{\mathrm{geo}}\circ\varphi_i^{-1}.$$

By hypothesis, the components of the metric and its inverse in chart (U_i,φ_i) are uniformly bounded. Since $(M,\mathbf{g})$ also has bounded geometry, the same holds in the geodesic chart $(U_j^{\mathrm{geo}},\varphi_j^{\mathrm{geo}})$. By Lemma 13.1.23, there exist positive constants c_0,C_0, independent of i, j, and the point considered, such that

$$c_0\|d\varphi_i(v)\|^2\le|v|_{\mathbf{g}}^2\le C_0\|d\varphi_i(v)\|^2$$

for every $v\in TM\restriction_{U_i}$, and

$$c_0\|d\varphi_j^{\mathrm{geo}}(v)\|^2\le|v|_{\mathbf{g}}^2\le C_0\|d\varphi_j^{\mathrm{geo}}(v)\|^2$$

for every $v\in TM\restriction_{U_j^{\mathrm{geo}}}$.

Now let

$$F_{ij}:=\varphi_i\circ(\varphi_j^{\mathrm{geo}})^{-1}.$$

Fix $p\in U_i\cap U_j^{\mathrm{geo}}$ and let $\xi\in\mathbb{R}^n$. Define

$$v:=(d\varphi_j^{\mathrm{geo}}|_p)^{-1}(\xi)\in T_pM.$$

Then

$$d\varphi_i|_p(v) = d(\varphi_i \circ (\varphi_j^{\mathrm{geo}})^{-1})|_{\varphi_j^{\mathrm{geo}}(p)}(\xi) = DF_{ij}\big(\varphi_j^{\mathrm{geo}}(p)\big)\,\xi.$$

Applying the preceding inequalities to this vector v gives

$$c_0\big\|DF_{ij}\big(\varphi_j^{\mathrm{geo}}(p)\big)\,\xi\big\|^2 \le |v|_{\mathbf{g}}^2 \le C_0\|\xi\|^2.$$

Therefore,

$$\big\|DF_{ij}\big(\varphi_j^{\mathrm{geo}}(p)\big)\,\xi\big\| \le \sqrt{\frac{C_0}{c_0}}\|\xi\|,$$

and hence

$$\big\|DF_{ij}\big(\varphi_j^{\mathrm{geo}}(p)\big)\big\|_{\mathrm{op}} \le \sqrt{\frac{C_0}{c_0}}.$$

Since the operator norm of DF_{ij} is uniformly bounded, the partial derivatives of order 1 are uniformly bounded.

For the inverse transition, define

$$G_{ij} := \varphi_j^{\mathrm{geo}} \circ \varphi_i^{-1},$$

Now taking $v = (d\varphi_i|_p)^{-1}(\xi)$ and applying the preceding two metric inequalities with the charts interchanged gives, for every $p \in U_i \cap U_j^{\mathrm{geo}}$,

$$\big\|DG_{ij}\big(\varphi_i(p)\big)\big\|_{\mathrm{op}} \le \sqrt{\frac{C_0}{c_0}}.$$

We turn to second derivatives. Applying item (c) of Lemma 2.4.12 to the charts (U_i, φ_i) and $(U_j^{\mathrm{geo}}, \varphi_j^{\mathrm{geo}})$ gives, for every $p \in U_i \cap U_j^{\mathrm{geo}}$,

$$\Gamma_{rs}^k(p) = \frac{\partial x^k}{\partial y^c}(p)\frac{\partial y^a}{\partial x^r}(p)\frac{\partial y^b}{\partial x^s}(p)\widetilde{\Gamma}_{ab}^c(p) + \frac{\partial x^k}{\partial y^c}(p)\frac{\partial^2 y^c}{\partial x^r \partial x^s}(p).$$

Multiplying by $\dfrac{\partial y^m}{\partial x^k}(p)$ and using the chain rule, we conclude that

$$\frac{\partial^2 y^m}{\partial x^r \partial x^s}(p) = \frac{\partial y^m}{\partial x^k}(p)\Gamma_{rs}^k(p) - \frac{\partial y^a}{\partial x^r}(p)\frac{\partial y^b}{\partial x^s}(p)\widetilde{\Gamma}_{ab}^m(p).$$

Since

$$y^m = \pi_m \circ \varphi_j^{\mathrm{geo}} = \pi_m \circ G_{ij} \circ \varphi_i$$

on $U_i \cap U_j^{\mathrm{geo}}$, we obtain

$$\frac{\partial^2 G_{ij}^m}{\partial x^r \partial x^s}\big(\varphi_i(p)\big) = \frac{\partial G_{ij}^m}{\partial x^k}\big(\varphi_i(p)\big)\,\Gamma_{rs}^k(p) - \frac{\partial G_{ij}^a}{\partial x^r}\big(\varphi_i(p)\big)\frac{\partial G_{ij}^b}{\partial x^s}\big(\varphi_i(p)\big)\,\widetilde{\Gamma}_{ab}^m(p).$$

The first derivatives of G_{ij} are already known to be uniformly bounded, as are the components of Γ and $\widetilde{\Gamma}$. Thus, the second derivatives of G_{ij} are uniformly bounded as well.

Interchanging the roles of the two charts in item (c) of Lemma 2.4.12 gives, for every $p \in U_i \cap U_j^{\mathrm{geo}}$,

$$\widetilde{\Gamma}_{ab}^{c}(p) = \frac{\partial y^c}{\partial x^k}(p)\frac{\partial x^r}{\partial y^a}(p)\frac{\partial x^s}{\partial y^b}(p)\,\Gamma_{rs}^{k}(p) + \frac{\partial y^c}{\partial x^k}(p)\frac{\partial^2 x^k}{\partial y^a \partial y^b}(p).$$

Multiplying by $\dfrac{\partial x^m}{\partial y^c}(p)$ gives

$$\frac{\partial^2 x^m}{\partial y^a \partial y^b}(p) = \frac{\partial x^m}{\partial y^c}(p)\,\widetilde{\Gamma}_{ab}^{c}(p) - \frac{\partial x^r}{\partial y^a}(p)\frac{\partial x^s}{\partial y^b}(p)\,\Gamma_{rs}^{m}(p).$$

Since

$$x^m = \pi_m \circ \varphi_i = \pi_m \circ F_{ij} \circ \varphi_j^{\mathrm{geo}}$$

on $U_i \cap U_j^{\mathrm{geo}}$, we obtain

$$\frac{\partial^2 F_{ij}^m}{\partial y^a \partial y^b}\big(\varphi_j^{\mathrm{geo}}(p)\big) = \frac{\partial F_{ij}^m}{\partial y^c}\big(\varphi_j^{\mathrm{geo}}(p)\big)\,\widetilde{\Gamma}_{ab}^{c}(p) - \frac{\partial F_{ij}^r}{\partial y^a}\big(\varphi_j^{\mathrm{geo}}(p)\big)\frac{\partial F_{ij}^s}{\partial y^b}\big(\varphi_j^{\mathrm{geo}}(p)\big)\,\Gamma_{rs}^{m}(p).$$

Thus, the second derivatives of F_{ij} are also uniformly bounded.

We now proceed by induction on the order of differentiation. Suppose that, for some $k \geq 2$, all partial derivatives of F_{ij} and G_{ij} of order at most k are already known to be uniformly bounded. We will show that those of order $k+1$ are uniformly bounded as well.

Consider the identity

$$\frac{\partial^2 F_{ij}^m}{\partial y^a \partial y^b}\big(\varphi_j^{\mathrm{geo}}(p)\big) = \frac{\partial F_{ij}^m}{\partial y^c}\big(\varphi_j^{\mathrm{geo}}(p)\big)\,\widetilde{\Gamma}_{ab}^{c}(p) - \frac{\partial F_{ij}^r}{\partial y^a}\big(\varphi_j^{\mathrm{geo}}(p)\big)\frac{\partial F_{ij}^s}{\partial y^b}\big(\varphi_j^{\mathrm{geo}}(p)\big)\,\Gamma_{rs}^{m}(p).$$

Composing with $(\varphi_j^{\mathrm{geo}})^{-1}$ gives, on the corresponding Euclidean open subset,

$$\frac{\partial^2 F_{ij}^m}{\partial y^a \partial y^b} = \frac{\partial F_{ij}^m}{\partial y^c}\,(\widetilde{\Gamma}_{ab}^{c} \circ (\varphi_j^{\mathrm{geo}})^{-1}) - \frac{\partial F_{ij}^r}{\partial y^a}\frac{\partial F_{ij}^s}{\partial y^b}\,(\Gamma_{rs}^{m} \circ \varphi_i^{-1} \circ F_{ij}).$$

Now let α be a multi-index with $|\alpha| \leq k-1$. Apply D^α to this identity. By the Leibniz rule, the first term on the right-hand side is a finite sum of products of derivatives of F_{ij} of order at most k and derivatives of $\widetilde{\Gamma}_{ab}^{c} \circ (\varphi_j^{\mathrm{geo}})^{-1}$ of order at most $k-1$, all uniformly bounded.

For the second term, use the Leibniz rule again and, when differentiating the composition

$$(\Gamma_{rs}^{m} \circ \varphi_i^{-1}) \circ F_{ij},$$

apply the Faà di Bruno formula of Theorem 1.1.19 in the form stated there. If

$$H := \Gamma_{rs}^{m} \circ \varphi_i^{-1}, \qquad F_{ij} = (F_{ij}^1, \ldots, F_{ij}^n),$$

then, for $1 \le |\beta| \le k-1$,

$$D^\beta (H \circ F_{ij})(y) = \sum_{\substack{\delta \in \mathbb{N}_0^n \\ 1 \le |\delta| \le |\beta|}} \sum_{\substack{\gamma_1, \dots, \gamma_{|\delta|} \in \mathbb{N}_0^n \\ \gamma_1 + \cdots + \gamma_{|\delta|} = \beta \\ |\gamma_\nu| > 0}} \sum_{\substack{\ell_1, \dots, \ell_{|\delta|} \in \{1, \dots, n\} \\ \#\{\nu \mid \ell_\nu = a\} = \delta_a}} c_{\beta, \delta, \gamma, \ell} \big(D^\delta H\big)(F_{ij}(y)) \prod_{\nu=1}^{|\delta|} D^{\gamma_\nu} F_{ij}^{\ell_\nu}(y). \tag{13.15}$$

The sum is finite, and its coefficients depend only on the multi-indices. Each summand satisfies $|\delta| \le |\beta| \le k-1$ and $1 \le |\gamma_\nu| \le |\beta| \le k-1$. Derivatives of H are bounded by the bounds already obtained for the Christoffel symbols in chart i, and derivatives of F_{ij} are bounded by the induction hypothesis. Consequently, every derivative

$$D^\beta \big((\Gamma_{rs}^m \circ \varphi_i^{-1}) \circ F_{ij} \big), \qquad |\beta| \le k-1,$$

is bounded by a constant independent of i, j. After applying the Leibniz rule to the three factors of the second term, no derivative of F_{ij} on the right-hand side has order greater than k. The left-hand side is $D^\alpha \partial_{y^a} \partial_{y^b} F_{ij}^m$; varying a, b and the multi-indices α with $|\alpha| = k-1$ gives every derivative of order $k+1$. We conclude that they are uniformly bounded.

For the inverse transition, begin with the identity

$$\frac{\partial^2 G_{ij}^m}{\partial x^r \partial x^s}\big(\varphi_i(p)\big) = \frac{\partial G_{ij}^m}{\partial x^k}\big(\varphi_i(p)\big)\, \Gamma_{rs}^k(p) - \frac{\partial G_{ij}^a}{\partial x^r}\big(\varphi_i(p)\big) \frac{\partial G_{ij}^b}{\partial x^s}\big(\varphi_i(p)\big)\, \widetilde{\Gamma}_{ab}^m(p),$$

which, after composition with φ_i^{-1}, takes the form

$$\frac{\partial^2 G_{ij}^m}{\partial x^r \partial x^s} = \frac{\partial G_{ij}^m}{\partial x^k} (\Gamma_{rs}^k \circ \varphi_i^{-1}) - \frac{\partial G_{ij}^a}{\partial x^r} \frac{\partial G_{ij}^b}{\partial x^s} (\widetilde{\Gamma}_{ab}^m \circ (\varphi_j^{\mathrm{geo}})^{-1} \circ G_{ij}),$$

and shows that all derivatives of order $k+1$ of G_{ij} are uniformly bounded as well. Here we use exactly (13.15) with $H = \widetilde{\Gamma}_{ab}^m \circ (\varphi_j^{\mathrm{geo}})^{-1}$ and with F_{ij} replaced by G_{ij}; the same bounds on $|\delta|$ and $|\gamma_\nu|$ ensure that only derivatives covered by the induction hypothesis occur.

By induction, for each $k \in \mathbb{N}_0$ there exists a constant $C_k > 0$, independent of i and j, such that

$$|D^\alpha F_{ij}| \le C_k \qquad \text{and} \qquad |D^\alpha G_{ij}| \le C_k$$

for every $|\alpha| \le k$. Together with the two cross-overlap bounds assumed in the statement, this proves (B1).

($\Rightarrow$) Now suppose that $(M, \mathbf{g})$ has bounded geometry and condition (B1) holds.

Let

$$\mathcal{A}^{\mathrm{geo}} = (U_j^{\mathrm{geo}}, \varphi_j^{\mathrm{geo}})_{j \in J}$$

be one of the uniform geodesic atlases for which (B1) holds. Fix $i \in I$ and $j \in J$ such that

$$U_i \cap U_j^{\mathrm{geo}} \neq \varnothing.$$

Write

$$\varphi_i = (x^1, \dots, x^n), \qquad \varphi_j^{\mathrm{geo}} = (y^1, \dots, y^n),$$

and let

$$F_{ij} := \varphi_i \circ (\varphi_j^{\mathrm{geo}})^{-1}, \qquad G_{ij} := \varphi_j^{\mathrm{geo}} \circ \varphi_i^{-1}.$$

By (B1), for each $k \in \mathbb{N}_0$ there exists a constant $C_k > 0$ such that all partial derivatives of F_{ij} and G_{ij} of order at most k are uniformly bounded.

Since $(M, \mathbf{g})$ has bounded geometry, Theorem 13.1.8 implies that, in the uniform geodesic atlas, the components of the metric and its inverse have uniformly bounded derivatives of every order. Denote by

$$\widetilde{g}_{ab}, \qquad \widetilde{g}^{ab}$$

the components of the metric and inverse metric in coordinates y. Then, for each $k \in \mathbb{N}_0$, there exists $A_k > 0$ such that

$$|D^\alpha \widetilde{g}_{ab}| \le A_k \qquad \text{and} \qquad |D^\alpha \widetilde{g}^{ab}| \le A_k$$

for every $|\alpha| \le k$.

Applying item (a) of Lemma 2.4.12 to the charts (U_i, φ_i) and $(U_j^{\mathrm{geo}}, \varphi_j^{\mathrm{geo}})$ gives, for every $p \in U_i \cap U_j^{\mathrm{geo}}$,

$$g_{rs}(p) = \frac{\partial y^a}{\partial x^r}(p) \frac{\partial y^b}{\partial x^s}(p)\, \widetilde{g}_{ab}(p).$$

Since

$$y^a = \pi_a \circ G_{ij} \circ \varphi_i$$

on $U_i \cap U_j^{\mathrm{geo}}$, composing with φ_i^{-1} gives

$$g_{rs} \circ \varphi_i^{-1} = \frac{\partial G_{ij}^a}{\partial x^r} \frac{\partial G_{ij}^b}{\partial x^s} \left(\widetilde{g}_{ab} \circ (\varphi_j^{\mathrm{geo}})^{-1} \circ G_{ij} \right).$$

Set

$$K_{ab} := \widetilde{g}_{ab} \circ (\varphi_j^{\mathrm{geo}})^{-1}.$$

The Leibniz rule for three factors gives, for every multi-index α,

$$D^\alpha (g_{rs} \circ \varphi_i^{-1}) = \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}_0^n \\ \alpha_1 + \alpha_2 + \alpha_3 = \alpha}} \frac{\alpha!}{\alpha_1! \alpha_2! \alpha_3!} D^{\alpha_1}(\partial_r G_{ij}^a) D^{\alpha_2}(\partial_s G_{ij}^b) \cdot D^{\alpha_3}(K_{ab} \circ G_{ij}). \tag{13.16}$$

For the last factor, apply (13.15) with $H = K_{ab}$ and with F_{ij} replaced by G_{ij}. Its summands therefore contain derivatives $D^\delta K_{ab}$ of order $|\delta| \le |\alpha_3|$ and products of derivatives of G_{ij} of positive orders summing to $|\alpha_3|$. The first two factors of (13.16) contain derivatives of G_{ij} of orders at most $|\alpha_1| + 1$ and $|\alpha_2| + 1$. All these quantities have uniform bounds by (B1) and the geodesic estimates. Thus, the derivative

$$D^\alpha (g_{rs} \circ \varphi_i^{-1})$$

is uniformly bounded. Therefore, for each $k \in \mathbb{N}_0$, there exists a constant $C_k' > 0$ such that

$$|D^\alpha (g_{rs} \circ \varphi_i^{-1})| \le C_k', \qquad \forall |\alpha| \le k.$$

Now using the identity

$$D^\alpha g_{rs} = \big(D^\alpha (g_{rs} \circ \varphi_i^{-1})\big) \circ \varphi_i,$$

we conclude that the partial derivatives of the components g_{rs} are uniformly bounded in every chart of $\mathcal{T}$. To bound the inverse-metric coefficients, apply item (b) of Lemma 2.4.12. For every $p \in U_i \cap U_j^{\mathrm{geo}}$,

$$g^{rs}(p) = \frac{\partial x^r}{\partial y^a}(p) \frac{\partial x^s}{\partial y^b}(p)\, \widetilde{g}^{ab}(p).$$

Since

$$x^r = \pi_r \circ F_{ij} \circ \varphi_j^{\mathrm{geo}}$$

on $U_i \cap U_j^{\mathrm{geo}}$, composing with $(\varphi_j^{\mathrm{geo}})^{-1}$ gives

$$(g^{rs} \circ \varphi_i^{-1} \circ F_{ij}) = \frac{\partial F_{ij}^r}{\partial y^a} \frac{\partial F_{ij}^s}{\partial y^b}\, (\widetilde{g}^{ab} \circ (\varphi_j^{\mathrm{geo}})^{-1}).$$

Now composing with G_{ij} yields

$$g^{rs} \circ \varphi_i^{-1} = \left(\frac{\partial F_{ij}^r}{\partial y^a} \circ G_{ij}\right) \left(\frac{\partial F_{ij}^s}{\partial y^b} \circ G_{ij}\right) (\widetilde{g}^{ab} \circ (\varphi_j^{\mathrm{geo}})^{-1} \circ G_{ij}).$$

For this identity, the Leibniz rule again distributes α among three factors. If, for example, $Q_a^r := \partial_{y^a} F_{ij}^r$ the formula (13.15), applied to $Q_a^r \circ G_{ij}$, contains derivatives of Q_a^r of order at most $|\alpha|$, that is, derivatives of F_{ij} of order at most $|\alpha|+1$, and derivatives of G_{ij} of order at most $|\alpha|$. The third factor is treated with $H = \widetilde{g}^{ab} \circ (\varphi_j^{\mathrm{geo}})^{-1}$. The bounds in (B1) and the geodesic bounds control all factors uniformly. We conclude that, for each multi-index α, there exists $C_\alpha'' > 0$, independent of i, j, such that

$$|D^\alpha (g^{rs} \circ \varphi_i^{-1})| \le C_\alpha''.$$

Consequently, the partial derivatives of the components g^{rs} are uniformly bounded as well.
This concludes the proof. □

13.3 Fractional Sobolev, Besov, and Triebel–Lizorkin spaces on Riemannian manifolds of bounded geometry

Geodesic trivializations allow us to define fractional Sobolev spaces on Riemannian manifolds of bounded geometry.

Definition 13.3.1. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let

$$\mathcal{T}^{\mathrm{geo}} = (U_i^{\mathrm{geo}}, \varphi_i^{\mathrm{geo}}, h_i^{\mathrm{geo}})_{i \in J}$$

be a geodesic trivialization as in Theorem 13.1.22. Let $s \in \mathbb{R}$ and $1 < p < \infty$. For $u \in \mathcal{D}'(M)$ and $i \in J$, the

distribution $(h_i^{\mathrm{geo}}u)\circ(\varphi_i^{\mathrm{geo}})^{-1}$ has compact support in $\varphi_i^{\mathrm{geo}}(U_i^{\mathrm{geo}})$. Denote its extension by zero by

$$\widetilde{(h_i^{\mathrm{geo}}u)\circ(\varphi_i^{\mathrm{geo}})^{-1}}\in\mathcal{D}'(\mathbb{R}^n)$$

. Define

$$H^{s,p}(M):=\left\{u\in\mathcal{D}'(M)\,\middle|\,\left(\sum_{i\in J}\left\|\widetilde{(h_i^{\mathrm{geo}}u)\circ(\varphi_i^{\mathrm{geo}})^{-1}}\right\|^p_{H^{s,p}(\mathbb{R}^n)}\right)^{\frac{1}{p}}<\infty\right\}.$$

$$\|u\|_{H^{s,p}(M)}:=\left(\sum_{i\in J}\left\|\widetilde{(h_i^{\mathrm{geo}}u)\circ(\varphi_i^{\mathrm{geo}})^{-1}}\right\|^p_{H^{s,p}(\mathbb{R}^n)}\right)^{\frac{1}{p}}.$$

Remark 13.3.2. The coordinate component of $h_i^{\mathrm{geo}}u$ is understood in the sense of Proposition 10.2.34. A tilde always indicates extension by zero from the coordinate open subset $\varphi_i^{\mathrm{geo}}(U_i^{\mathrm{geo}})$ to $\mathbb{R}^n$. This extension is well defined because the support of $h_i^{\mathrm{geo}}u$ lies in $\operatorname{supp}(h_i^{\mathrm{geo}})\Subset U_i^{\mathrm{geo}}$. Second countability of M and local finiteness of the cover imply that J is countable, so the sum in the definition is a well-defined series.

Later we will show that this definition is independent of the chosen geodesic trivialization. To formulate this independence, it is useful to introduce the same construction for a uniformly locally finite trivialization.

Definition 13.3.3. Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let $\mathcal{T}=(U_i,\varphi_i,h_i)_{i\in I}$ be an admissible trivialization. Let $s\in\mathbb{R}$ and $1<p<\infty$. For $u\in\mathcal{D}'(M)$, define

$$H^{s,p}_{\mathcal{T}}(M):=\left\{u\in\mathcal{D}'(M)\,\middle|\,\left(\sum_{i\in I}\left\|\widetilde{(h_iu)\circ\varphi_i^{-1}}\right\|^p_{H^{s,p}(\mathbb{R}^n)}\right)^{\frac{1}{p}}<\infty\right\},$$

with norm

$$\|u\|_{H^{s,p}_{\mathcal{T}}(M)}:=\left(\sum_{i\in I}\left\|\widetilde{(h_iu)\circ\varphi_i^{-1}}\right\|^p_{H^{s,p}(\mathbb{R}^n)}\right)^{\frac{1}{p}}.$$

The tilde indicates extension by zero from $\varphi_i(U_i)$ to $\mathbb{R}^n$; it is well defined because of the auxiliary cutoff function in (B3).

In general, the spaces $H^{s,p}_{\mathcal{T}}(M)$ depend on the chosen trivialization, and if M does not have bounded geometry, they may lack desirable functional properties. For Riemannian manifolds of bounded geometry and admissible trivializations, all these Sobolev spaces coincide and have good properties, as we will see below.

Lemma 13.3.4. (Uniform transfer between two localizations) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry. Let

$$\mathcal{T}=(U_i,\varphi_i,h_i)_{i\in I},\qquad \mathcal{T}^\sharp=(U_j^\sharp,\varphi_j^\sharp,h_j^\sharp)_{j\in J}$$

be two admissible trivializations of M. Let $(\chi_i)_{i\in I}$ and $(\chi_j^\sharp)_{j\in J}$ be families of auxiliary cutoff functions, so that

$$\operatorname{supp}(\chi_i)\Subset U_i,\qquad \operatorname{supp}(\chi_j^\sharp)\Subset U_j^\sharp,$$

and each function equals one on a neighborhood of the support of the corresponding partition-of-unity element. For $U_i \cap U_j^\sharp \neq \varnothing$, set

$$\Omega_{ij} := \varphi_i(U_i \cap U_j^\sharp), \qquad \Omega_{ji}^\sharp := \varphi_j^\sharp(U_i \cap U_j^\sharp)$$

and

$$\Phi_{ji} := \varphi_j^\sharp \circ \varphi_i^{-1} : \Omega_{ij} \longrightarrow \Omega_{ji}^\sharp.$$

The function

$$\eta_{ij} := (h_i \chi_j^\sharp) \circ \varphi_i^{-1}$$

has compact support in Ω_{ij}. Denote its extension by zero by $\widetilde{\eta}_{ij} \in C_c^\infty(\mathbb{R}^n)$.
For a distribution $v \in \mathcal{D}'(\mathbb{R}^n)$, the expression $v \circ \Phi_{ji}$ means the pullback of $v \restriction_{\Omega_{ji}^\sharp}$ by Φ_{ji}, defined by

$$\langle v \circ \Phi_{ji}, \varphi \rangle_{\Omega_{ij}} := \left\langle v, \widetilde{(\varphi \circ \Phi_{ji}^{-1}) \left| \det D\Phi_{ji}^{-1} \right|} \right\rangle_{\mathbb{R}^n}, \qquad \varphi \in C_c^\infty(\Omega_{ij}). \tag{13.17}$$

The tilde on the right-hand side denotes extension by zero from $\Omega_{ji}^\sharp$ to $\mathbb{R}^n$. This is smooth because the function being extended has compact support in $\Omega_{ji}^\sharp$. Define

$$P_{ij} v := \widetilde{\eta_{ij}(v \circ \Phi_{ji})}, \tag{13.18}$$

where the tilde on the right-hand side indicates extension by zero from Ω_{ij} to $\mathbb{R}^n$.
If $A^s(\mathbb{R}^n)$ is one of the spaces

$$H^{s,p}(\mathbb{R}^n), \qquad F_{p,q}^s(\mathbb{R}^n), \qquad B_{p,q}^s(\mathbb{R}^n)$$

in the Banach ranges of Lemma 6.4.3, there exists a constant $C > 0$, independent of i and j, such that

$$\|P_{ij} v\|_{A^s(\mathbb{R}^n)} \leq C \|v\|_{A^s(\mathbb{R}^n)}. \tag{13.19}$$

Moreover, for every $u \in \mathcal{D}'(M)$,

$$P_{ij} \left(\widetilde{(h_j^\sharp u) \circ (\varphi_j^\sharp)^{-1}} \right) = \widetilde{(h_i h_j^\sharp u) \circ \varphi_i^{-1}}. \tag{13.20}$$

Proof. The support of $h_i \chi_j^\sharp$ is contained in

$$\operatorname{supp}(h_i) \cap \operatorname{supp}(\chi_j^\sharp) \Subset U_i \cap U_j^\sharp.$$

Thus, η_{ij} vanishes on a relative neighborhood of $\partial\Omega_{ij}$, and its extension by zero is smooth. On Ω_{ij},

$$\eta_{ij} = (h_i \circ \varphi_i^{-1}) \left[(\chi_j^\sharp \circ (\varphi_j^\sharp)^{-1}) \circ \Phi_{ji} \right].$$

The Leibniz rule, the bounds in (B2) and (B3), and the Faà di Bruno formula of Theorem 1.1.19 show that,

for each $k \in \mathbb{N}_0$, there exists $C_k > 0$, independent of i and j, such that

$$\|\widetilde{\eta}_{ij}\|_{BC^k(\mathbb{R}^n)} \leq C_k.$$

The derivatives of Φ_{ji} and Φ_{ji}^{-1} are uniformly controlled by (B1) and Remark 13.1.20.
Let K be a common bound for the entries of $D\Phi_{ji}$ and $D\Phi_{ji}^{-1}$ on all overlaps. The Leibniz formula for the determinant gives

$$|\det D\Phi_{ji}| \leq n! K^n.$$

Moreover,

$$\frac{1}{|\det D\Phi_{ji}(x)|} = |\det D\Phi_{ji}^{-1}(\Phi_{ji}(x))| \leq n! K^n.$$

Thus, the Jacobians are bounded above and uniformly bounded away from zero. To apply the localized version of the coordinate-change lemma, we also verify the margins. Choose the cutoffs according to Lemma 13.1.15. The support of h_i lies in the region where $\chi_i = 1$, and the support of $\chi_j^\sharp$ is separated from the boundary of its chart by a fixed distance. The first-derivative bounds for the charts and their inverses convert these two margins into a ball of common radius $\rho > 0$ around each point of the support of η_{ij}, contained in Ω_{ij}, and an analogous margin in chart j. To verify the first assertion, follow a coordinate segment from that point: as long as its image remains in both charts, its speed in the second is uniformly bounded; a segment shorter than the margin divided by that bound cannot reach the boundary of the second chart. The margin in the first chart ensures that the segment is defined throughout this path.
Condition (B1) at order zero places all coordinate domains in a fixed Euclidean ball. We can cover the support of η_{ij} by a uniformly bounded number of balls of radius $\rho/4$, with subordinate cutoffs whose derivatives have uniform bounds. On each larger ball, of radius ρ, the hypotheses of the localized version of Lemma 6.4.3 hold. For order s, fix an integer $k > |s| + n + 1$; derivatives of the multipliers through order k and of the transitions through order $k+1$ are controlled by (B1)–(B3). The local estimate, summed over this uniformly finite cover, proves (13.19) with a common constant.
Finally, $\chi_j^\sharp h_j^\sharp = h_j^\sharp$. On Ω_{ij}, the definition of the pullback of distributions and compatibility of coordinate representations give

$$\eta_{ij}\left[\widetilde{\left((h_j^\sharp u)\circ(\varphi_j^\sharp)^{-1}\right)}\circ\Phi_{ji}\right] = (h_i\chi_j^\sharp h_j^\sharp u)\circ\varphi_i^{-1}.$$

Both sides have compact support in Ω_{ij}. Extending them by zero to $\mathbb{R}^n$ proves (13.20). □

Theorem 13.3.5. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let $\mathcal{T} = (U_i, \varphi_i, h_i)_{i\in I}$ be an admissible trivialization of M. Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then

$$H_{\mathcal{T}}^{s,p}(M) = H^{s,p}(M),$$

where $H^{s,p}(M)$ is defined as in Definition 13.3.1. More precisely, there exist constants $c, C > 0$, independent of u, such that

$$c\|u\|_{H^{s,p}(M)} \leq \|u\|_{H_{\mathcal{T}}^{s,p}(M)} \leq C\|u\|_{H^{s,p}(M)}.$$

Proof. Let

$$\mathcal{T}^{\mathrm{geo}} = (U_j^{\mathrm{geo}}, \varphi_j^{\mathrm{geo}}, h_j^{\mathrm{geo}})_{j\in J}$$

be the trivialization in Theorem 13.1.22. For $i \in I$ and $j \in J$, set

$$J_i := \{j \in J \mid U_i \cap U_j^{\mathrm{geo}} \neq \varnothing\}, \qquad I_j := \{i \in I \mid U_i \cap U_j^{\mathrm{geo}} \neq \varnothing\}.$$

The two cross-overlap bounds in (B1), transferred to the geodesic atlas by Remark 13.1.20, provide integers N_1, N_2 such that

$$\#J_i \le N_1, \qquad \#I_j \le N_2$$

for all indices.

The locally finite identity $\sum_{j\in J} h_j^{\mathrm{geo}} = 1$ can be multiplied by the distribution $h_i u$. After passing to chart i and extending each component by zero, we obtain in $\mathcal{D}'(\mathbb{R}^n)$

$$\widetilde{(h_i u)\circ\varphi_i^{-1}} = \sum_{j\in J_i} \widetilde{(h_i h_j^{\mathrm{geo}} u)\circ\varphi_i^{-1}}.$$

Lemma 13.3.4, applied with $\mathcal{T}^\sharp = \mathcal{T}^{\mathrm{geo}}$, gives a constant $C_0 > 0$, independent of i and j, such that

$$\left\|\widetilde{(h_i h_j^{\mathrm{geo}} u)\circ\varphi_i^{-1}}\right\|_{H^{s,p}(\mathbb{R}^n)} \le C_0 \left\|\widetilde{(h_j^{\mathrm{geo}} u)\circ(\varphi_j^{\mathrm{geo}})^{-1}}\right\|_{H^{s,p}(\mathbb{R}^n)}.$$

Set

$$a_j := \left\|\widetilde{(h_j^{\mathrm{geo}} u)\circ(\varphi_j^{\mathrm{geo}})^{-1}}\right\|_{H^{s,p}(\mathbb{R}^n)}.$$

Since each sum has at most N_1 terms,

$$\left\|\widetilde{(h_i u)\circ\varphi_i^{-1}}\right\|_{H^{s,p}(\mathbb{R}^n)}^p \le C_0^p N_1^{p-1} \sum_{j\in J_i} a_j^p.$$

Summing over i and rearranging the nonnegative series by Tonelli's Theorem B.6.17 yields

$$\begin{aligned}
\|u\|_{H_{\mathcal{T}}^{s,p}(M)}^p &\le C_0^p N_1^{p-1} \sum_{i\in I}\sum_{j\in J_i} a_j^p \\
&= C_0^p N_1^{p-1} \sum_{j\in J}\sum_{i\in I_j} a_j^p \\
&\le C_0^p N_1^{p-1} N_2 \|u\|_{H^{s,p}(M)}^p.
\end{aligned}$$

The factor N_1 controls the number of terms in each localization, and N_2 controls how many times a geodesic localization appears in the global sum.

For the reverse inequality, use $\sum_{i\in I} h_i = 1$. In the geodesic chart j,

$$\widetilde{(h_j^{\mathrm{geo}} u)\circ(\varphi_j^{\mathrm{geo}})^{-1}} = \sum_{i\in I_j} \widetilde{(h_j^{\mathrm{geo}} h_i u)\circ(\varphi_j^{\mathrm{geo}})^{-1}}.$$

Apply the same lemma with the two trivializations interchanged and repeat the preceding count. Thus,

$$\|u\|_{H^{s,p}(M)} \le C' \|u\|_{H^{s,p}_{\mathcal{T}}(M)}.$$

The two estimates prove equality of the spaces as subspaces of $\mathcal{D}'(M)$ and, for every u in that space,

$$(C')^{-1}\|u\|_{H^{s,p}(M)} \le \|u\|_{H^{s,p}_{\mathcal{T}}(M)} \le C_0 N_1^{1-\frac{1}{p}} N_2^{\frac{1}{p}} \|u\|_{H^{s,p}(M)}.$$

□

Having defined fractional Sobolev spaces on Riemannian manifolds, we turn to Besov and Triebel–Lizorkin spaces. On $\mathbb{R}^n$, priority is often given to Besov spaces, which at first sight seem simpler than Triebel–Lizorkin spaces. On a noncompact manifold, the situation changes: Besov spaces do not satisfy the strong localization principle with outer ℓ^p summation when the fine index differs from p, as proved in Theorem 6.4.4; see also [146, Theorem 2.4.7(i)]. We will therefore define Triebel–Lizorkin spaces first by localization and then construct Besov spaces by real interpolation.

Definition 13.3.6. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, let $\mathcal{T} = (U_i, \varphi_i, h_i)_{i\in I}$ be an admissible trivialization, and let $s \in \mathbb{R}$.

(a) If $1 \le p < \infty$ and $1 \le q \le \infty$, define

$$F^s_{p,q,\mathcal{T}}(M) := \left\{ u \in \mathcal{D}'(M) \,\middle|\, \|u\|_{F^s_{p,q,\mathcal{T}}(M)} < \infty \right\},$$

where

$$\|u\|_{F^s_{p,q,\mathcal{T}}(M)} := \left(\sum_{i\in I} \left\| \widetilde{(h_i u)\circ \varphi_i^{-1}} \right\|^p_{F^s_{p,q}(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

(b) If $1 \le p < \infty$, $1 \le q \le \infty$, and $-\infty < s_0 < s < s_1 < \infty$, write $s = (1-\Theta)s_0 + \Theta s_1$, with $0 < \Theta < 1$, and define

$$B^s_{p,q,\mathcal{T}}(M) := \left(F^{s_0}_{p,p,\mathcal{T}}(M), F^{s_1}_{p,p,\mathcal{T}}(M)\right)_{\Theta,q}.$$

In the first item, the tilde indicates extension by zero of the coordinate component from $\varphi_i(U_i)$ to $\mathbb{R}^n$. In the indicated ranges, these expressions are norms.

Remark 13.3.7. (a) If $\mathcal{T}$ is the geodesic trivialization in Theorem 13.1.22, we omit the reference to $\mathcal{T}$ and simply write

$$F^s_{p,q}(M) \qquad \text{and} \qquad B^s_{p,q}(M).$$

(b) In the range adopted here, the preceding expressions define norms. Triebel's theory also allows $0 < p < 1$ or $0 < q < 1$, but requires atomic methods for quasi-Banach spaces that will not be needed in this book.

(c) The definition of $F^s_{p,q,\mathcal{T}}(M)$ uses coordinate localization. For Besov spaces, however, we adopt the definition by real interpolation, since the same localization principle as for Triebel–Lizorkin spaces is not generally available.

(d) For every $s \in \mathbb{R}$ and every $1 \le p < \infty$,

$$B^s_{p,p,\mathcal{T}}(M) = F^s_{p,p,\mathcal{T}}(M)$$

and there exist constants $c_{s,p,\mathcal{T}}, C_{s,p,\mathcal{T}} > 0$ such that

$$c_{s,p,\mathcal{T}} \|u\|_{F^s_{p,p,\mathcal{T}}(M)} \le \|u\|_{B^s_{p,p,\mathcal{T}}(M)} \le C_{s,p,\mathcal{T}} \|u\|_{F^s_{p,p,\mathcal{T}}(M)}$$

for every u in the common space.

(e) The definition of $B^s_{p,q,\mathcal{T}}(M)$ is independent of the choice of s_0 and s_1, provided that

$$-\infty < s_0 < s < s_1 < \infty \qquad \text{and} \qquad s = (1-\Theta)s_0 + \Theta s_1.$$

(f) Alternatively, we could have defined

$$B^s_{p,p}(M) := (H^{s_0,p}(M), H^{s_1,p}(M))_{\Theta,p},$$

where $s_0, s_1 \in \mathbb{R}$, $1 < p < \infty$, $0 < \Theta < 1$, and $s = (1-\Theta)s_0 + \Theta s_1$.

Justification of items (d)–(f). To apply interpolation, we construct order-independent analysis and synthesis operators defining a retraction for all the spaces under consideration. Fix the auxiliary cutoff functions χ_i from (B3), set $\Omega_i := \varphi_i(U_i)$, and, for $r \in \mathbb{R}$, write

$$\mathbb{X}_r := \ell^p\big(I; F^r_{p,p}(\mathbb{R}^n)\big).$$

The analysis operator is

$$\mathcal{S}u := \Big(\widetilde{(h_i u)\circ\varphi_i^{-1}}\Big)_{i\in I}.$$

By definition, $\mathcal{S}\colon F^r_{p,p,\mathcal{T}}(M) \to \mathbb{X}_r$ is an isometry.
We define synthesis precisely. In chart i, let

$$a_i(x) := \sqrt{\det\big(\mathbf{g}^{(i)}(\varphi_i^{-1}(x))\big)}.$$

If $z \in \mathcal{D}'(\mathbb{R}^n)$ has compact support contained in Ω_i, the distribution $\mathcal{E}_i z \in \mathcal{D}'(M)$ is determined by

$$\langle \mathcal{E}_i z, \psi\rangle := \Big\langle z, a_i\big((\psi \restriction_{U_i})\circ\varphi_i^{-1}\big)\Big\rangle, \qquad \psi \in C^\infty_c(M).$$

This expression is well defined because it uses the test function only on a neighborhood of the compact support of z; this is the extension of the action established in Proposition 10.2.32. For $v = (v_i)_{i\in I}$, set

$$\mathcal{R}v := \sum_{i\in I} \mathcal{E}_i\big(\widetilde{\chi}_i v_i\big), \qquad \widetilde{\chi}_i := \widetilde{\chi_i\circ\varphi_i^{-1}}.$$

The sum defines a distribution: only finitely many terms contribute on the support of a test function. Since

$\chi_i = 1$ near $\operatorname{supp}(h_i)$ and $\sum_{i\in I} h_i = 1$, we obtain, first on each test function and therefore in $\mathcal{D}'(M)$,

$$\mathcal{R}\mathcal{S}u = \sum_{i\in I} h_i u = u.$$

We show that synthesis is continuous at every order. Localizing $\mathcal{R}v$ with h_j involves only the indices

$$I_j := \{i \in I \mid U_i \cap U_j \neq \varnothing\}.$$

The family I_j has uniformly bounded cardinality. Each summand, written in chart j, is obtained from $\widetilde{\chi}_i v_i$ by a coordinate change and multiplication by a smooth function with uniformly bounded derivatives. Lemma 13.3.4 gives, with a constant C_r independent of i and j,

$$\left\| \widetilde{\left(h_j \mathcal{E}_{\rangle}(\widetilde{\ }_{\rangle}\sqsubseteq_{\rangle})\right)} \circ \varphi_{|}^{-} \widetilde{\chi} \right\|_{F^r_{p,p}(\mathbb{R}^n)} \leq C_r \|v_i\|_{F^r_{p,p}(\mathbb{R}^n)}.$$

Hölder's inequality for the finite sum, followed by Tonelli, then shows that there exists $C_r' > 0$ such that

$$\|\mathcal{R}v\|_{F^r_{p,p,\mathcal{T}}(M)} \leq C_r' \|v\|_{\mathbb{X}_r}.$$

Thus, $F^r_{p,p,\mathcal{T}}(M)$ is a retract of $\mathbb{X}_r$, and the operators $\mathcal{S}$ and $\mathcal{R}$ do not depend on r. Moreover, $\mathcal{S}\mathcal{R}$ is a continuous projection whose closed image is $\mathcal{S}(F^r_{p,p,\mathcal{T}}(M))$. Since $\mathbb{X}_r$ is Banach and $\mathcal{S}$ is isometric, $F^r_{p,p,\mathcal{T}}(M)$ is Banach as well. This verifies that the pairs interpolated below are Banach pairs.

Now let $r_0 < r_1$, $0 < \eta < 1$, and $r = (1-\eta)r_0 + \eta r_1$. The Euclidean identity

$$(F^{r_0}_{p,p}(\mathbb{R}^n), F^{r_1}_{p,p}(\mathbb{R}^n))_{\eta,p} = F^r_{p,p}(\mathbb{R}^n)$$

follows from the dyadic decomposition, $B^t_{p,p}(\mathbb{R}^n) = F^t_{p,p}(\mathbb{R}^n)$, and Proposition 5.7.18; it also holds for $p = 1$. The same proposition gives

$$(\mathbb{X}_{r_0}, \mathbb{X}_{r_1})_{\eta,p} = \mathbb{X}_r.$$

Theorem 5.3.1, applied to $\mathcal{S}$ and $\mathcal{R}$, preserves the identity $\mathcal{R}\mathcal{S} = I$ on the interpolated spaces. Consequently,

$$(F^{r_0}_{p,p,\mathcal{T}}(M), F^{r_1}_{p,p,\mathcal{T}}(M))_{\eta,p} = F^r_{p,p,\mathcal{T}}(M) \tag{13.21}$$

with equivalence of norms. Taking $r_0 = s_0$, $r_1 = s_1$, and $\eta = \Theta$ proves item (d).

For item (e), consider two admissible pairs of endpoints. Choose $a < s < b$ so that all four endpoints lie in (a, b). By (13.21), each endpoint space is a space $(F^a_{p,p,\mathcal{T}}, F^b_{p,p,\mathcal{T}})_{\rho,p}$, with parameter ρ determined affinely by its order. The real reiteration Theorem 5.5.2 reduces both definitions of $B^s_{p,q,\mathcal{T}}(M)$ to the same space

$$(F^a_{p,p,\mathcal{T}}(M), F^b_{p,p,\mathcal{T}}(M))_{\rho_s,q}, \qquad \rho_s = \frac{s-a}{b-a}.$$

This proves independence of the endpoints, including equivalence of the resulting norms.

Finally, if $1 < p < \infty$, the same analysis and synthesis are continuous when $F^r_{p,p}(\mathbb{R}^n)$ is replaced by $H^{r,p}(\mathbb{R}^n)$; the proof uses exactly the preceding multiplication, coordinate-change, and overlap estimates. Item (a) of Theorem 6.3.11 identifies real interpolation of the Euclidean Bessel spaces with $B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n)$.

Interpolating the retraction gives

$$(H^{s_0,p}(M), H^{s_1,p}(M))_{\Theta,p} = F^s_{p,p}(M) = B^s_{p,p}(M),$$

which is item (f). □

These spaces are independent of the chosen trivialization (since the Riemannian manifold has bounded geometry), as we now show:

Theorem 13.3.8. Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let $\mathcal{T} = (U_i, \varphi_i, h_i)_{i\in I}$ be an admissible trivialization. Let $s \in \mathbb{R}$, $1 \le p < \infty$, and $1 \le q \le \infty$. Then

$$F^s_{p,q,\mathcal{T}}(M) = F^s_{p,q}(M) \qquad \text{and} \qquad B^s_{p,q,\mathcal{T}}(M) = B^s_{p,q}(M)$$

and there exist positive constants c_F, C_F, c_B, C_B, depending on s, p, q and the uniform data of the two trivializations but not on u, such that

$$c_F\|u\|_{F^s_{p,q}(M)} \le \|u\|_{F^s_{p,q,\mathcal{T}}(M)} \le C_F\|u\|_{F^s_{p,q}(M)}$$

and

$$c_B\|u\|_{B^s_{p,q}(M)} \le \|u\|_{B^s_{p,q,\mathcal{T}}(M)} \le C_B\|u\|_{B^s_{p,q}(M)}.$$

Proof. For the scale F, repeat the count in the proof of Theorem 13.3.5. If $\mathcal{T}^{\mathrm{geo}}$ is the geodesic trivialization, in $\mathcal{D}'(\mathbb{R}^n)$ we have

$$\widetilde{(h_i u)\circ\varphi_i^{-1}} = \sum_{j\in J_i} \widetilde{(h_i h_j^{\mathrm{geo}} u)\circ\varphi_i^{-1}}.$$

Lemma 13.3.4, with $A^s = F^s_{p,q}$, gives

$$\left\|\widetilde{(h_i h_j^{\mathrm{geo}} u)\circ\varphi_i^{-1}}\right\|_{F^s_{p,q}(\mathbb{R}^n)} \le C_0 \left\|\widetilde{(h_j^{\mathrm{geo}} u)\circ(\varphi_j^{\mathrm{geo}})^{-1}}\right\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

The constant C_0 depends on s, p, q and the uniform bounds for the cutoff functions, coordinate transitions, and their Jacobians, but not on i or j. If N_1, N_2 are the two cross multiplicities, outer ℓ^p summation gives

$$\|u\|^p_{F^s_{p,q,\mathcal{T}}(M)} \le C_0^p N_1^{p-1} N_2 \|u\|^p_{F^s_{p,q}(M)}.$$

Interchanging the trivializations gives the reverse estimate. The argument remains valid when $q = \infty$, since it applies to the local norms, which are nonnegative numbers, while the outer sum retains exponent $p < \infty$. For the scale B, fix $s_0 < s < s_1$ and $s = (1-\theta)s_0 + \theta s_1$. The identities on the common space $\mathcal{D}'(M)$ define compatible isomorphisms

$$F^{s_\nu}_{p,p,\mathcal{T}}(M) \longrightarrow F^{s_\nu}_{p,p}(M), \qquad \nu \in \{0,1\}.$$

The operator interpolation theorem for the K method 5.3.1, applied also to the inverse identity, gives

$$\left(F^{s_0}_{p,p,\mathcal{T}}, F^{s_1}_{p,p,\mathcal{T}}\right)_{\theta,q} = \left(F^{s_0}_{p,p}, F^{s_1}_{p,p}\right)_{\theta,q}$$

with norms controlled in both directions by the norms of the identities at the two endpoints and their inverses. These are precisely the constants c_B, C_B in the statement. This establishes the asserted equality of Besov spaces. The reiteration Theorem 5.5.2 also shows that the construction is independent of the auxiliary endpoints. □

13.3.1 Comparison with intrinsic definitions

Remark 13.3.9. (Heat on several components) In this chapter, M is not assumed connected. Its components $(M_\nu)_{\nu\in A}$ form an at most countable family, since they are open and M has a countable base. Each is complete by Remark 13.1.2. To apply the constructions of the heat chapter, use

$$L^p(M,\mathbf{E}) = \ell^p\big(A; L^p(M_\nu, \mathbf{E}|_{M_\nu})\big), \qquad 1 \le p < \infty,$$

and define the semigroup componentwise. It is contractive because every component semigroup is contractive. If $\mathbf{u} = (\mathbf{u}_\nu)_{\nu\in A}$, then

$$\|P_t^{\mathbf{E}}\mathbf{u} - \mathbf{u}\|^p_{L^p(M,\mathbf{E})} = \sum_{\nu\in A} \|P_t^{\mathbf{E}|_{M_\nu}}\mathbf{u}_\nu - \mathbf{u}_\nu\|^p_{L^p(M_\nu,\mathbf{E}\restriction_{M_\nu})} \longrightarrow 0 :$$

each summand tends to zero and is bounded by $2^p\|\mathbf{u}_\nu\|^p_{L^p(M_\nu,\mathbf{E}\restriction_{M_\nu})}$, a convergent series. Thus, strong continuity is preserved, including for $p = 1$. The projections onto the components are continuous and commute with the Bochner integrals defining the potentials. The intrinsic scales are therefore constructed with the same semigroup componentwise. The following estimates use constants independent of ν, since the geometric bounds are uniform on all of M. In the scalar case, use the trivial bundle.

The comparison will rely on three distinct mechanisms. The first is uniform transfer between charts, already established in Lemma 13.3.4. The second compares partial and covariant derivatives at integer orders. The third interpolates these integer-order identifications and compares them with the scale generated by the Laplacian. We begin by specifying the role of quadratic partitions.

Lemma 13.3.10. (Equivalence of quadratic localization) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, let $\mathcal{Q} = (U_i, \varphi_i, h_i)_{i\in I}$ be an admissible quadratic localization system, let $\rho_i := h_i^2$, and let $\mathcal{T}_\mathcal{Q} = (U_i, \varphi_i, \rho_i)_{i\in I}$ be the associated trivialization. For $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$, there exist constants $c_H, C_H, c_F, C_F > 0$, independent of u, such that

$$c_H\|u\|_{H^{s,p}_{\mathcal{T}_\mathcal{Q}}(M)} \le \left(\sum_{i\in I} \left\|\widetilde{(h_i u)\circ\varphi_i^{-1}}\right\|^p_{H^{s,p}(\mathbb{R}^n)}\right)^{\frac{1}{p}} \le C_H\|u\|_{H^{s,p}_{\mathcal{T}_\mathcal{Q}}(M)}$$

and

$$c_F\|u\|_{F^{s}_{p,q,\mathcal{T}_\mathcal{Q}}(M)} \le \left(\sum_{i\in I} \left\|\widetilde{(h_i u)\circ\varphi_i^{-1}}\right\|^p_{F^{s}_{p,q}(\mathbb{R}^n)}\right)^{\frac{1}{p}} \le C_F\|u\|_{F^{s}_{p,q,\mathcal{T}_\mathcal{Q}}(M)}.$$

When $q = p$, the second comparison also provides localization of $B^s_{p,p}(M)$.

Proof. Fix one of the two choices

$$A^s(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n) \qquad \text{or} \qquad A^s(\mathbb{R}^n) = F^s_{p,q}(\mathbb{R}^n)$$

and set

$$N_h(u) := \left(\sum_{i\in I} \left\| \widetilde{(h_i u)\circ \varphi_i^{-1}} \right\|^p_{A^s(\mathbb{R}^n)} \right)^{\frac{1}{p}},$$

$$N_\rho(u) := \left(\sum_{i\in I} \left\| \widetilde{(\rho_i u)\circ \varphi_i^{-1}} \right\|^p_{A^s(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

By definition, N_ρ is the norm associated with $\mathcal{T}_Q$. Since $\operatorname{supp}(h_i) \Subset U_i$, the function $h_i \circ \varphi_i^{-1}$ has an extension by zero $\widetilde{h_i \circ \varphi_i^{-1}} \in C^\infty_c(\mathbb{R}^n)$, and

$$\widetilde{(\rho_i u)\circ \varphi_i^{-1}} = \widetilde{h_i \circ \varphi_i^{-1}}\, \widetilde{(h_i u)\circ \varphi_i^{-1}}.$$

The multiplier theorem in Lemma 6.4.3 and the uniform bounds on (B2) give a constant $K_0 > 0$, independent of $i \in I$, such that

$$\left\| \widetilde{(\rho_i u)\circ \varphi_i^{-1}} \right\|_{A^s(\mathbb{R}^n)} \le K_0 \left\| \widetilde{(h_i u)\circ \varphi_i^{-1}} \right\|_{A^s(\mathbb{R}^n)}.$$

Raising to the power p and summing over $i \in I$ gives

$$N_\rho(u) \le K_0 N_h(u). \tag{13.22}$$

For the reverse estimate, use the locally finite identity

$$\sum_{j\in I} \rho_j = 1.$$

For each $i \in I$, define

$$I_i := \{ j \in I \mid U_i \cap U_j \neq \varnothing \}.$$

Uniform finiteness of the cover provides $L \in \mathbb{N}$ such that $\#I_i \le L$ for every $i \in I$. Therefore,

$$\widetilde{(h_i u)\circ \varphi_i^{-1}} = \sum_{j\in I_i} \widetilde{(h_i \rho_j u)\circ \varphi_i^{-1}} \quad \text{on } \mathcal{D}'(\mathbb{R}^n). \tag{13.23}$$

Fix $j \in I_i$ and let χ_j be the auxiliary cutoff function associated with ρ_j. On the overlap, set

$$\Omega_{ij} := \varphi_i(U_i \cap U_j), \qquad \Phi_{ji} := \varphi_j \circ \varphi_i^{-1}.$$

In accordance with (13.17), define

$$P^{h,\rho}_{ij} z := \big(\widetilde{(h_i \chi_j)\circ \varphi_i^{-1}} \big)(z \circ \Phi_{ji}).$$

The support of $(h_i \chi_j)\circ \varphi_i^{-1}$ is compact in Ω_{ij}. The proof of Lemma 13.3.4, which does not use the partition-

of-unity identity for this estimate, provides a constant $K_1 > 0$, independent of i and j, such that

$$\|P_{ij}^{h,\rho} z\|_{A^s(\mathbb{R}^n)} \le K_1 \|z\|_{A^s(\mathbb{R}^n)}. \tag{13.24}$$

Since $\chi_j \rho_j = \rho_j$,

$$P_{ij}^{h,\rho}\left(\widetilde{(\rho_j u)\circ\varphi_j^{-1}}\right) = \widetilde{(h_i \rho_j u)\circ\varphi_i^{-1}}.$$

The preceding identities imply

$$\left\|\widetilde{(h_i u)\circ\varphi_i^{-1}}\right\|_{A^s(\mathbb{R}^n)} \le K_1 \sum_{j\in I_i} \left\|\widetilde{(\rho_j u)\circ\varphi_j^{-1}}\right\|_{A^s(\mathbb{R}^n)}.$$

Hölder's inequality for a sum of at most L terms gives

$$\left(\sum_{j\in I_i} a_j\right)^p \le L^{p-1} \sum_{j\in I_i} a_j^p$$

for every family of nonnegative numbers $(a_j)_{j\in I_i}$. Applying this inequality and summing over $i \in I$ gives

$$\begin{aligned} N_h(u)^p &\le K_1^p L^{p-1} \sum_{i\in I}\sum_{j\in I_i} \left\|\widetilde{(\rho_j u)\circ\varphi_j^{-1}}\right\|_{A^s(\mathbb{R}^n)}^p \\ &= K_1^p L^{p-1} \sum_{j\in I} \#I_j \left\|\widetilde{(\rho_j u)\circ\varphi_j^{-1}}\right\|_{A^s(\mathbb{R}^n)}^p \\ &\le K_1^p L^p N_\rho(u)^p. \end{aligned}$$

The intermediate equality uses symmetry of the relation $U_i \cap U_j \neq \varnothing$. Consequently,

$$N_h(u) \le K_1 L N_\rho(u). \tag{13.25}$$

Estimates (13.22) and (13.25) prove the two comparisons. For $A^s = F^s_{p,p}$, also use the Euclidean identification

$$F^s_{p,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n)$$

of Theorem 6.3.14. □

Proposition 13.3.11. (Retraction, interpolation, and distributional duality) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $1 < p < \infty$, and $0 < \theta < 1$. If $s = (1-\theta)s_0 + \theta s_1$, then

$$[H^{s_0,p}(M), H^{s_1,p}(M)]_\theta = H^{s,p}(M) \tag{13.26}$$

and there exist constants $c_I, C_I > 0$, depending on s_0, s_1, θ, p and the uniform localization data, such that

$$c_I \|u\|_{H^{s,p}(M)} \le \|u\|_{[H^{s_0,p}(M), H^{s_1,p}(M)]_\theta} \le C_I \|u\|_{H^{s,p}(M)}.$$

Moreover, the identification of locally integrable functions with regular distributions, constructed using

$d\lambda_{\mathbf{g}}$, induces on $C_c^\infty(M)$ the bilinear pairing

$$\langle v, u\rangle_{\mathbf{g}} := T_v(u) = \int_M uv\, d\lambda_{\mathbf{g}}, \qquad u, v \in C_c^\infty(M). \tag{13.27}$$

For every $r \in \mathbb{R}$, this pairing has a unique continuous bilinear extension

$$\langle \cdot, \cdot \rangle_{r,p;\mathbf{g}} \colon H^{-r,p'}(M) \times H^{r,p}(M) \longrightarrow \mathbb{K},$$

and the operator

$$\mathcal{J}_{r,p;\mathbf{g}} \colon H^{-r,p'}(M) \longrightarrow \big(H^{r,p}(M)\big)', \qquad \mathcal{J}_{r,p;\mathbf{g}}(v)(u) := \langle v, u\rangle_{r,p;\mathbf{g}},$$

is a topological isomorphism. In particular,

$$\big(H^{r,p}(M)\big)' \cong H^{-r,p'}(M) \tag{13.28}$$

and there exist constants $c_D, C_D > 0$, depending on r, p and the uniform localization data, such that

$$c_D \|v\|_{H^{-r,p'}(M)} \le \|\mathcal{J}_{r,p;\mathbf{g}} v\|_{(H^{r,p}(M))'} \le C_D \|v\|_{H^{-r,p'}(M)}.$$

When $\mathbb{K} = \mathbb{C}$, the pairing (13.27) involves no conjugation: it is the distributional duality induced by the Riemann–Lebesgue measure, not the Hermitian inner product of $L^2(M)$.

Proof. Start with a geodesic trivialization and apply Lemma 13.1.19. This gives an admissible quadratic system

$$\mathcal{Q} = (U_i, \varphi_i, h_i)_{i\in I}, \qquad \sum_{i\in I} h_i^2 = 1,$$

and auxiliary cutoff functions $(\chi_i)_{i\in I}$ such that

$$\chi_i = 1 \quad \text{on a neighborhood of } \operatorname{supp}(h_i), \qquad \operatorname{supp}(\chi_i) \Subset U_i.$$

Write

$$\Omega_i := \varphi_i(U_i), \qquad a_i(x) := \sqrt{\det\big(\mathbf{g}^{(i)}(\varphi_i^{-1}(x))\big)}, \qquad x \in \Omega_i.$$

The function $\chi_i \circ \varphi_i^{-1}$ has compact support in Ω_i. Denote by

$$\widetilde{\chi_i} := \widetilde{\chi_i \circ \varphi_i^{-1}} \in C_c^\infty(\mathbb{R}^n)$$

its extension by zero from Ω_i to $\mathbb{R}^n$.

Before defining synthesis, we specify how to reconstruct a distribution from a chart. If $z \in \mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{supp}(z) \Subset \Omega_i$, define $\mathcal{E}_i z \in \mathcal{D}'(M)$ by

$$\langle \mathcal{E}_i z, \psi\rangle := \Big\langle z, a_i\big((\psi \restriction_{U_i}) \circ \varphi_i^{-1}\big)\Big\rangle_{\mathbb{R}^n}, \qquad \psi \in C_c^\infty(M). \tag{13.29}$$

Although the function in the second argument need not have compact support in Ω_i, only its restriction to a neighborhood of $\operatorname{supp}(z)$ is used. The expression is well defined by the extension of the action of a compactly supported distribution established in Proposition 10.2.32. The definition of coordinate components

of a distribution in Proposition 10.2.34 implies

$$\big((\mathcal{E}_i z)\restriction_{U_i}\big)\circ\varphi_i^{-1} = z\restriction_{\Omega_i} \quad \text{on } \mathcal{D}'(\Omega_i). \tag{13.30}$$

Indeed, for $\varphi\in C_c^\infty(\Omega_i)$,

$$\Big\langle\big((\mathcal{E}_i z)\restriction_{U_i}\big)\circ\varphi_i^{-1},\varphi\Big\rangle = \Big\langle \mathcal{E}_i z, \Big(\frac{\varphi}{a_i}\circ\varphi_i\Big)\Big\rangle = \langle z,\varphi\rangle.$$

If z is a locally integrable function with compact support in Ω_i, then $\mathcal{E}_i z$ is the regular distribution associated with the function $z\circ\varphi_i$ on U_i, extended by zero outside U_i. This construction avoids introducing notation for transporting distributions before specifying its meaning.

Define the analysis operator by

$$\mathcal{S}u := \Big(\widetilde{(h_i u)\circ\varphi_i^{-1}}\Big)_{i\in I}$$

and, for $v=(v_i)_{i\in I}$, the synthesis operator by

$$\mathcal{R}v := \sum_{i\in I} h_i\,\mathcal{E}_i(\widetilde{\chi}_i v_i). \tag{13.31}$$

The product $\widetilde{\chi}_i v_i$ has support contained in $\operatorname{supp}(\widetilde{\chi}_i)\Subset\Omega_i$, so $\mathcal{E}_i(\widetilde{\chi}_i v_i)$ is defined by (13.29). The sum in (13.31) is locally finite, since its ith summand has support contained in $\operatorname{supp}(h_i)$ and the family $(U_i)_{i\in I}$ is locally finite.

Definition 5.5.3 specifies the sequence spaces we will use. Lemma 13.3.10, together with independence of the trivialization, shows that, for each $r\in\mathbb{R}$,

$$\mathcal{S}\colon H^{r,p}(M)\longrightarrow \ell^p\big(I;H^{r,p}(\mathbb{R}^n)\big)$$

is continuous.

We prove continuity of $\mathcal{R}$. For $j\in I$, set

$$I_j := \{i\in I \mid U_i\cap U_j\neq\varnothing\}.$$

There exists $L\geq 1$ such that $\#I_j\leq L$ for every $j\in I$. If $i\in I_j$, write

$$\Omega_{ji} := \varphi_j(U_j\cap U_i), \qquad \Phi_{ij} := \varphi_i\circ\varphi_j^{-1}\colon \Omega_{ji}\longrightarrow \varphi_i(U_i\cap U_j),$$

and define

$$\eta_{ji} := (h_j h_i\chi_i)\circ\varphi_j^{-1}\in C_c^\infty(\Omega_{ji}).$$

Its support is compact in Ω_{ji} because

$$\operatorname{supp}(h_j h_i\chi_i)\subseteq\operatorname{supp}(h_j)\cap\operatorname{supp}(h_i)\Subset U_j\cap U_i.$$

Let $z\in\mathcal{D}'(\mathbb{R}^n)$. On the overlap, use the restriction of z to $\varphi_i(U_i\cap U_j)$ and define its distributional composition with Φ_{ij} by

$$\big\langle z\circ\Phi_{ij},\varphi\big\rangle_{\Omega_{ji}} := \Big\langle z, \widetilde{(\varphi\circ\Phi_{ij}^{-1})\,\big|\det D\Phi_{ij}^{-1}\big|}\Big\rangle_{\mathbb{R}^n}, \qquad \varphi\in C_c^\infty(\Omega_{ji}). \tag{13.32}$$

Here the tilde indicates extension by zero from $\varphi_i(U_i \cap U_j)$ to $\mathbb{R}^n$; the extension is smooth because the function's support is compactly contained in that open set. For a locally integrable function, this definition agrees with ordinary composition $x \mapsto z(\Phi_{ij}(x))$. Set

$$P_{ji} z := \widetilde{\eta_{ji}(z \circ \Phi_{ij})},$$

where the tilde indicates extension by zero from Ω_{ji} to $\mathbb{R}^n$. Compactness of the support of η_{ji} inside Ω_{ji} guarantees that this extension is a well-defined distribution. The transformation of the volume densities on the overlap is

$$a_j(x) = a_i(\Phi_{ij}(x)) \left| \det D\Phi_{ij}(x) \right|, \qquad x \in \Omega_{ji}. \tag{13.33}$$

Evaluating both sides against a test function on $C_c^\infty(\Omega_{ji})$, definition (13.29) and (13.33) give

$$\widetilde{\left(h_j h_i \, \mathcal{E}_{\rangle}(\widetilde{\ }_{\rangle} \sqsubseteq_{\rangle})\right)} \circ \varphi_{|}^{-} \overset{\wp}{\chi} = P_{ji} v_i. \tag{13.34}$$

The bounds in (B1)–(B3), the multiplier and coordinate-change theorem in Lemma 6.4.3, and the uniform Jacobian bounds obtained in Lemma 13.3.4 provide a constant $C_r > 0$, independent of i and j, such that

$$\|P_{ji} z\|_{H^{r,p}(\mathbb{R}^n)} \le C_r \|z\|_{H^{r,p}(\mathbb{R}^n)}. \tag{13.35}$$

By (13.34),

$$\left\| \widetilde{(h_j \mathcal{R} v) \circ \varphi_j^{-1}} \right\|_{H^{r,p}(\mathbb{R}^n)} \le C_r \sum_{i \in I_j} \|v_i\|_{H^{r,p}(\mathbb{R}^n)}.$$

Raise to the power p and use the fact that each sum contains at most L terms. Summing over $j \in I$ and counting overlaps again gives

$$\begin{aligned} &\sum_{j \in I} \left\| \widetilde{(h_j \mathcal{R} v) \circ \varphi_j^{-1}} \right\|_{H^{r,p}(\mathbb{R}^n)}^p \\ &\quad \le C_r^p L^{p-1} \sum_{j \in I} \sum_{i \in I_j} \|v_i\|_{H^{r,p}(\mathbb{R}^n)}^p \\ &\quad = C_r^p L^{p-1} \sum_{i \in I} \#I_i \, \|v_i\|_{H^{r,p}(\mathbb{R}^n)}^p \\ &\quad \le C_r^p L^p \sum_{i \in I} \|v_i\|_{H^{r,p}(\mathbb{R}^n)}^p. \end{aligned}$$

The two inequalities of Lemma 13.3.10 compare the left-hand side with the norm of $H^{r,p}(M)$ through c_H and C_H. Consequently,

$$\|\mathcal{R} v\|_{H^{r,p}(M)} \le C_r' \|v\|_{\ell^p(I; H^{r,p}(\mathbb{R}^n))}. \tag{13.36}$$

Since $\chi_i = 1$ on a neighborhood of $\operatorname{supp}(h_i)$,

$$\widetilde{\chi}_i \widetilde{(h_i u) \circ \varphi_i^{-1}} = \widetilde{(h_i u) \circ \varphi_i^{-1}}.$$

The identity (13.30), applied to the distribution $h_i u$, and the equality $\sum_{i\in I} h_i^2 = 1$ give

$$\mathcal{R}\mathcal{S}u = \sum_{i\in I} h_i^2 u = u.$$

Thus, $\mathcal{S}$ is a coretraction and $\mathcal{R}$ a retraction, and the same pair works at every real order. These identities also establish completeness before interpolation: the operator $\mathcal{S}\mathcal{R}$ is a continuous projection on the Banach space $\ell^p(I;H^{r,p}(\mathbb{R}^n))$, and its image is exactly $\mathcal{S}(H^{r,p}(M))$. This image is closed, being the kernel of $I-\mathcal{S}\mathcal{R}$. The norm comparisons for $\mathcal{S}$ and its inverse $\mathcal{R}$ identify $H^{r,p}(M)$ with that image; therefore, $H^{r,p}(M)$ is a Banach space.

This pair also proves density of $C_c^\infty(M)$ in $H^{r,p}(M)$. Finitely supported sequences with components in $C_c^\infty(\mathbb{R}^n)$ are dense in $\ell^p(I;H^{r,p}(\mathbb{R}^n))$. If a sequence of this kind approximates $\mathcal{S}u$, its image under $\mathcal{R}$ belongs to $C_c^\infty(M)$ and converges to u in $H^{r,p}(M)$ by (13.36).

Apply complex interpolation. Proposition 5.7.12 and the Euclidean identity of Theorem 6.3.11 give

$$\begin{aligned}[\ell^p(I;H^{s_0,p}(\mathbb{R}^n)),\ell^p(I;H^{s_1,p}(\mathbb{R}^n))]_\theta \\ = \ell^p(I;H^{s,p}(\mathbb{R}^n)).\end{aligned}$$

Interpolate $\mathcal{S}$ and $\mathcal{R}$ at the endpoints and use $\mathcal{R}\mathcal{S}=I$. The endpoint norms of these operators, their interpolated norms, and the localization bounds produce the constants $c_I, C_I > 0$ in the statement; in particular, for every u in the common space of (13.26),

$$c_I\|u\|_{H^{s,p}(M)} \le \|u\|_{[H^{s_0,p}(M),H^{s_1,p}(M)]_\theta} \le C_I\|u\|_{H^{s,p}(M)}.$$

We turn to duality. According to Definition 10.2.10, each $v\in L^1_{\mathrm{loc}}(M)$ determines the regular distribution

$$T_v(u) = \int_M uv\,d\lambda_{\mathbf{g}}, \qquad u\in C_c^\infty(M).$$

Thus, (13.27) is the restriction to smooth functions of the duality between $\mathcal{D}'(M)$ and $\mathcal{D}(M)$ induced by the Riemann–Lebesgue measure.

The metric criterion of Lemma 13.2.7 and the Faà di Bruno formula of Theorem 1.1.19 show that a_i, a_i^{-1}, and all their derivatives are bounded by constants independent of i. If $v\in\mathcal{D}'(M)$ and $u\in C_c^\infty(M)$, then

$$\begin{aligned}v(u) &= v\left(\sum_{i\in I} h_i^2 u\right) \\ &= \sum_{i\in I}\left\langle \widetilde{(h_i v)\circ\varphi_i^{-1}}, a_i\widetilde{((h_i u)\circ\varphi_i^{-1})}\right\rangle_{\mathbb{R}^n}.\end{aligned} \tag{13.37}$$

The sum is finite because u has compact support. The second equality is precisely the definition of the coordinate component of a distribution in Proposition 10.2.34. When v is smooth, apply the change-of-variables formula in each chart:

$$\int_{U_i}\eta\,d\lambda_{\mathbf{g}} = \int_{\Omega_i}(\eta\circ\varphi_i^{-1})(x)a_i(x)\,d\lambda_n(x).$$

The preceding identity then becomes

$$\int_M uv\,d\lambda_{\mathbf{g}} = \sum_{i\in I}\int_{\Omega_i} \big((h_i u)\circ\varphi_i^{-1}\big)\big((h_i v)\circ\varphi_i^{-1}\big)a_i\,dx. \tag{13.38}$$

Let $v \in H^{-r,p'}(M)$ and $u \in C_c^\infty(M)$. Euclidean duality from Proposition 6.1.3, uniform continuity of multiplication by a_i, and Hölder's inequality for sequences, applied to (13.37), give

$$\begin{aligned}|v(u)| &\le C\left(\sum_{i\in I}\left\|\widetilde{(h_i u)\circ\varphi_i^{-1}}\right\|^p_{H^{r,p}(\mathbb{R}^n)}\right)^{\frac{1}{p}}\\ &\quad\cdot\left(\sum_{i\in I}\left\|\widetilde{(h_i v)\circ\varphi_i^{-1}}\right\|^{p'}_{H^{-r,p'}(\mathbb{R}^n)}\right)^{\frac{1}{p'}}\\ &\le C'\|u\|_{H^{r,p}(M)}\|v\|_{H^{-r,p'}(M)}.\end{aligned}$$

By the density already proved, $u \mapsto v(u)$ has a unique continuous extension to $H^{r,p}(M)$. Define $\langle v,u\rangle_{r,p;\mathbf{g}}$ as the value of this extension. Then

$$\langle v,u\rangle_{r,p;\mathbf{g}} = v(u), \qquad v\in H^{-r,p'}(M),\quad u\in C_c^\infty(M), \tag{13.39}$$

and, for two smooth functions, the right-hand side is $\displaystyle\int_M uv\,d\lambda_{\mathbf{g}}$. This explicitly shows how the measure induces the pairing and proves continuity of $\mathcal{J}_{r,p;\mathbf{g}}$.

It remains to prove surjectivity. Let $\Lambda \in (H^{r,p}(M))'$. By (13.36), $\Lambda\circ\mathcal{R}$ is continuous on $\ell^p(I;H^{r,p}(\mathbb{R}^n))$. Euclidean duality and Lemma 5.5.4 provide

$$w=(w_i)_{i\in I}\in \ell^{p'}(I;H^{-r,p'}(\mathbb{R}^n))$$

such that

$$(\Lambda\circ\mathcal{R})(z)=\sum_{i\in I}\langle w_i,z_i\rangle_{\mathbb{R}^n} \tag{13.40}$$

for every $z=(z_i)_{i\in I}$, and

$$\|w\|_{\ell^{p'}(I;H^{-r,p'}(\mathbb{R}^n))}\le C\|\Lambda\|.$$

Since $\operatorname{supp}(\chi_i)\Subset U_i$, the function

$$(\chi_i\circ\varphi_i^{-1})a_i^{-1}$$

has a smooth extension by zero to $\mathbb{R}^n$. Denote it by

$$m_i := \widetilde{(\chi_i\circ\varphi_i^{-1})a_i^{-1}} \in C_c^\infty(\mathbb{R}^n).$$

Its derivatives are uniformly bounded. Define

$$v := \sum_{i\in I} h_i\,\mathcal{E}_i(m_i w_i). \tag{13.41}$$

The proof of (13.36), with $-r, p'$ in place of r, p and with the multipliers m_i, shows that $v \in H^{-r,p'}(M)$ and

$$\|v\|_{H^{-r,p'}(M)} \le C\|w\|_{\ell^{p'}(I;H^{-r,p'}(\mathbb{R}^n))} \le C'\|\Lambda\|. \tag{13.42}$$

If $u \in C_c^\infty(M)$, only finitely many terms of (13.41) act on u. The definition of $\mathcal{E}_i$ and the rule for multiplying a distribution by a smooth function give

$$\begin{aligned}
\langle v, u\rangle_{r,p;\mathbf{g}} &= v(u) \\
&= \sum_{i\in I} \left\langle m_i w_i, a_i \widetilde{\left((h_i u)\circ \varphi_i^{-1}\right)} \right\rangle_{\mathbb{R}^n} \\
&= \sum_{i\in I} \left\langle w_i, \widetilde{(\chi_i h_i u)\circ \varphi_i^{-1}} \right\rangle_{\mathbb{R}^n} \\
&= \sum_{i\in I} \left\langle w_i, \widetilde{(h_i u)\circ \varphi_i^{-1}} \right\rangle_{\mathbb{R}^n} \\
&= (\Lambda\circ\mathcal{R})(\mathcal{S}u) = \Lambda(u).
\end{aligned}$$

The third equality uses $\chi_i = 1$ near $\operatorname{supp}(h_i)$, and the penultimate equality uses (13.40) and $\mathcal{R}\mathcal{S} = I$. Density of $C_c^\infty(M)$ extends the equality to all of $H^{r,p}(M)$, so $\mathcal{J}_{r,p;\mathbf{g}}$ is surjective. If $\mathcal{J}_{r,p;\mathbf{g}}(v) = 0$, then v vanishes on $C_c^\infty(M)$ and is the zero distribution. Thus, it is also injective. The direct estimate provides C_D, and (13.42) provides c_D in (13.28). □

Corollary 13.3.12. (Functional connection with Amann's localization) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, let $\mathfrak{K}$ be the atlas constructed in Theorem 13.2.4 or an equivalent uniformly regular atlas, and let $\mathcal{Q}_{\mathfrak{K}} = (U_i, \varphi_i, h_i)_{i\in I}$ be the quadratic system of Proposition 13.2.6. For $s \in \mathbb{R}$ and $1 < p < \infty$, the analysis and synthesis operators defined by

$$\mathcal{S}_{\mathfrak{K}} u := \left(\widetilde{(h_i u)\circ\varphi_i^{-1}}\right)_{i\in I}, \qquad \mathcal{R}_{\mathfrak{K}} v := \sum_{i\in I} h_i\, \mathcal{E}_i(\widetilde{\chi}_i v_i)$$

are continuous, satisfy $\mathcal{R}_{\mathfrak{K}}\mathcal{S}_{\mathfrak{K}} = \mathrm{id}_{H^{s,p}(M)}$, and there exist constants $c_{s,p}, C_{s,p}, D_{s,p} > 0$ such that

$$c_{s,p}\|u\|_{H^{s,p}(M)} \le \|\mathcal{S}_{\mathfrak{K}} u\|_{\ell^p(I;H^{s,p}(\mathbb{R}^m))} \le C_{s,p}\|u\|_{H^{s,p}(M)} \tag{13.43}$$

and

$$\|\mathcal{R}_{\mathfrak{K}} v\|_{H^{s,p}(M)} \le D_{s,p}\|v\|_{\ell^p(I;H^{s,p}(\mathbb{R}^m))}.$$

The constants depend on m, s, p, the multiplicity, and bounds on coordinate transitions and cutoffs through the order required by the local multipliers; they do not depend on i, u, or v.

For $1 \le q \le \infty$, the localized norm with factors h_i and the norm of $F^s_{p,q}(M)$ defined using the ordinary partition $\rho_i = h_i^2$ also satisfy both inequalities of Lemma 13.3.10, with constants $c_F, C_F > 0$ independent of u; when $q = p$, the same assertion holds for $B^s_{p,p}(M)$.

Proof. Proposition 13.2.6 proves that $\mathcal{Q}_{\mathfrak{K}}$ is admissible. Lemma 13.3.10 provides the two inequalities in (13.43) and the Triebel–Lizorkin and Besov comparisons. The distributional construction of $\mathcal{E}_i$, local finiteness of the sum, the synthesis bound, and the identity $\mathcal{R}\mathcal{S} = I$ were proved in Proposition 13.3.11; its proof

uses only admissibility of the quadratic system and therefore applies unchanged to $\mathcal{Q}_{\tilde{\mathfrak{K}}}$. The stated dependencies are those of the multipliers, chart transitions, and the sum of at most L terms appearing in that proof. □

Amann's uniform regularity thus describes the same norms through normalized charts. Comparisons with the quadratic system allow Euclidean estimates to be used with constants independent of the chart. To do the same for sections, we must also control the bundle connection and the transition matrices of its frames.

Definition 13.3.13. (Bounded geometry of a vector bundle) Let $(M,\mathbf{g})$ be a Riemannian manifold of bounded geometry without boundary, and let $\mathbf{E}\to M$ be a smooth vector bundle of finite rank equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$, Hermitian when the bundle is complex, and a compatible connection $\nabla^{\mathbf{E}}$. Denote the curvature of $\nabla^{\mathbf{E}}$ by

$$\mathbf{R}^{\mathbf{E}}\in\Gamma\big(\Lambda^2T^*M\otimes\operatorname{End}(\mathbf{E})\big)$$

. We say that $(\mathbf{E},\mathbf{h}_{\mathbf{E}},\nabla^{\mathbf{E}})$ is a *bundle of bounded geometry* if, for each $k\in\mathbb{N}_0$, there exists $C_k^{\mathbf{E}}>0$ such that

$$\sup_{x\in M}\Big|(\nabla^{\Lambda^2T^*M\otimes\operatorname{End}(\mathbf{E})})^k\mathbf{R}^{\mathbf{E}}(x)\Big|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\le C_k^{\mathbf{E}}. \tag{13.44}$$

Here we use the product connection induced by the Levi–Civita connection and $\nabla^{\mathbf{E}}$.

Proposition 13.3.14. (Covariant derivatives and localization at integer orders) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, let $\mathbf{E}\to M$ be a bundle of bounded geometry of rank $r_{\mathbf{E}}$, and let

$$\mathcal{T}^{\mathrm{geo}}=(U_i,\varphi_i,h_i)_{i\in I}$$

be a geodesic trivialization. On U_i, use the synchronous frame $\mathbf{e}_i=(\mathbf{e}_{i,1},\dots,\mathbf{e}_{i,r_{\mathbf{E}}})$ of Proposition 8.2.26 and write

$$\mathbf{u}=\sum_{a=1}^{r_{\mathbf{E}}}u_i^a\mathbf{e}_{i,a}.$$

Set $W^{0,p}(M,\mathbf{E}):=L^p(M,\mathbf{E})$. If $m\in\mathbb{N}_0$ and $1\le p<\infty$, there exist constants $c_{m,p},C_{m,p}>0$ such that, for every $\mathbf{u}\in W^{m,p}(M,\mathbf{E})$,

$$\begin{aligned}c_{m,p}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}&\le\left(\sum_{i\in I}\sum_{a=1}^{r_{\mathbf{E}}}\Big\|\widetilde{(h_iu_i^a)\circ\varphi_i^{-1}}\Big\|_{W^{m,p}(\mathbb{R}^n)}^p\right)^{\frac1p}\\&\le C_{m,p}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.\end{aligned} \tag{13.45}$$

If also $1<p<\infty$, then for the trivial rank-one bundle we obtain

$$W^{m,p}(M)=H^{m,p}(M) \tag{13.46}$$

and there exist constants $c_{m,p}^{WH},C_{m,p}^{WH}>0$, depending on the uniform geometric data but not on $\mathbf{u}$, such that

$$c_{m,p}^{WH}\|\mathbf{u}\|_{W^{m,p}(M)}\le\|\mathbf{u}\|_{H^{m,p}(M)}\le C_{m,p}^{WH}\|\mathbf{u}\|_{W^{m,p}(M)}.$$

Proof. In the construction of Theorem 13.1.22, we have $U_i = B_{\mathbf{g}}(p_i, 2r)$. Since $3r < \operatorname{inj}(M)$ and $3r$ remains within the uniform radius of the metric criterion for bounded geometry, the normal chart extends to

$$\widehat{\varphi}_i \colon \widehat{U}_i := B_{\mathbf{g}}(p_i, 3r) \longrightarrow B_{\mathrm{euc}}(0, 3r), \qquad \widehat{\varphi}_i \restriction_{U_i} = \varphi_i.$$

The uniform bounds on the metric, transitions between geodesic charts, and covariant derivatives of $\mathbf{R}^{\mathbf{E}}$ verify precisely the hypotheses of Proposition 8.2.26. In particular, the synchronous frame $\mathbf{e}_i$ is defined on a neighborhood of $\overline{U_i}$, and its connection coefficients have bounds independent of i. Moreover, each (U_i, φ_i) is a regular coordinate ball in the sense of Chapter 9. We may therefore apply Lemma 9.1.44 to every section $\mathbf{v}$ with compact support in U_i.

The constants in that lemma depend, through order m, on the components of the metric and its inverse, the volume density, the bundle metric in the chosen frame, and the connection coefficients. In the present normal charts and synchronous frames, all these quantities and their derivatives are uniformly bounded by the metric criterion for bounded geometry and Proposition 8.2.26. Since the synchronous frame is orthonormal,

$$\left| \sum_{a=1}^{r_{\mathbf{E}}} z^a \mathbf{e}_{i,a}(x) \right|_{\mathbf{h}_{\mathbf{E}}(x)} = |z| \qquad (x \in U_i,\ z \in \mathbb{K}^{r_{\mathbf{E}}}),$$

and this identity allows the sum of the component norms in Lemma 9.1.44 to be replaced by their ℓ^p sum. Thus, there exist $c_m^{\mathrm{loc}}, C_m^{\mathrm{loc}} > 0$, independent of i, such that

$$\begin{aligned} c_m^{\mathrm{loc}} \left(\sum_{a=1}^{r_{\mathbf{E}}} \left\| \widetilde{v_i^a \circ \varphi_i^{-1}} \right\|_{W^{m,p}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} &\le \left(\sum_{q=0}^{m} \|\nabla^q \mathbf{v}\|_{L^p(U_i, T^{(0,q)}(TM)|_{U_i} \otimes \mathbf{E}|_{U_i})}^p \right)^{\frac{1}{p}} \\ &\le C_m^{\mathrm{loc}} \left(\sum_{a=1}^{r_{\mathbf{E}}} \left\| \widetilde{v_i^a \circ \varphi_i^{-1}} \right\|_{W^{m,p}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}. \end{aligned} \tag{13.47}$$

Apply this comparison to $\mathbf{v} = h_i \mathbf{u}$. For $q \in \{0, \dots, m\}$, the iterated Leibniz rule for the product connection is a finite sum, with coefficients depending only on q, of terms obtained by contracting

$$\nabla^{q-\ell} h_i \otimes \nabla^{\ell} \mathbf{u}, \qquad \ell \in \{0, \dots, q\}.$$

In particular,

$$|\nabla^q (h_i \mathbf{u})|_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}} \le C_q \sum_{\ell=0}^{q} |\nabla^{q-\ell} h_i|_{\mathbf{g}} \, |\nabla^{\ell} \mathbf{u}|_{\mathbf{g}, \mathbf{h}_{\mathbf{E}}}. \tag{13.48}$$

The derivatives of h_i are uniformly bounded. Since each point belongs to at most L supports, raising (13.48) to the power p, summing over $i \in I$, and integrating gives

$$\sum_{i \in I} \sum_{q=0}^{m} \|\nabla^q (h_i \mathbf{u})\|_{L^p(U_i, T^{(0,q)}(TM)|_{U_i} \otimes \mathbf{E}|_{U_i})}^p \le C \sum_{\ell=0}^{m} \|\nabla^{\ell} \mathbf{u}\|_{L^p(M, T^{(0,\ell)}(TM) \otimes \mathbf{E})}^p. \tag{13.49}$$

The first local inequality in (13.47), applied for each $i \in I$, proves the right-hand inequality of (13.45).

For the opposite comparison, use the locally finite identity

$$\mathbf{u} = \sum_{i \in I} h_i \mathbf{u}.$$

For each $q \in \{0, \dots, m\}$, we have, distributionally and pointwise when $\mathbf{u}$ is smooth,

$$\nabla^q \mathbf{u} = \sum_{i \in I} \nabla^q (h_i \mathbf{u}).$$

At each point there are at most L nonzero summands, so

$$|\nabla^q \mathbf{u}|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \le L^{p-1} \sum_{i \in I} |\nabla^q (h_i \mathbf{u})|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Integrate, sum over $q \in \{0, \dots, m\}$, and use the second inequality in (13.47). This gives the left-hand inequality of (13.45).

To extend the comparisons to a section $\mathbf{u} \in W^{m,p}(M, \mathbf{E})$, the Meyers–Serrin Theorem 9.1.48 provides $\mathbf{u}_j \in \Gamma(\mathbf{E}) \cap W^{m,p}(M, \mathbf{E})$ with $\mathbf{u}_j \to \mathbf{u}$ in $W^{m,p}(M, \mathbf{E})$. Denote by $\mathcal{S}\mathbf{v}$ the family of components $\widetilde{(h_i v_i^a) \circ \varphi_i^{-1}}$ appearing in (13.45). The upper bound, applied to $\mathbf{u}_j - \mathbf{u}_k$, gives

$$\|\mathcal{S}\mathbf{u}_j - \mathcal{S}\mathbf{u}_k\|_{\ell^p(I \times \{1,\dots,r_\mathbf{E}\}; W^{m,p}(\mathbb{R}^n))} \le C_{m,p} \|\mathbf{u}_j - \mathbf{u}_k\|_{W^{m,p}(M,\mathbf{E})}.$$

Completeness of this sequence space gives a limit V. For each fixed i, a, multiplication by h_i and local equivalence of norms give convergence to $\widetilde{(h_i u_i^a) \circ \varphi_i^{-1}}$ in $W^{m,p}(\mathbb{R}^n)$. Since projection onto a component is continuous, $V = \mathcal{S}\mathbf{u}$. Consequently, $\mathcal{S}\mathbf{u}_j \to \mathcal{S}\mathbf{u}$ in the norm of the entire family, not merely componentwise. Continuity of the norms allows limits to be taken in both inequalities applied to $\mathbf{u}_j$ and proves (13.45) for $\mathbf{u}$. The argument includes $p = 1$: the factor L^{p-1} is then one, and the sequence space remains complete. For the trivial rank-one bundle and $1 < p < \infty$, Theorem 6.3.10 uniformly identifies the local norms $W^{m,p}(\mathbb{R}^n)$ and $H^{m,p}(\mathbb{R}^n)$ using its Euclidean constants. Combining them with $c_{m,p}$ and $C_{m,p}$ from (13.45) gives precisely $c^{WH}_{m,p}$ and $C^{WH}_{m,p}$ on $C^\infty_c(M)$. To justify compactly supported approximation as well, enumerate the partition and set $\chi_J := \sum_{i=1}^{J} h_i$. Then $0 \le \chi_J \le 1$, each χ_J has compact support, and, on every fixed compact set, $\chi_J = 1$ for sufficiently large J. Uniform local finiteness and the bounds on $\nabla^q h_i$ give $\sup_{J \in \mathbb{N}} \|\nabla^q \chi_J\|_{L^\infty(M, T^{(0,q)}(TM))} < \infty$ for each q. For $0 \le q \le m$, the Leibniz rule gives a constant independent of J with

$$|\nabla^q_w((1 - \chi_J)\mathbf{u})|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \le C_q \sum_{\ell=0}^{q} |\nabla^{q-\ell}(1 - \chi_J)|_\mathbf{g} |\nabla^\ell_w \mathbf{u}|_{\mathbf{g},\mathbf{h}_\mathbf{E}}.$$

Every factor of the first type eventually vanishes on each fixed compact set and is uniformly bounded. The pth power of the right-hand side is dominated by $C \sum_{\ell=0}^{m} |\nabla^\ell_w \mathbf{u}|^p_{\mathbf{g},\mathbf{h}_\mathbf{E}} \in L^1(M)$. Theorem B.6.14, applied at each order and to their finite sum, proves $\chi_J \mathbf{u} \to \mathbf{u}$ in $W^{m,p}(M, \mathbf{E})$. To retain compact support in the smooth approximation, fix $\psi_J \in C^\infty_c(M)$ equal to one near $\operatorname{supp} \chi_J$. If $\mathbf{v}_{J,k} \to \chi_J \mathbf{u}$ in $W^{m,p}(M, \mathbf{E})$ is the Meyers–Serrin approximation, then

$$\|\psi_J \mathbf{v}_{J,k} - \chi_J \mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le C_J \|\mathbf{v}_{J,k} - \chi_J \mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

Choose $k(J)$ so that this last expression is less than 2^{-J}. The sections $\psi_J \mathbf{v}_{J,k(J)} \in \Gamma_c(\mathbf{E})$ converge to $\mathbf{u}$ in $W^{m,p}(M,\mathbf{E})$, proving the asserted density. Density of $C_c^\infty(M)$ in $H^{m,p}(M)$, proved in Proposition 13.3.11, and the preceding approximation show that both spaces are completions of the same subspace with respect to these two explicit comparisons. The identification preserves their realizations in $\mathcal{D}'(M)$ and proves (13.46). □

Corollary 13.3.15. (Uniform control of the Bochner commutator) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. For $\ell \in \mathbb{N}_0$, let

$$\mathbf{E}_\ell := T^{(0,\ell)}(TM) \otimes \mathbf{E}$$

carry the induced metric and product connection. Write

$$\nabla_\ell := \nabla^{\mathbf{E}_\ell} : \Gamma(\mathbf{E}_\ell) \longrightarrow \Gamma(\mathbf{E}_{\ell+1}), \qquad \nabla^m := \nabla_{\ell+m-1} \circ \cdots \circ \nabla_\ell,$$

with the new index in the first covariant factor, and set $\Delta_{B,\ell} = (\nabla_\ell)_h^* \nabla_\ell = -\operatorname{tr}_{\mathbf{g}}(\nabla_\ell^2)$, where the trace contracts the first two differentiation factors, for the positive-sign Bochner Laplacian. If $m \in \mathbb{N}$, then, for every $\mathbf{u} \in \Gamma_c(\mathbf{E}_\ell)$,

$$\Delta_{B,\ell+m}(\nabla^m \mathbf{u}) - \nabla^m(\Delta_{B,\ell}\mathbf{u}) = \sum_{j=0}^{m} \nabla^j \operatorname{Rm} * \nabla^{m-j}\mathbf{u}. \tag{13.50}$$

Here Rm represents $\mathbf{R}^M$ or $\mathbf{R}^{\mathbf{E}}$ according to the term, and $*$ absorbs the tensor contractions and their universal coefficients. Moreover, there exists $C_{\ell,m} > 0$ such that

$$\left|\Delta_{B,\ell+m}(\nabla^m \mathbf{u}) - \nabla^m(\Delta_{B,\ell}\mathbf{u})\right|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}} \le C_{\ell,m} \sum_{q=0}^{m} |\nabla^q \mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}. \tag{13.51}$$

Proof. Apply Lemma 9.1.41 to the section $\mathbf{u}$ of the bundle $\mathbf{E}_\ell$. That result is formulated for the positive Bochner Laplacians of the induced tensor bundles and gives (13.50) directly.
The curvature of $\mathbf{E}_\ell$ is the sum of the action of $\mathbf{R}^{\mathbf{E}}$ on the bundle factor and the action of $\mathbf{R}^M$ on each of the ℓ covariant factors. Bounded geometry of M and $\mathbf{E}$ bounds all covariant derivatives of these curvatures. Every term on the right-hand side of (13.50) is therefore bounded by a uniform multiple of $|\nabla^{m-j}\mathbf{u}|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}$. The finite sum gives (13.51). □

Proposition 13.3.16. (Perturbed chain of Bochner Laplacians) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, let $\mathbf{E} \to M$ be a bundle of bounded geometry, and let $N \in \mathbb{N}$. For $\ell \in \{0,\dots,N\}$, write

$$\mathbf{E}_\ell := T^{(0,\ell)}(TM) \otimes \mathbf{E}, \qquad \nabla_\ell := \nabla^{\mathbf{E}_\ell} : \Gamma(\mathbf{E}_\ell) \longrightarrow \Gamma(\mathbf{E}_{\ell+1}), \qquad L_{\ell,0} := (\nabla_\ell)_h^* \nabla_\ell = -\operatorname{tr}_{\mathbf{g}}(\nabla_\ell^2) \quad \text{on } \Gamma_c(\mathbf{E}_\ell),$$

, and let $L_{\ell,p}$ be the positive generator of the covariant semigroup on $L^p(M,\mathbf{E}_\ell)$. There exist self-adjoint sections

$$\mathbf{V}_\ell \in \Gamma\big(\operatorname{End}(\mathbf{E}_\ell)\big), \qquad \mathbf{V}_0 = 0,$$

and sections

$$\mathbf{W}_\ell \in \Gamma\big(\operatorname{Hom}(\mathbf{E}_\ell, \mathbf{E}_{\ell+1})\big), \qquad 0 \le \ell < N,$$

satisfying

$$\nabla_\ell (L_{\ell,0} + \mathbf{V}_\ell)\mathbf{u} - (L_{\ell+1,0} + \mathbf{V}_{\ell+1})\nabla_\ell \mathbf{u} = \mathbf{W}_\ell \mathbf{u}, \qquad \mathbf{u} \in \Gamma_c(\mathbf{E}_\ell). \tag{13.52}$$

For each $a \in \mathbb{N}_0$, there exists a constant $C_{N,a} > 0$ such that

$$\sup_{\ell\in\{0,\dots,N\}} \sup_{x\in M} |\nabla^a \mathbf{V}_\ell(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}} + \max_{\ell\in\{0,\dots,N-1\}} \sup_{x\in M} |\nabla^a \mathbf{W}_\ell(x)|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \le C_{N,a}. \tag{13.53}$$

There exists $\mu_N > 0$ such that, for $1 \le p < \infty$ and $\ell \in \{0,\dots,N\}$, the operator

$$B_{\ell,p} := \mu_N I + L_{\ell,p} + \mathbf{V}_\ell, \qquad \mathcal{D}(B_{\ell,p}) = \mathcal{D}(L_{\ell,p}), \tag{13.54}$$

satisfies

$$\|e^{-tB_{\ell,p}}\|_{\mathcal{L}(L^p(M,\mathbf{E}_\ell))} \le e^{-t}, \qquad t \ge 0.$$

In particular, $-(B_{\ell,p} - I)$ generates a contraction C_0-semigroup.

Proof. The bundle $\mathbf{E}_\ell$ is equipped with the product metric and connection. Its curvature is the sum of the action of $\mathbf{R}^\mathbf{E}$ on the factor $\mathbf{E}$ and the action of $\mathbf{R}^M$ on each of the ℓ covariant factors. Therefore, $\mathbf{E}_\ell$ has bounded geometry.

For a vector bundle $\mathbf{F} \to M$ equipped with a bundle metric $\mathbf{h}_\mathbf{F}$ and a compatible connection $\nabla^\mathbf{F}$, define the homomorphisms $\mathcal{V}_\mathbf{F} \in \Gamma(\operatorname{End}(T^*M \otimes \mathbf{F}))$ and $\mathcal{W}_\mathbf{F} \in \Gamma(\operatorname{Hom}(\mathbf{F}, T^*M \otimes \mathbf{F}))$ by

$$(\mathcal{V}_\mathbf{F}\mathbf{a})(\mathbf{X}) = \mathbf{a}(\operatorname{Ric}^\sharp \mathbf{X}) + 2\sum_{i=1}^n \mathbf{R}^\mathbf{F}(\mathbf{e}_i, \mathbf{X})\mathbf{a}(\mathbf{e}_i), \tag{13.55}$$

$$(\mathcal{W}_\mathbf{F}\mathbf{v})(\mathbf{X}) = \sum_{i=1}^n (\nabla_{\mathbf{e}_i} \mathbf{R}^\mathbf{F})(\mathbf{e}_i, \mathbf{X})\mathbf{v}. \tag{13.56}$$

Here $\mathbf{X} \in T_xM$ is arbitrary, $\mathbf{a} \in T_x^*M \otimes \mathbf{F}_x$, $\mathbf{v} \in \mathbf{F}_x$, and $(\mathbf{e}_i)_{i=1}^n$ is an orthonormal basis of T_xM. The sums are metric contractions and hence independent of the basis. The exact first-order identity proved in Lemma 9.1.41 states that

$$\nabla^\mathbf{F} \Delta_B^\mathbf{F} \mathbf{v} - \Delta_B^{T^*M\otimes\mathbf{F}}(\nabla^\mathbf{F}\mathbf{v}) = \mathcal{V}_\mathbf{F}(\nabla^\mathbf{F}\mathbf{v}) + \mathcal{W}_\mathbf{F}\mathbf{v}. \tag{13.57}$$

The homomorphism $\mathcal{V}_\mathbf{F}$ is self-adjoint. Indeed, $\operatorname{Ric}^\sharp$ is symmetric and, for the second term, the block taking component i to component j is $2\mathbf{R}^\mathbf{F}(\mathbf{e}_i, \mathbf{e}_j)$. Metric compatibility of the connection and antisymmetry of the curvature give

$$\big(\mathbf{R}^\mathbf{F}(\mathbf{e}_i, \mathbf{e}_j)\big)^* = -\mathbf{R}^\mathbf{F}(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{R}^\mathbf{F}(\mathbf{e}_j, \mathbf{e}_i),$$

which is exactly the relation between the adjoints of transposed blocks. If $\mathbf{F}$ has bounded geometry, all covariant derivatives of $\mathcal{V}_\mathbf{F}$ and $\mathcal{W}_\mathbf{F}$ are uniformly bounded, since both are obtained by contractions of the curvature of $\mathbf{F}$, of $\mathbf{R}^M$, and of its first covariant derivative.

Define recursively

$$\mathbf{V}_0 := 0, \qquad \mathbf{V}_{\ell+1} := I_{T^*M} \otimes \mathbf{V}_\ell + \mathcal{V}_{\mathbf{E}_\ell}, \qquad \mathbf{W}_\ell := \nabla \mathbf{V}_\ell + \mathcal{W}_{\mathbf{E}_\ell}. \tag{13.58}$$

Here we use the canonical identification $\mathbf{E}_{\ell+1} = T^*M \otimes \mathbf{E}_\ell$. For $\mathbf{u} \in \Gamma_c(\mathbf{E}_\ell)$, the Leibniz rule gives

$$\nabla_\ell(\mathbf{V}_\ell \mathbf{u}) = (I_{T^*M} \otimes \mathbf{V}_\ell)\nabla_\ell \mathbf{u} + (\nabla \mathbf{V}_\ell)\mathbf{u}.$$

Adding this identity to (13.57), with $\mathbf{F} = \mathbf{E}_\ell$, gives

$$\begin{aligned}\nabla_\ell(L_{\ell,0} + \mathbf{V}_\ell)\mathbf{u} - (L_{\ell+1,0} + \mathbf{V}_{\ell+1})\nabla_\ell \mathbf{u} \\ = (\nabla \mathbf{V}_\ell + \mathcal{W}_{\mathbf{E}_\ell})\mathbf{u} = \mathbf{W}_\ell \mathbf{u}.\end{aligned}$$

This is (13.52). Let us verify self-adjointness. The initial case is $\mathbf{V}_0 = 0$. If $\mathbf{V}_\ell$ is self-adjoint, then $I_{T^*M} \otimes \mathbf{V}_\ell$ is self-adjoint for the product metric of $\mathbf{E}_{\ell+1} = T^*M \otimes \mathbf{E}_\ell$; since $\mathcal{V}_{\mathbf{E}_\ell}$ is also self-adjoint, the recurrence (13.58) shows that $\mathbf{V}_{\ell+1}$ is self-adjoint. This completes the induction. To verify the bounds, fix $a \in \mathbb{N}_0$. Denote by $v_{\ell,a}$ a bound for $\|\nabla^a \mathbf{V}_\ell\|_\infty$, and by $A_{\ell,a}, D_{\ell,a}$ the bounds for $\|\nabla^a \mathcal{V}_{\mathbf{E}_\ell}\|_\infty$ and $\|\nabla^a \mathcal{W}_{\mathbf{E}_\ell}\|_\infty$ already obtained from the curvatures. The connection preserves the identity tensor, so (13.58) implies

$$v_{0,a} = 0, \qquad v_{\ell+1,a} \le C_{n,a}\big(v_{\ell,a} + A_{\ell,a}\big), \qquad \ell \in \{0, \dots, N-1\}.$$

This recurrence starts at zero and has exactly N steps. In particular, each $v_{\ell,a}$ is bounded by a finite sum of the $A_{j,a}$ with $j < \ell$, multiplied by powers of $C_{n,a}$. Moreover,

$$\|\nabla^a \mathbf{W}_\ell\|_\infty \le C_{n,a}\big(v_{\ell,a+1} + D_{\ell,a}\big).$$

Bounds at order $a+1$ follow from the same recurrence, independently of those at order a. Taking the maximum over the finitely many indices in the chain gives (13.53). This specifies which derivatives of the potentials enter each constant.

Theorem 12.3.2, applied to $\mathbf{E}_\ell$, states that $-L_{\ell,p}$ generates a contraction semigroup. Multiplication by $\mathbf{V}_\ell$ is a bounded operator on $L^p(M, \mathbf{E}_\ell)$, and its norm is bounded by the supremum norm of its values as operators on the fibers. If $\mathbf{h}_{\mathbf{E}_\ell}$ denotes the product metric induced by $\mathbf{g}$ and $\mathbf{h}_{\mathbf{E}}$, set

$$\|\mathbf{V}_\ell\|_{\infty,\mathrm{op}} := \operatorname*{ess\,sup}_{x \in M} \sup_{0 \neq v \in (\mathbf{E}_\ell)_x} \frac{|\mathbf{V}_\ell(x)v|_{\mathbf{h}_{\mathbf{E}_\ell}}}{|v|_{\mathbf{h}_{\mathbf{E}_\ell}}}.$$

Thus, the norm of the multiplication operator does not exceed

$$\|\mathbf{V}_\ell\|_{\infty,\mathrm{op}}.$$

Choose

$$\mu_N \ge 1 + \max_{\ell \in \{0,\dots,N\}} \|\mathbf{V}_\ell\|_{\infty,\mathrm{op}}.$$

Proposition B.4.17, applied to the generator $-L_{\ell,p}$ and perturbation $-\mathbf{V}_\ell$, gives

$$\|e^{-tB_{\ell,p}}\|_{\mathcal{L}(L^p(M,\mathbf{E}_\ell))} \le \exp\big(-t\big(\mu_N - \|\mathbf{V}_\ell\|_{\infty,\mathrm{op}}\big)\big) \le e^{-t}.$$

Finally,

$$e^{-t(B_{\ell,p}-I)} = e^t e^{-tB_{\ell,p}},$$

so $\|e^{-t(B_{\ell,p}-I)}\|_{\mathcal{L}(L^p(M,\mathbf{E}_\ell))} \le 1$. This is the contraction semigroup generated by $-(B_{\ell,p}-I)$. □

The preceding differential identity will also be used for sections without compact support. For this, we need to identify the operator's L^2 domain, not only its local expression.

Lemma 13.3.17. (Maximal domain of the perturbed Laplacian) Let $(\mathbf{F},\mathbf{h}_\mathbf{F},\nabla^\mathbf{F}) \to M$ be a bundle of bounded geometry over a manifold without boundary and with bounded geometry. Let $\mathbf{V}$ be a smooth, bounded, self-adjoint endomorphism, and let $B = L_{\mathbf{F},2}+\mathbf{V}+\mu I$ be the realization through the energy form. If $\mathbf{w},\mathbf{f} \in L^2(M,\mathbf{F})$ and $B\mathbf{w} = \mathbf{f}$ distributionally, then $\mathbf{w} \in \mathcal{D}(B)$ and the equality holds in L^2.

Proof. Local elliptic regularity gives $\mathbf{w} \in H^2_{\mathrm{loc}}$. Take the cutoffs χ_J constructed in the proof of Proposition 13.3.14. They have compact support, satisfy $0 \le \chi_J \le 1$, converge locally to one, and $|d\chi_J| \le C$ with a constant independent of J. Test the equation against $\chi_J^2\mathbf{w}$, using a smooth approximation on its support. The real part of integration by parts gives

$$\begin{aligned}\|\chi_J\nabla\mathbf{w}\|^2_{L^2(M,T^*M\otimes\mathbf{F})} &\le \|\mathbf{f}\|_{L^2(M,\mathbf{F})}\|\mathbf{w}\|_{L^2(M,\mathbf{F})}\\ &\quad + (\|\mathbf{V}\|_\infty + |\mu|)\|\mathbf{w}\|^2_{L^2(M,\mathbf{F})}\\ &\quad + 2\|\chi_J\nabla\mathbf{w}\|_{L^2(M,T^*M\otimes\mathbf{F})}\big\||d\chi_J|\mathbf{w}\big\|_{L^2(M,\mathbf{F})}.\end{aligned}$$

Young's inequality allows half of the energy term to be absorbed and gives a bound independent of J. Fatou implies $\nabla\mathbf{w} \in L^2$. Compactly supported density from the cited proposition then shows that $\mathbf{w}$ belongs to the form domain. For every test section $\boldsymbol{\varphi}$,

$$\int_M \langle\nabla\mathbf{w},\nabla\boldsymbol{\varphi}\rangle\, d\lambda_\mathbf{g} + \int_M \langle(\mathbf{V}+\mu I)\mathbf{w},\boldsymbol{\varphi}\rangle\, d\lambda_\mathbf{g} = \int_M \langle\mathbf{f},\boldsymbol{\varphi}\rangle\, d\lambda_\mathbf{g}.$$

All three terms are continuous for the form norm; density extends the equality to the entire form domain. The definition of the operator associated with a closed form gives $\mathbf{w} \in \mathcal{D}(B)$ and $B\mathbf{w} = \mathbf{f}$. □

The identification of fractional domains needed below will be obtained by interpolation. Before using functional calculus, it is useful to record interpolation of the covariant spaces at the two integer orders entering the square root.

Proposition 13.3.18. (Interpolation between orders zero and two for bundles) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let $\mathbf{F} \to M$ be a finite-rank bundle of bounded geometry. If $1 < p < \infty$, then

$$[L^p(M,\mathbf{F}), W^{2,p}(M,\mathbf{F})]_{1/2} = W^{1,p}(M,\mathbf{F}) \tag{13.59}$$

with equivalence of norms.

Proof. If $\mathbf{F}$ is real, perform interpolation in the complexified bundle and restrict to the real subspace at the end; the connection, metric, and localization operators complexify without changing their norms on real sections. Use the same geodesic trivialization $\mathcal{T}^{\mathrm{geo}} = (U_i,\varphi_i,h_i)_{i\in I}$ and synchronous frames $\mathbf{e}_i = (\mathbf{e}_{i,1},\dots,\mathbf{e}_{i,r})$ as in Proposition 13.3.14. As in the definition of an admissible trivialization, choose functions $\chi_i \in C^\infty_c(U_i)$

such that

$$\chi_i = 1 \quad \text{on a neighborhood of } \operatorname{supp}(h_i),$$

and whose derivatives in coordinates φ_i are uniformly bounded in i. Denote their extensions by zero by

$$\widetilde{\chi}_i := \widetilde{\chi_i \circ \varphi_i^{-1}} \in C_c^\infty(\mathbb{R}^n)$$

.

For $k \in \{0,1,2\}$, define the analysis operator

$$\mathcal{S}_{\mathbf{F}}\mathbf{u} := \left(\widetilde{(h_i u_i^a) \circ \varphi_i^{-1}}\right)_{(i,a)\in I\times\{1,\dots,r\}}, \qquad \mathbf{u} \restriction_{U_i} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a}.$$

Proposition 13.3.14 shows that $\mathcal{S}_{\mathbf{F}}$ is continuous from $W^{k,p}(M,\mathbf{F})$ to

$$\ell^p\left(I \times \{1,\dots,r\}; W^{k,p}(\mathbb{R}^n)\right),$$

with a norm depending on k,p and the uniform data, but not on the chart.

Construct a synthesis operator that works simultaneously at all three orders. If $v = (v_i^a)_{i,a}$ is a family with finitely many nonzero components and each $v_i^a \in C_c^\infty(\mathbb{R}^n)$, set

$$\mathcal{R}_{\mathbf{F}} v := \sum_{i\in I}\sum_{a=1}^{r} \left[\left((\widetilde{\chi}_i v_i^a) \restriction_{\varphi_i(U_i)}\right) \circ \varphi_i\right] \mathbf{e}_{i,a}, \tag{13.60}$$

where each summand is extended by zero outside U_i. The product with χ_i has compact support in U_i, so this extension is smooth. For a general family, the formula is first interpreted on each compact set: only uniformly finitely many charts contribute, and the sum is locally finite.

We prove continuity of synthesis. In a chart U_j, the components of the ith summand of (13.60) are obtained from $v_i = (v_i^1,\dots,v_i^r)$ through three operations: multiplication by a cutoff with uniformly bounded derivatives, a coordinate change $\varphi_i \circ \varphi_j^{-1}$, and multiplication by the transition matrix between the synchronous frames $\mathbf{e}_i$ and $\mathbf{e}_j$. The bounded-geometry bounds for synchronous frames and geodesic transitions imply that, for $k \in \{0,1,2\}$, these three operations are bounded on $W^{k,p}$ with a constant independent of i,j. If I_j denotes the set of indices whose charts meet U_j, there exists $L \geq 1$ with $\#I_j \leq L$. By the inequality

$$\left(\sum_{i\in I_j} a_i\right)^p \leq L^{p-1} \sum_{i\in I_j} a_i^p, \qquad a_i \geq 0,$$

the local equivalence of Proposition 13.3.14 gives

$$\|\mathcal{R}_{\mathbf{F}} v\|_{W^{k,p}(M,\mathbf{F})} \leq C_{k,p} \|v\|_{\ell^p(I\times\{1,\dots,r\};W^{k,p}(\mathbb{R}^n))}, \qquad k \in \{0,1,2\}. \tag{13.61}$$

By density, $\mathcal{R}_{\mathbf{F}}$ extends continuously to all three sequence spaces.

If $v = \mathcal{S}_{\mathbf{F}}\mathbf{u}$, the equality $\chi_i = 1$ near $\operatorname{supp}(h_i)$ makes the ith reconstructed term exactly $h_i\mathbf{u}$. Since $(h_i)_{i\in I}$ is a

partition of unity,

$$\mathcal{R}_{\mathbf{F}}\mathcal{S}_{\mathbf{F}}\mathbf{u} = \sum_{i\in I} h_i \mathbf{u} = \mathbf{u}. \tag{13.62}$$

Thus, the same analysis–synthesis pair is a coretraction–retraction at orders $0, 1, 2$.
By Theorem 6.3.14 and interpolation of the Euclidean Bessel scale,

$$[L^p(\mathbb{R}^n), W^{2,p}(\mathbb{R}^n)]_{1/2} = W^{1,p}(\mathbb{R}^n)$$

with equivalent norms. Proposition 5.7.12 allows us to take the outer ℓ^p sum and the finitely many bundle components. Now interpolate $\mathcal{S}_{\mathbf{F}}$ and $\mathcal{R}_{\mathbf{F}}$ between orders 0 and 2 and use (13.62). This gives both inclusions in (13.59), and the endpoint operator norms provide the equivalence constants. □

The second tool is a global elliptic L^p estimate with uniform constants. We prove it directly for the operators in the chain, making explicit the constants that will be used for the square root.

Proposition 13.3.19. (L^p domain of the chain operators) Under the hypotheses of Proposition 13.3.16, let $1 < p < \infty$. For each $\ell \in \{0, \dots, N\}$,

$$\mathcal{D}(B_{\ell,p}) = W^{2,p}(M, \mathbf{E}_\ell) \tag{13.63}$$

and there exist constants $c^{(2)}_{\ell,p}, C^{(2)}_{\ell,p} > 0$ such that

$$c^{(2)}_{\ell,p}\|\mathbf{u}\|_{W^{2,p}(M,\mathbf{E}_\ell)} \le \|B_{\ell,p}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \le C^{(2)}_{\ell,p}\|\mathbf{u}\|_{W^{2,p}(M,\mathbf{E}_\ell)}. \tag{13.64}$$

The constants can be chosen uniformly as ℓ ranges over the finite set $\{0, \dots, N\}$.

Proof. Fix ℓ and denote by

$$\mathcal{B}_\ell := \mu_N I + L_{\ell,0} + \mathbf{V}_\ell$$

the smooth differential expression whose realization is $B_{\ell,p}$. In a uniform geodesic chart and the synchronous frame of $\mathbf{E}_\ell$, this operator has the form

$$\mathcal{B}_i = -g_i^{ab}(x)\partial_a\partial_b + A_i^a(x)\partial_a + C_i(x). \tag{13.65}$$

The matrices A_i^a, C_i and all derivatives we will need are uniformly bounded; the same holds for g_i^{ab} and its derivatives. There are constants $0 < \lambda_0 \le \Lambda_0 < \infty$, independent of i, such that

$$\lambda_0|\xi|^2 \le g_i^{ab}(x)\xi_a\xi_b \le \Lambda_0|\xi|^2. \tag{13.66}$$

These bounds follow from the metric criterion for bounded geometry and the bounds of Proposition 13.3.16.
We begin with a constant-coefficient estimate. For a point x_0 in the chart, set

$$q_{x_0}(\xi) := g_i^{ab}(x_0)\xi_a\xi_b, \qquad P_{x_0} := -g_i^{ab}(x_0)\partial_a\partial_b.$$

If $|\alpha| \le 2$, the symbol

$$m_{\alpha,x_0}(\xi) := \frac{(i\xi)^\alpha}{1 + q_{x_0}(\xi)}$$

satisfies the hypotheses of Theorem 6.3.1 with constants independent of x_0 and i. Indeed, differentiating it r times gives a finite sum of quotients whose denominators are powers of $1 + q_{x_0}$ and whose numerators are polynomials in ξ with coefficients bounded by λ_0, Λ_0; using (13.66) gives

$$|D_\xi^\beta m_{\alpha,x_0}(\xi)| \le C_{\alpha,\beta} |\xi|^{-|\beta|}, \qquad \xi \neq 0,$$

for every multi-index required by Mikhlin. Therefore,

$$\|v\|_{W^{2,p}(\mathbb{R}^n)} \le C_p \|(I + P_{x_0})v\|_{L^p(\mathbb{R}^n)}, \qquad v \in C_c^\infty(\mathbb{R}^n), \tag{13.67}$$

with C_p independent of x_0 and i.

We will also use an elementary intermediate-order inequality. For $0 < \varepsilon \le 1$ and $j \in \{1, \dots, n\}$, the symbol

$$\frac{i\varepsilon\xi_j}{1 + \varepsilon^2|\xi|^2}$$

and its Mikhlin derivatives are uniformly bounded in ε. Applying it to $(I - \varepsilon^2\Delta)v$ and summing over j gives

$$\|Dv\|_{L^p(\mathbb{R}^n)} \le C_p \left(\varepsilon \|D^2 v\|_{L^p(\mathbb{R}^n)} + \varepsilon^{-1} \|v\|_{L^p(\mathbb{R}^n)} \right), \qquad v \in W^{2,p}(\mathbb{R}^n). \tag{13.68}$$

The first derivatives of g_i^{ab} have a uniform bound. Choose $r > 0$ independent of the chart and small enough that, on every Euclidean ball of radius $2r$ contained in a larger chart,

$$\|g_i^{ab} - g_i^{ab}(x_0)\|_{L^\infty(B_{\mathrm{euc}}(x_0, 2r))} \le \delta,$$

where $\delta > 0$ is fixed so that the corresponding term can be absorbed in (13.67). If $v \in C_c^\infty(B_{\mathrm{euc}}(x_0, 2r))$, write

$$(I + P_{x_0})v = \mathcal{B}_i v + (g_i^{ab} - g_i^{ab}(x_0))\partial_a\partial_b v - A_i^a \partial_a v + (I - C_i)v.$$

The constant-coefficient estimate, the choice of δ, and (13.68), with ε sufficiently small and then fixed, give

$$\|v\|_{W^{2,p}(\mathbb{R}^n)} \le C \left(\|\mathcal{B}_i v\|_{L^p(\mathbb{R}^n)} + \|v\|_{L^p(\mathbb{R}^n)} \right), \tag{13.69}$$

with a constant independent of the chart.

The compactly supported estimate implies an interior estimate without assuming a global second-order norm a priori. Let $r \le s < t \le 2r$, and choose $\zeta_{s,t} \in C_c^\infty(B_{\mathrm{euc}}(x_0, t))$ such that $\zeta_{s,t} = 1$ on $B_{\mathrm{euc}}(x_0, s)$ and

$$|D\zeta_{s,t}| \le C(t-s)^{-1}, \qquad |D^2\zeta_{s,t}| \le C(t-s)^{-2}.$$

If u is smooth on a neighborhood of $\overline{B_{\mathrm{euc}}(x_0, 2r)}$, apply (13.69) to $\zeta_{s,t}u$. The Leibniz rule gives

$$\|D^2 u\|_{L^p(B_s)} \le C\|\mathcal{B}_i u\|_{L^p(B_t)} + \frac{C}{t-s}\|Du\|_{L^p(B_t)} + C\left(1 + \frac{1}{(t-s)^2}\right)\|u\|_{L^p(B_t)}. \tag{13.70}$$

Here and in the next two formulas, we abbreviate $B_\rho = B_{\mathrm{euc}}(x_0,\rho)$.

We need to eliminate the first-order term on the right-hand side. The balls B_t, $r \le t \le 2r$, are Lipschitz domains obtained from one another by dilations with factors in the compact interval $[1,2]$. Extend $u\restriction_{B_t}$ by Theorem 3.6.2 and apply (13.68) to the extension. The second-order estimate for the extension operator also contains $\|Du\|_{L^p(B_t)}$; multiply by the small parameter in the interpolation inequality and absorb that term into the left-hand side. Since the radii lie in the fixed interval $[r,2r]$, the extension and absorption constants are uniform. We thus obtain

$$\|Du\|_{L^p(B_t)} \le \eta \|D^2 u\|_{L^p(B_t)} + C\eta^{-1}\|u\|_{L^p(B_t)}, \qquad 0<\eta\le\eta_0. \tag{13.71}$$

Take $\eta = \varepsilon(t-s)$, with $\varepsilon>0$ fixed and sufficiently small. If

$$\Phi(\rho) := \|D^2 u\|_{L^p(B_\rho)},$$

the preceding two estimates give numbers $0<\vartheta<1$ and $C>0$, independent of the chart, such that

$$\Phi(s) \le \vartheta\Phi(t) + C\|\mathcal{B}_i u\|_{L^p(B_{2r})} + \frac{C}{(t-s)^2}\|u\|_{L^p(B_{2r})}, \qquad r\le s<t\le 2r. \tag{13.72}$$

To iterate this inequality, choose $\rho\in(\sqrt{\vartheta},1)$ and set

$$r_k := 2r - r\rho^k, \qquad k\in\mathbb{N}_0.$$

Then $r_0 = r$, $r_k \uparrow 2r$, and $r_{k+1}-r_k = r(1-\rho)\rho^k$. Iterating (13.72) m times gives

$$\begin{aligned}\Phi(r) \le \vartheta^m\Phi(r_m) &+ C\sum_{k=0}^{m-1}\vartheta^k\|\mathcal{B}_i u\|_{L^p(B_{2r})}\\ &+ \frac{C}{r^2(1-\rho)^2}\sum_{k=0}^{m-1}\Big(\frac{\vartheta}{\rho^2}\Big)^k\|u\|_{L^p(B_{2r})}.\end{aligned}$$

Since u is smooth on a neighborhood of $\overline{B_{2r}}$, $\Phi(2r)<\infty$, so $\vartheta^m\Phi(r_m)\to 0$. Both geometric series converge, and, since r is the same in all charts, we arrive at

$$\|u\|_{W^{2,p}(B_r)} \le C\left(\|\mathcal{B}_i u\|_{L^p(B_{2r})} + \|u\|_{L^p(B_{2r})}\right). \tag{13.73}$$

The first-order component is recovered from (13.71); the order-zero terms are already on the right-hand side.

To obtain the global estimate, sum the interior estimates over a cover of bounded multiplicity. Use the geodesic trivialization of Theorem 13.1.22: the balls $V_i := B_{\mathbf{g}}(p_i,r)$ cover M, and the larger balls $U_i := B_{\mathbf{g}}(p_i,2r)$ have uniformly bounded multiplicity. In normal coordinates and synchronous frames, uniform comparison of covariant and Euclidean norms, together with (13.73), gives, for every smooth section $\mathbf{u}$ with $\mathbf{u},\mathcal{B}_\ell\mathbf{u}\in L^p$,

$$\|\mathbf{u}\|^p_{W^{2,p}(M,\mathbf{E}_\ell)} \le C\sum_{i\in I}\|\mathbf{u}\|^p_{W^{2,p}(V_i,\mathbf{E}_\ell)}$$

$$\le C \sum_{i\in I} \left(\|\mathcal{B}_\ell \mathbf{u}\|^p_{L^p(U_i,\mathbf{E}_\ell)} + \|\mathbf{u}\|^p_{L^p(U_i,\mathbf{E}_\ell)} \right)$$
$$\le C \left(\|\mathcal{B}_\ell \mathbf{u}\|^p_{L^p(M,\mathbf{E}_\ell)} + \|\mathbf{u}\|^p_{L^p(M,\mathbf{E}_\ell)} \right).$$

Consequently,

$$\|\mathbf{u}\|_{W^{2,p}(M,\mathbf{E}_\ell)} \le C \left(\|\mathcal{B}_\ell \mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} + \|\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \right). \tag{13.74}$$

All sums have nonnegative terms; the last inequality uses only the uniform multiplicity bound of (U_i). Thus, the argument is valid even when the global second-order norm is not known to be finite beforehand: the interior estimate makes each summand finite, and Tonelli allows them to be summed directly.
The realization $B_{\ell,p}$ has a bounded inverse given by

$$B^{-1}_{\ell,p}\mathbf{f} = \int_0^\infty e^{-tB_{\ell,p}}\mathbf{f}\, dt,$$

and estimate $\|e^{-tB_{\ell,p}}\| \le e^{-t}$ of Proposition 13.3.16 implies

$$\|B^{-1}_{\ell,p}\|_{\mathcal{L}(L^p(M,\mathbf{E}_\ell))} \le 1. \tag{13.75}$$

Now let $\mathbf{f} \in L^p(M,\mathbf{E}_\ell)$. Choose $\mathbf{f}_j \in \Gamma_c(\mathbf{E}_\ell)$ with $\mathbf{f}_j \to \mathbf{f}$ in L^p and set $\mathbf{u}_j = B^{-1}_{\ell,p}\mathbf{f}_j$. Consistency of the semigroups as p varies shows that, since $\mathbf{f}_j \in L^2 \cap L^p$,

$$\mathbf{u}_j = B^{-1}_{\ell,2}\mathbf{f}_j.$$

Local elliptic regularity applied to $\mathcal{B}_\ell \mathbf{u}_j = \mathbf{f}_j$ shows that $\mathbf{u}_j$ is smooth. Applying (13.74) to $\mathbf{u}_j - \mathbf{u}_k$ and using (13.75) gives

$$\|\mathbf{u}_j - \mathbf{u}_k\|_{W^{2,p}(M,\mathbf{E}_\ell)} \le C\|\mathbf{f}_j - \mathbf{f}_k\|_{L^p(M,\mathbf{E}_\ell)}.$$

By completeness, $\mathbf{u}_j$ converges in $W^{2,p}$ to a section $\mathbf{v}$. At the same time, $\mathbf{u}_j \to B^{-1}_{\ell,p}\mathbf{f}$ in L^p, so $\mathbf{v} = B^{-1}_{\ell,p}\mathbf{f}$. Thus, $\mathcal{D}(B_{\ell,p}) \subseteq W^{2,p}$ and

$$\|\mathbf{u}\|_{W^{2,p}(M,\mathbf{E}_\ell)} \le C\|B_{\ell,p}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)}, \qquad \mathbf{u} \in \mathcal{D}(B_{\ell,p}).$$

For the opposite inclusion, let $\mathbf{u} \in W^{2,p}(M,\mathbf{E}_\ell)$. Compactly supported density from Proposition 13.3.14 provides $\mathbf{u}_j \in \Gamma_c(\mathbf{E}_\ell)$ with $\mathbf{u}_j \to \mathbf{u}$ in $W^{2,p}$. The local expression (13.65) shows that $\mathcal{B}_\ell \mathbf{u}_j$ converges in L^p. Since $B_{\ell,p}$ is closed and agrees with $\mathcal{B}_\ell$ on compactly supported sections, it follows that $\mathbf{u} \in \mathcal{D}(B_{\ell,p})$. The evident continuity of the second-order differential expression gives the right-hand inequality in (13.64). This proves (13.63) and equivalence of norms. Since the chain contains only finitely many bundles and all geometric data involved have uniform bounds, the constants may be taken uniformly in ℓ. □

To pass from order two to order one, we need a bound on imaginary powers. We will prove it within the book using a Bellman function and the Poisson semigroup. The algebraic part is separated into the following lemma.

Lemma 13.3.20. (Bellman function for two Hilbert norms) Let $2 \le p < \infty$ and $q = p/(p-1)$. There exist constants $\delta_p, c_p, C_p > 0$ and a function $\mathcal{B}_p \colon H_1 \times H_2 \to [0,\infty)$, defined for any real Hilbert spaces H_1, H_2, with the following properties.

(a) $\mathcal{B}_p$ is of class C^1, is C^2 away from the hypersurface $|\xi|^p = |\eta|^q$ and the axes, and

$$0 \le \mathcal{B}_p(\xi,\eta) \le C_p\left(|\xi|^p + |\eta|^q\right).$$

(b) Its radial derivatives are nonnegative.

(c) At every point of second-order differentiability, there exists $\tau = \tau(\xi,\eta) > 0$ such that, for $a \in H_1$ and $b \in H_2$,

$$d^2\mathcal{B}_p(\xi,\eta)[(a,b),(a,b)] \geq c_p\big(\tau|a|^2 + \tau^{-1}|b|^2\big). \tag{13.76}$$

In particular,

$$d^2\mathcal{B}_p(\xi,\eta)[(a,b),(a,b)] \geq 2c_p|a||b|. \tag{13.77}$$

(d) With the same choice of τ,

$$|d\mathcal{B}_p(\xi,\eta)[(a,b)]|^2 \leq C_p\mathcal{B}_p(\xi,\eta)\big(\tau|a|^2 + \tau^{-1}|b|^2\big). \tag{13.78}$$

The constants depend only on p, not on the dimensions of H_1, H_2.

Proof. For $u, v \geq 0$, set

$$\beta_p(u,v) := u^p + v^q + \delta_p \begin{cases} u^2v^{2-q}, & u^p \leq v^q, \\ \dfrac{2}{p}u^p + \left(\dfrac{2}{q} - 1\right)v^q, & u^p \geq v^q. \end{cases}$$

On the boundary $u^p = v^q$, the two additional expressions agree. Indeed, there $v = u^{p-1}$ and $u^2v^{2-q} = u^p$, while $\frac{2}{p} + \frac{2}{q} - 1 = 1$. The first derivatives also agree:

$$2uv^{2-q} = 2u^{p-1}, \qquad (2-q)u^2v^{1-q} = (2-q)v^{q-1}.$$

Thus, β_p is C^1. Define

$$\mathcal{B}_p(\xi,\eta) := \beta_p(|\xi|,|\eta|).$$

The bound in item (a) follows from $u^2v^{2-q} \leq v^q$ when $u^p \leq v^q$ and from the definition in the other region. The radial derivatives are nonnegative if $0 < \delta_p \leq 1$.
We verify quantitative convexity. First suppose that $u^p \leq v^q$ and set

$$\tau := v^{2-q}.$$

The relation $u \leq v^{q-1}$ implies

$$u^{p-2} \leq v^{2-q}, \qquad u^2v^{-q} \leq v^{q-2}, \qquad uv^{1-q} \leq 1.$$

The second derivatives of the correction u^2v^{2-q} are

$$2v^{2-q}, \qquad 2(2-q)uv^{1-q}, \qquad -(2-q)(q-1)u^2v^{-q}.$$

Consequently,

$$\partial_{uu}\beta_p \geq 2\delta_p\tau,$$

$$\partial_{vv}\beta_p \geq (q-1)\big(q - \delta_p(2-q)\big)\tau^{-1}, \qquad |\partial_{uv}\beta_p| \leq 2\delta_p(2-q).$$

Choose $\delta_p > 0$ small enough that $q - \delta_p(2-q) \geq q/2$ and the mixed term can be absorbed using

$$2|ab| \leq \varepsilon\tau a^2 + \varepsilon^{-1}\tau^{-1}b^2.$$

Thus, for some $c_p > 0$,

$$\partial_{uu}\beta_p\, a^2 + 2\partial_{uv}\beta_p\, ab + \partial_{vv}\beta_p\, b^2 \geq c_p(\tau a^2 + \tau^{-1}b^2). \tag{13.79}$$

Moreover,

$$\frac{\partial_u\beta_p}{u} \geq 2\delta_p\tau, \qquad \frac{\partial_v\beta_p}{v} \geq q\tau^{-1}.$$

If $u^p \geq v^q$, set $\tau := u^{p-2}$. The correction is diagonal, and the relation $v \leq u^{p-1}$, together with $q-2 \leq 0$, gives

$$v^{q-2} \geq u^{2-p} = \tau^{-1}.$$

Hence,

$$\partial_{uu}\beta_p\, a^2 + \partial_{vv}\beta_p\, b^2 \geq c_p(\tau a^2 + \tau^{-1}b^2), \tag{13.80}$$

and the same bounds, with another constant, hold for $(\partial_u\beta_p)/u$ and $(\partial_v\beta_p)/v$.
For a radial function of two Hilbert-space variables, the Hessian decomposes into its radial and orthogonal parts. Suppose $u = |\xi| > 0$ and $v = |\eta| > 0$, and define the signed radial components

$$a_{\mathrm{r}} := \left\langle a, \frac{\xi}{u} \right\rangle_{H_1}, \qquad b_{\mathrm{r}} := \left\langle b, \frac{\eta}{v} \right\rangle_{H_2}.$$

The corresponding orthogonal decompositions are

$$a = a_{\parallel} + a_{\perp}, \qquad a_{\parallel} = a_{\mathrm{r}}\frac{\xi}{u}, \qquad \langle a_{\perp}, \xi\rangle_{H_1} = 0,$$

$$b = b_{\parallel} + b_{\perp}, \qquad b_{\parallel} = b_{\mathrm{r}}\frac{\eta}{v}, \qquad \langle b_{\perp}, \eta\rangle_{H_2} = 0.$$

Differentiating the norm along the line $\xi + ta$ gives

$$\frac{d}{dt}|\xi + ta|\Big|_{t=0} = a_{\mathrm{r}},$$
$$\frac{d^2}{dt^2}|\xi + ta|\Big|_{t=0} = \frac{|a|^2 - a_{\mathrm{r}}^2}{u} = \frac{|a_{\perp}|^2}{u}.$$

The formulas for $\eta + tb$ are analogous. The chain rule then gives, in each region of second-order differentiability,

$$d^2\mathcal{B}_p(\xi,\eta)[(a,b),(a,b)] = (\partial_{uu}\beta_p)a_{\mathrm{r}}^2 + 2(\partial_{uv}\beta_p)a_{\mathrm{r}}b_{\mathrm{r}} + (\partial_{vv}\beta_p)b_{\mathrm{r}}^2 + \frac{\partial_u\beta_p}{u}|a_{\perp}|^2 + \frac{\partial_v\beta_p}{v}|b_{\perp}|^2,$$

where the derivatives of β_p are evaluated at (u,v). Apply (13.79) to the real numbers $a_{\mathrm{r}}, b_{\mathrm{r}}$ in the first region, and (13.80) in the second. Together with the bounds for $(\partial_u\beta_p)/u$ and $(\partial_v\beta_p)/v$, and decreasing c_p if

necessary, this gives

$$d^2\mathcal{B}_p(\xi,\eta)[(a,b),(a,b)] \geq c_p\big(\tau(a_{\mathrm{r}}^2+|a_\perp|^2)+\tau^{-1}(b_{\mathrm{r}}^2+|b_\perp|^2)\big) = c_p\big(\tau|a|^2+\tau^{-1}|b|^2\big).$$

This proves (13.76). The inequality $\tau A^2+\tau^{-1}B^2 \geq 2AB$ gives (13.77).
The gradient bound remains. In the first region, the relations already used give

$$|\partial_u\beta_p| \leq C_p u\tau, \qquad |\partial_v\beta_p| \leq C_p v\tau^{-1},$$

and therefore

$$\frac{|\partial_u\beta_p|^2}{\tau}+|\partial_v\beta_p|^2\tau \leq C_p(u^p+v^q) \leq C_p\beta_p(u,v).$$

In the second region, write $v=u^{p-1}\rho$, with $0\leq\rho\leq 1$. Since $q-1=1/(p-1)$, we have

$$v^{2q-2}u^{p-2} = u^p\rho^{2(q-1)} \leq u^p,$$

and the explicit formulas for $\partial_u\beta_p$ and $\partial_v\beta_p$ give

$$|\partial_u\beta_p|^2\tau^{-1}+|\partial_v\beta_p|^2\tau \leq C_p(u^p+v^q).$$

The Cauchy–Schwarz inequality with weights τ,τ^{-1} proves (13.78). Points on the axes follow by taking limits. When $p=2$, we have $q=2$, and the same function reduces to a piecewise quadratic form whose pieces agree globally; the preceding argument remains valid with $\tau=1$. □

Theorem 13.3.21. (Laplace multipliers and imaginary powers of Bochner–Schrödinger operators) Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary, and let $\mathbf{F}\to M$ be a real or Hermitian vector bundle of finite rank equipped with a bundle metric and a compatible connection. Let $\mathbf{W}\in\Gamma(\mathrm{End}(\mathbf{F}))$ be smooth and self-adjoint, and suppose that

$$\langle \mathbf{W}(x)v,v\rangle \geq |v|^2 \quad \text{for every } x\in M, \quad v\in\mathbf{F}_x. \tag{13.81}$$

Let $B_2=L_{\mathbf{F},2}+\mathbf{W}$ be the self-adjoint realization on L^2, and suppose that its semigroup extends consistently to semigroups C_0 on $L^r(M,\mathbf{F})$, $1\leq r<\infty$, satisfying

$$\|e^{-sB_r}\|_{\mathcal{L}(L^r)} \leq e^{-s}, \qquad s\geq 0. \tag{13.82}$$

If $\mathbf{F}$ is real, the complex powers and multipliers below are understood on the complexification, and the estimates are subsequently restricted to real sections whenever the corresponding operator preserves that subspace. Then, for every $1<p<\infty$, the spectral powers B_2^{it} extend consistently to a C_0 group of bounded operators B_p^{it} on $L^p(M,\mathbf{F})$. There exist constants $C_p,\omega_p>0$, independent of the dimension of M and the rank of $\mathbf{F}$, such that

$$\|B_p^{it}\|_{\mathcal{L}(L^p(M,\mathbf{F}))} \leq C_p e^{\omega_p|t|}, \qquad t\in\mathbb{R}. \tag{13.83}$$

More generally, if $C_2:=B_2^{1/2}$ and, for $1<p<\infty$, $-C_p$ denotes the generator of the subordinated Poisson

semigroup on $L^p(M,\mathbf{F})$, and if $\psi \in L^\infty(0,\infty)$, then the spectral multiplier

$$m_\psi(C_2), \qquad m_\psi(\lambda) := \lambda \int_0^\infty \psi(s) e^{-s\lambda}\, ds, \quad \lambda \geq 1, \tag{13.84}$$

extends to L^p and

$$\|m_\psi(C_p)\|_{\mathcal{L}(L^p)} \leq C_p \|\psi\|_{L^\infty(0,\infty)}. \tag{13.85}$$

Proof. The proof is first carried out for compactly supported sections and then extended to L^p by density. If the bundle is complex, regard it as a real bundle with inner product given by the real part of the Hermitian metric; the norms are unchanged.

The Poisson semigroup. For $t > 0$ and $1 \leq r < \infty$, define

$$P_t^{(r)}\mathbf{f} := \frac{t}{2\sqrt{\pi}} \int_0^\infty s^{-3/2} e^{-\frac{t^2}{4s}} e^{-sB_r}\mathbf{f}\, ds. \tag{13.86}$$

The integral is a Bochner integral. The scalar subordination identity

$$e^{-t\sqrt{\lambda}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty s^{-3/2} e^{-\frac{t^2}{4s}} e^{-s\lambda}\, ds, \qquad \lambda > 0,$$

follows by the substitution $u = t/(2\sqrt{s})$ in the Gaussian integral. On L^2, the spectral theorem shows that $P_t^{(2)} = e^{-tC_2}$. Moreover, (13.82) and the same formula with $\lambda = 1$ give

$$\|P_t^{(r)}\|_{\mathcal{L}(L^r)} \leq e^{-t}. \tag{13.87}$$

Consistency of the heat semigroups implies consistency of the $P_t^{(r)}$. For $1 < r < \infty$, denote the generator of $(P_t^{(r)})_{t\geq 0}$ by $-C_r$. On L^2, we have $C_2 = B_2^{1/2}$, and consistency shows that the resolvents and Laplace calculi of the operators C_r agree on intersections of the spaces L^r.

Let $\mathbf{f} \in \Gamma_c(\mathbf{F})$. For $t > 0$, $\mathbf{U}(t) := P_t^{(2)}\mathbf{f}$ belongs to the domain of every power of B_2, since $\lambda^m e^{-t\sqrt{\lambda}}$ is bounded on $[1,\infty)$ for each m. Local elliptic regularity implies that $\mathbf{U}$ is smooth on $(0,\infty)\times M$, and spectral calculus gives

$$\partial_t^2 \mathbf{U} = B_2\mathbf{U} = -\operatorname{tr}_{\mathbf{g}}(\nabla^2\mathbf{U}) + \mathbf{W}\mathbf{U}. \tag{13.88}$$

Bilinear estimate. Let p, q be conjugate exponents. By symmetry, we may assume $p \geq 2$; if $p < 2$, interchange p, q and the two sections. Let $\mathbf{f} \in \Gamma_c(\mathbf{F})$ and $\mathbf{g} \in \Gamma_c(\mathbf{F})$, and set

$$\mathbf{U}(t) = P_t\mathbf{f}, \qquad \mathbf{V}(t) = P_t\mathbf{g}.$$

Apply Lemma 13.3.20 to the real fibers of $\mathbf{F}$ and write

$$b_\varepsilon(t,x) := \mathcal{B}_p\big((\mathbf{U}(t,x),\varepsilon),(\mathbf{V}(t,x),\varepsilon)\big) - \mathcal{B}_p((0,\varepsilon),(0,\varepsilon)), \qquad \varepsilon > 0.$$

Here we adjoin a parallel real direction to each fiber; thus, the two arguments of $\mathcal{B}_p$ never vanish. The function $\mathcal{B}_p$ is C^1 and piecewise C^2. On the set where the arguments have norm at least ε, its gradient is locally Lipschitz. Integrating by parts separately in the two regions $|\mathbf{U}_\varepsilon|^p < |\mathbf{V}_\varepsilon|^q$ and $|\mathbf{U}_\varepsilon|^p > |\mathbf{V}_\varepsilon|^q$, the interface terms cancel because the first derivatives of $\mathcal{B}_p$ agree there. The following chain rule therefore

holds almost everywhere and distributionally. Equivalently, it can be obtained by approximating $\mathcal{B}_p$ by convolution on compact sets away from the axes and passing to the limit.

Let

$$\mathscr{L} := \partial_t^2 + \operatorname{tr}_{\mathbf{g}} \nabla^2$$

be the scalar operator on $(0,\infty)\times M$. The Hessian formula for a radial function on a Hilbert fiber, together with (13.88), shows that the first-order terms in $\mathbf{U},\mathbf{V}$ are given by the Hessian of $\mathcal{B}_p$ applied to the time and covariant derivatives. The order-zero terms are

$$\frac{\partial_u \beta_p}{|\mathbf{U}_\varepsilon|}\langle \mathbf{W}\mathbf{U},\mathbf{U}\rangle + \frac{\partial_v \beta_p}{|\mathbf{V}_\varepsilon|}\langle \mathbf{W}\mathbf{V},\mathbf{V}\rangle,$$

and are nonnegative by (13.81) and radial monotonicity of the Bellman function. Item (c) of the lemma, applied to each time derivative and each spatial orthonormal direction, provides a measurable function $\tau = \tau(t,x) > 0$ such that

$$\mathscr{L} b_\varepsilon \geq c_p \Big[\tau \big(|\partial_t \mathbf{U}|^2 + |\nabla \mathbf{U}|^2 \big) + \tau^{-1} \big(|\partial_t \mathbf{V}|^2 + |\nabla \mathbf{V}|^2 \big) \Big]. \tag{13.89}$$

In particular, by $\tau A^2 + \tau^{-1} B^2 \geq 2AB$,

$$\mathscr{L} b_\varepsilon \geq 2c_p \big(|\partial_t \mathbf{U}|^2 + |\nabla \mathbf{U}|^2 \big)^{1/2} \big(|\partial_t \mathbf{V}|^2 + |\nabla \mathbf{V}|^2 \big)^{1/2} \tag{13.90}$$

almost everywhere.

Take the cutoff functions χ_j from Lemma 12.1.7. Thus, $0 \leq \chi_j \leq 1$, $\chi_j \to 1$ locally, and $\|d\chi_j\|_\infty \to 0$. Integrate $t\chi_j^2 \mathscr{L} b_\varepsilon$ over $M \times (a,R)$, with $0 < a < R < \infty$. For fixed j, a, R, first let $\varepsilon \downarrow 0$. On this finite time cylinder and on the compact support of $d\chi_j$, the sections $\mathbf{U},\mathbf{V}$ and their first derivatives are bounded. Since $\mathcal{B}_p$ is of class C^1, the function and its gradient are bounded on the compact set of arguments obtained by varying $0 < \varepsilon \leq 1$. This gives a dominating function independent of ε for the boundary and cutoff terms. Dominated convergence therefore permits passage to the limit in every term. Only then let $a \downarrow 0$ and $R \to \infty$. To justify these time limits, observe that, for compactly supported data, spectral calculus on L^2 gives

$$\|C_2^k P_t \mathbf{f}\|_{L^2(M,\mathbf{F})} \leq \sup_{\lambda \geq 1} \lambda^k e^{-t\lambda} \, \|\mathbf{f}\|_{L^2(M,\mathbf{F})},$$

and likewise for $\mathbf{g}$. In particular, terms multiplied by t vanish at $t = \infty$; near zero, use the fact that smooth compactly supported sections belong to the domain of every power of B_2. Time integration produces the value at $t = 0$. In the spatial variable, the divergence formula gives

$$\int_M \chi_j^2 \operatorname{tr}_{\mathbf{g}} \nabla^2 b_\varepsilon \, d\lambda_{\mathbf{g}} = -2 \int_M \chi_j \langle d\chi_j, db_\varepsilon \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}.$$

The gradient bound (13.78), the weighted inequality (13.89), and Young's inequality allow half of the weighted energy term to be absorbed, leaving

$$\begin{aligned} &\int_0^\infty \int_M t\chi_j^2 \big(|\partial_t \mathbf{U}|^2 + |\nabla \mathbf{U}|^2 \big)^{1/2} \big(|\partial_t \mathbf{V}|^2 + |\nabla \mathbf{V}|^2 \big)^{1/2} \, d\lambda_{\mathbf{g}} \, dt \\ &\qquad \leq C_p \int_M \chi_j^2 \big(|\mathbf{f}|^p + |\mathbf{g}|^q \big) \, d\lambda_{\mathbf{g}} \end{aligned}$$

$$+ C_p \|d\chi_j\|_\infty^2 \int_0^\infty t \left(\|\mathbf{U}(t)\|_{L^p(M,\mathbf{F})}^p + \|\mathbf{V}(t)\|_{L^q(M,\mathbf{F})}^q \right) dt. \tag{13.91}$$

After the limit $\varepsilon \downarrow 0$ on the finite cylinder, the bound in item (a) of the lemma leaves only a multiple of $|\mathbf{U}|^p + |\mathbf{V}|^q$ in the cutoff term. Now use (13.87):

$$\|\mathbf{U}(t)\|_{L^p(M,\mathbf{F})}^p \le e^{-pt} \|\mathbf{f}\|_{L^p(M,\mathbf{F})}^p, \qquad \|\mathbf{V}(t)\|_{L^q(M,\mathbf{F})}^q \le e^{-qt} \|\mathbf{g}\|_{L^q(M,\mathbf{F})}^q.$$

The time integral of the last term in (13.91) is finite and independent of j. As $j \to \infty$, that term disappears and Fatou gives

$$\int_0^\infty \int_M t\, |C_2 P_t \mathbf{f}|\, |C_2 P_t \mathbf{g}|\, d\lambda_{\mathbf{g}}\, dt \le C_p \left(\|\mathbf{f}\|_{L^p(M,\mathbf{F})}^p + \|\mathbf{g}\|_{L^q(M,\mathbf{F})}^q \right). \tag{13.92}$$

We used only the time derivatives $\partial_t P_t = -C_2 P_t$ from the stronger estimate obtained above. Replace $\mathbf{f}$ by $a\mathbf{f}$ and $\mathbf{g}$ by $a^{-1}\mathbf{g}$. Choose $a > 0$ so that $a^p \|\mathbf{f}\|_{L^p(M,\mathbf{F})}^p = a^{-q} \|\mathbf{g}\|_{L^q(M,\mathbf{F})}^q$. The left-hand side is unchanged, and we obtain the homogeneous form

$$\int_0^\infty \int_M t\, |C_2 P_t \mathbf{f}|\, |C_2 P_t \mathbf{g}|\, d\lambda_{\mathbf{g}}\, dt \le C_p \|\mathbf{f}\|_{L^p(M,\mathbf{F})} \|\mathbf{g}\|_{L^q(M,\mathbf{F})}. \tag{13.93}$$

Laplace multipliers. Let $\psi \in L^\infty(0,\infty)$. On L^2, define $m_\psi(C_2)$ by spectral calculus. Since $C_2 \ge I$,

$$\mathbf{g} = \int_0^\infty C_2 P_r \mathbf{g}\, dr$$

in L^2. For $\mathbf{f}, \mathbf{g} \in \Gamma_c(\mathbf{F})$, Fubini, self-adjointness, and the semigroup law give

$$\begin{aligned} \langle m_\psi(C_2)\mathbf{f}, \mathbf{g} \rangle &= \int_0^\infty \int_0^\infty \psi(s) \langle C_2 P_s \mathbf{f}, C_2 P_r \mathbf{g} \rangle\, dr\, ds \\ &= 2 \int_0^\infty \left(\int_0^{2t} \psi(s)\, ds \right) \langle C_2 P_t \mathbf{f}, C_2 P_t \mathbf{g} \rangle\, dt. \end{aligned}$$

The second equality uses the change of variable $t = (s+r)/2$, whose Jacobian is two. By (13.93),

$$|\langle m_\psi(C_2)\mathbf{f}, \mathbf{g} \rangle| \le 4 C_p \|\psi\|_\infty \|\mathbf{f}\|_{L^p(M,\mathbf{F})} \|\mathbf{g}\|_{L^q(M,\mathbf{F})}.$$

Duality between L^p and L^q and density prove (13.85).

Imaginary powers. For $\tau \in \mathbb{R}$, set

$$\psi_\tau(s) := \frac{s^{-2i\tau}}{\Gamma(1 - 2i\tau)}.$$

Since

$$\lambda \int_0^\infty s^{-2i\tau} e^{-s\lambda}\, ds = \Gamma(1 - 2i\tau) \lambda^{2i\tau},$$

the multiplier of (13.84) is $m_{\psi_\tau}(C_2) = C_2^{2i\tau} = B_2^{i\tau}$. Recall the elementary gamma-function bound we need. The beta–gamma identity and Euler's reflection formula give, for $y \in \mathbb{R}$,

$$|\Gamma(1 + iy)|^2 = \Gamma(1 + iy)\Gamma(1 - iy) = \frac{\pi |y|}{\sinh(\pi |y|)}, \tag{13.94}$$

with value one at $y = 0$ by continuity. The reflection formula used here follows from

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{t^{z-1}}{1+t}\,dt = \frac{\pi}{\sin(\pi z)}, \qquad 0 < \operatorname{Re} z < 1,$$

where the first equality is the substitution $t = x/(1-x)$ in the beta–gamma identity, and the second follows by integrating $w^{z-1}/(1+w)$ over a keyhole contour around the positive axis; the two sides of the cut differ by the factor $e^{2\pi i z}$, and the only interior pole is $w = -1$. The identity, obtained initially for $0 < \operatorname{Re} z < 1$, extends meromorphically to values where both sides are defined. Apply it to $z = iy$ and then use $\Gamma(1+iy) = iy\Gamma(iy)$. From (13.94), we deduce

$$\frac{1}{|\Gamma(1-2i\tau)|} \le C(1+|\tau|)^{-1/2} e^{\pi|\tau|} \le C e^{\pi|\tau|}. \tag{13.95}$$

The Laplace multiplier bound then gives (13.83), for example with $\omega_p = \pi$ after modifying C_p.
Strong continuity on τ remains. On L^2, it follows immediately from spectral calculus. If $p \neq 2$, choose $r \in (1,\infty)$ so that p lies strictly between 2 and r. For a compactly supported section, Hölder interpolates the L^p norm between L^2 and L^r; the L^2 factor tends to zero as $\tau \to \tau_0$, and the L^r factor remains bounded on compact intervals of τ by the estimate already proved. Density of $\Gamma_c(\mathbf{F})$ in L^p and local uniform boundedness of $B_p^{i\tau}$ extend continuity to all of L^p. The group law is verified on $L^2 \cap L^p$ by spectral calculus and then on L^p by density. □

The abstract step that converts the second-order information into a square root is recorded in the following lemma.

Lemma 13.3.22. (Square root by interpolation and imaginary powers) Let X be a reflexive complex Banach space, and let $-B$ be the generator of a C_0-semigroup satisfying $\|e^{-tB}\| \le e^{-t}$ for $t \ge 0$. Suppose that the negative powers B^{-s}, $s > 0$, are defined by the Laplace calculus of that semigroup and that the imaginary powers form a C_0 group satisfying

$$\|B^{it}\|_{\mathcal{L}(X)} \le M e^{\omega|t|}.$$

Equip $\mathcal{D}(B)$ with the norm $\|Bx\|_X$ and $\mathcal{D}(B^{1/2})$ with $\|B^{1/2}x\|_X$. Then

$$[X, \mathcal{D}(B)]_{1/2} = \mathcal{D}(B^{1/2}) \tag{13.96}$$

with equivalence of norms. If X is real, the same assertion holds after complexifying X and B and then restricting to the real subspace.

Proof. The identity is the case $a_0 = 0$, $a_1 = 1$ of domain interpolation. To prove it, fix an integer $N \ge 2$. For $x \in \mathcal{D}(B^N)$ and $\varepsilon > 0$, define on the strip $0 \le \operatorname{Re} z \le 1$

$$F_\varepsilon(z) := e^{\varepsilon(z-1/2)^2} B^{1/2-z} x.$$

Writing $B^{1/2-z}x = B^{1/2-z-N}B^N x$, the power with negative real part is expressed by the Laplace integral; this proves holomorphy in the interior. On the two boundary lines,

$$F_\varepsilon(it) = e^{\varepsilon(it-1/2)^2} B^{-it} B^{1/2} x,$$

$$BF_\varepsilon(1+it) = e^{\varepsilon(1/2+it)^2} B^{-it} B^{1/2} x.$$

The bound on imaginary powers and the Gaussian factor show that F_ε belongs to the Calderón space of the pair $(X, \mathcal{D}(B))$ and that

$$\|x\|_{[X,\mathcal{D}(B)]_{1/2}} \le C \|B^{1/2} x\|_X.$$

If $x \in \mathcal{D}(B^{1/2})$, set $x_\eta = (I+\eta B)^{-N} x$. Laplace functional calculus gives $x_\eta \in \mathcal{D}(B^N)$ and $B^{1/2} x_\eta \to B^{1/2} x$. The preceding estimate applied to differences proves that x_η converges to x in the interpolated space. Thus, $\mathcal{D}(B^{1/2}) \subseteq [X, \mathcal{D}(B)]_{1/2}$.

For the reverse inclusion, let $F \in \mathscr{F}(X, \mathcal{D}(B))$ and $x = F(1/2)$. Fix $\eta > 0$ and set $R_\eta = (I+\eta B)^{-N}$. The function

$$G_{\varepsilon,\eta}(z) := e^{\varepsilon(z-1/2)^2} B^z R_\eta F(z)$$

is holomorphic with values in X. On the boundary,

$$G_{\varepsilon,\eta}(it) = e^{\varepsilon(it-1/2)^2} B^{it} R_\eta F(it),$$

and

$$G_{\varepsilon,\eta}(1+it) = e^{\varepsilon(1/2+it)^2} B^{it} R_\eta B F(1+it).$$

The norms of R_η remain uniformly bounded because the resolvent comes from the same Laplace calculus. The three-lines lemma gives, with a constant independent of η,

$$\|B^{1/2} R_\eta x\|_X \le C \|F\|_{\mathscr{F}(X,\mathcal{D}(B))}.$$

As $\eta \downarrow 0$, $R_\eta x \to x$ in X. By reflexivity, extract a subsequence along which $B^{1/2} R_\eta x$ converges weakly. The graph of the closed operator $B^{1/2}$ is a closed subspace of $X \times X$ and, by convexity, is also weakly closed. It follows that $x \in \mathcal{D}(B^{1/2})$ and

$$\|B^{1/2} x\|_X \le C \|F\|_{\mathscr{F}(X,\mathcal{D}(B))}.$$

Taking the infimum over F proves the reverse inclusion and the second norm comparison. □

Corollary 13.3.23. (Internal first-order estimate for the chain) Under the hypotheses of Proposition 13.3.16, let $1 < p < \infty$. For each $\ell \in \{0, \dots, N\}$,

$$\mathcal{D}(B_{\ell,p}^{1/2}) = W^{1,p}(M, \mathbf{E}_\ell) \tag{13.97}$$

with equivalence of norms. In particular, there exist $\widetilde{a}_{\ell,p}, \widetilde{b}_{\ell,p} > 0$ such that

$$\widetilde{a}_{\ell,p} \left(\|\mathbf{z}\|_{L^p(M,\mathbf{E}_\ell)} + \|\nabla_\ell \mathbf{z}\|_{L^p(M,\mathbf{E}_{\ell+1})} \right) \le \|B_{\ell,p}^{1/2} \mathbf{z}\|_{L^p(M,\mathbf{E}_\ell)} \le \widetilde{b}_{\ell,p} \left(\|\mathbf{z}\|_{L^p(M,\mathbf{E}_\ell)} + \|\nabla_\ell \mathbf{z}\|_{L^p(M,\mathbf{E}_{\ell+1})} \right). \tag{13.98}$$

Proof. The choice of μ_N in Proposition 13.3.16 implies, pointwise,

$$\mu_N I + \mathbf{V}_\ell \ge I$$

as a self-adjoint endomorphism of $\mathbf{E}_\ell$. The same proposition gives

$$\|e^{-tB_{\ell,r}}\|_{\mathcal{L}(L^r)} \le e^{-t} \qquad (1 \le r < \infty).$$

Theorem 13.3.21 therefore applies to $B_{\ell,p}$ and provides the imaginary powers required in Lemma 13.3.22. Proposition 13.3.19 and Proposition 13.3.18 give

$$\mathcal{D}(B_{\ell,p}^{1/2}) = [L^p(M,\mathbf{E}_\ell), \mathcal{D}(B_{\ell,p})]_{1/2} = [L^p(M,\mathbf{E}_\ell), W^{2,p}(M,\mathbf{E}_\ell)]_{1/2} = W^{1,p}(M,\mathbf{E}_\ell).$$

The norm comparisons in these three results, together with the definition of the $W^{1,p}$ norm, produce (13.98). □

Proposition 13.3.24. (Power estimates for the Bochner chain) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, let $\mathbf{E} \to M$ be a bundle of bounded geometry, and let $N \in \mathbb{N}$ and $1 < p < \infty$. Use the operators $B_{\ell,p}$ of Proposition 13.3.16. For $k \in \mathbb{N}_0$ and $\ell \in \{0,\dots,N\}$, define

$$\mathscr{D}^p_{\ell,k} := B_{\ell,p}^{-\frac{k}{2}} L^p(M,\mathbf{E}_\ell), \qquad \|\mathbf{u}\|_{\mathscr{D}^p_{\ell,k}} := \|B_{\ell,p}^{\frac{k}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)}. \tag{13.99}$$

If also $k + \ell \le N$, then

$$\mathscr{D}^p_{\ell,k} = W^{k,p}(M,\mathbf{E}_\ell), \tag{13.100}$$

and there exist constants $c_{\ell,k,p}, C_{\ell,k,p} > 0$ such that

$$\begin{aligned} c_{\ell,k,p} \sum_{j=0}^{k} \|\nabla^j \mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+j})} &\le \|B_{\ell,p}^{\frac{k}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \\ &\le C_{\ell,k,p} \sum_{j=0}^{k} \|\nabla^j \mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+j})}. \end{aligned} \tag{13.101}$$

Proof. The negative power of (13.99) is understood through

$$B_{\ell,p}^{-\frac{s}{2}}\mathbf{f} = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty t^{\frac{s}{2}-1} e^{-tB_{\ell,p}}\mathbf{f}\, dt, \qquad s > 0. \tag{13.102}$$

Since $-(B_{\ell,p} - I)$ generates a contraction semigroup, Theorem B.4.19 guarantees that these powers are injective, their ranges are Banach spaces, and

$$\|B_{\ell,p}^{\frac{a}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \le \|B_{\ell,p}^{\frac{b}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)}, \qquad 0 \le a \le b, \tag{13.103}$$

for $\mathbf{u} \in \mathscr{D}^p_{\ell,b}$.

The first-order estimate follows by combining elliptic regularity with domain interpolation. Proposition 13.3.19 identifies the domain of $B_{\ell,p}$ with $W^{2,p}$, Theorem 13.3.21 provides its imaginary powers, and Proposition 13.3.18 identifies the interpolated space of order one. Corollary 13.3.23 therefore gives

$$\mathscr{D}^p_{\ell,1} = W^{1,p}(M,\mathbf{E}_\ell). \tag{13.104}$$

Moreover, there exist constants $\widetilde{a}_{\ell,p}, \widetilde{b}_{\ell,p} > 0$ such that

$$\widetilde{a}_{\ell,p}\left(\|\mathbf{z}\|_{L^p(M,\mathbf{E}_\ell)} + \|\nabla_\ell \mathbf{z}\|_{L^p(M,\mathbf{E}_{\ell+1})}\right) \le \|B_{\ell,p}^{\frac{1}{2}}\mathbf{z}\|_{L^p(M,\mathbf{E}_\ell)} \le \widetilde{b}_{\ell,p}\left(\|\mathbf{z}\|_{L^p(M,\mathbf{E}_\ell)} + \|\nabla_\ell \mathbf{z}\|_{L^p(M,\mathbf{E}_{\ell+1})}\right). \tag{13.105}$$

Thus, the starting point of the induction is established by the uniform elliptic regularity, Bellman function, and complex interpolation proved above.

We now develop the commutation step needed at higher orders. Write

$$S_\ell(t) := e^{-tB_{\ell,p}}, \qquad t \ge 0.$$

The covariant semigroups are consistent as p varies. The Dyson–Phillips series of Proposition B.4.17 shows that consistency is retained when the bounded potentials $\mathbf{V}_\ell$ and shift $\mu_N I$ are added.

Before commuting unbounded operators, fix a common regularizing core. Let $\varphi \in C_c^\infty((0,\infty))$ and define

$$\mathcal{R}_{\ell,\varphi}\mathbf{z} := \int_0^\infty \varphi(t) S_\ell(t)\mathbf{z}\, dt. \tag{13.106}$$

The integral is a Bochner integral. We need not assume that $S_\ell(t)\mathbf{z}$ is differentiable for arbitrary data in L^p. Extend φ by zero to $\mathbb{R}$. For small $h > 0$,

$$\frac{S_\ell(h) - I}{h}\mathcal{R}_{\ell,\varphi}\mathbf{z} = \int_0^\infty \frac{\varphi(t-h) - \varphi(t)}{h} S_\ell(t)\mathbf{z}\, dt \longrightarrow -\int_0^\infty \varphi'(t) S_\ell(t)\mathbf{z}\, dt.$$

Convergence is in L^p, by dominated convergence on a compact interval. The definition of the generator $-B_{\ell,p}$ proves the formula for $m = 1$; repeating it with $\varphi', \varphi'', \dots$ gives

$$B_{\ell,p}^m \mathcal{R}_{\ell,\varphi}\mathbf{z} = \int_0^\infty \varphi^{(m)}(t) S_\ell(t)\mathbf{z}\, dt, \qquad m \in \mathbb{N}_0. \tag{13.107}$$

Therefore, $\mathcal{R}_{\ell,\varphi}\mathbf{z} \in \bigcap_{m\in\mathbb{N}} \mathcal{D}(B_{\ell,p}^m)$.

Choose a family $(\varphi_\varepsilon)_{\varepsilon>0}$ of nonnegative functions in $C_c^\infty((0,\varepsilon))$ with integral one. Strong continuity of the semigroup implies

$$\mathcal{R}_{\ell,\varphi_\varepsilon}\mathbf{z} \longrightarrow \mathbf{z} \quad \text{on } L^p(M,\mathbf{E}_\ell).$$

This convergence holds in each space $\mathscr{D}_{\ell,\rho}^p$. Indeed, if $\mathbf{z} = B_{\ell,p}^{-\frac{\rho}{2}}\mathbf{f}$, Laplace functional calculus allows $B_{\ell,p}^{-\frac{\rho}{2}}$ to commute with $S_\ell(t)$, and

$$\|\mathcal{R}_{\ell,\varphi_\varepsilon}\mathbf{z} - \mathbf{z}\|_{\mathscr{D}_{\ell,\rho}^p} = \|\mathcal{R}_{\ell,\varphi_\varepsilon}\mathbf{f} - \mathbf{f}\|_{L^p(M,\mathbf{E}_\ell)} \longrightarrow 0.$$

As the common core, take

$$\mathscr{C}_{\ell,p} := \bigcap_{m\in\mathbb{N}} \mathcal{D}(B_{\ell,p}^m).$$

The preceding regularizations belong to this space and prove its density in every $\mathscr{D}_{\ell,\rho}^p$. Moreover, the power law shows that it is invariant under positive and negative powers of $B_{\ell,p}$ and under $S_\ell(t)$. This invariance will allow the same identity to be applied to $\mathbf{z}$ and $B_{\ell,p}^a\mathbf{z}$.

The geometric identities are first verified on regularizations of compactly supported data. Approximate $\mathbf{z} \in L^p(M,\mathbf{E}_\ell)$ by sections $\mathbf{z}_n \in \Gamma_c(\mathbf{E}_\ell)$. The sections $\mathcal{R}_{\ell,\varphi}\mathbf{z}_n$ belong simultaneously to L^2 and L^p and, by

(13.107), to the domains of every power of $B_{\ell,2}$. Since $B_{\ell,2}$ has the same elliptic principal symbol as the Bochner Laplacian, local elliptic regularity shows that these sections are smooth. The differential identity (13.52) holds locally for them. It also justifies the realizations involved: if $\mathbf{v} \in \bigcap_{m\in\mathbb{N}} \mathcal{D}(B_{\ell,2}^m)$, the first-order estimate gives $\nabla_\ell \mathbf{v}, \nabla_\ell B_{\ell,2}\mathbf{v} \in L^2$. By the differential identity,

$$B_{\ell+1,2}\nabla_\ell \mathbf{v} = \nabla_\ell B_{\ell,2}\mathbf{v} - \mathbf{W}_\ell \mathbf{v} \in L^2 \quad \text{en distribuciones.}$$

Lemma 13.3.17 implies $\nabla_\ell \mathbf{v} \in \mathcal{D}(B_{\ell+1,2})$. Applying the same argument to $S_\ell(t)\mathbf{v}$ and $B_{\ell,2}S_\ell(t)\mathbf{v}$ gives continuity in the graph norms needed to differentiate the product of semigroups. Formulas (13.107), the first-order estimate (13.105), and continuity of $\mathbf{W}_\ell$ allow passage to the limit $n \to \infty$. Thus, every identity established below on $\mathcal{R}_{\ell,\varphi}\mathbf{z}_n$ first passes to $\mathcal{R}_{\ell,\varphi}\mathbf{z}$. Finally, let $\varphi = \varphi_\varepsilon$ and $\varepsilon \to 0^+$. Convergence in all graph norms of finite order extends the identities to $\mathscr{C}_{\ell,p}$.

Functional calculus for Bessel powers shows that, for $\rho \geq 0$,

$$\|S_\ell(t)\mathbf{z}\|_{\mathscr{D}^p_{\ell,\rho}} \leq e^{-t}\|\mathbf{z}\|_{\mathscr{D}^p_{\ell,\rho}}. \tag{13.108}$$

Suppose that, for some $\rho \in \mathbb{N}_0$, multiplication by $\mathbf{W}_\ell$ is continuous from $\mathscr{D}^p_{\ell,\rho}$ to $\mathscr{D}^p_{\ell+1,\rho}$. For $t > 0$, define

$$\mathcal{K}_\ell(t)\mathbf{z} := \int_0^t S_{\ell+1}(t-\tau)\mathbf{W}_\ell S_\ell(\tau)\mathbf{z}\, d\tau. \tag{13.109}$$

The integral converges in $\mathscr{D}^p_{\ell+1,\rho}$ and

$$\|\mathcal{K}_\ell(t)\mathbf{z}\|_{\mathscr{D}^p_{\ell+1,\rho}} \leq C_\rho t e^{-t}\|\mathbf{z}\|_{\mathscr{D}^p_{\ell,\rho}}. \tag{13.110}$$

Let $\mathbf{z} \in \mathscr{C}_{\ell,p}$. On the smooth approximations described above, we may differentiate in L^2 the function

$$\tau \longmapsto S_{\ell+1}(t-\tau)\nabla_\ell S_\ell(\tau)\mathbf{z}, \qquad 0 < \tau < t.$$

The identity

$$\nabla_\ell B_{\ell,p} - B_{\ell+1,p}\nabla_\ell = \mathbf{W}_\ell,$$

which is (13.52) with the same shift on both bundles, gives

$$\frac{d}{d\tau}\left[S_{\ell+1}(t-\tau)\nabla_\ell S_\ell(\tau)\mathbf{z}\right] = -S_{\ell+1}(t-\tau)\mathbf{W}_\ell S_\ell(\tau)\mathbf{z}.$$

Integrate between 0 and t and pass to L^p by consistency. This gives

$$S_{\ell+1}(t)\nabla_\ell \mathbf{z} - \nabla_\ell S_\ell(t)\mathbf{z} = \mathcal{K}_\ell(t)\mathbf{z}. \tag{13.111}$$

The first-order estimate and (13.110) allow this identity to extend by density to $\mathscr{D}^p_{\ell,1}$; at higher orders it is interpreted in the $\mathscr{D}^p_{\ell+1,\rho}$ norm whenever both sides are defined. The expression on the right-hand side, however, is defined and continuous on all of $\mathscr{D}^p_{\ell,\rho}$.

Now consider the square root. For $\mathbf{z} \in \mathscr{C}_{\ell,p}$, the integral formula of Lemma B.4.20, applied first with trun-

cated limits, and (13.111) give

$$\begin{aligned}&\frac{1}{2\sqrt{\pi}}\int_\varepsilon^R t^{-\frac{3}{2}}\left(\nabla_\ell \mathbf{z} - S_{\ell+1}(t)\nabla_\ell \mathbf{z}\right) dt\\ &\quad = \nabla_\ell\left[\frac{1}{2\sqrt{\pi}}\int_\varepsilon^R t^{-\frac{3}{2}}\left(\mathbf{z} - S_\ell(t)\mathbf{z}\right) dt\right] - \frac{1}{2\sqrt{\pi}}\int_\varepsilon^R t^{-\frac{3}{2}}\mathcal{K}_\ell(t)\mathbf{z}\, dt.\end{aligned}$$

The first term on the right-hand side converges to $\nabla_\ell B^{\frac{1}{2}}_{\ell,p}\mathbf{z}$: the bracketed integral converges in $\mathscr{D}^p_{\ell,1}$ because $\mathbf{z}$ belongs to every domain, and we use (13.105). The second term converges in $\mathscr{D}^p_{\ell+1,\rho}$, since

$$\int_0^\infty t^{-\frac{3}{2}}(te^{-t})\, dt = \Gamma\left(\frac{1}{2}\right) < \infty.$$

Closedness of $B^{\frac{1}{2}}_{\ell+1,p}$ and the last assertion of Lemma B.4.20 show that $\nabla_\ell \mathbf{z} \in \mathscr{D}^p_{\ell+1,1}$ and that

$$\Theta_{\ell,1}\mathbf{z} := B^{\frac{1}{2}}_{\ell+1,p}\nabla_\ell \mathbf{z} - \nabla_\ell B^{\frac{1}{2}}_{\ell,p}\mathbf{z} = -\frac{1}{2\sqrt{\pi}}\int_0^\infty t^{-\frac{3}{2}}\mathcal{K}_\ell(t)\mathbf{z}\, dt. \tag{13.112}$$

By (13.110),

$$\|\Theta_{\ell,1}\mathbf{z}\|_{\mathscr{D}^p_{\ell+1,\rho}} \le C'_\rho \|\mathbf{z}\|_{\mathscr{D}^p_{\ell,\rho}}. \tag{13.113}$$

Since $\mathscr{C}_{\ell,p}$ is dense in $\mathscr{D}^p_{\ell,\rho}$, the integral on the right defines the unique extension of $\Theta_{\ell,1}$ to that space.
For $j \in \mathbb{N}$, write formally

$$\Theta_{\ell,j}\mathbf{z} := B^{\frac{j}{2}}_{\ell+1,p}\nabla_\ell \mathbf{z} - \nabla_\ell B^{\frac{j}{2}}_{\ell,p}\mathbf{z}. \tag{13.114}$$

Before using this expression, we must verify that its first term is defined. We do this simultaneously with the defect estimate.
Suppose that, for some $R \in \mathbb{N}_0$, multiplication by $\mathbf{W}_\ell$ is continuous from $\mathscr{D}^p_{\ell,\rho}$ to $\mathscr{D}^p_{\ell+1,\rho}$ for every $\rho \in \{0,\dots,R\}$. We claim that, if $j \ge 1$ and $j-1 \le R$, the following properties hold.

(i) For each $\mathbf{z} \in \mathscr{C}_{\ell,p}$, $\nabla_\ell \mathbf{z} \in \mathscr{D}^p_{\ell+1,j}$, and the identity (13.114) is well defined.

(ii) If $\rho \in \mathbb{N}_0$ and $\rho + j - 1 \le R$, then $\Theta_{\ell,j}$ extends uniquely to a continuous operator

$$\Theta_{\ell,j}\colon \mathscr{D}^p_{\ell,\rho+j-1} \longrightarrow \mathscr{D}^p_{\ell+1,\rho}$$

and there exists $C_{\ell,j,\rho,p} > 0$ such that

$$\|\Theta_{\ell,j}\mathbf{z}\|_{\mathscr{D}^p_{\ell+1,\rho}} \le C_{\ell,j,\rho,p}\|\mathbf{z}\|_{\mathscr{D}^p_{\ell,\rho+j-1}}. \tag{13.115}$$

For $j = 1$, both assertions follow from (13.112) and (13.113). Moreover, the square-root identity extends to the mapping

$$\nabla_\ell\colon \mathscr{D}^p_{\ell,2} \longrightarrow \mathscr{D}^p_{\ell+1,1}. \tag{13.116}$$

Indeed, if $\mathbf{z}_\nu \in \mathscr{C}_{\ell,p}$ converges to $\mathbf{z}$ in $\mathscr{D}^p_{\ell,2}$, the first-order estimate shows that $\nabla_\ell \mathbf{z}_\nu \to \nabla_\ell \mathbf{z}$ in L^p. On the other hand,

$$B^{\frac{1}{2}}_{\ell+1,p}\nabla_\ell \mathbf{z}_\nu = \nabla_\ell B^{\frac{1}{2}}_{\ell,p}\mathbf{z}_\nu + \Theta_{\ell,1}\mathbf{z}_\nu$$

is Cauchy in L^p: the first term is controlled by applying (13.105) to $B^{\frac{1}{2}}_{\ell,p}(\mathbf{z}_\nu - \mathbf{z}_\mu)$, and the second by (13.113) with $\rho = 0$. Closedness of $B^{\frac{1}{2}}_{\ell+1,p}$ proves (13.116) and permits passage to the limit in the identity.

Proceed by induction on j. Suppose both assertions have been proved through order $j-1$, with $j \geq 2$, and let $\mathbf{z} \in \mathscr{C}_{\ell,p}$. The induction hypothesis gives

$$B^{\frac{j-1}{2}}_{\ell+1,p} \nabla_\ell \mathbf{z} = \nabla_\ell B^{\frac{j-1}{2}}_{\ell,p} \mathbf{z} + \Theta_{\ell,j-1}\mathbf{z}. \tag{13.117}$$

The first term on the right belongs to $\mathscr{D}^p_{\ell+1,1}$ by (13.116), applied to $B^{\frac{j-1}{2}}_{\ell,p}\mathbf{z}$. The second also belongs to that space by (13.115) with order $j-1$ and $\rho = 1$; here we use $j - 1 \leq R$. Thus, the left-hand side of (13.117) belongs to the domain of $B^{\frac{1}{2}}_{\ell+1,p}$. This proves that $\nabla_\ell \mathbf{z} \in \mathscr{D}^p_{\ell+1,j}$.

Now apply $B^{\frac{1}{2}}_{\ell+1,p}$ to (13.117) and use the square-root identity on $B^{\frac{j-1}{2}}_{\ell,p}\mathbf{z}$. We obtain the recursion

$$\Theta_{\ell,j}\mathbf{z} = \Theta_{\ell,1}\big(B^{\frac{j-1}{2}}_{\ell,p}\mathbf{z}\big) + B^{\frac{1}{2}}_{\ell+1,p}\Theta_{\ell,j-1}\mathbf{z}. \tag{13.118}$$

If $\rho + j - 1 \leq R$, estimate the first sum using (13.113) in the $\mathscr{D}^p_{\ell+1,\rho}$ norm, and the second using the induction hypothesis in the $\mathscr{D}^p_{\ell+1,\rho+1}$ norm. Thus,

$$\begin{aligned}\|\Theta_{\ell,j}\mathbf{z}\|_{\mathscr{D}^p_{\ell+1,\rho}} &\leq C_\rho \|B^{\frac{j-1}{2}}_{\ell,p}\mathbf{z}\|_{\mathscr{D}^p_{\ell,\rho}} + \|\Theta_{\ell,j-1}\mathbf{z}\|_{\mathscr{D}^p_{\ell+1,\rho+1}} \\ &\leq C_{\ell,j,\rho,p}\|\mathbf{z}\|_{\mathscr{D}^p_{\ell,\rho+j-1}}.\end{aligned}$$

Density of $\mathscr{C}_{\ell,p}$ in the source space provides the unique extension and concludes the induction. In particular, taking $\rho = 0$ gives

$$\|\Theta_{\ell,j}\mathbf{z}\|_{L^p(M,\mathbf{E}_{\ell+1})} \leq C_{\ell,j,p}\|B^{\frac{j-1}{2}}_{\ell,p}\mathbf{z}\|_{L^p(M,\mathbf{E}_\ell)}. \tag{13.119}$$

We now deduce the connection's mapping property between domains. Let $\mathbf{z} \in \mathscr{D}^p_{\ell,j+1}$, and choose $\mathbf{z}_\nu \in \mathscr{C}_{\ell,p}$ with $\mathbf{z}_\nu \to \mathbf{z}$ in $\mathscr{D}^p_{\ell,j+1}$. The first-order estimate gives $\nabla_\ell \mathbf{z}_\nu \to \nabla_\ell \mathbf{z}$ in L^p. On the other hand,

$$B^{\frac{j}{2}}_{\ell+1,p}\nabla_\ell \mathbf{z}_\nu = \nabla_\ell B^{\frac{j}{2}}_{\ell,p}\mathbf{z}_\nu + \Theta_{\ell,j}\mathbf{z}_\nu.$$

The right-hand side is Cauchy in L^p: apply the first-order estimate to $B^{\frac{j}{2}}_{\ell,p}(\mathbf{z}_\nu - \mathbf{z}_\mu)$ for the first term and use (13.119) for the second. Closedness of $B^{\frac{j}{2}}_{\ell+1,p}$ implies $\nabla_\ell \mathbf{z} \in \mathscr{D}^p_{\ell+1,j}$ and permits passage to the limit in the identity. Moreover, the first-order estimate and (13.119) give

$$\begin{aligned}\|\nabla_\ell \mathbf{z}\|_{\mathscr{D}^p_{\ell+1,j}} &\leq C\|B^{\frac{j}{2}}_{\ell,p}\mathbf{z}\|_{\mathscr{D}^p_{\ell,1}} + C\|\mathbf{z}\|_{\mathscr{D}^p_{\ell,j-1}} \\ &\leq C'\|\mathbf{z}\|_{\mathscr{D}^p_{\ell,j+1}}.\end{aligned}$$

We have therefore proved the continuous mapping

$$\nabla_\ell \colon \mathscr{D}^p_{\ell,j+1} \longrightarrow \mathscr{D}^p_{\ell+1,j}. \tag{13.120}$$

We now prove (13.101) by induction on k. The case $k = 0$ is immediate, and the case $k = 1$ is contained in (13.104) and (13.105). Fix $k \in \{2, \dots, N\}$, suppose the assertion has been proved at all orders less than k, and fix

$$0 \le \ell \le N - k.$$

For $\rho \in \{0, \dots, k-2\}$ and $a \in \{0, \dots, \rho\}$, the Leibniz rule and (13.53) give

$$\|\nabla^a(\mathbf{W}_\ell \mathbf{z})\|_{L^p(M,\mathbf{E}_{\ell+a+1})} \le C_a \sum_{b=0}^{a} \|\nabla^b \mathbf{z}\|_{L^p(M,\mathbf{E}_{\ell+b})}. \tag{13.121}$$

Summing over a and applying the induction hypothesis at order ρ yields a continuous operator

$$\mathbf{W}_\ell \colon \mathscr{D}^p_{\ell,\rho} \longrightarrow \mathscr{D}^p_{\ell+1,\rho}, \qquad 0 \le \rho \le k-2. \tag{13.122}$$

The indices are admissible because $\ell + 1 \le N - (k-2)$. Consequently, we may use (13.120) and (13.119) with $j = k - 1$.
Let $\mathbf{u} \in \mathscr{D}^p_{\ell,k}$. By (13.120),

$$\nabla_\ell \mathbf{u} \in \mathscr{D}^p_{\ell+1,k-1}.$$

The inductive estimate at order $k-1$, applied to this section, and the identity defining $\Theta_{\ell,k-1}$ give

$$\begin{aligned}
\|\nabla^k \mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+k})} &\le C\|B_{\ell+1,p}^{\frac{k-1}{2}} \nabla_\ell \mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+1})} \\
&\le C\|\nabla_\ell B_{\ell,p}^{\frac{k-1}{2}} \mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+1})} + C\|\Theta_{\ell,k-1}\mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+1})} \\
&\le C\|B_{\ell,p}^{\frac{k}{2}} \mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} + C\|B_{\ell,p}^{\frac{k-2}{2}} \mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \\
&\le C'\|B_{\ell,p}^{\frac{k}{2}} \mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)}.
\end{aligned}$$

In the third line, we use (13.105) for the section $B_{\ell,p}^{\frac{k-1}{2}} \mathbf{u}$ and (13.119); in the last, we use (13.103). For lower-order derivatives, the induction hypothesis and the same monotonicity give

$$\sum_{j=0}^{k-1} \|\nabla^j \mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+j})} \le C\|B_{\ell,p}^{\frac{k-1}{2}} \mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \le C\|B_{\ell,p}^{\frac{k}{2}} \mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)}.$$

Therefore,

$$\sum_{j=0}^{k} \|\nabla^j \mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+j})} \le C\|B_{\ell,p}^{\frac{k}{2}} \mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)}. \tag{13.123}$$

In particular, this estimate proves the continuous inclusion $\mathscr{D}^p_{\ell,k} \subseteq W^{k,p}(M, \mathbf{E}_\ell)$
We prove the reverse estimate on smooth compactly supported sections. Proposition 12.3.4 shows that $\Gamma_c(\mathbf{E}_\ell) \subseteq \mathcal{D}(L_{\ell,p})$. Since the differential operator $L_{\ell,0} + \mathbf{V}_\ell$ preserves $\Gamma_c(\mathbf{E}_\ell)$, $B_{\ell,p}$ preserves it as well. For $r = 1$, we already have $\Gamma_c(\mathbf{E}_\ell) \subseteq \mathcal{D}(B_{\ell,p})$. If $\mathbf{u} \in \Gamma_c(\mathbf{E}_\ell)$ belongs to $\mathcal{D}(B^r_{\ell,p})$, then $B^r_{\ell,p}\mathbf{u} \in \Gamma_c(\mathbf{E}_\ell) \subseteq \mathcal{D}(B_{\ell,p})$; by definition of the domain of a composition, $\mathbf{u} \in \mathcal{D}(B^{r+1}_{\ell,p})$. Induction gives

$$\Gamma_c(\mathbf{E}_\ell) \subseteq \bigcap_{r \in \mathbb{N}} \mathcal{D}(B^r_{\ell,p}).$$

Monotonicity of the scale implies that every section in $\Gamma_c(\mathbf{E}_\ell)$ also belongs to the domains of the fractional powers appearing below. For $\mathbf{u} \in \Gamma_c(\mathbf{E}_\ell)$, apply the first-order estimate to $B_{\ell,p}^{\frac{k-1}{2}}\mathbf{u}$ and then add and subtract the defect $\Theta_{\ell,k-1}\mathbf{u}$:

$$\begin{aligned}\|B_{\ell,p}^{\frac{k}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} &\le C\|B_{\ell,p}^{\frac{k-1}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} + C\|\nabla_\ell B_{\ell,p}^{\frac{k-1}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+1})}\\ &\le C\|B_{\ell,p}^{\frac{k-1}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} + C\|B_{\ell+1,p}^{\frac{k-1}{2}}\nabla_\ell\mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+1})}\\ &\quad + C\|\Theta_{\ell,k-1}\mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+1})}\\ &\le C'\sum_{j=0}^{k}\|\nabla^j\mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+j})}.\end{aligned}$$

In the last inequality, apply the induction hypothesis at order $k-1$ to $\mathbf{u}$ and $\nabla_\ell\mathbf{u}$, and at order $k-2$ to the term controlling $\Theta_{\ell,k-1}\mathbf{u}$. Thus,

$$\|B_{\ell,p}^{\frac{k}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \le C\sum_{j=0}^{k}\|\nabla^j\mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+j})}, \qquad \mathbf{u}\in\Gamma_c(\mathbf{E}_\ell). \tag{13.124}$$

Let us verify the required density. Let $\mathcal{T}^{\mathrm{geo}} = (U_i,\varphi_i,h_i)_{i\in I}$ be a geodesic trivialization, so that $\sum_{i\in I} h_i = 1$. For finite $F \subseteq I$, set

$$\mathbf{u}_F := \sum_{i\in F} h_i\mathbf{u}.$$

Each $\mathbf{u}_F$ has compact support. The same Leibniz calculation and second overlap count used in Proposition 13.3.14 give

$$\|\mathbf{u}-\mathbf{u}_F\|_{W^{k,p}(M,\mathbf{E}_\ell)}^p \le C\sum_{i\in I\setminus F}\|h_i\mathbf{u}\|_{W^{k,p}(U_i,\mathbf{E}_\ell)}^p. \tag{13.125}$$

The series on the right converges by the direct localization inequality, so its tails tend to zero. Finally, the Meyers–Serrin Theorem 9.1.48 approximates each $\mathbf{u}_F$ by smooth sections. If $\chi_F \in C_c^\infty(M)$ equals one on a neighborhood of $\operatorname{supp}(\mathbf{u}_F)$, multiplying those approximations by χ_F produces a sequence in $\Gamma_c(\mathbf{E}_\ell)$ converging to $\mathbf{u}_F$ in $W^{k,p}$. Therefore, $\Gamma_c(\mathbf{E}_\ell)$ is dense in $W^{k,p}(M,\mathbf{E}_\ell)$.

Now let $\mathbf{u}\in W^{k,p}(M,\mathbf{E}_\ell)$, and choose $\mathbf{u}_n\in\Gamma_c(\mathbf{E}_\ell)$ with $\mathbf{u}_n\to\mathbf{u}$ in $W^{k,p}$. Estimate (13.124) shows that $(B_{\ell,p}^{\frac{k}{2}}\mathbf{u}_n)_n$ is Cauchy in $L^p(M,\mathbf{E}_\ell)$. Moreover, $\mathbf{u}_n\to\mathbf{u}$ in $L^p(M,\mathbf{E}_\ell)$. Positive powers in the Bessel scale are closed operators; consequently,

$$\mathbf{u}\in\mathscr{D}_{\ell,k}^p \quad\text{and}\quad B_{\ell,p}^{\frac{k}{2}}\mathbf{u} = \lim_{n\to\infty} B_{\ell,p}^{\frac{k}{2}}\mathbf{u}_n.$$

Passing to the limit gives

$$\|B_{\ell,p}^{\frac{k}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E}_\ell)} \le C\sum_{j=0}^{k}\|\nabla^j\mathbf{u}\|_{L^p(M,\mathbf{E}_{\ell+j})}. \tag{13.126}$$

Together with (13.123), this proves equality of spaces and both estimates at order k. The induction is complete. □

Theorem 13.3.25. (Bessel potentials and covariant derivatives) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, and let $(\mathbf{E},\mathbf{h}_{\mathbf{E}},\nabla^{\mathbf{E}})$ be a bundle of bounded geometry. If $m\in\mathbb{N}_0$ and $1<p<\infty$, then

$$H^{m,p}_{L_{\mathbf{E}}}(M,\mathbf{E})=W^{m,p}(M,\mathbf{E}).$$

There exist constants $c_{m,p},C_{m,p}>0$, independent of $\mathbf{u}$, such that

$$c_{m,p}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}\le\|\mathbf{u}\|_{H^{m,p}_{L_{\mathbf{E}}}(M,\mathbf{E})}\le C_{m,p}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}. \tag{13.127}$$

Proof. The case $m=0$ is immediate. Suppose $m\ge 1$ and apply Proposition 13.3.24 with $N=m$, $\ell=0$, and $k=m$. Since $\mathbf{V}_0=0$, there exist $a_{m,p},b_{m,p}>0$ such that

$$a_{m,p}\sum_{j=0}^{m}\|\nabla^j\mathbf{u}\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\le\|(\mu_m I+L_{\mathbf{E},p})^{\frac{m}{2}}\mathbf{u}\|_{L^p(M,\mathbf{E})}\le b_{m,p}\sum_{j=0}^{m}\|\nabla^j\mathbf{u}\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})}. \tag{13.128}$$

Lemma B.4.22, applied to the covariant semigroup, shows that

$$\operatorname{Ran}\big((\mu_m I+L_{\mathbf{E},p})^{-\frac{m}{2}}\big)=\operatorname{Ran}\big((I+L_{\mathbf{E},p})^{-\frac{m}{2}}\big)=H^{m,p}_{L_{\mathbf{E}}}(M,\mathbf{E}),$$

and, for $\mathbf{u}$ in this common range,

$$\|Q_{\mu_m,1}\|^{-1}_{\mathcal{L}(L^p(M,\mathbf{E}))}\|\mathbf{u}\|_{\operatorname{Ran}((I+L_{\mathbf{E},p})^{-\frac{m}{2}})}\le\|\mathbf{u}\|_{\operatorname{Ran}((\mu_m I+L_{\mathbf{E},p})^{-\frac{m}{2}})}\le\|Q_{1,\mu_m}\|_{\mathcal{L}(L^p(M,\mathbf{E}))}\|\mathbf{u}\|_{\operatorname{Ran}((I+L_{\mathbf{E},p})^{-\frac{m}{2}})},$$

where the operators $Q_{a,b}$ are those constructed in the proof of that lemma. On the other hand, the finite-sum norms ℓ^p and ℓ^1 satisfy

$$\begin{aligned}\left(\sum_{j=0}^{m}\|\nabla^j\mathbf{u}\|^p_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\right)^{\frac{1}{p}}&\le\sum_{j=0}^{m}\|\nabla^j\mathbf{u}\|_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\\&\le(m+1)^{1-\frac{1}{p}}\left(\sum_{j=0}^{m}\|\nabla^j\mathbf{u}\|^p_{L^p(M,T^{(0,j)}(TM)\otimes\mathbf{E})}\right)^{\frac{1}{p}}.\end{aligned} \tag{13.129}$$

The parenthesized expression is the norm of $W^{m,p}(M,\mathbf{E})$. The preceding comparisons prove equality of spaces and (13.127).

In the Hermitian case, the preceding estimates are interpreted on the underlying real bundle when the Bellman function is applied. The connection, potentials $\mathbf{V}_\ell$, operators $B_{\ell,p}$, and covariant derivatives commute with the complex structure, and the norm of the underlying real bundle agrees with the Hermitian norm. Theorem 13.3.21 also proves that the extensions obtained are complex-linear. Thus, the preceding comparisons retain the same constants in the complex case. □

To interpolate the scale defined through heat, we need to control powers of imaginary order. Self-adjointness resolves this on L^2. For other exponents, we can represent a section by scalar functions on the unit spheres of its fibers. This representation preserves the L^p norm up to a fixed factor and converts the covariant Laplacian into a shifted scalar Laplacian.

Lemma 13.3.26. (Imaginary powers of the covariant Laplacian) Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a finite-rank vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$, Hermitian in the complex case, and a compatible connection $\nabla^{\mathbf{E}}$. For $1<p<\infty$, let $L_{\mathbf{E},p}$ be the positive generator of covariant heat and set $B_p := I + L_{\mathbf{E},p}$. The spectral powers B_2^{it}, with $t\in\mathbb{R}$, extend consistently to bounded operators on $L^p(M,\mathbf{E})$. There exist $C_{p,\mathbf{E}},\omega_{p,\mathbf{E}}>0$, depending only on p, the rank, and the scalar field, such that

$$\|B_p^{it}\|_{\mathcal{L}(L^p(M,\mathbf{E}))}\le C_{p,\mathbf{E}}e^{\omega_{p,\mathbf{E}}|t|},\qquad t\in\mathbb{R}.$$

In the real case, these powers are understood on the complexification.

Proof. If the bundle is real, complexify it with the induced Hermitian metric and connection. The original norm is preserved on real sections, and the complex estimates restrict to them. It therefore suffices to treat a complex bundle with a Hermitian bundle metric. Write $r=\operatorname{rank}_{\mathbb{C}}\mathbf{E}$ and $d=2r$. Consider the manifold

$$\widehat{M}=\mathbb{S}(\mathbf{E}):=\{(x,\mathbf{v})\in\mathbf{E}\mid |\mathbf{v}|_{\mathbf{h}}=1\},\qquad \pi(x,\mathbf{v})=x.$$

The connection splits its tangent space into horizontal and vertical parts. Equip the horizontal part with the metric of M, each fiber with the unit-sphere metric, and declare the two parts orthogonal. Denote the resulting metric by $\widehat{\mathbf{g}}$.

This metric is complete. Indeed, π does not increase lengths. A Cauchy sequence in $\widehat{M}$ projects to a Cauchy sequence in M, converging to some x. Apart from finitely many terms, it lies over a compact set contained in a bundle-trivializing neighborhood of x. The inverse image of that compact set is compact because the fiber is a sphere. Thus, the original sequence has a convergent subsequence and, being Cauchy, converges. The same argument applies componentwise if the sphere bundle is disconnected.

Let $d\sigma_x$ be the invariant probability measure on the sphere of $\mathbf{E}_x$. The volume measure of $\widehat{\mathbf{g}}$, divided by the area of $\mathbb{S}^{d-1}$, is $d\lambda_{\mathbf{g}}(x)\,d\sigma_x(\mathbf{v})$; the preceding orthogonal decomposition verifies this in any local orthonormal frame. Use this normalized measure on $L^p(\widehat{M})$. Normalization does not change operator norms, since it multiplies the norms of arguments and images by the same factor. Define

$$(\mathcal{J}\mathbf{u})(x,\mathbf{v}):=\langle\mathbf{u}(x),\mathbf{v}\rangle_{\mathbf{h}}.$$

Use the Hermitian product linear in its first variable. Invariance of σ_x under fiber isometries gives

$$\|\mathcal{J}\mathbf{u}\|_{L^p(\widehat{M})}=c_{r,p}\|\mathbf{u}\|_{L^p(M,\mathbf{E})},\qquad c_{r,p}^p:=\int_{\mathbb{S}^{d-1}}|\langle\mathbf{e}_1,\mathbf{v}\rangle|^p\,d\sigma(\mathbf{v})>0. \tag{13.130}$$

Consequently, $\mathcal{J}$ has closed image. For $p=2$, we have $c_{r,2}=r^{-1/2}$.

We verify the relation between the Laplacians. If $\mathbf{X}$ is a field on M and $\mathbf{X}^H$ is its horizontal lift, the definition of parallel transport and metric compatibility imply

$$\mathbf{X}^H(\mathcal{J}\mathbf{u})=\mathcal{J}(\nabla_{\mathbf{X}}\mathbf{u}).$$

The fibers are totally geodesic for $\widehat{\mathbf{g}}$. To see this, transport two vertical fields $\mathbf{V},\mathbf{W}$ by the horizontal flow of

$\mathbf{X}^H$. This transport is an isometry of the spheres, so

$$\mathbf{X}^H\langle \mathbf{V},\mathbf{W}\rangle = \langle [\mathbf{X}^H,\mathbf{V}],\mathbf{W}\rangle + \langle \mathbf{V},[\mathbf{X}^H,\mathbf{W}]\rangle.$$

Substituting this identity into the Koszul formula gives $\langle \widehat{\nabla}_{\mathbf{V}}\mathbf{W},\mathbf{X}^H\rangle = 0$. For horizontal fields, every Koszul term paired with $\mathbf{Z}^H$ is the lift of the corresponding term on M. Hence,

$$\langle \widehat{\nabla}_{\mathbf{X}^H}\mathbf{Y}^H,\mathbf{Z}^H\rangle = \mathbf{g}(\nabla^M_{\mathbf{X}}\mathbf{Y},\mathbf{Z})\circ\pi.$$

The vertical component of the repeated derivative is computed with the same formula:

$$2\langle \widehat{\nabla}_{\mathbf{X}^H}\mathbf{X}^H,\mathbf{V}\rangle = -\mathbf{V}\big(\mathbf{g}(\mathbf{X},\mathbf{X})\circ\pi\big) - 2\langle [\mathbf{X}^H,\mathbf{V}],\mathbf{X}^H\rangle = 0.$$

The first term is zero because $\mathbf{V}$ is vertical; the second is zero because the bracket of a projectable field with a vertical field is vertical. Thus, taking the horizontal trace of the second derivative of $\mathcal{J}\mathbf{u}$ gives exactly $\mathcal{J}(\mathrm{tr}_{\mathbf{g}}\nabla^2\mathbf{u})$.

For the vertical trace, fix $\mathbf{v}$ in a fiber and a vector $\mathbf{w}$ tangent to its sphere. The spherical geodesic starting at $\mathbf{v}$ with velocity $\mathbf{w}$ satisfies $\boldsymbol{\gamma}''(0) = -|\mathbf{w}|^2\mathbf{v}$. The function $\langle \mathbf{u}(x),\boldsymbol{\gamma}(\tau)\rangle$ therefore has second derivative at zero equal to $-|\mathbf{w}|^2\langle \mathbf{u}(x),\mathbf{v}\rangle$. Take an orthonormal basis of the $d-1$ tangent vectors of the fiber and sum: the positive-sign vertical Laplacian is $(d-1)$ times $\mathcal{J}\mathbf{u}$. Adding the horizontal and vertical traces gives, for $\mathbf{u}\in\Gamma_c(\mathbf{E})$,

$$L_{\widehat{M}}\mathcal{J}\mathbf{u} = \mathcal{J}\big(L_{\mathbf{E},0}\mathbf{u} + (d-1)\mathbf{u}\big). \tag{13.131}$$

The differential identity also determines the closed realizations. Set $A := I + L_{\widehat{M}}$ and $D_2 := dI + L_{\mathbf{E},2}$. Completeness and the essential self-adjointness theorem of Chapter 12 allow each $\mathbf{u}\in\mathcal{D}(D_2)$ to be approximated by sections in $\Gamma_c(\mathbf{E})$ in the graph norm. Their images under $\mathcal{J}$ have compact support because π has compact fibers. Closedness of A and (13.131) give

$$\mathcal{J}\mathcal{D}(D_2)\subseteq\mathcal{D}(A), \qquad A\mathcal{J}\mathbf{u} = \mathcal{J}D_2\mathbf{u}.$$

If $\lambda>0$ and $\mathbf{f}\in L^2(M,\mathbf{E})$, apply this equality to $\mathbf{u} = (\lambda+D_2)^{-1}\mathbf{f}$. Uniqueness of the solution of the resolvent equation implies

$$(\lambda+A)^{-1}\mathcal{J}\mathbf{f} = \mathcal{J}(\lambda+D_2)^{-1}\mathbf{f}.$$

The resolvents are self-adjoint; the closed image of $\mathcal{J}$ and its orthogonal complement are invariant. Spectral calculus on that subspace then gives $A^{it}\mathcal{J} = \mathcal{J}D_2^{it}$ for $t\in\mathbb{R}$.

The scalar case of Theorem 13.3.21, applied to the operator $A = I + L_{\widehat{M},p}$ on the complete manifold $(\widehat{M},\widehat{\mathbf{g}})$, gives

$$\|A^{it}\|_{\mathcal{L}(L^p(\widehat{M}))}\le M_p e^{\omega_p|t|}, \qquad t\in\mathbb{R}.$$

For $\mathbf{u}\in L^2(M,\mathbf{E})\cap L^p(M,\mathbf{E})$, use spectral intertwining and (13.130); the factor $c_{r,p}$ appears on both sides and cancels. This yields

$$\|D_2^{it}\mathbf{u}\|_{L^p(M,\mathbf{E})}\le M_p e^{\omega_p|t|}\|\mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

Density of the intersection defines the extension D_p^{it} and ensures its consistency, where $D_p = dI + L_{\mathbf{E},p}$.

Finally, pass from the shift d to the shift 1. The integral representation of the resolvent and heat contrac-

tivity give $\|D_p^{-1}\| \le d^{-1}$. Set

$$K_p := (d-1)D_p^{-1}, \qquad G_p := -\sum_{j=1}^{\infty} \frac{K_p^j}{j}.$$

This series converges in operator norm, since $\|K_p\| \le (d-1)/d < 1$, and

$$\|G_p\| \le \sum_{j=1}^{\infty} \frac{((d-1)/d)^j}{j} = \log d.$$

On L^2, $G_2 = \log(I - K_2)$ and $B_2 = D_2(I - K_2)$, so $B_2^{it} = D_2^{it} e^{itG_2}$. The series are consistent on the spaces L^p, and hence

$$\|B_p^{it}\| \le M_p e^{\omega_p |t|} e^{|t| \|G_p\|} \le M_p e^{(\omega_p + \log d)|t|}.$$

We also obtain strong continuity in the parameter. If $p \neq 2$, choose $q \in (1,\infty)$ so that p lies strictly between 2 and q, and $\theta \in (0,1)$ with $1/p = \theta/2 + (1-\theta)/q$. For $\mathbf{u} \in \Gamma_c(\mathbf{E})$, Hölder gives

$$\|(B^{it} - B^{it_0})\mathbf{u}\|_{L^p(M,\mathbf{E})} \le \|(B^{it} - B^{it_0})\mathbf{u}\|_{L^2(M,\mathbf{E})}^{\theta} \|(B^{it} - B^{it_0})\mathbf{u}\|_{L^q(M,\mathbf{E})}^{1-\theta}.$$

The first factor tends to zero by spectral calculus; the second is bounded when t remains in a compact interval, by the estimate already proved with exponent q. Density of $\Gamma_c(\mathbf{E})$ in L^p and the local uniform operator bound extend convergence to every section in L^p. For $p = 2$, spectral continuity applies directly. □

Theorem 13.3.27. (Interpolation of Bessel potentials in bundles) Let $(M, \mathbf{g})$ be a complete Riemannian manifold without boundary, and let $\mathbf{E} \to M$ be a finite-rank vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$, Hermitian in the complex case, and a compatible connection $\nabla^{\mathbf{E}}$. Let $L_{\mathbf{E},p}$ be the positive generator of covariant heat and let $1 < p < \infty$. For real bundles, use complexification and restrict the interpolated spaces to real sections. If $0 \le s_0 < s_1$, $0 < \theta < 1$, and $s = (1-\theta)s_0 + \theta s_1$, then

$$[H^{s_0,p}_{L_{\mathbf{E}}}(M,\mathbf{E}), H^{s_1,p}_{L_{\mathbf{E}}}(M,\mathbf{E})]_\theta = H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E}) \tag{13.132}$$

and there exist constants $c_{s_0,s_1,\theta,p}, C_{s_0,s_1,\theta,p} > 0$ such that

$$c_{s_0,s_1,\theta,p} \|\mathbf{u}\|_{H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})} \le \|\mathbf{u}\|_{[H^{s_0,p}_{L_{\mathbf{E}}}(M,\mathbf{E}), H^{s_1,p}_{L_{\mathbf{E}}}(M,\mathbf{E})]_\theta} \le C_{s_0,s_1,\theta,p} \|\mathbf{u}\|_{H^{s,p}_{L_{\mathbf{E}}}(M,\mathbf{E})}.$$

The constants may also depend on the rank of the bundle, which is fixed.

Proof. Set

$$B := I + L_{\mathbf{E},p}, \qquad a_j := \frac{s_j}{2} \quad (j \in \{0,1\}), \qquad a := (1-\theta)a_0 + \theta a_1, \qquad \delta := a_1 - a_0.$$

For $r \ge 0$, the potential B^{-r} is injective and its range is the domain of B^r; moreover,

$$\|\mathbf{u}\|_{H^{2r,p}_{L_{\mathbf{E}}}(M,\mathbf{E})} = \|B^r \mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

It therefore suffices to interpolate the domains $\mathcal{D}(B^{a_j})$ equipped with these norms.
Lemma 13.3.26 proves that imaginary powers of B are bounded on $L^p(M,\mathbf{E})$ and that there exist $M_{p,\mathbf{E}}, \omega_{p,\mathbf{E}} >$

0 such that

$$\|B^{it}\|_{\mathcal{L}(L^p(M,\mathbf{E}))} \le M_{p,\mathbf{E}} e^{\omega_{p,\mathbf{E}}|t|}, \qquad t \in \mathbb{R}. \tag{13.133}$$

This is the analytic estimate allowing the interpolation argument of Chapter 5 to be transferred to the manifold. Use its strip $\mathcal{T} = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$.
Fix an integer $N > a_1$ and first consider $\mathbf{u} \in \mathcal{D}(B^N)$. For $\varepsilon > 0$ and $z \in \overline{\mathcal{T}}$, set $a(z) := (1-z)a_0 + za_1$ and

$$F_\varepsilon(z) := e^{\varepsilon(z-\theta)^2} B^{a-a(z)}\mathbf{u}.$$

Holomorphy is verified by writing $B^{a-a(z)}\mathbf{u} = B^{a-a(z)-N}B^N\mathbf{u}$: the exponent of the first factor has negative real part throughout the closed strip, and its Laplace integral is holomorphic in the interior. For $z = \sigma + it$, with $\sigma \in [0,1]$, it also factors into the negative real power and $B^{-it\delta}$. The former has norm at most one, and the latter satisfies (13.133). The Gaussian factor makes the family bounded on the strip and its boundary values tend to zero. We have $F_\varepsilon(\theta) = \mathbf{u}$. If $j \in \{0,1\}$ and $t \in \mathbb{R}$, then

$$B^{a_j} F_\varepsilon(j+it) = e^{\varepsilon(j+it-\theta)^2} B^{-it\delta} B^a \mathbf{u}. \tag{13.134}$$

By (13.133), the norm of the right-hand side is bounded by

$$M_{p,\mathbf{E}} \exp\bigl(\varepsilon(j-\theta)^2 - \varepsilon t^2 + \omega_{p,\mathbf{E}}\delta|t|\bigr)\|B^a\mathbf{u}\|_{L^p(M,\mathbf{E})}.$$

Continuity in each endpoint norm follows from the same factorization and strong continuity of the imaginary powers. This expression is bounded and tends to zero as $|t| \to \infty$. Thus, F_ε belongs to the Calderón space of the pair $(\mathcal{D}(B^{a_0}), \mathcal{D}(B^{a_1}))$, and

$$\|\mathbf{u}\|_{[\mathcal{D}(B^{a_0}),\mathcal{D}(B^{a_1})]_\theta} \le C\|B^a\mathbf{u}\|_{L^p(M,\mathbf{E})}. \tag{13.135}$$

Now let $\mathbf{u} \in \mathcal{D}(B^a)$ and set $\mathbf{u}_\eta = (I+\eta B)^{-N}\mathbf{u}$. Then $\mathbf{u}_\eta \in \mathcal{D}(B^N)$ and, since the resolvent commutes with B^a, we have $B^a\mathbf{u}_\eta \to B^a\mathbf{u}$ in $L^p(M,\mathbf{E})$. Estimate (13.135), applied to $\mathbf{u}_\eta - \mathbf{u}_{\eta'}$, shows that $(\mathbf{u}_\eta)$ is Cauchy in the interpolated space. Its limit agrees with $\mathbf{u}$ in $L^p(M,\mathbf{E})$ because the inclusions of both endpoint spaces in $L^p(M,\mathbf{E})$ are continuous. This proves the inclusion and estimate for every $\mathbf{u} \in \mathcal{D}(B^a)$.
We prove the reverse inclusion. Let $F \in \mathscr{F}(\mathcal{D}(B^{a_0}), \mathcal{D}(B^{a_1}))$ and write $\mathbf{u} = F(\theta)$. Regularize the values of F using operators that map them into the domains of the powers we will use. Fix an integer $N > a_1$, take $\eta > 0$, and set

$$R_\eta := (I+\eta B)^{-N}.$$

The operator R_η commutes with powers of B and maps $L^p(M,\mathbf{E})$ into the domain of B^N. Thus, the function

$$G_{\varepsilon,\eta}(z) := e^{\varepsilon(z-\theta)^2} B^{a(z)} R_\eta F(z)$$

is holomorphic with values in $L^p(M,\mathbf{E})$. On the boundary,

$$G_{\varepsilon,\eta}(j+it) = e^{\varepsilon(j+it-\theta)^2} B^{it\delta} R_\eta B^{a_j} F(j+it).$$

Uniform contractivity of R_η, the bound (13.133), and the Gaussian factor give a boundary bound indepen-

dent of η. The three-lines lemma, applied to the pair $(L^p(M,\mathbf{E}), L^p(M,\mathbf{E}))$, gives

$$\|B^a R_\eta \mathbf{u}\|_{L^p(M,\mathbf{E})} \le C\|F\|_{\mathscr{F}}. \tag{13.136}$$

As $\eta \to 0^+$, $R_\eta \mathbf{u} \to \mathbf{u}$ in $L^p(M,\mathbf{E})$. Since $1 < p < \infty$, from (13.136) we can extract a subsequence along which $B^a R_\eta \mathbf{u}$ converges weakly in $L^p(M,\mathbf{E})$. The graph of the closed operator B^a is a closed subspace and hence weakly closed. It follows that $\mathbf{u} \in \mathcal{D}(B^a)$ and

$$\|B^a \mathbf{u}\|_{L^p(M,\mathbf{E})} \le C\|F\|_{\mathscr{F}}.$$

Taking the infimum over functions F with $F(\theta) = \mathbf{u}$ proves the reverse inclusion in (13.135). The constant in that estimate and the constant in (13.135) are $c^{-1}_{s_0,s_1,\theta,p}$ and $C_{s_0,s_1,\theta,p}$, respectively.
Lemma B.4.22 allows any positive shift to be replaced by the shift 1 chosen in the definition of B. □

Theorem 13.3.28. (Uniform localization and Bessel potentials) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, let $s \in \mathbb{R}$, and let $1 < p < \infty$. If $L = \Delta_B = -\Delta$ is the positive realization of the scalar Laplacian, then, as spaces of distributions,

$$H^{s,p}(M) = F^s_{p,2}(M) = H^{s,p}_L(M).$$

More precisely, there exist constants $c_{s,p}, C_{s,p} > 0$ such that

$$c_{s,p}\|u\|_{H^{s,p}(M)} \le \|u\|_{H^{s,p}_L(M)} \le C_{s,p}\|u\|_{H^{s,p}(M)}, \tag{13.137}$$

and there exist $d_{s,p}, D_{s,p} > 0$ such that

$$d_{s,p}\|u\|_{F^s_{p,2}(M)} \le \|u\|_{H^{s,p}(M)} \le D_{s,p}\|u\|_{F^s_{p,2}(M)}. \tag{13.138}$$

The constants do not depend on u.

The interpolation scheme for this identification is that of [146, Section 7.4.5].

Proof. Begin with the identification of the two localized scales. Theorem 6.3.10 gives, for every $v \in \mathcal{S}'(\mathbb{R}^n)$,

$$d_{s,p}\|v\|_{F^s_{p,2}(\mathbb{R}^n)} \le \|v\|_{H^{s,p}(\mathbb{R}^n)} \le D_{s,p}\|v\|_{F^s_{p,2}(\mathbb{R}^n)}.$$

The constants depend only on n, s, p and are therefore the same in all charts. Apply the inequalities to $\widetilde{(h_i u)} \circ \varphi_i^{-1}$, raise to the power p, sum over i, and take the pth root. This proves (13.138); independence of the trivialization was proved earlier.
We now prove (13.137). First suppose $s > 0$ and choose an integer $m > s$. At order zero, Proposition 13.3.14 gives

$$H^{0,p}(M) = L^p(M) = H^{0,p}_L(M).$$

At order m, the same proposition and Theorem 13.3.25, applied to the trivial rank-one bundle, give

$$H^{m,p}(M) = W^{m,p}(M) = H^{m,p}_L(M) \tag{13.139}$$

and the comparisons in (13.45), (13.46), and (13.127) provide positive constants in both directions, independent of u. Set $\theta = \frac{s}{m}$. Proposition 13.3.11 gives

$$[H^{0,p}(M), H^{m,p}(M)]_\theta = H^{s,p}(M),$$

while Theorem 13.3.27 gives

$$[H_L^{0,p}(M), H_L^{m,p}(M)]_\theta = H_L^{s,p}(M).$$

The identity on $C_c^\infty(M)$ is continuous between the two pairs at the endpoints by (13.139). Theorem 5.7.5, applied to the identity and its inverse, proves equality and both estimates for $s > 0$. For $s = 0$, the assertion has already been obtained.

Finally, let $s < 0$ and write $t = -s > 0$. Applying the positive case with conjugate exponent p' and taking duals, Proposition 13.3.11 and the duality item of Theorem 12.4.9 give

$$H^{-t,p}(M) = \big(H^{t,p'}(M)\big)' = \big(H_L^{t,p'}(M)\big)' = H_L^{-t,p}(M).$$

In the real scalar case, both identifications use exactly the distributional pairing induced by $d\lambda_{\mathbf{g}}$. In the complex scalar case, the duality of Theorem 12.4.9 was written using the Hermitian product; composing the representing element with conjugation gives the bilinear distributional pairing of Proposition 13.3.11. The scalar semigroup and its potentials commute with conjugation, so this modification changes neither the space nor its norm. Thus, the equality holds in $\mathcal{D}'(M)$, not merely as an abstract Banach-space isomorphism. The norms of the isomorphisms and their duals produce the constants in (13.137).

The local comparisons and the passage to negative orders are justified by the preceding propositions. □

Corollary 13.3.29. (Agreement of the constructions at integer orders) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary and with bounded geometry, $m \in \mathbb{N}_0$, and $1 < p < \infty$. Then

$$W^{m,p}(M) = H^{m,p}(M) = F_{p,2}^m(M) = H_L^{m,p}(M)$$

and there exist constants $c_H, C_H, c_F, C_F, c_L, C_L > 0$, independent of u, such that

$$c_H\|u\|_{W^{m,p}(M)} \le \|u\|_{H^{m,p}(M)} \le C_H\|u\|_{W^{m,p}(M)},$$

$$c_F\|u\|_{W^{m,p}(M)} \le \|u\|_{F_{p,2}^m(M)} \le C_F\|u\|_{W^{m,p}(M)}, \qquad c_L\|u\|_{W^{m,p}(M)} \le \|u\|_{H_L^{m,p}(M)} \le C_L\|u\|_{W^{m,p}(M)}.$$

More generally, if $\mathbf{E} \to M$ is a bundle of bounded geometry, let $\mathbf{u}$ be a section of $\mathbf{E}$. Denote by $N_{m,p}(\mathbf{u})$ the localized component norm defined in (13.45), and by $c_{m,p}, C_{m,p}, c_{m,p}^L, C_{m,p}^L > 0$ the constants in (13.45) and (13.127). Then

$$c_{m,p}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le N_{m,p}(\mathbf{u}) \le C_{m,p}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})},$$
$$c_{m,p}^L\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le \|\mathbf{u}\|_{H_{L_{\mathbf{E}}}^{m,p}(M,\mathbf{E})} \le C_{m,p}^L\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

The constants do not depend on the section.

Proof. In the scalar case, combine Theorem 13.3.28, Proposition 13.3.14, and the Euclidean identification $H^{m,p}(\mathbb{R}^n) = F_{p,2}^m(\mathbb{R}^n)$. The vector-valued assertion follows directly by combining (13.45) and (13.127). □

Remark 13.3.30. (Uniformity and compactness) The preceding norm comparisons allow local estimates to be summed with uniform constants. When the integrability exponent changes, global embeddings also use the inclusion of outer sums $\ell^p \subseteq \ell^q$ for $p \leq q$; comparison of differential indices does not replace this condition on a manifold of infinite volume. The precise ranges are formulated in Theorem 14.6.2.

Nor does a strict gain in index imply global compactness. Already on $\mathbb{R}^n$, translates of a nonzero smooth compactly supported function preserve its Sobolev norms and can have pairwise disjoint supports; their distances in L^q remain equal to $2^{1/q}\|u\|_{L^q(\mathbb{R}^n)} > 0$ for $q < \infty$. Proposition 14.6.8 obtains compactness after multiplication by a compactly supported cutoff and for families supported in a common compact set. A global conclusion on a noncompact manifold also requires control of mass escaping to infinity, as explained in Remark 14.6.10.

14

Trace theorems in bounded geometry

A smooth function can be restricted to a submanifold without difficulty. For a Sobolev function, however, its values on a set of measure zero are not determined by its equivalence class. The trace resolves this problem when there is sufficient transverse regularity: it is defined by continuity, identifies the space receiving the restriction, and allows us to ask whether every datum in that space admits an extension.

The threshold comes from the Euclidean model. To transfer it to a noncompact manifold, we must control adapted coordinates, frames, and cutoff functions, and justify the sums reconstructing the trace and its right inverse. Fermi charts keep the submanifold visible; in the presence of a boundary, the collar also allows tangential derivatives to be separated from normal jets.

We begin with these geometric constructions and their uniform estimates. We then treat traces of functions and sections, comparison with covariant norms, and Sobolev embeddings. The exposition distinguishes restriction spaces on the half-space from extensions by zero and preserves the order of summation defining each function-space scale; see [72, 130, 6].

The boundary as analytic data

Restricting a smooth function to a submanifold seems elementary; for a Sobolev class, however, it is possible only when there is sufficient transverse regularity. The trace theorem precisely identifies what information survives at the boundary and which space receives it. On noncompact manifolds, the problem also requires collars, Fermi charts, and extensions to be constructed with uniform constants. This combination of adapted geometry and fractional regularity is one of the contemporary forms of global analysis [72, 127].

14.1 The Euclidean model and its compatibility properties

Let $n > k \geq 0$ and $c := n - k$. Identify $\mathbb{R}^k$ with $\{0\}^c \times \mathbb{R}^k \subseteq \mathbb{R}^c \times \mathbb{R}^k$, and, for a smooth function u, write

$$\operatorname{tr}_c u(x') := u(0, x'), \qquad x' \in \mathbb{R}^k.$$

When $k = 0$, $\mathbb{R}^0$ is a point with counting measure; its $H^{t,p}$, $F^t_{p,q}$, and $B^t_{p,q}$ spaces are identified with the scalar field, equipped with its absolute-value norm. This convention includes the boundary of a one-dimensional manifold.

The Euclidean result proved in Theorem 6.7.2 will be used in the following form.

Theorem 14.1.1. (Euclidean trace in arbitrary codimension) Let $1 < p < \infty$, $s > \frac{c}{p}$, and $c = n - k$. Restriction of Schwartz functions extends uniquely to a continuous surjective linear operator

$$\operatorname{tr}_c \colon H^{s,p}(\mathbb{R}^c \times \mathbb{R}^k) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(\mathbb{R}^k).$$

There exists a continuous linear operator

$$\operatorname{ex}_c \colon B^{s-\frac{c}{p}}_{p,p}(\mathbb{R}^k) \longrightarrow H^{s,p}(\mathbb{R}^n)$$

such that $\operatorname{tr}_c \operatorname{ex}_c = \mathrm{id}$. The operators can be chosen to be continuous also on the scales

$$\operatorname{tr}_c \colon F^s_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(\mathbb{R}^k), \qquad \operatorname{tr}_c \colon B^s_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-\frac{c}{p}}_{p,q}(\mathbb{R}^k),$$

for $1 \le q \le \infty$, with corresponding right extension operators.

Proof. For the last step when $k = 0$, we need evaluation at a point of $\mathbb{R}$. If $A^s_{p,q}$ denotes $F^s_{p,q}$ or $B^s_{p,q}$, Theorem 4.12.4 in dimension one gives

$$\sum_{j\in\mathbb{N}_0} \|\Delta_j u\|_\infty \le C \sum_{j\in\mathbb{N}_0} 2^{j/p} \|\Delta_j u\|_{L^p(\mathbb{R}^n)} \le C \left(\sum_{j\in\mathbb{N}_0} 2^{-j(s-1/p)} \right) \|u\|_{A^s_{p,q}}, \qquad s > 1/p.$$

In both scales, each $2^{js}\|\Delta_j u\|_{L^p(\mathbb{R}^n)}$ is bounded by the space norm. The series converges uniformly and defines a continuous representative whose evaluation at zero is a continuous trace. A fixed function $\varphi \in \mathcal{S}(\mathbb{R})$ with $\varphi(0) = 1$ gives the coretraction $a \mapsto a\varphi$, common to all orders. This also proves the Bessel case through $H^{s,p} = F^s_{p,2}$.

Write a point of $\mathbb{R}^n$ as $(z_1, \dots, z_c, x')$, with $x' \in \mathbb{R}^k$, and denote by τ_r the restriction $z_r = 0$ after eliminating the variables $z_1, \dots, z_{r-1}$. Theorem 6.7.2 and its coretraction give, for $r \in \{1, \dots, c\}$,

$$\tau_r \colon F^{s-\frac{r-1}{p}}_{p,p}(\mathbb{R}^{n-r+1}) \longrightarrow B^{s-\frac{r}{p}}_{p,p}(\mathbb{R}^{n-r}) = F^{s-\frac{r}{p}}_{p,p}(\mathbb{R}^{n-r}).$$

The equality in the last member is $B^\sigma_{p,p}(\mathbb{R}^d) = F^\sigma_{p,p}(\mathbb{R}^d)$. The condition $s > \frac{c}{p}$ implies

$$s - \frac{r-1}{p} > \frac{1}{p}, \qquad r \in \{1, \dots, c\},$$

so every operator is defined. Since $H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$, the first restriction is applied with second index 2, and its image already has second index p; the remaining $c-1$ restrictions are those above. The composition $\tau_c \cdots \tau_1$ is tr_c and satisfies

$$\|\operatorname{tr}_c u\|_{B^{s-\frac{c}{p}}_{p,p}(\mathbb{R}^k)} \le C \|u\|_{H^{s,p}(\mathbb{R}^n)}.$$

Starting from $F^s_{p,q}(\mathbb{R}^n)$, the first restriction has image $B^{s-\frac{1}{p}}_{p,p} = F^{s-\frac{1}{p}}_{p,p}$, so the preceding calculation applies again. Starting from $B^s_{p,q}(\mathbb{R}^n)$, every restriction preserves the index q and lowers the order by $\frac{1}{p}$. This gives the other two estimates.

Let e_r be the codimension-one coretractions, each inserting the variable z_r. In the construction of Theorem 6.7.2, a single family of profiles in the normal variable is fixed; thus, the same operators e_r act simultaneously on all scales in the statement. Choose those operators and define

$$\mathrm{ex}_c := e_1 e_2 \cdots e_c.$$

The composition is linear and continuous on each of the three scales. Moreover, canceling the factors from the outside inward gives

$$\mathrm{tr}_c \, \mathrm{ex}_c = \tau_c \cdots \tau_1 e_1 \cdots e_c = I.$$

Thus, tr_c is surjective and ex_c is a continuous right inverse. □

For restrictions computed in different charts to represent the same datum, the trace must commute with multiplication by cutoffs and with coordinate changes preserving the submanifold.

Lemma 14.1.2. (Compatibility properties of the Euclidean trace) Let $1 < p < \infty$, $1 \le q \le \infty$, and $s > \frac{c}{p}$. Consider any of the three continuous operators

$$\mathrm{tr}_c \colon H^{s,p}(\mathbb{R}^n) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(\mathbb{R}^k),$$
$$\mathrm{tr}_c \colon F^{s}_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(\mathbb{R}^k),$$
$$\mathrm{tr}_c \colon B^{s}_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-\frac{c}{p}}_{p,q}(\mathbb{R}^k).$$

The following properties hold in each case.

(a) Let $a \in C^\infty(\mathbb{R}^n)$. Suppose multiplication by a is continuous on the source space and multiplication by $a \restriction_{\mathbb{R}^k}$ is continuous on the target space. In the cases with fine index $q = \infty$, also assume that both multiplications remain continuous when s is replaced by $s - \varepsilon$ and the order of the target space by $s - \varepsilon - \frac{c}{p}$, for every sufficiently small $\varepsilon > 0$ with $s - \varepsilon > \frac{c}{p}$. Then

$$\mathrm{tr}_c(au) = (a \restriction_{\mathbb{R}^k}) \, \mathrm{tr}_c \, u.$$

(b) Let $\Phi \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a diffeomorphism preserving $\mathbb{R}^k$. Suppose composition with Φ and with $\Phi \restriction_{\mathbb{R}^k}$ is continuous on the source and target spaces, respectively. In the cases with fine index $q = \infty$, also assume the same continuity after replacing s by $s - \varepsilon$ and the target-space order by $s - \varepsilon - \frac{c}{p}$, for every sufficiently small $\varepsilon > 0$ with $s - \varepsilon > \frac{c}{p}$. Then

$$\mathrm{tr}_c(u \circ \Phi) = (\mathrm{tr}_c \, u) \circ (\Phi \restriction_{\mathbb{R}^k}).$$

(c) The same identities hold componentwise for functions taking values in a finite-dimensional vector space.

Proof. For $u \in \mathcal{S}(\mathbb{R}^n)$, the identities in (a) and (b) are pointwise equalities. First consider $H^{s,p}$ or one of the spaces $F^{s}_{p,q}$ and $B^{s}_{p,q}$ with $q < \infty$. In these cases, Corollary 6.3.9 gives density of $\mathcal{S}(\mathbb{R}^n)$ in the source space. Continuity of the trace, multiplication, and composition permits passage to the limit and proves the identities.

For $F^{s}_{p,\infty}$ and $B^{s}_{p,\infty}$, fix $\varepsilon \in (0, s - c/p)$ and write A for the scale under consideration. Corollary 6.3.9 provides $u_j \in \mathcal{S}(\mathbb{R}^n)$ such that

$$u_j \longrightarrow u \quad \text{in } A^{s-\varepsilon}_{p,\infty}(\mathbb{R}^n).$$

For example, first approximate in the space with fine index p and order $s - \varepsilon/2$, and then use its embedding into the displayed space. The loss of order permits this approximation even though Schwartz functions need not be dense in the original norm.

Traces of the same element taken at two admissible orders agree. Indeed, in codimension one they are defined by the same series of block restrictions in Theorem 6.7.2; its distributional limit is independent of the order used to estimate it. Composing these traces preserves compatibility in codimension c. The estimates place the limit in the target space of order $s - c/p$, even though the approximations converge only at the lower order.

Now apply the pointwise identities of the lemma to u_j. The additional continuities in (a) and (b), all at fine index ∞ and order $s - \varepsilon$, permit passage to the limit on both sides. Continuity of the trace at that order identifies their limits as distributions on $\mathbb{R}^k$. By the compatibility just proved, these limits are the traces of order s. Both sides belong to the target space of order $s - c/p$, so the equality holds there as well.

For (c), fix a basis of the value space and apply the scalar identities to each component. Since there are only finitely many components, all preceding operations and convergences are preserved. □

14.2 Pairs of bounded geometry and uniform Fermi estimates

In the Fermi charts of this section, write the c normal variables first, followed by the k tangential variables. Thus, N is represented by $\{0\}^c \times \mathbb{R}^k$, exactly as in the Euclidean model of the preceding section.

Definition 14.2.1. (Pair of bounded geometry) Let $(M, \mathbf{g})$ be a Riemannian manifold without boundary of dimension n, and let $N \subseteq M$ be an embedded submanifold without boundary, closed as a subset of M, of dimension k, with $0 \le k < n$, equipped with the induced metric. Set $c := n - k$. We say that (M, N) is a *pair of bounded geometry* if the following conditions hold.

(a) $(M, \mathbf{g})$ has bounded geometry in the sense of Definition 13.1.5.

(b) $\operatorname{inj}(N) > 0$.

(c) There exists $r_\perp > 0$ such that

$$\exp^{\perp}\colon \left\{(x, v) \in T^{\perp}N \,\middle|\, |v|_{\mathbf{g}} < r_\perp\right\} \longrightarrow M$$

is a diffeomorphism onto its image.

(d) For each $\ell \in \mathbb{N}_0$,

$$\sup_{x \in N} \left|(\nabla^{\mathbf{II}})^{\ell}\mathbf{II}(x)\right|_{\mathbf{g}} < \infty,$$

where the covariant derivatives and pointwise norm are those of Definition 2.11.5.

This is the bounded-geometry condition for the pair (M, N) in [72, Definition 18].

Remark 14.2.2. The third condition is global: it prevents normal tubes based at distinct points of N from meeting at arbitrarily small distance. A uniformly finite local version suffices for continuity of the trace; the global right-inverse construction we will use requires a uniform tubular neighborhood. This distinction corresponds to [72, Remark 33].

Lemma 14.2.3. (Gauss formula and induced geometry) Let $(M,\mathbf{g})$ be a Riemannian manifold without boundary, and let $N \subseteq M$ be an embedded submanifold without boundary, closed as a subset of M. If (M,N) is a pair of bounded geometry, then $(N,\mathbf{g}\restriction_N)$ is a manifold of bounded geometry. More precisely, for each $\ell \in \mathbb{N}_0$, there exists a constant C'_ℓ, determined by the bounds on $(\nabla^M)^j\mathbf{R}^M$ and $(\nabla^{\mathbf{II}})^j\mathbf{II}$ with $j \le \ell$, such that

$$\left|(\nabla^N)^\ell \mathbf{R}^N\right|_{\mathbf{g}\restriction_N} \le C'_\ell.$$

Proof. For fields $\mathbf{X},\mathbf{Y}$ tangent to N, the orthogonal decomposition of the ambient derivative is

$$\nabla^M_{\mathbf{X}}\mathbf{Y} = \nabla^N_{\mathbf{X}}\mathbf{Y} + \mathbf{II}(\mathbf{X},\mathbf{Y}).$$

If $\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}$ are tangent, substitute this identity into the definition of $\mathbf{R}^M(\mathbf{X},\mathbf{Y})\mathbf{Z}$, take tangential components, and pair with $\mathbf{W}$. Terms with two tangential derivatives form $\mathbf{R}^N$; normal terms are transformed using

$$\mathbf{g}(\nabla^M_{\mathbf{X}}\eta,\mathbf{W}) = -\mathbf{g}(\eta,\mathbf{II}(\mathbf{X},\mathbf{W})), \qquad \eta \in \Gamma(T^\perp N),$$

which follows by differentiating $\mathbf{g}(\eta,\mathbf{W}) = 0$. This gives the Gauss equation

$$\begin{aligned}\mathbf{g}(\mathbf{R}^N(\mathbf{X},\mathbf{Y})\mathbf{Z},\mathbf{W}) &= \mathbf{g}(\mathbf{R}^M(\mathbf{X},\mathbf{Y})\mathbf{Z},\mathbf{W}) \\ &\quad + \mathbf{g}(\mathbf{II}(\mathbf{X},\mathbf{W}),\mathbf{II}(\mathbf{Y},\mathbf{Z})) - \mathbf{g}(\mathbf{II}(\mathbf{X},\mathbf{Z}),\mathbf{II}(\mathbf{Y},\mathbf{W})).\end{aligned} \tag{14.1}$$

Apply $(\nabla^N)^\ell$ to (14.1), using compatibility with contractions from Proposition 2.4.13 and the Leibniz rule for the operator $*$ from Proposition 9.1.40. To specify the terms involved, define

$$\mathcal{I}_\ell := \left\{ (a,r,b_1,\dots,b_r) \;\middle|\; a,r \in \mathbb{N}_0,\ (b_1,\dots,b_r) \in \mathbb{N}_0^r,\ a + \sum_{j=1}^{r}(b_j+1) \le \ell \right\}.$$

The empty sum is zero when $r = 0$. We claim that $(\nabla^N)^\ell(\mathbf{R}^M\restriction_{TN})$ is a finite sum of contractions, with universal coefficients, of terms

$$\left((\nabla^M)^a\mathbf{R}^M\right)\restriction_N * (\nabla^{\mathbf{II}})^{b_1}\mathbf{II} * \cdots * (\nabla^{\mathbf{II}})^{b_r}\mathbf{II}, \qquad (a,r,b_1,\dots,b_r) \in \mathcal{I}_\ell. \tag{14.2}$$

The case $\ell = 0$ has a single factor: $a = r = 0$. Suppose the expression has been obtained at order ℓ. When a summand is differentiated, the Leibniz rule makes the connection act on exactly one factor. If it acts on the ambient curvature, the leading term replaces a by $a+1$. Comparing the ambient connection with the tangential and normal connections, the Gauss and Weingarten formulas also produce a factor $\mathbf{II}$: replace r by $r+1$ and add the exponent $b_{r+1} = 0$. If the derivative acts on factor number $j \in \{1,\dots,r\}$, it replaces b_j by b_j+1. In all three cases, the weight $a + \sum_{j=1}^{r}(b_j+1)$ increases by one. Metric contractions produce no

additional terms, since the connections preserve the metrics. Thus, all new indices belong to $\mathcal{I}_{\ell+1}$ and the number of terms still depends only on $n, \dim N, \ell+1$. This proves the expression for every $\ell \in \mathbb{N}_0$.
The derivatives of the quadratic Gauss term are

$$(\nabla^{\mathbf{II}})^a \mathbf{II} * (\nabla^{\mathbf{II}})^b \mathbf{II}, \qquad a+b=\ell.$$

The symbol $*$ denotes a universal contraction with the metric, and the number of summands depends only on n, k, ℓ. The hypotheses of Definition 14.2.1 bound every factor and give C'_ℓ. Since $\operatorname{inj}(N) > 0$ is already part of the definition, N has bounded geometry by Definition 13.1.5. □

Lemma 14.2.4. (Uniform estimates for families of differential equations) Let $P \subseteq \mathbb{R}^q$ and $V \subseteq \mathbb{R}^d$ be open, let $P_0 \Subset P$, and let Λ be an index set. Consider

$$\dot{y} = F_\lambda(t,a,y), \qquad y(0) = b_\lambda(a), \qquad a \in P_0.$$

Suppose F_λ is continuous in t, smooth in (a,y), and all its derivatives with respect to (a,y) are jointly continuous. Suppose b_λ is smooth and there exists a compact set $K \Subset V$ with the following property. For all multi-indices $\alpha \in \mathbb{N}_0^q$ and $\beta \in \mathbb{N}_0^d$, there exist constants independent of λ such that

$$\sup_{\lambda\in\Lambda} \sup_{(t,a,y)\in[0,1]\times P_0\times K} \left\| \partial_a^\alpha \partial_y^\beta F_\lambda(t,a,y) \right\| \le C_{\alpha,\beta},$$

$$\sup_{\lambda\in\Lambda} \sup_{a\in P_0} \left\| \partial_a^\alpha b_\lambda(a) \right\| \le C'_\alpha.$$

Here derivatives are ordinary partial derivatives in finite-dimensional spaces, taken componentwise. If the solutions $y_\lambda(t,a)$ exist for $0 \le t \le 1$ and remain in K, then, for each $m \in \mathbb{N}_0$, there exists $C''_m > 0$ such that

$$\sup_{\lambda\in\Lambda} \sup_{a\in P_0} \sup_{t\in[0,1]} \max_{\substack{\alpha\in\mathbb{N}_0^q \\ |\alpha|\le m}} \left\| \partial_a^\alpha y_\lambda(t,a) \right\| \le C''_m.$$

Proof. Fix λ and a parameter a in a neighborhood of P_0 where the solution exists. The integral equation is

$$y_\lambda(t,a) = b_\lambda(a) + \int_0^t F_\lambda(\tau, a, y_\lambda(\tau,a))\, d\tau.$$

On a compact neighborhood of the trajectory, $\partial_y F_\lambda$ is bounded. Subtracting the equations for two parameters and applying Grönwall first proves continuity of $a \mapsto y_\lambda(\cdot,a)$ in $C([0,1])$. For a coordinate direction, set $Z_h(t) := (y_\lambda(t, a+he_\nu) - y_\lambda(t,a))/h$. The fundamental theorem of calculus in the variables (a,y) gives

$$Z_h(t) = \frac{b_\lambda(a+he_\nu) - b_\lambda(a)}{h} + \int_0^t \big(A_h(\tau) Z_h(\tau) + f_h(\tau) \big)\, d\tau,$$

where A_h and f_h are averages of $\partial_y F_\lambda$ and $\partial_{a_\nu} F_\lambda$ along the segment joining the two arguments. By joint continuity, they converge uniformly to $\partial_y F_\lambda(\tau,a,y_\lambda)$ and $\partial_{a_\nu} F_\lambda(\tau,a,y_\lambda)$. Applying Grönwall again to the difference between this equation and its limiting equation proves uniform convergence of Z_h. Thus, $Z_\nu =$

$\partial_{a_\nu} y_\lambda$ exists and satisfies the variational equation

$$\dot{Z}_\nu = (\partial_y F_\lambda)(t,a,y_\lambda) Z_\nu + \partial_{a_\nu} F_\lambda(t,a,y_\lambda), \qquad Z_\nu(0,a) = \partial_{a_\nu} b_\lambda(a).$$

The hypotheses and Grönwall's Lemma 1.8.19 bound Z_ν uniformly, simultaneously for $\nu \in \{1,\dots,q\}$.
The same difference-quotient argument, applied to the variational equations already obtained, proves inductively the existence and continuity of all parameter derivatives. For their uniform bounds, suppose derivatives of order less than $m \geq 2$ have already been bounded. Fix a multi-index $\gamma \in \mathbb{N}_0^q$ with $|\gamma| = m$. Apply ∂_a^γ to the identity

$$\partial_t y_\lambda(t,a) = F_\lambda(t,a,y_\lambda(t,a)).$$

The multivariable Faà di Bruno formula of Theorem 1.1.19, applied componentwise to the function of the variables (a,y) and the mapping $a \mapsto (a, y_\lambda(t,a))$, separates exactly the term in which a derivative with respect to y falls on $\partial_a^\gamma y_\lambda$. This gives

$$\partial_t(\partial_a^\gamma y_\lambda) = (\partial_y F_\lambda)(t,a,y_\lambda)\partial_a^\gamma y_\lambda + Q_{\lambda,\gamma}(t,a),$$

$$\partial_a^\gamma y_\lambda(0,a) = \partial_a^\gamma b_\lambda(a).$$

Each summand of $Q_{\lambda,\gamma}$ is a product of a derivative

$$\partial_a^\alpha \partial_y^\beta F_\lambda(t,a,y_\lambda(t,a))$$

and derivatives $\partial_a^{\gamma_j} y_\lambda$ with $|\gamma_j| < m$. The number of summands depends only on m,q,d. The hypotheses and induction therefore provide a constant K_m such that

$$\|Q_{\lambda,\gamma}(t,a)\| \leq K_m$$

for all parameters considered. Another application of Grönwall's Lemma 1.8.19 bounds $\partial_a^\gamma y_\lambda$ uniformly in $[0,1]$. Since there are only finitely many multi-indices of order at most m, induction proves the estimate in the statement. □

Remark 14.2.5. Lemma 1.8.20 treats a single autonomous field and gives a more precise estimate along each trajectory. The preceding lemma allows time dependence, external parameters, and variable initial data. In particular, dependence on initial data is recovered by taking F_λ independent of a and $b_\lambda(a) = a$.

Proposition 14.2.6. (Uniform Fermi atlas and partition) Let (M,N) be a pair of bounded geometry in the sense of Definition 14.2.1; in particular, M and N are smooth manifolds without boundary of dimensions n and k, respectively. There exist $r_{\mathrm{F}} > 0$, a countable family of points $(p_i)_{i\in I_N} \subseteq N$, Fermi charts

$$\kappa_i \colon B^c_{\mathrm{euc}}(0,2r_{\mathrm{F}}) \times B^k_{\mathrm{euc}}(0,2r_{\mathrm{F}}) \longrightarrow U_i \subseteq M, \qquad i \in I_N,$$

geodesic charts $(\kappa_j)_{j\in I_0}$ whose images cover the complement of a smaller tube and satisfy $U_j \cap N = \varnothing$, and functions $h_\alpha \in C_c^\infty(U_\alpha)$, $\alpha \in I := I_N \sqcup I_0$, with the following properties.

(a) The cover $(U_\alpha)_{\alpha\in I}$ is uniformly locally finite in the sense of Definition 13.1.13.

(b) $\sum_{\alpha\in I} h_\alpha^2 = 1$ on M, and, for every multi-index β, there exists C_β such that

$$\sup_{\alpha\in I}\left\|D^\beta(h_\alpha\circ\kappa_\alpha)\right\|_{L^\infty(\operatorname{dom}\kappa_\alpha)} \le C_\beta.$$

(c) If $i \in I_N$, then

$$\kappa_i(z,u) = \exp^\perp\left(\exp_{p_i}^N(\lambda_i u), \sum_{a=1}^{c} z^a \boldsymbol{\nu}_{i,a}(\exp_{p_i}^N(\lambda_i u))\right), \tag{14.3}$$

where $\lambda_i\colon \mathbb{R}^k \longrightarrow T_{p_i}N$ is an isometry and $(\boldsymbol{\nu}_{i,a})_{a=1}^c$ is an orthonormal frame of the normal bundle obtained by parallel transport for the normal connection along radial geodesics of N. Moreover,

$$\kappa_i^{-1}(N\cap U_i) = \{0\}\times B^k_{\mathrm{euc}}(0, 2r_{\mathrm{F}}).$$

(d) In every chart, including Fermi charts, there exist $0 < c_0 \le C_0 < \infty$ such that

$$c_0\|\zeta\|^2 \le \sum_{a,b=1}^{n} g_{\alpha,ab}(x)\zeta^a\zeta^b \le C_0\|\zeta\|^2. \tag{14.4}$$

For each multi-index β, the functions $D^\beta g_{\alpha,ab}$ and $D^\beta \mathbf{g}_\alpha^{ab}$ are bounded by a constant independent of α.

(e) Transitions between these charts and the geodesic atlas of Theorem 13.1.22, and their inverses, have uniformly bounded derivatives of every order on the domains where they are used. Thus,

$$\mathcal{Q}_{\mathrm{F}} := (U_\alpha, \kappa_\alpha^{-1}, h_\alpha)_{\alpha\in I}$$

is an admissible quadratic localization system and $(U_\alpha, \kappa_\alpha^{-1}, h_\alpha^2)_{\alpha\in I}$ is the associated admissible trivialization.

Proof. Begin by obtaining estimates valid for any center in N. By Lemma 14.2.3, N has bounded geometry. Let $r_{0,N}$ and $r_{0,M}$ be radii for which condition (2) of Theorem 13.1.8 holds on N and M, respectively. Lemma 13.1.21 applies up to these radii. Choose

$$0 < r_{\mathrm{F}} < \min\left\{\frac{1}{8}\operatorname{inj}(N), \frac{r_\perp}{8}, \frac{1}{8}\operatorname{inj}(M), \frac{r_{0,N}}{5}, \frac{r_{0,M}}{5}\right\}.$$

For now, the index i denotes an arbitrary center $p_i \in N$, rather than a previously chosen net. The following estimates are independent of this center. Once all radius reductions have been fixed, we will choose the maximal net and verify the cover multiplicity.

To construct the normal frames, fix an orthonormal basis of $T_{p_i}^\perp N$ and parallel transport it, for the normal connection, along the radial geodesics of N starting at p_i. First specify all connections involved. Denote by ∇^M and ∇^N the Levi–Civita connections of M and N, by

$$\nabla^\perp_{\mathbf{X}}\boldsymbol{\eta} = (\nabla^M_{\mathbf{X}}\boldsymbol{\eta})^\perp$$

the normal connection, by $\nabla^{\mathbf{II}}$ the induced product connection on $T^*N \otimes T^*N \otimes T^\perp N$, and by ∇^F the connection induced by ∇^M on the pullback of TM under each variational mapping F below. Our conventions give the Gauss and Weingarten formulas

$$\nabla^M_{\mathbf{X}}\mathbf{Y} = \nabla^N_{\mathbf{X}}\mathbf{Y} + \mathbf{II}(\mathbf{X},\mathbf{Y}), \qquad \nabla^M_{\mathbf{X}}\eta = -\mathbf{A}_\eta\mathbf{X} + \nabla^\perp_{\mathbf{X}}\eta, \qquad \mathbf{g}(\mathbf{A}_\eta\mathbf{X},\mathbf{Y}) = \mathbf{g}(\mathbf{II}(\mathbf{X},\mathbf{Y}),\eta). \tag{14.5}$$

Taking the normal component in the definition of $\mathbf{R}^M(\mathbf{X},\mathbf{Y})\eta$ and using (14.5) gives the Ricci equation

$$\mathbf{g}(\mathbf{R}^\perp(\mathbf{X},\mathbf{Y})\eta,\zeta) = \mathbf{g}(\mathbf{R}^M(\mathbf{X},\mathbf{Y})\eta,\zeta) + \mathbf{g}([\mathbf{A}_\eta,\mathbf{A}_\zeta]\mathbf{X},\mathbf{Y}). \tag{14.6}$$

Indeed, the normal terms of $\nabla^M_{\mathbf{X}}(-\mathbf{A}_\eta\mathbf{Y})$ and $\nabla^M_{\mathbf{Y}}(-\mathbf{A}_\eta\mathbf{X})$ are $-\mathbf{II}(\mathbf{X},\mathbf{A}_\eta\mathbf{Y})$ and $-\mathbf{II}(\mathbf{Y},\mathbf{A}_\eta\mathbf{X})$; pairing them with ζ gives the commutator of (14.6). Derivatives of this identity are controlled by the following recurrence. At order zero, take exactly the two summands on its right-hand side. To pass from order q to $q+1$, apply the product connection to each contraction and use the exact rule

$$\nabla_{\mathbf{X}}\,\mathrm{contr}(\mathbf{T}_1\otimes\cdots\otimes\mathbf{T}_\ell) = \sum_{j=1}^{\ell}\mathrm{contr}(\mathbf{T}_1\otimes\cdots\otimes\nabla_{\mathbf{X}}\mathbf{T}_j\otimes\cdots\otimes\mathbf{T}_\ell).$$

If the derivative reaches the restriction of $(\nabla^M)^a\mathbf{R}^M$, the Gauss and Weingarten formulas produce $(\nabla^M)^{a+1}\mathbf{R}^M$ and contractions of $(\nabla^M)^a\mathbf{R}^M$ with $\mathbf{II}$. If it reaches a shape operator, the identity

$$\mathbf{g}\big((\nabla_{\mathbf{X}}\mathbf{A})_\eta\mathbf{Y},\mathbf{Z}\big) = \mathbf{g}\big((\nabla^{\mathbf{II}}_{\mathbf{X}}\mathbf{II})(\mathbf{Y},\mathbf{Z}),\eta\big)$$

—understood with the product connections on all arguments—replaces it by a derivative of $\mathbf{II}$. This rule determines a finite family at each order and proves by induction that $(\nabla^\perp)^q\mathbf{R}^\perp$ is a universal sum of contractions of $(\nabla^M)^a\mathbf{R}^M$ and $(\nabla^{\mathbf{II}})^b\mathbf{II}$, with $a,b\le q$, together with undifferentiated copies of $\mathbf{II}$ introduced by Gauss–Weingarten.

If $\omega^\perp_\mu$ and $F^\perp_{\nu\mu}$ are the matrices of the normal connection and its curvature, respectively, in a radial geodesic chart of N, radial gauge satisfies $\sum_{\mu=1}^{k} y^\mu\omega^\perp_\mu(y) = 0$ and the exact identity

$$\omega^\perp_\mu(y) = \int_0^1 s\sum_{\nu=1}^{k} y^\nu F^\perp_{\nu\mu}(sy)\,ds. \tag{14.7}$$

This follows by contracting the structure equation with the radial field and differentiating $s\omega^\perp_\mu(sy)$. For a multi-index α, the Leibniz rule and linearity of y^ν give the full formula

$$\begin{aligned}\partial^\alpha\omega^\perp_\mu(y) = {}& \int_0^1 s^{|\alpha|+1}\sum_{\nu=1}^{k} y^\nu(\partial^\alpha F^\perp_{\nu\mu})(sy)\,ds \\ & + \sum_{\substack{j\in\{1,\dots,k\}\\ \alpha_j>0}} \alpha_j\int_0^1 s^{|\alpha|}(\partial^{\alpha-e_j}F^\perp_{j\mu})(sy)\,ds.\end{aligned}$$

Control is established by simultaneous induction. At order zero, the tensor bound on $\mathbf{R}^\perp$ bounds its com-

ponents $F^{\perp}_{\nu\mu}$ in the orthonormal frame, and the radial formula bounds $\omega^{\perp}$. Suppose derivatives of $\omega^{\perp}$ of order less than q have been controlled. The component identity

$$(\nabla^{\perp}_{\lambda}F^{\perp})_{\nu\mu} = \partial_{\lambda}F^{\perp}_{\nu\mu} + [\omega^{\perp}_{\lambda}, F^{\perp}_{\nu\mu}] - \sum_{\kappa=1}^{k}\left(\Gamma^{\kappa}_{\lambda\nu}F^{\perp}_{\kappa\mu} + \Gamma^{\kappa}_{\lambda\mu}F^{\perp}_{\nu\kappa}\right)$$

and the higher-derivative expression in Lemma 9.1.28 express every ordinary derivative of $F^{\perp}$ of order q through $(\nabla^{\perp})^{a}F^{\perp}$, $a \le q$, derivatives of $\omega^{\perp}$ of order less than q, and already controlled derivatives of the Christoffel symbols of N. The preceding integral formula then bounds derivatives of $\omega^{\perp}$ of order q and closes the induction.

Frame bounds are interpreted through the connection, not through components in an arbitrary trivialization. Indeed,

$$\nabla^{\perp}_{\partial_{\mu}}\boldsymbol{\nu}_{i,a} = \sum_{b=1}^{c}(\omega^{\perp}_{\mu})^{b}{}_{a}\,\boldsymbol{\nu}_{i,b}.$$

Since the frame is orthonormal, this formula bounds its first covariant derivatives; differentiating it, the Leibniz rule and the bounds on $\partial^{\alpha}\omega^{\perp}$ bound derivatives of every order. Gauss–Weingarten converts these normal controls into controls for the ambient connection, adding only factors of $\mathbf{II}$ and its derivatives. In uniform ambient charts, the local expression for the connection finally converts these controls into component bounds. These are the initial data used in the Jacobi equation.

Now consider the normal exponential map as the mapping

$$\mathcal{F}\colon \mathcal{V}_{\perp} := \{(x,\xi) \in T^{\perp}N \mid |\xi|_{\mathbf{g}} < r_{\perp}\} \longrightarrow M, \qquad \mathcal{F}(x,\xi) = \exp^{M}_{x}(\xi).$$

By the tubular hypothesis, its restriction to $\mathcal{V}_{\perp}$ is a diffeomorphism onto its image. Set

$$x_i(u) := \exp^{N}_{p_i}(\lambda_i u), \qquad \boldsymbol{\xi}_i(z,u) := \sum_{a=1}^{c} z^{a}\boldsymbol{\nu}_{i,a}(x_i(u)),$$

and, for $0 \le t \le 1$,

$$F_i(z,u,t) := \mathcal{F}(x_i(u), t\boldsymbol{\xi}_i(z,u)). \tag{14.8}$$

Then $\kappa_i(z,u) = F_i(z,u,1)$.

Write

$$D := \nabla^{F_i}_{\partial_t}, \qquad \nabla_{\mathbf{n}_a} := \nabla^{F_i}_{\partial_{z^a}}, \qquad \nabla_{\mathbf{t}_\mu} := \nabla^{F_i}_{\partial_{u^\mu}},$$

and use the ordered alphabet

$$\mathscr{A} = \{\mathbf{n}_1, \dots, \mathbf{n}_c, \mathbf{t}_1, \dots, \mathbf{t}_k\}.$$

Let

$$\mathbf{T} := dF_i(\partial_t), \qquad \mathbf{J}_{\mathfrak{a}} := dF_i(\partial_{\mathfrak{a}}), \qquad \mathfrak{a} \in \mathscr{A}.$$

Since the domain's coordinate fields commute and ∇^{M} is torsion-free,

$$D\mathbf{T} = 0, \qquad \nabla_{\mathfrak{a}}\mathbf{T} = D\mathbf{J}_{\mathfrak{a}}, \qquad \nabla_{\mathfrak{a}}D\mathbf{Z} - D\nabla_{\mathfrak{a}}\mathbf{Z} = \mathbf{R}^{M}(\mathbf{J}_{\mathfrak{a}}, \mathbf{T})\mathbf{Z}. \tag{14.9}$$

Each $\mathbf{J}_{\mathfrak{a}}$ satisfies the Jacobi equation

$$D^2\mathbf{J}_{\mathfrak{a}} + \mathbf{R}^M(\mathbf{J}_{\mathfrak{a}}, \mathbf{T})\mathbf{T} = 0.$$

Fix words with their exact order retained. For a nonempty word $I = (\mathfrak{a}_1, \dots, \mathfrak{a}_m)$, define

$$\mathbf{Y}_I := \nabla_{\mathfrak{a}_1} \cdots \nabla_{\mathfrak{a}_{m-1}} \mathbf{J}_{\mathfrak{a}_m}. \tag{14.10}$$

The letters of I are not permuted. Introduce the operator

$$\mathcal{L}\mathbf{Z} := D^2\mathbf{Z} + \mathbf{R}^M(\mathbf{Z}, \mathbf{T})\mathbf{T}.$$

From (14.9), $D\mathbf{T} = 0$, and the covariant Leibniz rule, we obtain

$$\mathcal{L}(\nabla_{\mathfrak{a}}\mathbf{Z}) = \nabla_{\mathfrak{a}}(\mathcal{L}\mathbf{Z}) + \mathscr{C}_{\mathfrak{a}}(\mathbf{Z}), \tag{14.11}$$

where

$$\begin{aligned}\mathscr{C}_{\mathfrak{a}}(\mathbf{Z}) = &-2\mathbf{R}^M(\mathbf{J}_{\mathfrak{a}}, \mathbf{T})D\mathbf{Z} - (\nabla^M_{\mathbf{T}}\mathbf{R}^M)(\mathbf{J}_{\mathfrak{a}}, \mathbf{T})\mathbf{Z} - \mathbf{R}^M(D\mathbf{J}_{\mathfrak{a}}, \mathbf{T})\mathbf{Z} \\ &- (\nabla^M_{\mathbf{J}_{\mathfrak{a}}}\mathbf{R}^M)(\mathbf{Z}, \mathbf{T})\mathbf{T} - \mathbf{R}^M(\mathbf{Z}, D\mathbf{J}_{\mathfrak{a}})\mathbf{T} - \mathbf{R}^M(\mathbf{Z}, \mathbf{T})D\mathbf{J}_{\mathfrak{a}}.\end{aligned} \tag{14.12}$$

In particular, the sources $\mathbf{Q}_I$ are defined by the finite recurrence, compatible with the tensor spaces,

$$\mathbf{Q}_{(\mathfrak{a})} := 0, \qquad \mathbf{Q}_{(\mathfrak{a},I)} := \nabla_{\mathfrak{a}}\mathbf{Q}_I + \mathscr{C}_{\mathfrak{a}}(\mathbf{Y}_I), \qquad \mathcal{L}\mathbf{Y}_I = \mathbf{Q}_I. \tag{14.13}$$

The recurrence is closed and indexes every term. Consider formal expressions obtained by contracting, using $\mathbf{g}$, factors $(\nabla^M)^q\mathbf{R}^M$, $\mathbf{T}$, $\mathbf{Y}_J$, and $D\mathbf{Y}_J$, with a single free vector index. Define the formal derivation $\mathfrak{d}_{\mathfrak{a}}^{F_i}$ by the Leibniz rule on products and contractions and by

$$\mathfrak{d}_{\mathfrak{a}}^{F_i}\mathbf{T} = D\mathbf{J}_{\mathfrak{a}}, \qquad \mathfrak{d}_{\mathfrak{a}}^{F_i}\mathbf{Y}_J = \mathbf{Y}_{(\mathfrak{a},J)}, \qquad \mathfrak{d}_{\mathfrak{a}}^{F_i}(D\mathbf{Y}_J) = D\mathbf{Y}_{(\mathfrak{a},J)} + \mathbf{R}^M(\mathbf{J}_{\mathfrak{a}}, \mathbf{T})\mathbf{Y}_J,$$

$$\mathfrak{d}_{\mathfrak{a}}^{F_i}\big((\nabla^M)^q\mathbf{R}^M\big) = \nabla^M_{\mathbf{J}_{\mathfrak{a}}}\big((\nabla^M)^q\mathbf{R}^M\big) = (\nabla^M)^{q+1}\mathbf{R}^M(\mathbf{J}_{\mathfrak{a}}, \cdot).$$

In the last formula, $\cdot$ represents the preceding arguments in their original order, and the new argument is placed first; remaining arguments are differentiated separately by the Leibniz rule. Metric compatibility prevents additional summands when contractions are differentiated. Let $\mathfrak{C}^{F_i}_{\mathfrak{a},I}$ be the multiset of the six summands, with their coefficients, of $\mathscr{C}_{\mathfrak{a}}(\mathbf{Y}_I)$ in (14.12). Define finite multisets by

$$\mathfrak{A}^{F_i}_{(\mathfrak{a})} := \varnothing, \qquad \mathfrak{A}^{F_i}_{(\mathfrak{a},I)} := \mathfrak{d}_{\mathfrak{a}}^{F_i}\mathfrak{A}^{F_i}_I \sqcup \mathfrak{C}^{F_i}_{\mathfrak{a},I}.$$

Here differentiation of a multiset includes all summands produced by the Leibniz rule and preserves coefficients and multiplicities. Induction on (14.13) gives the exact identity

$$\mathbf{Q}_I = \sum_{\mathfrak{b} \in \mathfrak{A}^{F_i}_I} \mathfrak{b}. \tag{14.14}$$

Assign weight zero to $\mathbf{T}$ and weight $|J|$ to every occurrence of $\mathbf{Y}_J$ or $D\mathbf{Y}_J$. For $|I| = m \geq 2$, every summand of

(14.14) satisfies

$$\sum_{\text{occurrences of } \mathbf{Y}_J \text{ or } D\mathbf{Y}_J} |J| = m. \tag{14.15}$$

Moreover, it contains one or more factors $(\nabla^M)^q \mathbf{R}^M$, with $0 \le q \le m-1$; each field carries at most one derivative D, and every field in the source has length at most $m-1$. Indeed, the six summands of $\mathscr{C}_{\mathfrak{a}}(\mathbf{Y}_I)$ have weight $1+|I|$, and every rule of $\mathfrak{d}_{\mathfrak{a}}^{F_i}$ increases the weight by one, never produces $D^2\mathbf{Y}_J$, and increases the order of a curvature derivative by at most one. In particular, the equation of length m contains no field of length m outside the unknown term $\mathbf{Y}_I$.

The initial data are also determined by an exact recurrence. Let

$$\mathbf{X}_\mu := dx_i(\partial_{u^\mu}), \qquad \mathbf{H}_I := \mathbf{Y}_I|_{t=0}, \qquad \mathbf{K}_I := D\mathbf{Y}_I|_{t=0}.$$

For words of length one,

$$\mathbf{H}_{(\mathbf{n}_a)} = 0, \qquad \mathbf{K}_{(\mathbf{n}_a)} = \boldsymbol{\nu}_{i,a}(x_i(u)), \tag{14.16}$$

$$\mathbf{H}_{(\mathbf{t}_\mu)} = \mathbf{X}_\mu, \qquad \mathbf{K}_{(\mathbf{t}_\mu)} = \nabla^M_{\mathbf{X}_\mu}\boldsymbol{\xi}_i = -\mathbf{A}_{\boldsymbol{\xi}_i}\mathbf{X}_\mu + \nabla^\perp_{\mathbf{X}_\mu}\boldsymbol{\xi}_i. \tag{14.17}$$

The last equality fixes the sign and displays separately the second fundamental form and the normal connection. In our coordinates,

$$\mathbf{K}_{(\mathbf{t}_\mu)} = -\sum_{a=1}^{c} z^a \mathbf{A}_{\boldsymbol{\nu}_{i,a}}\mathbf{X}_\mu + \sum_{a=1}^{c} z^a \nabla^\perp_{\mathbf{X}_\mu}\boldsymbol{\nu}_{i,a}.$$

The second summand may be nonzero: radial gauge makes it vanish only in the radial direction of the chart of N.

Denote by $\nabla^0_{\mathfrak{a}}$ the induced connection on $t=0$: for $\mathfrak{a}=\mathbf{t}_\mu$, it is the derivative $\nabla^M_{\mathbf{X}_\mu}$, whereas for $\mathfrak{a}=\mathbf{n}_a$, it is ordinary differentiation with respect to z^a in the fixed space $T_{x_i(u)}M$. For every word I,

$$\mathbf{H}_{(\mathfrak{a},I)} = \nabla^0_{\mathfrak{a}}\mathbf{H}_I, \qquad \mathbf{K}_{(\mathfrak{a},I)} = \nabla^0_{\mathfrak{a}}\mathbf{K}_I - \mathbf{R}^M(\mathbf{H}_{(\mathfrak{a})}, \boldsymbol{\xi}_i)\mathbf{H}_I. \tag{14.18}$$

The second identity is precisely $D\nabla_{\mathfrak{a}} = \nabla_{\mathfrak{a}}D - \mathbf{R}^M(\mathbf{J}_{\mathfrak{a}}, \mathbf{T})$ evaluated at $t=0$.

Formulas (14.16)–(14.18), expanded recursively using (14.5), express all $\mathbf{H}_I, \mathbf{K}_I$ as finite universal sums of contractions of

$$(\nabla^M)^q\mathbf{R}^M, \qquad (\nabla^{\mathbf{II}})^q\mathbf{II}, \qquad (\nabla^\perp)^q\boldsymbol{\nu}_{i,a}, \qquad (\nabla^N)^q\mathbf{X}_\mu,$$

together with powers of z. Thus, ambient curvature appears in the sources $\mathbf{Q}_I$, whereas $\mathbf{II}$ and the normal connection appear precisely in the initial data. The bounded-geometry bounds for M and N, the hypotheses on $\mathbf{II}$, and (14.7) bound all these data uniformly.

We now prove the bounds by induction on word length. Since $\mathbf{T}$ is parallel and $|\mathbf{T}|_{\mathbf{g}} = |\boldsymbol{\xi}_i|_{\mathbf{g}} \le 2r_{\mathrm{F}}$, in a parallel frame along $t \mapsto F_i(z,u,t)$, equation $\mathcal{L}\mathbf{Y}_I = \mathbf{Q}_I$ is a first-order linear system for $(\mathbf{Y}_I, D\mathbf{Y}_I)$ with uniformly bounded matrix. For $m=1$, the source is zero and the data (14.16)–(14.17) are uniformly bounded. If words of length less than m are controlled, (14.13) bounds $\mathbf{Q}_I$ for $|I|=m$, and (14.18) bounds $\mathbf{H}_I, \mathbf{K}_I$. Grönwall's Lemma 1.8.19 gives, with a constant independent of i,z,u,

$$\sup_{t\in[0,1]} \left(|\mathbf{Y}_I(t)|_{\mathbf{g}} + |D\mathbf{Y}_I(t)|_{\mathbf{g}}\right) \le C_m \left(|\mathbf{H}_I|_{\mathbf{g}} + |\mathbf{K}_I|_{\mathbf{g}} + \int_0^1 |\mathbf{Q}_I(s)|_{\mathbf{g}}\, ds \right).$$

More precisely, C_m depends only on the dimension, r_{F}, the bounds on $(\nabla^M)^q\mathbf{R}^M$ for $q \le m-1$, the bounds on $(\nabla^{\mathbf{II}})^q\mathbf{II}$ and the normal connection appearing in the data of order m, and the constants already constructed for words of length less than m; it does not depend on the chart index or (z,u). This explicitly completes the induction for every tangential and normal word.

We pass from covariant to ordinary derivatives without identifying the two. Let $\mathbf{P}_1,\dots,\mathbf{P}_n$ be an orthonormal frame on $t=0$, formed from a controlled tangential frame of N and $\boldsymbol{\nu}_{i,1},\dots,\boldsymbol{\nu}_{i,c}$, and extended by $D\mathbf{P}_r = 0$. For an ordered word $I = (\mathfrak{a}_1,\dots,\mathfrak{a}_m)$, set

$$\partial_I := \partial_{\mathfrak{a}_1}\cdots\partial_{\mathfrak{a}_m}, \qquad \nabla_I := \nabla_{\mathfrak{a}_1}\cdots\nabla_{\mathfrak{a}_m}, \qquad \partial_\varnothing = \nabla_\varnothing = \mathrm{id}.$$

The first letter is the outer operator; this word convention does not change the order of the tensor factors of ∇^m. Also set

$$\mathbf{P}_{r,I} := \nabla_{\mathfrak{a}_1}\cdots\nabla_{\mathfrak{a}_m}\mathbf{P}_r.$$

These words satisfy the triangular recurrence

$$D\mathbf{P}_{r,\varnothing} = 0, \qquad D\mathbf{P}_{r,(\mathfrak{a},I)} = \nabla_{\mathfrak{a}}(D\mathbf{P}_{r,I}) - \mathbf{R}^M(\mathbf{J}_{\mathfrak{a}},\mathbf{T})\mathbf{P}_{r,I}.$$

Their data at $t=0$ are controlled by the frames of TN and $T^\perp N$; the preceding induction and Grönwall's Lemma 1.8.19 bound all $\mathbf{P}_{r,I}$. Finally, if I_S denotes the ordered subword of I whose positions belong to S, the iterated Leibniz rule gives the exact identity

$$\partial_I\langle \mathbf{Z},\mathbf{P}_r\rangle = \sum_{S\subseteq\{1,\dots,|I|\}} \left\langle \nabla_{I_S}\mathbf{Z}, \mathbf{P}_{r,I_{S^c}}\right\rangle. \tag{14.19}$$

The term with $S=\{1,\dots,|I|\}$ contains the highest-order covariant derivative; the others contain lower orders. This triangular relation proves bounds for all ordinary derivatives of the variational fields and, using the geodesic coordinates of M already controlled by Theorem 13.1.8, for all ordinary derivatives of F_i and $\kappa_i = F_i(\cdot,1)$.

On $z=0$,

$$d\kappa_i(\partial_{z^a}) = \boldsymbol{\nu}_{i,a}, \qquad d\kappa_i(\partial_{u^\mu}) = \mathbf{X}_\mu.$$

The two subspaces are orthogonal; the normal block is orthonormal, and the tangential block is the differential of a controlled geodesic chart of N. Thus, there exists $\sigma_0>0$, independent of i,u, such that the smallest singular value of $D\kappa_i(0,u)$ is at least σ_0. The uniform bound for $D^2\kappa_i$ gives

$$\|D\kappa_i(z,u) - D\kappa_i(0,u)\| \le C|z|.$$

Decreasing r_{F} once, impose $2Cr_{\mathrm{F}} \le \frac{\sigma_0}{2}$; then

$$\frac{\sigma_0}{2}|v| \le |D\kappa_i(z,u)v| \le C|v|$$

on all domains.

Bounds for the inverses follow by an explicit induction. If ρ is a coordinate representation of κ_i and $\lambda = \rho^{-1}$, order one is already controlled by

$$D\lambda(y) = D\rho(\lambda(y))^{-1}.$$

Suppose derivatives of λ of orders less than m are uniformly bounded, and fix a multi-index α with $|\alpha| = m \geq 2$. Apply the Faà di Bruno formula of Theorem 1.1.19 componentwise to the identity $\rho \circ \lambda = \mathrm{id}$. For each component a, we obtain

$$0 = D^\alpha(\rho^a \circ \lambda)(y) = \sum_{b=1}^{n} (\partial_b \rho^a)(\lambda(y)) D^\alpha \lambda^b(y) + R^a_\alpha(y),$$

where R^a_α is a finite sum of products of the form

$$c\,(D^\beta \rho^a)(\lambda(y)) \prod_{\ell=1}^{|\beta|} D^{\gamma_\ell} \lambda^{j_\ell}(y), \qquad 2 \leq |\beta| \leq m, \quad \sum_{\ell=1}^{|\beta|} \gamma_\ell = \alpha, \quad |\gamma_\ell| > 0.$$

Since $|\beta| \geq 2$, each $|\gamma_\ell| \leq m-1$; by the induction hypothesis, all factors containing derivatives of λ are uniformly bounded. The derivative bounds already obtained for ρ then bound $R_\alpha = (R^a_\alpha)_{a=1}^n$. In matrix form, the preceding equality is

$$D\rho(\lambda(y)) D^\alpha \lambda(y) = -R_\alpha(y),$$

and hence

$$D^\alpha \lambda(y) = -D\rho(\lambda(y))^{-1} R_\alpha(y).$$

The uniform bound for $D\rho^{-1}$ gives the bound of order m and completes the induction.

On the overlap of two Fermi charts, uniqueness of the normal-exponential representation gives a more precise formula than a mere composition. If

$$\tau_{ji} := x_j^{-1} \circ x_i, \qquad \boldsymbol{\nu}_{i,a}(x_i(u)) = \sum_{b=1}^{c} O_{ji}(u)^b{}_a \, \boldsymbol{\nu}_{j,b}(x_j(\tau_{ji}(u))),$$

then $O_{ji}(u) \in O(c)$ and

$$\kappa_j^{-1} \circ \kappa_i(z,u) = \big(O_{ji}(u) z, \tau_{ji}(u)\big). \tag{14.20}$$

In particular, the transition preserves the zero section and normal fibers and is linear in the normal variable. Moreover,

$$O_{ji}(u)^b{}_a = \mathbf{g}\big(\boldsymbol{\nu}_{j,b}, \boldsymbol{\nu}_{i,a}\big)_{x_i(u)},$$

so (14.19) and the bounds for both frames control all derivatives of O_{ji}; its inverse is its transpose. Transitions between Fermi charts and geodesic charts of M, and their inverses, are controlled by the Faà di Bruno formula of Theorem 1.1.19 and the preceding induction.

The Fermi metric coefficients are

$$g_{AB} = \mathbf{g}(\mathbf{J}_A, \mathbf{J}_B)\big|_{t=1}.$$

The Leibniz rule and (14.19) bound all their derivatives. The two-sided bound for $D\kappa_i$ proves (14.4). Differentiating $G^{-1}G = I$ gives, for $\alpha \neq 0$,

$$(\partial^\alpha G^{-1})G = - \sum_{\substack{\beta \in \mathbb{N}_0^n \\ \beta \leq \alpha,\ \beta \neq \alpha}} \binom{\alpha}{\beta} (\partial^\beta G^{-1})(\partial^{\alpha-\beta} G),$$

which closes the inverse-metric bounds by induction. Likewise,

$$0 < c \le |\det D\kappa_i| \le C, \qquad 0 < c \le \sqrt{\det(\mathbf{g})} \le C,$$

and all derivatives of both Jacobians are bounded: the determinant is polynomial in the entries, and the square root is differentiated on a compact interval contained in $(0,\infty)$.

Now fix $r := r_{\mathrm{F}}$ after all preceding reductions; take it also small enough that $20r < r_{0,M}$ and $10r < r_{0,N}$. Choose a maximal family of centers $p_i \in N$ whose intrinsic balls $B_N(p_i, r/2)$ are disjoint. The balls $B_N(p_i, r)$ cover N: otherwise, another disjoint ball could be added. The family is countable because N is second countable. Its multiplicity, even for balls of radius $4r$, is bounded by comparing volumes of the disjoint balls with the volume of a ball of radius $5r$.

Denote by $U_r(N)$ the image of normal vectors of length less than r. Also choose a maximal ρ-separated net in $M \setminus U_r(N)$, where $0 < \rho < r/8$. Its balls of radius ρ cover this complement, and we use balls of radius 2ρ as charts. These do not meet N. Indeed, since N is closed and M complete, every point at distance less than r from N has a minimizing segment to N orthogonal to N and belongs to $U_r(N)$. The Fermi charts cover $U_{2r}(N)$; consequently, the two families cover M and leave an overlap band.

To compare the two families, count overlaps through volumes of tubular sets over N in the ambient manifold. Use the disjoint sets

$$D_i := \exp^{\perp}\left(\{(x,v) \mid x \in B_N(p_i, r/2),\ |v| < r/2\}\right).$$

Injectivity of the tube makes them disjoint. The Fermi Jacobian bounds and volume bounds on N give, with a uniform constant,

$$\lambda_{\mathbf{g}}(D_i) \ge c\, r^c\, \lambda_{\mathbf{g}|_N}(B_N(p_i, r/2)) \ge c' r^n.$$

If a Fermi chart meets an ambient ball $B_M(x, 2\rho)$, its center is at distance less than $4r + 2\rho$ from x and $D_i \subset B_M(x, 6r)$. The volume of this last ball does not exceed Cr^n. By disjointness, the number of such Fermi charts is at most C/c'. In the reverse direction, the disjoint balls of radius $\rho/2$ based at interior centers whose charts meet a Fermi chart lie in an ambient ball of radius $7r$; each has volume at least $c\rho^n$. This bounds their number by $C(r/\rho)^n$, a fixed constant. The same reasoning compares the Fermi family with any uniform geodesic net at the fixed scale. Between Fermi charts, the normal footpoint of an overlap point lies in both balls of N, so their centers are at distance less than $4r$ in N. Between interior charts, use geodesic packing. This proves uniform local finiteness and both cross-overlap bounds.

Finally, choose model functions

$$\eta_\perp \in C_c^\infty(B^c(0, 2r_{\mathrm{F}})), \qquad \eta_N \in C_c^\infty(B^k(0, 2r_{\mathrm{F}})),$$

equal to one on the balls of radius r_{F}, and use $\eta_\perp(z)\eta_N(u)$ in Fermi charts; in charts away from the submanifold, use a fixed geodesic cutoff. Let $\widetilde{\eta}_\alpha$ be their transports to M. The covering property and uniform multiplicity give

$$1 \le S := \sum_{\alpha \in I} \widetilde{\eta}_\alpha^2 \le C.$$

Only uniformly finitely many summands contribute in any chart. The transition bounds and the Faà di Bruno formula of Theorem 1.1.19 control all derivatives of S and $S^{-\frac{1}{2}}$. Thus,

$$h_\alpha := \widetilde{\eta}_\alpha S^{-\frac{1}{2}}$$

satisfies $\sum_{\alpha\in I} h_\alpha^2 = 1$ and all estimates in (b). To construct auxiliary cutoffs without losing uniformity, choose once and for all functions

$$\widehat{\eta}_\perp \in C_c^\infty(B^c(0,2r_{\mathrm{F}})), \qquad \widehat{\eta}_N \in C_c^\infty(B^k(0,2r_{\mathrm{F}})),$$

with values in $[0,1]$, equal to one on neighborhoods of $\operatorname{supp}\eta_\perp$ and $\operatorname{supp}\eta_N$, respectively. In a Fermi chart, transport the product $\widehat{\eta}_\perp(z)\widehat{\eta}_N(u)$; in the other charts, likewise use a single geodesic profile equal to one near the support of the profile used for $\widetilde{\eta}_\alpha$. The resulting functions χ_α belong to $C_c^\infty(U_\alpha)$, satisfy $0 \le \chi_\alpha \le 1$, and equal one on a neighborhood of $\operatorname{supp} h_\alpha$, since $\operatorname{supp} h_\alpha \subseteq \operatorname{supp}\widetilde{\eta}_\alpha$. Because only these fixed profiles were used, their coordinate derivatives have bounds independent of α; these are exactly the estimates in (B3).

We now have adapted charts, frames, and cutoffs with uniform constants. In the next section, we can apply the Euclidean trace and sum its estimates over this cover. □

14.3 Trace on an interior submanifold

Let $(U_\alpha, \kappa_\alpha^{-1}, h_\alpha)_{\alpha\in I}$ be the quadratic localization system of Proposition 14.2.6. For $i \in I_N$, write

$$\kappa_i^N(u) := \kappa_i(0,u), \qquad U_i^N := U_i \cap N, \qquad h_i^N := h_i \restriction_N .$$

The charts κ_i^N are geodesic charts of N, charts indexed by I_0 do not meet N, and

$$\sum_{i\in I_N} (h_i^N)^2 = 1 \qquad \text{on } N. \tag{14.21}$$

By Theorems 13.3.5 and 13.3.8, together with Lemma 13.3.10, for each $s,t \in \mathbb{R}$ and $1 < p < \infty$ there exist constants $c_M, C_M, c_N, C_N > 0$ such that, for every $u \in H^{s,p}(M)$ and every $v \in B^t_{p,p}(N)$,

$$c_M \|u\|_{H^{s,p}(M)} \le \left(\sum_{\alpha\in I} \|(h_\alpha u)\circ\kappa_\alpha\|^p_{H^{s,p}(\mathbb{R}^n)} \right)^{\frac{1}{p}} \le C_M \|u\|_{H^{s,p}(M)}$$

and

$$c_N \|v\|_{B^t_{p,p}(N)} \le \left(\sum_{i\in I_N} \|(h_i^N v)\circ\kappa_i^N\|^p_{B^t_{p,p}(\mathbb{R}^k)} \right)^{\frac{1}{p}} \le C_N \|v\|_{B^t_{p,p}(N)}.$$

The four constants depend on s,t,p and the uniform data of the pair (M,N), but not on u, v, or the chart indices.

Theorem 14.3.1. (Bessel trace on a submanifold) Let (M,N) be a pair of bounded geometry in the sense of Definition 14.2.1; in particular, M and N are smooth manifolds without boundary of dimensions n and k, respectively. Let $c = n-k$, $1 < p < \infty$, and $s > \frac{c}{p}$. There exists a unique continuous linear operator

$$\mathrm{Tr}_N : H^{s,p}(M) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(N)$$

agreeing with pointwise restriction on $C_c^\infty(M)$. It is surjective and admits a continuous linear operator

$$\mathrm{Ex}_{M,N} : B_{p,p}^{s-\frac{c}{p}}(N) \longrightarrow H^{s,p}(M)$$

such that

$$\mathrm{Tr}_N \,\mathrm{Ex}_{M,N} = \mathrm{id}_{B_{p,p}^{s-\frac{c}{p}}(N)}.$$

Proof. Set $t := s - \frac{c}{p}$. For $u \in H^{s,p}(M)$, define the localizations

$$u_i := \big(h_i u\big) \circ \kappa_i, \qquad i \in I_N,$$

regarded as distributions on $\mathbb{R}^n$ by extension by zero outside the coordinate domain. This is legitimate because $\mathrm{supp}(h_i) \Subset U_i$. By admissibility and the localization principle,

$$\sum_{i \in I_N} \|u_i\|_{H^{s,p}(\mathbb{R}^n)}^p \le C \|u\|_{H^{s,p}(M)}^p. \tag{14.22}$$

Define

$$\mathrm{Tr}_N u := \sum_{i \in I_N} h_i^N \left[\big(\mathrm{tr}_c\, u_i\big) \circ (\kappa_i^N)^{-1} \right]. \tag{14.23}$$

The sum is locally finite. Moreover, each $\mathrm{tr}_c\, u_i$ belongs to $B_{p,p}^t(\mathbb{R}^k)$ by Theorem 14.1.1.
We prove the global estimate. For $j \in I_N$, let

$$A(j) := \left\{ i \in I_N \,\middle|\, U_i^N \cap U_j^N \neq \varnothing \right\}.$$

The cardinality of $A(j)$ is bounded by the multiplicity L. Localizing (14.23) with h_j^N and transporting by κ_j^N to $\mathbb{R}^k$ gives

$$\big(h_j^N \,\mathrm{Tr}_N u\big) \circ \kappa_j^N = \sum_{i \in A(j)} a_{ij} \left[\big(\mathrm{tr}_c\, u_i\big) \circ \Phi_{ij} \right], \tag{14.24}$$

where

$$a_{ij} := (h_j^N h_i^N) \circ \kappa_j^N, \qquad \Phi_{ij} := (\kappa_i^N)^{-1} \circ \kappa_j^N$$

on the overlap, after inserting cutoffs equal to one on the supports involved. Proposition 14.2.6 uniformly bounds all derivatives of a_{ij}, Φ_{ij}, and Φ_{ij}^{-1}. The localized version of Lemma 6.4.3, applied on $B_{p,p}^t$, and Theorem 14.1.1 give

$$\left\| \big(h_j^N \,\mathrm{Tr}_N u\big) \circ \kappa_j^N \right\|_{B_{p,p}^t(\mathbb{R}^k)}^p \le C L^{p-1} \sum_{i \in A(j)} \|u_i\|_{H^{s,p}(\mathbb{R}^n)}^p. \tag{14.25}$$

Sum over j. Each i appears in at most L sets $A(j)$; by (14.22),

$$\|\mathrm{Tr}_N u\|_{B_{p,p}^t(N)}^p \le C \sum_{i \in I_N} \|u_i\|_{H^{s,p}(\mathbb{R}^n)}^p \le C' \|u\|_{H^{s,p}(M)}^p. \tag{14.26}$$

Thus, Tr_N is well defined and continuous. If $u \in C_c^\infty(M)$, Lemma 14.1.2 and (14.21) show that

$$\operatorname{Tr}_N u = \sum_{i \in I_N} (h_i^N)^2 u \restriction_N = u \restriction_N .$$

Density of $C_c^\infty(M)$ in $H^{s,p}(M)$, obtained through the trivialization and Proposition 6.1.3, proves uniqueness. We now construct a right inverse. Choose a fixed function

$$\psi \in C_c^\infty\left(B^c_{\mathrm{euc}}(0, 2r_F) \times B^k_{\mathrm{euc}}(0, 2r_F)\right)$$

equal to one on a common neighborhood of all supports $(h_i \circ \kappa_i)$; this is possible because these supports were obtained from the same rescaled cutoff function. For $f \in B^t_{p,p}(N)$, set

$$f_i := (h_i^N f) \circ \kappa_i^N$$

and define

$$\operatorname{Ex}_{M,N} f := \sum_{i \in I_N} h_i \left[\left(\psi \operatorname{ex}_c f_i \right) \circ \kappa_i^{-1} \right], \tag{14.27}$$

extending each summand by zero outside U_i. The support of ψ makes this extension smooth at the test-function level and, by duality, well defined at the distributional level.

The continuity proof repeats (14.24)–(14.26) with the spaces interchanged. Uniform continuity of multipliers and coordinate changes, the multiplicity L, and continuity of ex_c give

$$\begin{aligned} \| \operatorname{Ex}_{M,N} f \|^p_{H^{s,p}(M)} &\le C \sum_{i \in I_N} \| \psi \operatorname{ex}_c f_i \|^p_{H^{s,p}(\mathbb{R}^n)} \\ &\le C' \sum_{i \in I_N} \| f_i \|^p_{B^t_{p,p}(\mathbb{R}^k)} \le C'' \| f \|^p_{B^t_{p,p}(N)}. \end{aligned} \tag{14.28}$$

Finally, $\psi(0,u) = 1$ on the relevant supports, and $\operatorname{tr}_c \operatorname{ex}_c = \mathrm{id}$. Applying (14.23) to (14.27) and using (14.21) again gives

$$\operatorname{Tr}_N \operatorname{Ex}_{M,N} f = \sum_{i \in I_N} (h_i^N)^2 f = f.$$

This proves both surjectivity and existence of the right inverse. □

Theorem 14.3.2. (Traces in the Besov and Triebel–Lizorkin scales) Let (M,N) be a pair of bounded geometry in the sense of Definition 14.2.1; in particular, M and N are smooth manifolds without boundary of dimensions n and k, respectively. Let $c = n - k$, $1 < p < \infty$, $1 \le q \le \infty$, and $s > \frac{c}{p}$. Then there exist retractions

$$\operatorname{Tr}_N \colon F^s_{p,q}(M) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(N), \qquad \operatorname{Tr}_N \colon B^s_{p,q}(M) \longrightarrow B^{s-\frac{c}{p}}_{p,q}(N),$$

both with continuous linear right extension operators.

Proof. Set $t = s - \frac{c}{p}$. Retain the cutoffs and Fermi charts from the proof of Theorem 14.3.1. For $u \in F^s_{p,q}(M)$, define directly, in the distributional sense, the components

$$u_i = (h_i u) \circ \kappa_i, \qquad i \in I_N.$$

Formula (14.23) defines the trace. If $A(j)$ is the set of indices whose charts meet chart j of N, the chart-change identities of Lemma 14.1.2, the uniform multipliers, and the Euclidean estimate give

$$\left\|(h_j^N \operatorname{Tr}_N u)\circ \kappa_j^N\right\|_{B^t_{p,p}(\mathbb{R}^k)} \le C \sum_{i\in A(j)} \|u_i\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

The cardinality of $A(j)$ is at most the uniform multiplicity L. Raising to the power p, summing over j, and observing that each i appears in at most L sets $A(j)$ gives

$$\|\operatorname{Tr}_N u\|^p_{B^t_{p,p}(N)} \le C \sum_{i\in I_N} \|u_i\|^p_{F^s_{p,q}(\mathbb{R}^n)} \le C'\|u\|^p_{F^s_{p,q}(M)}.$$

The strong localization principle, Theorem 6.4.4, justifies the last inequality and convergence of the reconstruction.

For $f \in B^t_{p,p}(N)$, set $f_i = (h_i^N f)\circ \kappa_i^N$ and use (14.27). The synthesis estimate, uniform multiplicity, and Euclidean coretraction give

$$\begin{aligned}\|\operatorname{Ex}_{M,N} f\|^p_{F^s_{p,q}(M)} &\le C \sum_{i\in I_N} \|\psi \operatorname{ex}_c f_i\|^p_{F^s_{p,q}(\mathbb{R}^n)}\\ &\le C' \sum_{i\in I_N} \|f_i\|^p_{B^t_{p,p}(\mathbb{R}^k)} \le C''\|f\|^p_{B^t_{p,p}(N)}.\end{aligned}$$

This yields the retraction and coretraction in the Triebel–Lizorkin scale.

For Besov, choose $s_0 < s < s_1$, still with $s_0 > \frac{c}{p}$, and $s = (1-\theta)s_0+\theta s_1$. The preceding operators, constructed using the same charts, cutoffs, and Euclidean coretractions, are continuous at both endpoints. Theorem 5.3.1 and the identities

$$(F^{s_0}_{p,p}(M), F^{s_1}_{p,p}(M))_{\theta,q} = B^s_{p,q}(M),$$

$$(B^{s_0-\frac{c}{p}}_{p,p}(N), B^{s_1-\frac{c}{p}}_{p,p}(N))_{\theta,q} = B^{s-\frac{c}{p}}_{p,q}(N)$$

give continuity of both Tr_N and $\operatorname{Ex}_{M,N}$ in the Besov scale.

Finally, in all scales, compatibility of the trace with multipliers and $\operatorname{tr}_c \operatorname{ex}_c = I$ allows the direct calculation

$$\operatorname{Tr}_N \operatorname{Ex}_{M,N} f = \sum_{i\in I_N} (h_i^N)^2 f = f.$$

Equality in distributions also justifies the case $q = \infty$, where density in the space norm has not been assumed. □

For noninteger orders $s = m + \sigma$, with $0 < \sigma < 1$, we will use the norm obtained by localizing Euclidean Slobodeckij norms and summing their pth powers. Proposition 14.5.4 proves independence of these localizations and their equivalence with the intrinsic difference norm defined through parallel transport.

Corollary 14.3.3. ($W^{s,p}$ and integer-order scales) Let (M,N) be a pair of bounded geometry consisting of smooth manifolds without boundary, let $c = \dim M - \dim N$, and let $1 < p < \infty$. If $s > \frac{c}{p}$ and $s \notin \mathbb{N}$, then

$$\operatorname{Tr}_N \colon W^{s,p}(M) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(N)$$

is a retraction. If $m \in \mathbb{N}$ and $m > \frac{c}{p}$, then

$$\mathrm{Tr}_N \colon W^{m,p}(M) \longrightarrow B_{p,p}^{m-\frac{c}{p}}(N)$$

is a retraction.

Proof. For $s \notin \mathbb{N}$, use the global comparison $W^{s,p}(M) = B^s_{p,p}(M)$ of Proposition 14.5.4 and apply the Besov retraction in the preceding theorem with $q = p$. For $m \in \mathbb{N}$, Proposition 13.3.14 gives $W^{m,p}(M) = H^{m,p}(M)$; then apply Theorem 14.3.1. □

14.4 Manifolds with boundary and bounded geometry

On a manifold with boundary, geodesic balls centered at boundary points are unavailable. The appropriate local models are half-spaces, obtained through a uniform geodesic collar.

Definition 14.4.1. (Uniform collar and uniform geodesic collar) Let M be a smooth manifold with nonempty boundary. A *uniform collar* of radius $r_\partial > 0$ is a diffeomorphism

$$\mathcal{C} \colon \partial M \times [0, r_\partial) \longrightarrow U_\partial,$$

where U_∂ is an open neighborhood of ∂M in M and $\mathcal{C}(x, 0) = x$. If $(M, \mathbf{g})$ is Riemannian, $\boldsymbol{\nu}$ is the inward unit normal, and

$$\mathcal{C}(x, t) = \exp_x(t\boldsymbol{\nu}_x),$$

we say the collar is *uniform geodesic*.

Definition 14.4.2. (Bounded geometry with boundary) Let $(M, \mathbf{g})$ be a smooth Riemannian manifold of dimension n with nonempty boundary, and let $\boldsymbol{\nu}$ be the inward unit normal. We say that M has *bounded geometry with nonempty boundary* if the following conditions hold.

(a) There exists $r_\partial > 0$ such that

$$\mathcal{C} \colon \partial M \times [0, r_\partial) \longrightarrow M, \qquad \mathcal{C}(x, t) := \exp_x(t\boldsymbol{\nu}_x), \tag{14.29}$$

is a uniform geodesic collar.

(b) $\mathrm{inj}(\partial M) > 0$ for the induced metric.

(c) For each $0 < r < r_\partial$, there exists $i(r) > 0$ such that, for every

$$x \in M \setminus \mathcal{C}(\partial M \times [0, r)),$$

the mapping $\exp_x$ is defined and is a diffeomorphism from $B_{\mathbf{g}_x}(0, i(r)) \subseteq T_xM$ onto its image.

(d) For each $m \in \mathbb{N}_0$,

$$\sup_{x \in M} |(\nabla^M)^m \mathbf{R}^M(x)|_{\mathbf{g}} < \infty.$$

(e) If $\mathbf{II}$ is the second fundamental form of ∂M, then, for each $m \in \mathbb{N}_0$,

$$\sup_{x \in \partial M} \left| (\nabla^{\mathbf{II}})^m \mathbf{II}(x) \right|_{\mathbf{g}} < \infty.$$

This is the intrinsic formulation of [128, Definition 2.2]; the derivatives in the last item are those of Definition 2.11.5.

Proposition 14.4.3. (Induced geometry on the boundary) Let $(M, \mathbf{g})$ be a Riemannian manifold of bounded geometry with nonempty boundary. Then $(\partial M, \mathbf{g}\restriction_{\partial M})$ is a manifold without boundary and with bounded geometry.

Proof. The second item of the definition gives $\operatorname{inj}(\partial M) > 0$. The Gauss equation (14.1), applied to $N = \partial M$, expresses $\mathbf{R}^{\partial M}$ as the tangential restriction of $\mathbf{R}^M$ plus quadratic contractions of $\mathbf{II}$. Differentiating it ℓ times, calculation (14.2) expresses $(\nabla^{\partial M})^\ell \mathbf{R}^{\partial M}$ as a finite sum of contractions of $(\nabla^M)^a \mathbf{R}^M$ and $(\nabla^{\mathbf{II}})^b \mathbf{II}$, with $a, b \le \ell$. The fourth and fifth items bound all these terms uniformly. Definition 13.1.5 concludes the proof. □

Remark 14.4.4. Schick also proves that the preceding formulation is equivalent to uniform bounds on the metric and its inverse in boundary normal coordinates [128, Theorem 2.5]. Amann assumes bounded geometry already induced on ∂M and proves that his formulation is equivalent to Schick's; see [6, Definition 2.1 and Theorem 2.2]. In the following proofs, we will use the five items of the definition and the preceding proposition directly.

Proposition 14.4.5. (Uniform geodesic–Fermi atlas and collar) Let M be a manifold of bounded geometry with nonempty boundary. There exist fixed radii $r_B, r_I > 0$, a uniformly locally finite cover, and charts of two types:

$$\kappa_i^{\partial} \colon B^{n-1}_{\mathrm{euc}}(0, 2r_B) \times [0, 2r_B) \longrightarrow U_i^{\partial}, \qquad i \in I_{\partial}, \tag{14.30}$$

and interior geodesic charts $\kappa_j^0 \colon B^n_{\mathrm{euc}}(0, 2r_I) \longrightarrow U_j^0$, $j \in I_0$, such that

$$\kappa_i^{\partial}(u, t) = \mathcal{C}\big(\exp_{p_i}^{\partial M}(\lambda_i u), t\big).$$

In these charts, the uniform estimates (14.4) hold for the metric, its inverse, and all their derivatives. There is also a family $h_\alpha \in C_c^\infty(U_\alpha)$ such that

$$\sum_{\alpha \in I} h_\alpha^2 = 1,$$

and its local representations have uniformly bounded derivatives. The family $(U_\alpha, \kappa_\alpha^{-1}, h_\alpha)$ is an admissible quadratic localization system, and $(U_\alpha, \kappa_\alpha^{-1}, h_\alpha^2)$ is the associated admissible trivialization. Moreover, for some $0 < r_0 < r_B$,

$$\partial_t^\ell (h_i \circ \kappa_i^{\partial})(u, t) = 0, \qquad i \in I_{\partial}, \quad \ell \ge 1, \quad 0 \le t < r_0. \tag{14.31}$$

Proof. By Proposition 14.4.3, ∂M has bounded geometry. First consider boundary geodesic charts centered at an arbitrary point, with a common radius smaller than the uniform radius in Theorem 13.1.8. The collar formula gives a Fermi chart at every center. We obtain its estimates before choosing the net of centers; thus,

radius reductions will not affect any covering property already fixed.

In geodesic coordinates of ∂M, set

$$x_i(u) := \exp_{p_i}^{\partial M}(\lambda_i u)$$

and consider the scaled family

$$\widehat{F}_i(r,u,s) := \exp_{x_i(u)}^{M}(sr\,\boldsymbol{\nu}_{x_i(u)}), \qquad 0 \le s \le 1.$$

Then $\kappa_i^{\partial}(u,r) = \widehat{F}_i(r,u,1)$. Apply to this family the ordered-word induction (14.10)–(14.18), with one normal letter $\mathbf{n} = \partial_r$ and tangential letters $\mathbf{t}_\mu = \partial_{u^\mu}$. The connections are ∇^M, the Levi–Civita connection of ∂M, the normal connection of $\partial M \subset M$, and the pullback connection under $\widehat{F}_i$.

To verify the sign explicitly, if

$$\mathbf{J}_X(t) := d\mathcal{C}_{(x,t)}(X,0), \qquad \gamma_x(t) = \mathcal{C}(x,t),$$

then

$$\frac{D^2\mathbf{J}_X}{dt^2} + \mathbf{R}^M(\mathbf{J}_X,\dot\gamma_x)\dot\gamma_x = 0, \qquad \mathbf{J}_X(0) = X, \qquad \frac{D\mathbf{J}_X}{dt}(0) = \nabla^M_X\boldsymbol{\nu} = -\mathbf{A}_{\boldsymbol{\nu}}X.$$

Indeed,

$$\nabla^M_X\boldsymbol{\nu} = -\mathbf{A}_{\boldsymbol{\nu}}X + \nabla^\perp_X\boldsymbol{\nu}, \qquad \mathbf{g}(\nabla^M_X\boldsymbol{\nu},\boldsymbol{\nu}) = \frac{1}{2}X\mathbf{g}(\boldsymbol{\nu},\boldsymbol{\nu}) = 0,$$

and the normal bundle has rank one, so $\nabla^\perp_X\boldsymbol{\nu} = 0$.

For the scaled family, if $\mathbf{X}_\mu = dx_i(\partial_{u^\mu})$, the length-one data are exactly

$$\mathbf{H}_{(\mathbf{n})} = 0, \qquad \mathbf{K}_{(\mathbf{n})} = \boldsymbol{\nu},$$

and

$$\mathbf{H}_{(\mathbf{t}_\mu)} = \mathbf{X}_\mu, \qquad \mathbf{K}_{(\mathbf{t}_\mu)} = -r\mathbf{A}_{\boldsymbol{\nu}}\mathbf{X}_\mu.$$

The data for every longer word are

$$\mathbf{H}_{(\mathfrak{a},I)} = \nabla^0_{\mathfrak{a}}\mathbf{H}_I, \qquad \mathbf{K}_{(\mathfrak{a},I)} = \nabla^0_{\mathfrak{a}}\mathbf{K}_I - \mathbf{R}^M(\mathbf{H}_{(\mathfrak{a})}, r\boldsymbol{\nu})\mathbf{H}_I.$$

Thus, derivatives of $\mathbf{A}_{\boldsymbol{\nu}}$, or equivalently covariant derivatives of $\mathbf{II}$, appear in the initial data with the sign fixed above. The interior sources are exactly

$$\mathbf{Q}_{(\mathfrak{a})} = 0, \qquad \mathbf{Q}_{(\mathfrak{a},I)} = \nabla_{\mathfrak{a}}\mathbf{Q}_I + \mathscr{C}_{\mathfrak{a}}(\mathbf{Y}_I),$$

with $\mathscr{C}_{\mathfrak{a}}$ given by (14.12), which collects the commutation terms. Consequently, at order m, the induction uses only $(\nabla^M)^q\mathbf{R}^M$, $q \le m-1$, derivatives of $\mathbf{II}$ present in the data, and word fields of length less than m.

The Grönwall-based induction of Proposition 14.2.6 bounds all covariant words of $\widehat{F}_i$. At this point, retain that covariant formulation: ambient normal coordinates centered at a point of ∂M are unnecessary. Ordinary derivatives of metric coefficients are obtained by applying the Leibniz rule to inner products of variational fields, as we do next.

Throughout the collar,

$$g_{tt} = 1, \qquad g_{\mu t} = 0.$$

The first identity follows because $t \mapsto \mathcal{C}(x,t)$ is a unit-speed geodesic. For the second, if $\mathbf{J}_\mu = d\kappa_i^\partial(\partial_{u^\mu})$, then

$$\frac{d}{dt}\mathbf{g}(\mathbf{J}_\mu, \partial_t) = \mathbf{g}(\nabla_{\partial_t}\mathbf{J}_\mu, \partial_t) = \mathbf{g}(\nabla_{\mathbf{J}_\mu}\partial_t, \partial_t) = \frac{1}{2}\mathbf{J}_\mu \mathbf{g}(\partial_t, \partial_t) = 0,$$

and orthogonality holds at $t = 0$. Moreover, $g_{\mu\nu} = \mathbf{g}(\mathbf{J}_\mu, \mathbf{J}_\nu)$, so the preceding induction controls all derivatives of the metric.

At $t = 0$, the differential is the orthogonal sum of the differential of the geodesic chart of ∂M and the unit normal. To measure it in different fibers, parallel transport each Jacobi field to its initial fiber. The integral equation and bounds already obtained give

$$\left|P_{t\to 0}\mathbf{J}_X(t) - X\right| \le Ct|X|,$$

with C independent of the center. Choose $r_B > 0$ so that $8Cr_B < 1/2$, $8r_B$ lies within the original collar, and all larger tangential charts lie within the uniform radius of ∂M. Consequently, the Fermi metric is uniformly comparable to $\mathbf{g}|_{\partial M} + dt^2$ in the collar of width $8r_B$. Formula $g_{\mu\nu} = \langle \mathbf{J}_\mu, \mathbf{J}_\nu \rangle$ and the Leibniz rule bound all its derivatives. Differentiating $G^{-1}G = I$ gives the corresponding inverse bounds. In particular, there exist uniform constants $c, C > 0$ such that

$$c\, d\lambda_{\mathbf{g}|_{\partial M}}\, dt \le d\lambda_{\mathbf{g}} \le C\, d\lambda_{\mathbf{g}|_{\partial M}}\, dt.$$

The Jacobians and their derivatives are also bounded, since the determinant is polynomial in the entries and the square root is evaluated on a compact interval in $(0, \infty)$.

The normal variable is preserved exactly under boundary chart transitions. If $\tau_{ji} = x_j^{-1} \circ x_i$, uniqueness of collar coordinates gives

$$(\kappa_j^\partial)^{-1} \circ \kappa_i^\partial(u,t) = (\tau_{ji}(u), t). \tag{14.32}$$

Thus, transitions between boundary charts and their inverses have uniform estimates and preserve t exactly. In choosing r_B, also require that all its fixed multiples used below lie within the uniform boundary radius. Then choose a maximal r_B-separated net in ∂M. Its balls of radius $r_B/2$ are disjoint, those of radius r_B cover ∂M, and the Fermi charts cover $\mathcal{C}(\partial M \times [0, 2r_B))$.

For the interior part, set

$$K := M \setminus \mathcal{C}(\partial M \times [0, r_B/2)).$$

The third item of Definition 14.4.2 provides the radius $i(r_B/4)$. Choose

$$0 < r_I < \min\{r_B/16,\ i(r_B/4)/20\},$$

and, if necessary, reduce it to apply the normal-coordinate estimates obtained from curvature in Chapter 13. Take a maximal r_I-separated net in K. The interior charts are exponentials of radius $2r_I$ centered at this net. Although K has an artificial boundary, the exponentials used are those of M and are defined on full tangent balls by the preceding hypothesis. The Gauss lemma and the first-exit-time argument identify their images with metric balls of that radius. Balls of radius r_I cover K, and the larger charts do not meet ∂M.

We also justify separation from the smaller collar. In the controlled collar, $|dt|_{\mathbf{g}} = 1$. A curve crossing from $t \ge r_B/2$ to $t < r_B/4$ has length at least $r_B/4$. A curve coming from outside the collar must first cross this same

interval of values of t. It cannot leave the collar toward infinity in finite length: comparison with $\mathbf{g}|_{\partial M} + dt^2$ and completeness of ∂M prevent this. Since $2r_I < r_B/8$, all interior charts remain outside $\mathcal{C}(\partial M \times [0, r_B/4))$. The union of the two families covers M; the band $r_B/2 \le t < r_B$ lies in regions covered by both.

Multiplicity among boundary charts is estimated on ∂M, since an overlap point has the same normal footpoint. Among interior charts, compare disjoint balls of radius $r_I/2$ with a ball of radius $5r_I$, within the radius $i(r_B/4)$. For mixed overlaps, an interior chart meeting a boundary chart lies in the controlled collar, and its projection onto ∂M lies in a ball of radius Cr_B. The projection is uniformly Lipschitz by the collar metric comparison. Thus, all disjoint interior balls involved lie in a half-tube of volume at most $C' r_B^n$, and each has volume at least $c' r_I^n$. Their number is bounded by $(C'/c')(r_B/r_I)^n$. In the reverse direction, the boundary centers of charts meeting an interior chart lie in a ball of ∂M of radius $C'' r_B$; packing on ∂M bounds their number. Both nets are countable by second countability. This proves uniform local finiteness and the cross-overlap bounds.

On a mixed overlap, the connection transformation identity gives, for the transition Φ between the two charts,

$$\partial_a \partial_b \Phi^k = \Gamma^c_{ab} \partial_c \Phi^k - \widetilde{\Gamma}^k_{ij}(\Phi) \partial_a \Phi^i \partial_b \Phi^j.$$

Metric comparison bounds $D\Phi$ and its inverse. Successive differentiation of this formula bounds all higher derivatives, since the metrics and their inverses have uniform jets in both systems. Order zero is controlled by their fixed-radius domains. This also gives estimates for mixed transitions, without extending a boundary chart to a normal ball centered outside M.

Fix $0 < r_0 < r_B/4$. For the partition, choose cutoffs $\eta_i^\partial(u)$ on ∂M and a function $\chi(t)$ equal to one for $0 \le t < r_0$. In boundary charts use $\eta_i^\partial(u)\chi(t)$; interior cutoffs vanish on the smaller collar. Normalizing by

$$S^{\frac{1}{2}} = \left(\sum_{\alpha \in I} \widetilde{\eta}_\alpha^2 \right)^{\frac{1}{2}},$$

gives $h_\alpha = \widetilde{\eta}_\alpha S^{-\frac{1}{2}}$. On $0 \le t < r_0$, every summand of S contains the same factor $\chi(t)^2 = 1$, and no interior cutoffs contribute; therefore, $h_i \circ \kappa_i^\partial$ is independent of t in that region. This proves (14.31). The remaining bounds follow through the quadratic normalization used in Proposition 14.2.6. For (B3), before transporting the cutoffs choose slightly wider model functions equal to one on neighborhoods of the supports of $\eta_i^\partial(u)\chi(t)$ and the interior cutoffs, respectively. Their transports provide functions $\chi_\alpha \in C_c^\infty(U_\alpha)$ equal to one near $\operatorname{supp}(h_\alpha)$; since they come from a finite family of fixed profiles, all their coordinate derivatives are bounded by constants independent of α. □

Proposition 14.4.6. (The compact case with boundary) Every compact Riemannian manifold with smooth nonempty boundary satisfies Definition 14.4.2.

Proof. The boundary is a compact Riemannian manifold without boundary; Theorem 13.1.11 gives, in particular, $\operatorname{inj}(\partial M) > 0$. Apply Theorem 2.12.3 to ∂M in the smooth double of M, equipped with a smooth extension of $\mathbf{g}$. Compactness of ∂M provides $r_\partial > 0$ such that the normal exponential is injective on normal vectors of length less than r_∂. Its restriction to the inward normal is therefore the required uniform geodesic collar.

Fix $0 < r < r_\partial$ and set $K := M \setminus \mathcal{C}(\partial M \times [0, r))$. This set is compact and contained in the interior of M. Each

$x \in K$ has a normal ball $B_{\mathbf{g}_x}(0, r_x)$; by smooth dependence of the exponential map, the same conclusion, with radius $\frac{r_x}{2}$, holds for centers in a neighborhood of x. A finite subcover of K and the minimum of these radii give a number $i(r) > 0$ independent of the center, as required by the third item of the definition.
For each $j \in \mathbb{N}_0$, the continuous functions $x \mapsto |\nabla^j \mathbf{R}^M(x)|_{\mathbf{g}}$ and $y \mapsto |\nabla^j \mathbf{II}(y)|_{\mathbf{g}}$ attain a maximum on M and ∂M, respectively. All five items are verified. □

Corollary 14.4.7. (Uniform regularity in the presence of boundary) Let $(M, \mathbf{g})$ be a manifold of bounded geometry with nonempty boundary. The Fermi and interior atlas of Proposition 14.4.5, after fixed rescaling of its domains, defines a uniformly regular Riemannian structure in the sense of Definitions 13.2.1 and 13.2.3. Its constants depend only on the dimension, collar width, interior and boundary injectivity radii, and bounds on $\nabla^j \mathbf{R}^M$ and $\nabla^j \mathbf{II}$ through the required order. Proposition 13.2.6, applied to this normalized atlas, produces an admissible quadratic localization system with the uniform bounds established there.

Proof. Use the radii r_B, r_I of the proposition. Choose $b_B, b_I > 0$ such that $b_B\sqrt{n} < r_B$ and $b_I\sqrt{n} < r_I$. On the boundary and in the interior, respectively, define

$$\bar{\kappa}_i^{-1}(x) := \kappa_i^{\partial}\big(b_B(x_2, \dots, x_n), 2r_B x_1\big), \qquad x \in Q_+^n,$$

$$\bar{\kappa}_j^{-1}(x) := \kappa_j^0(b_I x), \qquad x \in Q^n.$$

These are two fixed linear transformations with bounded inverses; thus, they preserve all uniform bounds, although the constants may depend on b_B, b_I, r_B.
Restriction to cubes is accompanied by a new choice of centers. On ∂M, choose a net of spacing δ_B small enough that the images of tangential cubes of half the side length cover ∂M. This follows from the first-exit-time argument: the metric lower bound gives a uniform minimum length for leaving the smaller cube. The boundary half-boxes thus cover $0 \le t < r_B$. On $K = M \setminus \mathcal{C}(\partial M \times [0, r_B/2))$, choose a net of spacing δ_I less than the minimum length required to leave an interior cube of half the side length. Its half-sized boxes cover K. The interior radii remain less than r_I, so the charts do not meet $t = 0$. These two families cover M after shrinking by the same factor 1/2. The volume arguments of the proposition, applied at the new fixed scales, preserve uniform multiplicity.
The proof of the cited proposition gives, for each $j \in \mathbb{N}_0$, a constant $C_{\mathrm{tr}}(j) > 0$, independent of the charts, bounding all coordinate transitions in BUC^j. It also gives constants $c_g, C_g > 0$ such that

$$c_g|\xi|^2 \le g_\kappa(x)(\xi, \xi) \le C_g|\xi|^2$$

for every chart, every point of its normalized domain, and every $\xi \in \mathbb{R}^n$. For each j, it also provides constants $G_j, G_j' > 0$, independent of the chart, satisfying

$$Q_\alpha := \bar{\kappa}_\alpha(U_\alpha) = \begin{cases} Q_+^n, & \alpha \in I_\partial, \\ Q^n, & \alpha \in I_0, \end{cases} \qquad I := I_\partial \sqcup I_0,$$

and

$$\|(g_{\bar{\kappa}_\alpha})_{ab}\|_{BUC^j(Q_\alpha)} \le G_j, \qquad \|(g_{\bar{\kappa}_\alpha}^{-1})^{ab}\|_{BUC^j(Q_\alpha)} \le G_j', \qquad \alpha \in I.$$

Uniform continuity of derivatives of order j follows by applying the mean value theorem to the bounds of

order $j+1$ in the slightly larger charts used in that proof. Now apply Proposition 13.2.6 to the normalized atlas. This gives cutoffs $\pi_\alpha \in C_c^\infty(U_\alpha,[0,1])$ and a fixed coordinate cutoff $\widehat{\chi}$ such that

$$\sum_{\alpha\in I}\pi_\alpha^2 = 1, \qquad \widehat{\chi}=1 \quad \text{en } \operatorname{supp}(\pi_\alpha\circ\bar{\kappa}_\alpha^{-1}).$$

For each $j\in\mathbb{N}_0$, there exists $D_j>0$, independent of α, with

$$\|\pi_\alpha\circ\bar{\kappa}_\alpha^{-1}\|_{BUC^j(Q_\alpha)}\le D_j, \qquad \alpha\in I.$$

The constant D_j depends only on j, the multiplicity N, the shrinking factor $\rho=\frac{1}{2}$, and $C_{\mathrm{tr}}(0),\dots,C_{\mathrm{tr}}(j)$. These cutoffs are constructed after reselecting the centers and restricting to cubes, so they may differ from the cutoffs h_β in the quadratic localization system of Proposition 14.4.5. □

Theorem 14.4.8. (Full comparison with boundary) A Riemannian manifold with smooth nonempty boundary is uniformly regular if and only if it has bounded geometry with boundary in the sense of Definition 14.4.2.

The implication from bounded geometry to uniform regularity is Corollary 14.4.7, whose proof uses the Fermi atlas already constructed. Conversely, the results in [6, Theorems 2.2 and 3.5, §§2–3] state that a uniformly regular Riemannian atlas recovers a uniform geodesic collar, interior and boundary injectivity radii, and curvature bounds, and that this formulation is equivalent to Schick's through the second-fundamental-form bounds. The intrinsic equivalence with the second-fundamental-form formulation is also found in [128, §2, Definition 2.2 and Theorem 2.5].

Definition 14.4.9. (Localized spaces on a manifold with boundary) Let M be a Riemannian manifold of bounded geometry with nonempty boundary in the sense of Definition 14.4.2, let $\mathbb{R}^n_+ := \mathbb{R}^{n-1}\times[0,\infty)$, and fix the quadratic localization system of Proposition 14.4.5.

(a) For $s\in\mathbb{R}$ and $1<p<\infty$, define

$$H^{s,p}(\mathbb{R}^n_+) := \left\{ U\restriction_{\mathbb{R}^n_+} \,\middle|\, U\in H^{s,p}(\mathbb{R}^n)\right\},$$

with quotient norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n_+)} := \inf_{\substack{U\in H^{s,p}(\mathbb{R}^n)\\ U\restriction_{\mathbb{R}^n_+}=u}} \|U\|_{H^{s,p}(\mathbb{R}^n)}. \tag{14.33}$$

Here and below, distributional restriction is taken to the open set $\mathbb{R}^{n-1}\times(0,\infty)$; we retain the notation $\mathbb{R}^n_+=\mathbb{R}^{n-1}\times[0,\infty)$ because trace values on $t=0$ will be considered later. The restriction mapping is a continuous surjection, and this norm is, by definition, the quotient norm modulo its kernel. In particular, it does not depend on a chosen extension, and $H^{s,p}(\mathbb{R}^n_+)$ is a Banach space.

In all restriction scales of this chapter, a distribution on M means a distribution on $\operatorname{Int}M$. No independent distributions supported on ∂M are added: boundary values will be obtained through trace

operators. The space $H^{s,p}(M)$ consists of those distributions u for which the norm

$$\|u\|_{H^{s,p}(M)} := \left(\sum_{i \in I_\partial} \|(h_i u) \circ \kappa_i^\partial\|^p_{H^{s,p}(\mathbb{R}^n_+)} + \sum_{j \in I_0} \|(h_j u) \circ \kappa_j^0\|^p_{H^{s,p}(\mathbb{R}^n)} \right)^{\frac{1}{p}}. \tag{14.34}$$

is finite.

(b) If $s \in \mathbb{R}$, $1 \le p < \infty$, and $1 \le q \le \infty$, define

$$F^s_{p,q}(\mathbb{R}^n_+) := \left\{ U \restriction_{\mathbb{R}^n_+} \middle| U \in F^s_{p,q}(\mathbb{R}^n) \right\}$$

and

$$\|u\|_{F^s_{p,q}(\mathbb{R}^n_+)} := \inf_{\substack{U \in F^s_{p,q}(\mathbb{R}^n) \\ U \restriction_{\mathbb{R}^n_+} = u}} \|U\|_{F^s_{p,q}(\mathbb{R}^n)}. \tag{14.35}$$

Again, this is the quotient norm of the restriction operator and does not depend on a particular extension. The space $F^s_{p,q}(M)$ consists of the distributions for which

$$\left(\sum_{i \in I_\partial} \|(h_i u) \circ \kappa_i^\partial\|^p_{F^s_{p,q}(\mathbb{R}^n_+)} + \sum_{j \in I_0} \|(h_j u) \circ \kappa_j^0\|^p_{F^s_{p,q}(\mathbb{R}^n)} \right)^{\frac{1}{p}}$$

is finite; this expression is its norm.

(c) For $s \in \mathbb{R}$, $1 \le p < \infty$, and $1 \le q \le \infty$, also define the Euclidean restriction space

$$B^s_{p,q}(\mathbb{R}^n_+) := \left\{ U \restriction_{\mathbb{R}^n_+} \middle| U \in B^s_{p,q}(\mathbb{R}^n) \right\}$$

with quotient norm

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n_+)} := \inf_{\substack{U \in B^s_{p,q}(\mathbb{R}^n) \\ U \restriction_{\mathbb{R}^n_+} = u}} \|U\|_{B^s_{p,q}(\mathbb{R}^n)}. \tag{14.36}$$

Provisionally fix $s_0 < s < s_1$ and $\theta = \frac{s - s_0}{s_1 - s_0}$. Define

$$B^s_{p,q}(M; s_0, s_1) := \left(F^{s_0}_{p,p}(M), F^{s_1}_{p,p}(M) \right)_{\theta,q}. \tag{14.37}$$

Independence of s_0, s_1, which is not assumed in this definition, is proved in Corollary 14.4.15 using a common analysis–synthesis pair. After that corollary, the endpoints will be omitted and the space written simply as $B^s_{p,q}(M)$.

Compositions with parametrizations are understood in the sense of Proposition 10.2.34; in boundary charts, use the restriction norm just defined. On ∂M, use the definitions without boundary from the preceding chapter.

Lemma 14.4.10. (Localized operators between boundary models) Let $s \in \mathbb{R}$ and choose $k \in \mathbb{N}$ with $k > |s| + n + 1$. Consider $H^{s,p}$ for $1 < p < \infty$, $F^s_{p,q}$ for $1 \le p < \infty$, $1 \le q \le \infty$, and $B^s_{p,q}$ for $1 \le p < \infty$, $1 \le q \le \infty$.

(a) Rychkov's cone formula defines a single linear operator

$$\mathbf{E}_+ \colon \mathcal{A}(\mathbb{R}^n_+) \longrightarrow \mathcal{A}(\mathbb{R}^n), \qquad \mathcal{A} \in \{H^{s,p}, F^s_{p,q}, B^s_{p,q}\},$$

such that $(\mathbf{E}_+ u)\restriction_{\mathbb{R}^n_+} = u$. There exist constants $C^H_+(s,p)$, $C^F_+(s,p,q)$, and $C^B_+(s,p,q)$, depending additionally only on n and the fixed kernels, bounding the three norms of $\mathbf{E}_+$.

(b) Let $U', V' \subseteq \mathbb{R}^{n-1}$ be open, $\vartheta \colon U' \longrightarrow V'$ a diffeomorphism of class C^{k+1}, $\lambda > 0$, and $\Psi(x',t) := (\vartheta(x'), \lambda t)$. Suppose there exist $\lambda_-, \lambda_+ > 0$ such that

$$\lambda_- \le \lambda \le \lambda_+, \qquad c_J \le |\det D\vartheta(x')| \le C_J \quad (x' \in U'),$$
$$\|D^\ell \vartheta\|_{L^\infty(U')} + \|D^\ell \vartheta^{-1}\|_{L^\infty(V')} \le A, \qquad \ell \in \{1, \dots, k+1\}.$$

and $\eta \in C^{k+1}_c(U' \times [0,\infty))$ satisfies $\|\eta\|_{BC^{k+1}(U'\times[0,\infty))} \le A_\eta$. Let K_η be the tangential projection of $\operatorname{supp}\eta$. Also fix $R, \delta > 0$ such that

$$\operatorname{supp}\eta \subseteq \overline{B}^{n-1}_{\mathrm{euc}}(0,R) \times [0,R],$$
$$\operatorname{dist}(K_\eta, \mathbb{R}^{n-1} \setminus U') \ge \delta,$$
$$\operatorname{dist}(\vartheta(K_\eta), \mathbb{R}^{n-1} \setminus V') \ge \delta.$$

These distances concern the artificial chart boundaries; the support may meet $t = 0$. The initial formula

$$T^{+\leftarrow+}_{\eta,\Psi} u := \begin{cases} \eta\,(u \circ \Psi), & \text{on } U' \times [0,\infty), \\ 0, & \text{elsewhere in } \mathbb{R}^n_+, \end{cases}$$

defines continuous operators

$$T^{+\leftarrow+}_{\eta,\Psi} \colon \mathcal{A}(\mathbb{R}^n_+) \longrightarrow \mathcal{A}(\mathbb{R}^n_+)$$

in all three scales. Their norms are bounded by constants $M_H, M_F(q), M_B(q)$ depending on $n, s, p, k, c_J, C_J, A, A_\eta, \lambda_-, \lambda_+, R, \delta$ and, in the last two scales, also on q.

(c) Let $\mathbb{X}, \mathbb{Y} \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$, let $U \subseteq \mathbb{Y}$ and $V \subseteq \mathbb{X}$ be relatively open, let $\Phi \colon U \longrightarrow V$ be a diffeomorphism, and let $\zeta \in C^{k+1}_c(U)$. If a half-space occurs and the relevant support meets $t = 0$, assume $\mathbb{X} = \mathbb{Y} = \mathbb{R}^n_+$ and that (ζ, Φ) has the form in (b). In the other cases, assume the relevant supports have distance at least $d > 0$ from every boundary present: $\operatorname{dist}(\operatorname{supp}\zeta, \partial\mathbb{R}^n_+) \ge d$ if $\mathbb{Y} = \mathbb{R}^n_+$ and $\operatorname{dist}(\Phi(\operatorname{supp}\zeta), \partial\mathbb{R}^n_+) \ge d$ if $\mathbb{X} = \mathbb{R}^n_+$. Then

$$T^{\mathbb{Y}\leftarrow\mathbb{X}}_{\zeta,\Phi} \colon \mathcal{A}(\mathbb{X}) \longrightarrow \mathcal{A}(\mathbb{Y})$$

is continuous. For uniform bounds, include the diameter of the supports and their positive separation from the relative boundaries of U and V, together with the coordinate-change bounds required in the localized version of Lemma 6.4.3. If $\widetilde{T}$ denotes the localized full-space operator associated with the same data, r_+ the restriction, and $\mathbf{E}_+$ the operator in (a), the four types are

$(\mathbb{X}, \mathbb{Y})$	$T^{\mathbb{Y}\leftarrow\mathbb{X}}_{\zeta,\Phi}$
$(\mathbb{R}^n, \mathbb{R}^n)$	$\widetilde{T}$
$(\mathbb{R}^n_+, \mathbb{R}^n)$	$\widetilde{T}\mathbf{E}_+$
$(\mathbb{R}^n, \mathbb{R}^n_+)$	$r_+\widetilde{T}$
$(\mathbb{R}^n_+, \mathbb{R}^n_+)$	$r_+\widetilde{T}\mathbf{E}_+$.

In each case, on smooth functions the formula means $T^{\mathbb{Y}\leftarrow\mathbb{X}}_{\zeta,\Phi}u = \zeta(u\circ\Phi)$ on U and zero outside U. The preceding constants also depend on d and bounds for Φ, Φ^{-1}, their Jacobians, and their derivatives through order $k+1$, but not on u.

All these assertions include $s<0$. At endpoint fine indices, the distributional formula specifies the operator.

Proof. Specialize the construction of Lemma 6.5.1 to $\omega = 0$. With its cone kernels, set

$$\mathbf{E}_+ u := \sum_{j\in\mathbb{N}_0} \psi_j * \bigl(\mathbf{1}_{\mathbb{R}^n_+}(\varphi_j * u)\bigr).$$

The identity (6.11) proves $r_+\mathbf{E}_+ = I$. Estimates (6.14) and (6.13) give $C^F_+(s,p,q)$; for $q=2$ and $H^{s,p} = F^s_{p,2}$, they give $C^H_+(s,p)$. For Besov, take the ℓ^q norm outside L^p in the same estimates, giving $C^B_+(s,p,q)$. Fix the Rychkov kernels once, with all vanishing moments required at positive levels. To estimate a particular order s, only finitely many $N > |s|+1$ of these moments are used; the kernels do not change as s varies. This gives a single operator at every order, including $s<0$.

We record the density that will be used. Let $Q_{\mathcal{A}}$ be the restriction quotient mapping from $\mathcal{A}(\mathbb{R}^n)$ to $\mathcal{A}(\mathbb{R}^n_+)$. If $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{A}(\mathbb{R}^n)$ and $u = Q_{\mathcal{A}}U$, choose $U_j\in\mathcal{S}$ with $U_j\longrightarrow U$; since $\|Q_{\mathcal{A}}\|\le 1$, we have $Q_{\mathcal{A}}U_j\longrightarrow u$. Thus, restrictions of Schwartz functions are dense in $H^{s,p}$, in $F^s_{p,q}$ when $q<\infty$, and in $B^s_{p,q}$ when $p,q<\infty$. This conclusion is neither obtained nor needed at the other indices.

For (b), let $a_1,\dots,a_{k+2}$ be the unique coefficients satisfying

$$\sum_{\nu=1}^{k+2} a_\nu(-\nu)^j = 1, \qquad 0\le j\le k+1,$$

and define

$$\widetilde{\eta}(x',t) := \begin{cases} \eta(x',t), & t\ge 0,\\ \displaystyle\sum_{\nu=1}^{k+2} a_\nu \eta(x',-\nu t), & t<0. \end{cases}$$

The Vandermonde identities match derivatives through order $k+1$. Moreover, $\widetilde{\eta}\in C^{k+1}_c(U'\times\mathbb{R})$ and, with

$$C_k := 1 + \sum_{\nu=1}^{k+2} |a_\nu| \max\{1,\nu^{k+1}\},$$

we have $\|\widetilde{\eta}\|_{BC^{k+1}(U'\times\mathbb{R})}\le C_k A_\eta$. The diffeomorphism $\widetilde{\Psi}(x',t) := (\vartheta(x'),\lambda t)$ preserves the half-spaces. Moreover,

$$D\widetilde{\Psi} = \begin{pmatrix} D\vartheta & 0\\ 0 & \lambda\end{pmatrix}, \qquad D\widetilde{\Psi}^{-1} = \begin{pmatrix} D\vartheta^{-1} & 0\\ 0 & \lambda^{-1}\end{pmatrix}, \qquad |\det D\widetilde{\Psi}| = \lambda|\det D\vartheta|.$$

Consequently, derivatives of $\widetilde{\Psi}$ and $\widetilde{\Psi}^{-1}$ through order $k+1$, and upper and lower bounds for their Jacobians, are controlled by constants depending only on $c_J, C_J, A, \lambda_-, \lambda_+$. The support of $\widetilde{\eta}$ remains in a compact set of diameter controlled by R and separated from the boundary of $U'\times\mathbb{R}$ by δ. The corresponding image margin is also uniform. The localized version of Lemma 6.4.3, with these distances and bounds, and item

(a) give

$$\|r_+ T_{\widetilde{\eta},\widetilde{\Psi}} \mathbf{E}_+ u\|_{H^{s,p}(\mathbb{R}^n_+)} \le C^H_{\mathrm{loc}} C^H_+(s,p) \|u\|_{H^{s,p}(\mathbb{R}^n_+)},$$
$$\|r_+ T_{\widetilde{\eta},\widetilde{\Psi}} \mathbf{E}_+ u\|_{F^s_{p,q}(\mathbb{R}^n_+)} \le C^F_{\mathrm{loc}}(q) C^F_+(s,p,q) \|u\|_{F^s_{p,q}(\mathbb{R}^n_+)},$$
$$\|r_+ T_{\widetilde{\eta},\widetilde{\Psi}} \mathbf{E}_+ u\|_{B^s_{p,q}(\mathbb{R}^n_+)} \le C^B_{\mathrm{loc}}(q) C^B_+(s,p,q) \|u\|_{B^s_{p,q}(\mathbb{R}^n_+)}.$$

The restriction r_+ has norm at most one by the quotient definition. Thus, we may take

$$M_H := C^H_{\mathrm{loc}} C^H_+(s,p), \quad M_F(q) := C^F_{\mathrm{loc}}(q) C^F_+(s,p,q), \quad M_B(q) := C^B_{\mathrm{loc}}(q) C^B_+(s,p,q).$$

The third bound follows directly from the almost-diagonal Besov estimate in Lemma 6.4.3; the outer sum ℓ^p and integration in the K method are not interchanged. These formulas display every stated dependence. In ranges where restrictions of Schwartz functions are dense, they also prove uniqueness of the extension; at the endpoints, the composition $r_+ \widetilde{T} \mathbf{E}_+$ is its distributional definition.

For (c), if the supports are separated from $t = 0$, the effective open sets are open in $\mathbb{R}^n$ and the localized operator $\widetilde{T}$ from the full-space lemma applies without crossing a boundary. Precomposing with $\mathbf{E}_+$ when $\mathbb{X} = \mathbb{R}^n_+$ and postcomposing with r_+ when $\mathbb{Y} = \mathbb{R}^n_+$ gives exactly the four factorizations in the table. If the support meets $t = 0$, the construction in (b) provides $\widetilde{T}$. In H and F, the norms are the product of the localized constant with C^H_+ or C^F_+ whenever $\mathbf{E}_+$ occurs. In B, the same procedure gives the localized constant $C^B_{\mathrm{loc}}(q)$, multiplied by $C^B_+(s,p,q)$ precisely in the two types whose domain is $\mathbb{R}^n_+$; the restriction r_+ has norm at most one. The almost-diagonal argument of the full-space lemma includes $s < 0$, proving the assertions for negative orders through the same local estimates. □

Proposition 14.4.11. (Independence of the boundary atlas) Let M be a Riemannian manifold of bounded geometry with nonempty boundary, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two atlases satisfying Proposition 14.4.5. For each space in Definition 14.4.9, there exist constants $c^{\mathcal{A}}_{12}, C^{\mathcal{A}}_{12} > 0$, one pair for each function-space scale $\mathcal{A}$, depending on its indices and the uniform data of both atlases but not on u, such that

$$c^{\mathcal{A}}_{12} \|u\|_{\mathcal{A},\mathcal{T}_1} \le \|u\|_{\mathcal{A},\mathcal{T}_2} \le C^{\mathcal{A}}_{12} \|u\|_{\mathcal{A},\mathcal{T}_1}.$$

For $B^s_{p,q}$, provisionally fix the same interpolation endpoints in both atlases; their independence is proved in Corollary 14.4.15.

The retraction–coretraction argument corresponds to [6, Theorems 4.2 and 4.4].

Proof. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two systems. Insert the identity $\sum_{\beta \in I_2} h^2_{2,\beta} = 1$ into each localization $h_{1,\alpha} u$. The two cross-overlap bounds leave at most L summands per index, where L depends only on the uniform data of both atlases.

Between two boundary charts, using the same collar normal parameter makes the Fermi transition

$$(x', t) \longmapsto \big(\vartheta_{\alpha\beta}(x'), t\big).$$

The bounds in Proposition 14.4.5 provide numbers $c_J, C_J, A, A_\eta > 0$, independent of α, β, verifying the hy-

potheses of Lemma 14.4.10(b). Thus, each boundary–boundary summand has norm at most M_H, $M_F(q)$, or $M_B(q)$ times the source localization norm, according to the scale; these assertions include $s<0$. For two interior charts, use Lemma 6.4.3 on $\mathbb{R}^n$. On a mixed overlap, the interior cutoff supports are separated from $t=0$ by a uniform distance; item (c) of the preceding lemma again reduces the estimate to the full-space case. Denote by $M_{\mathcal{A}}$ the largest constant for scale $\mathcal{A}$. It depends on n,s,p, also on q in F and B, and on the uniform data of $\mathcal{T}_1,\mathcal{T}_2$, but not on the indices.

The inequality $\left(\sum_{\ell=1}^{L} a_\ell\right)^p \le L^{p-1}\sum_{\ell=1}^{L} a_\ell^p$ and a second overlap count give

$$\|u\|_{H^{s,p}_{\mathcal{T}_1}(M)} \le LM_H\|u\|_{H^{s,p}_{\mathcal{T}_2}(M)}.$$

Interchanging the atlases gives $\|u\|_{H^{s,p}_{\mathcal{T}_2}(M)} \le L'M'_H\|u\|_{H^{s,p}_{\mathcal{T}_1}(M)}$; taking $c^H_{12} := (LM_H)^{-1}$ and $C^H_{12} := L'M'_H$ yields the assertion in H. In F, the same count gives explicitly

$$c^F_{12}(q) := (LM_F(q))^{-1}, \qquad C^F_{12}(q) := L'M'_F(q).$$

For B, fix provisional endpoints $a<s<b$ and $\theta = \frac{s-a}{b-a}$. If

$$c_t\|u\|_{F^t_{p,p,\mathcal{T}_1}(M)} \le \|u\|_{F^t_{p,p,\mathcal{T}_2}(M)} \le C_t\|u\|_{F^t_{p,p,\mathcal{T}_1}(M)}, \qquad t\in\{a,b\},$$

are the inequalities just proved with $c_t := (L_tM_F(t,p))^{-1}$ and $C_t := L'_tM'_F(t,p)$, Theorem 5.3.1 gives

$$c_a^{1-\theta}c_b^{\theta}\|u\|_{B^s_{p,q}(M;a,b),\mathcal{T}_1} \le \|u\|_{B^s_{p,q}(M;a,b),\mathcal{T}_2} \le C_a^{1-\theta}C_b^{\theta}\|u\|_{B^s_{p,q}(M;a,b),\mathcal{T}_1}.$$

Thus, for these endpoints one may take $c^B_{12} := c_a^{1-\theta}c_b^{\theta}$ and $C^B_{12} := C_a^{1-\theta}C_b^{\theta}$. Subsequent independence of a,b allows them to be omitted. □

Corollary 14.4.12. (Functional connection with Amann's theory in the presence of boundary) Let M be a Riemannian manifold of bounded geometry with nonempty boundary, let $\tilde{\mathfrak{K}} = (U_\alpha,\varphi_\alpha)_{\alpha\in I}$ be the normalized atlas constructed in Corollary 14.4.7, and let $(\pi_\alpha)_\alpha$ be the new quadratic system obtained by applying Proposition 13.2.6 to $\tilde{\mathfrak{K}}$. Set

$$X_\alpha := \begin{cases} \mathbb{R}^n, & U_\alpha\cap\partial M = \varnothing,\\ \mathbb{R}^n_+, & U_\alpha\cap\partial M \neq \varnothing, \end{cases}$$

and, at a boundary index, postcompose the chart with the linear isometry

$$\tau(x_1,\dots,x_n) := (x_2,\dots,x_n,x_1),$$

identifying the model Q^n_+ with normal variable first with the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1}\times[0,\infty)$ used in Definition 14.4.9. Retain the symbol φ_α for the postcomposed chart and write $\Omega_\alpha := \varphi_\alpha(U_\alpha)\subseteq X_\alpha$. For $s\in\mathbb{R}$ and $1<p<\infty$, define

$$\mathbb{H}^{s,p}_{\partial}(\tilde{\mathfrak{K}}) := \left\{ v=(v_\alpha)_{\alpha\in I} \,\middle|\, \sum_{\alpha\in I}\|v_\alpha\|^p_{H^{s,p}(X_\alpha)} < +\infty \right\}$$

with norm given by the pth root of the preceding sum. For $s \in \mathbb{R}$, $1 \le p < \infty$, and $1 \le q \le \infty$, likewise set

$$\mathbb{F}^{s}_{p,q,\partial}(\mathfrak{K}) := \left\{ v = (v_\alpha)_{\alpha \in I} \,\middle|\, \sum_{\alpha \in I} \|v_\alpha\|^{p}_{F^{s}_{p,q}(X_\alpha)} < +\infty \right\}.$$

Both spaces carry the norm given by the pth root of the corresponding sum. Let $\chi \in C^\infty_c(Q^n, [0,1])$ be the fixed cutoff chosen in Proposition 13.2.6. Define

$$\widehat{\chi}_\alpha := \begin{cases} \chi \restriction_{\Omega_\alpha}, & X_\alpha = \mathbb{R}^n, \\ (\chi \circ \tau^{-1}) \restriction_{\Omega_\alpha}, & X_\alpha = \mathbb{R}^n_+. \end{cases}$$

Then $\widehat{\chi}_\alpha$ equals one on a neighborhood of $\operatorname{supp}(\pi_\alpha \circ \varphi_\alpha^{-1})$ and, for each $j \in \mathbb{N}_0$,

$$\sup_{\alpha \in I} \|\widehat{\chi}_\alpha\|_{BUC^j(\Omega_\alpha)} \le \|\chi\|_{BUC^j(Q^n)} < +\infty,$$

because τ is a linear isometry. Let $\widetilde{\chi}_\alpha$ be the extension by zero of $\widehat{\chi}_\alpha$ from Ω_α to X_α. Since its support stays separated from the nongeometric faces of Ω_α, this extension belongs to $C^\infty_c(X_\alpha)$, including when $X_\alpha = \mathbb{R}^n_+$ and the support meets $t = 0$. For a component v_α of either sequence space, define $\mathcal{E}_\alpha(\widetilde{\chi}_\alpha v_\alpha)$ as the distribution whose coordinate representation on U_α is $(\widetilde{\chi}_\alpha v_\alpha) \restriction_{\Omega_\alpha}$ and which is extended by zero outside U_α. Likewise, the tilde in $\widetilde{(\pi_\alpha u) \circ \varphi_\alpha^{-1}}$ denotes extension by zero from Ω_α to X_α. This is legitimate because $\operatorname{supp}(\pi_\alpha \circ \varphi_\alpha^{-1}) \Subset \Omega_\alpha$. The operators, whose formulas do not depend on s, p, q,

$$\mathcal{S}_{\partial,\mathfrak{K}} u := \left(\widetilde{(\pi_\alpha u) \circ \varphi_\alpha^{-1}} \right)_{\alpha \in I}, \qquad \mathcal{R}_{\partial,\mathfrak{K}} v := \sum_{\alpha \in I} \pi_\alpha \, \mathcal{E}_\alpha(\widetilde{\chi}_\alpha v_\alpha)$$

are continuous in both the H and F scales, and

$$\mathcal{R}_{\partial,\mathfrak{K}} \mathcal{S}_{\partial,\mathfrak{K}} = \mathrm{id}$$

on each of these spaces. There exist constants $c^\partial_{s,p}, C^\partial_{s,p}, D^\partial_{s,p} > 0$ such that

$$c^\partial_{s,p} \|u\|_{H^{s,p}(M)} \le \|\mathcal{S}_{\partial,\mathfrak{K}} u\|_{\mathbb{H}^{s,p}_\partial(\mathfrak{K})} \le C^\partial_{s,p} \|u\|_{H^{s,p}(M)}, \tag{14.38}$$

$$\|\mathcal{R}_{\partial,\mathfrak{K}} v\|_{H^{s,p}(M)} \le D^\partial_{s,p} \|v\|_{\mathbb{H}^{s,p}_\partial(\mathfrak{K})}. \tag{14.39}$$

Moreover, there exist $c^F_{s,p,q}, C^F_{s,p,q}, D^F_{s,p,q} > 0$ such that

$$c^F_{s,p,q} \|u\|_{F^{s}_{p,q}(M)} \le \|\mathcal{S}_{\partial,\mathfrak{K}} u\|_{\mathbb{F}^{s}_{p,q,\partial}(\mathfrak{K})} \le C^F_{s,p,q} \|u\|_{F^{s}_{p,q}(M)}, \tag{14.40}$$

$$\|\mathcal{R}_{\partial,\mathfrak{K}} v\|_{F^{s}_{p,q}(M)} \le D^F_{s,p,q} \|v\|_{\mathbb{F}^{s}_{p,q,\partial}(\mathfrak{K})}. \tag{14.41}$$

For a fixed integer $k > |s| + n + 1$, the constants depend only on n, s, p, k, the two overlap bounds, q in scale F, and uniform coordinate-transition and cutoff bounds through order $k+1$; they do not depend on α, u, or v.

Proof. Denote by $\mathcal{T}_F = (V_\beta, \kappa_\beta, h_\beta)_{\beta \in J}$ the quadratic localization system fixed in Definition 14.4.9. The construction of $\mathfrak{K}$ takes place in the same Fermi and interior chart families, through the two fixed linear trans-

formations in the proof of Corollary 14.4.7. Denote the normal rescaling factor by $b_N := 2r_B$. Each support of π_α or h_β lies in a common shrunken box. The packing argument of Proposition 14.4.5, applied in both directions, gives

$$L := \max\left\{\sup_{\alpha\in I} \#\left\{\beta\in J \,\middle|\, \operatorname{supp}\pi_\alpha \cap \operatorname{supp} h_\beta \neq \varnothing\right\}, \sup_{\beta\in J} \#\left\{\alpha\in I \,\middle|\, \operatorname{supp} h_\beta \cap \operatorname{supp}\pi_\alpha \neq \varnothing\right\}\right\} < +\infty.$$

Fix $k > |s| + n + 1$. The BUC^{k+1} bounds for transitions and cutoffs, the upper and lower Jacobian bounds, and Lemma 14.4.10 provide numbers $M_{\to,\mathcal{A}}, M_{\leftarrow,\mathcal{A}} > 0$, for $\mathcal{A} = H^{s,p}$ or $F^s_{p,q}$, bounding, respectively, each localized operator passing from $\mathcal{T}_F$ to Amann's localization system and each operator in the reverse direction. For two boundary charts, the transition preserves the normal parameter within each system. Comparing $\mathcal{T}_F$ with the rescaled charts in the proof of Corollary 14.4.7 introduces the fixed normal factor b_N; thus, the transition has the form $(x', t) \mapsto (\vartheta(x'), \lambda t)$, with $\lambda \in \{1, b_N, b_N^{-1}\}$. Use item (b) with $\lambda_- := \min\{1, b_N, b_N^{-1}\}$ and $\lambda_+ := \max\{1, b_N, b_N^{-1}\}$. On a mixed overlap, the support is at uniform distance from $t = 0$ and item (c) is used; between interior charts, use Lemma 6.4.3. In particular, both bounds remain valid for $s < 0$. The numbers depend only on n, s, p, k, also on q in scale F, the Jacobian bounds, the BUC^{k+1} bounds for transitions and cutoffs, and the uniform separation of mixed overlaps.

Insert $\sum_{\beta\in J} h_\beta^2 = 1$ into every localization $\pi_\alpha u$. If A_α denotes its norm in $H^{s,p}(X_\alpha)$ and B_β the corresponding summand of (14.34), then

$$A_\alpha \le M_{\to,H} \sum_{\{\beta\in J \,|\, \operatorname{supp}\pi_\alpha\cap\operatorname{supp} h_\beta\neq\varnothing\}} B_\beta.$$

The inequality $(\sum_{j=1}^{L} a_j)^p \le L^{p-1}\sum_{j=1}^{L} a_j^p$ and the two overlap counts give

$$\|\mathcal{S}_{\partial,\mathfrak{K}} u\|_{\mathbb{H}^{s,p}_\partial(\mathfrak{K})} \le L M_{\to,H} \|u\|_{H^{s,p}(M)}.$$

In the reverse direction, insert $\sum_{\alpha\in I}\pi_\alpha^2 = 1$ into each $h_\beta u$ to obtain

$$\|u\|_{H^{s,p}(M)} \le L M_{\leftarrow,H} \|\mathcal{S}_{\partial,\mathfrak{K}} u\|_{\mathbb{H}^{s,p}_\partial(\mathfrak{K})}.$$

Consequently, in (14.38) we may take

$$c^\partial_{s,p} := (L M_{\leftarrow,H})^{-1}, \qquad C^\partial_{s,p} := L M_{\to,H}.$$

The same calculation with $F^s_{p,q}$ norms gives (14.40) with

$$c^F_{s,p,q} := (L M_{\leftarrow,F})^{-1}, \qquad C^F_{s,p,q} := L M_{\to,F}.$$

For synthesis, localizing $\mathcal{R}_{\partial,\mathfrak{K}} v$ using h_β involves at most L indices α. The same three transition cases provide uniform constants $M_{\sin,H} > 0$ and $M_{\sin,F}(q) > 0$ such that each summand has norm at most $M_{\sin,H}\|v_\alpha\|_{H^{s,p}(X_\alpha)}$ in H, and the analogous bound with $M_{\sin,F}(q)$ in F. Raising to the power p, summing

first over β, and counting overlaps in the reverse direction gives

$$\|\mathcal{R}_{\partial,\mathfrak{K}} v\|^p_{H^{s,p}(M)} \le L^p M^p_{\sin,H} \sum_{\alpha\in I} \|v_\alpha\|^p_{H^{s,p}(X_\alpha)}.$$

Thus, we may take $D^{\partial}_{s,p} := LM_{\sin,H}$ and, in (14.41), $D^{F}_{s,p,q} := LM_{\sin,F}(q)$. The sum is well defined by local finiteness.

The retraction identity is distributional and does not depend on density. Indeed, if $u \in \mathcal{D}'(M)$, then $\widetilde{\chi}_\alpha = 1$ on a neighborhood of the coordinate support of π_α implies, by definition of multiplication and extension of distributions,

$$\mathcal{E}_\alpha\left(\widetilde{\chi}_\alpha \widetilde{(\pi_\alpha u)\circ\varphi_\alpha^{-1}}\right) = \pi_\alpha u.$$

The family is locally finite and $\sum_{\alpha\in I} \pi_\alpha^2 = 1$; therefore, in $\mathcal{D}'(M)$,

$$\mathcal{R}_{\partial,\mathfrak{K}}\mathcal{S}_{\partial,\mathfrak{K}} u = \sum_{\alpha\in I} \pi_\alpha^2 u = u.$$

Finally, this retraction gives global density exactly in the dense ranges. If u belongs to $H^{s,p}(M)$, or to $F^s_{p,q}(M)$ with $q<\infty$, first truncate $\mathcal{S}_{\partial,\mathfrak{K}} u$ to finitely many indices and then approximate each component by the restriction of a Schwartz function, as proved using the quotient in Lemma 14.4.10. Synthesis of each finite family belongs to $C^\infty_c(M)$ and converges to u by (14.39) or (14.41). The case $q=\infty$ is outside the density range. □

Remark 14.4.13. (The two localization systems) Corollary 14.4.7 constructs the normalized atlas and its own cutoffs π_α; it does not identify them with the cutoffs h_β of $\mathcal{T}_F$. Corollary 14.4.12 then uses the constants L, $M_{\to,\mathcal{A}}$, and $M_{\leftarrow,\mathcal{A}}$ to prove that both systems produce the same scales H and F with norms comparable through explicit inequalities. Thus, uniform regularity provides a connection at the function-space level and does not replace the Fermi localization already used in the trace and embedding proofs.

Proposition 14.4.14. (Exactness under interpolation of the localized scale) Let M be a Riemannian manifold of bounded geometry with nonempty boundary. Let $1\le p<\infty$, $a<b$, $0<\rho<1$, and $r=(1-\rho)a+\rho b$. There exist constants $\lambda_{a,b,r}, \Lambda_{a,b,r}>0$ such that

$$\lambda_{a,b,r}\|u\|_{F^r_{p,p}(M)} \le \|u\|_{(F^a_{p,p}(M),F^b_{p,p}(M))_{\rho,p}} \le \Lambda_{a,b,r}\|u\|_{F^r_{p,p}(M)}. \tag{14.42}$$

The pair $\mathcal{S}_{\partial,\mathfrak{K}}$, $\mathcal{R}_{\partial,\mathfrak{K}}$ from the preceding corollary is the same for all values of a,b,r.

Proof. Write $Y_t := \mathbb{F}^t_{p,p,\partial}(\mathfrak{K})$. Use the identity $F^t_{p,p}(\mathbb{R}^n) = B^t_{p,p}(\mathbb{R}^n)$ and the Besov analysis and synthesis of Lemma 6.3.8. This synthesis is bounded for every $1\le p<\infty$, including $p=1$, and its formulas do not depend on t. The two operators realize $F^t_{p,p}(\mathbb{R}^n)$ as a retract of

$$b^t_p := \left\{ w=(w_j)_{j\ge0} \in \left(L^p(\mathbb{R}^n)\right)^{\mathbb{N}_0} \,\middle|\, \sum_{j\in\mathbb{N}_0} 2^{jtp}\|w_j\|^p_{L^p(\mathbb{R}^n)} < +\infty \right\}.$$

More precisely, there exist operators $\mathfrak{a}, \mathfrak{r}$, independent of t, such that, for each $t \in \mathbb{R}$,

$$\mathfrak{a}\colon F^t_{p,p}(\mathbb{R}^n) \longrightarrow b^t_p, \qquad \mathfrak{r}\colon b^t_p \longrightarrow F^t_{p,p}(\mathbb{R}^n), \qquad \mathfrak{r}\mathfrak{a} = I_{F^t_{p,p}(\mathbb{R}^n)}.$$

There also exist $A_t, B_t > 0$, depending only on n, p, t and the fixed resolution, such that

$$\|\mathfrak{a}u\|_{b^t_p} \le A_t \|u\|_{F^t_{p,p}(\mathbb{R}^n)}, \qquad \|\mathfrak{r}w\|_{F^t_{p,p}(\mathbb{R}^n)} \le B_t \|w\|_{b^t_p}.$$

In particular, $B_t^{-1}\|u\|_{F^t_{p,p}(\mathbb{R}^n)} \le \|\mathfrak{a}u\|_{b^t_p} \le A_t\|u\|_{F^t_{p,p}(\mathbb{R}^n)}$; no isometric equality between Triebel–Lizorkin and Besov norms is used. In a coordinate j,

$$K\big(\tau, w_j; 2^{ja}L^p(\mathbb{R}^n), 2^{jb}L^p(\mathbb{R}^n)\big) = \min\{2^{ja}, \tau 2^{jb}\}\|w_j\|_{L^p(\mathbb{R}^n)}.$$

If $k(\tau, w)$ is the ℓ^p norm of these coordinate functionals, the definitions give

$$k(\tau, w) \le K(\tau, w; b^a_p, b^b_p) \le 2k(\tau, w).$$

Moreover,

$$\int_0^\infty \Big[\tau^{-\rho}\min\{2^{ja}, \tau 2^{jb}\}\Big]^p \frac{d\tau}{\tau} = c^p_{\rho,p} 2^{jrp}, \qquad c^p_{\rho,p} := \frac{1}{p\rho} + \frac{1}{p(1-\rho)}.$$

This equality follows by integrating separately before and after $\tau = 2^{j(a-b)}$. Tonelli applies because the interpolation index is precisely p. Interpolating the same dyadic analysis–synthesis pair gives $m_{\mathbb{R}}, M_{\mathbb{R}} > 0$ such that

$$m_{\mathbb{R}}\|w\|_{F^r_{p,p}(\mathbb{R}^n)} \le \|w\|_{(F^a_{p,p}(\mathbb{R}^n), F^b_{p,p}(\mathbb{R}^n))_{\rho,p}} \le M_{\mathbb{R}}\|w\|_{F^r_{p,p}(\mathbb{R}^n)}.$$

On the half-space, r_+ has norm at most one and the same $\mathbf{E}_+$ satisfies $r_+\mathbf{E}_+ = I$ at orders a, b. If e_a, e_b are its norms, interpolation of $\mathbf{E}_+$ gives the following lower inequality. For the upper inequality, choose $U \in F^r_{p,p}(\mathbb{R}^n)$ with $r_+U = w$, apply the full-space inequality to U, and then take the infimum over all such extensions. Thus, the quotient definition gives

$$m_{\mathbb{R}} e_a^{-(1-\rho)} e_b^{-\rho}\|w\|_{F^r_{p,p}(\mathbb{R}^n_+)} \le \|w\|_{(F^a_{p,p}(\mathbb{R}^n_+), F^b_{p,p}(\mathbb{R}^n_+))_{\rho,p}} \le M_{\mathbb{R}}\|w\|_{F^r_{p,p}(\mathbb{R}^n_+)}.$$

Taking the outer ℓ^p sum and using Tonelli again gives $m_Y, M_Y > 0$ with

$$m_Y\|v\|_{Y_r} \le \|v\|_{(Y_a, Y_b)_{\rho,p}} \le M_Y\|v\|_{Y_r}. \tag{14.43}$$

One may take m_Y as the minimum of the lower constants for the two models and M_Y as twice the maximum of their upper constants.

Abbreviate $C_t := C^F_{t,p,p}$ and $D_t := D^F_{t,p,p}$. Interpolating the common operators, using their retraction identity and (14.43), gives (14.42) with

$$\lambda_{a,b,r} := \frac{m_Y}{D_r C_a^{1-\rho} C_b^{\rho}}, \qquad \Lambda_{a,b,r} := M_Y C_r D_a^{1-\rho} D_b^{\rho}.$$

These constants depend only on the indices and uniform atlas data, not on u. □

Corollary 14.4.15. (Independence of the Besov endpoints) Let M be a Riemannian manifold of bounded geometry with nonempty boundary. Let $1 \le p < \infty$, $1 \le q \le \infty$, and let $P = (s_0, s_1, \theta)$, $Q = (t_0, t_1, \eta)$ be two triples such that $s_0 < s < s_1$, $t_0 < s < t_1$, $\theta, \eta \in (0,1)$, and

$$s = (1-\theta)s_0 + \theta s_1 = (1-\eta)t_0 + \eta t_1.$$

There exist constants $c_{P,Q}, C_{P,Q} > 0$ such that

$$c_{P,Q}\|u\|_{B^s_{p,q}(M;s_0,s_1)} \le \|u\|_{B^s_{p,q}(M;t_0,t_1)} \le C_{P,Q}\|u\|_{B^s_{p,q}(M;s_0,s_1)}.$$

Thus, the space and its topology are independent of the auxiliary endpoints. Henceforth, denote it simply by $B^s_{p,q}(M)$.

Proof. Choose $a < \min\{s_0, t_0\}$ and $b > \max\{s_1, t_1\}$. For $x \in \{s_0, s_1, t_0, t_1\}$, set $\rho_x = \frac{x-a}{b-a}$ and

$$Z_x := (F^a_{p,p}(M), F^b_{p,p}(M))_{\rho_x, p}.$$

The proposition provides $\lambda_x, \Lambda_x > 0$ with

$$\lambda_x\|u\|_{F^x_{p,p}(M)} \le \|u\|_{Z_x} \le \Lambda_x\|u\|_{F^x_{p,p}(M)}.$$

Let $W_s := (F^a_{p,p}(M), F^b_{p,p}(M))_{\frac{s-a}{b-a}, q}$. Theorem 5.5.2 provides $r_P, R_P, r_Q, R_Q > 0$ such that

$$r_P\|u\|_{W_s} \le \|u\|_{(Z_{s_0}, Z_{s_1})_{\theta,q}} \le R_P\|u\|_{W_s},$$

and the same inequalities with Q in place of P. Interpolating the four preceding inequalities gives

$$\alpha_P\|u\|_{W_s} \le \|u\|_{B^s_{p,q}(M;s_0,s_1)} \le \beta_P\|u\|_{W_s},$$

where

$$\alpha_P := \frac{r_P}{\Lambda_{s_0}^{1-\theta}\Lambda_{s_1}^{\theta}}, \qquad \beta_P := \frac{R_P}{\lambda_{s_0}^{1-\theta}\lambda_{s_1}^{\theta}}.$$

Define α_Q, β_Q by the same formula, with Q in place of P. Then we may take

$$c_{P,Q} := \frac{\alpha_Q}{\beta_P}, \qquad C_{P,Q} := \frac{\beta_Q}{\alpha_P}.$$

The outer ℓ^p sum was commuted with interpolation only in the intermediate step whose index is p; the arbitrary index q enters afterward through reiteration. □

Before taking traces on the half-space, we must verify that the datum does not depend on the chosen full-space representative.

Lemma 14.4.16. (The trace depends only on restriction to the half-space) Let $1 < p < \infty$, $j \in \mathbb{N}_0$, and $s > j + 1/p$. If $U \in H^{s,p}(\mathbb{R}^n)$ vanishes on $t > 0$, then $\mathrm{tr}_1(\partial_t^\ell U) = 0$ for $0 \le \ell \le j$. For $j = 0$, the same assertion holds in $F^s_{p,q}$ and $B^s_{p,q}$, $1 \le q \le \infty$, provided that $s > 1/p$.

Proof. For $h>0$, set $U_h(x',t)=U(x',t+h)$. This distribution vanishes on $t>-h$. A smooth function $\eta_h(t)$ supported in $(-h/2,h/2)$ and equal to one near zero satisfies $\eta_h\partial_t^\ell U_h=0$. Compatibility with multipliers from Lemma 14.1.2 gives $\operatorname{tr}_1(\partial_t^\ell U_h)=0$. Translations converge strongly to the identity on $H^{s,p}$; continuity of differentiation and the trace permits $h\to 0^+$. The same argument applies to $F^s_{p,q}$ and $B^s_{p,q}$ when the fine index is finite. For $q=\infty$, choose $0<\varepsilon<s-1/p$ and work in the space of order $s-\varepsilon$ and fine index 1. Compatibility of traces at different orders identifies the distributional limit with the trace at order s. $\square$

Theorem 14.4.17. (Trace and jets on the half-space) Let $1<p<\infty$, $j\in\mathbb{N}_0$, and $s>j+\frac{1}{p}$. The operator

$$\boldsymbol{\gamma}_j U:=\big(U\restriction_{t=0},\partial_t U\restriction_{t=0},\dots,\partial_t^j U\restriction_{t=0}\big)$$

extends to a retraction

$$\boldsymbol{\gamma}_j\colon H^{s,p}(\mathbb{R}^n_+)\longrightarrow\prod_{\ell=0}^{j}B^{\,s-\ell-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}). \tag{14.44}$$

There exists a continuous linear coretraction $\boldsymbol{e}_j$ for which $\boldsymbol{\gamma}_j\boldsymbol{e}_j=I$. For $j=0$ and $1\le q\le\infty$, one may replace $H^{s,p}$ by $F^s_{p,q}$ and $B^s_{p,q}$, with target spaces $B^{s-\frac{1}{p}}_{p,p}$ and $B^{s-\frac{1}{p}}_{p,q}$, respectively.

Proof. Let $u\in H^{s,p}(\mathbb{R}^n_+)$ and choose an extension $U\in H^{s,p}(\mathbb{R}^n)$. Define $\gamma_\ell u:=\operatorname{tr}_1(\partial_t^\ell U)$. Two extensions differ by a distribution vanishing on $t>0$; Lemma 14.4.16 shows that the definition is independent of U. Continuity of differentiation, the Euclidean trace, and the infimum over extensions give, for $\ell\le j$,

$$\|\gamma_\ell u\|_{B^{\,s-\ell-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\le C_\ell\inf_{\substack{U\in H^{s,p}(\mathbb{R}^n)\\ r_+U=u}}\|U\|_{H^{s,p}(\mathbb{R}^n)}=C_\ell\|u\|_{H^{s,p}(\mathbb{R}^n_+)}.$$

Summing the finitely many estimates proves continuity of $\boldsymbol{\gamma}_j$.

To prescribe the $j+1$ boundary data simultaneously, construct a normal profile for each derivative and estimate the sum of the extensions. Let $(\Delta'_k)_{k\ge0}$ be a dyadic resolution in the tangential variables. Choose $\vartheta\in\mathcal{S}(\mathbb{R})$ with compactly supported Fourier transform and $\vartheta(0)\neq0$. The finite matrix

$$\Big(\frac{d^m}{dt^m}\big(t^r\vartheta(t)\big)\Big|_{t=0}\Big)_{0\le m,r\le j}$$

is triangular with diagonal $r!\vartheta(0)$ and hence invertible. Thus, there exist linear combinations $\varphi_0,\dots,\varphi_j$ of the functions $\vartheta,t\vartheta,\dots,t^j\vartheta$ such that

$$\varphi_\ell^{(m)}(0)=\delta_{\ell m},\qquad 0\le\ell,m\le j. \tag{14.45}$$

All φ_ℓ are Schwartz functions with compactly supported Fourier transforms. For a datum g, define on the full space

$$(\mathbf{E}_\ell g)(x',t):=\sum_{k=0}^{\infty}2^{-k\ell}\varphi_\ell(2^kt)\Delta'_kg(x'). \tag{14.46}$$

We record the discrete estimate controlling superposition in the normal variable. For every $\varphi\in\mathcal{S}(\mathbb{R})$, every

nonnegative sequence $(a_k)_{k\ge 0}$, and $1<p<\infty$,

$$\int_{\mathbb{R}} \left(\sum_{k\in\mathbb{N}_0} 2^{\frac{2k}{p}} |\varphi(2^k t)|^2 a_k^2 \right)^{\frac{p}{2}} dt \le C_{\varphi,p} \sum_{k\in\mathbb{N}_0} a_k^p. \tag{14.47}$$

Indeed, decompose $\mathbb{R}\setminus\{0\}$ into the annuli $A_m := \{t\in\mathbb{R} \mid 2^{-m-1}<|t|\le 2^{-m}\}$, $m\in\mathbb{Z}$, and extend a_k by zero for $k<0$. If $t\in A_m$, boundedness of φ for $k\le m$ and its decay of order $N>\frac{1}{p}$ for $k>m$ give

$$2^{\frac{k}{p}}|\varphi(2^k t)| \le C_N 2^{\frac{m}{p}} b_{k-m}, \qquad b_r := \begin{cases} 2^{\frac{r}{p}}, & r\le 0,\\ 2^{-(N-\frac{1}{p})r}, & r>0. \end{cases}$$

The sequence b belongs to $\ell^1(\mathbb{Z})$. Since the ℓ^2 norm does not exceed the ℓ^1 norm, the integral over A_m, whose measure is comparable to 2^{-m}, is bounded by $C(\sum_{k\in\mathbb{Z}} b_{k-m}a_k)^p$. Summing over m, Young's inequality for sequences from Proposition B.8.4 proves (14.47).

For $k\ge 1$, the Fourier transform of the kth summand of (14.46) lies in an annulus of radius comparable to 2^k: the tangential frequency already lies in that annulus, and the normal frequency lies in a ball of radius $C2^k$. Finite interaction of blocks and Lemma 6.3.6 give the synthesis estimate

$$\|\mathbf{E}_\ell g\|_{F^s_{p,2}(\mathbb{R}^n)} \le C \left\| \left(\sum_{k\in\mathbb{N}_0} 2^{2k(s-\ell)} |\varphi_\ell(2^k t)\Delta'_k g(x')|^2 \right)^{\frac12} \right\|_{L^p(\mathbb{R}^{n-1}\times\mathbb{R})}.$$

Apply (14.47) for each x', with $a_k(x') = 2^{k(s-\ell-\frac{1}{p})}|\Delta'_k g(x')|$, and integrate in x'. This gives

$$\|\mathbf{E}_\ell g\|^p_{H^{s,p}(\mathbb{R}^n)} \le C \sum_{k\in\mathbb{N}_0} 2^{kp(s-\ell-\frac{1}{p})} \|\Delta'_k g\|^p_{L^p(\mathbb{R}^{n-1})} = C\|g\|^p_{B^{s-\ell-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}. \tag{14.48}$$

In particular, the series in (14.46) converges in $H^{s,p}(\mathbb{R}^n)$. Define

$$\boldsymbol{e}_j(g_0,\dots,g_j) := \left(\sum_{\ell=0}^{j} \mathbf{E}_\ell g_\ell \right) \restriction_{\mathbb{R}^n_+}.$$

The quotient norm of the restriction space and (14.48) show that this operator is continuous. For Schwartz data, we can differentiate termwise at (14.46); by (14.45),

$$\partial_t^m \mathbf{E}_\ell g|_{t=0} = \delta_{m\ell} \sum_{k\in\mathbb{N}_0} \Delta'_k g = \delta_{m\ell} g, \qquad 0\le m\le j.$$

Density of $\mathcal{S}$ in each $B^{s-\ell-\frac{1}{p}}_{p,p}$, together with the already proved continuity of the trace operators, extends this identity to all data. Consequently, $\boldsymbol{\gamma}_j \boldsymbol{e}_j = I$, and

$$\|\boldsymbol{e}_j(g_0,\dots,g_j)\|_{H^{s,p}(\mathbb{R}^n_+)} \le C \sum_{\ell=0}^{j} \|g_\ell\|_{B^{s-\ell-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

For $j = 0$, assertions F and B are those of Theorem 14.1.1 with $c = 1$, restricted to the half-space. □

Theorem 14.4.18. (Trace and normal derivatives at the boundary) Let M be a manifold of dimension n with nonempty boundary and bounded geometry, and let $1 < p < \infty$, $j \in \mathbb{N}_0$, and $s > j + \frac{1}{p}$. For $u \in C_c^\infty(M)$, define

$$\partial_{\boldsymbol{\nu}}^{\ell} u(x) := \frac{d^\ell}{dt^\ell} u(\mathcal{C}(x,t))\Big|_{t=0}, \qquad 0 \le \ell \le j.$$

Then

$$\mathbf{Tr}_{\partial M, j} u := \big(u \restriction_{\partial M}, \partial_{\boldsymbol{\nu}} u, \dots, \partial_{\boldsymbol{\nu}}^{j} u\big)$$

extends to a retraction

$$\mathbf{Tr}_{\partial M, j} \colon H^{s,p}(M) \longrightarrow \prod_{\ell=0}^{j} B_{p,p}^{s-\ell-\frac{1}{p}}(\partial M). \tag{14.49}$$

In particular,

$$\mathrm{Tr}_{\partial M} \colon H^{s,p}(M) \longrightarrow B_{p,p}^{s-\frac{1}{p}}(\partial M)$$

is surjective and has a continuous linear right extension.

Proof. Let $(U_i^\partial, \kappa_i^\partial, h_i)_{i \in I_\partial}$ be the boundary part of the system in Proposition 14.4.5, and write

$$\kappa_i^{\partial,0}(u) := \kappa_i^\partial(u, 0), \qquad h_i^\partial := h_i \restriction_{\partial M}.$$

By (14.31), on a neighborhood of $t = 0$,

$$\partial_t^\ell \big((h_i u) \circ \kappa_i^\partial\big) \restriction_{t=0} = (h_i^\partial \circ \kappa_i^{\partial,0})\big((\partial_{\boldsymbol{\nu}}^\ell u) \circ \kappa_i^{\partial,0}\big). \tag{14.50}$$

No lower-order Leibniz terms appear; this is the reason for condition (14.31).
Apply $\boldsymbol{\gamma}_j$ of Theorem 14.4.17 to each $(h_i u) \circ \kappa_i^\partial$. If $A(\beta)$ is the set of charts meeting boundary chart β, compatibility with multipliers and tangential changes gives, for $\ell \in \{0, \dots, j\}$,

$$\big\| (h_\beta^\partial\, \partial_{\boldsymbol{\nu}}^\ell u) \circ \kappa_\beta^{\partial,0} \big\|_{B_{p,p}^{s-\ell-\frac{1}{p}}(\mathbb{R}^{n-1})} \le C \sum_{i \in A(\beta)} \big\| (h_i u) \circ \kappa_i^\partial \big\|_{H^{s,p}(\mathbb{R}^n_+)}.$$

The cardinality of $A(\beta)$ is uniformly bounded. Raising to the power p, summing over β, and using the fact that each chart appears in uniformly finitely many sets $A(\beta)$ gives

$$\sum_{\ell=0}^{j} \|\partial_{\boldsymbol{\nu}}^\ell u\|_{B_{p,p}^{s-\ell-\frac{1}{p}}(\partial M)} \le C \|u\|_{H^{s,p}(M)}. \tag{14.51}$$

This defines the continuous extension of $\mathbf{Tr}_{\partial M, j}$.
For the coretraction, let $g = (g_0, \dots, g_j)$ be boundary data and set

$$g_{\ell,i} := (h_i^\partial g_\ell) \circ \kappa_i^{\partial,0}.$$

Choose $\chi_T \in C_c^\infty(B^{n-1}(0, 2r_B))$ equal to one on the tangential supports and $\chi_N \in C_c^\infty([0, 2r_B))$ equal to one

on $[0,r_0]$. Set $\psi(u,t)=\chi_T(u)\chi_N(t)$. Then $\psi \boldsymbol{e}_j(g_{0,i},\dots,g_{j,i})$ is supported in the chart, and its jet at $t=0$ is the local datum. Transport, extend by zero, and reconstruct:

$$\mathbf{Ex}_{\partial M,j} g := \sum_{i\in I_\partial} h_i \left[\psi\, \boldsymbol{e}_j(g_{0,i},\dots,g_{j,i})\right]\circ(\kappa_i^\partial)^{-1}. \tag{14.52}$$

Localizing $\mathbf{Ex}_{\partial M,j} g$ in chart β involves only the indices in $A(\beta)$. Uniform bounds for multipliers, chart transitions, and $\boldsymbol{e}_j$ bound this localization by the sum of the norms of $g_{\ell,i}$, with $i\in A(\beta)$ and $\ell\in\{0,\dots,j\}$. Raise to the power p and count overlaps again to obtain

$$\|\mathbf{Ex}_{\partial M,j} g\|_{H^{s,p}(M)} \le C \sum_{\ell=0}^{j} \|g_\ell\|_{B_{p,p}^{s-\ell-\frac{1}{p}}(\partial M)}.$$

The functions h_i are constant in the normal direction near the boundary, $\psi=1$ there, and $\boldsymbol{\gamma}_j \boldsymbol{e}_j = I$. Thus, taking the jet of (14.52) gives $\sum_{i\in I_\partial} (h_i^\partial)^2 g_\ell = g_\ell$ in each component. Hence,

$$\mathbf{Tr}_{\partial M,j}\mathbf{Ex}_{\partial M,j} = I.$$

□

Theorem 14.4.19. (F, B, and W traces at the boundary) Let M be a Riemannian manifold of dimension n with bounded geometry and nonempty boundary. Let $1<p<\infty$, $1\le q\le\infty$, and $s>\frac{1}{p}$. Then

$$\mathrm{Tr}_{\partial M}\colon F^s_{p,q}(M)\longrightarrow B^{s-\frac{1}{p}}_{p,p}(\partial M), \qquad \mathrm{Tr}_{\partial M}\colon B^s_{p,q}(M)\longrightarrow B^{s-\frac{1}{p}}_{p,q}(\partial M)$$

are retractions. If $s\notin\mathbb{N}$,

$$\mathrm{Tr}_{\partial M}\colon W^{s,p}(M)\longrightarrow B^{s-\frac{1}{p}}_{p,p}(\partial M)$$

is also a retraction; for $m\in\mathbb{N}$, $m>\frac{1}{p}$, the same holds with $W^{m,p}(M)$ and $B^{m-\frac{1}{p}}_{p,p}(\partial M)$.

Proof. Take $j=0$ in formulas (14.49) and (14.52). If $u_i=(h_i u)\circ\kappa_i^\partial$, the Euclidean half-space trace, tangential coordinate changes, and uniform overlap count give

$$\|\mathrm{Tr}_{\partial M}\, u\|^p_{B^{s-\frac{1}{p}}_{p,p}(\partial M)} \le C\sum_{i\in I_\partial}\|u_i\|^p_{F^s_{p,q}(\mathbb{R}^n_+)} \le C'\|u\|^p_{F^s_{p,q}(M)}.$$

For $g\in B^{s-\frac{1}{p}}_{p,p}(\partial M)$, the same count applied to the reconstruction formula, now with the Euclidean coretraction, gives

$$\|\mathrm{Ex}_{\partial M,0}\, g\|_{F^s_{p,q}(M)} \le C\|g\|_{B^{s-\frac{1}{p}}_{p,p}(\partial M)}.$$

Since the cutoffs are constant in the normal direction where the trace is computed, the identity $\sum_{i\in I_\partial}(h_i^\partial)^2=1$ implies $\mathrm{Tr}_{\partial M}\,\mathrm{Ex}_{\partial M,0}=I$.

For the Besov scale, choose $s_0<s<s_1$ with $s_0>\frac{1}{p}$ and apply Theorem 5.3.1 to the same operators at the

endpoints. The interpolation identities

$$(F^{s_0}_{p,p}(M), F^{s_1}_{p,p}(M))_{\theta,q} = B^{s}_{p,q}(M)$$

and

$$(B^{s_0-\frac{1}{p}}_{p,p}(\partial M), B^{s_1-\frac{1}{p}}_{p,p}(\partial M))_{\theta,q} = B^{s-\frac{1}{p}}_{p,q}(\partial M)$$

give the second retraction.

If $s \notin \mathbb{N}$, use restriction $W^{s,p}(\mathbb{R}^n_+)$ norms in boundary charts, never extension by zero across $t = 0$. Proposition 14.5.4 proves that $W^{s,p}(M) = B^{s}_{p,p}(M)$. This identification reduces the assertion to the Besov retraction with $q = p$; the target space remains $B^{s-\frac{1}{p}}_{p,p}(\partial M)$. For $m \in \mathbb{N}$, the boundary equivalence of Proposition 14.5.3, applied to the trivial bundle, reduces the assertion to Theorem 14.4.18 with $j = 0$. □

14.5 Version for vector bundles of bounded geometry

The trace of a section takes values in the restricted bundle. To construct it through components, choose frames adapted to the charts and parallel transport. Bounds on their transition matrices ensure uniform constants in the scalar estimates; compatibility with frame changes allows local traces to be assembled into a section of the restricted bundle.

We use the notion of a bundle of bounded geometry from Definition 13.3.13. If M has boundary, the same expression is understood up to ∂M: require the bound (13.44) throughout M, using the induced product connection on $\Lambda^2 T^*M \otimes \mathrm{End}(\mathbf{E})$. Uniform Fermi charts allow these bounds to be formulated with ordinary derivatives up to the boundary.

Proposition 14.5.1. (Uniform synchronous frames) Let M be a Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry.

(a) If M has no boundary, its uniform geodesic charts admit synchronous orthonormal frames $\mathbf{e}_i = (\mathbf{e}_{i,1}, \dots, \mathbf{e}_{i,r})$ such that the transition matrices, their inverses, and all their derivatives are bounded by constants independent of the indices.

(b) If (M, N) is a pair of bounded geometry consisting of manifolds without boundary, the frames can be constructed in the Fermi charts of Proposition 14.2.6: first transport a basis of $\mathbf{E}_{p_i}$ along radial geodesics of N, then along normal geodesics of N in M. Their restrictions to N are uniform synchronous frames of $\mathbf{E}\restriction_N$.

(c) If M has nonempty boundary, in a boundary chart $\kappa^{\partial}_i(u,t)$ the frame may be chosen parallel along the curves $t \mapsto \kappa^{\partial}_i(u,t)$. If $\nabla^{\mathbf{E}} = d + \mathbf{A}_i$ in this frame, then

$$\mathbf{A}_i(\partial_t) = 0. \tag{14.53}$$

In all three cases, the components of $\mathbf{A}_i$ and all their ordinary derivatives are uniformly bounded.

Proof. Item (a), including the bounds on transition matrices and connection coefficients, is Proposition 8.2.26. The radial formula producing these bounds was established in the chapter on connections, in Proposition 8.2.25.

For (b), first apply Proposition 8.2.26 to the restricted bundle $\mathbf{E}\restriction_N$. Its curvature is

$$\mathbf{R}^{\mathbf{E}\restriction_N}(X,Y) = \mathbf{R}^{\mathbf{E}}(X,Y), \qquad X, Y \in TN,$$

, and derivatives for the product connection of N are finite sums of restrictions of derivatives of $\mathbf{R}^{\mathbf{E}}$ and contractions with $\mathbf{II}$. Lemma 14.2.3 and the pair hypotheses therefore provide the required bounds.

From the frame obtained over N, extend each vector by parallel transport along

$$s \longmapsto \exp^{\perp}_{x_i(u)}\Big(s \sum_{a=1}^{n-k} z^a \boldsymbol{\nu}_{i,a}(x_i(u))\Big).$$

Write frames as columns and use the convention

$$\nabla^{\mathbf{E}} \mathbf{e}_i = \mathbf{e}_i \boldsymbol{\Omega}_i, \qquad \mathscr{F}_i = d\boldsymbol{\Omega}_i + \boldsymbol{\Omega}_i \wedge \boldsymbol{\Omega}_i.$$

Transport imposes the normal radial gauge

$$\sum_{a=1}^{n-k} z^a \Omega_{i,a}(z,u) = 0.$$

Contracting the curvature equation with the radial field gives the exact identities

$$\Omega_{i,b}(z,u) = \int_0^1 s \sum_{a=1}^{n-k} z^a \mathscr{F}_{i,ab}(sz,u)\, ds, \tag{14.54}$$

$$\Omega_{i,\mu}(z,u) = \Omega_{i,\mu}(0,u) + \int_0^1 \sum_{a=1}^{n-k} z^a \mathscr{F}_{i,a\mu}(sz,u)\, ds. \tag{14.55}$$

Here $k = \dim N$, normal indices a, b belong to $\{1,\dots,n-k\}$, and $\mu \in \{1,\dots,k\}$ is tangential. The first identity follows by integrating

$$\sum_{a=1}^{n-k} z^a \mathscr{F}_{i,ab} = \sum_{a=1}^{n-k} z^a \partial_a \Omega_{i,b} + \Omega_{i,b}$$

along the radius; the second follows from

$$\sum_{a=1}^{n-k} z^a \mathscr{F}_{i,a\mu} = \sum_{a=1}^{n-k} z^a \partial_a \Omega_{i,\mu}.$$

Commutators containing $\displaystyle\sum_{a=1}^{n-k} z^a \Omega_{i,a}$ vanish by radial gauge.

We prove the connection and curvature-component bounds simultaneously. Denote by A_m and F_m common bounds for the supremum norms of partial derivatives of order at most m of $\Omega_{i,A}$ and $\mathscr{F}_{i,AB}$, respectively, where $i \in I$ and $A, B \in \{1,\dots,n\}$. At order zero, F_0 is controlled by $\|\mathbf{R}^{\mathbf{E}}\|_{\infty}$, uniform comparison of coordinate

bases, and orthonormality of the frame. The two integral formulas then give A_0, using the frame bounds over N.

Suppose A_{m-1} and F_{m-1} have been obtained, with $m \in \mathbb{N}$. Apply Lemma 9.1.28 to the tensor $\mathbf{R}^{\mathbf{E}}$ taking values in $\mathrm{End}(\mathbf{E})$. Solving for its leading term expresses each partial derivative of order m as the corresponding component of $\nabla^m \mathbf{R}^{\mathbf{E}}$ plus a finite sum of partial derivatives of curvature of order less than m. The coefficients of this sum use only derivatives of Ω_i and the Christoffel symbols of order at most $m-1$. The former are controlled by A_{m-1} and the latter by Proposition 14.2.6. This gives F_m from A_{m-1}, F_{m-1} and $\|\nabla^m \mathbf{R}^{\mathbf{E}}\|_\infty$.

Now differentiate the integral formulas. For $\alpha = (\alpha_z, \alpha_u) \in \mathbb{N}_0^n$ and $|\alpha| = m$, applying the Leibniz rule to the integrand without the outer factor s gives

$$D^\alpha \left(z^a \mathscr{F}_{i,aB}(sz,u) \right) = z^a s^{|\alpha_z|} (D^\alpha \mathscr{F}_{i,aB})(sz,u) + \alpha_a s^{|\alpha_z|-1} (D^{\alpha - e_a} \mathscr{F}_{i,aB})(sz,u).$$

Omit the second term if $\alpha_a = 0$; thus, no negative power of s is ever integrated. If $r_\perp$ is the common normal radius and A_m^N bounds derivatives of the initial frame over N, then

$$A_m \le A_m^N + C_{n,m}\left(r_\perp F_m + m F_{m-1} \right) + A_{m-1}.$$

We have obtained F_m before using it to compute A_m. The recurrence starts from A_0, F_0 and provides bounds at every order, without requiring an order-m connection bound in the step establishing it.

In a boundary chart, the same construction along $t \mapsto \kappa_i^\partial(u,t)$ gives

$$\mathbf{\Omega}_i(\partial_t) = 0.$$

The curvature equation reduces exactly to

$$\partial_t \Omega_{i,\mu} = \mathscr{F}_{i,t\mu}, \qquad \Omega_{i,\mu}(u,t) = \Omega_{i,\mu}(u,0) + \int_0^t \mathscr{F}_{i,t\mu}(u,\tau)\, d\tau.$$

For purely tangential derivatives of order m, differentiate this integral m times in u; the order-m bound for $\mathscr{F}$ is obtained first by the lemma of Chapter 9, exactly as above. If at least one normal derivative occurs, use $\partial_t \Omega_{i,\mu} = \mathscr{F}_{i,t\mu}$: a total derivative of order m of the connection is expressed through a derivative of order $m-1$ of curvature. The initial case is the order-zero integral and the frame value at $t = 0$. These two cases cover every multi-index $\alpha \in \mathbb{N}_0^n$ and complete the recurrence in boundary charts.

It remains to control the transition matrices. On an overlap, after expressing both frames in the same coordinates, write

$$\mathbf{e}_i = \mathbf{e}_j G_{ji}.$$

Since the frames are orthonormal, $G_{ji} \in O(r)$ in the real case and $G_{ji} \in U(r)$ in the complex case; thus, $G_{ji}^{-1} = G_{ji}^{\mathsf{T}}$ in the former and $G_{ji}^{-1} = G_{ji}^{*}$ in the latter. If $\widetilde{\Omega}_j$ is the pullback of Ω_j under the chart transition, compatibility of connections gives the exact equation

$$dG_{ji} = G_{ji}\mathbf{\Omega}_i - \widetilde{\Omega}_j G_{ji}. \tag{14.56}$$

The operator norm of G_{ji} is one. To pass from order m to order $m+1$, let $\alpha \in \mathbb{N}_0^n$, $|\alpha| = m$, and $A \in \{1, \dots, n\}$.

The Leibniz rule gives

$$D^\alpha \partial_A G_{ji} = \sum_{\substack{\beta\in\mathbb{N}_0^n \\ \beta\le\alpha}} \binom{\alpha}{\beta} \left[(D^\beta G_{ji})(D^{\alpha-\beta}\Omega_{i,A}) - (D^\beta \widetilde{\Omega}_{j,A})(D^{\alpha-\beta} G_{ji}) \right].$$

Every derivative of G_{ji} on the right has order at most m. Derivatives of the connection forms are bounded by the preceding step, and those of the pullback by Faà di Bruno and the chart-transition bounds. If G_m is a common bound through order m and D_m controls these forms, we may take $G_{m+1} = G_m + 2^{m+1} D_m G_m$ after incorporating the matrix-norm constants into D_m. Starting from $G_0 = 1$, this bounds every derivative. The identity $G_{ji}^{-1} = G_{ji}^*$, or the transpose in the real case, gives the same bounds for the inverse. At $z = 0$, we recover the synchronous frame constructed over N, also proving the restriction assertion. This concludes the proof. □

Definition 14.5.2. (Function spaces of sections) Let M be a Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Fix an admissible quadratic localization system

$$\mathcal{Q} = (U_i, \varphi_i, h_i)_{i\in I}$$

and, on each U_i, one of the synchronous frames $\mathbf{e}_i = (\mathbf{e}_{i,1}, \dots, \mathbf{e}_{i,r})$ constructed in Proposition 14.5.1. Let $\mathbf{u} \in \mathcal{D}'(M, \mathbf{E})$; when there is a boundary, this means $\mathcal{D}'(\operatorname{Int} M, \mathbf{E}|_{\operatorname{Int} M})$, as in the scalar definition. In chart i, denote by u_i^a the distributional components of $\mathbf{u}$ in frame $\mathbf{e}_i$, defined by Proposition 10.2.34. Thus, the component of $h_i\mathbf{u}$ is $h_i u_i^a$. Set $\mathbb{X}_i = \mathbb{R}^n$ in an interior chart and $\mathbb{X}_i = \mathbb{R}^n_+$ in a boundary chart.
For $s \in \mathbb{R}$ and $1 < p < \infty$, define

$$H^{s,p}(M, \mathbf{E}) := \left\{ \mathbf{u} \in \mathcal{D}'(M, \mathbf{E}) \,\middle|\, \|\mathbf{u}\|_{H^{s,p}(M,\mathbf{E})} < \infty \right\},$$

where

$$\|\mathbf{u}\|_{H^{s,p}(M,\mathbf{E})} := \left(\sum_{i\in I} \sum_{a=1}^{r} \left\| (h_i u_i^a) \circ \varphi_i^{-1} \right\|_{H^{s,p}(\mathbb{X}_i)}^p \right)^{\frac{1}{p}}. \tag{14.57}$$

For $1 \le p < \infty$ and $1 \le q \le \infty$, define $F^s_{p,q}(M, \mathbf{E})$ analogously using the norm

$$\left(\sum_{i\in I} \sum_{a=1}^{r} \left\| (h_i u_i^a) \circ \varphi_i^{-1} \right\|_{F^s_{p,q}(\mathbb{X}_i)}^p \right)^{\frac{1}{p}}.$$

Finally, if $s = (1-\theta)s_0 + \theta s_1$, with $s_0 < s < s_1$, define

$$B^s_{p,q}(M, \mathbf{E}) := \left(F^{s_0}_{p,p}(M, \mathbf{E}), F^{s_1}_{p,p}(M, \mathbf{E}) \right)_{\theta,q}.$$

On the half-space, always use the restriction quotient norms of Definition 14.4.9. Definitions on N and ∂M use the restricted frames and constructions without boundary.

In boundary charts, extensions by zero are taken only across artificial faces. The face representing ∂M belongs to the model $\mathbb{R}^n_+$ and is treated by the extension operator of Chapter 6. Fixed coordinate cutoffs leave a common margin from these artificial faces; thus, multiplication and extension constants are the same throughout the

family. In an interior chart, apply Lemma 13.1.15.

Proposition 14.5.3. (Intrinsic character and integer-order equivalence) Let M be a Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ be two admissible quadratic localization systems equipped with the synchronous frames of Proposition 14.5.1. For each scale $A \in \{H, F, B\}$, write $\|\mathbf{u}\|_{A_{\mathcal{Q}}}$ for the localized norm constructed with $\mathcal{Q}$. For every admissible set of indices, there exist $c_A, C_A > 0$, independent of $\mathbf{u}$, such that

$$c_A \|\mathbf{u}\|_{A_{\mathcal{Q}}} \le \|\mathbf{u}\|_{A_{\widetilde{\mathcal{Q}}}} \le C_A \|\mathbf{u}\|_{A_{\mathcal{Q}}}.$$

The constants depend on the function-space indices, multiplicity, and uniform transition, cutoff, and frame bounds, but not on the charts. For $m \in \mathbb{N}_0$ and $1 < p < \infty$, $H^{m,p}(M, \mathbf{E}) = W^{m,p}(M, \mathbf{E})$ and there exist $c_{m,p}, C_{m,p} > 0$, independent of $\mathbf{u}$, such that

$$c_{m,p} \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le \|\mathbf{u}\|_{H^{m,p}(M,\mathbf{E})} \le C_{m,p} \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

More precisely, if $N_{m,p}(\mathbf{u})$ denotes the localized norm of (14.57), there exist $c'_{m,p}, C'_{m,p} > 0$ such that

$$c'_{m,p} \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le N_{m,p}(\mathbf{u}) \le C'_{m,p} \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})},$$

where the covariant norm is

$$\left(\sum_{\ell=0}^{m} \|(\nabla^{\mathbf{E}})^{\ell} \mathbf{u}\|^{p}_{L^p(M, T^{(0,\ell)}(TM) \otimes \mathbf{E})} \right)^{\frac{1}{p}}. \tag{14.58}$$

Comparison of covariant and partial derivatives holds more generally for $1 \le p < \infty$. If $\mathbf{u} = \sum_{a=1}^{r_{\mathbf{E}}} u_i^a \mathbf{e}_{i,a}$ on U_i, set

$$N^W_{m,p}(\mathbf{u}) := \left(\sum_{i \in I} \sum_{a=1}^{r_{\mathbf{E}}} \|(h_i u_i^a) \circ \varphi_i^{-1}\|^{p}_{W^{m,p}(\mathbb{X}_i)} \right)^{1/p}. \tag{14.59}$$

There exist $c^W_{m,p}, C^W_{m,p} > 0$ such that

$$c^W_{m,p} \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})} \le N^W_{m,p}(\mathbf{u}) \le C^W_{m,p} \|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

Smooth sections with compact support in M, which may meet the boundary, are dense in these spaces for every $1 \le p < \infty$. Identification with $H^{m,p}$ holds in the range $1 < p < \infty$.

Proof. On an overlap, component vectors are related by $u_i = G_{ij} u_j$. Proposition 14.5.1 bounds all derivatives of G_{ij}, and Lemma 6.4.3 controls matrix multiplication and coordinate change simultaneously. Uniform multiplicity and the finite-sum inequality used in Proposition 14.4.11 produce the constants c_A, C_A in the stated inequalities. The argument holds in F, and real interpolation transfers it to B.

For the integer-order comparison, use Lemma 9.1.28 in each regular chart. If $q \in \mathbb{N}$, $a \in \{1, \dots, r_{\mathbf{E}}\}$, and

$(i_1,\dots,i_q)\in\{1,\dots,n\}^q$, its formula is

$$((\nabla^{\mathbf{E}})^q\mathbf{u})^a_{i_1\cdots i_q}=\partial_{i_1}\cdots\partial_{i_q}u^a+\sum_{b=1}^{r_{\mathbf{E}}}\sum_{\substack{\beta\in\mathbb{N}_0^n\\|\beta|\le q-1}}(B_\beta)^a_{b,i_1\cdots i_q}\partial^\beta u^b.$$

The coefficients in that lemma use connection derivatives through order $q-1$; their supremum norms are uniform by the frame proposition. Let P_q be the sum of the L^p norms of partial derivatives of all components through order q, and let V_q be the sum of the L^p norms of covariant derivatives through that order in the chart considered. Uniform comparison of metrics and density gives $P_0\le K_0V_0$. If $P_{q-1}\le K_{q-1}V_{q-1}$, solving for the leading term in the preceding formula gives

$$P_q\le P_{q-1}+C_qV_q+C_qP_{q-1}\le\big(C_q+(1+C_q)K_{q-1}\big)V_q.$$

Thus, $K_q=C_q+(1+C_q)K_{q-1}$ can be chosen independently of the chart. The reverse inequality follows from the same formula without solving for the leading term: $V_q\le C_q'P_q$. Coefficients in both comparisons are those in the lemma of Chapter 9, whose bounds are uniform in the trivializations considered.
Apply the comparisons to $h_i\mathbf{u}$. The Leibniz rule for the product connection controls its derivatives by

$$|\nabla^q(h_i\mathbf{u})|\le C_q\sum_{b=0}^{q}|\nabla^{q-b}h_i||\nabla^b\mathbf{u}|,\qquad q\in\{0,\dots,m\}.$$

The supremum norms of cutoff derivatives and uniform multiplicity allow raising to the power p, summing over $i\in I$, and integrating. For the reverse inequality, use $\mathbf{u}=\sum_{i\in I}h_i(h_i\mathbf{u})$ and the same Leibniz rule; uniformly finitely many summands contribute at each point. Equivalence between a sum of finitely many norms and their ℓ^p sum adds only factors depending on $m,n,r_{\mathbf{E}},p$. All these steps hold for $1\le p<\infty$: only triangle inequalities, the Leibniz rule, and sums of bounded multiplicity were used. They prove both bounds in $N^W_{m,p}$, including $p=1$. For $1<p<\infty$, Theorem 6.3.10 identifies Euclidean integer-order norms with Bessel norms; on the half-space, apply it after the extension and restriction of Chapter 6. This provides the two asserted global constants. These triangular formulas also hold for weak derivatives by Lemma 9.1.44. Thus, the comparisons apply directly to each given section without presupposing an approximation already satisfying trace conditions.
We specify the density used later. The series $\mathbf{u}=\sum_{i\in I}h_i(h_i\mathbf{u})$ converges in $W^{m,p}$: its tail norms are controlled by tails of the localized ℓ^p sum. In each of the finitely many charts of a partial sum, extend components to the full space (using $\mathcal{E}_m$ from Theorem 3.6.4 in a boundary chart), regularize, and multiply by a fixed auxiliary cutoff. Restriction to the half-space gives approximations smooth up to $t=0$. Reconstruction yields sections in $\Gamma_c(M,\mathbf{E})$ converging in $W^{m,p}$. Their supports are not required to avoid the boundary. This approximation holds for every $1\le p<\infty$ by continuity of translation and regularization in L^p, applied to the finitely many derivatives. For $1<p<\infty$, the same construction works in $H^{m,p}$ and, together with the preceding comparisons, proves equality of the two spaces as subspaces of distributions on $\operatorname{Int}M$. □

The integer-order comparison identifies the derivatives entering the norm. At a noninteger order, we still need to compare values in different fibers. The error between parallel transport and component differences is first order in the distance and is therefore integrable in the fractional seminorm.

Proposition 14.5.4. (Uniform Slobodeckij localization) Let M and $\mathbf{E} \to M$ have bounded geometry, with or without boundary, let $1 < p < \infty$ and $s = m + \sigma$, where $m \in \mathbb{N}_0$ and $0 < \sigma < 1$. In a uniform quadratic system with synchronous frames, set

$$\|\mathbf{u}\|_{W^{s,p}_{\mathcal{Q}}(M,\mathbf{E})} := \left(\sum_{i \in I} \sum_{a=1}^{r_{\mathbf{E}}} \left\| (h_i u_i^a) \circ \varphi_i^{-1} \right\|^p_{W^{s,p}(\mathbb{X}_i)} \right)^{1/p}.$$

In interior charts, extend by zero outside the chart; in boundary charts, extend by zero only across artificial faces and take the restriction norm on the half-space. These norms define a space $W^{s,p}(M,\mathbf{E})$ independently of the chosen system, and

$$W^{s,p}(M,\mathbf{E}) = B^s_{p,p}(M,\mathbf{E}) \quad \text{con normas equivalentes.}$$

If M has no boundary, this space agrees with the one defined using $\mathbf{u} \in W^{m,p}(M,\mathbf{E})$ and the seminorm

$$[\nabla^m_w \mathbf{u}]^p_{\sigma,p;r} := \int_M \int_{0<d_{\mathbf{g}}(x,y)<r} \frac{|\nabla^m_w \mathbf{u}(x) - P^{\mathbf{E}_m}_{y\to x} \nabla^m_w \mathbf{u}(y)|^p}{d_{\mathbf{g}}(x,y)^{n+\sigma p}} \, d\lambda_{\mathbf{g}}(y) \, d\lambda_{\mathbf{g}}(x), \tag{14.60}$$

where $\mathbf{E}_m = T^{(0,m)}(TM) \otimes \mathbf{E}$, transport is taken along the unique short geodesic, and $r > 0$ is a sufficiently small uniform radius. Replacing r by another positive radius below the uniform radius gives an equivalent norm.

Proof. In each Euclidean model, the identification $W^{s,p} = B^s_{p,p} = F^s_{p,p}$ has constants depending only on n, s, p. It also holds for restriction norms on the half-space, by Euclidean extension and the comparisons of Chapter 6. Proposition 14.4.14 and its version without boundary in the preceding chapter show that the localized space $F^s_{p,p}$ agrees with $B^s_{p,p}$. For sections, use the same analysis and synthesis on every component: the rank is finite, matrix transitions are uniform multipliers, and the sums over charts, components, and blocks here have the same exponent p. Tonelli and the estimates in those proofs apply without changing their order. This gives equivalence with $B^s_{p,p}$ and, by Proposition 14.5.3, independence of the system. This part of the argument includes the boundary case.

Now suppose $\partial M = \varnothing$. Fix normal charts larger than those used by the cutoffs, and choose r smaller than their uniform margin. If one point lies in the support of h_i and its distance from the other is less than r, the short geodesic remains inside the larger chart. In the synchronous frame, the transport matrix $G_i(x,y)$ satisfies

$$\|G_i(x,y) - I\| + \|G_i(x,y)^{-1} - I\| \le C d_{\mathbf{g}}(x,y).$$

Indeed, integrate the transport equation; its coefficients are the bounded connection matrices evaluated on the curve velocity. Coordinate length does not exceed a constant times Riemannian length. Grönwall gives the inequality with C independent of i. The same proof applies to each tensor bundle $\mathbf{E}_m$.

Fix one of these bundles $\mathbf{E}_m$ and a section $\mathbf{v}$ of it; write v_i for its component vector. In both directions we obtain

$$|\mathbf{v}(x) - P_{y\to x}\mathbf{v}(y)| \le C\big(|v_i(x) - v_i(y)| + d_{\mathbf{g}}(x,y)|v_i(y)|\big),$$
$$|v_i(x) - v_i(y)| \le C\big(|\mathbf{v}(x) - P_{y\to x}\mathbf{v}(y)| + d_{\mathbf{g}}(x,y)|\mathbf{v}(y)|\big).$$

Distances and measures are uniformly comparable to their coordinate counterparts. The contribution of

the error terms is bounded by $C\|\mathbf{v}\|^p_{L^p(M,\mathbf{E}_m)}$, since

$$\sup_{y\in M}\int_{d_{\mathbf{g}}(x,y)<r} d_{\mathbf{g}}(x,y)^{-n+p(1-\sigma)}\,d\lambda_{\mathbf{g}}(x)\le C\int_0^r t^{p(1-\sigma)-1}\,dt<\infty.$$

This is the local comparison between the intrinsic seminorm and the component seminorm, now with a uniform constant.

To localize, we use the identity

$$h_i(x)\mathbf{v}(x)-h_i(y)P_{y\to x}\mathbf{v}(y)=h_i(x)\big(\mathbf{v}(x)-P_{y\to x}\mathbf{v}(y)\big)+(h_i(x)-h_i(y))P_{y\to x}\mathbf{v}(y).$$

We have $|h_i(x)-h_i(y)|\le Cd_{\mathbf{g}}(x,y)$, and at most $2L$ supports contribute to each pair. We raise to the power p, sum, and apply the preceding integral estimate. Conversely, we reconstruct $\mathbf{v}=\sum_{i\in I}h_i(h_i\mathbf{v})$ and use the same finite-sum estimate. The parts of the extension by zero lying outside the chart are separated from the support by a uniform distance $\delta>0$; their contribution is bounded by $C\delta^{-\sigma p}\|\mathbf{v}\|^p_{L^p(M,\mathbf{E}_m)}$. Thus the local norms and the $\|\mathbf{v}\|_{L^p(M,\mathbf{E}_m)}+[\mathbf{v}]_{\sigma,p;r}$ norm are equivalent.

For $s=m+\sigma$, we use the triangular expression for covariant derivatives in Lemma 9.1.28, both directly and to solve for the partial derivatives of order m. Each lower-order term contains a derivative of order at most $m-1$ and a uniformly smooth coefficient. It is controlled in $W^{\sigma,p}$ by $W^{1,p}\hookrightarrow W^{\sigma,p}$ and the multiplier estimate of Lemma 9.4.15. These terms can therefore be absorbed into $\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}$. The comparison at order σ applied to $\nabla^m_w\mathbf{u}$, together with the Leibniz rule for the cutoffs, proves equivalence with $\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}+[\nabla^m_w\mathbf{u}]_{\sigma,p;r}$. The local formulas hold for weak derivatives; density in the fractional norm has not been assumed.

Finally, if $0<r_1<r_2$ are two admissible radii, the difference between the integrals is taken over $r_1\le d_{\mathbf{g}}(x,y)<r_2$. The isometry of parallel transport and the uniform bound on the volume of $B_{\mathbf{g}}(y,r_2)$ bound this difference by $C_{r_1,r_2}\|\nabla^m_w\mathbf{u}\|^p_{L^p(M,\mathbf{E}_m)}$. This proves independence of the radius without requiring M to have finite total volume. □

Proposition 14.5.5. (Interpolation and complex Sobolev duality for bundles) Let $(M,\mathbf{g})$ be a manifold without boundary of bounded geometry, and let $(\mathbf{E},\mathbf{h}_{\mathbf{E}},\nabla^{\mathbf{E}})\to M$ be a complex Hermitian bundle of bounded geometry; we adopt the convention that $\mathbf{h}_{\mathbf{E}}$ is linear in the first variable. If $s_0<s_1$, $0<\theta<1$, $s_\theta=(1-\theta)s_0+\theta s_1$, and $1<p<\infty$, then

$$[H^{s_0,p}(M,\mathbf{E}),H^{s_1,p}(M,\mathbf{E})]_\theta=H^{s_\theta,p}(M,\mathbf{E}) \tag{14.61}$$

with equivalent norms. These identifications are induced by the identity on test sections and are independent of the admissible quadratic localization system and synchronous frames used to define the norms.

If $s\in\mathbb{R}$, $1<p<\infty$, and $p'=\frac{p}{p-1}$, the complex-bilinear pairing

$$\mathbf{E}_x\times\overline{\mathbf{E}}_x\longrightarrow\mathbb{C},\qquad (e,\overline{f})\longmapsto\mathbf{h}_{\mathbf{E}}(e,f),$$

followed by integration extends uniquely to a continuous bilinear pairing

$$\langle\cdot,\cdot\rangle_{s,p;\mathbf{E}}\colon H^{s,p}(M,\mathbf{E})\times H^{-s,p'}(M,\overline{\mathbf{E}})\longrightarrow\mathbb{C}. \tag{14.62}$$

The map

$$\mathcal{J}_{s,p,\mathbf{E}}\colon H^{-s,p'}(M,\overline{\mathbf{E}}) \longrightarrow (H^{s,p}(M,\mathbf{E}))', \qquad \mathcal{J}_{s,p,\mathbf{E}}(\mathbf{v})(\mathbf{u}) := \langle \mathbf{u},\mathbf{v}\rangle_{s,p;\mathbf{E}}, \tag{14.63}$$

is a complex-linear topological isomorphism. Pointwise conjugation also induces a complex-linear isometric isomorphism

$$\mathcal{C}_{s,p,\mathbf{E}}\colon \overline{H^{s,p}(M,\mathbf{E})} \longrightarrow H^{s,p}(M,\overline{\mathbf{E}}), \qquad \mathcal{C}_{s,p,\mathbf{E}}(\overline{\mathbf{u}}) := \overline{\mathbf{u}}. \tag{14.64}$$

Thus, equivalently,

$$\overline{H^{-s,p'}(M,\mathbf{E})} \cong (H^{s,p}(M,\mathbf{E}))'.$$

In the real case all bars are omitted.

These identifications are compatible with (14.61): when the orders s_0, s_1 are interpolated, the spaces in the second entry interpolate between orders $-s_0, -s_1$ and yield order $-s_\theta$ (interchanging the endpoints if necessary); the resulting pairing is the unique extension of the same integral pairing on $\Gamma_c(\mathbf{E})\times\Gamma_c(\overline{\mathbf{E}})$. Pointwise conjugation likewise commutes with these identifications.

Proof. Fix an admissible quadratic localization system $\mathcal{Q} = (U_i, \varphi_i, h_i)_{i\in I}$, with $\sum_{i\in I} h_i^2 = 1$, and unitary synchronous frames $\mathbf{e}_i = (\mathbf{e}_{i,1},\dots,\mathbf{e}_{i,r})$. Choose cutoffs $\chi_i \in C_c^\infty(U_i)$ equal to one on a neighborhood of $\operatorname{supp} h_i$, and use the conjugate frame $\overline{\mathbf{e}}_i$ in $\overline{\mathbf{E}}$. If

$$\mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a}, \qquad \mathbf{v} = \sum_{a=1}^{r} v_i^a \overline{\mathbf{e}}_{i,a},$$

and $\mathbf{v} = \overline{\mathbf{w}}$ is smooth, then $v_i^a = \overline{w_i^a}$ and $\mathbf{h}_{\mathbf{E}}(\mathbf{u},\mathbf{w}) = \sum_{a=1}^{r} u_i^a v_i^a$. Set

$$a_i(x) := \sqrt{\det\big((\varphi_i^{-1})^*\mathbf{g}\big)(x)}.$$

The functions a_i and a_i^{-1} and the cutoffs involved satisfy uniform multiplier bounds of every order.

Let $\langle z,\varphi\rangle_{\mathbb{R}^n}$ denote the Euclidean *bilinear* duality between $H^{-s,p'}(\mathbb{R}^n)$ and $H^{s,p}(\mathbb{R}^n)$. For $\mathbf{u}\in H^{s,p}(M,\mathbf{E})$ and $\mathbf{v}\in H^{-s,p'}(M,\overline{\mathbf{E}})$, define

$$\langle \mathbf{u},\mathbf{v}\rangle_{s,p;\mathbf{E}} := \sum_{i\in I}\sum_{a=1}^{r} \Big\langle (h_i v_i^a)\circ\varphi_i^{-1}, a_i\,(h_i u_i^a)\circ\varphi_i^{-1}\Big\rangle_{\mathbb{R}^n}. \tag{14.65}$$

Since multiplication by a_i is uniformly bounded on $H^{s,p}$, Euclidean duality and Hölder's inequality for sequences give

$$|\langle \mathbf{u},\mathbf{v}\rangle_{s,p;\mathbf{E}}| \le C\|\mathbf{u}\|_{H^{s,p}(M,\mathbf{E})}\|\mathbf{v}\|_{H^{-s,p'}(M,\overline{\mathbf{E}})}.$$

The series in (14.65) is therefore absolutely convergent. If $\mathbf{u},\mathbf{w}\in\Gamma_c(\mathbf{E})$ and $\mathbf{v}=\overline{\mathbf{w}}$, change of variables and $\sum_{i\in I} h_i^2 = 1$ give

$$\langle \mathbf{u},\overline{\mathbf{w}}\rangle_{s,p;\mathbf{E}} = \sum_{i\in I}\int_M h_i^2\,\mathbf{h}_{\mathbf{E}}(\mathbf{u},\mathbf{w})\,d\lambda_{\mathbf{g}} = \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},\mathbf{w})\,d\lambda_{\mathbf{g}}.$$

Smooth compactly supported sections are dense in both spaces: this follows by approximation with finitely supported sequences having Schwartz components in the retraction–coretraction used below. Thus the extension is unique.

Conjugation changes the components u_i^a into $\overline{u_i^a}$ and commutes with the cutoffs and real coordinate changes. Since conjugation is an isometry on each Euclidean space $H^{s,p}(\mathbb{R}^n,\mathbb{C}^r)$, the rule (14.64) is a linear isometric isomorphism; its linearity uses $\lambda\overline{\mathbf{u}}=\overline{\overline{\lambda}\mathbf{u}}$ in the conjugate domain.

We now construct a retraction that will prove interpolation and then the surjectivity of (14.63) without presupposing global duality itself. For each $r_0\in\mathbb{R}$, write

$$\mathbb{H}_{\mathbf{E}}^{r_0,p}:=\ell^p\bigl(I;H^{r_0,p}(\mathbb{R}^n,\mathbb{C}^r)\bigr)$$

and define, using the same formulas for every r_0,

$$\mathsf{A}_{\mathbf{E}}\mathbf{u}:=\bigl((h_i\mathbf{u}_i)\circ\varphi_i^{-1}\bigr)_{i\in I},$$

$$\mathsf{B}_{\mathbf{E}}(Z_i):=\sum_{i\in I}h_i\sum_{a=1}^{r}\Bigl[\bigl((\chi_i\circ\varphi_i^{-1})Z_i\bigr)\circ\varphi_i\Bigr]^a\mathbf{e}_{i,a},$$

where each summand is extended by zero. The uniform multiplier and coordinate-change theorems and finite multiplicity show that

$$\mathsf{A}_{\mathbf{E}}\colon H^{r_0,p}(M,\mathbf{E})\longrightarrow\mathbb{H}_{\mathbf{E}}^{r_0,p},\qquad \mathsf{B}_{\mathbf{E}}\colon\mathbb{H}_{\mathbf{E}}^{r_0,p}\longrightarrow H^{r_0,p}(M,\mathbf{E})$$

are continuous; moreover, $\mathsf{B}_{\mathbf{E}}\mathsf{A}_{\mathbf{E}}\mathbf{u}=\sum_{i\in I}h_i^2\mathbf{u}=\mathbf{u}$.

Apply these bounds with $r_0=s_0,s_1$. Theorem 6.3.11, applied to the r components, and Proposition 5.7.12 identify

$$[\mathbb{H}_{\mathbf{E}}^{s_0,p},\mathbb{H}_{\mathbf{E}}^{s_1,p}]_\theta=\mathbb{H}_{\mathbf{E}}^{s_\theta,p}$$

with equivalent norms. The complex interpolation theorem for operators 5.7.5, applied to $\mathsf{A}_{\mathbf{E}}$ and $\mathsf{B}_{\mathbf{E}}$, preserves the identity $\mathsf{B}_{\mathbf{E}}\mathsf{A}_{\mathbf{E}}=I$ and proves (14.61). Since the formulas for both operators are independent of r_0, this identification is the identity on test sections. Independence of the quadratic localization system also follows from Proposition 14.5.3.

Let $\Lambda\in(H^{s,p}(M,\mathbf{E}))'$. Euclidean bilinear duality and $(\ell^p(X_i))'=\ell^{p'}(X_i')$ provide

$$W=(W_i)_{i\in I}\in\ell^{p'}\Bigl(I;H^{-s,p'}(\mathbb{R}^n,\mathbb{C}^r)\Bigr)$$

such that

$$(\Lambda\circ\mathsf{B}_{\mathbf{E}})(Z)=\sum_{i\in I}\langle W_i,Z_i\rangle_{\mathbb{R}^n},\qquad \|W\|_{\ell^{p'}(I;H^{-s,p'}(\mathbb{R}^n,\mathbb{C}^r))}\le\|\Lambda\|_{(H^{s,p}(M,\mathbf{E}))'}\|\mathsf{B}_{\mathbf{E}}\|_{\mathcal{L}(\mathbb{H}_{\mathbf{E}}^{s,p},H^{s,p}(M,\mathbf{E}))}.\tag{14.66}$$

Define an $\overline{\mathbf{E}}$-valued distribution $\mathbf{v}$ by

$$\langle\mathbf{u},\mathbf{v}\rangle:=\sum_{i\in I}\Bigl\langle W_i,(h_i\mathbf{u}_i)\circ\varphi_i^{-1}\Bigr\rangle_{\mathbb{R}^n},\qquad\mathbf{u}\in\Gamma_c(\mathbf{E}).\tag{14.67}$$

The sum is finite for each $\mathbf{u}$.

We verify directly, without using the duality being proved, that $\mathbf{v}\in H^{-s,p'}(M,\overline{\mathbf{E}})$. For each j, let L_{ij} be the

local operator that sends the components of a section $\mathbf{u}$ in the frame $\mathbf{e}_j$ to

$$\mathsf{L}_{ij}(\mathbf{u}_j) := \big(h_i h_j \mathbf{u}_i\big) \circ \varphi_i^{-1}, \qquad U_i \cap U_j \neq \varnothing,$$

inserting χ_j before extending by zero. It is a composition of a uniform coordinate change, a uniform transition matrix, and a uniform multiplier. Thus

$$\mathsf{L}_{ij} \colon H^{s,p}(\mathbb{R}^n, \mathbb{C}^r) \longrightarrow H^{s,p}(\mathbb{R}^n, \mathbb{C}^r)$$

has norm bounded independently of i, j. Its bilinear transpose $\mathsf{L}_{ij}^{\mathsf{T}}$ satisfies the same bound between the spaces of order $-s$ and exponent p'. If v_j denotes the component vector of $\mathbf{v}$ in the frame $\overline{\mathbf{e}}_j$, definition (14.67) gives, in $\mathcal{D}'(\mathbb{R}^n, \mathbb{C}^r)$,

$$a_j\,(h_j v_j) \circ \varphi_j^{-1} = \sum_{\substack{i \in I \\ U_i \cap U_j \neq \varnothing}} \mathsf{L}_{ij}^{\mathsf{T}} W_i. \tag{14.68}$$

Indeed, both sides have the same action on every test function: this equality is precisely (14.67) applied to $h_j \mathbf{u}$ and written in the chart j. Uniform multiplicity, the bound on the transposes, and multiplication by a_j^{-1} give

$$\|\mathbf{v}\|_{H^{-s,p'}(M,\overline{\mathbf{E}})}^{p'} \leq C \sum_{j \in I} \left(\sum_{\substack{i \in I \\ U_i \cap U_j \neq \varnothing}} \|W_i\|_{H^{-s,p'}(\mathbb{R}^n, \mathbb{C}^r)} \right)^{p'} \leq C' \sum_{i \in I} \|W_i\|_{H^{-s,p'}(\mathbb{R}^n, \mathbb{C}^r)}^{p'}.$$

Thus $\mathbf{v}$ belongs to the asserted space, and its norm is bounded by a multiple of $\|\Lambda\|_{(H^{s,p}(M,\mathbf{E}))'}$.
Finally, for $\mathbf{u} \in H^{s,p}(M, \mathbf{E})$,

$$\Lambda(\mathbf{u}) = \Lambda(\mathsf{B}_{\mathbf{E}} \mathsf{A}_{\mathbf{E}} \mathbf{u}) = \sum_{i \in I} \langle W_i, (h_i \mathbf{u}_i) \circ \varphi_i^{-1} \rangle_{\mathbb{R}^n} = \langle \mathbf{u}, \mathbf{v} \rangle_{s,p;\mathbf{E}}.$$

Therefore $\mathsf{J}_{s,p,\mathbf{E}}$ is surjective, and the preceding construction proves continuity of its inverse. If $\mathsf{J}_{s,p,\mathbf{E}}(\mathbf{v}) = 0$, then $\mathbf{v}$ vanishes on $\Gamma_c(\mathbf{E})$; since $\mathbf{h}_{\mathbf{E}}$ identifies $\mathbf{E}$ linearly with $(\overline{\mathbf{E}})^*$, this means that $\mathbf{v}$ is the zero distribution. Thus the map is also injective. Its continuity has already been proved using (14.65).
Finally, the local formulas defining the pairing, conjugation, $\mathsf{A}_{\mathbf{E}}$, and $\mathsf{B}_{\mathbf{E}}$ are the same at every order. Density of $\Gamma_c(\mathbf{E})$ and $\Gamma_c(\overline{\mathbf{E}})$ and uniqueness of the continuous extension identify the pairing at the interpolated order with (14.62). This proves the asserted compatibility with interpolation and conjugation. The representative has been constructed using local duality and the analysis and synthesis operators. □

Theorem 14.5.6. (Identification of heat potentials with the local spaces) Let $(M, \mathbf{g})$ be a manifold without boundary of bounded geometry, and let $(\mathbf{E}, \mathbf{h}_{\mathbf{E}}, \nabla^{\mathbf{E}}) \to M$ be a finite-rank bundle of bounded geometry. For $s \in \mathbb{R}$ and $1 < p < \infty$, the identity on test sections induces the identifications

$$H_{L_{\mathbf{E}}}^{s,p}(M, \mathbf{E}) = H^{s,p}(M, \mathbf{E}) = F_{p,2}^{s}(M, \mathbf{E}). \tag{14.69}$$

In particular, for any admissible quadratic system $\mathcal{Q} = (U_i, \varphi_i, h_i)_{i \in I}$ with synchronous frames, there exist

$c, C > 0$ such that

$$c\|\mathbf{u}\|_{H^{s,p}_{L^{\mathbf{E}}}(M,\mathbf{E})} \le \left(\sum_{i\in I} \sum_{a=1}^{r_{\mathbf{E}}} \left\| \widetilde{(h_i u_i^a) \circ \varphi_i^{-1}} \right\|^p_{F^s_{p,2}(\mathbb{R}^n)} \right)^{1/p} \le C\|\mathbf{u}\|_{H^{s,p}_{L^{\mathbf{E}}}(M,\mathbf{E})}. \tag{14.70}$$

The constants depend on $n, r_{\mathbf{E}}, s, p$, the uniform bounds of the localization system, and finitely many bounded-geometry bounds for M and $\mathbf{E}$. They do not depend on the chart index, the number of charts, or the section $\mathbf{u}$. For $s < 0$, the first equality also gives a canonical injective realization of the completion defined by heat potentials as a space of $\mathbf{E}$-valued distributions.

Proof. The Euclidean identification $H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$ in Theorem 6.3.10 has constants depending only on n, s, p. Apply it to each component and sum the pth powers over $i \in I$ and $a \in \{1, \dots, r_{\mathbf{E}}\}$. This proves the second equality in (14.69) and its two norm comparisons for every real order.

First consider $s \ge 0$. For $s = 0$, all three norms are equivalent to the $L^p(M, \mathbf{E})$ norm, by uniform comparison of the measures and fiber norms. If $s = m$ is a positive integer, Theorem 13.3.25 and Proposition 14.5.3 give

$$H^{m,p}_{L^{\mathbf{E}}}(M, \mathbf{E}) = W^{m,p}(M, \mathbf{E}) = H^{m,p}(M, \mathbf{E}).$$

Both equalities preserve the section as a distribution, and the corresponding norms are bounded by one another.

For a positive noninteger order, choose $m \in \mathbb{N}$ with $m > s$ and set $\theta = s/m \in (0, 1)$. At the endpoints $0, m$ we have the same compatible pair of spaces, with equivalent norms. Complex interpolation of the identity and its inverse, by Theorem 5.7.5, preserves this equivalence. On the one hand, Theorem 13.3.27 gives

$$[L^p(M, \mathbf{E}), H^{m,p}_{L^{\mathbf{E}}}(M, \mathbf{E})]_\theta = H^{s,p}_{L^{\mathbf{E}}}(M, \mathbf{E}).$$

On the other hand, Proposition 14.5.5 identifies the same interpolated space with $H^{s,p}(M, \mathbf{E})$. The identity of the initial spaces therefore yields the first equality sought. The imaginary-power bound that makes this step possible is the one in Lemma 13.3.26, applied to the connection Laplacian of $\mathbf{E}$.

Negative order remains. Write $s = -a$, with $a > 0$, and $p' = p/(p-1)$. The conjugate bundle $\overline{\mathbf{E}}$ has the same geometric bounds as $\mathbf{E}$. The positive-order result already proved identifies

$$H^{a,p'}_{L^{\overline{\mathbf{E}}}}(M, \overline{\mathbf{E}}) = H^{a,p'}(M, \overline{\mathbf{E}})$$

with equivalent norms. Take the complex-linear duals of both sides and use Theorem 12.4.9 and Proposition 14.5.5, respectively. We obtain an isomorphism between $H^{-a,p}_{L^{\mathbf{E}}}(M, \mathbf{E})$ and $H^{-a,p}(M, \mathbf{E})$. On test sections this isomorphism is the identity: both dualities agree with the same bilinear integral pairing, using the metric to pair $\mathbf{E}$ with $\overline{\mathbf{E}}$. Density of test sections in the localized positive-order space uniquely determines the corresponding distribution. It also proves injectivity of this realization. In the real case conjugations are omitted.

Let us make the uniformity precise. Integer-order equivalence uses bounds on the metric, its inverse, the connection coefficients, and the cutoffs in the larger charts, together with the overlap multiplicity. The internal power estimate for the Bochner chain adds bounds on the potentials $\mathbf{V}_\ell$ and $\mathbf{W}_\ell$ of the finite chain corresponding to the integer m. These bounds are expressed in terms of the supremum norms of finitely many derivatives of the curvatures. Interpolation adds the constants of the two endpoint norms and those for imaginary powers; the latter depend only on p and the rank. For $s < 0$, the same data are used with $a = -s$

and p'. None of these steps introduces a sum of constants over all charts: overlaps are always counted using the same uniform bound. This gives the constants in (14.70). □

Remark 14.5.7. (Choice of operator in the presence of a boundary) The hypothesis $\partial M = \varnothing$ in the preceding theorem is necessary to identify the operator without imposing additional conditions. On a manifold with boundary, the localized space $H^{s,p}(M,\mathbf{E})$ is defined by restriction in boundary charts. Powers of a Dirichlet or Neumann realization incorporate the conditions of that realization. For example, on the half-space, for the scalar Dirichlet Laplacian on L^2,

$$\mathcal{D}(I + L_D) = H^{2,2}(\mathbb{R}^n_+) \cap H^{1,2}_0(\mathbb{R}^n_+),$$

whereas the localized norm of order two admits functions in $H^{2,2}(\mathbb{R}^n_+)$ with nonzero trace. Thus, before comparing a heat scale with the spaces of this chapter in the presence of a boundary, one must fix the realization and describe its trace conditions.

Theorem 14.5.8. (Trace of sections on a submanifold) Let (M,N) be a bounded-geometry pair of smooth manifolds without boundary, where M has dimension n and N has dimension k, $c = n-k$, and let $\mathbf{E} \to M$ be a vector bundle of bounded geometry. If $1 < p < \infty$ and $s > \frac{c}{p}$, restriction of smooth sections extends to a retraction

$$\mathbf{Tr}^{\mathbf{E}}_N : H^{s,p}(M,\mathbf{E}) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(N, \mathbf{E}\restriction_N). \tag{14.71}$$

Moreover, for $1 \le q \le \infty$, there are retractions

$$\mathbf{Tr}^{\mathbf{E}}_N : F^s_{p,q}(M,\mathbf{E}) \longrightarrow B^{s-\frac{c}{p}}_{p,p}(N, \mathbf{E}\restriction_N), \qquad \mathbf{Tr}^{\mathbf{E}}_N : B^s_{p,q}(M,\mathbf{E}) \longrightarrow B^{s-\frac{c}{p}}_{p,q}(N, \mathbf{E}\restriction_N). \tag{14.72}$$

Each operator has a continuous linear right inverse.

Proof. In the Fermi frame $\mathbf{e}_i$, write $h_i \mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a}$ and apply tr_c to each component. By part (c) of Lemma 14.1.2, the local estimate is the finite sum of the scalar estimates in Theorem 14.1.1. On an overlap, parts (a) and (b) of the same lemma give

$$\mathrm{tr}_c(G_{ij} u_j \circ \Phi_{ji}) = (G_{ij}\restriction_{z=0})(\mathrm{tr}_c\, u_j) \circ (\Phi_{ji}\restriction_{z=0}). \tag{14.73}$$

This is the transformation law for the components of a section of $\mathbf{E}\restriction_N$; hence the local traces define an intrinsic operator. The overlap count in (14.26) proves its continuity.

To construct the right inverse, write $h_i^N \mathbf{f} = \sum_{a=1}^{r} f_i^a \mathbf{e}_{i,a}\restriction_N$ locally, apply ex_c componentwise, and reconstruct by

$$\mathbf{Ex}^{\mathbf{E}}_{M,N} \mathbf{f} = \sum_{i \in I_N} \sum_{a=1}^{r} h_i \left[\psi\, \mathrm{ex}_c\, f_i^a\right] \circ \kappa_i^{-1}\, \mathbf{e}_{i,a}. \tag{14.74}$$

The uniform bounds on the transition matrices prove continuity as in (14.28). Upon restriction, $\mathrm{tr}_c \mathrm{ex}_c = I$ and $\sum_{i \in I_N} (h_i^N)^2 = 1$, so $\mathbf{Tr}^{\mathbf{E}}_N \mathbf{Ex}^{\mathbf{E}}_{M,N} = I$. For $F^s_{p,q}$, use the same formulas with the corresponding Euclidean bounds and sum in ℓ^p over the charts. For $q = \infty$, the local formula defines the operator on the entire space,

and the compatibility relations are understood in the sense of distributions.
For $B^{s}_{p,q}$, choose $s_0 < s < s_1$, with $s_0 > c/p$, and apply the common operators to the two spaces $F^{s_i}_{p,p}(M,\mathbf{E})$. We interpolate these operators, rather than the norms of each chart separately:

$$(F^{s_0}_{p,p}(M,\mathbf{E}), F^{s_1}_{p,p}(M,\mathbf{E}))_{\theta,q} = B^{s}_{p,q}(M,\mathbf{E}),$$
$$(B^{s_0-c/p}_{p,p}(N,\mathbf{E}|_N), B^{s_1-c/p}_{p,p}(N,\mathbf{E}|_N))_{\theta,q} = B^{s-c/p}_{p,q}(N,\mathbf{E}|_N).$$

The second equality follows by reiteration of the $F_{p,p}$ scale. Continuity and the retraction identity are preserved. In particular, $B^{s}_{p,q}(M,\mathbf{E})$ is not identified with an ℓ^p sum of local Besov norms when $q \neq p$. □

Theorem 14.5.9. (Trace of sections on a manifold with boundary) Let M be a manifold with nonempty boundary of bounded geometry, and let $\mathbf{E} \to M$ be a vector bundle of bounded geometry. If $j \in \mathbb{N}_0$, $1 < p < \infty$, and $s > j + \frac{1}{p}$, let $\widetilde{\boldsymbol{\nu}}$ denote the collar vector field determined by $\widetilde{\boldsymbol{\nu}}_{\mathcal{C}(x,t)} = \partial_t \mathcal{C}(x,t)$. Then

$$\mathbf{Tr}^{\mathbf{E}}_{\partial M,j}\mathbf{u} := \big(\mathbf{u}\restriction_{\partial M}, \nabla^{\mathbf{E}}_{\widetilde{\boldsymbol{\nu}}}\mathbf{u}\restriction_{\partial M}, \dots, (\nabla^{\mathbf{E}}_{\widetilde{\boldsymbol{\nu}}})^{j}\mathbf{u}\restriction_{\partial M}\big) \tag{14.75}$$

extends to a retraction

$$\mathbf{Tr}^{\mathbf{E}}_{\partial M,j} \colon H^{s,p}(M,\mathbf{E}) \longrightarrow \prod_{\ell=0}^{j} B^{s-\ell-\frac{1}{p}}_{p,p}(\partial M, \mathbf{E}\restriction_{\partial M}). \tag{14.76}$$

For $j = 0$ and $1 \le q \le \infty$, one also obtains the retractions

$$\mathbf{Tr}^{\mathbf{E}}_{\partial M,0} \colon F^{s}_{p,q}(M,\mathbf{E}) \longrightarrow B^{s-\frac{1}{p}}_{p,p}(\partial M, \mathbf{E}\restriction_{\partial M}), \qquad \mathbf{Tr}^{\mathbf{E}}_{\partial M,0} \colon B^{s}_{p,q}(M,\mathbf{E}) \longrightarrow B^{s-\frac{1}{p}}_{p,q}(\partial M, \mathbf{E}\restriction_{\partial M}). \tag{14.77}$$

All these operators have continuous linear right inverses.

Proof. Choose the frames in part (c) of Proposition 14.5.1. By (14.53), if $\mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a}$, then

$$(\nabla^{\mathbf{E}}_{\widetilde{\boldsymbol{\nu}}})^{\ell}\mathbf{u} = \sum_{a=1}^{r} (\partial_t^{\ell} u_i^a)\mathbf{e}_{i,a} \quad \text{in the boundary chart.} \tag{14.78}$$

On an overlap of two such frames, the connection transformation formula gives $\partial_t G_{ij} = 0$, since both normal connection matrices vanish. Consequently, the jet components transform by $G_{ij}|_{t=0}$ without lower-order terms. Moreover, the cutoffs h_i are constant in t near the boundary by (14.31); hence no Leibniz terms appear when localizing the jets. Applying the retraction $\boldsymbol{\gamma}_j$ of Theorem 14.4.17 componentwise, summing over the overlaps, and using the bounds on the matrices $G_{ij}\restriction_{\partial M}$ yields

$$\sum_{\ell=0}^{j} \|(\nabla^{\mathbf{E}}_{\widetilde{\boldsymbol{\nu}}})^{\ell}\mathbf{u}\restriction_{\partial M}\|_{B^{s-\ell-\frac{1}{p}}_{p,p}(\partial M, \mathbf{E}\restriction_{\partial M})} \le C\|\mathbf{u}\|_{H^{s,p}(M,\mathbf{E})}.$$

The coretraction is obtained by applying $\boldsymbol{e}_j$ to the component vectors of the data, multiplying by the fixed cutoff ψ, and summing with factors h_i using formula (14.52). The cutoff ψ equals one in the region where the jets are computed and allows each summand to be extended by zero outside its chart. The identities

(14.78), $\boldsymbol{\gamma}_j \boldsymbol{e}_j = I$, and $\sum_{i \in I_\partial} (h_i^\partial)^2 = 1$ show that the jet of the extension is the prescribed datum. The continuity estimates are the scalar estimates summed over finitely many components and a cover of uniform multiplicity. For $j = 0$, the estimate in $F^s_{p,q}$ follows directly from the Euclidean trace and the same counting argument in ℓ^p. The formula remains valid for $q = \infty$ by the distributional interpretation of the trace. For Besov spaces, fix $s_0 < s < s_1$ with $s_0 > 1/p$ and interpolate the same operators between the global spaces $F^{s_i}_{p,p}(M, \mathbf{E})$ and $B^{s_i - 1/p}_{p,p}(\partial M, \mathbf{E}|_{\partial M})$. The definition of Besov spaces and reiteration, exactly as in the preceding theorem, give (14.77) without interchanging an ℓ^p sum over charts with the fine index q. Independence of the frames follows from Proposition 14.5.3 and (14.73). □

Remark 14.5.10. (Covariant traces and normal jets) The full covariant trace $\mathbf{Tr}_\ell \mathbf{u} = ((\nabla^{\mathbf{E}})^\ell \mathbf{u}) \restriction_{\partial M}$ takes values in $\big(T^{(0,\ell)}(TM) \otimes \mathbf{E}\big) \restriction_{\partial M}$. Write $\widetilde{\boldsymbol{\nu}}$ for the unit normal field extended along the collar geodesics. Since $\nabla^M_{\widetilde{\boldsymbol{\nu}}} \widetilde{\boldsymbol{\nu}} = 0$, the tensor differentiation rule of Chapter 9 gives, for $\ell \in \mathbb{N}_0$,

$$(\nabla^{\ell+1}\mathbf{u})(\widetilde{\boldsymbol{\nu}}^{\otimes(\ell+1)}) = \nabla^{\mathbf{E}}_{\widetilde{\boldsymbol{\nu}}}\big((\nabla^\ell \mathbf{u})(\widetilde{\boldsymbol{\nu}}^{\otimes \ell})\big) - \sum_{a=1}^{\ell} (\nabla^\ell \mathbf{u})(\widetilde{\boldsymbol{\nu}}, \dots, \underbrace{\nabla^M_{\widetilde{\boldsymbol{\nu}}} \widetilde{\boldsymbol{\nu}}}_{\text{position } a}, \dots, \widetilde{\boldsymbol{\nu}}).$$

The sum vanishes. The case $\ell = 0$ is the definition of $\nabla^{\mathbf{E}}_{\widetilde{\boldsymbol{\nu}}} \mathbf{u}$; if the contraction of order ℓ agrees with the iterated normal derivative, the preceding equality proves the same at order $\ell + 1$. Thus

$$(\nabla^\ell \mathbf{u})(\widetilde{\boldsymbol{\nu}}^{\otimes \ell}) = (\nabla^{\mathbf{E}}_{\widetilde{\boldsymbol{\nu}}})^\ell \mathbf{u}, \qquad \ell \in \mathbb{N}_0.$$

The normal jet is therefore the contraction of the full trace with $\boldsymbol{\nu}^{\otimes \ell}$.

Let us make the converse assertion precise. If $\mathbf{S}$ is an $\mathbf{E}$-valued covariant tensor field, the induced connections satisfy

$$\begin{aligned} &\big[(\nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]})\mathbf{S}\big](\mathbf{V}_1, \dots, \mathbf{V}_r) \\ &\quad = \mathbf{R}^{\mathbf{E}}(\mathbf{X}, \mathbf{Y})\big(\mathbf{S}(\mathbf{V}_1, \dots, \mathbf{V}_r)\big) \\ &\qquad - \sum_{a=1}^{r} \mathbf{S}(\mathbf{V}_1, \dots, \mathbf{R}^M(\mathbf{X}, \mathbf{Y})\mathbf{V}_a, \dots, \mathbf{V}_r). \end{aligned} \tag{14.79}$$

This identity follows directly from the definitions of the induced connections and from $\mathbf{R}^M, \mathbf{R}^{\mathbf{E}}$. Moreover, for $\mathbf{X}$ tangent to the boundary, the Gauss and Weingarten formulas in Proposition 2.11.3 give

$$\nabla^M_{\mathbf{X}} \mathbf{Y} = \nabla^{\partial M}_{\mathbf{X}} \mathbf{Y} + \mathbf{II}(\mathbf{X}, \mathbf{Y}), \qquad \nabla^M_{\mathbf{X}} \boldsymbol{\nu} = -\mathbf{A}_{\boldsymbol{\nu}} \mathbf{X}.$$

Now suppose that the normal jets of orders $\ell \in \{0, \dots, m-1\}$ vanish. At order zero this says that $\mathbf{u} \restriction_{\partial M} = 0$. Let $\ell \in \{1, \dots, m-1\}$, and suppose that all covariant traces of orders less than ℓ vanish. The component of $\nabla^\ell \mathbf{u}$ with all arguments normal is zero by the preceding identity. In any other component, choose a tangential argument $\mathbf{X}$ and move it to the first position by finitely many adjacent transpositions.

For each transposition, apply (14.79) to the corresponding lower-order tensor. The outer derivatives are distributed over its products by Proposition 9.1.40. The added terms are contractions of the form

$$\nabla^a \mathbf{R}^M * \nabla^{\ell-2-a} \mathbf{u} \quad \text{or} \quad \nabla^a \mathbf{R}^{\mathbf{E}} * \nabla^{\ell-2-a} \mathbf{u}, \qquad a \in \{0, \dots, \ell-2\}.$$

For $\ell = 1$ there are no transpositions or curvature terms. For $\ell \geq 2$, all factors containing $\mathbf{u}$ have order less than ℓ, and their traces vanish by the induction hypothesis. In the remaining component, the tangential derivative is in the first position. Evaluating the derivative of this tensor gives the tangential derivative of a trace of order $\ell - 1$ and terms obtained by differentiating its arguments. The former is zero; the latter, described by Gauss and Weingarten, are components of the same trace of order $\ell - 1$ and also vanish. Thus all components vanish at order ℓ, completing the induction up to $m - 1$. The reverse implication follows by contracting each full trace with $\boldsymbol{\nu}^{\otimes \ell}$.

To pass to $\mathbf{u} \in W^{m,p}(M, \mathbf{E})$, with $1 \leq p < \infty$, first observe that Theorem 3.7.1, applied to the components of $\nabla^{\ell}\mathbf{u}$ in the boundary models, and comparison (14.59) give, on setting $\mathbf{E}_{\ell} := T^{(0,\ell)}(TM) \otimes \mathbf{E}$,

$$\|\mathbf{Tr}_{\ell}\mathbf{u}\|_{L^p(\partial M, \mathbf{E}_{\ell}|_{\partial M})} \leq C_{m,p}\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}, \qquad 0 \leq \ell < m. \tag{14.80}$$

Indeed, each tensor $\nabla^{\ell}\mathbf{u}$ belongs to $W^{1,p}$, the charts and cutoffs satisfy uniform bounds, and the sum of the pth powers of the local estimates is controlled by the multiplicity of the cover. The constant depends on m, p, the dimension and rank, and the uniform bounds on the geometry, frames, and cutoffs up to the orders used; it does not depend on the section. This argument includes $p = 1$ and, in dimension one, counting measure on the boundary. Contraction with $\boldsymbol{\nu}^{\otimes \ell}$ yields continuous normal jets taking values in $L^p(\partial M, \mathbf{E}|_{\partial M})$. For $p > 1$, they agree with the Besov jets already constructed, since they agree on smooth sections and both operators are continuous.

Now approximate the localizations of $\mathbf{u}$ by sections smooth up to the boundary. The approximations need not have vanishing jets. The commutation and tangential differentiation identities are written before imposing this condition; each side is a finite combination of tangential derivatives of traces and multiplication by uniformly smooth coefficients. The traces are continuous in the L^p spaces of (14.80), and tangential derivatives are continuous in distributions. The identities therefore pass to the limit, after which the preceding induction applies to the jets of $\mathbf{u}$. Thus vanishing of the normal jets of orders less than m is equivalent to vanishing of all covariant traces of those orders.

This equivalence describes the boundary conditions. To identify $W_0^{m,p}$, it remains to check that there is no obstruction to approximation at infinity either. Summability of the localizations and the uniform cutoffs also reduce this question to the half-space.

Theorem 14.5.11. (Characterization of $W_0^{m,p}$ in bounded geometry) Let M be a manifold of bounded geometry with nonempty boundary, and let $\mathbf{E} \to M$ be a vector bundle of bounded geometry equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$; in the complex case the metric is Hermitian. If $m \in \mathbb{N}$ and $1 \leq p < \infty$, then

$$W_0^{m,p}(M, \mathbf{E}) = \ker \mathbf{Tr}^{\mathbf{E}}_{\partial M, m-1} = \{\mathbf{u} \in W^{m,p}(M, \mathbf{E}) | \mathbf{Tr}_{\ell}\mathbf{u} = 0,\ 0 \leq \ell < m\}.$$

Here $W_0^{m,p}(M, \mathbf{E})$ is the closure, in the $W^{m,p}$ norm, of smooth sections whose support is a compact subset of $\operatorname{Int} M$. For $p = 1$, traces and jets are interpreted using the L^1-valued operators of the preceding remark; no identification with the Bessel scale is required.

Proof. Smooth sections compactly supported in the interior have all traces equal to zero. Continuity of the jet operator established in (14.80) proves the inclusion of $W_0^{m,p}$ in the kernel. The last equality in the

statement is the preceding remark. It therefore suffices to show that a section $\mathbf{u}$ whose normal jets of all orders less than m vanish belongs to $W_0^{m,p}$.

Take the quadratic partition (h_i) and the normally parallel frames used in the trace theorem. Write $h_i\mathbf{u} = \sum_{a=1}^{r_{\mathbf{E}}} v_i^a \mathbf{e}_{i,a}$ and extend each $v_i^a \circ \varphi_i^{-1}$ by zero across the artificial faces of the chart. In a boundary chart this yields a function in $W^{m,p}(\mathbb{R}^n_+)$ with bounded support. Since the cutoff is constant in the normal variable near the boundary and the frame is parallel in that direction, its traces satisfy

$$\gamma_0 \partial_t^\ell (v_i^a \circ \varphi_i^{-1}) = 0, \qquad 0 \le \ell < m.$$

The trace commutes with tangential derivatives: the identity is proved for smooth functions and passes to the limit in distributions by continuity of the traces. Consequently, we also have $\gamma_0 D^\beta (v_i^a \circ \varphi_i^{-1}) = 0$ for every $|\beta| \le m-1$.

The half-space criterion in Theorem 3.7.1 implies that each component belongs to $W_0^{m,p}(\mathbb{R}^n_+)$. We can approximate it in $W^{m,p}$ by functions in $C_c^\infty(\mathbb{R}^n_+)$. Multiply these approximations by a fixed cutoff equal to one on the component's support and supported inside the artificial faces of the chart. This multiplication preserves convergence and allows the functions to be reconstructed on M. Each reconstructed approximation is compactly supported in the interior. In interior charts, proceed in the same way using Euclidean regularization. Local equivalence of norms and finite rank thus show that $h_i\mathbf{u} \in W_0^{m,p}(M,\mathbf{E})$ for each i.

To combine the approximations, exhaust the countable index set by finite sets I_J and define

$$\mathbf{u}_J := \sum_{i \in I_J} h_i (h_i \mathbf{u}).$$

Each summand belongs to $W_0^{m,p}$, since multiplication by h_i is continuous and preserves the interior support of the approximations. By the synthesis estimate and integer-order equivalence of norms,

$$\|\mathbf{u} - \mathbf{u}_J\|_{W^{m,p}(M,\mathbf{E})} \le C \left(\sum_{i \in I \setminus I_J} \sum_{a=1}^{r_{\mathbf{E}}} \|v_i^a \circ \varphi_i^{-1}\|^p_{W^{m,p}(\mathbb{X}_i)} \right)^{1/p} \longrightarrow 0.$$

The right-hand side is the tail of a convergent series. Since $W_0^{m,p}$ is closed, it follows that $\mathbf{u} \in W_0^{m,p}$. This proof does not use compactness of M. □

14.6 Sobolev embeddings in bounded geometry

Without boundary, Corollary 13.3.12 formulates the local transfer using a uniformly regular atlas, with constants $c_{s,p}, C_{s,p}, D_{s,p}$. With boundary, Corollary 14.4.12 gives the analysis–synthesis pair on the mixed models $\mathbb{R}^n$ and $\mathbb{R}^n_+$, with constants $c^\partial_{s,p}, C^\partial_{s,p}, D^\partial_{s,p}$. The arguments in this section retain the admissible quadratic localization system fixed when defining the global spaces—with geodesic charts when there is no boundary, and geodesic and Fermi charts when the boundary is nonempty—because it simultaneously incorporates the bundle frames and control of the boundary face. Amann's language thus provides a functional bridge without replacing the covariant estimates already established.

On a compact manifold, local Sobolev estimates are combined using a finite cover. On a noncompact manifold, finiteness must be replaced by three uniform controls: the charts have a common model and radius, the constants

of the local operators are independent of the chart, and the cover has bounded multiplicity. The outer sum defining the global spaces also retains the spatial exponent. Thus, when the integrability exponent increases from p_0 to p_1, the inclusion $\ell^{p_0} \hookrightarrow \ell^{p_1}$ appears and $p_0 \le p_1$ is required. This restriction does not occur on a compact manifold, which has finite measure.

Lemma 14.6.1. (Uniform transfer of a local embedding) Let M be a Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $1 < p_0 \le p_1 < \infty$ and, for $\nu \in \{0,1\}$, let $A_\nu(\mathbb{R}^n)$ be one of the spaces

$$H^{s_\nu,p_\nu}(\mathbb{R}^n) \qquad \text{or} \qquad F^{s_\nu}_{p_\nu,q_\nu}(\mathbb{R}^n),$$

, with $1 \le q_\nu \le \infty$ in the second case. Suppose that the identity inclusion satisfies

$$\|v\|_{A_1(\mathbb{R}^n)} \le C_{\mathrm{euc}} \|v\|_{A_0(\mathbb{R}^n)}. \tag{14.81}$$

Then the corresponding continuous embedding holds:

$$A_0(M,\mathbf{E}) \hookrightarrow A_1(M,\mathbf{E}).$$

The global constant depends on C_{euc}, the rank of $\mathbf{E}$, and the constants c_A, C_A of Proposition 14.5.3, but not on the section.

Proof. We begin with the boundary model. The spaces on $\mathbb{R}^n_+$ carry the quotient norm of restriction. If $v \in A_0(\mathbb{R}^n_+)$ and $V \in A_0(\mathbb{R}^n)$ is an extension of v, then

$$\|v\|_{A_1(\mathbb{R}^n_+)} \le \|V\|_{A_1(\mathbb{R}^n)} \le C_{\mathrm{euc}} \|V\|_{A_0(\mathbb{R}^n)}.$$

Taking the infimum over all extensions proves

$$\|v\|_{A_1(\mathbb{R}^n_+)} \le C_{\mathrm{euc}} \|v\|_{A_0(\mathbb{R}^n_+)}. \tag{14.82}$$

The constant is the same as on the full space.
Now fix an admissible quadratic localization system $\mathcal{Q} = (U_i, \varphi_i, h_i)_{i\in I}$ and the synchronous frames used in Definition 14.5.2. If $\mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a}$ in U_i, set

$$a_{i,a} := \|(h_i u_i^a) \circ \varphi_i^{-1}\|_{A_0(\mathbb{X}_i)}, \qquad b_{i,a} := \|(h_i u_i^a) \circ \varphi_i^{-1}\|_{A_1(\mathbb{X}_i)}.$$

Estimate (14.81), or (14.82), gives $b_{i,a} \le C_{\mathrm{euc}} a_{i,a}$ with the same constant for all indices. Since $p_0 \le p_1$, the elementary inclusion $\ell^{p_0}(I \times \{1,\dots,r\}) \hookrightarrow \ell^{p_1}(I \times \{1,\dots,r\})$ yields

$$\|\mathbf{u}\|_{A_1(M,\mathbf{E})} = \left(\sum_{i\in I} \sum_{a=1}^{r} b_{i,a}^{p_1} \right)^{\frac{1}{p_1}} \le C_{\mathrm{euc}} \left(\sum_{i\in I} \sum_{a=1}^{r} a_{i,a}^{p_1} \right)^{\frac{1}{p_1}} \le C_{\mathrm{euc}} \left(\sum_{i\in I} \sum_{a=1}^{r} a_{i,a}^{p_0} \right)^{\frac{1}{p_0}} = C_{\mathrm{euc}} \|\mathbf{u}\|_{A_0(M,\mathbf{E})}.$$

This calculation has been carried out in a fixed system. Its intrinsic nature follows from Proposition 14.5.3. Recall where the geometric hypotheses enter there. On an overlap, insert the quadratic partition, multiply

each component by an auxiliary cutoff equal to one on the support being transported, and compose with the coordinate change. Uniform derivative bounds for the cutoffs, transition matrices, and coordinate changes control multipliers and compositions; Jacobian bounds compare the measures. Finally, uniform multiplicity bounds both the number of summands entering a chart and the number of times a localization appears in the global sum. Thus c_A and C_A are also independent of i. □

Theorem 14.6.2. (Embeddings between the real-order scales) Let M be an n-dimensional Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $s_0, s_1 \in \mathbb{R}$ and $1 < p_0 \le p_1 < \infty$.

(a) If

$$s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}, \tag{14.83}$$

then, for all $1 \le q_0, q_1 \le \infty$,

$$F^{s_0}_{p_0,q_0}(M,\mathbf{E}) \hookrightarrow F^{s_1}_{p_1,q_1}(M,\mathbf{E})$$

and

$$B^{s_0}_{p_0,q_0}(M,\mathbf{E}) \hookrightarrow B^{s_1}_{p_1,q_1}(M,\mathbf{E}).$$

(b) If

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} \tag{14.84}$$

and $1 \le q_0 \le q_1 \le \infty$, then the same embeddings hold in the scales F and B.

(c) If

$$s_0 - \frac{n}{p_0} \ge s_1 - \frac{n}{p_1},$$

then

$$H^{s_0,p_0}(M,\mathbf{E}) \hookrightarrow H^{s_1,p_1}(M,\mathbf{E}).$$

All embeddings hold both without boundary and with boundary.

Proof. First suppose that (14.84). Part (a) of Theorem 6.6.1, applied with fine index q_0, gives

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1,q_0}(\mathbb{R}^n).$$

Part (c) of the same theorem and $q_0 \le q_1$ then give

$$F^{s_1}_{p_1,q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^n).$$

Lemma 14.6.1 transfers the composition to M.
If (14.83) holds, define

$$\sigma_1 := s_0 - \frac{n}{p_0} + \frac{n}{p_1}.$$

Then $\sigma_1 > s_1$. The result already proved, with the same fine index q_0, and part (b) of Theorem 6.6.1 yield

$$F^{s_0}_{p_0,q_0}(M,\mathbf{E}) \hookrightarrow F^{\sigma_1}_{p_1,q_0}(M,\mathbf{E}) \hookrightarrow F^{s_1}_{p_1,q_1}(M,\mathbf{E}).$$

This proves the assertions for the Triebel–Lizorkin scale. Taking $q_0 = q_1 = 2$ and using the Euclidean identification $H^{s,p} = F^s_{p,2}$ in Theorem 6.3.10, summed over the charts as in the proof of the lemma, proves part (c).

We turn to Besov spaces. When the indices are equal, choose $\delta > 0$. The F result already proved, applied at both endpoints, gives the continuous operators

$$F^{s_0-\delta}_{p_0,p_0}(M,\mathbf{E}) \hookrightarrow F^{s_1-\delta}_{p_1,p_1}(M,\mathbf{E}), \qquad F^{s_0+\delta}_{p_0,p_0}(M,\mathbf{E}) \hookrightarrow F^{s_1+\delta}_{p_1,p_1}(M,\mathbf{E}),$$

since $p_0 \le p_1$ is also the inequality between the fine indices of these pairs. Interpolate the identity by Theorem 5.3.1. The definition of the Besov scale gives

$$B^{s_0}_{p_0,q_0}(M,\mathbf{E}) \hookrightarrow B^{s_1}_{p_1,q_0}(M,\mathbf{E}).$$

Monotonicity of the second index for the real method, Corollary 5.2.6, allows q_0 to be replaced by $q_1 \ge q_0$ in the target space. If the differential indices satisfy the strict inequality, factor once again through $B^{\sigma_1}_{p_1,q_0}(M,\mathbf{E})$. Fix $a < s_1 < \sigma_1 < b$ and write, using Theorem 5.5.2 for independence of the auxiliary endpoints in the definition of the scale,

$$B^{\sigma_1}_{p_1,q_0}(M,\mathbf{E}) = (X_0,X_1)_{\theta_\sigma,q_0}, \qquad B^{s_1}_{p_1,q_1}(M,\mathbf{E}) = (X_0,X_1)_{\theta_s,q_1},$$

where $X_0 = F^a_{p_1,p_1}(M,\mathbf{E})$, $X_1 = F^b_{p_1,p_1}(M,\mathbf{E})$, and $0 < \theta_s < \theta_\sigma < 1$. The F result already proved gives $X_1 \hookrightarrow X_0$. Lemma 5.2.5, separating positive and negative dyadic indices, shows that

$$(X_0,X_1)_{\theta_\sigma,q_0} \hookrightarrow (X_0,X_1)_{\theta_s,q_1}$$

for all q_0, q_1: the geometric factor $t^{\theta_\sigma-\theta_s}$ appears in the tail corresponding to $t \le 1$, whereas for $t \ge 1$ we use the bound of the X_0 norm by the interpolation norm, proved together with completeness of the K method. This completes the proof. □

Corollary 14.6.3. (Integer-order Sobolev embeddings) Let M be an n-dimensional Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $m_0, m_1 \in \mathbb{N}_0$ and $1 < p_0 \le p_1 < \infty$. If

$$m_0 - \frac{n}{p_0} \ge m_1 - \frac{n}{p_1},$$

then

$$W^{m_0,p_0}(M,\mathbf{E}) \hookrightarrow W^{m_1,p_1}(M,\mathbf{E}). \tag{14.85}$$

In particular, let $j \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $1 < p < \infty$.

(a) If $mp < n$ and $p^*_m := \frac{np}{n-mp}$, then

$$W^{j+m,p}(M,\mathbf{E}) \hookrightarrow W^{j,q}(M,\mathbf{E}) \qquad \text{for } p \le q \le p^*_m.$$

(b) If $mp = n$, the same embedding holds for every $p \le q < \infty$.

(c) If $mp > n$, the same embedding holds for every $p \le q < \infty$.

Proof. Proposition 14.5.3 identifies $H^{m_\nu,p_\nu}(M,\mathbf{E})$ with $W^{m_\nu,p_\nu}(M,\mathbf{E})$ and provides constants $c_{m_\nu,p_\nu}, C_{m_\nu,p_\nu} > 0$ such that, for every $\mathbf{u} \in W^{m_\nu,p_\nu}(M,\mathbf{E})$ and $\nu \in \{0,1\}$,

$$c_{m_\nu,p_\nu}\|\mathbf{u}\|_{W^{m_\nu,p_\nu}(M,\mathbf{E})} \le \|\mathbf{u}\|_{H^{m_\nu,p_\nu}(M,\mathbf{E})} \le C_{m_\nu,p_\nu}\|\mathbf{u}\|_{W^{m_\nu,p_\nu}(M,\mathbf{E})}.$$

Thus (14.85) is part (c) of Theorem 14.6.2. The remaining three assertions follow by writing

$$j + m - \frac{n}{p} \ge j - \frac{n}{q}.$$

In the subcritical case this inequality is equivalent to $q \le p_m^*$; in the critical and supercritical cases it holds for every finite exponent $q \ge p$. □

Corollary 14.6.4. (Subcritical and critical cases) Let M be an n-dimensional Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $s > 0$ and $1 < p < \infty$.

(a) If $sp < n$ and $p_s^* := \frac{np}{n-sp}$, then

$$H^{s,p}(M,\mathbf{E}) \hookrightarrow L^q(M,\mathbf{E}) \qquad \text{for } p \le q \le p_s^*.$$

(b) If $sp = n$, then

$$H^{s,p}(M,\mathbf{E}) \hookrightarrow L^q(M,\mathbf{E}) \qquad \text{for every } p \le q < \infty.$$

Proof. Apply part (c) of Theorem 14.6.2 with $s_1 = 0$ and $p_1 = q$. Proposition 14.5.3, at order zero, identifies $H^{0,q}(M,\mathbf{E})$ with $L^q(M,\mathbf{E})$. If $sp < n$, the condition on the indices is equivalent to $q \le p_s^*$; if $sp = n$, it is strict for every $q < \infty$. □

To treat the supercritical case, we need a uniform formulation of Hölder sections. In the absence of a boundary, the intrinsic norm of Definition 9.2.6 is already available. Near the boundary, we use the localized formulation of Definition 9.4.32, replacing the finite sum by a uniform supremum.

Definition 14.6.5. (Uniform Hölder sections up to the boundary) Let M be a Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $k \in \mathbb{N}_0$ and $0 < \alpha \le 1$. Fix an admissible quadratic localization system $\mathcal{Q} = (U_i, \varphi_i, h_i)_{i\in I}$ and synchronous frames $\mathbf{e}_i$. Let $\mathbf{\Omega}_i := \varphi_i(U_i)$ denote the bounded coordinate domain; in a boundary chart it is contained in $\mathbb{R}^n_+$. If

$$h_i\mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a},$$

, set

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})} := \sup_{i\in I} \sum_{a=1}^{r} \|u_i^a \circ \varphi_i^{-1}\|_{C^{k,\alpha}(\overline{\mathbf{\Omega}_i})}. \tag{14.86}$$

Here we use the extension norm of $C^{k,\alpha}(\overline{\mathbf{\Omega}_i})$ from Definition 3.5.1. Define $\Gamma^{k,\alpha}(\mathbf{E})$ to be the space of sections of class C^k up to the boundary for which (14.86) is finite.

Proposition 14.6.6. (Independence and characterization of the uniform Hölder norm) Let M be a Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. If $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are two admissible quadratic localization systems equipped with synchronous frames, there exist $c_{\mathcal{Q},\widetilde{\mathcal{Q}}}, C_{\mathcal{Q},\widetilde{\mathcal{Q}}} > 0$, independent of $\mathbf{u}$, such that

$$c_{\mathcal{Q},\widetilde{\mathcal{Q}}} \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})} \le \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\widetilde{\mathcal{Q}}}(\mathbf{E})} \le C_{\mathcal{Q},\widetilde{\mathcal{Q}}} \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})}.$$

The constants depend on k, α, the multiplicity, and the uniform bounds on coordinate changes, cutoffs, and frames, but not on the chart indices. If M has no boundary, there exist $c_{\mathrm{int}}, C_{\mathrm{int}} > 0$ such that

$$c_{\mathrm{int}} \|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})} \le C_{\mathrm{int}} \|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})},$$

where the first norm is the intrinsic norm of (9.48). If M is compact with boundary and $\mathcal{A}$ are the finite data of Definition 9.4.32, there exist $c_\partial, C_\partial > 0$ such that

$$c_\partial \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{A}}(\mathbf{E})} \le \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})} \le C_\partial \|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{A}}(\mathbf{E})}.$$

Proof. Let $\mathcal{Q} = (U_i, \varphi_i, h_i)_{i\in I}$ and $\widetilde{\mathcal{Q}} = (\widetilde{U}_j, \widetilde{\varphi}_j, \widetilde{h}_j)_{j\in J}$ be two admissible quadratic localization systems equipped with synchronous frames $\mathbf{e}_i$ and $\widetilde{\mathbf{e}}_j$. Insert $\sum_{i\in I} h_i^2 = 1$ into each localization of the second system:

$$\widetilde{h}_j \mathbf{u} = \sum_{\substack{i\in I \\ U_i \cap \widetilde{U}_j \neq \varnothing}} \widetilde{h}_j h_i (h_i \mathbf{u}). \tag{14.87}$$

There are at most L summands, with L independent of j. On an overlap, the component of each summand in the frame $\widetilde{\mathbf{e}}_j$ is obtained from a component of $h_i\mathbf{u}$ by three operations: composition with the coordinate change, multiplication by an entry of the frame transition matrix, and multiplication by the two cutoffs. Also insert auxiliary cutoff functions equal to one on the supports that occur, so that all expressions are compactly supported in the exact domain of the coordinate change.

Proposition 14.4.5 and Proposition 14.5.1 bound the derivatives of the coordinate changes, their inverses, the cutoffs, and the transition matrices uniformly in i, j. The Faà di Bruno formula in Theorem 1.1.19 and the Leibniz rule then give, for each local component f, if D_{ij} is the exact coordinate domain of the term under consideration and $D_i := \Phi_{ij}(D_{ij})$,

$$\|b_{ij}(f \circ \Phi_{ij})\|_{C^{k,\alpha}(\overline{D_{ij}})} \le C_{k,\alpha} \|f\|_{C^{k,\alpha}(\overline{D_i})}, \tag{14.88}$$

where b_{ij} contains the preceding multipliers and Φ_{ij} is the coordinate change. For the seminorm of order k, use

$$[fg]_{C^{0,\alpha}(\overline{D_{ij}})} \le \|f\|_{C^0(\overline{D_{ij}})} [g]_{C^{0,\alpha}(\overline{D_{ij}})} + \|g\|_{C^0(\overline{D_{ij}})} [f]_{C^{0,\alpha}(\overline{D_{ij}})}$$

and the uniform Lipschitz bound on Φ_{ij} over the cutoff's support. In boundary charts, the coordinate changes and multipliers extend to Euclidean neighborhoods of the corresponding compact sets; thus (14.88) descends to the extension norms upon taking the infimum over extensions.

Apply (14.88) to (14.87), sum over the r components, and use the fact that there are at most L terms. This

yields

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\widetilde{\mathcal{Q}}}(\mathbf{E})} \le LC_{k,\alpha}\|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})}.$$

Interchanging the two systems proves independence.

Now suppose that M has no boundary. The proof of Proposition 9.2.11, applied in geodesic charts and synchronous frames, has uniform constants: they depend only on the constants c_g, C_g in the uniform metric inequalities, the bounds on the connection coefficients, and the derivatives of the cutoffs, all independent of the chart. This gives

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})} \le C\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})}.$$

For the reverse inequality, use $\mathbf{u} = \sum_{i\in I} h_i(h_i\mathbf{u})$. Each section $h_i(h_i\mathbf{u})$ extends by zero outside U_i, and its intrinsic Hölder norm is bounded by the right-hand side of (14.86) with a uniform constant. At any point, at most L summands contribute. When comparing two points, only indices whose terms are nonzero at one of them contribute, giving at most $2L$ summands. The triangle inequality for parallel transport then controls the Hölder seminorm, while the Leibniz rule controls the uniform norms of the derivatives. Thus

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C'\|\mathbf{u}\|_{\Gamma^{k,\alpha}_{\mathcal{Q}}(\mathbf{E})}.$$

If M is compact with boundary and the index set has N elements, the maximum norm and the sum of the local norms satisfy

$$\max_{i\in\{1,\dots,N\}} a_i \le \sum_{i=1}^{N} a_i \le N \max_{i\in\{1,\dots,N\}} a_i, \qquad a_i \ge 0.$$

The independence already proved allows the same data to be chosen in both definitions. □

Theorem 14.6.7. (Sobolev–Morrey embeddings in bounded geometry) Let M be an n-dimensional Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and $0 < \alpha < 1$.

(a) If $s > 0$ and

$$k + \alpha < s - \frac{n}{p},$$

, then

$$H^{s,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{k,\alpha}(\mathbf{E}).$$

(b) If $m \in \mathbb{N}$ and

$$k + \alpha \le m - \frac{n}{p}, \tag{14.89}$$

, then

$$W^{m,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{k,\alpha}(\mathbf{E}).$$

(c) If $0 < s < 1$, $sp > n$, and $\alpha = s - \frac{n}{p}$, then

$$B^{s}_{p,p}(M,\mathbf{E}) \hookrightarrow \Gamma^{0,\alpha}(\mathbf{E}).$$

In each case, elements of the source space have a unique representative in the target space. If M has

boundary, the restriction of the representative in part (b) agrees with the trace of the section.

Proof. Fix the quadratic localization system and synchronous frames of the preceding proposition. For a section $\mathbf{u}$ of the source space, write

$$h_i\mathbf{u} = \sum_{a=1}^{r} u_i^a \mathbf{e}_{i,a}, \qquad f_i^a := u_i^a \circ \varphi_i^{-1}.$$

Thus f_i^a is the ath component of the already localized section $h_i\mathbf{u}$; no second factor h_i is introduced. In an interior chart, Theorem 6.6.3 gives, under the strict hypothesis in part (a),

$$\|f_i^a\|_{C^{k,\alpha}(\overline{\mathbf{\Omega}_i})} \le C\|f_i^a\|_{H^{s,p}(\mathbb{R}^n)}. \tag{14.90}$$

The constant depends only on n, s, p, k, α. In a boundary chart, choose an extension $F_i^a \in H^{s,p}(\mathbb{R}^n)$ of $f_i^a \in H^{s,p}(\mathbb{R}^n_+)$. The same theorem controls the Hölder norm of F_i^a; restricting to $\overline{\mathbf{\Omega}_i}$ and taking the infimum over extensions gives (14.90) with the quotient norm on the right-hand side. By Proposition 14.6.6,

$$\begin{aligned}\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} &\le C \sup_{\substack{i\in I\\ a\in\{1,\dots,r_{\mathbf{E}}\}}} \|f_i^a\|_{C^{k,\alpha}(\overline{\mathbf{\Omega}_i})} \\ &\le C'\left(\sum_{i\in I}\sum_{a=1}^{r}\|f_i^a\|^p_{H^{s,p}(\mathbb{X}_i)}\right)^{\frac{1}{p}} = C'\|\mathbf{u}\|_{H^{s,p}(M,\mathbf{E})}.\end{aligned}$$

This proves (a).

For (b), use the limiting exponent in Theorem 3.5.7. In an interior chart, f_i^a is supported in a compact subset of a fixed model, and its extension by zero belongs to $W^{m,p}(\mathbb{R}^n)$. In a boundary chart, two different extensions are required. First extend f_i^a by zero from the coordinate region to $\mathbb{R}^n_+$; this crosses only the lateral part, since the support of h_i is separated from it, and Lemma 3.6.8 preserves the Sobolev norm. Then apply the operator $\mathcal{E}_m\colon W^{m,p}(\mathbb{R}^n_+) \longrightarrow W^{m,p}(\mathbb{R}^n)$ of Theorem 3.6.4. Finally, multiply by a fixed cutoff equal to one on the compact set where the original function may be nonzero. The result belongs to $W_0^{m,p}(B_{\mathrm{euc}}(0,R))$, agrees with f_i^a on the coordinate region, and satisfies

$$\|T_i f_i^a\|_{W^{m,p}(B_{\mathrm{euc}}(0,R))} \le C\|f_i^a\|_{W^{m,p}(\mathbb{R}^n_+)}. \tag{14.91}$$

The radius R, cutoff, and constant can be chosen independently of i: Proposition 14.4.5 uses domains of a common radius, the cutoff supports lie in a common smaller coordinate region, and the operator $\mathcal{E}_m$ is always the same. The Euclidean theorem and (14.91) give

$$\|f_i^a\|_{C^{k,\alpha}(\overline{\mathbf{\Omega}_i})} \le C'\|f_i^a\|_{W^{m,p}(\mathbb{X}_i)}$$

in both types of charts. Take the supremum, bound it by the ℓ^p sum, and use the inequalities with constants $c_{m,p}, C_{m,p}$ in Proposition 14.5.3. This yields

$$\|\mathbf{u}\|_{\Gamma^{k,\alpha}(\mathbf{E})} \le C\|\mathbf{u}\|_{W^{m,p}(M,\mathbf{E})}.$$

For (c), the Euclidean identification $B^s_{p,p} = W^{s,p}$ in Theorem 6.3.14 and Lemma 6.6.2 give, in each chart,

$$\|f_i^a\|_{C^{0,\alpha}(\overline{\mathbf{\Omega}_i})} \le C\|f_i^a\|_{B^s_{p,p}(\mathbb{X}_i)}.$$

On the half-space, proceed with quotient norms as in (a). The definition of $B^s_{p,p}(M,\mathbf{E})$ by real interpolation, the retraction and coretraction argument of Proposition 14.5.3, and the Euclidean identification $B^s_{p,p} = F^s_{p,p}$ imply

$$B^s_{p,p}(M,\mathbf{E}) = F^s_{p,p}(M,\mathbf{E})$$

and provide constants $c_{BF}, C_{BF} > 0$ such that

$$c_{BF}\|\mathbf{u}\|_{F^s_{p,p}(M,\mathbf{E})} \le \|\mathbf{u}\|_{B^s_{p,p}(M,\mathbf{E})} \le C_{BF}\|\mathbf{u}\|_{F^s_{p,p}(M,\mathbf{E})}.$$

By the localized definition of the F norm, if

$$N_{\mathrm{loc}}(\mathbf{u}) := \left(\sum_{i\in I}\sum_{a=1}^{r}\|f_i^a\|^p_{F^s_{p,p}(\mathbb{X}_i)}\right)^{\frac{1}{p}},$$

, the same constants satisfy

$$c_{BF}N_{\mathrm{loc}}(\mathbf{u}) \le \|\mathbf{u}\|_{B^s_{p,p}(M,\mathbf{E})} \le C_{BF}N_{\mathrm{loc}}(\mathbf{u}).$$

The estimate using the supremum proves (c).
For each i, the representatives of $f_i^1,\dots,f_i^r$ reconstruct a section $\mathbf{s}_i$ in U_i that agrees almost everywhere with $h_i\mathbf{u}$. Since the support of h_i lies in a smaller coordinate region, the section $h_i\mathbf{s}_i$ extends by zero outside U_i. The sum

$$\widetilde{\mathbf{u}} := \sum_{i\in I} h_i\mathbf{s}_i$$

is locally finite, defines a section of class C^k up to the boundary, and agrees with $\sum_{i\in I} h_i^2\mathbf{u} = \mathbf{u}$ as a distribution. The preceding uniform estimates and bounded multiplicity of the cover show that $\widetilde{\mathbf{u}} \in \Gamma^{k,\alpha}(\mathbf{E})$ with the bound already obtained. If two continuous representatives agree almost everywhere, the open set where they differ must be empty, since every nonempty relative ball or half-ball has positive Riemannian measure. This proves uniqueness. In the presence of a boundary, compatibility with the trace follows by approximation in $W^{m,p}(M,\mathbf{E})$ by smooth sections: the estimate just proved gives convergence in $\Gamma^{k,\alpha}(\mathbf{E})$, while Theorem 14.5.9 gives convergence of the restrictions in the Besov space on the boundary. The two distributional limits agree. □

Proposition 14.6.8. (Compactness after localization) Let M be an n-dimensional Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $m_0, m_1 \in \mathbb{N}_0$ and $1 \le p_0, p_1 < \infty$, and suppose that

$$m_0 > m_1, \qquad m_0 - \frac{n}{p_0} > m_1 - \frac{n}{p_1}.$$

. For each $\chi \in C_c^\infty(M)$, the operator

$$\mathbf{u} \longmapsto \chi\mathbf{u}$$

is compact from $W^{m_0,p_0}(M,\mathbf{E})$ to $W^{m_1,p_1}(M,\mathbf{E})$.

Proof. The support of χ meets only finitely many members of a uniformly locally finite cover. If $(\mathbf{u}_j)_{j\in\mathbb{N}}$ is bounded in $W^{m_0,p_0}(M,\mathbf{E})$, the inequalities with constants $c^W_{m_0,p_0}, C^W_{m_0,p_0}$ in Proposition 14.5.3 show that, in each of these charts and for each component, the localizations of $\chi\mathbf{u}_j$ are bounded in a Euclidean space W^{m_0,p_0} and supported in a fixed compact set. In an interior chart, Theorem 3.4.2, applied to a ball containing this support, provides a subsequence converging in W^{m_1,p_1}. The localized functions belong to $W_0^{m_0,p_0}$ of the ball, since their support is separated from its boundary. Multiplying by a cutoff equal to one on the common support and extending by zero gives convergence on all of $\mathbb{R}^n$. In a boundary chart, extend by zero only across the artificial faces to the half-space. The support remains in a compact subset of the closed half-space, possibly meeting $t = 0$; Theorem 3.6.6 gives a convergent subsequence there. Both theorems apply precisely because of the two hypotheses $m_0 > m_1$ and the strict inequality between the indices, and both allow all finite exponents in the statement.

In the range $1 < p_0 \le p_1 < \infty$, one also obtains the following factorization through Bessel spaces. The strict inequality between the differential indices allows us to define

$$\sigma_1 := m_0 - \frac{n}{p_0} + \frac{n}{p_1} > m_1.$$

Theorem 6.6.1 gives the continuous embedding $H^{m_0,p_0} \hookrightarrow H^{\sigma_1,p_1}$, while Theorem 6.6.4, applied on a bounded open set containing the fixed support, gives the compact embedding $H^{\sigma_1,p_1} \hookrightarrow H^{m_1,p_1}$. The integer-order identifications thus provide a subsequence converging in W^{m_1,p_1} in an interior chart. Since all functions are supported in a fixed compact set, multiplying by a cutoff equal to one there and extending by zero turns this local convergence into convergence in the full Euclidean space. In a boundary chart, first apply lateral extension by zero and then the half-space operator, as in (14.91); the same factorization argument applies to the extensions supported in a fixed compact set.

Only finitely many charts and components are involved. Successive extractions yield a single subsequence converging in all of them. The synthesis estimate, which uses multipliers, coordinate changes, and uniform multiplicity, reconstructs convergence in $W^{m_1,p_1}(M,\mathbf{E})$. Thus the operator is compact. □

Proposition 14.6.9. (Interpolation inequality with a small parameter) Let M be an n-dimensional Riemannian manifold of bounded geometry, with or without boundary, and let $\mathbf{E} \to M$ be a bundle of bounded geometry. Let $t, s, s_0 \in \mathbb{N}_0$ and $p_0, p, q \in (1,\infty)$ satisfy

$$t < s < s_0, \qquad \max\{p_0, q\} \le p,$$

and

$$t - \frac{n}{q} \le s - \frac{n}{p} < s_0 - \frac{n}{p_0}. \tag{14.92}$$

. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|\mathbf{u}\|_{W^{s,p}(M,\mathbf{E})} \le \varepsilon \|\mathbf{u}\|_{W^{s_0,p_0}(M,\mathbf{E})} + C_\varepsilon \|\mathbf{u}\|_{W^{t,q}(M,\mathbf{E})} \tag{14.93}$$

for every $\mathbf{u} \in W^{s_0,p_0}(M,\mathbf{E}) \cap W^{t,q}(M,\mathbf{E})$.

Proof. First establish an inequality on the fixed local model. Let K denote the common compact set containing the supports of the localization functions. The strict inequality on the right in (14.92) allows us to set

$$\sigma := s_0 - \frac{n}{p_0} + \frac{n}{p} > s.$$

Theorem 6.6.1 gives

$$H^{s_0,p_0}(\mathbb{R}^n) \hookrightarrow H^{\sigma,p}(\mathbb{R}^n).$$

. Choose a bounded Lipschitz domain Ω and a cutoff $\eta \in C_c^\infty(\Omega)$ equal to one on a neighborhood of K. Theorem 6.6.4 shows that the restriction of a bounded family in $H^{\sigma,p}(\mathbb{R}^n)$ has a subsequence converging in $H^{s,p}(\Omega)$. Since each function under consideration is supported in K, it agrees on the whole space with the extension by zero of $\eta v \restriction_\Omega$. Continuity of multiplication by η and of this extension, whose support remains separated from $\partial\Omega$, proves that

$$\{v \in W^{s_0,p_0}(\mathbb{R}^n) \mid \operatorname{supp} v \subseteq K\} \hookrightarrow W^{s,p}(\mathbb{R}^n)$$

is compact. Here the identifications $H^{s_0,p_0} = W^{s_0,p_0}$ and $H^{s,p} = W^{s,p}$ are at integer orders. On the other hand, since $t < s$ and $q \le p$, first lose regularity while keeping the exponent p fixed and then apply Hölder's inequality on the bounded domain containing K; this gives the continuous embedding into $W^{t,q}(\mathbb{R}^n)$ of the subspace of functions in $W^{s,p}(\mathbb{R}^n)$ supported in K.
We claim that, for each $\delta > 0$, there exists $C_\delta > 0$ such that

$$\|v\|_{W^{s,p}(\mathbb{R}^n)} \le \delta \|v\|_{W^{s_0,p_0}(\mathbb{R}^n)} + C_\delta \|v\|_{W^{t,q}(\mathbb{R}^n)} \tag{14.94}$$

for every $v \in W^{s_0,p_0}(\mathbb{R}^n)$ supported in K. If this failed for some δ, there would be a sequence $(v_j) \subset W^{s_0,p_0}(\mathbb{R}^n)$ supported in K which, after normalization, would satisfy

$$\|v_j\|_{W^{s,p}(\mathbb{R}^n)} = 1, \qquad \|v_j\|_{W^{s_0,p_0}(\mathbb{R}^n)} \le \delta^{-1}, \qquad \|v_j\|_{W^{t,q}(\mathbb{R}^n)} \longrightarrow 0.$$

By compactness, a subsequence would converge in $W^{s,p}$ to a limit of norm one. By continuity of $W^{s,p}_K \hookrightarrow W^{t,q}$, the same limit would be zero in $W^{t,q}$ and hence the zero distribution. This contradiction proves (14.94).
In a boundary chart, the compact inclusion used in the contradiction is obtained by first extending by zero from the half-ball to the half-space, only across the lateral part, then applying $\mathcal{E}_{s_0}$ and multiplying by a fixed cutoff. The continuous inclusion of the space of order s into that of order t is proved directly on the half-space by Hölder's inequality, since all functions are supported in K. Thus the same contradiction gives (14.94) with half-space norms. Since the domains, compact set K, cutoffs, and extension operator are common to the entire family, C_δ is independent of the chart. Apply the inequality to each localized component and take the ℓ^p norm. The triangle inequality in ℓ^p gives

$$\|(b_{i,a})\|_{\ell^p} \le \delta \|(a_{i,a})\|_{\ell^p} + C_\delta \|(c_{i,a})\|_{\ell^p},$$

where a, b, c are the local norms of orders and exponents (s_0, p_0), (s, p), and (t, q), respectively. The hypotheses $p_0 \le p$ and $q \le p$ imply

$$\|(a_{i,a})\|_{\ell^p} \le \|(a_{i,a})\|_{\ell^{p_0}}, \qquad \|(c_{i,a})\|_{\ell^p} \le \|(c_{i,a})\|_{\ell^q}.$$

Proposition 14.5.3 compares these three sums with the global covariant norms. Choose δ so that the first coefficient, after incorporating the constants c_A, C_A of Proposition 14.5.3, is ε. This gives (14.93). □

Remark 14.6.10. (Local compactness and loss of global compactness) Proposition 14.6.8 does not imply compactness of the global embedding. Even on $M = \mathbb{R}^n$, if $0 \neq v \in C_c^\infty(\mathbb{R}^n)$ and $v_j(x) := v(x - je_1)$, the sequence (v_j) is bounded in all Sobolev spaces of a fixed order. Choosing a subsequence with disjoint supports gives, for $j \neq \ell$,

$$\|v_j - v_\ell\|_{W^{m,p}(\mathbb{R}^n)}^p = 2\|v\|_{W^{m,p}(\mathbb{R}^n)}^p.$$

Thus there is no convergent subsequence. Bounded geometry prevents the local constants from degenerating, but does not prevent a sequence from escaping to infinity. Inequality (14.93) is compatible with this phenomenon: it is a global estimate with a small parameter, not a statement of global compactness.

Remark 14.6.11. (Completeness and weighted embeddings) Completeness alone does not provide a uniform coordinate radius or the preceding unweighted embeddings. Fix $0 < \varepsilon < 1$. A ball $B_{\mathbf{g}}(x, R)$ is called $(1, \varepsilon)$-admissible if it has a chart $\varphi\colon B_{\mathbf{g}}(x, R) \longrightarrow \Omega \subseteq \mathbb{R}^n$ whose metric coefficients $g_{ij}(z)$ satisfy

$$(1-\varepsilon)\sum_{i=1}^n (\xi^i)^2 \le \sum_{i,j=1}^n g_{ij}(z)\xi^i\xi^j \le (1+\varepsilon)\sum_{i=1}^n (\xi^i)^2 \qquad (z \in \Omega,\ \xi \in \mathbb{R}^n)$$

and

$$R \max_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta|=1}} \max_{i,j \in \{1,\dots,n\}} \|\partial^\beta g_{ij}\|_{L^\infty(\varphi(B_{\mathbf{g}}(x,R)))} \le \varepsilon.$$

In the first condition, ξ represents the components of a tangent vector in the chart; hence this condition is equivalent to

$$\sqrt{1-\varepsilon}\,\|d\varphi_p(v)\| \le |v|_{\mathbf{g}} \le \sqrt{1+\varepsilon}\,\|d\varphi_p(v)\| \qquad (p \in B_{\mathbf{g}}(x,R),\ v \in T_pM).$$

The second condition additionally controls the first coordinate derivatives of the metric. If $R'(x)$ is the supremum of these radii, set $R_\varepsilon(x) := \min\left\{1, \dfrac{R'(x)}{2}\right\}$. The local constants can be retained inside the integral through weights formed from powers of R_ε. With the convention

$$\|\mathbf{u}\|_{L^a(M,\mathbf{E};w)} := \left(\int_M |\mathbf{u}|_{\mathbf{h}_\mathbf{E}}^a\, w\, d\lambda_{\mathbf{g}}\right)^{\frac{1}{a}},$$

for $1 < r < n$ and $r^* = \frac{nr}{n-r}$, one obtains an estimate of the form

$$\|\mathbf{u}\|_{L^{r^*}(M,\mathbf{E};R_\varepsilon^{2r^*})} \le C\|\mathbf{u}\|_{W^{1,r}(M,\mathbf{E})}$$

on a complete manifold, under the local control of the connection needed for sections; see [7, Theorem 7.5]. When R_ε is uniformly bounded away from zero, there exists $\rho > 0$ such that $\rho \le R_\varepsilon(x) \le 1$ for every $x \in M$, and hence

$$\rho^{2r^*} \le R_\varepsilon(x)^{2r^*} \le 1.$$

These inequalities recover the unweighted mechanism of this section.

Part IV

Index theory

Introduction to Part IV

One of the deepest questions in elliptic theory is how much can be known about an operator without knowing all its coefficients. For an elliptic operator on a closed manifold, lower-order terms influence the particular solutions but do not alter certain fundamental global information. Among this information is the index, defined as the difference between the dimension of the kernel and the dimension of the cokernel.

The first step will be to extend the differential calculus to the pseudodifferential calculus. Its symbols allow parametrices to be constructed and the regularity and singularities of solutions to be described precisely. Fredholm theory turns a parametrix modulo compact operators into a stable integer. This integer still has an analytic origin, but the principal symbol has an interpretation independent of the full operator: it determines a K-theory class on the cotangent bundle.

The second half of this part develops the topological tools needed to interpret that class. Bott periodicity, the Thom isomorphism, and the morphisms associated with embeddings allow the problem to be transported to Euclidean space. The Atiyah–Singer theorem finally asserts that the index obtained by this procedure agrees with the analytic index. The question uniting these chapters is simple to state, although its answer requires several languages: how can a differential equation retain, in its leading term, topological information about the entire manifold?

From elliptic analysis to the index

Fredholm discovered that certain integral equations on function spaces retained an essentially finite-dimensional structure. With the development of elliptic theory and the pseudodifferential calculus, a similar idea emerged for differential operators. In the 1960s, Atiyah and Singer proved that the analytic index could be calculated from topological data of the principal symbol. The result brought together ideas from Hodge, Hirzebruch, Thom, Bott, Grothendieck, and operator theory. Since then, the index has become one of the most fruitful bridges between analysis, geometry, and topology.

15

Pseudodifferential operators

The starting point of the pseudodifferential calculus is a difficulty that already arises when one tries to invert a constant-coefficient elliptic operator. If

$$P(D) = \sum_{|\alpha| \le m} c_\alpha D^\alpha,$$

, the Fourier transform turns the equation $P(D)u = f$ into $p(\xi)\widehat{u}(\xi) = \widehat{f}(\xi)$. When $p(\xi)$ is invertible, the formal solution is obtained by multiplication by $p(\xi)^{-1}$; this inverse, however, is almost never a polynomial and therefore does not correspond to a differential operator. Variable coefficients introduce a further difficulty: the symbol depends on both the point x and the frequency ξ, and composition no longer reduces to the ordinary product of functions.

Pseudodifferential operators enlarge the class of differential operators just enough to solve this problem. Their symbols allow separate control of variation in the base and behavior at high frequencies. Composition is described by an asymptotic expansion, the adjoint remains in the same class, and ellipticity leads to an inverse modulo operators with smooth kernels. This last property turns symbolic inversion into elliptic regularity and, on a closed manifold, into a Fredholm statement.

The analytic ingredients were prepared in earlier chapters. The Fourier transform and tempered distributions of Chapter 4 provide the local model; the Bessel potentials of Chapter 6 show that natural multipliers need not be polynomial; the principal symbol and ellipticity of Chapter 11 identify the part to be inverted. For example,

$$J^s = (I - \Delta)^{\frac{s}{2}}, \qquad \widehat{J^s u}(\xi) = \langle \xi \rangle^s \widehat{u}(\xi),$$

is an operator of order s, whereas $J^{-2} = (I - \Delta)^{-1}$ gains two derivatives without being differential. The calculus we construct explains this phenomenon within a stable theory and then transfers it to manifolds and vector bundles.

We begin on $\mathbb{R}^n$ with symbol classes, oscillatory integrals, and Kohn–Nirenberg quantization. The study of kernels will distinguish locality, pseudolocality, and proper support, and yield the composition and adjoint formulas. We then globalize the calculus, identify the kernels as distributions conormal to the diagonal, and compare local quantization with a Riemannian quantization based on the exponential map. Parametrices will then lead to microlocal regularity and the wavefront set.

The last part of the chapter separates two global situations. On manifolds of bounded geometry, we construct a uniform calculus with finite propagation that preserves Sobolev estimates over an infinite cover. On a closed manifold, smoothing remainders are compact and the parametrix yields Fredholm operators. Finally, the parameter-dependent calculus will allow us to construct Seeley's complex powers, compare them with the spectral calculus, and relate the heat kernel to the McKean–Singer formula.

The symbolic and microlocal theory will rely primarily on Hörmander [84], Taylor [140], and Shubin [135]. For geometric quantization we follow the idea of Bokobza–Haggiag [23]; the construction of complex powers is due to Seeley [132], and the transition to the analytic index uses the exposition of Bleecker and Booß–Bavnbek [21]. The proof of the Atiyah–Singer theorem is left to later chapters.

Hörmander, Seeley, and the maturation of the symbolic calculus

The pseudodifferential calculus emerged by enlarging Fourier multipliers to a class stable under composition and capable of constructing approximate inverses for elliptic operators. Hörmander organized microlocal analysis around symbols, kernels, and wavefront sets; Seeley showed how the same language constructs complex powers of elliptic operators. The strength of the theory lies in its simultaneous treatment of three scales: local behavior in the base variable, homogeneity in frequency, and global regularity of solutions.

15.1 From the Fourier inverse to symbol classes

15.1.1 Conventions and the first model

With the unitary normalization of the Fourier transform fixed in Definition 4.11.1,

$$\widehat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx, \qquad u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi)\, d\xi.$$

Moreover, D_j denotes the partial derivative ∂_j, not the operator $-i\partial_j$. By Proposition 4.11.3, $\mathcal{F}(D^\alpha u)(\xi) = (i\xi)^\alpha \widehat{u}(\xi)$. These conventions determine all factors that will appear in quantization.

First consider $P(D) = \sum_{|\alpha|\le m} c_\alpha D^\alpha$. If $u, f \in \mathcal{S}(\mathbb{R}^n)$ and $P(D)u = f$, then

$$p(\xi)\widehat{u}(\xi) = \widehat{f}(\xi), \qquad p(\xi) := \sum_{|\alpha|\le m} c_\alpha (i\xi)^\alpha.$$

When $p(\xi) \neq 0$, formal inversion uses $p(\xi)^{-1}$. The polynomial p comes from a differential operator, but its inverse is not polynomial. If the coefficients now depend on x, the function $p(x,\xi)^{-1}$ depends simultaneously on position and frequency. The symbol class must control derivatives in both variables while allowing each derivative with respect to ξ to lower the order.

We write $\langle \xi \rangle = (1 + \|\xi\|^2)^{\frac{1}{2}}$. All real bundles and vector spaces will be complexified during the Fourier calculus; at the end, operators preserving the real structure will be restricted back to it.

15.1.2 Symbols of type $(1,0)$

The preceding heuristic suggests the two estimates a symbol must satisfy. Derivatives in x measure variation of the coefficients, while each derivative in ξ should lower the order. The class of type $(1,0)$ formalizes both controls

simultaneously.

Definition 15.1.1. (Symbol class) Let $n, r, s \in \mathbb{N}$ and $m \in \mathbb{R}$. The global uniform class

$$S^m_{1,0}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$$

consists of smooth maps $a\colon \mathbb{R}^n_x \times \mathbb{R}^n_\xi \longrightarrow \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)$ such that, for all multi-indices α, β, there exists $C_{\alpha,\beta} > 0$ with

$$\|D_x^\alpha D_\xi^\beta a(x,\xi)\| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. If $r = s = 1$, we simply write $S^m_{1,0}(\mathbb{R}^n)$.
For an open set $U \subseteq \mathbb{R}^n$, the local class

$$S^m_{1,0}(U \times \mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$$

consists of smooth maps with the same target such that, for every compact set $K \Subset U$ and all multi-indices α, β, there exists $C_{K,\alpha,\beta} > 0$ satisfying

$$\|D_x^\alpha D_\xi^\beta a(x,\xi)\| \le C_{K,\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in K \times \mathbb{R}^n.$$

We abbreviate this class as $S^m_{1,0}(U; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$ and omit the target space only in the scalar case. When the order and all spaces are fixed within a block of text, the subscript $(1,0)$ may be abbreviated.

For $n, r, s \in \mathbb{N}$ and $m \in \mathbb{R}$, the natural topology on $S^m_{1,0}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$ is the Fréchet topology defined by the seminorms

$$q_{N,m}(a) := \max_{|\alpha|+|\beta| \le N} \sup_{(x,\xi) \in \mathbb{R}^{2n}} \langle \xi \rangle^{-m+|\beta|} \|D_x^\alpha D_\xi^\beta a(x,\xi)\|, \qquad N \in \mathbb{N}_0.$$

On an open set U, use a compact exhaustion (K_j) and the seminorms obtained by restricting x to K_j. For $U \subseteq \mathbb{R}^n$ open, define the class of **smoothing symbols** by

$$\begin{aligned} &S^{-\infty}(U \times \mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)) \\ &\quad := \bigcap_{\mu \in \mathbb{R}} S^\mu_{1,0}(U \times \mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)). \end{aligned} \tag{15.1}$$

Equivalently, a belongs to this intersection if, for every compact set $K \Subset U$, all multi-indices α, β, and every $N \in \mathbb{N}_0$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\|D_x^\alpha D_\xi^\beta a(x,\xi)\| \le C_{K,\alpha,\beta,N} \langle \xi \rangle^{-N}, \qquad (x,\xi) \in K \times \mathbb{R}^n. \tag{15.2}$$

For the global uniform class, the same estimate is required for every $x \in \mathbb{R}^n$, with a constant independent of x, and we write

$$S^{-\infty}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)) := \bigcap_{\mu \in \mathbb{R}} S^\mu_{1,0}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)). \tag{15.3}$$

In any of the global or local spaces just defined, the inclusions from order m_1 into order m_2 are continuous when $m_1 \le m_2$, differentiation $D_x^\alpha D_\xi^\beta$ takes order m to $m - |\beta|$, and matrix multiplication adds orders whenever the target spaces allow composition.

The reason for privileging $(\rho,\delta)=(1,0)$ can already be seen in these three properties: frequency differentiation lowers the order, position differentiation does not increase it, and the Taylor expansions appearing in composition produce strictly decreasing orders. The general classes $S^m_{\rho,\delta}(U\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$ replace $m-|\beta|$ by $m-\rho|\beta|+\delta|\alpha|$; they will not be needed here.

Definition 15.1.2. (Euclidean elliptic symbol) Let $U\subseteq\mathbb{R}^n$ be open, and let $r\in\mathbb{N}$ and $m\in\mathbb{R}$. A matrix-valued symbol $a\in S^m_{1,0}(U;\mathrm{End}(\mathbb{C}^r))$ is **elliptic** if, for every compact set $K\Subset U$, there exist $R_K,C_K>0$ such that $a(x,\xi)$ is invertible and

$$\|a(x,\xi)^{-1}\|\le C_K\langle\xi\rangle^{-m}$$

when $x\in K$ and $\|\xi\|\ge R_K$. It is **uniformly elliptic** if R_K and C_K can be chosen independently of K.

15.1.3 Classical symbols and asymptotic expansions

The class $S^m_{1,0}$ controls the size of a symbol but does not distinguish successive contributions from its different degrees. To construct a parametrix, we first need to separate the leading term and correct it order by order. Classical symbols provide precisely this homogeneous expansion at high frequencies.

Definition 15.1.3. (Classical symbol) Let $U\subseteq\mathbb{R}^n$ be open, and let $r,s\in\mathbb{N}$ and $m\in\mathbb{R}$. A symbol $a\in S^m_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$ is **classical** or **polyhomogeneous** if there exist smooth maps

$$a_{m-j}\colon U\times(\mathbb{R}^n\setminus\{0\})\longrightarrow\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s),\qquad j\in\mathbb{N}_0,$$

, homogeneous in the sense that $a_{m-j}(x,t\xi)=t^{m-j}a_{m-j}(x,\xi)$ for $t>0$, and a function $\vartheta\in C^\infty(\mathbb{R}^n)$ that vanishes near 0 and equals one for $\|\xi\|\ge1$, such that, for every $N\ge1$,

$$a-\sum_{j=0}^{N-1}\vartheta a_{m-j}\in S^{m-N}_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s)).$$

We write $a\in S^m_{\mathrm{cl}}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$ and call a_m the **principal homogeneous component**.

The presence of ϑ does not alter the homogeneous information: two cutoffs yield symbols whose difference is compactly supported in ξ and belongs to $S^{-\infty}(U\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$. The component a_m is unique. If b_m is another principal component, then $a_m-b_m\in S^{m-1}_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$ for $\|\xi\|\ge1$. For $\omega\in\mathbb{S}^{n-1}$ and $t\ge1$, homogeneity gives

$$t^m\|a_m(x,\omega)-b_m(x,\omega)\|\le C_K\,t^{m-1}.$$

Dividing by t^m and letting $t\to\infty$ gives $a_m=b_m$. The same inductive argument proves uniqueness of all homogeneous components.

Definition 15.1.4. (Asymptotic expansion) Let $U\subseteq\mathbb{R}^n$ be open, let $r,s\in\mathbb{N}$, and let $m_0\ge m_1\ge\cdots$ be real numbers with $m_j\to-\infty$ and

$$a_j\in S^{m_j}_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s)).$$

For $a \in S^{m_0}_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$, we write

$$a \sim \sum_{j=0}^{\infty} a_j$$

if, for every $N \geq 1$,

$$a - \sum_{j=0}^{N-1} a_j \in S^{m_N}_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s)).$$

When several m_j coincide, the conclusion is understood after grouping terms of equal order.

Differentiation may be performed term by term: if $a \sim \sum_{j=0}^{\infty} a_j$, then $D_x^\alpha D_\xi^\beta a \sim \sum_{j=0}^{\infty} D_x^\alpha D_\xi^\beta a_j$, with orders $m_j - |\beta|$. The coefficients of a general expansion are not unique as functions; what is determined is the class of the term of order m_j modulo symbols of the next order. For classical symbols, homogeneity removes this ambiguity.

Lemma 15.1.5. (Asymptotic summation of symbols) Let $U \subseteq \mathbb{R}^n$ be open and let $r,s \in \mathbb{N}$. In the setting of Definition 15.1.4, there exists

$$a \in S^{m_0}_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$$

with $a \sim \sum_{j=0}^{\infty} a_j$. In particular, if h_{m_0-j} is smooth on $U\times(\mathbb{R}^n\setminus\{0\})$ with values in $\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s)$ and homogeneous of degree $m_0 - j$ for each $j \in \mathbb{N}_0$, then, for a radial cutoff ϑ as in Definition 15.1.3, one can choose $a \in S^{m_0}_{\mathrm{cl}}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$ with

$$a \sim \sum_{j=0}^{\infty} \vartheta h_{m_0-j}.$$

The lattice of degrees $m_0 - j$ in the last clause is essential to the notion in Definition 15.1.3. A homogeneous family of merely decreasing real degrees admits a general asymptotic sum, but defines only a homogeneous expansion with that set of degrees; it is classical in the sense adopted here when those degrees belong to $m_0 - \mathbb{N}_0$, inserting zero components at the missing degrees.

Proof. Throughout this proof, U, r, s remain fixed, and we write

$$S^q := S^q_{1,0}\big(U \times \mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s)\big).$$

Since $m_j \to -\infty$ and the sequence is nonincreasing, each order can occur only finitely many times. For each value μ that occurs, the set

$$I_\mu := \{j \in \mathbb{N}_0 \mid m_j = \mu\}$$

is finite. Group the terms corresponding to I_μ and replace the group by the finite sum $\sum_{j\in I_\mu} a_j$. This is exactly the convention fixed after Definition 15.1.4. It suffices to treat the case

$$m_0 > m_1 > m_2 > \cdots.$$

Fix an exhaustion $K_1 \Subset K_2 \Subset \cdots$ of U and a function $\chi \in C^\infty(\mathbb{R}^n)$ vanishing on $\{\|\xi\| \leq 1\}$ and equal to one on

$\{\|\xi\| \geq 2\}$. For a compact set $K \Subset U$, write

$$q^K_{L,\mu}(c) := \max_{|\alpha|+|\beta|\leq L} \sup_{x\in K,\,\xi\in\mathbb{R}^n} \langle\xi\rangle^{-\mu+|\beta|} \|D_x^\alpha D_\xi^\beta c(x,\xi)\|.$$

We will choose a sequence $\varepsilon_j \to 0^+$ and set

$$b_j(x,\xi) := \chi(\varepsilon_j\xi)a_j(x,\xi).$$

If $N < j$, then $m_j < m_N$. The Leibniz rule and the fact that $\chi(\varepsilon\xi)$ is supported where $\|\xi\| \geq \varepsilon^{-1}$ show that, for fixed K, L, N, j with $N < j$,

$$q^K_{L,m_N}\big(\chi(\varepsilon\,\cdot)a_j\big) \longrightarrow 0 \qquad (\varepsilon \longrightarrow 0^+). \tag{15.4}$$

Indeed, each Leibniz term contains, besides a seminorm of a_j, the factor $\langle\xi\rangle^{m_j-m_N}$; when a derivative hits the cutoff, powers of ε appear, and on its support $\|\xi\|$ is comparable to ε^{-1}, giving the same power of $\varepsilon^{m_N-m_j}$ or a better one.

For each $j \geq 1$, there are only finitely many conditions

$$q^{K_p}_{p,m_N}(b_j) \leq 2^{-j}, \qquad 0 \leq N < j, \qquad 1 \leq p \leq j.$$

Convergence (15.4) allows ε_j to be chosen so that all of them hold and, by decreasing it if necessary, so that $\varepsilon_j < \varepsilon_{j-1}$. Choose $\varepsilon_0 > 0$ arbitrarily.

For every compact set $K \Subset U$ and every L, the conditions with $N = 0$, applied on a K_p containing K and with $p \geq L$, show that $\displaystyle\sum_{j=0}^{\infty} b_j$ converges in the seminorm q^K_{L,m_0}. Consequently,

$$a := \sum_{j=0}^{\infty} b_j$$

belongs to S^{m_0}. Now fix N. The term b_N belongs to S^{m_N}, and the preceding conditions, this time with target order m_N, show that $\displaystyle\sum_{j>N} b_j$ converges in all seminorms of S^{m_N}. Thus, for each N,

$$a - \sum_{j=0}^{N-1} a_j = \sum_{j=N}^{\infty} b_j + \sum_{j=0}^{N-1} (b_j - a_j).$$

The first sum belongs to S^{m_N}. Each difference $b_j - a_j$ has bounded support in ξ and therefore belongs to $S^{-\infty}(U\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$; this proves the asymptotic expansion. Ungrouping the finite groups recovers the interpretation specified in the definition. For the last clause, apply the same construction with radial cutoffs to $a_j = \vartheta h_{m_0-j}$; the remainder after the first N terms belongs to S^{m_0-N}, which is exactly the expansion in Definition 15.1.3. □

15.1.4 Oscillatory integrals and quantization

So far we have worked with symbol functions. To turn them into operators, we must give meaning to an integral that, except at very negative orders, is not absolutely convergent. Oscillation of the phase will compensate for

growth of the symbol.

Let $n, r, s \in \mathbb{N}$, $m < -n$, $a \in S^m_{1,0}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$, and $u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^r)$. Then the integral

$$(2\pi)^{-n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, dy\, d\xi$$

is absolutely convergent. At higher orders, oscillation of the exponential takes the place of absolute integrability. We make this replacement precise by regularization, rather than by an implicit convention.

Let $c \in C^\infty_c(\mathbb{R}^{2n})$ equal one near $(0,0)$. If $q(x,y,\xi)$ is smooth and all its derivatives have polynomial growth in (y,ξ), define, whenever the limit exists,

$$\mathrm{Os} \iint_{\mathbb{R}^n_y \times \mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} q(x,y,\xi)\, dy\, d\xi := \lim_{\varepsilon \to 0^+} \iint_{\mathbb{R}^n_y \times \mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} c(\varepsilon y, \varepsilon\xi) q(x,y,\xi)\, dy\, d\xi.$$

Existence will be proved under growth bounds of fixed order. Specifically, suppose there exist $\mu, \tau \in \mathbb{R}$ such that, for each compact set K in the variable x and all multi-indices γ, α, β,

$$\|D_x^\gamma D_y^\alpha D_\xi^\beta q(x,y,\xi)\| \le C_{K,\gamma,\alpha,\beta} \langle y\rangle^\tau \langle \xi\rangle^{\mu-|\beta|}, \qquad x \in K.$$

The exponents μ and τ do not depend on the number of derivatives. The phase is the real function $(x-y)\cdot\xi$. The identities

$$e^{i(x-y)\cdot\xi} = \langle \xi\rangle^{-2N} (1-\Delta_y)^N e^{i(x-y)\cdot\xi}$$

and

$$e^{i(x-y)\cdot\xi} = \langle x-y\rangle^{-2L} (1-\Delta_\xi)^L e^{i(x-y)\cdot\xi}$$

transfer derivatives to the amplitude. Write $c_\varepsilon(y,\xi) = c(\varepsilon y, \varepsilon\xi)$. Integrating by parts first in y and then in ξ shows that the regularized integral equals the integral of $e^{i(x-y)\cdot\xi} R_\varepsilon(x,y,\xi)$, where

$$R_\varepsilon = \langle x-y\rangle^{-2L} (1-\Delta_\xi)^L \big[\langle \xi\rangle^{-2N} (1-\Delta_y)^N (c_\varepsilon q) \big].$$

These operations are valid before taking the limit because the cutoff has compact support. For $0 < \varepsilon \le 1$, its derivatives satisfy

$$|D_y^\alpha D_\xi^\beta c_\varepsilon(y,\xi)| \le C_{\alpha,\beta} \langle \xi\rangle^{-|\beta|}.$$

Indeed, each derivative contributes a factor of ε, and on the support of the corresponding term, $\varepsilon\langle\xi\rangle$ is bounded independently of ε. The Leibniz rule and the bounds on q give, for $x \in K$,

$$\|R_\varepsilon(x,y,\xi)\| \le C_K \langle y\rangle^{\tau-2L} \langle \xi\rangle^{\mu-2N}.$$

Here we use that $\langle x-y\rangle$ and $\langle y\rangle$ are uniformly comparable as x ranges over K. Choose $2N > \mu + n$ and $2L > \tau + n$; the last expression is then integrable in (y,ξ). For each fixed (y,ξ), the cutoff equals one and all its positive-order derivatives vanish when ε is sufficiently small. By dominated convergence, the limit is the absolutely convergent integral obtained by replacing $c_\varepsilon q$ with q in R_ε. This expression is independent of the cutoff c.

Differentiating with respect to x up to order d also produces powers of ξ of degree at most d. The same proof, with $2N > \mu + d + n$, gives an integrable majorant for all these derivatives, uniformly on K. Convergence of the integrand and its derivatives is also uniform in $x \in K$ for fixed (y,ξ). This yields convergence in every $C^d(K)$ norm

and smooth dependence of the oscillatory integral on x.

For the following quantization, $q(x,y,\xi) = a(x,\xi)u(y)$ satisfies these bounds with $\mu = m$ and $\tau = 0$, since u is a Schwartz function. The same holds for $b(x,y,\xi)u(y)$ for the fixed-order amplitudes introduced below. In the composition and reduction formulas, apply the argument to the integration variables, leaving the remaining variables as parameters; symbol estimates and Peetre's inequality control the weights appearing there. The bounds are local in the parameter variables. Polynomial growth of each derivative, without common control of the exponents, will not be used as a sufficient hypothesis for existence.

Together with asymptotic summation, regularization allows us to pass from formal expansions to well-defined operators. An equivalent construction, with remainders estimated seminorm by seminorm, can be found in [135, Sections 1 and 3].

Definition 15.1.6. (Kohn–Nirenberg quantization) Let $n, r, s \in \mathbb{N}$, $m \in \mathbb{R}$, and $a \in S^m_{1,0}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$. Define

$$\mathrm{Op}(a)u(x) := (2\pi)^{-n} \operatorname{Os} \iint_{\mathbb{R}^n_y \times \mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,\xi)u(y)\, dy\, d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^r).$$

Equivalently,

$$\mathrm{Op}(a)u(x) = (2\pi)^{-\frac{n}{2}} \operatorname{Os} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\, d\xi.$$

The second formula is first obtained when the integrals are absolutely convergent, using Fourier inversion and Fubini's theorem, and then by regularization. The factor $(2\pi)^{-n}$ in the first formula results from the two occurrences of the unitary normalization; it cannot be replaced by $(2\pi)^{-\frac{n}{2}}$.

Proposition 15.1.7. (Continuity of quantization) Let $n, r, s \in \mathbb{N}$, $m \in \mathbb{R}$, and

$$a \in S^m_{1,0}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)).$$

. Then $\mathrm{Op}(a)$ is continuous from $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^r)$ to $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^s)$. By transposition it extends uniquely to a continuous operator

$$\mathrm{Op}(a)\colon \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^r) \longrightarrow \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^s).$$

If $a \in S^{-\infty}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$, it maps $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^r)$ into $C^\infty(\mathbb{R}^n, \mathbb{C}^s)$; if, in addition, a decays rapidly also in x, it maps $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^r)$ into $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^s)$.

Proof. Differentiating the second formula with respect to x produces sums of terms with factors $\xi^\gamma D_x^{\alpha-\gamma} a(x,\xi)$. To control a power x^β, use $x_j e^{ix\cdot\xi} = -iD_{\xi_j} e^{ix\cdot\xi}$ and integrate by parts in ξ. After expanding by the Leibniz rule, each Schwartz seminorm of $\mathrm{Op}(a)u$ is bounded by a finite sum of symbol seminorms of a and Schwartz seminorms of u. Choose the number of integrations by parts greater than $m + n$ to make the variable ξ integrable. This proves existence, membership in $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^s)$, and continuity simultaneously. The transposed operator is defined by $\langle \mathrm{Op}(a)u, \varphi\rangle := \langle u, \mathrm{Op}(a)^t\varphi\rangle$, where the transpose on the right-hand side is obtained by interchanging the kernel variables; the proposition on amplitudes proved below shows that it too preserves the corresponding Schwartz spaces. If $a \in S^{-\infty}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$, the decay in ξ can be chosen arbitrarily large, allowing differentiation under the integral against a tempered distribution and yielding smoothness. Additional decay in x gives all Schwartz seminorms. □

15.1.5 Amplitudes and reduction to symbols

Kohn–Nirenberg quantization uses a symbol evaluated at the output variable x. Composing operators, taking adjoints, or changing coordinates naturally introduces coefficients that also depend on the input variable y. Before closing the calculus under these operations, we must show that such amplitudes can be reduced back to symbols.

Definition 15.1.8. (Amplitude) Let $n, r, s \in \mathbb{N}$ and $m \in \mathbb{R}$. An **amplitude of order** m is a smooth map

$$b\colon \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\xi \longrightarrow \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^s)$$

such that, for all multi-indices α, γ, β, there exists $C_{\alpha,\gamma,\beta} > 0$ with

$$\|D_x^\alpha D_y^\gamma D_\xi^\beta b(x,y,\xi)\| \le C_{\alpha,\gamma,\beta} \langle \xi \rangle^{m-|\beta|}.$$

Its operator, initially on $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^r)$, is defined by

$$I(b)u(x) := (2\pi)^{-n} \operatorname{Os} \iint_{\mathbb{R}^n_y \times \mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} b(x,y,\xi) u(y)\, dy\, d\xi.$$

A symbol is the special amplitude independent of y. The reverse direction contains the argument that will recur in composition, adjoints, and coordinate changes.

Theorem 15.1.9. (Amplitude reduction) Let $n, r, s \in \mathbb{N}$ and $m \in \mathbb{R}$. For each amplitude b of order m with values in $\operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^s)$, there exists

$$a \in S^m_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^s))$$

such that $I(b) = \operatorname{Op}(a)$. The symbol can be chosen with expansion

$$a(x,\xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha b(x,y,\xi)\big|_{y=x}.$$

The operator determines the symbol modulo $S^{-\infty}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^s))$.

Proof. Throughout this proof, n, r, s, m remain fixed, and we write

$$S^q := S^q_{1,0}\big(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^s)\big).$$

Exact reduction is obtained through the partial Fourier transform of the kernel in the variable $x - y$. With this chapter's normalization, define

$$a(x,\xi) := (2\pi)^{-n} \operatorname{Os} \iint_{\mathbb{R}^n_y \times \mathbb{R}^n_\eta} e^{-iy\cdot\eta} b(x, x+y, \xi+\eta)\, dy\, d\eta. \tag{15.5}$$

This oscillatory integral exists. To see this, use, with $L, K \in \mathbb{N}$,

$$e^{-iy\cdot\eta} = \langle \eta \rangle^{-2K} (1 - \Delta_y)^K e^{-iy\cdot\eta}, \qquad e^{-iy\cdot\eta} = \langle y \rangle^{-2L} (1 - \Delta_\eta)^L e^{-iy\cdot\eta},$$

applying the first identity first, and then the second to each resulting term. After integration by parts, the

derivatives fall on the amplitude and the weight in η. Peetre's inequality in Lemma 6.1.1 gives, for every $\mu \in \mathbb{R}$,

$$\langle \xi+\eta\rangle^{\mu} \le 2^{\frac{|\mu|}{2}} \langle \xi\rangle^{\mu}\langle \eta\rangle^{|\mu|}.$$

Choosing $L > \frac{n}{2}$ and $2K > n+\big|m-|\beta|\big|$ makes the resulting integrand absolutely integrable, and after applying $D_x^{\alpha}D_{\xi}^{\beta}$ it is bounded by

$$C_{\alpha,\beta}\langle \xi\rangle^{m-|\beta|}.$$

The same argument for any number of derivatives proves that $a \in S^m$. Inserting compactly supported cutoffs before applying Fubini's theorem, the kernel formula and the changes $y-x \mapsto y$ and $\eta-\xi \mapsto \eta$ give $I(b) = \mathrm{Op}(a)$; the preceding estimates allow the cutoffs to be removed. Thus (15.5) directly constructs the symbol associated with the amplitude.

Now fix $N \ge 1$ and expand the amplitude in y about x:

$$b(x,y,\xi) = \sum_{|\alpha|<N} \frac{(y-x)^{\alpha}}{\alpha!} D_y^{\alpha} b(x,x,\xi) + \sum_{|\alpha|=N} (y-x)^{\alpha} r_{\alpha}(x,y,\xi),$$

where

$$r_{\alpha}(x,y,\xi) = \frac{N}{\alpha!}\int_0^1 (1-t)^{N-1} D_y^{\alpha} b(x,x+t(y-x),\xi)\,dt.$$

Since $(y-x)^{\alpha}e^{i(x-y)\cdot\xi} = i^{|\alpha|}D_{\xi}^{\alpha}e^{i(x-y)\cdot\xi}$, integration by parts in ξ turns each term of the finite sum into

$$\frac{i^{-|\alpha|}}{\alpha!} D_{\xi}^{\alpha} D_y^{\alpha} b(x,x,\xi).$$

Equivalently, inserting this Taylor expansion into (15.5), the identity

$$y^{\alpha}e^{-iy\cdot\eta} = i^{|\alpha|}D_{\eta}^{\alpha}e^{-iy\cdot\eta}$$

produces the same terms. In the remainder, the N integrations by parts make D_{η}^{α} act on $r_{\alpha}(x,x+y,\xi+\eta)$. Each derivative in η is a derivative in the covariable of b and lowers the order by one. The estimate used for (15.5), now applied to these derivatives, shows directly that the reduced remainder belongs to S^{m-N}. Since N is arbitrary, this gives the asserted expansion without a second appeal to amplitude reduction.

In the global model, the partial transform defining (15.5) recovers the symbol from the kernel. After localization on an open set, two choices may differ by a part of the kernel separated from the diagonal; integration by parts in the covariable shows that its transform belongs to $S^{-\infty}(\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$. Thus the operator determines the symbol modulo this global smoothing class. □

15.1.6 Fundamental examples

Before developing the algebraic calculus, it is useful to recognize some familiar operators within the new class. The following examples show how symbol order records loss or gain of derivatives while distinguishing locality, nonlocality, and smoothing.

Example 15.1.10. (Differential operators) Let $U \subseteq \mathbb{R}^n$ be open, and let $r,s \in \mathbb{N}$ and $A_{\alpha} \in$

$C^\infty(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$. For

$$P=\sum_{|\alpha|\le m}A_\alpha(x)D^\alpha$$

, the Fourier symbol is

$$p(x,\xi)=\sum_{|\alpha|\le m}A_\alpha(x)(i\xi)^\alpha,\qquad P=\mathrm{Op}(p).$$

If $U=\mathbb{R}^n$ and the coefficients and all their derivatives are bounded,

$$p\in S^m_{1,0}(\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s));$$

; locally it always belongs to $S^m_{1,0}(U;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$. The principal Fourier component is

$$p_m(x,\xi)=i^m\sum_{|\alpha|=m}A_\alpha(x)\xi^\alpha=i^m\sigma_m(P)(x,\xi),$$

where $\sigma_m(P)$ is exactly the intrinsic symbol of Definition 11.1.1. This reconciles the previously fixed differential convention with Fourier quantization. Multiplication by i^m does not change ellipticity.
The operator is local: its kernel is a finite sum of transverse derivatives of the delta distribution on the diagonal,

$$\mathbf{K}_P(x,y)=\sum_{|\alpha|\le m}A_\alpha(x)D_x^\alpha\delta(x-y).$$

Its support is therefore contained in the diagonal, where all its singularities are concentrated. The Fourier and localization estimates already proved in the Sobolev chapters give $P\colon H^{t+m}_{\mathrm{loc}}(U,\mathbb{C}^r)\longrightarrow H^t_{\mathrm{loc}}(U,\mathbb{C}^s)$; if $U=\mathbb{R}^n$ and the coefficients and their derivatives are globally bounded, one obtains $P\colon H^{t+m}(\mathbb{R}^n,\mathbb{C}^r)\longrightarrow H^t(\mathbb{R}^n,\mathbb{C}^s)$.

Example 15.1.11. (Multipliers and Bessel potentials) If a depends only on ξ, then $\mathrm{Op}(a)u=\mathcal{F}^{-1}(a\widehat{u})$. For $s\in\mathbb{R}$, the estimates preceding Definition 6.1.2 show that $\langle\xi\rangle^s\in S^s_{\mathrm{cl}}(\mathbb{R}^n)$, with principal component $\|\xi\|^s$ for $\xi\neq0$. Thus

$$J^s=\mathrm{Op}(\langle\xi\rangle^s),\qquad J^{-s}=\mathrm{Op}(\langle\xi\rangle^{-s}).$$

The first has order s and the second order $-s$; neither is local unless its symbol is a polynomial. When $s>0$, the kernel of J^{-s} is the Bessel kernel in Proposition 6.1.4.
Their principal symbols are $\|\xi\|^s$ and $\|\xi\|^{-s}$, respectively. Directly from the Fourier definition, for every $t\in\mathbb{R}$,

$$J^s\colon H^{t+s}(\mathbb{R}^n)\longrightarrow H^t(\mathbb{R}^n),\qquad J^{-s}\colon H^t(\mathbb{R}^n)\longrightarrow H^{t+s}(\mathbb{R}^n)\quad(s>0)$$

are isomorphisms. The kernels are smooth when $x\neq y$ and, except in the local polynomial cases, have a conormal singularity at $x=y$; for $s>0$, the kernel of J^{-s} progressively loses this singularity as s increases, but does not become C^∞ at a general finite order.

Example 15.1.12. (The fractional Laplacian and low frequencies) Let $s>0$. The multiplier of $(-\Delta)^{\frac{s}{2}}$ is $\|\xi\|^s$. If s is not an even integer, it is not smooth at $\xi=0$ and therefore does not belong to the global inhomogeneous class $S^s_{1,0}(\mathbb{R}^n)$ just defined. If ϑ vanishes near zero and equals one for $\|\xi\|\ge1$, then $\vartheta(\xi)\|\xi\|^s\in S^s_{\mathrm{cl}}(\mathbb{R}^n)$. The remaining part has compact frequency support and maps compactly supported distributions to smooth

functions. Thus the fractional Laplacian is microlocally a classical pseudodifferential operator of order s, although its global homogeneous treatment retains additional information at frequency zero. This distinction disappears for $(I-\Delta)^{\frac{s}{2}}$, whose symbol $\langle\xi\rangle^s$ is smooth on all of $\mathbb{R}^n$.
For $s>0$, the multiplier $\|\xi\|^s$ maps $H^{t+s}(\mathbb{R}^n)$ continuously into $H^t(\mathbb{R}^n)$ and has principal symbol $\|\xi\|^s$. It is nonlocal when s is not an even integer. If $0<s<2$, its kernel off the diagonal is represented, up to a dimensional constant, by $\|x-y\|^{-n-s}$, and its action is interpreted in the principal-value sense through the difference $u(x)-u(y)$. Thus the kernel's singularity lies on the diagonal, but its support does not: this is the distinction between nonlocality and pseudolocality.

Example 15.1.13. (Riesz and Hilbert transforms) The Riesz transform $R_j = D_j(-\Delta)^{-\frac{1}{2}}$ has symbol $\frac{i\xi_j}{\|\xi\|}$ away from $\xi=0$, homogeneous of degree zero. In dimension one, the Hilbert transform with principal-value kernel $\pi^{-1}(x-y)^{-1}$ has multiplier $-i\operatorname{sign}(\xi)$. Both are nonlocal operators of order zero at high frequencies; a cutoff at frequency zero produces representatives in $S^0_{\mathrm{cl}}(\mathbb{R}^n)$ with the same microlocal behavior.
These multipliers are also their principal symbols. Since they are bounded, both operators act continuously on every $H^t(\mathbb{R}^n)$. Away from the diagonal, the Riesz kernel is a multiple of $\frac{x_j-y_j}{\|x-y\|^{n+1}}$, and the Hilbert kernel is $\pi^{-1}\operatorname{pv}(x-y)^{-1}$. They are smooth away from $x=y$, singular on the diagonal, and not supported there.

Example 15.1.14. (Smoothing operators and exact inverses) The symbol $e^{-\|\xi\|^2}$ belongs to $S^{-\infty}(\mathbb{R}^n;\operatorname{Hom}(\mathbb{C},\mathbb{C}))$, and its operator is convolution with a Gaussian; it maps $\mathcal{S}'(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$. On the other hand, $I-\Delta$ has multiplier $1+\|\xi\|^2$, and its exact inverse on $\mathcal{S}'(\mathbb{R}^n)$ is $\operatorname{Op}(\langle\xi\rangle^{-2})$. This last example contains the general mechanism of elliptic symbolic inversion without error terms.
The Gaussian operator has order $-\infty$, zero principal symbol in every finite-order quotient, and a smooth nonlocal kernel, and it maps $H^t(\mathbb{R}^n)$ into $H^N(\mathbb{R}^n)$ for all $t,N\in\mathbb{R}$. The inverse $(I-\Delta)^{-1}$ has order -2 and principal symbol $\|\xi\|^{-2}$ and is an isomorphism $H^t(\mathbb{R}^n)\longrightarrow H^{t+2}(\mathbb{R}^n)$. Its Bessel kernel is smooth off the diagonal and has the singularity there corresponding to order -2; it is an exact inverse and, in particular, a parametrix without a remainder.

15.2 Kernels, proper support, and symbolic calculus

The symbol describes the operator in frequency, whereas the kernel shows how two points in space interact. This second description is essential for distinguishing local and nonlocal operators, formulating proper support, and proving pseudolocality. It also explains why composition, adjoints, and coordinate changes remain within the calculus.

15.2.1 The kernel and the diagonal

The Schwartz kernel theorem for vector bundles, Theorem 10.2.57, represents a continuous operator by a distribution on the product. In the Euclidean scalar model, Definition 15.1.6 gives

$$\mathbf{K}_{\operatorname{Op}(a)}(x,y) = (2\pi)^{-n}\operatorname{Os}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}a(x,\xi)\,d\xi.$$

The possible singularity is concentrated at $x = y$. If $x \neq y$, the operator

$$L_{x,y} := \frac{1}{i\|x-y\|^2}(x-y)\cdot D_\xi$$

satisfies $L_{x,y}e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi}$. After integrating by parts N times, derivatives of order N with respect to ξ reduce the symbol to order $m-N$. For $N > m+n$, the resulting integral is absolutely convergent. The same procedure may be repeated after differentiating in x and y. Thus $\mathbf{K}_{\mathrm{Op}(a)}$ is smooth off the diagonal

$$\Delta_{\mathbb{R}^n} := \{(x,x) \mid x \in \mathbb{R}^n\}.$$

An operator A is called **pseudolocal** if, for every distribution u on which it is defined,

$$\operatorname{sing\,supp}(Au) \subseteq \operatorname{sing\,supp}(u).$$

This property does not say that A is local: it only prevents it from creating singularities at points where the datum is smooth.

Thus a pseudodifferential operator can be nonlocal because its kernel is not supported on the diagonal, while remaining pseudolocal because that kernel is smooth off the diagonal. Extending the operator to general distributions and controlling supports also requires a global condition on the projections of its kernel.

Definition 15.2.1. (Proper support) Let N and M be smooth manifolds without boundary, let $\mathbf{E} \to N$ and $\mathbf{F} \to M$ be smooth vector bundles of finite rank, let

$$A\colon \Gamma_c(N,\mathbf{E}) \longrightarrow \Gamma(M,\mathbf{F})$$

be a continuous operator, also regarded as taking values in $\mathcal{D}'(M,\mathbf{F})$, and let $\mathbf{K}_A$ be its kernel on $M\times N$. We say that A is **properly supported** if both projections

$$\mathrm{pr}_M\colon \operatorname{supp}\mathbf{K}_A \longrightarrow M, \qquad \mathrm{pr}_N\colon \operatorname{supp}\mathbf{K}_A \longrightarrow N$$

are proper maps. Equivalently, for every compact subset K of either factor, the intersection of $\operatorname{supp}\mathbf{K}_A$ with the preimage of K has compact projection onto the other factor.

In this chapter's pseudodifferential calculus we take $N = M$; the formulation with two manifolds simply records that proper support is a condition on the kernel and its two projections.

This condition has two distinct consequences. Projection onto the source, pr_N, controls the support of Au when u has compact support. Projection onto the target, pr_M, allows Au to be defined for a distribution without compact support: local evaluation against a test function involves only a compact part of the distribution. In particular, a properly supported operator extends continuously

$$A\colon \Gamma_c(\mathbf{E}) \longrightarrow \Gamma_c(\mathbf{F}), \qquad A\colon \mathcal{D}'(N,\mathbf{E}) \longrightarrow \mathcal{D}'(M,\mathbf{F}),$$

if its local expression is pseudodifferential. The second extension is defined by duality with the properly supported transpose.

Proposition 15.2.2. (Decomposition with proper support) Let $U \subseteq \mathbb{R}^n$ be open, and let $r, s \in \mathbb{N}$ and $a \in S^m_{1,0}(U; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$. For the local operator $A = \mathrm{Op}(a)$, there exist a properly supported pseudodifferential operator A_0 of the same order and an operator R with smooth kernel such that $A = A_0 + R$. Moreover, A_0 has the same principal symbol as A.

Proof. Choose a function $\chi(x, y)$ equal to one on a neighborhood of the diagonal and supported in a properly supported neighborhood of it. Define $\mathbf{K}_{A_0} := \chi \mathbf{K}_A$. Then A_0 is properly supported. The amplitude $\chi(x, y)a(x, \xi)$ reduces to a symbol by Theorem 15.1.9. In its expansion, the term with $\alpha = 0$ is $a(x, \xi)$, and for $|\alpha| \geq 1$,

$$D_\xi^\alpha D_y^\alpha \left(\chi(x, y)a(x, \xi)\right)\big|_{y=x} = 0,$$

since χ is identically one on a neighborhood of the diagonal. Thus the reduced symbol differs from a by an element of $S^{-\infty}(U \times \mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$ and, in particular, has the same principal symbol. On the other hand, $(1 - \chi)\mathbf{K}_A$ is supported outside a neighborhood of the diagonal. The integrations by parts at the beginning of this section show that it is smooth. Thus $R = A - A_0$ has the asserted kernel. □

Definition 15.2.3. (Smoothing operator) Let $U \subseteq \mathbb{R}^n$ be open and let $r, s \in \mathbb{N}$. The class

$$\Psi^{-\infty}_{\mathrm{loc}}(U; \mathbb{C}^r, \mathbb{C}^s)$$

consists of continuous linear operators

$$A \colon C^\infty_c(U, \mathbb{C}^r) \longrightarrow C^\infty(U, \mathbb{C}^s)$$

whose Schwartz kernel, written relative to Lebesgue measure in the input variable, belongs to

$$C^\infty(U \times U; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)).$$

Its elements are called **local smoothing operators**. Define

$$\Psi^{-\infty}(U; \mathbb{C}^r, \mathbb{C}^s) := \left\{ A \in \Psi^{-\infty}_{\mathrm{loc}}(U; \mathbb{C}^r, \mathbb{C}^s) \;\middle|\; \mathbf{K}_A \text{ is properly supported} \right\}. \tag{15.6}$$

Thus the notation without the subscript loc includes proper support.
On all of $\mathbb{R}^n$, if

$$a \in S^{-\infty}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)),$$

, the global Kohn–Nirenberg operator $\mathrm{Op}(a)$ has smooth kernel and, by Proposition 15.1.7, acts continuously from $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^r)$ to $C^\infty(\mathbb{R}^n, \mathbb{C}^s)$. This global statement imposes no proper-support condition; the operator belongs to $\Psi^{-\infty}(\mathbb{R}^n; \mathbb{C}^r, \mathbb{C}^s)$ only if its kernel also satisfies that condition.

Proposition 15.2.4. (Smoothing symbols and kernels) Let $U \subseteq \mathbb{R}^n$ be open, let $r, s \in \mathbb{N}$, and let A be a local pseudodifferential operator from $C^\infty_c(U, \mathbb{C}^r)$ to $C^\infty(U, \mathbb{C}^s)$. The following conditions are equivalent:

(a) every localization of A has a symbol in $S^{-\infty}(U \times \mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$;

(b) $\mathbf{K}_A$ is smooth near the diagonal;

(c) $\mathbf{K}_A$ belongs to $C^\infty(U\times U;\operatorname{Hom}(\mathbb{C}^r,\mathbb{C}^s))$, that is, $A\in\Psi^{-\infty}_{\mathrm{loc}}(U;\mathbb{C}^r,\mathbb{C}^s)$.

If A is properly supported, these conditions are also equivalent to $A\in\Psi^{-\infty}(U;\mathbb{C}^r,\mathbb{C}^s)$.

Proof. If a localization has symbol in $S^{-\infty}(U\times\mathbb{R}^n;\operatorname{Hom}(\mathbb{C}^r,\mathbb{C}^s))$, applying a total of q derivatives in x and y to the kernel produces only polynomial factors in ξ of degree at most q. Estimate (15.2) with $N>n+q$ yields an integrable majorant in $\mathbb{R}^n_\xi$; differentiation under the integral is therefore allowed, and the kernel is smooth near the diagonal. Every pseudodifferential kernel is smooth off the diagonal, as proved at the beginning of this section, so (b) implies (c).
Conversely, suppose (c) holds and choose $\chi(x,y)$ equal to one on a neighborhood of the diagonal, with properly supported support inside a coordinate localization. The partial transform

$$a_\chi(x,\xi):=\int_U e^{-i(x-y)\cdot\xi}\chi(x,y)\mathbf{K}_A(x,y)\,dy$$

is a rapidly decaying symbol. For multi-indices α,β, first use $(D_x+D_y)e^{-i(x-y)\cdot\xi}=0$ and integration by parts in y to obtain

$$D_x^\alpha D_\xi^\beta a_\chi(x,\xi)=(-i)^{|\beta|}\int_U e^{-i(x-y)\cdot\xi}(D_x+D_y)^\alpha\big((x-y)^\beta\chi(x,y)\mathbf{K}_A(x,y)\big)\,dy.$$

Since $(1-\Delta_y)^L e^{-i(x-y)\cdot\xi}=\langle\xi\rangle^{2L}e^{-i(x-y)\cdot\xi}$, a second integration by parts gives, for each compact set K in the variable x,

$$\sup_{x\in K}\|D_x^\alpha D_\xi^\beta a_\chi(x,\xi)\|\le C_{K,\alpha,\beta,L}\langle\xi\rangle^{-2L}.$$

Proper support of χ ensures that the variable y ranges over a compact set when $x\in K$. Since L is arbitrary, $a_\chi\in S^{-\infty}(U\times\mathbb{R}^n;\operatorname{Hom}(\mathbb{C}^r,\mathbb{C}^s))$. The corresponding operator has kernel $\chi\mathbf{K}_A$. The part $(1-\chi)\mathbf{K}_A$ is smooth for every pseudodifferential operator by the off-diagonal smoothness proved above. Thus every localization has a smoothing symbol, giving (a). The last assertion follows directly from Definition 15.2.3 upon adding proper support. □

Theorem 15.2.5. (Pseudolocality) If A is pseudodifferential and properly supported, then

$$\operatorname{sing\,supp}(Au)\subseteq\operatorname{sing\,supp}(u)$$

for every distribution u.

Proof. Let $x_0\notin\operatorname{sing\,supp}(u)$. Choose $\chi\in C_c^\infty$ equal to one on a neighborhood of the closure of an open set $W\ni x_0$ and supported in a region where u is smooth. Write $u=\chi u+(1-\chi)u$. The first term is a smooth compactly supported function, and $A(\chi u)$ is smooth by continuity of quantization. If $x\in W$ and $y\in\operatorname{supp}(1-\chi)$, then $x\neq y$; hence the pairs involved in $A((1-\chi)u)$ remain off the diagonal, where the kernel is smooth. Proper support allows y to be restricted to a compact set when evaluating locally, so pairing this smooth kernel with $(1-\chi)u$ depends smoothly on x. Thus Au is smooth on W. □

15.2.2 Composition

Amplitude reduction now closes the calculus under composition. The ordinary product of the symbols gives the leading term; mixed derivatives in x and ξ describe lower-order corrections.

Let $r_0, r_1, r_2 \in \mathbb{N}$, $A = \operatorname{Op}(a)$, and $B = \operatorname{Op}(b)$, with

$$a \in S^m_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})), \qquad b \in S^\ell_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^{r_0}, \mathbb{C}^{r_1})).$$

As operators on the Schwartz class, A and B can be composed without additional hypotheses. Substituting the integral formulas, first inserting compactly supported cutoffs and regrouping the phases, yields an oscillatory integral with an amplitude. After the change $z = x + y$ and $\eta = \zeta - \xi$, the composition symbol can be written as

$$(a\#b)(x,\xi) = (2\pi)^{-n} \operatorname{Os} \iint_{\mathbb{R}^n_y \times \mathbb{R}^n_\eta} e^{-iy\cdot\eta} a(x, \xi+\eta) b(x+y, \xi)\, dy\, d\eta.$$

The Taylor expansion of $b(x+y,\xi)$ in $y = 0$, followed by integration by parts in η, gives the following result.

Theorem 15.2.6. (Symbolic composition) Let $n, r_0, r_1, r_2 \in \mathbb{N}$ and $m, \ell \in \mathbb{R}$, and let

$$a \in S^m_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})), \qquad b \in S^\ell_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^{r_0}, \mathbb{C}^{r_1})).$$

As operators from $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^{r_0})$ to $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^{r_2})$,

$$\operatorname{Op}(a)\operatorname{Op}(b) = \operatorname{Op}(a\#b), \qquad a\#b \in S^{m+\ell}_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^{r_0}, \mathbb{C}^{r_2})),$$

and

$$a\#b \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!} (D_\xi^\alpha a)(D_x^\alpha b).$$

In particular, $(a\#b)_{m+\ell} = a_m b_\ell$ for classical symbols. The same formulas apply to operators localized on an open set and to their extensions to distributions whenever proper support ensures that the composition is defined.

Proof. The preceding integral formula is first obtained with compactly supported cutoffs, where Fubini's theorem applies, and the cutoffs are then removed using oscillatory integral estimates. For $N \geq 1$, write

$$b(x+y,\xi) = \sum_{|\alpha|<N} \frac{y^\alpha}{\alpha!} D_x^\alpha b(x,\xi) + \sum_{|\alpha|=N} y^\alpha r_\alpha(x,y,\xi).$$

Here we use the integral Taylor remainder

$$r_\alpha(x,y,\xi) = \frac{N}{\alpha!} \int_0^1 (1-t)^{N-1} D_x^\alpha b(x+ty,\xi)\, dt, \qquad |\alpha| = N.$$

Since $y^\alpha e^{-iy\cdot\eta} = i^{|\alpha|} D_\eta^\alpha e^{-iy\cdot\eta}$, integration by parts makes $i^{-|\alpha|} D_\eta^\alpha$ act on $a(x,\xi+\eta)$. Integrating $e^{-iy\cdot\eta}$ against a function independent of y recovers evaluation at $\eta = 0$, producing the displayed terms.
The remaining part is a finite sum, for $|\alpha| = N$, of integrals in $t \in [0,1]$ whose amplitudes are

$$\frac{N i^{-N}}{\alpha!} (1-t)^{N-1} (D_\xi^\alpha a)(x,\xi+\eta)(D_x^\alpha b)(x+ty,\xi).$$

These amplitudes have order $m + \ell - N$ uniformly in t. The reduction estimate proved in Theorem 15.1.9

provides, for each seminorm of order P, another of finite order P' and a constant independent of t such that

$$q_{P,m+\ell-N}(r_N^{\#}) \le C_{P,N,m,\ell}\, q_{P',m}(a) q_{P',\ell}(b).$$

On an open set, use the same inequality on $K \Subset U$, replacing K on the right-hand side by a slightly larger compact set. Thus $r_N^{\#} \in S^{m+\ell-N}_{1,0}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^{r_0}, \mathbb{C}^{r_2}))$. Since N is arbitrary, Lemma 15.1.5 completes the expansion. □

Corollary 15.2.7. (Commutators) Let $a \in S^m_{1,0}(\mathbb{R}^n)$ and $b \in S^\ell_{1,0}(\mathbb{R}^n)$ be scalar symbols, and suppose composition is defined in both orders. Then $[\mathrm{Op}(a), \mathrm{Op}(b)]$ has order at most $m+\ell-1$. More precisely,

$$a\#b - b\#a = \frac{1}{i}\{a,b\} + r, \qquad r \in S^{m+\ell-2}_{1,0}(\mathbb{R}^n),$$

where

$$\{a,b\} := \sum_{j=1}^{n} \left(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j} \right).$$

If, in addition, a and b are classical, the commutator is classical and its homogeneous component of order $m+\ell-1$ is

$$\frac{1}{i}\{a_m, b_\ell\} := \frac{1}{i}\sum_{j=1}^{n} \left(\frac{\partial a_m}{\partial \xi_j}\frac{\partial b_\ell}{\partial x_j} - \frac{\partial a_m}{\partial x_j}\frac{\partial b_\ell}{\partial \xi_j} \right).$$

If $r \in \mathbb{N}$,

$$a \in S^m_{\mathrm{cl}}\big(\mathbb{R}^n; M_{r\times r}(\mathbb{C})\big), \qquad b \in S^\ell_{\mathrm{cl}}\big(\mathbb{R}^n; M_{r\times r}(\mathbb{C})\big),$$

then the term of order $m+\ell$ is $[a_m, b_\ell]$. If it vanishes, the commutator has order at most $m+\ell-1$, and its homogeneous component of that degree is

$$[a_m, b_{\ell-1}] + [a_{m-1}, b_\ell] + \frac{1}{i}\sum_{j=1}^{n} \left(\frac{\partial a_m}{\partial \xi_j}\frac{\partial b_\ell}{\partial x_j} - \frac{\partial b_\ell}{\partial \xi_j}\frac{\partial a_m}{\partial x_j} \right).$$

In particular, if a_m and b_ℓ are scalar multiples of the identity, the two algebraic commutators in this formula vanish, leaving only the Poisson term.

Proof. The composition theorem gives

$$a\#b \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!}(D_\xi^\alpha a)(D_x^\alpha b), \qquad b\#a \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!}(D_\xi^\alpha b)(D_x^\alpha a).$$

In the scalar case, take the terms with $|\alpha| < 2$ in both expansions. The remainders belong to $S^{m+\ell-2}_{1,0}$, and the products without derivatives cancel because $ab = ba$. Thus

$$a\#b - b\#a = \frac{1}{i}\sum_{j=1}^{n}\big[(D_{\xi_j}a)(D_{x_j}b) - (D_{\xi_j}b)(D_{x_j}a)\big] + r = \frac{1}{i}\{a,b\} + r,$$

with $r \in S^{m+\ell-2}_{1,0}$. Each term in the sum has order at most $m+\ell-1$, proving the assertion without assuming

classicality.

If the scalar symbols are classical, at high frequencies we may write $a = a_m + a_{<m}$ and $b = b_\ell + b_{<\ell}$, where the remainders have orders $m-1$ and $\ell-1$. Poisson terms containing either remainder have order at most $m+\ell-2$. Thus the homogeneous component of order $m+\ell-1$ is $\frac{1}{i}\{a_m, b_\ell\}$. Composition of the classical expansions likewise gives the homogeneous components of lower orders.

For matrix-valued symbols, the same grouping must preserve the order of the factors. Degree $m+\ell$ is $a_m b_\ell - b_\ell a_m = [a_m, b_\ell]$. If it vanishes, the next degree is the sum of the derivative-free terms

$$[a_m, b_{\ell-1}] + [a_{m-1}, b_\ell]$$

and the terms with one derivative displayed in the statement. If both principal symbols are scalar multiples of the identity, these algebraic commutators vanish. □

15.2.3 Transpose and formal adjoint

Composition shows that the calculus is stable under chaining operators. The second operation needed for elliptic theory is the adjoint. At the kernel level, transposition interchanges x and y; in the Hermitian case one also takes the fiberwise adjoint. The resulting amplitude depends on y and is reduced by Theorem 15.1.9. We denote by $\operatorname{Op}(a)_h^*$ the formal Hermitian adjoint determined by the pairing of Schwartz sections.

Theorem 15.2.8. (Symbol of the adjoint) Let $n, r, s \in \mathbb{N}$, $m \in \mathbb{R}$, and

$$a \in S^m_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^s)).$$

. With respect to the inner products on $L^2(\mathbb{R}^n, \mathbb{C}^r)$ and $L^2(\mathbb{R}^n, \mathbb{C}^s)$ defined by Lebesgue measure and the standard Hermitian products,

$$\operatorname{Op}(a)_h^* = \operatorname{Op}(a^{(*)}),$$

where

$$a^{(*)} \in S^m_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^s, \mathbb{C}^r))$$

and

$$a^{(*)}(x,\xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \big(a(x,\xi)^*\big).$$

In particular,

$$a^{(*)} - a^* \in S^{m-1}_{1,0}(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^s, \mathbb{C}^r)).$$

If a is classical, then so is $a^{(*)}$, and $(a^{(*)})_m = a_m^*$.

Proof. For $u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^r)$ and $v \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^s)$, the kernel formula and regularized Fubini theorem give

$$\langle \operatorname{Op}(a)u, v\rangle = (2\pi)^{-n} \operatorname{Os} \iiint_{\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\xi} \Big\langle u(y), e^{i(y-x)\cdot\xi} a(x,\xi)^* v(x) \Big\rangle \, dx\, dy\, d\xi.$$

Thus the adjoint amplitude is $a(y,\xi)^*$ after renaming the variables. Its reduction to a symbol is exactly the asserted expansion. The term with $\alpha = 0$ is a^*, and the remainder has order at most $m-1$. If a is classical,

substituting its homogeneous expansion and grouping terms of equal degree yields a classical expansion for $a^{(*)}$: each derivative in ξ lowers the degree by one, and at degree m only a^*_m remains. □

For a differential operator of integer order m, the principal Fourier component is $a_m = i^m \sigma_m(P)$. Thus

$$(a_m)^* = (-i)^m \sigma_m(P)^* = i^m \big((-1)^m \sigma_m(P)^*\big).$$

Removing the conversion factor i^m gives exactly

$$\sigma_m(P_h^*) = (-1)^m \sigma_m(P)^*$$

in Proposition 11.1.6. The apparent sign difference comes solely from the convention $D_j = \partial_j$ fixed at the beginning of the chapter.

Corollary 15.2.9. (The ideal of smoothing operators) Let $U \subseteq \mathbb{R}^n$ be open. If $R \in \Psi^{-\infty}(U;\mathbb{C}^r,\mathbb{C}^s)$, then $R_h^* \in \Psi^{-\infty}(U;\mathbb{C}^s,\mathbb{C}^r)$. If $A\colon C_c^\infty(U,\mathbb{C}^s) \to C^\infty(U,\mathbb{C}^q)$ and $B\colon C_c^\infty(U,\mathbb{C}^p) \to C^\infty(U,\mathbb{C}^r)$ are pseudodifferential operators with support conditions making the compositions well defined, then

$$AR \in \Psi^{-\infty}_{\mathrm{loc}}(U;\mathbb{C}^r,\mathbb{C}^q), \qquad RB \in \Psi^{-\infty}_{\mathrm{loc}}(U;\mathbb{C}^p,\mathbb{C}^s).$$

The compositions belong to the classes without the subscript loc when their kernels are properly supported; this holds, in particular, when the factors are properly supported.

Proof. At the symbol level, each term in the composition and adjoint expansions contains derivatives of a symbol of arbitrarily negative order. It therefore remains in $S^{-\infty}(U\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q))$, with target spaces adjusted in each composition. The kernel formulation gives the same conclusion: composition integrates a smooth kernel against a kernel singular only on the diagonal, and proper support allows the integral to be taken and differentiated. □

15.2.4 Coordinate changes

The preceding formulas were obtained in Euclidean coordinates. For them to define a theory on manifolds, we must check that a change of chart transforms a pseudodifferential operator into another of the same order and that the principal component follows the expected cotangent rule. Let $\kappa\colon U \longrightarrow V$ be a diffeomorphism between Euclidean open sets, and let $T_\kappa u := u \circ \kappa$. If $A = \mathrm{Op}(a)$ acts on V, consider

$$\widetilde{A} := T_\kappa A T_\kappa^{-1}.$$

For $x, y \in U$ sufficiently close, define

$$B(x,y) := \int_0^1 D\kappa\big(y + t(x-y)\big)\, dt.$$

Hadamard's lemma 1.4.10, with base point y, gives the exact identity $\kappa(x) - \kappa(y) = B(x,y)(x-y)$ and also $B(x,x) = D\kappa(x)$. After the change of variables $z = \kappa(y)$ and $\xi = B(x,y)^T \eta$, the transformed operator has amplitude

$$\widetilde{b}(x,y,\xi) = a\big(\kappa(x), B(x,y)^{-T}\xi\big)\, \frac{|\det D\kappa(y)|}{|\det B(x,y)|}.$$

Away from the diagonal the kernel is smooth, so we may insert a cutoff and assume that $B(x,y)$ is invertible.

Theorem 15.2.10. (Invariance under coordinate changes) The operator $\widetilde{A}$ is pseudodifferential of the same order as A. If a is classical of order m, the transformed principal symbol is

$$\widetilde{a}_m(x,\xi) = a_m\big(\kappa(x), D\kappa(x)^{-T}\xi\big).$$

The full symbol is obtained by reducing $\widetilde{b}$ using Theorem 15.1.9 and depends on higher derivatives of κ.

Proof. The preceding formula shows that $\widetilde{b}$ is an amplitude of order m on each pair of compact coordinate sets. This follows from the chain rule: each derivative in ξ produces a factor B^{-T} and a derivative in the covariable of a, hence lowers the order; derivatives in x and y produce only smooth coefficients and spatial derivatives of a. Theorem 15.1.9 produces the full symbol; weights in that reduction are controlled by Peetre's inequality in Lemma 6.1.1. At $y = x$, the ratio of determinants equals one and $B(x,x) = D\kappa(x)$. The $\alpha = 0$ term of the reduction gives the principal formula, while each term with $|\alpha| \geq 1$ contains a derivative in ξ and has order at most $m-1$. □

This formula proves the two assertions needed globally. The full symbol is not an intrinsic function, because it changes by terms involving higher derivatives of the diffeomorphism. The principal component, however, transforms exactly by the cotangent action: if $\eta \in T^*_{\kappa(x)}V$ and $\xi = D\kappa(x)^T\eta$, then $\eta = D\kappa(x)^{-T}\xi$. It therefore defines a global object on the cotangent bundle.

Amplitude reduction controls the composition and adjoint remainders, while coordinate changes distinguish the chart-dependent full symbol from its cotangent principal component. This organization is presented in detail in [140, Chapter II, Sections 3–5] and, in a later, more compact treatment, in [141, Sections 0.2–0.4].

15.3 Pseudodifferential operators on manifolds

Invariance under coordinate changes allows us to leave the Euclidean model. The global definition will be local near the diagonal, supplemented by a proper-support condition when we wish to act on distributions without compact support.

Throughout this section, M is a finite-dimensional smooth manifold without boundary. It is not assumed compact, complete, or Riemannian until the subsection on geometric quantization. Let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of finite rank. The real case follows by complexification and restriction.

15.3.1 Local definition and independence of choices

Let (U,κ) be a chart, and let $\mathbf{e} = (\mathbf{e}_1,\dots,\mathbf{e}_r)$ and $\mathbf{f} = (\mathbf{f}_1,\dots,\mathbf{f}_s)$ be local frames of $\mathbf{E}$ and $\mathbf{F}$. If $\chi, \psi \in C^\infty_c(U)$, the localized operator $\chi A \psi$ is represented, using the chart and frames, by a matrix of operators between functions compactly supported in $\kappa(U)$. Extension by zero after shrinking the supports allows it to be regarded on $\mathbb{R}^n$.

Definition 15.3.1. (Pseudodifferential operator on a manifold) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of finite rank. For $m \in \mathbb{R}$, a continuous linear operator

$$A : \Gamma_c(M,\mathbf{E}) \longrightarrow \Gamma(M,\mathbf{F})$$

belongs to $\Psi^m_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F})$ if the following two conditions hold:

(a) the kernel $\mathbf{K}_A$ is smooth off Δ_M;

(b) for every chart (U,κ), every pair of local frames of $\mathbf{E}$ and $\mathbf{F}$ over U, and all $\chi,\psi\in C_c^\infty(U)$, each entry of the Euclidean representation of $\chi A\psi$ is $\mathrm{Op}(a_{ba})$ with $a_{ba}\in S^m_{1,0}(\kappa(U))$, modulo an operator with smooth kernel.

If the representations can be chosen classical, write $\Psi^m_{\mathrm{cl,loc}}(M;\mathbf{E},\mathbf{F})$. Denote by

$$\Psi^m(M;\mathbf{E},\mathbf{F})\subseteq\Psi^m_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F})$$

the subclass of properly supported operators. This latter notation will be used for composition and global action on distributions.

Definition 15.3.2. (Smoothing operators on a manifold) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E},\mathbf{F}\to M$ be smooth complex vector bundles of finite rank. Define

$$\Psi^{-\infty}_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F}):=\left\{A\in\Psi^m_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F})\text{ for some }m\in\mathbb{R}\;\middle|\;\mathbf{K}_A\text{ is smooth on }M\times M\right\}.$$

Proposition 15.2.4, applied in charts and frames, shows that an element of this class locally belongs to every order. The properly supported class is

$$\Psi^{-\infty}(M;\mathbf{E},\mathbf{F}):=\left\{A\in\Psi^{-\infty}_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F})\;\middle|\;\mathbf{K}_A\text{ is properly supported}\right\}. \tag{15.7}$$

If M is closed, the two classes coincide. If M is noncompact, smoothness of the kernel implies neither proper support nor, after a Riemannian metric and Hermitian bundle metrics are chosen, compactness of the operator $L^2(M,\mathbf{E})\longrightarrow L^2(M,\mathbf{F})$.

The definition uses charts, frames, and cutoffs, but these choices merely express the kernel locally. The following result shows that it suffices to verify the definition in a trivializing atlas and that the resulting class is intrinsic.

Theorem 15.3.3. (Independence of charts, frames, and cutoffs) Let M be a finite-dimensional smooth manifold without boundary, let $\mathbf{E},\mathbf{F}\to M$ be smooth complex vector bundles of finite rank, and let $m\in\mathbb{R}$. Definition 15.3.1 is independent of the atlas, local frames, and cutoffs. Moreover, every $A\in\Psi^m_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F})$ can be written as $A=A_0+R$, where $A_0\in\Psi^m(M;\mathbf{E},\mathbf{F})$ and $R\in\Psi^{-\infty}_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F})$.

Proof. Consider two overlapping charts. After inserting cutoffs whose supports remain inside the overlap, the representations are conjugate by the transition diffeomorphism. Theorem 15.2.10 shows that conjugation preserves the order and the classical class. A change of frame in $\mathbf{E}$ or $\mathbf{F}$ multiplies the operator on the right or left by a smooth matrix. Theorem 15.2.6, applied to symbols independent of ξ, again shows that the order is preserved. Changing a cutoff yields a difference supported where the two variables are distinct, or another smooth multiplication; in the first case the kernel is smooth, and in the second the composition calculus applies.

For the last assertion, choose a properly supported neighborhood $\mathcal{U}$ of the diagonal and a cutoff $\rho\in C^\infty(M\times$

M) equal to one near Δ_M with $\operatorname{supp}\rho \subseteq \mathcal{U}$. Existence follows from a locally finite cover by relatively compact open sets and a partition of unity. The kernel $\rho\mathbf{K}_A$ defines A_0 and is properly supported; $(1-\rho)\mathbf{K}_A$ is smooth because it is separated from the diagonal. □

The distinction between the two classes is not merely typographical. On a noncompact manifold, the purely local definition does not control kernel behavior when both variables leave every compact set. Without proper support, a composition may be undefined even when both factors are locally pseudodifferential. On a closed manifold, every projection of a closed subset of $M \times M$ is proper, so the distinction disappears.

15.3.2 The global principal symbol

Once independence of local choices has been established, the principal homogeneous components can be glued over the cotangent bundle. The result is the global object that will preserve composition, adjoints, and ellipticity.

Let $\pi\colon T^*M \longrightarrow M$ be the projection and 0_M the zero section. In a chart and local frames, a classical operator $A \in \Psi^m_{\mathrm{cl,loc}}(M;\mathbf{E},\mathbf{F})$ has a principal component $a_m(x,\xi)$. Theorem 15.2.10 and the frame transformation show that, on overlaps, these components satisfy exactly the transformation rule for a bundle homomorphism

$$\pi^*\mathbf{E} \longrightarrow \pi^*\mathbf{F}$$

over $T^*M \setminus 0_M$.

Definition 15.3.4. (Pseudodifferential principal symbol) Let M be a finite-dimensional smooth manifold without boundary, let $\mathbf{E},\mathbf{F} \to M$ be smooth complex vector bundles of finite rank, and let $\pi\colon T^*M \to M$. For $A \in \Psi^m_{\mathrm{cl,loc}}(M;\mathbf{E},\mathbf{F})$, the **principal Fourier symbol** of A is the smooth section

$$\sigma^{\Psi}_m(A) \in \Gamma\big(T^*M \setminus 0_M, \operatorname{Hom}(\pi^*\mathbf{E},\pi^*\mathbf{F})\big)$$

whose local expression is a_m. It satisfies

$$\sigma^{\Psi}_m(A)(x,t\xi) = t^m \sigma^{\Psi}_m(A)(x,\xi), \qquad t>0.$$

For a differential operator P of order m, the exact compatibility with Chapter 11 is

$$\sigma^{\Psi}_m(P)(x,\xi) = i^m \sigma_m(P)(x,\xi). \tag{15.8}$$

The factor i^m must not be omitted from this equality, because the convention $D_j = \partial_j$ was fixed. Since this factor is a nonzero scalar, the two notions of ellipticity coincide. The earlier composition and adjoint rules are also recovered exactly, as checked after Theorem 15.2.8.

Example 15.3.5. (The Dirac operator as a pseudodifferential operator) Let M be a Riemannian manifold without boundary, and let $D = c \circ \nabla^{\mathbf{S}}$ be a Dirac-type operator constructed in Section 8.3. Every differential operator of order one is a properly supported classical pseudodifferential operator; thus

$$D \in \Psi^1_{\mathrm{cl}}(M;\mathbf{S},\mathbf{S}).$$

Combining (15.8) with (11.2) gives, with this chapter's Fourier convention,

$$\sigma_1^{\Psi}(D)(x,\xi) = i\,c(\xi). \tag{15.9}$$

For $\xi \neq 0$, this homomorphism is invertible and

$$\big(\sigma_1^{\Psi}(D)(x,\xi)\big)^{-1} = \frac{i\,c(\xi)}{|\xi|_{\mathbf{g}}^2},$$

since $(i\,c(\xi))^2 = |\xi|_{\mathbf{g}}^2 I$. If $\mathbf{S}$ is graded, the same formula restricted to $\mathbf{S}_x^+ \to \mathbf{S}_x^-$ is the pseudodifferential symbol of D^+.

Definition 15.3.6. (Space of principal symbols) Let M be a finite-dimensional smooth manifold without boundary, let $\mathbf{E},\mathbf{F} \to M$ be smooth complex vector bundles of finite rank, and let $\pi\colon T^*M \to M$. Let $\mathcal{S}^m_{\mathrm{hom}}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))$ be the space of smooth sections of $\mathrm{Hom}(\pi^*\mathbf{E},\pi^*\mathbf{F})$ over $T^*M \setminus 0_M$ that are homogeneous of degree m under positive fiber dilations.

Theorem 15.3.7. (Exact sequence of the principal symbol) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E},\mathbf{F},\mathbf{G} \to M$ be smooth complex vector bundles of finite rank. For every $m \in \mathbb{R}$, there is an exact sequence

$$0 \longrightarrow \Psi^{m-1}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F}) \longrightarrow \Psi^{m}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F}) \xrightarrow{\sigma^{\Psi}_m} \mathcal{S}^m_{\mathrm{hom}}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F})) \longrightarrow 0.$$

Moreover, for every $\ell \in \mathbb{R}$, if $A \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ and $B \in \Psi^{\ell}_{\mathrm{cl}}(M;\mathbf{G},\mathbf{E})$, then

$$\sigma^{\Psi}_{m+\ell}(AB) = \sigma^{\Psi}_m(A) \circ \sigma^{\Psi}_{\ell}(B)$$

and, when Hermitian bundle metrics and a density have been fixed to define the adjoint,

$$\sigma^{\Psi}_m(A_h^*) = \sigma^{\Psi}_m(A)^*.$$

Proof. Well-definedness and the two compatibility statements follow from Theorems 15.2.10, 15.2.6, and 15.2.8. If A has order at most $m-1$, its homogeneous component of degree m vanishes. Conversely, if this component vanishes in every chart, the definition of a classical symbol shows that each local representative satisfies estimates of order $m-1$; hence $A \in \Psi^{m-1}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$.

For surjectivity, choose an auxiliary metric on T^*M, a cutoff vanishing on a neighborhood of 0_M, a locally finite trivializing cover, and a partition of unity. In each chart, quantize the given homogeneous component multiplied by the cutoff, and sum using a quadratic system of localization functions. The sum is locally finite and can be made properly supported by inserting a cutoff near the diagonal. On overlaps, Theorem 15.2.10 shows that the terms of degree m glue to the original section; discrepancies have order at most $m-1$. The resulting global operator has the prescribed symbol. □

15.3.3 The conormal bundle of the diagonal

The definition by symbols still depends on a local quantization. There is an equivalent description using only the kernel: classical pseudodifferential operators are exactly those whose kernels are conormal to the diagonal. To formulate this equivalence, we first fix the order and density conventions for a general submanifold.

Let X be a smooth manifold without boundary of dimension d, let $Y \subseteq X$ be an embedded submanifold without boundary of codimension k, and let $\mathbf{G} \to X$ be a smooth complex vector bundle of finite rank $r_{\mathbf{G}}$. Its normal and conormal bundles are

$$NY := \operatorname{coker}\big(TY \hookrightarrow TX|_Y\big), \qquad N^*Y := \{(y,\eta) \in T^*X|_Y \mid \eta|_{T_yY} = 0\}.$$

Let $\pi_Y \colon N^*Y \to Y$ denote the projection and $\boldsymbol{\mathcal{D}}_1(Y)$ the bundle of weight-one densities from Definition 2.14.14. A chart $(y,z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$ is **adapted to** Y if Y is given by $z = 0$. In this chart, conormal covectors are precisely $(y,0;0,\theta)$.

Definition 15.3.8. (Conormal distribution) A distribution $u \in \mathcal{D}'(X,\mathbf{G})$ is **classical conormal to** Y **of order** m, written

$$u \in I^m(X,Y;\mathbf{G}),$$

if it is smooth on $X \setminus Y$ and, in every localization in an adapted chart and a trivialization of G, can be written modulo a smooth section as

$$u = \left[(2\pi)^{-k} \operatorname{Os}\int_{\mathbb{R}^k_\theta} e^{iz\cdot\theta} b(y,z,\theta)\, d\theta\right] |dy\, dz|, \qquad b \in S^{m+\frac{d-2k}{4}}_{\mathrm{cl}}\big(\mathbb{R}^d_{(y,z)} \times \mathbb{R}^k_\theta; \mathbb{C}^{r_{\mathbf{G}}}\big). \tag{15.10}$$

Here $|dy\,dz|$ records the coordinate density implicit in the convention defining distributions as functionals on test sections of the dual bundle. This factor is suppressed in Euclidean formulas but must be retained when analyzing coordinate changes. After localization, the amplitude is taken compactly supported in the spatial variables. The number m is the **conormal order**; the shift $\frac{d-2k}{4}$ adapts Hörmander's convention to the operator order used on the diagonal.

The definition is independent of adapted coordinates, as the following calculation shows. Let

$$\widetilde{y} = \varphi(y,z), \qquad \widetilde{z} = \psi(y,z)$$

be two adapted systems related by the diffeomorphism $\Phi = (\varphi,\psi)$. The identity $\psi(y,0) = 0$ and Hadamard's lemma 1.4.10, applied with y as parameter, give the exact factorization

$$\psi_a(y,z) = \sum_{b=1}^{k} C_{ab}(y,z) z_b, \qquad C_{ab}(y,z) := \int_0^1 \partial_{z_b}\psi_a(y,tz)\, dt. \tag{15.11}$$

In particular, $C(y,0) = D_z\psi(y,0)$. Since

$$D\Phi(y,0) = \begin{pmatrix} D_y\varphi(y,0) & D_z\varphi(y,0) \\ 0 & C(y,0) \end{pmatrix}$$

is invertible, so are $A(y) := D_y\varphi(y,0)$ and $C(y,0)$. After shrinking the chart, $C(y,z)$ remains invertible. For each

fixed (y,z), the linear substitution

$$\theta = C(y,z)^{T}\widetilde{\theta}, \qquad d\theta = |\det C(y,z)|\, d\widetilde{\theta} \tag{15.12}$$

is exact and satisfies $z\cdot\theta = \widetilde{z}\cdot\widetilde{\theta}$. If $T(y,z)$ denotes the matrix taking the components of G in the original frame to those in the new frame, the transformed amplitude is

$$\widetilde{b}(\widetilde{y},\widetilde{z},\widetilde{\theta}) := T(y,z)\,b\big(y,z,C(y,z)^{T}\widetilde{\theta}\big)\,|\det C(y,z)|\,|\det D\Phi(y,z)|^{-1}, \qquad (y,z)=\Phi^{-1}(\widetilde{y},\widetilde{z}). \tag{15.13}$$

Indeed, the last determinant is exactly the one that converts $|dy\,dz|$ into $|d\widetilde{y}\,d\widetilde{z}|$. To check the order, first apply derivatives in $\widetilde{\theta}$ and then spatial derivatives. The Leibniz and chain rules express each derivative of $b\big(\Phi^{-1}(\widetilde{y},\widetilde{z}),C^{T}\widetilde{\theta}\big)$ as a finite sum of terms

$$A_{\alpha,\beta,\gamma;\delta,\nu,\rho}(\widetilde{y},\widetilde{z})\,\widetilde{\theta}^{\rho}\,(D_{y,z}^{\delta}D_{\theta}^{\nu}b)\big(y,z,C(y,z)^{T}\widetilde{\theta}\big),$$

where $(y,z)=\Phi^{-1}(\widetilde{y},\widetilde{z})$ and

$$|\nu| = |\gamma| + |\rho|, \qquad |\delta| + |\rho| \le |\alpha| + |\beta|.$$

Indeed, a spatial derivative hitting b increases $|\delta|$; if it hits $C^{T}\widetilde{\theta}$, it creates both a linear factor in $\widetilde{\theta}$ and a derivative in θ; if it hits a coefficient, it merely differentiates that coefficient. This proves both relations by induction on $|\alpha|+|\beta|$. Uniform invertibility of C on each localized compact set gives $c\langle\widetilde{\theta}\rangle \le \langle C^{T}\widetilde{\theta}\rangle \le C\langle\widetilde{\theta}\rangle$. The factor $\widetilde{\theta}^{\rho}$ is offset by the $|\rho|$ additional derivatives in θ. Since the remaining factors in (15.13) are smooth, for all multi-indices α,β,γ there is a constant $C_{\alpha,\beta,\gamma}>0$ such that

$$\big\| D_{\widetilde{y}}^{\alpha} D_{\widetilde{z}}^{\beta} D_{\widetilde{\theta}}^{\gamma}\widetilde{b}\big\| \le C_{\alpha,\beta,\gamma}\langle\widetilde{\theta}\rangle^{m+\frac{d-2k}{4}-|\gamma|}.$$

Each homogeneous component transforms by the same formula; thus (15.13) belongs to the same classical class. Applying the calculation to Φ^{-1} proves the reverse implication.

The principal coefficient alone is not a canonical function. If $\mu = m+\frac{d-2k}{4}$ and b_{μ} is the principal homogeneous component, then

$$b_{\mu}(y,0,\theta)|dy| \tag{15.14}$$

defines a homogeneous section of degree μ of

$$\pi_{Y}^{*}\big(G\restriction_{Y}\otimes\boldsymbol{\mathcal{D}}_{1}(Y)\big) \quad \text{on } N^{*}Y\setminus 0_{Y},$$

modulo the next degree. This is the **principal conormal symbol** in the density convention fixed above. To verify invariance, restrict (15.13) to $z=0$. The triangular form of $D\Phi(y,0)$ implies

$$|\det D\Phi(y,0)| = |\det A(y)|\,|\det C(y,0)|,$$

and hence

$$\widetilde{b}_{\mu}\big(\varphi(y,0),0,\widetilde{\theta}\big) = T(y,0)\,b_{\mu}\big(y,0,C(y,0)^{T}\widetilde{\theta}\big)\,|\det A(y)|^{-1} \quad \bmod \{\text{conormal symbols of degree at most } \mu-1\}. \tag{15.15}$$

Since $|d\widetilde{y}| = |\det A(y)|\,|dy|$ on Y, multiplying (15.15) by $|d\widetilde{y}|$ cancels the sole remaining determinant. In turn,

$\theta = C(y,0)^{T}\widetilde{\theta}$ is exactly the coordinate transformation rule in $N^{*}Y$. Thus $b_{\mu}(y,0,\theta)|dy|$ defines the asserted global section. This verification explains why omitting the density line would produce a noncanonical object.

The normal phases used in adapted charts are related by the homogeneous linear substitution (15.12). Since no additional phase variables are introduced, the changes are fully described by the cotangent rule and the preceding densities. With this convention, the normalization agrees with [84, Theorems 18.2.8–18.2.11]: in ambient dimension d and codimension k, a conormal distribution of order m is represented by an amplitude of order $m + \frac{d-2k}{4}$.

For $X = M \times M$ and $Y = \Delta_M$, the tangent space at (x,x) consists of pairs (v,v). A covector (ξ,η) annihilates it if and only if $\xi(v) + \eta(v) = 0$ for every $v \in T_x M$. Thus

$$N^{*}\Delta_M = \{(x,\xi;x,-\xi) \mid x \in M,\ \xi \in T_x^{*}M\}. \tag{15.16}$$

This is the geometric reason that the cotangent bundle, rather than the tangent bundle, parametrizes the kernel's oscillations.

In this case $d = 2n$ and $k = n$, so the shift in Definition 15.3.8 is $\frac{2n-2n}{4} = 0$. A distribution $\mathbf{K} \in I^{m}(M \times M, \Delta_M; \mathbf{F} \boxtimes \mathbf{E}^{*})$ therefore has the local expression, modulo a smooth section,

$$(2\pi)^{-n}\,\mathrm{Os}\int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi}\, b(x,y,\xi)\, d\xi, \qquad b \text{ classical amplitude of order } m.$$

The bundle $\boldsymbol{\mathcal{D}}_1$ of Definition 2.14.14 supplies the required density factor. To relate it to an operator kernel, temporarily fix a positive density ν on M and use ν both to identify smooth sections with distributions and to write integration in the input variable. The principal conormal symbol of the kernel on the diagonal then takes values in

$$\mathrm{Hom}(\mathbf{E},\mathbf{F}) \otimes \boldsymbol{\mathcal{D}}_1(\Delta_M).$$

The identification $\Delta_M \cong M$ and division by ν produce a section of $\mathrm{Hom}(\pi^{*}\mathbf{E}, \pi^{*}\mathbf{F})$. If $\widetilde{\nu} = f\nu$, the kernel coefficient relative to integration in the input variable acquires $f(y)^{-1}$, which cancels with the new integration density $f(y)\nu(y)$. Pairing in the output variable, in contrast, multiplies the kernel distribution by $f(x)$; its conormal symbol on the diagonal therefore acquires $f \restriction_{\Delta_M}$, and the final division by $\widetilde{\nu} = f\nu$ cancels exactly this factor. The normalized section is consequently independent of ν. In a chart, taking the coordinate density, this section is precisely the homogeneous component of the Kohn–Nirenberg amplitude. Thus normalization by the density preserves the operator order and produces an intrinsic principal conormal symbol.

Theorem 15.3.9. (Characterization by conormal kernels) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of finite rank. Let

$$A \colon \Gamma_c(M,\mathbf{E}) \longrightarrow \Gamma(M,\mathbf{F})$$

be a continuous properly supported operator. The following conditions are equivalent:

(a) $A \in \Psi^{m}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$;

(b) $\mathbf{K}_A \in I^{m}(M \times M, \Delta_M; \mathbf{F} \boxtimes \mathbf{E}^{*})$.

The equivalence preserves the principal symbol: the homogeneous component of the conormal amplitude, restricted to $N^{*}\Delta_M$ by (15.16) and normalized by the density as explained above, is $\sigma^{\Psi}_{m}(A)$.

Proof. Suppose (a) holds. In a chart and local frames, the kernel formula for $\mathrm{Op}(a)$ is an oscillatory representation with amplitude $a(x,\xi)$. Chart changes transform the phase and amplitude as in the proof of Theorem 15.2.10; the principal component follows the cotangent rule. Off the diagonal, the integrations by parts in the first subsection of 15.2 yield smoothness. Thus (b) holds.
Conversely, suppose (b) holds. Each local conormal representation defines an operator with amplitude of order m. Theorem 15.1.9 reduces it to a classical symbol of that order. The remaining smooth term is smoothing by Proposition 15.2.4. This gives (a). In both directions, the $\alpha = 0$ term of amplitude reduction preserves the principal homogeneous component; the other terms contain at least one derivative in ξ and lower the order. This proves the last assertion. □

Combining the preceding theorem with Theorem 15.1.9 gives a precise equivalence among the three descriptions:

$$\begin{aligned} \text{classical symbol of order } m &\iff \text{classical amplitude of order } m,\\ \text{classical amplitude of order } m &\iff \text{conormal kernel of order } m \text{ to } \Delta_M. \end{aligned}$$

The first equivalence is exact at the operator level and determines the full symbols, in each chart and frames, modulo the corresponding class $S^{-\infty}(U\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s))$; the second is exact modulo smooth kernels. The principal symbol is exactly the same global object in all three formulations.

15.4 Riemannian quantization via the exponential map

The preceding global definition is obtained by localizing Kohn–Nirenberg quantization. Once a metric and bundle connections are fixed, one can also construct a geometric quantization: the difference $x-y$ is replaced by the vector $\exp_x^{-1}(y)$, and fibers are compared by parallel transport. This formulation does not remove coordinates from the proofs, but clearly separates the geometric choices from the symbolic part.

Now suppose that $(M,\mathbf{g})$ is Riemannian and without boundary, but not necessarily complete or compact. Let $\mathbf{E},\mathbf{F}$ be complex vector bundles with Hermitian bundle metrics and compatible connections. There is an open neighborhood $\mathcal{V}$ of the zero section of TM on which

$$(x,v)\longmapsto(x,\exp_x v)$$

is a diffeomorphism onto a neighborhood $\mathcal{U}$ of Δ_M. This assertion is local around each $(x,0)$ and is globalized by shrinking a locally finite cover; it requires no uniform injectivity radius. Write

$$\ell(x,y):=\exp_x^{-1}(y)\in T_xM,\qquad (x,y)\in\mathcal{U}.$$

Let $\tau^{\mathbf{E}}_{x\leftarrow y}\colon \mathbf{E}_y\longrightarrow\mathbf{E}_x$ be parallel transport along the radial geodesic $t\mapsto\exp_x(t\ell(x,y))$. Choose a properly supported $\chi\in C^\infty(M\times M)$, supported in $\mathcal{U}$ and equal to one near the diagonal. The metric induces Lebesgue measures dv_x on T_xM and $d\xi_x$ on T_x^*M.

By the construction in Subsection 2.7.1, the Levi–Civita connection determines the horizontal–vertical decomposition of $T(T^*M)$, while the connections on $\mathbf{E}$ and $\mathbf{F}$ induce the pullback connection on $\pi^*\mathrm{Hom}(\mathbf{E},\mathbf{F})\to T^*M$. Differentiating with this connection in horizontal and vertical directions gives the derivatives ∇^{H} and ∇^{V} defined in (2.67). Write $\langle\xi\rangle_{\mathbf{g}}=(1+\|\xi\|_{\mathbf{g}}^2)^{\frac12}$.

Fix a chart $(U,x^1,\dots,x^n)$, the induced coordinates (x,ξ) determined by $\xi=\xi_j dx^j$, and frames $\mathbf{e}=(\mathbf{e}_1,\dots,\mathbf{e}_r)$ and

$\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_s)$ of $\mathbf{E}$ and $\mathbf{F}$. Adopt the conventions

$$\nabla^{\mathbf{E}}_{\partial_i} \mathbf{e}_\alpha = \sum_{\gamma=1}^{r} \mathbf{e}_\gamma (\omega_i^{\mathbf{E}})_{\gamma\alpha}, \qquad \nabla^{\mathbf{F}}_{\partial_i} \mathbf{f}_b = \sum_{c=1}^{s} \mathbf{f}_c (\omega_i^{\mathbf{F}})_{cb}, \qquad \nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

Proposition 2.7.27 derived the sign in the formula

$$H_i = D_{x^i} + \sum_{j,k=1}^{n} \Gamma_{ij}^k(x) \xi_k D_{\xi_j};$$

from the covector parallel-transport equation; this is identity (2.65). In particular, if $\mathbf{a} \in \Gamma(\pi^* \operatorname{Hom}(\mathbf{E}, \mathbf{F})|_{T^*U})$ and we also denote its local matrix by $\mathbf{a}(x, \xi)$, Proposition 2.7.28 gives

$$\nabla^{\mathrm{V}}_j \mathbf{a} = D_{\xi_j} \mathbf{a}, \qquad \nabla^{\mathrm{H}}_i \mathbf{a} = H_i \mathbf{a} + \omega_i^{\mathbf{F}} \mathbf{a} - \mathbf{a} \omega_i^{\mathbf{E}};$$

these are identities (2.68).

To iterate these operations intrinsically, introduce the bundles

$$\mathscr{T}^{p,q} := \pi^* \left((T^*M)^{\otimes p} \otimes (TM)^{\otimes q} \otimes \operatorname{Hom}(\mathbf{E}, \mathbf{F}) \right).$$

The Levi–Civita connection on the tensor factors, the connection on $\operatorname{Hom}(\mathbf{E}, \mathbf{F})$, and the horizontal–vertical decomposition induce

$$\nabla^{\mathrm{H}} \colon \Gamma(\mathscr{T}^{p,q}) \longrightarrow \Gamma(\mathscr{T}^{p+1,q}), \qquad \nabla^{\mathrm{V}} \colon \Gamma(\mathscr{T}^{p,q}) \longrightarrow \Gamma(\mathscr{T}^{p,q+1}).$$

As in Chapter 9, the new index is added at the beginning of the corresponding horizontal or vertical block. If T has components $T_{i_1 \cdots i_p;}{}^{j_1 \cdots j_q}$, then

$$\begin{aligned} \left((\nabla^{\mathrm{H}} T)_{i i_1 \cdots i_p;} \right)^{j_1 \cdots j_q} &= H_i T_{i_1 \cdots i_p;}{}^{j_1 \cdots j_q} + \omega_i^{\mathbf{F}} T_{i_1 \cdots i_p;}{}^{j_1 \cdots j_q} - T_{i_1 \cdots i_p;}{}^{j_1 \cdots j_q} \omega_i^{\mathbf{E}} \\ &\quad - \sum_{a=1}^{p} \Gamma_{i i_a}^k T_{i_1 \cdots k \cdots i_p;}{}^{j_1 \cdots j_q} + \sum_{b=1}^{q} \Gamma_{ik}^{j_b} T_{i_1 \cdots i_p;}{}^{j_1 \cdots k \cdots j_q}, \end{aligned} \tag{15.17}$$

$$\left((\nabla^{\mathrm{V}} T)_{i_1 \cdots i_p;} \right)^{j j_1 \cdots j_q} = D_{\xi_j} T_{i_1 \cdots i_p;}{}^{j_1 \cdots j_q}. \tag{15.18}$$

In particular, $\nabla^{\mathrm{H}} a$ takes values in $\pi^* T^*M \otimes \pi^* \operatorname{Hom}(\mathbf{E}, \mathbf{F})$, whereas $\nabla^{\mathrm{V}} a$ takes values in $\pi^* TM \otimes \pi^* \operatorname{Hom}(\mathbf{E}, \mathbf{F})$. The subscript on ∇^{V}_j in the preceding directional formula labels the vertical direction $\mathbf{d}x^j$; it is not a covariant index of the vertical tensor. Define

$$(\nabla^{\mathrm{H}})^p (\nabla^{\mathrm{V}})^q a \in \Gamma(\mathscr{T}^{p,q})$$

by applying first the q vertical derivatives and then the p horizontal derivatives, always using the preceding tensorial recurrence.

To record the order of differentiation without assuming commutativity, let $\mathfrak{W}_p := \{1, \dots, n\}^p$, with $\mathfrak{W}_0 := \{\varnothing\}$. For words $I = (i_1, \dots, i_p) \in \mathfrak{W}_p$ and $J = (j_1, \dots, j_q) \in \mathfrak{W}_q$, set

$$\nabla^{\mathrm{H}}_I := \nabla^{\mathrm{H}}_{i_1} \cdots \nabla^{\mathrm{H}}_{i_p}, \qquad \nabla^{\mathrm{V}}_J := \nabla^{\mathrm{V}}_{j_1} \cdots \nabla^{\mathrm{V}}_{j_q}. \tag{15.19}$$

These expressions are deliberately directional compositions, not components of a tensorial derivative. The word

preserves the order; in particular, I is not replaced by a multi-index, nor are letters commuted. Only to count the vertical derivatives do we write $\beta(J) := e_{j_1} + \cdots + e_{j_q} \in \mathbb{N}_0^n$.

Proposition 15.4.1. (Horizontal–vertical commutators) In a coordinate chart and the preceding local frames, the directional compositions satisfy

$$[\nabla_j^{\mathrm{V}}, \nabla_\ell^{\mathrm{V}}]a = 0, \tag{15.20}$$

$$[\nabla_i^{\mathrm{H}}, \nabla_j^{\mathrm{V}}]a = -\Gamma_{ik}^{j} \nabla_k^{\mathrm{V}} a, \tag{15.21}$$

$$[\nabla_i^{\mathrm{H}}, \nabla_j^{\mathrm{H}}]a = \mathbf{R}^{\mathbf{F}}(\partial_i, \partial_j)a - a\mathbf{R}^{\mathbf{E}}(\partial_i, \partial_j) + R_{ij\ell}{}^{k} \xi_k \nabla_\ell^{\mathrm{V}} a. \tag{15.22}$$

In particular, the order of a word can be changed only at the cost of the curvature and vertical-order terms displayed here.

Proof. The first identity follows from commutation of the derivatives D_{ξ_j}. Since $H_i = D_{x^i} + \Gamma_{i\ell}^{k} \xi_k D_{\xi_\ell}$,

$$[H_i, D_{\xi_j}] = -\Gamma_{i\ell}^{j} D_{\xi_\ell}, \qquad [H_i, H_j] = R_{ij\ell}{}^{k} \xi_k D_{\xi_\ell}.$$

Adding the connection matrices gives curvature $\mathbf{R}^{\mathbf{F}}(\partial_i, \partial_j)a - a\mathbf{R}^{\mathbf{E}}(\partial_i, \partial_j)$ on $\operatorname{Hom}(\mathbf{E}, \mathbf{F})$. These three identities yield (15.20)–(15.22). □

The following result follows the triangular scheme of Lemma 9.1.28; here we must also record the vertical derivatives created by (2.65).

Lemma 15.4.2. (Local expression for iterated geometric derivatives) Let $(M, \mathbf{g})$ be a smooth Riemannian manifold of dimension n without boundary, not necessarily complete or compact, and let $\pi\colon T^*M \to M$. Let $\mathbf{E}, \mathbf{F} \to M$ be complex vector bundles of ranks r, s equipped with bundle metrics and compatible connections. The horizontal decomposition of $T(T^*M)$ is determined by the Levi–Civita connection, and the derivatives on $\pi^* \operatorname{Hom}(\mathbf{E}, \mathbf{F})$ are those in (2.67). Fix a chart $(U, x^1, \dots, x^n)$ and local frames of $\mathbf{E}$ and $\mathbf{F}$, and let

$$\mathbf{a} \in \Gamma(\pi^* \operatorname{Hom}(\mathbf{E}, \mathbf{F})|_{T^*U}).$$

Regard the coefficients as elements of $\operatorname{End}_{\mathbb{C}}(M_{s\times r}(\mathbb{C}))$. For $I \in \mathfrak{W}_p$ and $J \in \mathfrak{W}_q$, there exist coefficients $C_{\mu,\nu}^{I,J}(x, \xi)$ indexed by the explicit finite set

$$\mathcal{A}_{I,J} := \big\{ (\mu, \beta(J) + \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \;\big|\; |\mu| + |\gamma| \le p \big\}, \tag{15.23}$$

such that

$$\nabla_I^{\mathrm{H}} \nabla_J^{\mathrm{V}} \mathbf{a} = \sum_{(\mu,\nu) \in \mathcal{A}_{I,J}} C_{\mu,\nu}^{I,J}(x, \xi) \big(D_x^{\mu} D_\xi^{\nu} \mathbf{a}(x, \xi) \big). \tag{15.24}$$

If $\nu = \beta(J) + \gamma$, the coefficient $C_{\mu,\nu}^{I,J}$ is polynomial in ξ of degree at most $|\gamma|$. For $p \ge 1$, its coefficients are finite sums of products and compositions formed from $D_x^{\delta} \Gamma_{ij}^{k}$, $D_x^{\delta} \omega_i^{\mathbf{E}}$, and $D_x^{\delta} \omega_i^{\mathbf{F}}$, where $|\delta| \le p - 1$; for $p = 0$ none of these data occur.

More precisely, let $L_B(T) := BT$, $R_D(T) := TD$, and

$$\mathcal{H}_i C := D_{x^i} C + \sum_{j,k=1}^{n} \Gamma_{ij}^{k} \xi_k D_{\xi_j} C$$

for a coefficient taking values in $\mathrm{End}_{\mathbb{C}}(M_{s\times r}(\mathbb{C}))$. With the convention that a coefficient whose index does not belong to $\mathcal{A}_{I,J}$ is zero, the coefficients are determined by

$$C_{0,\beta(J)}^{\varnothing,J} = \mathrm{Id}_{M_{s\times r}(\mathbb{C})} \tag{15.25}$$

and, if $I^+ = (i, i_1, \dots, i_p)$,

$$C_{\mu,\nu}^{I^+,J} = \mathcal{H}_i C_{\mu,\nu}^{I,J} + L_{\omega_i^{\mathrm{F}}} \circ C_{\mu,\nu}^{I,J} - R_{\omega_i^{\mathrm{E}}} \circ C_{\mu,\nu}^{I,J} + \mathbf{1}_{\{\mu_i \geq 1\}} C_{\mu-e_i,\nu}^{I,J} + \sum_{j,k=1}^{n} \mathbf{1}_{\{\nu_j \geq 1\}} \Gamma_{ij}^{k} \xi_k C_{\mu,\nu-e_j}^{I,J}. \tag{15.26}$$

Proof. If $|I| + |J| = 1$, the two possibilities are, by (2.68),

$$\nabla_j^{\mathrm{V}} \mathbf{a} = D_{\xi_j} \mathbf{a}$$

and

$$\nabla_i^{\mathrm{H}} \mathbf{a} = D_{x^i} \mathbf{a} + \sum_{j,k=1}^{n} \Gamma_{ij}^{k} \xi_k D_{\xi_j} \mathbf{a} + L_{\omega_i^{\mathrm{F}}}(\mathbf{a}) - R_{\omega_i^{\mathrm{E}}}(\mathbf{a}).$$

In the first case the only coefficient is the identity with $(\mu, \nu) = (0, e_j)$. In the second, the coefficients are the identity for $(e_i, 0)$, the endomorphism $(\sum_{k=1}^{n} \Gamma_{ij}^{k} \xi_k)\,\mathrm{Id}_{M_{s\times r}(\mathbb{C})}$ for $(0, e_j)$, and $L_{\omega_i^{\mathrm{F}}} - R_{\omega_i^{\mathrm{E}}}$ for $(0,0)$. Their respective degrees are 0, 1, and 0, and no derivatives of the connection data appear.

Begin with the vertical induction. For $q = 1$, we have already obtained $\nabla_j^{\mathrm{V}} \mathbf{a} = D_{\xi_j} \mathbf{a}$. If $\nabla_J^{\mathrm{V}} \mathbf{a} = D_\xi^{\beta(J)} \mathbf{a}$ and $J^+ = (j, j_1, \dots, j_q)$, the first identity in (2.68) gives

$$\nabla_{J^+}^{\mathrm{V}} \mathbf{a} = D_{\xi_j} \left(D_\xi^{\beta(J)} \mathbf{a} \right) = D_\xi^{\beta(J)+e_j} \mathbf{a}.$$

No additional coefficient appears because the linear coordinates on each fiber have trivial vertical connection. This proves (15.25) for every word J. If a vertical letter is added to J while keeping a horizontal word I, the convention (15.19) places that differentiation before the horizontal ones: replace the initial datum $C_{0,\beta(J)}^{\varnothing,J}$ by $C_{0,\beta(J)+e_j}^{\varnothing,J^+}$, then apply, in the same order, the horizontal recurrences corresponding to the letters of I. This gives the vertical step for any I, with differentiation ∇_j^{V} applied before the horizontal derivatives.

Now fix J and assume (15.24) for a word I of length p. For brevity, let $A_{\mu,\nu} := D_x^\mu D_\xi^\nu \mathbf{a}$. Applying an additional horizontal derivative to each summand and using the Leibniz rule gives the terms

$$\begin{aligned} \nabla_i^{\mathrm{H}} \left(C_{\mu,\nu}^{I,J} A_{\mu,\nu} \right) &= (D_{x^i} C_{\mu,\nu}^{I,J}) A_{\mu,\nu} + \sum_{j,k=1}^{n} \Gamma_{ij}^{k} \xi_k (D_{\xi_j} C_{\mu,\nu}^{I,J}) A_{\mu,\nu} \\ &\quad + C_{\mu,\nu}^{I,J} D_x^{\mu+e_i} D_\xi^\nu \mathbf{a} + \sum_{j,k=1}^{n} \Gamma_{ij}^{k} \xi_k C_{\mu,\nu}^{I,J} D_x^\mu D_\xi^{\nu+e_j} \mathbf{a} \end{aligned}$$

$$+ L_{\omega_i^{\mathbf{F}}}\big(C^{I,J}_{\mu,\nu}A_{\mu,\nu}\big) - R_{\omega_i^{\mathbf{E}}}\big(C^{I,J}_{\mu,\nu}A_{\mu,\nu}\big).$$

The first two expressions differentiate the coefficient, the third differentiates $\mathbf{a}$ spatially, the fourth comes from the vertical part of the horizontal lift, and the last two are left and right connection multiplication, respectively. Regrouping by the derivative $D_x^{\mu}D_\xi^{\nu}\mathbf{a}$ gives exactly (15.26).

Let us verify the four inductive properties. The first three terms of the recurrence preserve (μ,ν); the fourth replaces μ by $\mu+e_i$; the last replaces $\nu=\beta(J)+\gamma$ by $\nu+e_j$. In all three cases, $|\mu|+|\gamma|\le p+1$. Differentiation in x preserves the degree in ξ; the operator $\xi_k D_{\xi_j}$ does not increase it either; connection multiplication preserves it; and the term creating D_{ξ_j} multiplies by a function linear in ξ, so the new degree is at most $|\gamma|+1$. Differentiating an earlier coefficient in x may introduce derivatives of order p of Γ, $\omega^{\mathbf{E}}$, or $\omega^{\mathbf{F}}$; the other terms introduce only undifferentiated data. This is precisely the bound $(p+1)-1$. Finally, (15.23) is finite, and each sum in (15.26) contains finitely many terms. This proves the step $p\longmapsto p+1$ and hence the lemma. The inductions $q\longmapsto q+1$ for the vertical datum and $p\longmapsto p+1$ for the horizontal letters cover, in particular, each total level $p+q=N+1$ from earlier levels. □

The preceding relation is triangular and can be inverted without losing control of the indices.

Lemma 15.4.3. (Inverse local triangular formula) Let $(M,\mathbf{g})$ be a smooth Riemannian manifold of dimension n without boundary, not necessarily complete or compact. Let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of finite rank with bundle metrics and compatible connections; use the Levi–Civita connection for the horizontal lift. Fix a chart U, frames of $\mathbf{E}$ and $\mathbf{F}$ over U, and $\mathbf{a}\in\Gamma(\pi^*\mathrm{Hom}(\mathbf{E},\mathbf{F})|_{T^*U})$. For each $\alpha,\beta\in\mathbb{N}_0^n$, there exist coefficients $\widetilde{C}^{\alpha,\beta}_{I,J}(x,\xi)$ such that

$$D_x^{\alpha}D_\xi^{\beta}\mathbf{a} = \sum_{(I,J)\in\widetilde{\mathcal{A}}_{\alpha,\beta}} \widetilde{C}^{\alpha,\beta}_{I,J}(x,\xi)\big(\nabla^{\mathrm{H}}_I\nabla^{\mathrm{V}}_J\mathbf{a}\big), \tag{15.27}$$

where the admissible finite set is

$$\widetilde{\mathcal{A}}_{\alpha,\beta} := \bigcup_{h=0}^{|\alpha|}\ \bigcup_{q=|\beta|}^{|\beta|+|\alpha|-h} \mathfrak{W}_h\times\mathfrak{W}_q. \tag{15.28}$$

For $I\in\mathfrak{W}_h$ and $J\in\mathfrak{W}_q$, the coefficient can be chosen polynomial in ξ of degree at most $q-|\beta|$ and formed from derivatives in x of Γ, $\omega^{\mathbf{E}}$, and $\omega^{\mathbf{F}}$ of order at most $|\alpha|-1$ when $|\alpha|\ge 1$.

Proof. Order partial derivatives by their *horizontal complexity* $|\alpha|$; the number of vertical derivatives does not enter the first component of this ordering. If $|\alpha|=0$, choose a word J_β containing each index j exactly β_j times. The vertical induction in the preceding proof gives $D_\xi^{\beta}\mathbf{a}=\nabla^{\mathrm{V}}_{J_\beta}\mathbf{a}$, which is (15.27).

Suppose the inverse formula has been constructed for every derivative of horizontal complexity less than p and every vertical multi-index. Let $|\alpha|=p$ and choose a word I_α containing i exactly α_i times. In (15.24), applied to (I_α,J_β), there is a unique term with $|\mu|=p$: to obtain it, each of the p horizontal derivatives must choose D_{x^i} acting on $\mathbf{a}$. Its coefficient is the identity, its index is (α,β), and all other terms have $|\mu|\le p-1$. Thus

$$D_x^{\alpha}D_\xi^{\beta}\mathbf{a} = \nabla^{\mathrm{H}}_{I_\alpha}\nabla^{\mathrm{V}}_{J_\beta}\mathbf{a} - \sum_{(\mu,\nu)\in\mathcal{A}_{I_\alpha,J_\beta}\setminus\{(\alpha,\beta)\}} C^{I_\alpha,J_\beta}_{\mu,\nu}D_x^{\mu}D_\xi^{\nu}\mathbf{a}. \tag{15.29}$$

In each summand substitute the induction hypothesis corresponding to (μ,ν). After summing coefficients

with the same words, this defines the exact recurrence

$$\widetilde{C}^{\alpha,\beta}_{I,J} = \mathbf{1}_{\{(I,J)=(I_\alpha,J_\beta)\}}\,\mathrm{Id}_{M_{s\times r}(\mathbb{C})} - \sum_{(\mu,\nu)\in\mathcal{A}_{I_\alpha,J_\beta}\setminus\{(\alpha,\beta)\}} C^{I_\alpha,J_\beta}_{\mu,\nu}\circ\widetilde{C}^{\mu,\nu}_{I,J}, \tag{15.30}$$

where the sum is restricted to $\mathcal{A}_{I_\alpha,J_\beta}$ and an undefined coefficient is interpreted as zero.
It remains to check the bounds. Write $\nu=\beta+\gamma$; the preceding lemma gives $|\mu|+|\gamma|\le p$. If a word produced by substituting $D_x^\mu D_\xi^\nu\mathbf{a}$ has lengths h and q, the induction hypothesis gives $h\le|\mu|$ and

$$|\nu|\le q\le|\nu|+|\mu|-h\le|\beta|+p-h.$$

This is exactly (15.28). The degree of the product of coefficients is at most $|\gamma|+(q-|\nu|)=q-|\beta|$. Its connection jets have order at most $p-1$, since the first factor has this bound and the inductive coefficients use at most $|\mu|-1$. The sums are finite. This proves the step from horizontal complexity $p-1$ to p. □

Lemma 15.4.4. (Tensorial derivatives and ordered words) In a chart and local frames, each component of $(\nabla^{\mathrm{H}})^p(\nabla^{\mathrm{V}})^q a$ is the directional word with the same indices plus a finite sum of words with fewer than p horizontal letters. The coefficients are universal polynomials in the Christoffel symbols and their derivatives of order at most $p-1$. Conversely, each word of lengths (p,q) is the corresponding component of the iterated tensor plus a sum of the same type. On each compact set, the two families of components are therefore equivalent without loss of order.

Proof. For $(p,q)=(1,0)$ or $(0,1)$ there is no correction. The first nontrivial identities are

$$\big((\nabla^{\mathrm{H}})^2 a\big)_{ij} = \nabla^{\mathrm{H}}_i(\nabla^{\mathrm{H}}_j a) - \Gamma^k_{ij}\nabla^{\mathrm{H}}_k a$$

and

$$\big(\nabla^{\mathrm{H}}\nabla^{\mathrm{V}} a\big)_{i;}{}^{j} = \nabla^{\mathrm{H}}_i(\nabla^{\mathrm{V}}_j a) + \Gamma^j_{ik}\nabla^{\mathrm{V}}_k a.$$

Formulas (15.17)–(15.18) show inductively that adding an index leaves as leading term the word obtained by adding the same letter, and that all corrections have smaller horizontal length. The triangular matrix has identity diagonal; solving inductively gives the converse assertion. Uniform bounds on compact sets follow from bounds on the Christoffel jets. □

Definition 15.4.5. (Full geometric symbol) Let $(M,\mathbf{g})$ be a smooth Riemannian manifold without boundary, not necessarily complete or compact, let $\pi\colon T^*M\to M$, and let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of respective finite ranks $r_{\mathbf{E}},r_{\mathbf{F}}$, with bundle metrics $\mathbf{h}_{\mathbf{E}},\mathbf{h}_{\mathbf{F}}$ and compatible connections. On T^*M use the Levi–Civita horizontal lift, and on $\pi^*\mathrm{Hom}(\mathbf{E},\mathbf{F})$ the pullback connection induced by the connections on $\mathbf{E}$ and $\mathbf{F}$. A **geometric symbol of order** m is a section

$$\mathbf{a}\in\Gamma\big(\pi^*\mathrm{Hom}(\mathbf{E},\mathbf{F})\big)$$

such that, for every compact set $K\Subset M$ and all $p,q\in\mathbb{N}_0$, there exists $C_{K,p,q}>0$ with

$$\big\|(\nabla^{\mathrm{H}})^p(\nabla^{\mathrm{V}})^q\mathbf{a}(x,\xi)\big\|_{\mathbf{g},\mathbf{h}_{\mathbf{E}},\mathbf{h}_{\mathbf{F}}}\le C_{K,p,q}\langle\xi\rangle_{\mathbf{g}}^{m-q},\qquad x\in K,\quad \xi\in T^*_xM. \tag{15.31}$$

The class is denoted by $S^m_{\mathbf{g},\nabla}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))$.
In (15.31), the notation $(\nabla^{\mathrm{H}})^p(\nabla^{\mathrm{V}})^q\mathbf{a}$ denotes the tensor in $\mathscr{T}^{p,q}$ defined by the preceding intrinsic recurrence, and the norm is induced by $\mathbf{g}$ and the bundle metrics. The words (15.19) are retained only as auxiliary compositions for the local inductions.
The symbol is **classical** if there exist sections $\mathbf{a}_{m-j}$ on $T^*M\setminus 0_M$, homogeneous of degree $m-j$, such that, for a radial cutoff ϑ vanishing near 0_M and equal to one at high frequencies,

$$\mathbf{a}-\sum_{j=0}^{N-1}\vartheta\mathbf{a}_{m-j}\in S^{m-N}_{\mathbf{g},\nabla}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))\qquad(N\geq 1).$$

We then write $\mathbf{a}\in S^m_{\mathrm{cl},\mathbf{g},\nabla}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))$.

The comparison with local classes is quantitative. Fix a matrix norm $\|\cdot\|_0$ in the chosen frames, and denote by $\|\cdot\|_{\mathrm{op},0}$ the induced norm on the corresponding endomorphism spaces. For a tensor $T\in\mathscr{T}^{p,q}$, let $\|T\|_{0,\mathrm{comp}}$ denote the Euclidean product norm of the matrix norms of its components in the fixed chart and frames; for $(p,q)=(0,0)$ it agrees with $\|\cdot\|_0$. For $K\Subset U$, $N\in\mathbb{N}_0$, and $m\in\mathbb{R}$, define

$$p^{\mathrm{loc}}_{K,N,m}(a):=\max_{|\alpha|+|\beta|\leq N}\sup_{x\in K,\,\xi\in\mathbb{R}^n}\langle\xi\rangle^{-m+|\beta|}\|D_x^\alpha D_\xi^\beta a(x,\xi)\|_0,\tag{15.32}$$

$$p^{\mathrm{geom}}_{K,N,m}(a):=\max_{p+q\leq N}\sup_{x\in K,\,\xi\in T^*_xM}\langle\xi\rangle_{\mathbf{g}}^{-m+q}\|(\nabla^{\mathrm{H}})^p(\nabla^{\mathrm{V}})^q a(x,\xi)\|_{\mathbf{g},\mathbf{h}_{\mathbf{E}},\mathbf{h}_{\mathbf{F}}},\tag{15.33}$$

$$p^{\mathrm{word}}_{K,N,m}(a):=\max_{\substack{p+q\leq N\\ I\in\mathfrak{W}_p,\,J\in\mathfrak{W}_q}}\sup_{x\in K,\,\xi\in T^*_xM}\langle\xi\rangle_{\mathbf{g}}^{-m+q}\|\nabla^{\mathrm{H}}_I\nabla^{\mathrm{V}}_J a(x,\xi)\|_{\mathbf{g},\mathbf{h}_{\mathbf{E}},\mathbf{h}_{\mathbf{F}}}.\tag{15.34}$$

The tensor norms are induced by $\mathbf{g}$ and the bundle metrics.

Proposition 15.4.6. (Quantitative equivalence of symbol seminorms) Let $(M,\mathbf{g})$ be a smooth Riemannian manifold without boundary, not necessarily complete or compact, and let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of finite rank with bundle metrics $\mathbf{h}_{\mathbf{E}},\mathbf{h}_{\mathbf{F}}$ and compatible connections. Fix a chart (U,x), local frames of $\mathbf{E}$ and $\mathbf{F}$, and a symbol $\mathbf{a}\in\Gamma(\pi^*\mathrm{Hom}(\mathbf{E},\mathbf{F})|_{T^*U})$. For each $K\Subset U$, $N\in\mathbb{N}_0$, and $m\in\mathbb{R}$, there exists $C_{K,N,m}\geq 1$ such that

$$C^{-1}_{K,N,m}p^{\mathrm{loc}}_{K,N,m}(\mathbf{a})\leq p^{\mathrm{word}}_{K,N,m}(\mathbf{a})\leq C_{K,N,m}p^{\mathrm{loc}}_{K,N,m}(\mathbf{a}),\qquad C^{-1}_{K,N,m}p^{\mathrm{word}}_{K,N,m}(\mathbf{a})\leq p^{\mathrm{geom}}_{K,N,m}(\mathbf{a})\leq C_{K,N,m}p^{\mathrm{word}}_{K,N,m}(\mathbf{a}).\tag{15.35}$$

In particular, the same order N suffices in both inequalities: no derivatives are lost. The constants depend on K,N,m, the chart and frames, comparison of the Euclidean and Riemannian norms, the bundle metrics, and bounds on K for derivatives of Γ, $\omega^{\mathbf{E}}$, and $\omega^{\mathbf{F}}$ of order at most $N-1$; when $N=0$ they do not depend on these jets.

Proof. Compactness of K provides constants $c_K,C_K,c'_K,C'_K>0$ with

$$c_K\langle\xi\rangle\leq\langle\xi\rangle_{\mathbf{g}}\leq C_K\langle\xi\rangle,\qquad c'_K\|T\|_{0,\mathrm{comp}}\leq\|T\|_{\mathbf{g},\mathbf{h}_{\mathbf{E}},\mathbf{h}_{\mathbf{F}}}\leq C'_K\|T\|_{0,\mathrm{comp}}.$$

The polynomial and jet bounds in Lemma 15.4.2 imply that, for $p+q\le N$, there exists $B_{K,N}>0$ such that

$$\|C^{I,J}_{\mu,\beta(J)+\gamma}(x,\xi)\|_{\mathrm{op},0}\le B_{K,N}\langle\xi\rangle^{|\gamma|},\qquad x\in K,\quad \xi\in\mathbb{R}^n.$$

Since $|\mu|+|\beta(J)+\gamma|\le p+q\le N$, each partial derivative of (15.24) is included in $p^{\mathrm{loc}}_{K,N,m}$. Moreover,

$$\langle\xi\rangle^{|\gamma|}\|D_x^{\mu}D_\xi^{\beta(J)+\gamma}\mathbf{a}\|_0\le p^{\mathrm{loc}}_{K,N,m}(\mathbf{a})\langle\xi\rangle^{m-q}.$$

Summing over the finitely many indices in (15.23) and using both norm comparisons proves the bound of p^{word} by p^{loc} in (15.35).

For the reverse direction, Lemma 15.4.3 gives a constant $\widetilde{B}_{K,N}>0$ such that, if $(I,J)\in\mathfrak{W}_h\times\mathfrak{W}_q$ contributes to $D_x^\alpha D_\xi^\beta\mathbf{a}$ with $|\alpha|+|\beta|\le N$, then

$$\|\widetilde{C}^{\alpha,\beta}_{I,J}(x,\xi)\|_{\mathrm{op},0}\le\widetilde{B}_{K,N}\langle\xi\rangle^{q-|\beta|},\qquad h+q\le|\alpha|+|\beta|\le N.$$

By definition of the word seminorm,

$$\langle\xi\rangle^{q-|\beta|}\|\nabla^{\mathrm{H}}_I\nabla^{\mathrm{V}}_J\mathbf{a}\|_{\mathbf{g},\mathbf{h}_{\mathbf{E}},\mathbf{h}_{\mathbf{F}}}\le C'_{K,N,m}p^{\mathrm{word}}_{K,N,m}(\mathbf{a})\langle\xi\rangle^{m-|\beta|}.$$

The finite sum in (15.27) bounds p^{loc} by p^{word}. Recurrences (15.26) and (15.30) show exactly the asserted dependence of the constants. Finally, Lemma 15.4.4, applied for $p+q\le N$, compares p^{geom} with p^{word} at the same orders. Increasing the constant once gives all four stated inequalities. □

Proposition 15.4.6 shows that, in every relatively compact chart (U,κ) and local bundle frames, $S^m_{\mathbf{g},\nabla}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))$ coincides with $S^m_{1,0}(\kappa(U)\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^{r_{\mathbf{E}}},\mathbb{C}^{r_{\mathbf{F}}}))$, and that both families of seminorms generate the same Fréchet topology. It also shows that replacing the metric, connections, frames, or charts by other smooth data does not change the local class: apply (15.35) to each choice and compare through the local family. On a noncompact base, all constants depend on K, so this equivalence is uniform only on compact sets.

Definition 15.4.7. (Riemannian quantization) Let $(M,\mathbf{g})$ be a smooth Riemannian manifold of dimension n without boundary, not necessarily complete or compact. Let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of finite rank with bundle metrics and compatible connections; let $\chi\in C^\infty(M\times M)$ be the properly supported cutoff fixed at the beginning of this section, and use parallel transport in $\mathbf{E}$ in the input variable. For $a\in S^m_{\mathbf{g},\nabla}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))$ and $\mathbf{u}\in\Gamma_c(M,\mathbf{E})$, define

$$\mathrm{Op}_{\mathbf{g},\nabla,\chi}(a)\mathbf{u}(x):=(2\pi)^{-n}\,\mathrm{Os}\int_{T^*_xM}\int_{T_xM}e^{-i\langle\xi,v\rangle}\chi(x,\exp_x v)\,a(x,\xi)\cdot\tau^{\mathbf{E}}_{x\leftarrow\exp_x v}\mathbf{u}(\exp_x v)\,dv_x\,d\xi_x.$$

The formula can be written using Riemannian measure. If

$$\exp_x^*(d\lambda_{\mathbf{g}})(v)=j_x(v)\,dv_x,$$

, the integrand over M contains the factor $j_x(\ell(x,y))^{-1}d\lambda_{\mathbf{g}}(y)$. Thus Definition 15.4.7 is compatible with the global convention $d\lambda_{\mathbf{g}}$: the factor $j_x(\ell(x,y))^{-1}$ is the Jacobian relating the two measures.

This construction is the pseudodifferential quantization associated with the linearization ℓ, not the geometric quantization of Hamiltonian systems. It follows the intrinsic idea of Bokobza–Haggiag [23, Chapters III–VI]: a linearization, a cutoff, and a transport turn a cotangent symbol into a kernel near the diagonal.

Theorem 15.4.8. (Equivalence of local and Riemannian quantization) Let $(M,\mathbf{g})$ be a smooth Riemannian manifold without boundary, not necessarily complete or compact. Let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of finite rank with bundle metrics and compatible connections, and let χ be a properly supported cutoff as in Definition 15.4.7. The following two assertions hold.

(i) If $a\in S^m_{\mathrm{cl},\mathbf{g},\nabla}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))$, then

$$\mathrm{Op}_{\mathbf{g},\nabla,\chi}(a)\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$$

, and its principal symbol is the homogeneous component a_m.

(ii) If $A\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$, there exists a classical geometric symbol a such that

$$A-\mathrm{Op}_{\mathbf{g},\nabla,\chi}(a)\in\Psi^{-\infty}(M;\mathbf{E},\mathbf{F}).$$

If the smooth kernel of this remainder is allowed to be added to the quantization, reconstruction is exact.

Proof. Throughout this proof, $M,\mathbf{E},\mathbf{F}$ and the geometric data remain fixed; we abbreviate $\Psi^q:=\Psi^q(M;\mathbf{E},\mathbf{F})$, $\Psi^{-\infty}:=\Psi^{-\infty}(M;\mathbf{E},\mathbf{F})$, and $S^q_{\mathbf{g},\nabla}:=S^q_{\mathbf{g},\nabla}(T^*M;\mathrm{Hom}(\mathbf{E},\mathbf{F}))$. For (i), localize the operator with functions supported in a chart, choosing the chart so that the segments appearing below remain inside it. Let $\mathbf{g}_0(x):=\det(\mathbf{g}(x))$, and let $T(x,y)$ be the matrix of $\tau^{\mathbf{E}}_{x\leftarrow y}$. Since $\ell(x,x)=0$, the parameter-dependent version of Hadamard's lemma 1.4.10 gives, componentwise, the exact identity

$$\ell^a(x,y)=\sum_{b=1}^{n}C^a_b(x,y)(y^b-x^b),\qquad C^a_b(x,y):=\int_0^1\partial_{y^b}\ell^a(x,x+t(y-x))\,dt. \tag{15.36}$$

Since $D_y\ell(x,x)=\mathrm{Id}_{T_xM}$, we have $C(x,x)=I$. After shrinking the localization support, $C(x,y)$ is invertible, and C, C^{-1}, and every finite family of their derivatives are bounded.
In cotangent coordinates, $d\xi_x=\mathbf{g}_0(x)^{-\frac12}d\xi$, whereas $d\lambda_{\mathbf{g}}(y)=\mathbf{g}_0(y)^{\frac12}dy$. The formula preceding the theorem and (15.36) turn the localized operator into

$$(2\pi)^{-n}\,\mathrm{Os}\iint_{\mathbb{R}^n_y\times\mathbb{R}^n_\xi}e^{-i(y-x)\cdot C(x,y)^T\xi}h(x,y)a(x,\xi)T(x,y)u(y)\,dy\,d\xi,$$

where, including the two localization functions in h,

$$h(x,y)=\chi(x,y)j_x(\ell(x,y))^{-1}\left(\frac{\mathbf{g}_0(y)}{\mathbf{g}_0(x)}\right)^{\frac12}.$$

Now make the exact change

$$\eta=C(x,y)^T\xi,\qquad \xi=L(x,y)\eta,\qquad L(x,y):=C(x,y)^{-T}. \tag{15.37}$$

Its Jacobian is $d\xi = |\det C(x,y)|^{-1} d\eta$. Consequently, the full local amplitude is

$$b(x,y,\eta) = h(x,y)|\det C(x,y)|^{-1} a\big(x, L(x,y)\eta\big) T(x,y). \tag{15.38}$$

Let us verify its estimates. The chain rules, applying derivatives in η first and then spatial derivatives, produce a finite sum of terms

$$B_{\alpha,\beta,\gamma;\mu,\nu,\rho}(x,y)\eta^{\rho}(D_x^{\mu}D_{\xi}^{\nu}a)\big(x,L(x,y)\eta\big), \tag{15.39}$$

when computing $D_x^{\alpha}D_y^{\beta}D_{\eta}^{\gamma}[a(x,L\eta)]$. The indices in each term satisfy

$$|\nu| = |\gamma| + |\rho|, \qquad |\mu| + |\rho| \le |\alpha| + |\beta|,$$

and the coefficients $B_{\alpha,\beta,\gamma;\mu,\nu,\rho}$ are products of derivatives of L. These assertions follow inductively: a spatial derivative hitting a produces $D_x a$; one hitting $L\eta$ simultaneously produces a linear factor in η and an additional derivative in ξ; and one hitting a coefficient merely differentiates that coefficient. This enumerates the three Leibniz possibilities and preserves both displayed relations.

On the spatial compact set under consideration, there exist $c, C > 0$ such that $c\langle\eta\rangle \le \langle L(x,y)\eta\rangle \le C\langle\eta\rangle$. Proposition 15.4.6 and (15.39) give, for all α,β,γ, a constant $C_{\alpha,\beta,\gamma} > 0$ with

$$\big\|D_x^{\alpha}D_y^{\beta}D_{\eta}^{\gamma}a(x,L\eta)\big\| \le C_{\alpha,\beta,\gamma}\langle\eta\rangle^{m-|\gamma|}.$$

Indeed, the factor η^{ρ} in each summand is exactly offset by the $|\rho|$ additional derivatives in ξ. All derivatives of h, $|\det C|^{-1}$, and T are bounded on the compact set, so

$$\|D_x^{\alpha}D_y^{\beta}D_{\eta}^{\gamma}b(x,y,\eta)\| \le C'_{\alpha,\beta,\gamma}\langle\eta\rangle^{m-|\gamma|}. \tag{15.40}$$

This proves, derivative by derivative, that b is an amplitude of order m. If a is classical, substituting $L\eta$ into each homogeneous component preserves its degree, and the same estimate applied to each remainder proves that b is classical. Theorem 15.1.9 reduces (15.38) to a local classical symbol of order m.

On the diagonal, $C(x,x) = L(x,x) = I$, $T(x,x) = I$, $j_x(0) = 1$, and $\mathbf{g}_0(x)^{\frac{1}{2}}\mathbf{g}_0(x)^{-\frac{1}{2}} = 1$ hold, and the localization cutoffs can be taken equal to one. Thus $b_m(x,x,\eta) = a_m(x,\eta)$. The zero-index term in amplitude-to-symbol reduction preserves this component; every remaining term contains at least one derivative in η and has order at most $m-1$. This proves (i), including the statement on the principal symbol.

For (ii), we construct all terms, not just the first. Set $R_0 := A$ and $r_m := \sigma_m^{\Psi}(R_0)$. A radial cutoff, vanishing near the zero section and equal to one for large covectors, extends r_m to a classical geometric symbol $a^{(0)}$ of order m. By (i), $\sigma_m^{\Psi}(\mathrm{Op}_{\mathbf{g},\nabla,\chi}(a^{(0)})) = r_m$; hence

$$R_1 := R_0 - \mathrm{Op}_{\mathbf{g},\nabla,\chi}(a^{(0)}) \in \Psi_{\mathrm{cl}}^{m-1}.$$

The precise induction statement is as follows: for some $N \ge 1$, we have constructed $a^{(j)} \in S_{\mathrm{cl},\mathbf{g},\nabla}^{m-j}$, $0 \le j < N$, so that

$$R_N := A - \mathrm{Op}_{\mathbf{g},\nabla,\chi}\left(\sum_{j=0}^{N-1} a^{(j)}\right) \in \Psi_{\mathrm{cl}}^{m-N}. \tag{15.41}$$

The case $N = 1$ has just been proved. Assume (15.41) and let $r_{m-N} := \sigma_{m-N}^{\Psi}(R_N)$. Extend r_{m-N} by a radial

cutoff to $a^{(N)} \in S^{m-N}_{\mathrm{cl},\mathbf{g},\nabla}$. Part (i) gives

$$\sigma^{\Psi}_{m-N}\big(\mathrm{Op}_{\mathbf{g},\nabla,\chi}(a^{(N)})\big) = r_{m-N}.$$

By exactness of the principal symbol,

$$R_{N+1} := R_N - \mathrm{Op}_{\mathbf{g},\nabla,\chi}(a^{(N)}) \in \Psi^{m-N-1}_{\mathrm{cl}},$$

which is precisely (15.41) with $N+1$. This completes the induction.
Lemma 15.1.5, applied in charts and combined using a locally finite partition, provides $a \in S^{m}_{\mathrm{cl},\mathbf{g},\nabla}$ such that, for every $N \geq 1$,

$$a - \sum_{j=0}^{N-1} a^{(j)} \in S^{m-N}_{\mathbf{g},\nabla}. \tag{15.42}$$

Part (i), now at order $m-N$, turns the left-hand side of (15.42) into an operator in Ψ^{m-N}. Adding it to R_N gives

$$A - \mathrm{Op}_{\mathbf{g},\nabla,\chi}(a) \in \Psi^{m-N} \qquad \text{for every } N \geq 1.$$

The intersection of these classes is $\Psi^{-\infty}$ by the characterization through smooth kernels. This proves (ii). □

The theorem distinguishes two levels that must not be confused: the principal component is recovered in a single step, modulo $\Psi^{m-1}(M;\mathbf{E},\mathbf{F})$; equivalence modulo $\Psi^{-\infty}(M;\mathbf{E},\mathbf{F})$ requires constructing all terms and applying asymptotic summation.

Proposition 15.4.9. (Dependence on choices) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E},\mathbf{F} \to M$ be complex vector bundles of finite rank. Consider two geometric quantization rules obtained by choosing Riemannian metrics, bundle metrics, compatible connections, linearizations, and properly supported cutoffs as in Definition 15.4.7. For a symbol of order m, changing the cutoff changes the quantization by an operator in $\Psi^{-\infty}(M;\mathbf{E},\mathbf{F})$. Changing the other data modifies the full symbol but preserves the class $\Psi^{m}(M;\mathbf{E},\mathbf{F})$, the principal symbol, and quantization modulo $\Psi^{m-1}(M;\mathbf{E},\mathbf{F})$. After successive corrections of lower-order terms, the two quantizations correspond modulo $\Psi^{-\infty}(M;\mathbf{E},\mathbf{F})$.

Proof. Throughout this proof, $M,\mathbf{E},\mathbf{F}$ remain fixed; write $\Psi^{q} := \Psi^{q}(M;\mathbf{E},\mathbf{F})$ and $\Psi^{-\infty} := \Psi^{-\infty}(M;\mathbf{E},\mathbf{F})$. Work in a relatively compact chart and write $v = y - x$. We use the exact factorizations in Hadamard's lemma 1.4.10, with x as parameter. If a smooth matrix-valued function $F(x,v)$ satisfies $F(x,0) = 0$, then

$$F(x,v) = \sum_{j=1}^{n} v_j F_j(x,v), \qquad F_j(x,v) = \int_0^1 \partial_{v_j} F(x,tv)\,dt. \tag{15.43}$$

If, in addition, $D_v F(x,0) = 0$, then

$$F(x,v) = \sum_{|\alpha|=2} v^{\alpha} F_{\alpha}(x,v), \qquad F_{\alpha}(x,v) = \frac{2}{\alpha!}\int_0^1 (1-t)\partial_v^{\alpha} F(x,tv)\,dt. \tag{15.44}$$

In general, if the vertical derivatives of orders less than L vanish at $v=0$, the same proof gives

$$F(x,v)=\sum_{|\alpha|=L} v^{\alpha}F_{\alpha,L}(x,v), \qquad F_{\alpha,L}(x,v)=\frac{L}{\alpha!}\int_0^1 (1-t)^{L-1}\partial_v^{\alpha}F(x,tv)\,dt. \tag{15.45}$$

These identities also provide the required bounds. For example, for K compact and $N\in\mathbb{N}_0$, there exists $C_{K,N,L}>0$ such that

$$\max_{|\rho|+|\sigma|\le N}\sup_{x\in K,\ v\in\overline{B}_{\mathrm{euc}}(0,\varepsilon)}\|D_x^{\rho}D_v^{\sigma}F_{\alpha,L}(x,v)\|\le C_{K,N,L}\max_{|\rho'|+|\sigma'|\le N+L}\sup_{x\in K,\ v\in\overline{B}_{\mathrm{euc}}(0,\varepsilon)}\|D_x^{\rho'}D_v^{\sigma'}F(x,v)\|.$$

Each transverse monomial translates exactly into a change of symbol order. For the normalized linear phase,

$$v^{\alpha}e^{-i\eta\cdot v}=i^{|\alpha|}D_{\eta}^{\alpha}e^{-i\eta\cdot v}, \qquad \int_{\mathbb{R}^n_{\eta}} e^{-i\eta\cdot v}v^{\alpha}c(\eta)\,d\eta=(-i)^{|\alpha|}\int_{\mathbb{R}^n_{\eta}} e^{-i\eta\cdot v}D_{\eta}^{\alpha}c(\eta)\,d\eta. \tag{15.46}$$

The second identity is understood first with a cutoff in η and then through the oscillatory procedure already established. Thus a factor v^{α} applied to an amplitude of order r produces an amplitude of order $r-|\alpha|$.

First let χ_0 and χ_1 be two admissible cutoffs. Their difference vanishes on a neighborhood of $v=0$, so all its vertical derivatives at $v=0$ are zero. For each L, formulas (15.45) and (15.46) express the difference kernel through an amplitude of order $m-L$. Given any prescribed order, choose L larger; the difference belongs to every order and therefore has smooth kernel by the characterization of smoothing operators.

Now consider two connections on $\mathbf{E}$ and their radial transports $T_0(x,x+v)$ and $T_1(x,x+v)$. Both equal the identity at $v=0$. Define

$$F(x,v):=T_1(x,x+v)-T_0(x,x+v).$$

Applied to this function F, identity (15.43) gives

$$T_1(x,x+v)-T_0(x,x+v)=\sum_{j=1}^{n} v_jT_j(x,v), \qquad T_j(x,v)=\int_0^1 \partial_{v_j}F(x,tv)\,dt. \tag{15.47}$$

The coefficients T_j satisfy the preceding bounds for each finite family of derivatives. Integration by parts in (15.46) shows that the difference between the quantizations, with all other data fixed, has amplitude of order $m-1$.

The same factorization controls Jacobian factors and measures. In the coordinates of (15.38), the scalar combination corresponding to any geometric data is

$$q(x,y)=j_x(\ell(x,y))^{-1}\left(\frac{g_0(y)}{g_0(x)}\right)^{\frac{1}{2}}|\det C(x,y)|^{-1}, \qquad q(x,x)=1.$$

For a nonexponential linearization, q denotes the exact factor obtained from the Jacobian of $y\longmapsto\ell(x,y)$ and the chosen densities; identity $D_y\ell(x,x)=I$ again gives $q(x,x)=1$. Thus, for two choices, q_1-q_0 satisfies (15.43) its coefficients and all their derivatives up to order N are bounded by a constant $C_{K,N}$ on each compact set. Each term of the difference again has order at most $m-1$ after (15.46).

It remains to compare two linearizations ℓ_0 and ℓ_1. Both satisfy

$$\ell_{\kappa}(x,x)=0, \qquad D_y\ell_{\kappa}(x,x)=I, \qquad \kappa\in\{0,1\}.$$

Consequently, $G(x,v) := \ell_1(x,x+v) - \ell_0(x,x+v)$ satisfies both hypotheses of (15.44), and

$$G(x,v) = \sum_{|\alpha|=2} v^\alpha G_\alpha(x,v), \qquad G_\alpha(x,v) = \frac{2}{\alpha!}\int_0^1 (1-t)\partial_v^\alpha G(x,tv)\,dt. \tag{15.48}$$

Define $\ell_t = \ell_0 + tG$. Shrinking the neighborhood ensures that all ℓ_t, $0 \le t \le 1$, are linearizations and that the matrices C_t in (15.36) are uniformly invertible. The fundamental theorem of calculus gives the exact identity

$$e^{-i\langle \xi,\ell_1\rangle} - e^{-i\langle \xi,\ell_0\rangle} = -i\int_0^1 e^{-i\langle \xi,\ell_t\rangle}\langle \xi, G\rangle\,dt \tag{15.49}$$

In each integrand make the change $\eta = C_t(x,y)^T\xi$. By (15.48), the factor $\langle \xi, G\rangle$ becomes a sum of monomials v^α, $|\alpha| = 2$, multiplied by a function linear in η and smooth coefficients uniformly bounded in t. Before integration by parts, the amplitude has order $m+1$; two applications of (15.46) bring it to order $m-1$. The Leibniz rules used in (15.39) give, for all ρ,σ,γ, a constant $C_{K,\rho,\sigma,\gamma} > 0$ independent of t such that the final amplitude d_t satisfies

$$\|D_x^\rho D_y^\sigma D_\eta^\gamma d_t(x,y,\eta)\| \le C_{K,\rho,\sigma,\gamma}\langle \eta\rangle^{m-1-|\gamma|}. \tag{15.50}$$

Integration in $t \in [0,1]$ preserves the estimate.

A change of metric splits into a change of its exponential linearization, already treated, and a change of the factors q; a change of connection on $\mathbf{E}$ reduces to (15.47). The connection on $\mathbf{F}$ enters the symbol seminorms, not the integral transporting the input section, and Proposition 15.4.6 shows that changing it preserves the symbol class. Combining (15.47), the factorization of q, and (15.50) gives, for each finite family of derivatives, the estimate for an amplitude of order $m-1$. Thus all the quantizations considered have the same principal symbol; this is a consequence of the factorizations, not a hypothesis.

Finally, let Q_0, Q_1 be two such quantization rules. What we have proved says that $Q_0(a) - Q_1(a) \in \Psi^{m-1}$ for every symbol of order m. Take the principal symbol of this difference, choose a correction of order $m-1$ for one of the two symbols, and subtract. If after N corrections the difference belongs to Ψ^{m-N}, its principal symbol of degree $m-N$ determines the next correction, and the new difference belongs to Ψ^{m-N-1}. An asymptotic sum of the corrections produces two corresponding symbols whose operator difference belongs to Ψ^{m-N} for every N, that is, to $\Psi^{-\infty}$. □

This separates what is canonical from what is auxiliary. The class $\Psi^m(M;\mathbf{E},\mathbf{F})$, principal symbol, ellipticity, and operator class modulo lower orders are canonical. The directional microlocal invariant will be added after it has been constructed by Fourier methods and its invariance proved. The full symbol, specific quantization, metric, connections, cutoff, and transport depend on choices. Calling the Riemannian formula *intrinsic* means that all these data are geometric and global, not that the result is independent of them term by term.

On a curved manifold, replacing partial derivatives by covariant derivatives in the flat composition formula requires additional lower-order terms. These terms contain jets of the exponential map, the connection, and curvature. The particularly simple formulas in [23, Chapter VI] require a locally affine linearization; the existence of such a linearization is equivalent to a local affine structure. For the general calculus, the local formulas in Theorems 15.2.6 and 15.2.8, together with the equivalence just proved, provide expansions containing all these geometric terms.

15.5 Elliptic inversion and microlocal analysis

The symbolic calculus was constructed to invert elliptic operators without requiring an exact inverse. We first invert the principal component and successively correct lower-order terms. The final remainder will be smoothing; this arbitrary gain of derivatives is the source of both elliptic regularity and its microlocal version.

Definition 15.5.1. (Parametrix) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E},\mathbf{F} \to M$ be smooth complex vector bundles of finite rank. Let $A \in \Psi^m(M;\mathbf{E},\mathbf{F})$. An operator $B \in \Psi^{-m}(M;\mathbf{F},\mathbf{E})$ is a **right parametrix** if $AB - I_{\mathbf{F}} \in \Psi^{-\infty}(M;\mathbf{F},\mathbf{F})$, a **left parametrix** if $BA - I_{\mathbf{E}} \in \Psi^{-\infty}(M;\mathbf{E},\mathbf{E})$, and a **two-sided parametrix** if both conditions hold. For the purely local calculus, replace all three classes by their variants with the subscript loc.

Ellipticity allows inversion of an operator at high frequencies but does not guarantee an algebraic or functional inverse. The distinction is essential: the first assertion produces a parametrix modulo smoothing operators; the second also requires control of the kernel and cokernel in a specific realization. We first develop the symbolic construction and then measure directional regularity using the wavefront set. All proofs use the convention $D_j = \partial_j$ and hence the factors $i^{-|\alpha|}$ in Theorem 15.2.6.

15.5.1 Criteria for ellipticity

We begin by collecting several formulations that will be used interchangeably. For a matrix T, let $s_{\min}(T)$ denote its smallest singular value.

Proposition 15.5.2. (Equivalent formulations of ellipticity) Let $U \subseteq \mathbb{R}^n$ be open, and let $r \in \mathbb{N}$ and $m \in \mathbb{R}$. Let

$$a \in S^m_{\mathrm{cl}}\big(U;\mathrm{End}(\mathbb{C}^r)\big), \qquad a \sim \sum_{j=0}^{\infty} a_{m-j}.$$

The following conditions are equivalent, with constants that may depend on each compact set $K \Subset U$:

(a) a is elliptic in the sense of Definition 15.1.2;

(b) there exist $c_K, R_K > 0$ such that

$$\|a(x,\xi)v\| \geq c_K \langle \xi \rangle^m \|v\|, \qquad x \in K, \quad \|\xi\| \geq R_K;$$

(c) there exist $c_K, R_K > 0$ such that

$$|\det a(x,\xi)| \geq c_K \langle \xi \rangle^{mr}, \qquad x \in K, \quad \|\xi\| \geq R_K;$$

(d) $a_m(x,\xi)$ is invertible for every $x \in U$ and every $\xi \neq 0$;

(e) there exists $b_0 \in S^{-m}_{1,0}(U;\mathrm{End}(\mathbb{C}^r))$ such that

$$ab_0 - I_r \in S^{-1}_{1,0}(U;\mathrm{End}(\mathbb{C}^r)), \qquad b_0 a - I_r \in S^{-1}_{1,0}(U;\mathrm{End}(\mathbb{C}^r)).$$

Moreover, in the high-frequency region where a is invertible, its inverse satisfies $a^{-1} \in S^{-m}_{1,0}(U;\mathrm{End}(\mathbb{C}^r))$.

Proof. Throughout this proof, U, r, m remain fixed, and we write

$$S^q := S^q_{1,0}\big(U; \mathrm{End}(\mathbb{C}^r)\big).$$

The identity

$$s_{\min}(a(x,\xi)) = \|a(x,\xi)^{-1}\|^{-1}$$

proves equivalence of (a) and (b). If (b) holds, all singular values are bounded below by $c_K\langle\xi\rangle^m$, so their product gives (c). Conversely, the estimates for S^m imply

$$\|\operatorname{adj} a(x,\xi)\| \le C_K\langle\xi\rangle^{m(r-1)}.$$

By the formula $a^{-1} = (\det a)^{-1}\operatorname{adj} a$, condition (c) yields the bound in (a).
Suppose (d) holds. On the compact set $K\times\mathbb{S}^{n-1}$, the function $s_{\min}(a_m)$ has a positive minimum. By definition of the classical expansion, there exists $C_K > 0$ such that, uniformly for $x \in K$, $\omega \in \mathbb{S}^{n-1}$, and $t \ge 1$,

$$\left\| t^{-m}a(x,t\omega) - a_m(x,\omega)\right\| \le C_K t^{-1}.$$

This inequality gives (b). Conversely, if $a_m(x_0,\omega_0)v = 0$ for some $v \neq 0$, the same convergence contradicts the lower bound in (b) applied to $(x_0, t\omega_0)$.
If (d) holds, choose a radial cutoff ϑ vanishing near the origin and equal to one at large frequencies, and set

$$b_0(x,\xi) = \vartheta(\xi)a_m(x,\xi)^{-1}.$$

Homogeneity of degree $-m$ and local compactness of the sphere show that $b_0 \in S^{-m}$. Since $a - \vartheta a_m \in S^{m-1}$ at high frequencies, both products in (e) are I_r modulo S^{-1}. If (e) holds, then $ab_0 = I_r + r$ with $r \in S^{-1}$. For large frequencies, $\|r\| < \frac{1}{2}$; thus a is invertible and

$$\|a^{-1}\| \le \|(I_r + r)^{-1}\|\,\|b_0\| \le C_K\langle\xi\rangle^{-m}.$$

Finally, in the region where a is invertible,

$$D_{x_j}(a^{-1}) = -a^{-1}(D_{x_j}a)a^{-1}, \qquad D_{\xi_j}(a^{-1}) = -a^{-1}(D_{\xi_j}a)a^{-1}.$$

Write $\gamma = (\alpha,\beta) \in \mathbb{N}_0^{2n}$ and $D^\gamma = D_x^\alpha D_\xi^\beta$. Applying D^γ to $aa^{-1} = I_r$, separating the summand in which no derivative hits a, and multiplying on the left by a^{-1} gives the exact recurrence

$$D^\gamma(a^{-1}) = -a^{-1}\sum_{0<\delta\le\gamma}\binom{\gamma}{\delta}(D^\delta a)D^{\gamma-\delta}(a^{-1}), \qquad |\gamma| \ge 1. \tag{15.51}$$

The base case $|\gamma| = 0$ is the bound $\|a^{-1}\| \le C_K\langle\xi\rangle^{-m}$ already proved. Suppose that, for every $\gamma' = (\alpha',\beta')$ with $|\gamma'| < q$,

$$\|D^{\gamma'}a^{-1}(x,\xi)\| \le C_{K,\gamma'}\langle\xi\rangle^{-m-|\beta'|}.$$

Let $|\gamma| = q$ and write $\delta = (\alpha_1,\beta_1)$. Since $0 < \delta \le \gamma$, the factor $D^{\gamma-\delta}(a^{-1})$ is covered by the induction hypothesis.

Each summand in (15.51) is then bounded by

$$C_{K,\gamma,\delta}\langle\xi\rangle^{-m}\langle\xi\rangle^{m-|\beta_1|}\langle\xi\rangle^{-m-(|\beta|-|\beta_1|)} = C_{K,\gamma,\delta}\langle\xi\rangle^{-m-|\beta|}.$$

The sum contains finitely many multi-indices δ; thus

$$\|D_x^{\alpha} D_{\xi}^{\beta} a^{-1}(x,\xi)\| \le C_{K,\alpha,\beta}\langle\xi\rangle^{-m-|\beta|},$$

which completes the induction and proves the remaining assertion. □

Definition 15.5.3. (Elliptic set and characteristic set) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E},\mathbf{F} \to M$ be smooth complex vector bundles of the same finite rank. Let $A \in \Psi^{m}_{\mathrm{cl,loc}}(M;\mathbf{E},\mathbf{F})$. Its **elliptic set** is

$$\mathrm{Ell}(A) := \big\{(x,\xi) \in T^*M \setminus 0_M \mid \sigma^{\Psi}_m(A)(x,\xi) \text{ is an isomorphism}\big\},$$

and its **characteristic set** is

$$\mathrm{Char}(A) := (T^*M \setminus 0_M) \setminus \mathrm{Ell}(A).$$

The operator is **elliptic** if $\mathrm{Char}(A) = \varnothing$.

This ellipticity is local in the base variable. On a noncompact manifold, it does not imply uniform ellipticity in the sense of Definition 15.1.2. Parameter ellipticity, which requires uniform inversion of a family $\lambda I - A$ in a sector, will be defined only when constructing complex powers.

Homogeneity of the principal symbol shows that $\mathrm{Ell}(A)$ is open and conic, whereas $\mathrm{Char}(A)$ is closed and conic. In a chart, membership in $\mathrm{Ell}(A)$ is equivalent to validity of the lower bound in Proposition 15.5.2 on a neighborhood in the base and a conic neighborhood of the covariable. For a differential operator P, identity (15.8) shows that this definition agrees with differential ellipticity despite the factor i^m.

15.5.2 Construction of a two-sided parametrix

The preceding equivalences reduce ellipticity to uniform inversion of the principal component at high frequencies. Starting from this inverse, right and left corrections are constructed recursively. We first write the recurrence in coordinates, where the order of each summand can be tracked explicitly. Let $a \sim \sum_{j=0}^{\infty} a_{m-j}$ be an elliptic classical symbol. First seek an expansion

$$b^{\mathrm{d}} \sim \sum_{k=0}^{\infty} b^{\mathrm{d}}_{-m-k}$$

such that $a \# b^{\mathrm{d}} \sim I_r$; the letter d recalls that $\mathrm{Op}(b^{\mathrm{d}})$ lies to the right of $\mathrm{Op}(a)$. The degree-zero term imposes

$$b^{\mathrm{d}}_{-m} = a_m^{-1}.$$

For $N \geq 1$, the component of degree $-N$ in $a \# b^{\mathrm{d}}$ is

$$\sum_{\substack{j,k\geq 0,\ \alpha\in\mathbb{N}_0^n \\ j+k+|\alpha|=N}} \frac{i^{-|\alpha|}}{\alpha!}(D_\xi^\alpha a_{m-j})(D_x^\alpha b^{\mathrm{d}}_{-m-k}).$$

Separating the unique summand containing the unknown b^{d}_{-m-N} gives the explicit recurrence

$$b^{\mathrm{d}}_{-m-N} = -a_m^{-1} \sum_{\substack{j,k\geq 0,\ \alpha\in\mathbb{N}_0^n \\ j+k+|\alpha|=N,\ k<N}} \frac{i^{-|\alpha|}}{\alpha!}(D_\xi^\alpha a_{m-j})(D_x^\alpha b^{\mathrm{d}}_{-m-k}). \tag{15.52}$$

To construct a left parametrix, write $b^{\mathrm{i}} \# a \sim I_r$. Now $b^{\mathrm{i}}_{-m} = a_m^{-1}$ and

$$b^{\mathrm{i}}_{-m-N} = -\left(\sum_{\substack{j,k\geq 0,\ \alpha\in\mathbb{N}_0^n \\ j+k+|\alpha|=N,\ k<N}} \frac{i^{-|\alpha|}}{\alpha!}(D_\xi^\alpha b^{\mathrm{i}}_{-m-k})(D_x^\alpha a_{m-j}) \right) a_m^{-1}. \tag{15.53}$$

In particular, the first corrections are

$$\begin{aligned} b^{\mathrm{d}}_{-m-1} &= -a_m^{-1}\left(a_{m-1}a_m^{-1} + i^{-1}\sum_{\ell=1}^{n}(D_{\xi_\ell}a_m)(D_{x_\ell}a_m^{-1}) \right), \\ b^{\mathrm{i}}_{-m-1} &= -\left(a_m^{-1}a_{m-1} + i^{-1}\sum_{\ell=1}^{n}(D_{\xi_\ell}a_m^{-1})(D_{x_\ell}a_m) \right) a_m^{-1}. \end{aligned}$$

The order of factors matters for matrix-valued symbols. The factors $i^{-|\alpha|}$ likewise cannot be omitted: they come exactly from the convention $D = \partial$ fixed at the beginning of the chapter.

Theorem 15.5.4. (Existence and equivalence of parametrices) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of the same finite rank. Let $A \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{F})$. The following conditions are equivalent:

(a) A is elliptic;

(b) there exists $B \in \Psi^{-m}_{\mathrm{cl}}(M; \mathbf{F}, \mathbf{E})$ such that

$$AB = I_{\mathbf{F}} - R_{\mathbf{F}}, \qquad BA = I_{\mathbf{E}} - R_{\mathbf{E}}, \qquad R_{\mathbf{E}} \in \Psi^{-\infty}(M; \mathbf{E}, \mathbf{E}), \qquad R_{\mathbf{F}} \in \Psi^{-\infty}(M; \mathbf{F}, \mathbf{F});$$

(c) the principal symbol of A has a homogeneous inverse of degree $-m$ on $T^*M \setminus 0_M$.

The operator B can be chosen properly supported and is a two-sided parametrix of A. Two two-sided parametrices differ by an operator in $\Psi^{-\infty}(M; \mathbf{F}, \mathbf{E})$.

Proof. On an open set $U \subseteq \mathbb{R}^n$ and in frames of common rank r, formulas (15.52) and (15.53) define homogeneous terms of degrees $-m-k$. Let us verify this inductively. For $k = 0$, $a_m^{-1}(x, t\xi) = t^{-m} a_m(x,\xi)^{-1}$. Suppose b^{d}_{-m-k} is homogeneous of degree $-m-k$ for $k < N$. In each summand of (15.52), $D_\xi^\alpha a_{m-j}$ has degree

$m-j-|\alpha|$ and $D_x^\alpha b^{\mathrm{d}}_{-m-k}$ retains degree $-m-k$; since $j+k+|\alpha|=N$, their product has degree $-N$. Multiplication by a_m^{-1} lowers the degree by m and gives exactly $-m-N$. The sum is finite because $j,k,|\alpha|\le N$. The same occurs in the left recurrence: the sum in parentheses has degree $-N$, and the factor a_m^{-1} on the right gives degree $-m-N$. This proves both inductive steps simultaneously. Lemma 15.1.5 produces symbols $b^{\mathrm{d}}, b^{\mathrm{i}} \in S^{-m}_{\mathrm{cl}}(U;\mathrm{End}(\mathbb{C}^r))$ with these expansions. Theorem 15.2.6 and the recurrences give

$$a\#b^{\mathrm{d}} - I_r \in S^{-\infty}(U\times\mathbb{R}^n;\mathrm{End}(\mathbb{C}^r)), \qquad b^{\mathrm{i}}\#a - I_r \in S^{-\infty}(U\times\mathbb{R}^n;\mathrm{End}(\mathbb{C}^r)).$$

Quantizing and arranging proper support yields a right parametrix B_{d} and a left parametrix B_{i}. If $AB_{\mathrm{d}} = I_{\mathbf{F}} - R_{\mathrm{d}}$ and $B_{\mathrm{i}}A = I_{\mathbf{E}} - R_{\mathrm{i}}$, then

$$B_{\mathrm{i}} - B_{\mathrm{d}} = B_{\mathrm{i}}(I_{\mathbf{F}} - AB_{\mathrm{d}}) - (I_{\mathbf{E}} - B_{\mathrm{i}}A)B_{\mathrm{d}} = B_{\mathrm{i}}R_{\mathrm{d}} - R_{\mathrm{i}}B_{\mathrm{d}}.$$

Corollary 15.2.9 shows that this difference is smoothing. Thus either parametrix is two-sided modulo $\Psi^{-\infty}(M;\mathbf{E},\mathbf{E})$ in the left defect and $\Psi^{-\infty}(M;\mathbf{F},\mathbf{F})$ in the right defect.

Globally, (c) supplies the homogeneous section $\mathbf{a}_m^{-1} := \sigma^\Psi_m(A)^{-1}$. Surjectivity of the exact sequence in Theorem 15.3.7 allows it to be quantized as an operator B_0 of order $-m$; then $AB_0 - I_{\mathbf{F}}$ and $B_0A - I_{\mathbf{E}}$ have order -1. Set $B_0^{\mathrm{d}} = B_0^{\mathrm{i}} = B_0$.

The right induction statement is

$$B^{\mathrm{d}}_{<N} := \sum_{k=0}^{N-1} B^{\mathrm{d}}_k, \qquad B^{\mathrm{d}}_k \in \Psi^{-m-k}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{E}), \qquad AB^{\mathrm{d}}_{<N} - I_{\mathbf{F}} \in \Psi^{-N}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{F}).$$

The base case $N=1$ is the choice of B_0. Assuming the assertion for N, define the global homogeneous section

$$\mathbf{e}^{\mathrm{d}}_N := \sigma^\Psi_{-N}\big(AB^{\mathrm{d}}_{<N} - I_{\mathbf{F}}\big) \in \mathcal{S}^{-N}_{\mathrm{hom}}\big(T^*M;\mathrm{End}(\mathbf{F})\big)$$

and

$$\mathbf{c}^{\mathrm{d}}_{-m-N} := -\mathbf{a}_m^{-1}\mathbf{e}^{\mathrm{d}}_N \in \mathcal{S}^{-m-N}_{\mathrm{hom}}\big(T^*M;\mathrm{Hom}(\mathbf{F},\mathbf{E})\big).$$

These are global sections: $\mathbf{e}^{\mathrm{d}}_N$ is the principal symbol of a global operator, and the bundle composition $\mathbf{a}_m^{-1}\circ\mathbf{e}^{\mathrm{d}}_N$ transforms $\pi^*\mathbf{F}$ into $\pi^*\mathbf{E}$. Surjectivity of the exact sequence provides $B^{\mathrm{d}}_N \in \Psi^{-m-N}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{E})$ with principal symbol $\mathbf{c}^{\mathrm{d}}_{-m-N}$. The product rule for principal symbols gives

$$\begin{aligned}\sigma^\Psi_{-N}\big(A(B^{\mathrm{d}}_{<N}+B^{\mathrm{d}}_N) - I_{\mathbf{F}}\big) &= \mathbf{e}^{\mathrm{d}}_N + \sigma^\Psi_m(A)\mathbf{c}^{\mathrm{d}}_{-m-N}\\ &= \mathbf{e}^{\mathrm{d}}_N - \sigma^\Psi_m(A)\mathbf{a}_m^{-1}\mathbf{e}^{\mathrm{d}}_N = 0.\end{aligned}$$

By exactness, the new error belongs to $\Psi^{-N-1}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{F})$; this completes the right step.

For the left side, suppose $B^{\mathrm{i}}_{<N}A - I_{\mathbf{E}} \in \Psi^{-N}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$ and set

$$\mathbf{e}^{\mathrm{i}}_N := \sigma^\Psi_{-N}\big(B^{\mathrm{i}}_{<N}A - I_{\mathbf{E}}\big) \in \mathcal{S}^{-N}_{\mathrm{hom}}\big(T^*M;\mathrm{End}(\mathbf{E})\big), \qquad \mathbf{c}^{\mathrm{i}}_{-m-N} := -\mathbf{e}^{\mathrm{i}}_N\mathbf{a}_m^{-1}.$$

Now $\mathbf{c}^{\mathrm{i}}_{-m-N}$ is a global section of $\mathrm{Hom}(\pi^*\mathbf{F},\pi^*\mathbf{E})$ and can be quantized as $B^{\mathrm{i}}_N \in \Psi^{-m-N}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{E})$. In this case,

$$\sigma^\Psi_{-N}\big((B^{\mathrm{i}}_{<N}+B^{\mathrm{i}}_N)A - I_{\mathbf{E}}\big) = \mathbf{e}^{\mathrm{i}}_N + \mathbf{c}^{\mathrm{i}}_{-m-N}\sigma^\Psi_m(A) = 0,$$

so the left error also drops from $-N$ to $-N-1$. In a chart, the components of the two choices of $\mathbf{c}_{-m-N}$ are exactly the right-hand sides of (15.52) and (15.53); the preceding global formulas prove, rather than presuppose, that these expressions glue.

Lemma 15.1.5, applied over a locally finite cover and followed by adjustment of proper support, sums the two families into operators B_{d} and B_{i}. The inductive steps show that $AB_{\mathrm{d}} - I_{\mathbf{F}}$ belongs to $\Psi^{-N}(M;\mathbf{F},\mathbf{F})$ and $B_{\mathrm{i}}A - I_{\mathbf{E}}$ to $\Psi^{-N}(M;\mathbf{E},\mathbf{E})$ for each N, so both are smoothing. The algebraic identity above shows that $B_{\mathrm{i}} - B_{\mathrm{d}} \in \Psi^{-\infty}(M;\mathbf{F},\mathbf{E})$ and that either operator is two-sided. This reproduces the local recurrences globally and proves (c)⇒(b). The implication (b)⇒(c) follows by taking principal symbols in $AB = I_{\mathbf{F}} - R_{\mathbf{F}}$ and $BA = I_{\mathbf{E}} - R_{\mathbf{E}}$. Equivalence of (a) and (c) is Proposition 15.5.2 applied in charts. If B and $\widetilde{B}$ are two-sided, the same identity above, with $B_{\mathrm{i}} = B$ and $B_{\mathrm{d}} = \widetilde{B}$, shows that $B - \widetilde{B}$ is smoothing. □

Remark 15.5.5. (A parametrix need not be an inverse) The theorem's identities hold modulo smooth kernels, not exactly. A parametrix exists because of high-frequency behavior; an inverse also requires injectivity and surjectivity in the space where A is realized. For example, on a closed connected manifold, the Laplacian is elliptic and has a parametrix, but constant functions belong to its kernel. It is therefore not invertible on all of $C^{\infty}(M)$. If a realization of A has a true inverse, that inverse agrees with any parametrix modulo the correction eliminating the smoothing remainders; this additional assertion is not part of symbolic ellipticity.

Corollary 15.5.6. (Qualitative elliptic regularity) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E},\mathbf{F} \to M$ be smooth complex vector bundles of the same finite rank. Let $A \in \Psi^{m}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic and properly supported. If $\mathbf{u} \in \mathcal{D}'(M,\mathbf{E})$ and $A\mathbf{u}$ is smooth on an open set $V \subseteq M$, then $\mathbf{u}$ is smooth on V. In particular,

$$\operatorname{singsupp}(A\mathbf{u}) = \operatorname{singsupp}(\mathbf{u}).$$

Proof. Let B be a two-sided parametrix. From $BA = I_{\mathbf{E}} - R_{\mathbf{E}}$ we obtain $\mathbf{u} = B(A\mathbf{u}) + R_{\mathbf{E}}\mathbf{u}$. The first summand is smooth on V by pseudolocality of B, and the second is smooth on all of M. This proves the first assertion. The reverse inclusion for singular supports is Theorem 15.2.5. □

15.5.3 The wavefront set on a Euclidean open set

Qualitative elliptic regularity detects the points where a distribution is smooth. To also record cotangent directions in which smoothness fails, we must localize in the base and examine Fourier decay within cones.

Write

$$\mathcal{E}'(\mathbb{R}^n) := \{v \in \mathcal{D}'(\mathbb{R}^n) \mid \operatorname{supp} v \text{ is compact}\}.$$

A compactly supported distribution v is tempered, and its Fourier transform is the smooth function

$$\widehat{v}(\xi) = (2\pi)^{-\frac{n}{2}} \langle v(x), e^{-ix\cdot\xi} \rangle.$$

Continuity of v on functions supported in a fixed compact set shows that $\widehat{v}$ and all its derivatives have at most polynomial growth. The wavefront set records the directions where this growth improves to rapid decay after localization in the spatial variable.

We say that two open cones $\Gamma', \Gamma \subseteq \mathbb{R}^n \setminus \{0\}$ satisfy $\Gamma' \Subset_{\mathrm{c}} \Gamma$ if

$$\overline{\Gamma' \cap \mathbb{S}^{n-1}} \subseteq \Gamma \cap \mathbb{S}^{n-1}.$$

Lemma 15.5.7. (Stability under shrinking the cone) Let $v \in \mathcal{E}'(\mathbb{R}^n)$ and suppose that, for every N, there exists C_N such that

$$|\widehat{v}(\xi)| \le C_N \langle \xi \rangle^{-N}, \qquad \xi \in \Gamma,$$

where Γ is an open cone. If $\psi \in C_c^\infty(\mathbb{R}^n)$ and $\Gamma' \Subset_{\mathrm{c}} \Gamma$, then $\widehat{\psi v}$ decays rapidly in Γ'.

Proof. With the unitary normalization,

$$\widehat{\psi v}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\psi}(\xi - \eta)\widehat{v}(\eta)\, d\eta.$$

The function $\widehat{v}$ has polynomial growth, and $\widehat{\psi}$ is a Schwartz function. For $\xi \in \Gamma'$, split the integral into $\eta \in \Gamma$ and $\eta \notin \Gamma$. In the first region, use the arbitrary decay of $\widehat{v}$ and Peetre's inequality in Lemma 6.1.1. In the second, angular separation of the cones gives a constant $c > 0$ such that

$$|\xi - \eta| \ge c(|\xi| + |\eta|)$$

if $\xi \in \Gamma'$ and $\eta \notin \Gamma$. The decay of $\widehat{\psi}$ then offsets any power of ξ as well as the polynomial growth of $\widehat{v}$. Both integrals are bounded by $C_N' \langle \xi \rangle^{-N}$ for every N. □

Definition 15.5.8. (Euclidean wavefront set) Let $U \subseteq \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(U)$. A point $(x_0, \xi_0) \in U \times (\mathbb{R}^n \setminus \{0\})$ is a **regular direction** of u if there exist $\varphi \in C_c^\infty(U)$ equal to one on a neighborhood of x_0 and an open cone Γ containing ξ_0 such that, for every $N \in \mathbb{N}_0$,

$$\sup_{\xi \in \Gamma} \langle \xi \rangle^N |\widehat{\varphi u}(\xi)| < \infty.$$

The **wavefront set** of u is the complement of the regular directions:

$$\mathrm{WF}(u) \subseteq U \times (\mathbb{R}^n \setminus \{0\}).$$

Lemma 15.5.7 shows that the definition is independent of the cutoff: if ψ is supported in the region where $\varphi = 1$, then $\psi u = \psi(\varphi u)$, and it suffices to shrink the cone.

Proposition 15.5.9. (Elementary properties of the wavefront set) Let $U \subseteq \mathbb{R}^n$ be open. For $u, v \in \mathcal{D}'(U)$ and $f \in C^\infty(U)$, the following hold:

(i) $\mathrm{WF}(u)$ is closed in $T^*U \setminus 0_U$ and conic in the covariable;

(ii)
$$\mathrm{WF}(u + v) \subseteq \mathrm{WF}(u) \cup \mathrm{WF}(v), \qquad \mathrm{WF}(f u) \subseteq \mathrm{WF}(u);$$

(iii) the projection $\pi \colon T^*U \longrightarrow U$ satisfies

$$\pi\big(\mathrm{WF}(u)\big) = \operatorname{sing\,supp}(u).$$

Proof. If a cutoff and a cone establish regularity at (x_0, ξ_0), the same function works for base points in a neighborhood of x_0, and the same cone works for directions near ξ_0. Thus the set of regular directions is open. Invariance under positive dilations is immediate and proves (i).
For a sum, choose a common cutoff and intersect two cones of regularity. For fu, take a cutoff ψ supported where the original cutoff equals one, and apply Lemma 15.5.7 to the smooth multiplier ψf. This proves (ii).
If u is smooth near x_0, a localization φu belongs to C_c^∞ and its transform is Schwartz in every direction. Thus x_0 does not belong to the projection of the wavefront set. Conversely, suppose no nonzero covariable over x_0 belongs to the wavefront set. Compactness of $\mathbb{S}^{n-1}$ allows finitely many cones of regularity to be chosen covering the sphere. Choose a single function φ equal to one near x_0 and supported in the intersection of the regions where the corresponding cutoffs equal one. The preceding lemma then shows that $\widehat{\varphi u}$ decays faster than any power in every direction. Thus $\xi^\alpha \widehat{\varphi u} \in L^1$ for every α. Fourier inversion in the distributional sense implies $\varphi u \in C^\infty$. Hence u is smooth near x_0, proving (iii). □

15.5.4 Invariance and the global definition

The Euclidean definition is expressed in frequencies, but the direction it records belongs to the cotangent bundle. We must check that a diffeomorphism transports these directions by its cotangent map and preserves rapid decay in the corresponding cones. The precise assertion is as follows.

Proposition 15.5.10. (Invariance under diffeomorphisms) Let $\kappa\colon U \longrightarrow V$ be a diffeomorphism between Euclidean open sets, and let $u \in \mathcal{D}'(U)$. For the transformed distribution $\kappa_* u \in \mathcal{D}'(V)$,

$$\mathrm{WF}(\kappa_* u) = \left\{ \left(\kappa(x), D\kappa(x)^{-T}\xi\right) \mid (x,\xi) \in \mathrm{WF}(u) \right\}.$$

Proof. Localize u on a compact set where κ is the chart under consideration. By Fourier inversion of the localized distribution, the Fourier transform of a localization of $\kappa_* u$ can be written as an integral in (x,ξ) with phase

$$x \cdot \xi - \kappa(x) \cdot \eta.$$

Its stationary points in x satisfy $\xi = D\kappa(x)^T \eta$. Suppose (x_0, ξ_0) is regular, and set $\eta_0 = D\kappa(x_0)^{-T}\xi_0$. Shrink the neighborhood of x_0 and the cones of ξ_0, η_0 so that every solution of the stationary relation remains within the cone where the localized transform of u decays rapidly. Use this decay in that region. In the complement, the gradient $\xi - D\kappa(x)^T\eta$ is bounded away from zero; repeated integration by parts with

$$L = \frac{1 - i(\xi - D\kappa(x)^T\eta)\cdot D_x}{1 + \|\xi - D\kappa(x)^T\eta\|^2}$$

produces any negative power of $\langle \eta \rangle$. Jacobians and cutoffs are smooth compactly supported amplitudes and do not change the estimates. Thus $(\kappa(x_0), \eta_0)$ is regular for $\kappa_* u$. Applying the same argument to κ^{-1} gives the reverse inclusion and proves equality. □

Definition 15.5.11. (Global wavefront set) Let M be a finite-dimensional smooth manifold without boundary, let $\mathbf{E} \to M$ be a smooth complex vector bundle of finite rank, and let $\mathbf{u} \in \mathcal{D}'(M, \mathbf{E})$. In a chart $\kappa\colon U \to \mathbb{R}^n$

and a local frame, write $\mathbf{u} = (u^1, \dots, u^r)$. Declare that $(x,\xi) \in T^*M \setminus 0_M$ does not belong to $\mathrm{WF}(\mathbf{u})$ if

$$\big(\kappa(x), D\kappa(x)^{-T}\xi\big) \notin \bigcup_{a=1}^{r} \mathrm{WF}(u^a).$$

Proposition 15.5.10 proves independence of the chart. A frame change multiplies the components by a smooth invertible matrix; Proposition 15.5.9(ii), applied in both directions, proves independence of the frame. Thus $\mathrm{WF}(\mathbf{u})$ is a closed conic subset of $T^*M \setminus 0_M$, and

$$\pi\big(\mathrm{WF}(\mathbf{u})\big) = \operatorname{sing\,supp}(\mathbf{u}).$$

The definition is also independent of the choice of a positive density for representing distributions in coordinates: two choices differ by a nonvanishing smooth function. In the Riemannian case, as elsewhere in Riemannian analysis, we use the measure $d\lambda_{\mathbf{g}}$. In this precise sense, the wavefront set is canonical: it is independent of the atlas, frame, metric, and auxiliary density.

15.5.5 Pseudodifferential action and microlocal regularity

Pseudolocality says that a pseudodifferential operator creates no singularities in the base. The wavefront set refines this assertion: the operator creates no new singular directions either. When the symbol is elliptic at a covector, a microlocal parametrix also recovers the datum's singular direction from that of its image.

Theorem 15.5.12. (Pseudodifferential operators create no wavefronts) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of finite rank. Let $A \in \Psi^m(M; \mathbf{E}, \mathbf{F})$ be properly supported. Then, for every $\mathbf{u} \in \mathcal{D}'(M, \mathbf{E})$,

$$\mathrm{WF}(A\mathbf{u}) \subseteq \mathrm{WF}(\mathbf{u}).$$

In particular, for every multi-index α and every smooth function f,

$$\mathrm{WF}(D^\alpha \mathbf{u}) \subseteq \mathrm{WF}(\mathbf{u}), \qquad \mathrm{WF}(f\mathbf{u}) \subseteq \mathrm{WF}(\mathbf{u})$$

in coordinates.

Proof. The assertion is local. In coordinates, fix $(x_0, \xi_0) \notin \mathrm{WF}(\mathbf{u})$ and write $A = \mathrm{Op}(a)$ modulo a smoothing operator. Choose $\psi \in C_c^\infty$ equal to one near x_0 and supported where a cutoff establishing regularity of (x_0, ξ_0) equals one, and decompose $\mathbf{u} = \psi\mathbf{u} + (1-\psi)\mathbf{u}$. The second summand yields a smooth function near x_0 because the kernel of A is smooth off the diagonal. For $\mathbf{v} = \psi\mathbf{u} \in \mathcal{E}'$ and $\varphi \in C_c^\infty$ supported near x_0, the quantization formula gives

$$\widehat{\varphi A \mathbf{v}}(\zeta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathcal{F}_{x\to\theta}\big(\varphi(x) a(x,\eta)\big)(\zeta - \eta)\widehat{\mathbf{v}}(\eta)\, d\eta. \tag{15.54}$$

Integrating by parts in x gives, for every L,

$$\big\| \mathcal{F}_{x\to\theta}\big(\varphi a(\cdot,\eta)\big)(\theta) \big\| \le C_L \langle\theta\rangle^{-L} \langle\eta\rangle^m.$$

Take a cone Γ where $\widehat{\mathbf{v}}$ decays rapidly and a cone $\Gamma' \Subset_{\mathrm{c}} \Gamma$ containing ξ_0. In (15.54), separate $\eta \in \Gamma$ and $\eta \notin \Gamma$. In the first part, decay of $\widehat{\mathbf{v}}$ absorbs the order m; in the second, use $|\zeta - \eta| \geq c(|\zeta| + |\eta|)$ for $\zeta \in \Gamma'$. Since L is arbitrary and $\widehat{\mathbf{v}}$ has polynomial growth, the right-hand side decays rapidly in Γ'. This proves the inclusion. The final two assertions are special cases, since derivatives and smooth multiplication are pseudodifferential operators. □

To invert only near a covector, it is useful to fix some terminology. Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be smooth complex vector bundles of finite rank. We say that $R \in \Psi^s_{\mathrm{loc}}(M; \mathbf{E}, \mathbf{F})$ is **microlocally smoothing** at $q = (x_0, \xi_0) \in T^*M \setminus 0_M$ if, in some chart and local frames of $\mathbf{E}$ and $\mathbf{F}$, the symbol of a localization of R satisfies, on a product $V \times \Gamma$ containing q,

$$\|D_x^\alpha D_\xi^\beta r(x,\xi)\| \leq C_{N,\alpha,\beta} \langle \xi \rangle^{-N}$$

for every N and large frequencies. The proof of the preceding theorem, using these bounds in the corresponding conic part and cone separation in its complement, gives

$$q \notin \mathrm{WF}(R\mathbf{u}) \qquad \text{for every } \mathbf{u} \in \mathcal{D}'(M, \mathbf{E}). \tag{15.55}$$

Let us check that this condition is independent of the chart. Let $\kappa \colon U \to \widetilde{U}$ be a coordinate change, write

$$\mathcal{B}_\kappa(x,y) := \int_0^1 D\kappa\big(y + t(x-y)\big)\, dt$$

, and suppose r satisfies the preceding bounds near $(\kappa(x_0), \eta_0)$, where $\eta_0 = D\kappa(x_0)^{-T}\xi_0$. After inserting a cutoff $\rho(x,y)$ equal to one near the diagonal, the formula in Theorem 15.2.10 gives the transformed amplitude

$$\widetilde{b}(x,y,\xi) = \rho(x,y) r\big(\kappa(x), \mathcal{B}_\kappa(x,y)^{-T}\xi\big) \frac{|\det D\kappa(y)|}{|\det \mathcal{B}_\kappa(x,y)|}. \tag{15.56}$$

Frame changes merely add smooth matrices on the left and right of this expression.

By continuity of $\mathcal{B}_\kappa^{-T}$ on unit directions, one can choose cones $\Gamma_2 \Subset_{\mathrm{c}} \Gamma_1$ around ξ_0 and a neighborhood U_0 of x_0 such that

$$\mathcal{B}_\kappa(x,y)^{-T}\xi \in \Gamma, \qquad c\langle \xi \rangle \leq \langle \mathcal{B}_\kappa(x,y)^{-T}\xi \rangle \leq C\langle \xi \rangle$$

for $x, y \in U_0$, $\xi \in \Gamma_1$, and large frequencies. Fix an angular cutoff ϑ supported in Γ_1 and equal to one on Γ_2. The chain rule shows that each derivative of $\vartheta(\xi)\widetilde{b}(x,y,\xi)$ is a finite sum of terms

$$c_{\mu,\nu,\lambda}(x,y)\, \xi^\mu (D_z^\nu D_\eta^\lambda r)\big(\kappa(x), \mathcal{B}_\kappa(x,y)^{-T}\xi\big),$$

and terms of the same form with a derivative of ϑ. The coefficients $c_{\mu,\nu,\lambda}$ and all their derivatives are bounded after shrinking U_0. Given L, apply the bound on r with exponent $L + |\mu|$; the preceding comparability gives, for all α, γ, β,

$$\|D_x^\alpha D_y^\gamma D_\xi^\beta (\vartheta \widetilde{b})(x,y,\xi)\| \leq C_{L,\alpha,\gamma,\beta} \langle \xi \rangle^{-L} \tag{15.57}$$

on $U_0 \times U_0 \times \Gamma_2$.

When reducing (15.56), each term

$$\frac{i^{-|\delta|}}{\delta!} D_\xi^\delta D_y^\delta \widetilde{b}(x,y,\xi)\big|_{y=x}$$

satisfies (15.57) for any L. If s is a global order of r, the remainder after summing all terms with $|\delta| < J$ satisfies symbol estimates of order $s - J$ by Theorem 15.1.9. For a prescribed bound $\langle\xi\rangle^{-L}$, choose $J > s + L$; the remainder then also satisfies that bound in the smaller cone. Thus the transformed symbol satisfies estimates of order $-L$ on $U_0 \times \Gamma_2$ for every L. Applying the same calculation to κ^{-1} proves equivalence of the two conditions. The smooth frame and density factors already considered change none of the bounds. This proves independence of the chart and auxiliary trivializations.

Proposition 15.5.13. (Microlocal parametrix) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of the same finite rank. If $q \in \mathrm{Ell}(A)$ for $A \in \Psi^m_{\mathrm{cl,loc}}(M; \mathbf{E}, \mathbf{F})$, there exist $B \in \Psi^{-m}_{\mathrm{cl,loc}}(M; \mathbf{F}, \mathbf{E})$ and operators $R_{\mathbf{E}}, R_{\mathbf{F}}$ microlocally smoothing at q such that

$$BA = I_{\mathbf{E}} - R_{\mathbf{E}}, \qquad AB = I_{\mathbf{F}} - R_{\mathbf{F}}$$

microlocally at q.

Proof. Choose a chart $\kappa\colon U \longrightarrow \Omega \subseteq \mathbb{R}^n$ and local frames of common rank r. In this proof write

$$S^j := S^j_{1,0}\big(\Omega; \mathrm{End}(\mathbb{C}^r)\big).$$

Choose neighborhoods $V_2 \Subset V_1 \Subset \Omega$ of the base point and cones $\Gamma_2 \Subset_{\mathrm{c}} \Gamma_1$ around the covariable of q so that a_m is invertible on $V_1 \times \Gamma_1$. Let $\chi(x,\xi)$ be a cutoff of degree zero at large frequencies, supported in $V_1 \times \Gamma_1$ and equal to one on $V_2 \times \Gamma_2$. Set

$$b^{\mathrm{d}}_{-m} = b^{\mathrm{i}}_{-m} = \chi a_m^{-1}.$$

Since all derivatives of χ vanish on the smaller product, there $a \# b^{\mathrm{d}}_{-m} - I_{\mathbf{F}} \in S^{-1}$ and $b^{\mathrm{i}}_{-m} \# a - I_{\mathbf{E}} \in S^{-1}$.
The right induction statement is that, after constructing b^{d}_{-m-k} for $0 \le k < N$,

$$a \# \sum_{k=0}^{N-1} b^{\mathrm{d}}_{-m-k} - I_{\mathbf{F}} \in S^{-N} \qquad \text{on } V_2 \times \Gamma_2.$$

Let e^{d}_N be the error's homogeneous component of degree $-N$, computed on $V_1 \times \Gamma_1$. Define

$$b^{\mathrm{d}}_{-m-N} = -\chi a_m^{-1} e^{\mathrm{d}}_N.$$

On $V_2 \times \Gamma_2$, the new component of degree $-N$ is $e^{\mathrm{d}}_N + a_m b^{\mathrm{d}}_{-m-N} = 0$; the composition formula therefore gives an error in S^{-N-1}. This is exactly the step in (15.52) and proves the right induction.
For the left side, if e^{i}_N is the component of degree $-N$ of

$$\Big(\sum_{k=0}^{N-1} b^{\mathrm{i}}_{-m-k}\Big) \# a - I_{\mathbf{E}},$$

computed on $V_1 \times \Gamma_1$, take $b^{\mathrm{i}}_{-m-N} = -\chi e^{\mathrm{i}}_N a_m^{-1}$. The equality $e^{\mathrm{i}}_N + b^{\mathrm{i}}_{-m-N} a_m = 0$ again reduces the error to S^{-N-1} and reproduces (15.53).
Lemma 15.1.5 sums both families into symbols $b^{\mathrm{d}}, b^{\mathrm{i}} \in S^{-m}$. For each N, both errors belong to S^{-N} on

$V_2 \times \Gamma_2$, so they are microlocally smoothing at q. At the operator level,

$$B_{\mathrm{i}} - B_{\mathrm{d}} = B_{\mathrm{i}}(I_{\mathbf{F}} - AB_{\mathrm{d}}) - (I_{\mathbf{E}} - B_{\mathrm{i}}A)B_{\mathrm{d}}.$$

The composition expansion shows that composing a finite-order symbol with one belonging to S^{-N} on the smaller product for every N preserves this property, after shrinking the cones once more. Thus $B_{\mathrm{i}} - B_{\mathrm{d}}$ is microlocally smoothing, and $B := B_{\mathrm{d}}$ also satisfies the left identity. Quantizing, inserting spatial cutoffs, and arranging proper support yields the two asserted identities; these changes add only smooth kernels near q. □

Theorem 15.5.14. (Microlocal elliptic regularity) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of the same finite rank. Let $A \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{F})$ be properly supported. For every $\mathbf{u} \in \mathcal{D}'(M, \mathbf{E})$,

$$\mathrm{WF}(A\mathbf{u}) \subseteq \mathrm{WF}(\mathbf{u}) \subseteq \mathrm{WF}(A\mathbf{u}) \cup \mathrm{Char}(A).$$

Equivalently,

$$\mathrm{WF}(A\mathbf{u}) \cap \mathrm{Ell}(A) = \mathrm{WF}(\mathbf{u}) \cap \mathrm{Ell}(A). \tag{15.58}$$

If A is elliptic, then $\mathrm{WF}(A\mathbf{u}) = \mathrm{WF}(\mathbf{u})$.

Proof. The first inclusion is Theorem 15.5.12. Now let $q \in \mathrm{Ell}(A)$ and suppose $q \notin \mathrm{WF}(A\mathbf{u})$. A microlocal parametrix gives $\mathbf{u} = B(A\mathbf{u}) + R_{\mathbf{E}}\mathbf{u}$ near q. The first summand has no wavefront at q by the preceding theorem, and neither does the second by (15.55). Thus $q \notin \mathrm{WF}(\mathbf{u})$, which is the second inclusion. The remaining assertions reformulate these two inclusions. □

The wavefront set can also be defined entirely in the language of the pseudodifferential calculus.

Theorem 15.5.15. (Characterization by operators of order zero) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E} \to M$ be a smooth complex vector bundle of finite rank. Let $\mathbf{u} \in \mathcal{D}'(M, \mathbf{E})$ and $q \in T^*M \setminus 0_M$. The following are equivalent:

(a) $q \notin \mathrm{WF}(\mathbf{u})$;

(b) there exists $Q \in \Psi^0_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{E})$, properly supported and elliptic at q, such that $Q\mathbf{u} \in \Gamma(M, \mathbf{E})$.

Consequently,

$$\mathrm{WF}(\mathbf{u}) = \bigcap \big\{ \mathrm{Char}(Q) \mid Q \in \Psi^0_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{E}),\ Q\mathbf{u} \in \Gamma(M, \mathbf{E}) \big\}. \tag{15.59}$$

The intersection may be restricted to operators whose principal symbol is a scalar multiple of the identity of $\mathbf{E}$.

Proof. Suppose (a) holds. In a chart and local frame, choose a function φ equal to one near the base point and a cone Γ in which $\widehat{\varphi \mathbf{u}}$ decays rapidly. Let χ be a spatial cutoff equal to one on a smaller neighborhood, and ϑ a homogeneous cutoff of degree zero, angularly supported in Γ and equal to one near the ray of q at large frequencies. Quantize

$$q_0(x, \xi) = \chi(x)\vartheta(\xi) I_{\mathbf{E}}$$

and arrange proper support. In the region where q_0 contributes, the function $\vartheta\widehat{\varphi\mathbf{u}}$ decays rapidly. Fourier inversion and estimate (15.54) show that $Q\mathbf{u}$ is smooth; the parts separated from the diagonal are also smooth. Moreover, the principal symbol of Q is the identity near q, so Q is elliptic there. This proves (b). If (b) holds, microlocal elliptic regularity applied to Q gives

$$\mathrm{WF}(\mathbf{u}) \cap \mathrm{Ell}(Q) = \mathrm{WF}(Q\mathbf{u}) \cap \mathrm{Ell}(Q) = \varnothing.$$

Since $q \in \mathrm{Ell}(Q)$, we obtain (a). Formula (15.59) is the set-theoretic expression of this equivalence. The construction in the first paragraph already uses a scalar symbol times $I_{\mathbf{E}}$, proving the last assertion. □

15.5.6 The wavefront set of the kernel

The conormal description of the kernel and the wavefront-set definition must agree on the diagonal. The following result identifies the possible directions exactly and explains, in microlocal terms, why a pseudodifferential operator relates a covector only to its negative in the input variable.

Theorem 15.5.16. (Wavefront set of a pseudodifferential kernel) Let M be a finite-dimensional smooth manifold without boundary, and let $\mathbf{E}, \mathbf{F} \to M$ be smooth complex vector bundles of finite rank. If $A \in \Psi^{m}_{\mathrm{cl,loc}}(M; \mathbf{E}, \mathbf{F})$ and $\mathbf{K}_A$ is its Schwartz kernel, then

$$\mathrm{WF}(\mathbf{K}_A) \subseteq N^{*}\Delta_M \setminus 0_{M\times M} = \big\{(x,\xi;x,-\xi) \mid (x,\xi) \in T^{*}M \setminus 0_M\big\}. \tag{15.60}$$

Proof. By Theorem 15.3.9, in a chart $\kappa\colon U \longrightarrow \Omega \subseteq \mathbb{R}^n$ and local frames of ranks $r_{\mathbf{E}}, r_{\mathbf{F}}$, the kernel has the form, modulo a smooth function,

$$K(x,y) = (2\pi)^{-n}\,\mathrm{Os}\int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi}\, b(x,y,\xi)\,d\xi, \qquad b\colon \Omega\times\Omega\times\mathbb{R}^n_\xi \longrightarrow \mathrm{Hom}(\mathbb{C}^{r_{\mathbf{E}}}, \mathbb{C}^{r_{\mathbf{F}}}),$$

where b is a classical amplitude of order m. We already know that K is smooth if $x \neq y$. Now localize near a point of the diagonal with $\rho \in C_c^{\infty}$ and take the Fourier transform in (x,y). Up to a constant depending only on n,

$$\widehat{\rho K}(\zeta,\eta) = \int_{\mathbb{R}^n} \mathcal{F}_{(x,y)\to(\theta,\omega)}\big(\rho b(\cdot,\cdot,\xi)\big)(\zeta-\xi,\eta+\xi)\,d\xi. \tag{15.61}$$

Integration by parts in x and y gives, for every N,

$$\big\|\mathcal{F}_{x,y}(\rho b)(\theta,\omega,\xi)\big\| \le C_N \langle(\theta,\omega)\rangle^{-N}\langle\xi\rangle^{m}.$$

Consider a closed cone of covectors (ζ,η) disjoint from the conormal bundle, that is, one in which

$$|\zeta+\eta| \ge c\big(|\zeta|+|\eta|\big).$$

With $q = \frac{\zeta-\eta}{2}$ and $\xi = q+\tau$, we have the identity

$$|\zeta-\xi|^2 + |\eta+\xi|^2 = 2|\tau|^2 + \tfrac{1}{2}|\zeta+\eta|^2.$$

In the cone under consideration, $|q| \le C|\zeta+\eta|$. Inserting the preceding estimate into (15.61), choosing N

arbitrarily large, and integrating in τ gives, for every L,

$$|\widehat{\rho K}(\zeta,\eta)| \le C_L \langle(\zeta,\eta)\rangle^{-L}.$$

By Definition 15.5.8, no covector with $\zeta+\eta\neq 0$ belongs to the wavefront set over the diagonal. The remaining covectors are exactly $(x,\xi;x,-\xi)$, in accordance with (15.16). Invariance under coordinate changes and multiplication by the smooth factors arising from frames and a density globalize the inclusion. □

If one adopts the convention of changing sign in the input covariable,

$$\mathrm{WF}'(\mathbf{K}_A) := \big\{(x,\xi;y,\eta)\,|\,(x,\xi;y,-\eta)\in \mathrm{WF}(\mathbf{K}_A)\big\},$$

the inclusion (15.60) becomes

$$\mathrm{WF}'(\mathbf{K}_A) \subseteq \big\{(x,\xi;x,\xi)\,|\,(x,\xi)\in T^*M\setminus 0_M\big\}.$$

Thus a pseudodifferential kernel transmits a singularity only at the same point and in the same cotangent direction. This assertion contains qualitative pseudolocality and, together with the microlocal parametrix, explains the elliptic equality (15.58).

The symbolic construction of the parametrix and the conormal inclusion for the kernel's wavefront set correspond, under the translation $D=\partial$, to Theorems 18.1.9, 18.1.16, and 18.1.24 of [84]. These proofs make explicit the recurrence, control of remainders, and passage from Fourier analysis to the global formulation.

15.6 Action on Sobolev spaces, uniform calculus, and Fredholm theory

The symbolic calculus acquires analytic content when translated into estimates between Sobolev spaces. In this section we write

$$H^s(M,\mathbf{E}) := H^{s,2}(M,\mathbf{E}), \qquad s\in\mathbb{R},$$

and use the bundle metrics and measure $d\lambda_{\mathbf{g}}$ to define the inner products on $L^2(M,\mathbf{E})$ and $L^2(M,\mathbf{F})$. In the complex case, we retain the previously fixed convention: the Hermitian inner product is linear in its first variable. We first prove the Euclidean order-zero estimate, then transfer it to manifolds, carefully distinguish the uniform noncompact case from the closed case, and deduce the Fredholm property.

15.6.1 The Calderón–Vaillancourt estimate

The decisive point is that an order-zero symbol defines a bounded operator between $L^2(\mathbb{R}^n,\mathbb{C}^r)$ and $L^2(\mathbb{R}^n,\mathbb{C}^s)$ even though its kernel need not be absolutely integrable. We give a proof using wave packets, because it exhibits finitely many seminorms and uses neither interpolation nor spectral theory.

Theorem 15.6.1. (Calderón–Vaillancourt) Let $n,r,s\in\mathbb{N}$, let

$$a\in S^0_{1,0}\big(\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^s)\big)$$

, and let $N_{\mathrm{CV}} := 4n+4$. There exists a constant $C=C(n,r,s)>0$ such that

$$\|\mathrm{Op}(a)u\|_{L^2(\mathbb{R}^n,\mathbb{C}^s)} \le C\, q_{N_{\mathrm{CV}},0}(a)\,\|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^r)} \tag{15.62}$$

for every $u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^r)$. Consequently, $\mathrm{Op}(a)$ extends uniquely to a continuous operator from $L^2(\mathbb{R}^n, \mathbb{C}^r)$ to $L^2(\mathbb{R}^n, \mathbb{C}^s)$.

Proof. It suffices to treat the scalar case: in finite rank, apply the argument to each matrix entry. Fix a nonzero $g \in C_c^\infty(\mathbb{R}^n)$ and define

$$g_{z,\zeta}(x) := e^{ix\cdot\zeta} g(x-z), \qquad (Wu)(z,\zeta) := \langle u, g_{z,\zeta}\rangle_{L^2(\mathbb{R}^n)}, \qquad (z,\zeta) \in \mathbb{R}^{2n}.$$

Plancherel's identity, applied first in ζ and then in z, gives Moyal's identity

$$\|Wu\|^2_{L^2(\mathbb{R}^{2n})} = (2\pi)^n \|g\|^2_{L^2(\mathbb{R}^n)} \|u\|^2_{L^2(\mathbb{R}^n)}. \tag{15.63}$$

By polarization, $W^*W = c_g I$, where $c_g = (2\pi)^n \|g\|^2_{L^2(\mathbb{R}^n)}$.
Consider the matrix kernel of $W\,\mathrm{Op}(a)W^*$:

$$\mathcal{M}_a(z,\zeta;w,\eta) := \langle \mathrm{Op}(a) g_{w,\eta}, g_{z,\zeta}\rangle.$$

The quantization formula of Definition 15.1.6 writes it as an oscillatory integral in (x,y,θ) with phase

$$\Phi = x\cdot(\theta-\zeta) + y\cdot(\eta-\theta)$$

and amplitude

$$(2\pi)^{-n} a(x,\theta) g(y-w)\overline{g(x-z)}.$$

Set $L = n+1$. Since

$$(1-\Delta_x)^L e^{i\Phi} = \langle \theta-\zeta\rangle^{2L} e^{i\Phi}, \qquad (1-\Delta_y)^L e^{i\Phi} = \langle \theta-\eta\rangle^{2L} e^{i\Phi},$$

, integrate by parts L times in x and in y. Derivatives in x hit a and $g(x-z)$; derivatives in y hit only $g(y-w)$. On the supports of these two factors,

$$\langle z-w\rangle \le C_g \langle x-y\rangle.$$

To introduce this weight without changing the oscillatory integral, expand the polynomial $\langle z-w\rangle^{2L} = (1+|z-w|^2)^L$ using $z-w = (x-y)-(x-z)+(y-w)$. The factors in $x-z$ and $y-w$ are incorporated into the smooth compactly supported functions and their derivatives. The remaining factors are monomials $(x-y)^\gamma$ with $|\gamma| \le 2L$; use

$$(x-y)^\gamma e^{i\Phi} = i^{-|\gamma|} D_\theta^\gamma e^{i\Phi}$$

to integrate by parts in θ. Derivatives of the weights $\langle\theta-\zeta\rangle^{-2L}$ and $\langle\theta-\eta\rangle^{-2L}$ do not worsen their decay, and those hitting the symbol are controlled by $q_{4L,0}(a)$. This yields

$$|\mathcal{M}_a(z,\zeta;w,\eta)| \le C_g q_{4L,0}(a) \langle z-w\rangle^{-2L} \int_{\mathbb{R}^n} \langle\theta-\zeta\rangle^{-2L}\langle\theta-\eta\rangle^{-2L}\, d\theta.$$

Since $2L > n$, the elementary convolution inequality, together with Peetre's weight comparison in Lemma 6.1.1, gives

$$\int_{\mathbb{R}^n} \langle\theta-\zeta\rangle^{-2L}\langle\theta-\eta\rangle^{-2L}\, d\theta \le C_L \langle\zeta-\eta\rangle^{-2L}.$$

This inequality is proved by splitting the domain into the regions $\|\theta-\zeta\| \geq \frac{\|\zeta-\eta\|}{2}$ and $\|\theta-\eta\| \geq \frac{\|\zeta-\eta\|}{2}$. Thus

$$|\mathcal{M}_a(z,\zeta;w,\eta)| \leq C q_{4n+4,0}(a)\langle z-w\rangle^{-2n-2}\langle \zeta-\eta\rangle^{-2n-2}. \tag{15.64}$$

The right-hand side is integrable in (w,η) uniformly in (z,ζ), and also in (z,ζ) uniformly in (w,η). Proposition B.8.3, applied with $X = Y = \mathbb{R}^{2n}$, Lebesgue measure, and $r = s = 1$, shows that the integral operator with kernel $\mathcal{M}_a$ is bounded on $L^2(\mathbb{R}^{2n})$. Finally,

$$\operatorname{Op}(a) = c_g^{-2} W^*\big(W \operatorname{Op}(a) W^*\big) W,$$

and (15.63) yield (15.62). Density of $\mathcal{S}(\mathbb{R}^n,\mathbb{C}^r)$ in $L^2(\mathbb{R}^n,\mathbb{C}^r)$ gives the unique extension. □

The integer $4n+4$ is a sufficient choice using only finitely many derivatives. For other choices and proofs, see [140, 84, 135].

15.6.2 Action on Sobolev scales

Recall that

$$J^t = \operatorname{Op}(\langle\xi\rangle^t)\colon H^t(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

is an isometric isomorphism with the unitary normalization fixed in Definition 4.11.1 and Theorem 4.11.10.

Theorem 15.6.2. (Euclidean Sobolev mapping) Let $n, r, q \in \mathbb{N}$, $m, \sigma \in \mathbb{R}$, and

$$a \in S^m_{1,0}\big(\mathbb{R}^n; \operatorname{Hom}(\mathbb{C}^r,\mathbb{C}^q)\big).$$

. Then

$$\operatorname{Op}(a)\colon H^{\sigma+m}(\mathbb{R}^n,\mathbb{C}^r) \longrightarrow H^{\sigma}(\mathbb{R}^n,\mathbb{C}^q)$$

is continuous. Its norm is controlled by finitely many seminorms of a in $S^m_{1,0}(\mathbb{R}^n;\operatorname{Hom}(\mathbb{C}^r,\mathbb{C}^q))$.

Proof. The composition calculus of Theorem 15.2.6 shows that

$$B := J^{\sigma} \operatorname{Op}(a) J^{-(\sigma+m)}$$

is a pseudodifferential operator of order zero. The seminorms of a fixed number of terms in its symbol and of the remainder are controlled by finitely many seminorms of a. Theorem 15.6.1 gives

$$\|\operatorname{Op}(a)u\|_{H^{\sigma}(\mathbb{R}^n,\mathbb{C}^q)} = \|J^{\sigma}\operatorname{Op}(a)u\|_{L^2(\mathbb{R}^n,\mathbb{C}^q)} \leq C\|J^{\sigma+m}u\|_{L^2(\mathbb{R}^n,\mathbb{C}^r)} = C\|u\|_{H^{\sigma+m}(\mathbb{R}^n,\mathbb{C}^r)}.$$

□

Before passing to manifolds, it is useful to isolate the other term that appears when localizing an operator: a smooth compactly supported kernel. Unlike the pseudodifferential part, this term improves regularity by arbitrarily many derivatives. The following estimate specifies how many kernel derivatives suffice to control each Sobolev mapping.

Lemma 15.6.3. (Sobolev mapping of a smooth compactly supported kernel) Let $U, V \subseteq \mathbb{R}^n$ be bounded open sets, and let $K \Subset U \times V$, $r, s \in \mathbb{N}$, and

$$k \in C^\infty\big(U \times V; \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^s)\big), \qquad \operatorname{supp} k \subseteq K.$$

. For $\sigma, \tau \in \mathbb{R}$, define

$$a_\sigma := \max\{0, \lceil \sigma \rceil\}, \qquad b_\tau := \max\{0, \lceil -\tau \rceil\}, \qquad N(n, \sigma, \tau) := a_\sigma + b_\tau. \tag{15.65}$$

For $u \in H^\tau(V, \mathbb{C}^r)$, choose an extension $\widetilde{u} \in H^\tau(\mathbb{R}^n, \mathbb{C}^r)$ of u. For each $x \in U$, define $T_k u(x) \in \mathbb{C}^s$ by

$$\langle T_k u(x), w \rangle_{\mathbb{C}^s} := \langle \widetilde{u}, k(x, \cdot)^* w \rangle_{H^\tau(\mathbb{R}^n, \mathbb{C}^r), H^{-\tau}(\mathbb{R}^n, \mathbb{C}^r)}, \qquad w \in \mathbb{C}^s, \tag{15.66}$$

where $k(x, \cdot)^* w$ is extended by zero outside V and the pairing is linear in the first variable, in accordance with this chapter's Hermitian convention. This definition is independent of the chosen extension and determines a continuous linear operator

$$T_k \colon H^\tau(V, \mathbb{C}^r) \longrightarrow H^\sigma(U, \mathbb{C}^s).$$

The map $x \mapsto T_k u(x)$ is smooth and, for every multi-index α,

$$\big\langle D_x^\alpha (T_k u)(x), w \big\rangle_{\mathbb{C}^s} = \big\langle \widetilde{u}, \big(D_x^\alpha k\big)(x, \cdot)^* w \big\rangle_{H^\tau(\mathbb{R}^n, \mathbb{C}^r), H^{-\tau}(\mathbb{R}^n, \mathbb{C}^r)}. \tag{15.67}$$

If u has a representative in $L^1_{\mathrm{loc}}(V, \mathbb{C}^r)$, then

$$(T_k u)(x) = \int_V k(x, y) u(y)\, dy, \qquad x \in U.$$

Moreover, there is a constant $C = C(n, \sigma, \tau, U, V, K, r, s) > 0$ independent of k such that

$$\|T_k\|_{\mathcal{L}(H^\tau(V, \mathbb{C}^r), H^\sigma(U, \mathbb{C}^s))} \le C \max_{|\alpha| + |\beta| \le N(n, \sigma, \tau)} \sup_{(x, y) \in K} \|D_x^\alpha D_y^\beta k(x, y)\|. \tag{15.68}$$

The choice of N in (15.65) is sufficient; smaller values may be valid but are not needed for later applications.

Proof. Spaces on open sets are understood with the quotient restriction norm of Definition 6.1.5, applied componentwise. Let $K_U := \operatorname{pr}_1(K) \Subset U$ and $K_V := \operatorname{pr}_2(K) \Subset V$. For $x \in U$ and $w \in \mathbb{C}^s$,

$$k(x, \cdot)^* w \in C_c^\infty(V, \mathbb{C}^r), \qquad \operatorname{supp}\big(k(x, \cdot)^* w\big) \subseteq K_V.$$

Its extension by zero belongs to $C_c^\infty(\mathbb{R}^n, \mathbb{C}^r)$ and, by Proposition 6.1.6, to $H^{-\tau}(\mathbb{R}^n, \mathbb{C}^r)$. The right-hand side of (15.66) is therefore well defined. Since the Hermitian inner product is linear in its first variable, this expression is antilinear in w. If $(e_j)_{j=1}^s$ is an orthonormal basis of $\mathbb{C}^s$, the vector

$$\sum_{j=1}^{s} \big\langle \widetilde{u}, k(x, \cdot)^* e_j \big\rangle_{H^\tau(\mathbb{R}^n, \mathbb{C}^r), H^{-\tau}(\mathbb{R}^n, \mathbb{C}^r)}\, e_j$$

satisfies (15.66) for every $w \in \mathbb{C}^s$. Nondegeneracy of the Hermitian inner product shows that it is the unique vector with this property.

Let us check independence of the extension. If $\widetilde{u}_1$ and $\widetilde{u}_2$ restrict to u on V, then $d := \widetilde{u}_1 - \widetilde{u}_2$ vanishes as a distribution on V. Since $k(x,\cdot)^* w$ has compact support contained in V, the Sobolev pairing here agrees with the distributional action, and

$$\langle d, k(x,\cdot)^* w \rangle_{H^{\tau}(\mathbb{R}^n,\mathbb{C}^r),H^{-\tau}(\mathbb{R}^n,\mathbb{C}^r)} = 0.$$

Thus (15.66) is independent of $\widetilde{u}$.

For each $w \in \mathbb{C}^s$, the map

$$x \longmapsto k(x,\cdot)^* w$$

is smooth from U into the Fréchet space $C^{\infty}_{K_V}(\mathbb{R}^n,\mathbb{C}^r)$ of smooth functions supported in K_V. The inclusion of this space into $H^{-\tau}(\mathbb{R}^n,\mathbb{C}^r)$ is continuous. Composing with the pairing against $\widetilde{u}$ gives a smooth function of x and allows differentiation within the duality. Taking w successively in an orthonormal basis of $\mathbb{C}^s$ shows that $x \mapsto T_k u(x)$ is smooth and gives (15.67) for every multi-index α.

If $u \in L^1_{\mathrm{loc}}(V,\mathbb{C}^r)$, pairing with a test function recovers the Lebesgue integral. By definition of the adjoint and the Hermitian convention,

$$\begin{aligned} \langle T_k u(x), w \rangle_{\mathbb{C}^s} &= \int_V \big\langle u(y), k(x,y)^* w \big\rangle_{\mathbb{C}^r}\, dy \\ &= \int_V \big\langle k(x,y) u(y), w \big\rangle_{\mathbb{C}^s}\, dy. \end{aligned}$$

Since this holds for every $w \in \mathbb{C}^s$, the duality definition agrees with the usual integral asserted in the statement.

It remains to prove the estimate. Fix $\varepsilon > 0$ and choose the extension so that

$$\|\widetilde{u}\|_{H^{\tau}(\mathbb{R}^n,\mathbb{C}^r)} \le \|u\|_{H^{\tau}(V,\mathbb{C}^r)} + \varepsilon.$$

If $|\alpha| \le a_\sigma$ and $\|w\|_{\mathbb{C}^s} = 1$, duality, $b_\tau \ge -\tau$, and (15.67) give

$$\begin{aligned} \big|\big\langle D_x^{\alpha}(T_k u)(x), w \big\rangle_{\mathbb{C}^s}\big| &\le \|\widetilde{u}\|_{H^{\tau}(\mathbb{R}^n,\mathbb{C}^r)} \big\| \big(D_x^{\alpha} k\big)(x,\cdot)^* w \big\|_{H^{-\tau}(\mathbb{R}^n,\mathbb{C}^r)} \\ &\le \|\widetilde{u}\|_{H^{\tau}(\mathbb{R}^n,\mathbb{C}^r)} \big\| \big(D_x^{\alpha} k\big)(x,\cdot)^* w \big\|_{H^{b_\tau}(\mathbb{R}^n,\mathbb{C}^r)}. \end{aligned}$$

For $q \in \mathbb{N}_0$, the polynomial comparison

$$\langle \xi \rangle^{2q} \le C_{n,q} \sum_{|\beta| \le q} |\xi^{\beta}|^2$$

and Plancherel's theorem 4.11.10, applied componentwise, imply

$$\|f\|^2_{H^q(\mathbb{R}^n,\mathbb{C}^r)} \le C_{n,q} \sum_{|\beta| \le q} \|D^{\beta} f\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^r)}.$$

Since all preceding test functions are supported in K_V, this inequality with $q = b_\tau$ gives

$$\|\big(D_x^\alpha k\big)(x,\cdot)^* w\|_{H^{b_\tau}(\mathbb{R}^n,\mathbb{C}^r)} \le C_0 \max_{|\beta|\le b_\tau} \sup_{(z,y)\in K} \|D_z^\alpha D_y^\beta k(z,y)\|. \tag{15.69}$$

Here C_0 depends on n, b_τ, r, s and the volume of K_V, but not on x, k, u or w.
Taking the supremum over unit vectors w, integrating in $x \in K_U$, summing over $|\alpha| \le a_\sigma$, and applying the same Plancherel comparison with $q = a_\sigma$ yields

$$\|T_k u\|_{H^{a_\sigma}(\mathbb{R}^n,\mathbb{C}^s)} \le C_1 \|\widetilde{u}\|_{H^\tau(\mathbb{R}^n,\mathbb{C}^r)} \max_{\substack{|\alpha|\le a_\sigma \\ |\beta|\le b_\tau}} \sup_{(x,y)\in K} \|D_x^\alpha D_y^\beta k(x,y)\|.$$

In this formula, extend $T_k u$ by zero outside U; the extension is smooth because $\operatorname{supp}(T_k u) \subseteq K_U \Subset U$. Since $a_\sigma \ge \sigma$, the inclusion $H^{a_\sigma}(\mathbb{R}^n,\mathbb{C}^s) \hookrightarrow H^\sigma(\mathbb{R}^n,\mathbb{C}^s)$ and restriction to U are continuous. Finally, $|\alpha| + |\beta| \le a_\sigma + b_\tau = N(n,\sigma,\tau)$; let $\varepsilon \to 0^+$ to obtain (15.68). □

The local version requires a distinction between compact support and local regularity. For a compact set $K \Subset M$, set

$$H^t_K(M,\mathbf{E}) := \{\mathbf{u} \in H^t_{\mathrm{loc}}(M,\mathbf{E}) \mid \operatorname{supp}\mathbf{u} \subseteq K\}, \qquad H^t_{\mathrm{comp}}(M,\mathbf{E}) := \bigcup_{K\Subset M} H^t_K(M,\mathbf{E}),$$

with the usual inductive topology. The topology of $H^t_{\mathrm{loc}}(M,\mathbf{E})$ is defined by the seminorms $\|\chi\mathbf{u}\|_{H^t(M,\mathbf{E})}$, $\chi \in C^\infty_c(M)$, in accordance with Definition 6.1.7.

Proposition 15.6.4. (Local and compact-support versions) Let $(M,\mathbf{g})$ be a finite-dimensional smooth Riemannian manifold without boundary, not necessarily complete, and let $\mathbf{E},\mathbf{F} \to M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and metric connections. Let $m, s \in \mathbb{R}$ and $A \in \Psi^m_{\mathrm{loc}}(M;\mathbf{E},\mathbf{F})$. Then

$$A\colon H^{s+m}_{\mathrm{comp}}(M,\mathbf{E}) \longrightarrow H^s_{\mathrm{loc}}(M,\mathbf{F})$$

is continuous. If A is properly supported, there are also continuous extensions

$$A\colon H^{s+m}_{\mathrm{comp}}(M,\mathbf{E}) \longrightarrow H^s_{\mathrm{comp}}(M,\mathbf{F}), \qquad A\colon H^{s+m}_{\mathrm{loc}}(M,\mathbf{E}) \longrightarrow H^s_{\mathrm{loc}}(M,\mathbf{F}).$$

Proof. Let $\chi, \psi \in C^\infty_c(M)$. In finitely many charts and frames, $\chi A \psi$ is a matrix of Euclidean operators of order m modulo a smooth kernel. Theorem 15.6.2 controls the pseudodifferential part, while Lemma 15.6.3 controls the smooth kernel and shows that it gains arbitrarily many derivatives. This proves the first mapping after choosing $\psi = 1$ near the datum's support.
If A is properly supported, the image of a compact set has support in another compact set. For the local mapping, given χ, proper support allows $\psi \in C^\infty_c(M)$ to be chosen so that

$$\chi A = \chi A \psi$$

on a neighborhood of the support of χ. The estimate already proved applies to ψu. These estimates are precisely continuity for the stated inductive and projective topologies. □

Corollary 15.6.5. (Local Sobolev elliptic regularity) Let $(M,\mathbf{g})$ be a finite-dimensional smooth Riemannian manifold without boundary, not necessarily complete, and let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and metric connections. Let $m,s\in\mathbb{R}$, and let $A\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic and properly supported. If $\mathbf{u}\in\mathcal{D}'(M,\mathbf{E})$ and $A\mathbf{u}\in H^s_{\mathrm{loc}}(M,\mathbf{F})$, then $\mathbf{u}\in H^{s+m}_{\mathrm{loc}}(M,\mathbf{E})$. In particular, for every $t\in\mathbb{R}$,

$$\mathbf{u}\in H^t_{\mathrm{loc}}(M,\mathbf{E})\quad\Longleftrightarrow\quad A\mathbf{u}\in H^{t-m}_{\mathrm{loc}}(M,\mathbf{F}).$$

Proof. Let $B\in\Psi^{-m}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{E})$ be a parametrix. The identity $BA=I-R$, with $R\in\Psi^{-\infty}(M;\mathbf{E},\mathbf{E})$, gives $\mathbf{u}=B(A\mathbf{u})+R\mathbf{u}$. Proposition 15.6.4 maps the first term into $H^{s+m}_{\mathrm{loc}}(M,\mathbf{E})$, and the second is smooth. This proves the first assertion. The remaining implication in the equivalence is the local mapping property of A. For differential operators, this recovers the regularity of Theorem 11.4.2; the pseudodifferential calculus adds the localization in codirections of Theorem 15.5.14. □

In particular, if M is closed, every proper-support condition is automatic and a finite cover gives the following result.

Corollary 15.6.6. (Mapping on a closed manifold) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, and let $\mathbf{E},\mathbf{F}\to M$ be complex Hermitian bundles of finite rank equipped with metric connections. Let $m,s\in\mathbb{R}$ and $A\in\Psi^m(M;\mathbf{E},\mathbf{F})$. Then

$$A_s\colon H^{s+m}(M,\mathbf{E})\longrightarrow H^s(M,\mathbf{F})$$

is continuous. The norm of A_s is controlled by finitely many symbol seminorms in a finite atlas and finitely many seminorms of the smoothing kernel.

Proof. Since M is compact, the constant function 1 belongs to $C^\infty_c(M)$. In the topology of $H^t_{\mathrm{loc}}(M,\mathbf{E})$ defined before Proposition 15.6.4, the seminorm corresponding to $\chi=1$ is

$$\|1\,\mathbf{u}\|_{H^t(M,\mathbf{E})}=\|\mathbf{u}\|_{H^t(M,\mathbf{E})}.$$

Conversely, continuity of multiplication by smooth functions, obtained in coordinates using Lemma 6.1.8, shows that each seminorm $\|\chi\mathbf{u}\|_{H^t(M,\mathbf{E})}$ is bounded by a multiple of $\|\mathbf{u}\|_{H^t(M,\mathbf{E})}$. Thus

$$H^t_{\mathrm{loc}}(M,\mathbf{E})=H^t(M,\mathbf{E})$$

as topological spaces. Moreover, the support of every distribution on M is a closed subset of a compact space. In the definition of $H^t_{\mathrm{comp}}(M,\mathbf{E})$ we may take $K=M$, giving

$$H^t_{\mathrm{comp}}(M,\mathbf{E})=H^t(M,\mathbf{E})$$

also as topological spaces.
Proposition 15.6.4, with these two identifications and $t=s+m$, gives the continuous mapping

$$A_s\colon H^{s+m}(M,\mathbf{E})\longrightarrow H^s(M,\mathbf{F}).$$

For the assertion on the norm, compactness of M allows a finite atlas, unitary trivializations, and finitely

many cutoffs to be chosen. In the proof of Proposition 15.6.4, each part localized near the diagonal is estimated by Theorem 15.6.2; its constant depends on finitely many seminorms of the local symbol. The parts away from the diagonal have smooth kernel by Proposition 15.2.4, and Lemma 15.6.3 controls them through finitely many seminorms of that kernel. Since only finitely many localizations occur, the norm of A_s is controlled by the finite collection of seminorms in the statement. □

15.6.3 Uniform pseudodifferential calculus with finite propagation

The preceding results are local, and on a closed manifold a finite cover suffices to combine their constants. On a noncompact manifold, the same estimate may deteriorate as the chart changes. To prevent this, we fix a bounded-geometry localization system, require uniform symbol bounds, and also control propagation of the remainders.

On an infinite cover, requiring each local representative to be a symbol is insufficient: the constants in its estimates might grow with the chart. There is no unique extension of the pseudodifferential calculus to open manifolds. We consider the calculus with finite propagation and uniform local bounds, which suffices for later applications and is compatible with the model of [57, Section 5]. Chapter 13 already constructed geodesic charts, quadratic systems, and synchronous frames with all derivatives controlled; Chapter 14 proved independence of section norms from those choices. We develop only the new pseudodifferential information.

Fix an admissible geodesic quadratic system

$$\mathcal{Q} = (U_i, \varphi_i, h_i, \mathbf{e}_i^{\mathbf{E}}, \mathbf{e}_i^{\mathbf{F}})_{i\in I}, \qquad U_i = B_{\mathbf{g}}(p_i, 2r), \qquad \sum_{i\in I} h_i^2 = 1,$$

with the synchronous frames of Proposition 8.2.26. The radius r and separation of the centers are those of Theorem 13.1.22. For $\rho \geq 0$, set

$$I_{\mathcal{Q}}(i,\rho) := \{j \in I \mid d_{\mathbf{g}}(U_i, U_j) \leq \rho\}, \qquad N_{\mathcal{Q}}(\rho) := \sup_{i\in I} \#I_{\mathcal{Q}}(i,\rho).$$

Lemma 15.6.7. (Uniform number of interactions at bounded distance) Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary of bounded geometry, and let $\mathcal{Q}$ be the preceding admissible geodesic quadratic system, of radius $r>0$ and finite overlap multiplicity. For every $\rho \geq 0$, we have $N_{\mathcal{Q}}(\rho) < \infty$. More precisely, if

$$L_{\mathcal{Q}} := \sup_{i\in I} \#\{j \in I \mid U_i \cap U_j \neq \varnothing\}$$

and

$$k(\rho) := \left\lceil \frac{\rho + 4r}{r} \right\rceil + 2,$$

, then

$$N_{\mathcal{Q}}(\rho) \leq 1 + L_{\mathcal{Q}} + \cdots + L_{\mathcal{Q}}^{k(\rho)}. \tag{15.70}$$

Proof. If $d_{\mathbf{g}}(U_i, U_j) \leq \rho$, then $d_{\mathbf{g}}(p_i, p_j) \leq \rho + 4r$. Completeness and Hopf–Rinow provide a minimizing geodesic between the centers. Divide it into at most $\left\lceil \frac{\rho+4r}{r} \right\rceil + 1$ segments of length at most r. Each subdivision point belongs to some ball $B_{\mathbf{g}}(p_v, r)$ of the cover. Two centers chosen at consecutive subdivision points are at distance at most $3r$, so their balls of radius $2r$ intersect. Adding i and j at the endpoints gives a chain of at most $k(\rho)$ intersections joining U_i to U_j. From any chart there are at most $L_{\mathcal{Q}}$ choices at each step. Counting all chains of lengths from zero to $k(\rho)$ gives (15.70). This count permits repetitions and therefore

remains an upper bound. □

The preceding lemma ensures that, for a fixed propagation radius, each chart interacts with only a uniformly bounded number of charts. With this control, we can choose a diagonal representation of the operator and define seminorms that separately measure its symbolic part and smooth remainder.

Set $B := B_{\mathrm{euc}}(0, 2r)$. The functions in the geodesic system can be chosen with $\operatorname{supp}(h_i \circ \varphi_i^{-1}) \subset B_{\mathrm{euc}}\left(0, \frac{3r}{2}\right)$, as in the proof of Theorem 13.1.22. If $\mathbf{u} \in \Gamma_c(\mathbf{E})$, let $\mathbf{u}_i \colon U_i \to \mathbb{C}^{r_{\mathbf{E}}}$ denote its column of components in the frame $\mathbf{e}_i^{\mathbf{E}}$, and define local analysis and synthesis by

$$\mathcal{A}_i^{\mathbf{E}} \mathbf{u} := \widetilde{(h_i \mathbf{u}_i) \circ \varphi_i^{-1}},$$
$$\mathcal{S}_i^{\mathbf{F}} v := h_i \sum_{b=1}^{r_{\mathbf{F}}} (v^b \circ \varphi_i) \mathbf{e}_{i,b}^{\mathbf{F}}.$$

The tilde denotes extension by zero; this is smooth because the support of h_i is uniformly separated from the boundary of U_i. The auxiliary cutoffs of (B3) are inserted before extending an expression that does not already contain the factor h_i.

The locally finite sum

$$w_{\mathcal{Q}}(x, y) := \sum_{i \in I} h_i(x) h_i(y)$$

has uniformly bounded derivatives near the diagonal, and $w_{\mathcal{Q}}(x, x) = 1$. Indeed, at most $L_{\mathcal{Q}}$ summands occur in each chart, and their derivatives are controlled by (B2). The mean value theorem in charts and uniform comparison of Euclidean and Riemannian distances give numbers $\delta_{\mathcal{Q}} > 0$ and $c_{\mathcal{Q}} > 0$ such that

$$w_{\mathcal{Q}}(x, y) \geq \frac{1}{2}, \qquad d_{\mathbf{g}}(x, y) \leq 2\delta_{\mathcal{Q}}, \tag{15.71}$$

and all derivatives of $w_{\mathcal{Q}}^{-1}$ are bounded in that region by constants independent of the chart. Decrease $\delta_{\mathcal{Q}}$ so that it is less than a quarter of the normal radius and the $2\delta_{\mathcal{Q}}$-neighborhood of each $\operatorname{supp}(h_i)$ remains in U_i.

Fix a function $\omega_{\mathcal{Q}} \in C^\infty(M \times M, [0, 1])$ equal to one when $d_{\mathbf{g}}(x, y) \leq \frac{\delta_{\mathcal{Q}}}{3}$ and zero when $d_{\mathbf{g}}(x, y) \geq \frac{2\delta_{\mathcal{Q}}}{3}$. It is constructed by applying a fixed cutoff to $d_{\mathbf{g}}(x, y)^2$ within the normal radius; its covariant derivatives are uniformly bounded. In the chart i, write ω_i for its representative and set

$$(\mathrm{Op}_{\omega_i}(a_i) v)(x) := (2\pi)^{-n} \operatorname{Os} \iint_{B_y \times \mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} \omega_i(x, y) a_i(x, \xi) v(y)\, dy\, d\xi, \qquad x \in B.$$

This formula uses the same chart and frames for x and y. Define

$$Q_i^{\mathcal{Q}}(a_i) := \mathcal{S}_i^{\mathbf{F}} \mathrm{Op}_{\omega_i}(a_i) \mathcal{A}_i^{\mathbf{E}}. \tag{15.72}$$

Its kernel is supported where $x, y \in \operatorname{supp}(h_i)$ and $d_{\mathbf{g}}(x, y) \leq \frac{2\delta_{\mathcal{Q}}}{3}$. In the usual abbreviated notation, (15.72) is precisely

$$h_i\, \varphi_i^* \mathrm{Op}_{\omega_i}(a_i) (\varphi_i^{-1})^* h_i,$$

with the identifications through $\mathbf{e}_i^{\mathbf{E}}, \mathbf{e}_i^{\mathbf{F}}$ written in $\mathcal{A}_i^{\mathbf{E}}$ and $\mathcal{S}_i^{\mathbf{F}}$.

For a family $(a_i)_{i\in I}$, define the symbol seminorm of the representation by

$$q^{\mathrm{u},\mathcal{Q}}_{N,m}((a_i)) := \sup_{i\in I} \max_{|\alpha|+|\beta|\le N} \sup_{x\in B,\,\xi\in\mathbb{R}^n} \langle\xi\rangle^{-m+|\beta|}\,\|D_x^\alpha D_\xi^\beta a_i(x,\xi)\|. \tag{15.73}$$

An operator with kernel $\mathbf{K}_R$ has propagation at most ρ if

$$\operatorname{supp}\mathbf{K}_R \subseteq \{(x,y)\in M\times M \mid d_{\mathbf{g}}(x,y)\le\rho\}.$$

If R has smooth kernel of propagation at most ρ, then

$$\mathbf{K}_R \in C^\infty\left(M_x\times M_y;\mathbf{F}_x\boxtimes\mathbf{E}_y^*\right) = C^\infty\left(M_x\times M_y;\operatorname{Hom}(\mathbf{E}_y,\mathbf{F}_x)\right).$$

Equip this bundle with the product connection induced by the connection of $\mathbf{F}$ in the variable x and the dual connection of $\mathbf{E}$ in the variable y. The expression $(\nabla_x)^p(\nabla_y)^q\mathbf{K}_R$ denotes iterated tensorial derivatives in each factor, with the new index first in its block; the two product connections commute. In the following expression, the norm is induced by $\mathbf{g}$, $\mathbf{h}_\mathbf{F}$, and $\mathbf{h}_\mathbf{E}$ on $(T^*_xM)^{\otimes p}\otimes(T^*_yM)^{\otimes q}\otimes\operatorname{Hom}(\mathbf{E}_y,\mathbf{F}_x)$. Set

$$s^{\mathrm{u}}_N(R) := \max_{p+q\le N}\sup_{(x,y)\in M\times M}\|(\nabla_x)^p(\nabla_y)^q\mathbf{K}_R(x,y)\|_{\mathbf{g},\mathbf{h}_\mathbf{F},\mathbf{h}_\mathbf{E}}. \tag{15.74}$$

Definition 15.6.8. (Uniform diagonal representation) Let $(M,\mathbf{g})$ be complete, without boundary, and of bounded geometry, and let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics $\mathbf{h}_\mathbf{E},\mathbf{h}_\mathbf{F}$ and compatible connections of bounded geometry. A **uniform diagonal representation of order** m **and residual radius** ρ of an operator $A\colon\Gamma_c(M,\mathbf{E})\to\Gamma(M,\mathbf{F})$ is a pair $\mathfrak{r}=((a_i)_{i\in I},R)$ such that

$$A = A_{\mathrm{near}} + R, \qquad A_{\mathrm{near}} := \sum_{i\in I} Q_i^{\mathcal{Q}}(a_i), \tag{15.75}$$

$q^{\mathrm{u},\mathcal{Q}}_{N,m}((a_i))<\infty$ for every N, and R has propagation at most ρ and satisfies $s^{\mathrm{u}}_N(R)<\infty$ for every N. Denote the set of these representations by $\operatorname{Rep}^{\mathcal{Q}}_{m,\rho}(A)$ and define their seminorms by

$$P^{\mathcal{Q}}_{m,N}(\mathfrak{r}) := q^{\mathrm{u},\mathcal{Q}}_{N,m}((a_i)) + s^{\mathrm{u}}_N(R). \tag{15.76}$$

The operator belongs to $\Psi^m_{\mathrm{u},\rho}(M;\mathbf{E},\mathbf{F})$ if this set is nonempty, and $\Psi^m_{\mathrm{u}}(M;\mathbf{E},\mathbf{F}) := \bigcup_{\rho\ge0}\Psi^m_{\mathrm{u},\rho}(M;\mathbf{E},\mathbf{F})$.

If $\mathbf{E}=\mathbf{F}$, the identity belongs to the calculus with $a_i(x,\xi)=I$ and $R=0$, since $\operatorname{Op}_{\omega_i}(I)=I$ on the support of h_i because $\omega_i=1$ on the diagonal, and

$$\sum_{i\in I} Q_i^{\mathcal{Q}}(I)u = \sum_{i\in I} h_i^2 u = u.$$

This is the reason for using the same chart in both variables and a quadratic system.

The quantities (15.76) depend on the representation. To obtain operator seminorms, fix $\rho\ge\delta_{\mathcal{Q}}$ and set

$$\|A\|^{\mathrm{u},\mathcal{Q}}_{m,N;\rho} := \inf_{\mathfrak{r}\in\operatorname{Rep}^{\mathcal{Q}}_{m,\rho}(A)} P^{\mathcal{Q}}_{m,N}(\mathfrak{r}), \tag{15.77}$$

with value $+\infty$ if no such representation exists. On $\Psi^m_{\mathrm{u},\rho}(M;\mathbf{E},\mathbf{F})$ these are seminorms: addition and scalar multi-

plication are performed componentwise in representations, and choosing representations approximating the two infima gives the triangle inequality and homogeneity. They are the quotient seminorms of the linear map

$$((a_i),R)\longmapsto \sum_{i\in I} Q_i^{\mathcal{Q}}(a_i)+R.$$

The quotient seminorms separate points. If they all vanish for an operator A, then for each $j\ge 1$ one can choose a representation of A whose first j seminorms sum to less than 2^{-j}. On any fixed pair of charts, its kernels then converge to zero distributionally: for the pseudodifferential part this follows from continuity of quantization on test functions, and for the remainder from uniform convergence of the kernel. All representations have the same distributional kernel $\mathbf{K}_A$; by uniqueness, $\mathbf{K}_A=0$ and $A=0$. Thus each stage $\Psi^m_{\mathrm{u},\rho}(M;\mathbf{E},\mathbf{F})$, with $\rho\ge\delta_{\mathcal{Q}}$, is a Hausdorff metrizable locally convex space, since the family of seminorms is indexed by $N\in\mathbb{N}_0$.

These stages are considered only as locally convex spaces; completeness, and hence the Fréchet property, would require proving that the chosen representation space is complete and that the kernel of the quotient map is closed in that realization. If $\rho_1\le\rho_2$, inclusion of the first stage into the second is continuous, since a representation of radius ρ_1 is also one of radius ρ_2; its image need not be closed. In particular, the filtered union is not identified with a strict inductive limit or an LF-space.

The global structure used here is bornological and is fixed explicitly: a family $\mathscr{B}\subset\Psi^m_{\mathrm{u}}(M;\mathbf{E},\mathbf{F})$ is called **bounded in the uniform calculus** if there exists $\rho\ge\delta_{\mathcal{Q}}$ such that $\mathscr{B}\subset\Psi^m_{\mathrm{u},\rho}(M;\mathbf{E},\mathbf{F})$ and

$$\sup_{A\in\mathscr{B}}\|A\|^{\mathrm{u},\mathcal{Q}}_{m,N;\rho}<\infty\qquad\text{for every } N\in\mathbb{N}_0.$$

This definition does not identify such families with the bounded sets of an inductive-limit topology. Only when $\mathcal{Q}$ and ρ are fixed will we omit these superscripts from q^{u}, P, and $\|\cdot\|^{\mathrm{u}}_{m,N;\rho}$.

Define $\Psi^{-\infty}_{\mathrm{u}}(M;\mathbf{E},\mathbf{F})$ to consist of finite-propagation operators $R\colon\Gamma_c(M,\mathbf{E})\to\Gamma(M,\mathbf{F})$ whose kernel

$$\mathbf{K}_R\in C^\infty\big(M\times M;\operatorname{Hom}(\mathbf{E}_y,\mathbf{F}_x)\big)$$

satisfies (15.74) for every $N\in\mathbb{N}_0$. In the synchronous frames of the system $\mathcal{Q}$, this condition is equivalent to requiring all localized kernels $h_i(x)\mathbf{K}_R(x,y)h_j(y)$ to have partial derivatives of every order bounded by constants independent of (i,j); the equivalence follows from uniform bounds on the connections, frame changes, and functions h_i. This is the required compatibility with the localization system. Finite propagation is global, not microlocal; by Hopf–Rinow, every operator in the calculus maps $\Gamma_c(M,\mathbf{E})$ into $\Gamma_c(M,\mathbf{F})$. If M is noncompact, membership in $\Psi^{-\infty}_{\mathrm{u}}(M;\mathbf{E},\mathbf{F})$ does not imply compactness of the extension $L^2(M,\mathbf{E})\to L^2(M,\mathbf{F})$; Remark 15.6.16 gives a counterexample.

Lemma 15.6.9. (Exact symbol of a uniformly smooth kernel) Let $r_{\mathbf{E}},r_{\mathbf{F}}\in\mathbb{N}$, let $B=B_{\mathrm{euc}}(0,2r)\subset\mathbb{R}^n$, and let

$$K_i\in C^\infty_c\big(B\times B;\operatorname{Hom}(\mathbb{C}^{r_{\mathbf{E}}},\mathbb{C}^{r_{\mathbf{F}}})\big),\qquad i\in I,$$

be a family of kernels supported in a fixed compact subset of $B\times B$, with uniformly bounded derivatives. Their exact symbols

$$r_i(x,\xi):=\int_{B_y}e^{-i(x-y)\cdot\xi}K_i(x,y)\,dy$$

form a bounded family in $S^{-P}_{1,0}(B\times\mathbb{R}^n;\operatorname{Hom}(\mathbb{C}^{r_{\mathbf{E}}},\mathbb{C}^{r_{\mathbf{F}}}))$ for each $P\in\mathbb{R}$. In particular, the definition of

$\Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{E},\mathbf{F})$ agrees with the intersection of all orders when a common radius is fixed and the preceding diagonal representations are used.

Proof. The identity avoiding spurious terms from differentiation of the phase is, for all multi-indices α,β,

$$D_x^\alpha D_\xi^\beta r_i(x,\xi) = (-i)^{|\beta|}\int_{B_y} e^{-i(x-y)\cdot\xi}(D_x+D_y)^\alpha\left[(x-y)^\beta K_i(x,y)\right]dy. \tag{15.78}$$

To prove it, D_ξ^β produces exactly $(-i)^{|\beta|}(x-y)^\beta$. Moreover, $(D_x+D_y)e^{-i(x-y)\cdot\xi}=0$; integrating the term D_y by parts shows that differentiating the integral in x is equivalent to applying D_x+D_y to the remaining factor. Iterating this equality gives (15.78).

Now use

$$\langle\xi\rangle^{2L}e^{-i(x-y)\cdot\xi} = (1-\Delta_y)^L e^{-i(x-y)\cdot\xi}$$

and transfer $(1-\Delta_y)^L$ by integration by parts to the last factor of (15.78). Since the supports remain in a fixed compact set, this yields

$$\langle\xi\rangle^{2L}\|D_x^\alpha D_\xi^\beta r_i(x,\xi)\| \le C_{\alpha,\beta,L}\max_{|\mu|+|\nu|\le|\alpha|+|\beta|+2L}\|D_x^\mu D_y^\nu K_i\|_{L^\infty(B\times B;\mathrm{Hom}(\mathbb{C}^{r_{\mathbf{E}}},\mathbb{C}^{r_{\mathbf{F}}}))},$$

with a constant independent of i. Given P, choosing $2L \ge P+|\beta|$ proves the estimates for $S^{-P}_{1,0}(B\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^{r_{\mathbf{E}}},\mathbb{C}^{r_{\mathbf{F}}}))$. Comparison of local partial and covariant derivatives, using the bounds on Christoffel symbols and connection matrices, identifies these estimates with (15.74).

If $R\in\Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{E},\mathbf{F})$, take $a_i=0$ in (15.75), so the operator belongs to every order with the same radius. For the converse, we do not assume that one representation works simultaneously at all orders. Let A be a finite-propagation operator belonging to $\Psi_{\mathrm{u}}^{\mu}(M;\mathbf{E},\mathbf{F})$ for every $\mu\in\mathbb{R}$, and fix $p,q\in\mathbb{N}_0$. Choose $\mu<-n-p-q-2$ and a representation of A of that order,

$$A=\sum_{i\in I}Q_i^{\mathcal{Q}}(a_i^{(\mu)})+R_\mu.$$

After applying p derivatives in the output variable and q in the input variable to the kernel of each diagonal piece, the oscillatory integral terms are bounded by $C_{p,q}\langle\xi\rangle^{\mu+p+q}$. This function is integrable in $\mathbb{R}^n$, and uniform multiplicity of the cover gives a chart-independent bound for the sum of pieces. The kernel of R_μ is already uniformly smooth. Consequently, the same distributional kernel $\mathbf{K}_A$ is represented by a kernel of class $C^{p,q}$ with the corresponding derivatives uniformly bounded. As p,q varies, the resulting representations are compatible because they all represent the unique distributional kernel $\mathbf{K}_A$. Thus $\mathbf{K}_A$ is smooth and satisfies (15.74) for every N; together with finite propagation of A, this proves $A\in\Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{E},\mathbf{F})$. □

The preceding representations control a family of symbols chart by chart. To construct parametrices, we must sum formal expansions using frequency cutoffs that can be chosen uniformly in the chart index. The following lemma is the uniform version of symbolic asymptotic summation.

Lemma 15.6.10. (Uniform asymptotic summation) Let $r_{\mathbf{E}},r_{\mathbf{F}}\in\mathbb{N}$, let $\mathcal{I}$ be an index set, and let $m_0>m_1>\cdots$

be real numbers with $m_j \to -\infty$. In this lemma, abbreviate

$$S^{\mu} := S^{\mu}_{1,0}\big(B \times \mathbb{R}^n; \mathrm{Hom}(\mathbb{C}^{r_{\mathrm{E}}}, \mathbb{C}^{r_{\mathrm{F}}})\big).$$

For each $j \in \mathbb{N}_0$, let $(a_\nu^{(j)})_{\nu \in \mathcal{I}}$ be a family in S^{m_j} such that

$$q^{\mathrm{u}}_{N,m_j}((a_\nu^{(j)})) < \infty \qquad \text{for all } N, j.$$

There is a family $(a_\nu)_{\nu\in\mathcal{I}} \subset S^{m_0}$, bounded in the seminorms of S^{m_0}, such that, for every $N \geq 1$ and every $L \in \mathbb{N}_0$,

$$q^{\mathrm{u}}_{L,m_N}\left((a_\nu) - \sum_{j=0}^{N-1} (a_\nu^{(j)})\right) < \infty. \tag{15.79}$$

If orders repeat, first group the families of equal order.

Proof. Take $\chi \in C^\infty(\mathbb{R}^n)$ vanishing for $\|\xi\| \leq 1$ and equal to one for $\|\xi\| \geq 2$. If $j > N$, the Leibniz rule and the support of the derivatives of $\chi(\varepsilon\xi)$ give

$$q^{\mathrm{u}}_{L,m_N}\big((\chi(\varepsilon\cdot)a_\nu^{(j)})_\nu\big) \leq C_{L,N,j}\,\varepsilon^{m_N - m_j} q^{\mathrm{u}}_{L,m_j}((a_\nu^{(j)})), \qquad 0 < \varepsilon \leq 1. \tag{15.80}$$

Indeed, a term in which D_ξ^γ hits the cutoff is supported in $\varepsilon^{-1} \leq \|\xi\| \leq 2\varepsilon^{-1}$ if $|\gamma| > 0$ and in $\|\xi\| \geq \varepsilon^{-1}$ if $\gamma = 0$; in either case,

$$\varepsilon^{|\gamma|}\langle\xi\rangle^{m_j - m_N + |\gamma|} \leq C_{N,j,\gamma}\,\varepsilon^{m_N - m_j}.$$

The constant is uniform in ν because the seminorm in the last expression of (15.80) is uniform.
Choose $\varepsilon_j \in (0,1]$ decreasing. Once $\varepsilon_0, \dots, \varepsilon_{j-1}$ are fixed, the positive power of (15.80) allows ε_j to be chosen so that

$$q^{\mathrm{u}}_{L,m_N}\big((\chi(\varepsilon_j\cdot)a_\nu^{(j)})_\nu\big) \leq 2^{-j} \tag{15.81}$$

for all pairs $0 \leq N < j$ and $0 \leq L \leq j$. Define

$$a_\nu(x,\xi) := \sum_{j=0}^{\infty} \chi(\varepsilon_j\xi) a_\nu^{(j)}(x,\xi).$$

For a fixed seminorm q^{u}_{L,m_0}, the terms with $j > \max\{L, 0\}$ form, by (15.81), a series dominated by $\displaystyle\sum_{j=0}^{\infty} 2^{-j}$; the remaining terms are finite in number. This proves convergence in all seminorms of S^{m_0}, uniformly in ν.
Now fix $N \geq 1$. We have the exact identity

$$a_\nu - \sum_{j=0}^{N-1} a_\nu^{(j)} = \sum_{j=0}^{N-1} (\chi(\varepsilon_j\xi) - 1) a_\nu^{(j)} + \sum_{j=N}^{\infty} \chi(\varepsilon_j\xi) a_\nu^{(j)}.$$

The first sum has uniformly bounded support in ξ and belongs to every symbol class. In the second, the terms with $N \leq j \leq \max\{N, L\}$ are finite in number and belong to S^{m_N}; the remainder converges in q^{u}_{L,m_N} by (15.81). This proves (15.79) for each L. □

Definition 15.6.11. (Uniform classicality and ellipticity) Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary of bounded geometry; let $\mathbf{E},\mathbf{F}\to M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and compatible connections of bounded geometry; fix an admissible system $\mathcal{Q}$ as above, set $r_{\mathbf{E}}:=\operatorname{rk}\mathbf{E}$ and $r_{\mathbf{F}}:=\operatorname{rk}\mathbf{F}$, and let $m\in\mathbb{R}$. An operator $A\in\Psi^m_{\mathrm{u}}(M;\mathbf{E},\mathbf{F})$ is **uniformly classical** if it admits a diagonal representation whose symbols have expansions

$$a_i\sim\sum_{j=0}^{\infty}a^{(i)}_{m-j}$$

whose homogeneous components, restricted to $\|\xi\|=1$, have all derivatives bounded by constants independent of i, and if the remainders after N components are uniformly bounded in

$$S^{m-N}_{1,0}\big(B\times\mathbb{R}^n;\operatorname{Hom}(\mathbb{C}^{r_{\mathbf{E}}},\mathbb{C}^{r_{\mathbf{F}}})\big).$$

We write $A\in\Psi^m_{\mathrm{u,cl}}(M;\mathbf{E},\mathbf{F})$.
Its principal symbol is the global section whose matrices in a chart are obtained from

$$\sigma^{\Psi}_m(A)(x,\xi)=\sum_{i\in I}h_i(x)^2\big[a^{(i)}_m(x,\xi)\big]_{\mathbf{e}^{\mathbf{E}}_i,\mathbf{e}^{\mathbf{F}}_i},\tag{15.82}$$

where each summand is transported to the corresponding fiber and covariable. The sum is finite at every point. Independence of the representation and system will be proved below by the exact transformation of kernels near the diagonal.
If $\mathbf{E}$ and $\mathbf{F}$ have the same rank, an operator $A\in\Psi^m_{\mathrm{u,cl}}(M;\mathbf{E},\mathbf{F})$ is **uniformly elliptic** if there exists $c>0$ such that

$$\big\|\sigma^{\Psi}_m(A)(x,\xi)v\big\|_{\mathbf{h}_{\mathbf{F}}(x)}\ge c\|v\|_{\mathbf{h}_{\mathbf{E}}(x)},\qquad x\in M,\quad \|\xi\|_{\mathbf{g}}=1,\quad v\in\mathbf{E}_x.\tag{15.83}$$

By homogeneity, for $\xi\neq 0_x$ this is equivalent to

$$\big\|\sigma^{\Psi}_m(A)(x,\xi)^{-1}\big\|_{\mathcal{L}((\mathbf{F}_x,\mathbf{h}_{\mathbf{F}}(x)),(\mathbf{E}_x,\mathbf{h}_{\mathbf{E}}(x)))}\le c^{-1}\|\xi\|_{\mathbf{g}}^{-m}.$$

The definition was formulated relative to an admissible geodesic system $\mathcal{Q}$. Before using the calculus, we must check that changing charts, frames, or cutoffs preserves the operator class and its bounded families. We therefore compare two admissible systems $\mathcal{Q}$ and $\mathcal{Q}^{\sharp}$, including their frames. The cross bounds in (B1), used in Lemma 13.3.4 and Proposition 14.5.3, provide the two finite cross multiplicities; let $L_{\times}$ denote their maximum. To specify how many derivatives are used, for $N\in\mathbb{N}_0$ and $\mu\in\mathbb{R}$ define

$$L_N:=n+N+1,\qquad K_{N,\mu}:=n+N+1+\lceil|\mu|\rceil,\qquad \nu(N,\mu,n):=N+2L_N+2K_{N,\mu}+4.$$

In the exact amplitude reduction formula (15.5), L_N integrations using $1-\Delta_\eta$ make the spatial variable integrable, and $K_{N,\mu}$ integrations using $1-\Delta_y$ make the frequency variable integrable after N derivatives are taken. Thus the order-N seminorm of the reduced symbol uses at most $\nu(N,\mu,n)$ derivatives of the amplitude. This number is not optimal, but it is fixed and independent of the chart.

Let $I^{\sharp}$ denote the index set of the system $\mathcal{Q}^{\sharp}$. Take a representation $\mathfrak{r}=((a_i),R)\in\operatorname{Rep}^{\mathcal{Q}}_{m,\rho}(A)$. Choose $0<\delta<$

$\frac{1}{4}\min\{\delta_{\mathcal{Q}}, \delta_{\mathcal{Q}^\sharp}\}$ and a uniform cutoff $\zeta(x,y)$ equal to one for $d_{\mathbf{g}}(x,y) \le \delta$ and zero for $d_{\mathbf{g}}(x,y) \ge 2\delta$. Split exactly as

$$\mathbf{K}_{A_{\mathrm{near}}} = \zeta \mathbf{K}_{A_{\mathrm{near}}} + (1-\zeta)\mathbf{K}_{A_{\mathrm{near}}}.$$

The second summand is separated from the diagonal. The integrations by parts in the covariable used in the pseudolocality proof, with the uniform seminorms of (a_i), show that it is a uniformly smooth kernel of propagation at most $\frac{2\delta_{\mathcal{Q}}}{3}$; incorporate it into R.

On the support of the first summand, $w_{\mathcal{Q}^\sharp} \ge \frac{1}{2}$ by (15.71). Thus the identity

$$\zeta \mathbf{K}_{A_{\mathrm{near}}}(x,y) = \sum_{a \in I^\sharp} h_a^\sharp(x) \frac{\zeta(x,y)\mathbf{K}_{A_{\mathrm{near}}}(x,y)}{w_{\mathcal{Q}^\sharp}(x,y)} h_a^\sharp(y) \tag{15.84}$$

redistributes the kernel exactly into pieces using the same chart $U_a^\sharp$ for x and y. For fixed a, only the indices

$$J(a) := \{i \in I \mid U_i \cap U_a^\sharp \neq \varnothing\}, \qquad \#J(a) \le L_\times.$$

contribute.

Write $\Phi_{ia} := \varphi_i \circ (\varphi_a^\sharp)^{-1}$ on the overlap and

$$B_{ia}(x,y) := \int_0^1 D\Phi_{ia}(y + t(x-y))\, dt.$$

Then $\Phi_{ia}(x) - \Phi_{ia}(y) = B_{ia}(x,y)(x-y)$, and, decreasing δ if necessary, B_{ia} is invertible with uniformly bounded inverse. Let $G_{ia}^{\mathbf{E}}(y)$ and $G_{ia}^{\mathbf{F}}(x)$ be the frame-change matrices from the new frames to the old ones. After the change $\xi = B_{ia}(x,y)^T \eta$, the contribution of index i to the term a in (15.84) has, before reduction, amplitude

$$c_{ai}(x,y,\xi) = \Theta_{ai}(x,y) G_{ia}^{\mathbf{F}}(x) a_i\big(\Phi_{ia}(x), B_{ia}(x,y)^{-T}\xi\big) G_{ia}^{\mathbf{E}}(y)^{-1} \frac{|\det D\Phi_{ia}(y)|}{|\det B_{ia}(x,y)|}, \tag{15.85}$$

where Θ_{ai} is the product of ζ, the functions h_i in both variables, the cutoff ω_i, the necessary auxiliary cutoffs, and $w_{\mathcal{Q}^\sharp}^{-1}$. All these factors are expressed in the chart a and have uniformly bounded derivatives. The formula separately includes the coordinate transition, both frame changes, and the Jacobian in the integration variable.

Exact reduction in Theorem 15.1.9, applied to $\sum_{i \in J(a)} c_{ai}$, produces a symbol $a_a^\sharp$ in the single chart a. The Leibniz rule, the Faà di Bruno formula of Theorem 1.1.19, the number of derivatives fixed by ν, and $\#J(a) \le L_\times$ give a representation $\mathfrak{r}^\sharp$ of A for which

$$P_{m,N}^{\mathcal{Q}^\sharp}(\mathfrak{r}^\sharp) \le C_{N,m} L_\times P_{m,\nu(N,m,n)}^{\mathcal{Q}}(\mathfrak{r}). \tag{15.86}$$

The constant depends on bounds up to order $\nu(N,m,n)+2$ for transitions and their inverses, frame changes and their inverses, cutoffs, bundle metrics, and densities, but not on a or i. The new remainder has propagation at most $\rho^{\mathrm{com}} := \max\{\rho, \delta_{\mathcal{Q}}, \delta_{\mathcal{Q}^\sharp}\}$.

Now take the infimum of (15.86) over all representations of radius at most ρ. This gives the intrinsic comparison

$$\|A\|_{m,N;\rho^{\mathrm{com}}}^{\mathrm{u},\mathcal{Q}^\sharp} \le C_{N,m} L_\times \|A\|_{m,\nu(N,m,n);\rho}^{\mathrm{u},\mathcal{Q}}. \tag{15.87}$$

Interchanging $\mathcal{Q}$ and $\mathcal{Q}^\sharp$ gives the reverse inequality after using the common radius ρ^{com}. This proves quantitative independence of the class, its filtration by radii, its bounded sets, and its locally convex topology. Applied to the

principal homogeneous components of (15.85), the same formula gives the exact cotangent rule and shows that (15.82) is independent of the representation and system.

Having proved independence of choices, we must verify that the uniform calculus preserves the operations that motivated its construction. Write A_h^{*} for the formal Hermitian adjoint of the pseudodifferential operator A, in accordance with the preceding local theorems. Reserve $T^{\dagger}$ for the Hilbert space adjoint of a bounded operator between two Hilbert spaces whose inner products are already fixed. For a representation of residual radius ρ, write

$$\widehat{\rho} := \max\left\{\rho, \frac{2\delta_{\mathcal{Q}}}{3}\right\};$$

this bounds propagation of the full operator, including its diagonal part. Given $N \in \mathbb{N}_0$ and $\mu, \lambda \in \mathbb{R}$, define the derivative count and interaction factor

$$\mathfrak{n}_N(\mu,\lambda) := \nu(N,\mu+\lambda,n) + \nu(N,\mu,n) + \nu(N,\lambda,n), \tag{15.88}$$

$$\mathfrak{M}_{\mathcal{Q}}(\rho,\sigma) := (1+L_{\mathcal{Q}})^{2}\big(1+N_{\mathcal{Q}}(\rho+\sigma)\big)^{2}. \tag{15.89}$$

The second factor records both overlaps of diagonal pieces and bounded-distance interactions of residual kernels; applications require only that it be finite and depend on the stated data.

Proposition 15.6.12. (Properties of the uniform calculus) Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary of bounded geometry, and let $\mathbf{E},\mathbf{F},\mathbf{G} \to M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and compatible connections of bounded geometry. Let $m,\ell \in \mathbb{R}$, $\rho_A, \rho_B \geq \delta_{\mathcal{Q}}$, $A \in \Psi^{m}_{\mathrm{u},\rho_A}(M;\mathbf{E},\mathbf{F})$, and $B \in \Psi^{\ell}_{\mathrm{u},\rho_B}(M;\mathbf{G},\mathbf{E})$. Then

(i) $AB \in \Psi^{m+\ell}_{\mathrm{u},\rho_{AB}}(M;\mathbf{G},\mathbf{F})$ with
$$\rho_{AB} := \rho_A + \rho_B;$$

(ii) $A_h^{*} \in \Psi^{m}_{\mathrm{u},\rho_A}(M;\mathbf{F},\mathbf{E})$;

(iii) the classes $\Psi^{-\infty}_{\mathrm{u}}(M;\mathbf{E},\mathbf{F})$ are stable under adjoints and form a two-sided ideal between the corresponding spaces within the finite-propagation calculus.

In particular, composition is the continuous bilinear map of stages

$$\begin{aligned}\Psi^{m_1}_{\mathrm{u},\rho_1}(M;\mathbf{F},\mathbf{G}) \times \Psi^{m_2}_{\mathrm{u},\rho_2}(M;\mathbf{E},\mathbf{F}) &\longrightarrow \Psi^{m_1+m_2}_{\mathrm{u},\rho_1+\rho_2}(M;\mathbf{E},\mathbf{G}),\\ (A,B) &\longmapsto A\circ B,\end{aligned} \tag{15.90}$$

for $\rho_1,\rho_2 \geq \delta_{\mathcal{Q}}$, and the adjoint is a continuous map from $\Psi^{m}_{\mathrm{u},\rho}(M;\mathbf{E},\mathbf{F})$ to $\Psi^{m}_{\mathrm{u},\rho}(M;\mathbf{F},\mathbf{E})$. For each N, there exists $C_{N,m,\ell}$ independent of local indices such that

$$\|AB\|^{\mathrm{u},\mathcal{Q}}_{m+\ell,N;\rho_{AB}} \leq C_{N,m,\ell}\,\mathfrak{M}_{\mathcal{Q}}(\rho_A,\rho_B)\,\|A\|^{\mathrm{u},\mathcal{Q}}_{m,\mathfrak{n}_N(m,\ell);\rho_A}\,\|B\|^{\mathrm{u},\mathcal{Q}}_{\ell,\mathfrak{n}_N(m,\ell);\rho_B}. \tag{15.91}$$

Moreover, with $\mathfrak{a}_N(m) := \nu(N,m,n) + N + 2$,

$$\|A_h^{*}\|^{\mathrm{u},\mathcal{Q}}_{m,N;\rho_A} \leq C_{N,m}\,\|A\|^{\mathrm{u},\mathcal{Q}}_{m,\mathfrak{a}_N(m);\rho_A}. \tag{15.92}$$

Proof. Take diagonal representations $\mathfrak{r}_A = ((a_i), R_A)$ and $\mathfrak{r}_B = ((b_j), R_B)$. Since $\rho_A, \rho_B \geq \delta_{\mathcal{Q}}$, we have $\widehat{\rho}_A = \rho_A$ and $\widehat{\rho}_B = \rho_B$. The kernel formula for composition is

$$\mathbf{K}_{AB}(x,y) = \int_{M_z} \mathbf{K}_A(x,z)\mathbf{K}_B(z,y)\, d\lambda_{\mathbf{g}}(z).$$

If the integrand is nonzero, then $d_{\mathbf{g}}(x,y) \leq \widehat{\rho}_A + \widehat{\rho}_B$, proving the propagation bound. Split as

$$AB = A_{\mathrm{near}}B_{\mathrm{near}} + A_{\mathrm{near}}R_B + R_A B_{\mathrm{near}} + R_A R_B. \tag{15.93}$$

Consider the first summand first. Its kernel is the sum of

$$K_{ij}(x,y) := \int_{M_z} \mathbf{K}_{Q_i(a_i)}(x,z)\mathbf{K}_{Q_j(b_j)}(z,y)\, d\lambda_{\mathbf{g}}(z).$$

This occurs only if $U_i \cap U_j \neq \varnothing$; hence, for fixed i, there are at most $L_{\mathcal{Q}}$ values of j. Choose a cutoff ζ vanishing outside $d_{\mathbf{g}}(x,y) < \frac{\delta_{\mathcal{Q}}}{2}$ and equal to one near the diagonal. The part $(1-\zeta)K_{ij}$ is separated from the diagonal. Integration by parts in both covariables, before and after integration in z, gives all bounds in (15.74); at most $L_{\mathcal{Q}}$ summands occur per chart. Incorporate this part into the remainder.
For ζK_{ij}, use the exact identity (15.84) again:

$$\zeta(x,y) \sum_{(i,j)\in I\times I} K_{ij}(x,y) = \sum_{a\in I} h_a(x) \frac{\zeta(x,y)\sum_{(i,j)\in I\times I} K_{ij}(x,y)}{w_{\mathcal{Q}}(x,y)} h_a(y). \tag{15.94}$$

After shrinking the support of ζ once, the choice of $\delta_{\mathcal{Q}}$ ensures that, if the summand a,i,j is nonzero, all three points x,z,y belong to U_a. Moreover, $U_i \cap U_a \neq \varnothing$ and $U_j \cap U_a \neq \varnothing$; for fixed a there are at most $L_{\mathcal{Q}}^2$ pairs (i,j).
Write $\Phi_{ia} = \varphi_i \circ \varphi_a^{-1}$ and

$$B_{ia}(x,z) = \int_0^1 D\Phi_{ia}(z + t(x-z))\, dt.$$

In the chart a, the change $\eta = B_{ia}(x,z)^{-T}\xi$ transforms the first piece into the amplitude

$$c_{ia}(x,z,\xi) = \Theta_{ia}(x,z) G_{ia}^{\mathbf{F}}(x) a_i\big(\Phi_{ia}(x), B_{ia}(x,z)^{-T}\xi\big) G_{ia}^{\mathbf{E}}(z)^{-1} \frac{|\det D\Phi_{ia}(z)|}{|\det B_{ia}(x,z)|}. \tag{15.95}$$

Here Θ_{ia} contains the corresponding cutoffs; for the second piece, with Φ_{ja}, one obtains the same formula in the variables (z,y) and with frame changes $G_{ja}^{\mathbf{E}}, G_{ja}^{\mathbf{G}}$. Both intermediate matrices take values in the same frame $\mathbf{e}_a^{\mathbf{E}}$. This transformation takes place within the single chart a; it is not a quantization with base variables in different charts. Uniform bounds on transitions, frames, and Jacobians control all derivatives of both amplitudes. Exact reduction gives symbols $\widetilde{a}_{ia}(x,\xi)$ and $\widetilde{b}_{ja}(x,\xi)$, both defined in the chart a. Theorem 15.2.6, now applied in that single chart, yields

$$\widetilde{a}_{ia} \# \widetilde{b}_{ja} \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha \widetilde{a}_{ia}\, D_x^\alpha \widetilde{b}_{ja}.$$

Differentiating this formula up to order N, including reduction of both amplitudes, uses at most $\mathfrak{n}_N(m,\ell)$

derivatives of (a_i) and (b_j). Summing the at most $L_{\mathcal{Q}}^2$ pairs gives a symbol c_a in the same chart a and a uniformly smooth part.

The remaining three terms in (15.93) are uniformly smooth. For $R_A R_B$, this is checked by differentiation under the integral; for each output chart, the integration domain is covered by at most $N_{\mathcal{Q}}(\widehat{\rho}_A + \widehat{\rho}_B)$ charts, each of uniformly bounded volume. In $A_{\mathrm{near}} R_B$, apply the pseudodifferential piece, in its single chart, to derivatives of the smooth kernel of R_B; in $R_A B_{\mathrm{near}}$, proceed in the other variable. The same count and Calderón–Vaillancourt after conjugation by powers of $\langle D \rangle$ give, for the constructed representation,

$$P^{\mathcal{Q}}_{m+\ell,N}(\mathfrak{r}_{AB}) \le C_{N,m,\ell}\, \mathfrak{M}_{\mathcal{Q}}(\widehat{\rho}_A, \widehat{\rho}_B) P^{\mathcal{Q}}_{m,\mathfrak{n}_N(m,\ell)}(\mathfrak{r}_A) P^{\mathcal{Q}}_{\ell,\mathfrak{n}_N(m,\ell)}(\mathfrak{r}_B). \tag{15.96}$$

Its remainder has radius at most ρ_{AB}. Choosing representations separately approximating the two infima and passing to those infima proves (15.91).

The adjoint of $Q_i^{\mathcal{Q}}(a_i)$ still uses the chart i in both variables. Theorem 15.2.8 and exact reduction give a symbol $a_i^{(*)}$ with

$$a_i^{(*)}(x,\xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha (a_i(x,\xi)^*).$$

In nonunitary frames, insert the bundle metric matrices and the local density of $d\lambda_{\mathbf{g}}$. Their derivatives and those of their inverses are uniformly bounded. The difference between the exact adjoint kernel and the truncated quantization of $a_i^{(*)}$ is uniformly smooth and preserves propagation. This first proves the analogue of (15.92) for $P^{\mathcal{Q}}_{m,N}$ and then, taking the infimum, the asserted inequality. Finally, $\mathbf{K}_{R_h^*}(x,y) = \mathbf{K}_R(y,x)^*$ preserves the radius and all seminorms (15.74). The arguments for the preceding three smooth terms, with one factor already smoothing, prove the two-sided ideal property. □

Stability under composition now leads to elliptic inversion. We first invert the principal symbol in each chart, correct lower-order terms by a uniform recurrence, and sum these corrections without losing control of propagation.

Theorem 15.6.13. (Uniform parametrix) Let $(M, \mathbf{g})$ be a complete Riemannian manifold without boundary of bounded geometry, and let $\mathbf{E}, \mathbf{F} \to M$ be complex vector bundles of the same finite rank equipped with Hermitian bundle metrics and compatible connections of bounded geometry. If $m \in \mathbb{R}$ and $A \in \Psi^m_{\mathrm{u,cl}}(M; \mathbf{E}, \mathbf{F})$ is uniformly elliptic, there exists $B \in \Psi^{-m}_{\mathrm{u}}(M; \mathbf{F}, \mathbf{E})$ such that

$$BA - I \in \Psi^{-\infty}_{\mathrm{u}}(M; \mathbf{E}, \mathbf{E}), \qquad AB - I \in \Psi^{-\infty}_{\mathrm{u}}(M; \mathbf{F}, \mathbf{F}).$$

Proof. Fix a diagonal representation of A of residual radius ρ_A and use the convention $\widehat{\rho}_A$ introduced before Proposition 15.6.12. Localize the full symbol of A in one chart at a time. More precisely, apply redistribution (15.84) with two cutoffs equal to one on $\mathrm{supp}(h_i)$ and reduce the amplitude in the chart i. Let $a^{(i)}$ denote the resulting symbol. Its homogeneous families satisfy

$$a^{(i)}(x,\xi) \sim \sum_{k=0}^{\infty} a^{(i)}_{m-k}(x,\xi),$$

and $a^{(i)}_m$ is the matrix, in the frame i, of the global symbol $\sigma^{\Psi}_m(A)$. Thus inversion is applied to the local matrix of the global principal symbol and uses uniform ellipticity directly.

For $j, N \in \mathbb{N}_0$, set

$$A_{j,N} := \sup_{i \in I} \max_{|\gamma| \le N} \sup_{\substack{x \in B,\ \xi \in \mathbb{R}^n \\ \|\xi\| = 1}} \|D^\gamma a^{(i)}_{m-j}(x,\xi)\|, \qquad D^\gamma := D_x^{\gamma_x} D_\xi^{\gamma_\xi}.$$

Let I_N be the same quantity for $(a^{(i)}_m)^{-1}$. Differentiating $(a^{(i)}_m)^{-1} a^{(i)}_m = I$, isolating the term $\eta = 0$, and multiplying on the right by $(a^{(i)}_m)^{-1}$ gives, for $|\gamma| \ge 1$, the exact multi-index recurrence

$$D^\gamma (a^{(i)}_m)^{-1} = - \sum_{0 < \eta \le \gamma} \binom{\gamma}{\eta} D^{\gamma - \eta} (a^{(i)}_m)^{-1} (D^\eta a^{(i)}_m)(a^{(i)}_m)^{-1}. \tag{15.97}$$

In particular, $I_0 \le c^{-1}$ after uniform comparison of the fiber norms, and

$$I_N \le I_0 + C_N I_0 A_{0,N} \sum_{p=0}^{N-1} I_p. \tag{15.98}$$

This recurrence proves inductively that $I_N < \infty$ for every N and specifies which derivatives of the principal symbol occur.

The formal components of the left and right inverses are defined by

$$b^{\mathrm{L},(i)}_{-m} = b^{\mathrm{R},(i)}_{-m} := (a^{(i)}_m)^{-1}, \tag{15.99}$$

$$b^{\mathrm{L},(i)}_{-m-j} := - \left[\sum_{\substack{k,q \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n \\ k+q+|\alpha| = j,\ q<j}} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha b^{\mathrm{L},(i)}_{-m-q} D_x^\alpha a^{(i)}_{m-k} \right] (a^{(i)}_m)^{-1}, \quad j \ge 1, \tag{15.100}$$

$$b^{\mathrm{R},(i)}_{-m-j} := -(a^{(i)}_m)^{-1} \left[\sum_{\substack{k,q \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n \\ k+q+|\alpha| = j,\ q<j}} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha a^{(i)}_{m-k} D_x^\alpha b^{\mathrm{R},(i)}_{-m-q} \right]. \tag{15.101}$$

Here we have isolated $b_{-m-j} a_m$ in $b \# a$ and $a_m b_{-m-j}$ in $a \# b$, respectively; the other summands have $q < j$. Define $B^{\mathrm{L}}_{j,N}$ and $B^{\mathrm{R}}_{j,N}$ as the supremum, over i, $x \in B$, $\|\xi\| = 1$, and $|\gamma| \le N$, of the derivatives of the corresponding components. The Leibniz rule and (15.97) give the quantified bounds

$$B^{\mathrm{L}}_{j,N} \le C_{j,N} I_{N+j} \sum_{\substack{k,q \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n \\ k+q+|\alpha| = j,\ q<j}} B^{\mathrm{L}}_{q,N+j} A_{k,N+j}, \tag{15.102}$$

$$B^{\mathrm{R}}_{j,N} \le C_{j,N} I_{N+j} \sum_{\substack{k,q \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n \\ k+q+|\alpha| = j,\ q<j}} A_{k,N+j} B^{\mathrm{R}}_{q,N+j}. \tag{15.103}$$

Induction on j proves that both quantities are finite for every j, N. Each component has degree $-m-j$; homogeneity turns bounds on the sphere into uniform symbol estimates.

For the first Borel sum, fix radial cutoffs ϑ, χ vanishing on $\|\xi\| \le 1$ and equal to one on $\|\xi\| \ge 2$.

Lemma 15.6.10 chooses sequences $\varepsilon_j^{\mathrm{L}}, \varepsilon_j^{\mathrm{R}} \to 0^+$ for which

$$\widetilde{b}_{\mathrm{L}}^{(i)}(x,\xi) := \sum_{j=0}^{\infty} \chi(\varepsilon_j^{\mathrm{L}}\xi)\vartheta(\xi) b_{-m-j}^{\mathrm{L},(i)}(x,\xi), \tag{15.104}$$

$$\widetilde{b}_{\mathrm{R}}^{(i)}(x,\xi) := \sum_{j=0}^{\infty} \chi(\varepsilon_j^{\mathrm{R}}\xi)\vartheta(\xi) b_{-m-j}^{\mathrm{R},(i)}(x,\xi) \tag{15.105}$$

are families in $S^{-m}_{1,0}(B\times\mathbb{R}^n;\mathrm{Hom}(\mathbb{C}^{r_{\mathrm{F}}},\mathbb{C}^{r_{\mathrm{E}}}))$ and, after subtracting the first N summands, belong uniformly to the analogous class of order $-m-N$. These sums fix the expansion to be realized by the global inverse. Their principal term defines, in the adopted diagonal model,

$$b_0^{(i)}(x,\xi) := \vartheta(\xi)(a_m^{(i)}(x,\xi))^{-1}, \qquad B_0 := \sum_{i\in I} Q_i^{\mathcal{Q}}(b_0^{(i)}). \tag{15.106}$$

In particular, diagonal quantization of $\widetilde{b}_{\mathrm{L}}^{(i)}$ or $\widetilde{b}_{\mathrm{R}}^{(i)}$ differs from B_0 at order $-m-1$. The equality $\sum_{i\in I} h_i^2 = 1$ shows that the principal symbol of B_0 is $\sigma_m^{\Psi}(A)^{-1}$. Take $\rho_{B_0} := \delta_{\mathcal{Q}}$ and

$$R_0 := I - B_0A \in \Psi^{-1}_{\mathrm{u},\varrho_0}(M;\mathbf{E},\mathbf{E}), \qquad S_0 := I - AB_0 \in \Psi^{-1}_{\mathrm{u},\varrho_0}(M;\mathbf{F},\mathbf{F}), \qquad \varrho_0 := \widehat{\rho}_A + \widehat{\rho}_{B_0}.$$

We now specify the bounds on powers, including their radii. Let

$$\mathfrak{C}_N(\rho,\sigma;\mu,\lambda) := C_{N,\mu,\lambda}\,\mathfrak{M}_{\mathcal{Q}}(\rho,\sigma), \tag{15.107}$$

where $C_{N,\mu,\lambda}$ is the constant in (15.91). For $j \ge 1$, set

$$r_{j,N} := \|R_0^j\|^{\mathrm{u},\mathcal{Q}}_{-j,N;j\varrho_0}, \qquad s_{j,N} := \|S_0^j\|^{\mathrm{u},\mathcal{Q}}_{-j,N;j\varrho_0},$$
$$\ell_{j,N} := \|R_0^jB_0\|^{\mathrm{u},\mathcal{Q}}_{-m-j,N;j\varrho_0+\rho_{B_0}}, \qquad d_{j,N} := \|B_0S_0^j\|^{\mathrm{u},\mathcal{Q}}_{-m-j,N;\rho_{B_0}+j\varrho_0},$$

and $b_{0,N} := \|B_0\|^{\mathrm{u},\mathcal{Q}}_{-m,N;\rho_{B_0}}$, $a_N := \|A\|^{\mathrm{u},\mathcal{Q}}_{m,N;\rho_A}$. For $j=0$, adopt $\ell_{0,N} = d_{0,N} = b_{0,N}$. Exact cancellation of the principal symbols of B_0A and AB_0, followed by amplitude reduction with one additional derivative, gives the base

$$r_{1,N} + s_{1,N} \le C_{N,m}\,\mathfrak{M}_{\mathcal{Q}}(\widehat{\rho}_A,\widehat{\rho}_{B_0})\left(1 + b_{0,\mathfrak{n}_{N+1}(-m,m)}a_{\mathfrak{n}_{N+1}(-m,m)}\right). \tag{15.108}$$

The composition construction followed by passage to the infimum gives, for $j \ge 1$,

$$r_{j+1,N} \le \mathfrak{C}_N(j\varrho_0,\varrho_0;-j,-1)\,r_{j,\mathfrak{n}_N(-j,-1)}\,r_{1,\mathfrak{n}_N(-j,-1)}, \tag{15.109}$$

$$s_{j+1,N} \le \mathfrak{C}_N(j\varrho_0,\varrho_0;-j,-1)\,s_{j,\mathfrak{n}_N(-j,-1)}\,s_{1,\mathfrak{n}_N(-j,-1)}, \tag{15.110}$$

$$\ell_{j,N} \le \mathfrak{C}_N(j\varrho_0,\rho_{B_0};-j,-m)\,r_{j,\mathfrak{n}_N(-j,-m)}\,b_{0,\mathfrak{n}_N(-j,-m)}, \tag{15.111}$$

$$d_{j,N} \le \mathfrak{C}_N(\rho_{B_0},j\varrho_0;-m,-j)\,b_{0,\mathfrak{n}_N(-m,-j)}\,s_{j,\mathfrak{n}_N(-m,-j)}. \tag{15.112}$$

In particular, all four families of quantities are finite. The formulas simultaneously show growth of the radius, the increase $N \mapsto \mathfrak{n}_N$ in the derivative count, and the explicit appearance of $N_{\mathcal{Q}}(j\varrho_0+\varrho_0)$ within $\mathfrak{M}_{\mathcal{Q}}$. For R_0^N and S_0^N, the first two recurrences, iterated from $r_{1,N}$ and $s_{1,N}$, are the required quantified bounds;

in particular,

$$R_0^N \in \Psi_{\mathrm{u},N\rho_0}^{-N}(M;\mathbf{E},\mathbf{E}), \qquad S_0^N \in \Psi_{\mathrm{u},N\rho_0}^{-N}(M;\mathbf{F},\mathbf{F}). \tag{15.113}$$

For $N \geq 1$, introduce the finite sums

$$C_N^{\mathrm{L}} := \sum_{j=0}^{N-1} R_0^j B_0, \qquad C_N^{\mathrm{R}} := B_0 \sum_{j=0}^{N-1} S_0^j. \tag{15.114}$$

The identities $B_0 A = I - R_0$ and $A B_0 = I - S_0$ give exactly

$$I - C_N^{\mathrm{L}} A = R_0^N, \qquad I - A C_N^{\mathrm{R}} = S_0^N. \tag{15.115}$$

Now use the representations produced recursively in the proof of Proposition 15.6.12. Let $(c_{j,i}^{\mathrm{L}})_i$ and $(c_{j,i}^{\mathrm{R}})_i$ be their symbol families near the diagonal for $R_0^j B_0$ and $B_0 S_0^j$. Bounds (15.111) and (15.112), before taking the infimum, verify the hypotheses of uniform asymptotic summation. With two further cutoff sequences, we obtain explicitly

$$b_{\mathrm{L}}^{(i)}(x,\xi) := \sum_{j=0}^{\infty} \chi(\delta_j^{\mathrm{L}}\xi) c_{j,i}^{\mathrm{L}}(x,\xi), \qquad B_{\mathrm{L}} := \sum_{i\in I} Q_i^{\mathcal{Q}}(b_{\mathrm{L}}^{(i)}), \tag{15.116}$$

$$b_{\mathrm{R}}^{(i)}(x,\xi) := \sum_{j=0}^{\infty} \chi(\delta_j^{\mathrm{R}}\xi) c_{j,i}^{\mathrm{R}}(x,\xi), \qquad B_{\mathrm{R}} := \sum_{i\in I} Q_i^{\mathcal{Q}}(b_{\mathrm{R}}^{(i)}). \tag{15.117}$$

Here only symbols of the parts near the diagonal are summed. The residual kernels of $R_0^j B_0$ and $B_0 S_0^j$, whose radii grow with j, are not summed; this distinction keeps propagation fixed for B_{L} and B_{R}.

For each N, the defining property of the two Borel sums and the finitely many remainders of terms with $j < N$ give

$$B_{\mathrm{L}} - C_N^{\mathrm{L}} \in \Psi_{\mathrm{u}}^{-m-N}(M;\mathbf{F},\mathbf{E}), \qquad B_{\mathrm{R}} - C_N^{\mathrm{R}} \in \Psi_{\mathrm{u}}^{-m-N}(M;\mathbf{F},\mathbf{E}). \tag{15.118}$$

The radius of a representation of these differences may depend on N; the four operators B_{L}, B_{R}, $B_{\mathrm{L}}A - I$, and $AB_{\mathrm{R}} - I$, however, have fixed propagations. Combining (15.118) with (15.115) gives, for every N,

$$B_{\mathrm{L}} A - I \in \Psi_{\mathrm{u}}^{-N}(M;\mathbf{E},\mathbf{E}), \qquad A B_{\mathrm{R}} - I \in \Psi_{\mathrm{u}}^{-N}(M;\mathbf{F},\mathbf{F}).$$

The two remainders are fixed finite-propagation operators belonging to $\Psi_{\mathrm{u}}^{-N}(M;\mathbf{E},\mathbf{E})$ and $\Psi_{\mathrm{u}}^{-N}(M;\mathbf{F},\mathbf{F})$, respectively, for every N. Apply the converse implication in Lemma 15.6.9: given p,q, choose $N > n+p+q+2$ and a representation of order $-N$ of the corresponding remainder. Absolute convergence of derivatives of its diagonal kernel and the uniformly smooth kernel of that representation provide the covariant bounds of orders p,q. Uniqueness of the distributional kernel makes these representations compatible as p,q varies; hence the two remainders belong to $\Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{E},\mathbf{E})$ and $\Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{F},\mathbf{F})$, respectively.

If $B_{\mathrm{L}} A = I - R_{\mathbf{E}}$ and $A B_{\mathrm{R}} = I - R_{\mathbf{F}}$, then

$$B_{\mathrm{L}} - B_{\mathrm{R}} = B_{\mathrm{L}} R_{\mathbf{F}} - R_{\mathbf{E}} B_{\mathrm{R}} \in \Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{F},\mathbf{E}).$$

Taking $B = B_{\mathrm{L}}$ and using the ideal property also on the right gives

$$BA - I \in \Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{E},\mathbf{E}), \qquad AB - I \in \Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{F},\mathbf{F}). \tag{15.119}$$

□

The preceding parametrix remains a symbolic construction. To obtain global regularity, we must relate uniform seminorms to the Sobolev norms defined through the localization system. Lemma 15.6.3 controls the smooth remainders, while the diagonal part reduces to the Euclidean theorem.

Theorem 15.6.14. (Uniform Sobolev mapping) Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary of dimension n and bounded geometry; let $\mathbf{E},\mathbf{F} \to M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and compatible connections of bounded geometry, and fix the preceding admissible system $\mathcal{Q}$. If $m,s \in \mathbb{R}$, $\rho \geq \delta_{\mathcal{Q}}$, and $A \in \Psi_{\mathrm{u},\rho}^{m}(M;\mathbf{E},\mathbf{F})$, then

$$A\colon H^{s+m}(M,\mathbf{E}) \longrightarrow H^{s}(M,\mathbf{F})$$

is continuous. There exist an integer $N = N(n,m,s)$ and a constant C_s, depending on the uniform data of the system but not on a chart index or on A, such that

$$\|A\|_{\mathcal{L}(H^{s+m}(M,\mathbf{E}),H^{s}(M,\mathbf{F}))} \leq C_s\bigl(1 + N_{\mathcal{Q}}(\rho)\bigr)\|A\|_{m,N;\rho}^{\mathrm{u},\mathcal{Q}}. \tag{15.120}$$

In particular, every $R \in \Psi_{\mathrm{u}}^{-\infty}(M;\mathbf{E},\mathbf{F})$ acts continuously from $H^{t}(M,\mathbf{E})$ to $H^{s}(M,\mathbf{F})$ for all $s,t \in \mathbb{R}$. All these estimates are uniform on bounded families in the calculus.

Proof. Let $\mathfrak{r} = ((a_i),R)$ be a diagonal representation of A of residual radius ρ. Introduce the analysis, diagonal block, and synthesis operators

$$\begin{aligned}
\mathbf{A}_{\mathbf{E}}\mathbf{u} &:= (\mathcal{A}_i^{\mathbf{E}}\mathbf{u})_{i\in I},\\
\mathbf{T}_{(a_i)}(v_i) &:= (\mathrm{Op}_{\omega_i}(a_i)v_i)_{i\in I},\\
\mathbf{S}_{\mathbf{F}}(v_i) &:= \sum_{i\in I} \mathcal{S}_i^{\mathbf{F}} v_i.
\end{aligned}$$

The part near the diagonal factors exactly as

$$A_{\mathrm{near}} = \mathbf{S}_{\mathbf{F}}\mathbf{T}_{(a_i)}\mathbf{A}_{\mathbf{E}}. \tag{15.121}$$

Analysis and synthesis are the retraction and coretraction operators of the preceding chapter: their continuity was established in (13.36) and their equivalence with global norms in Lemma 13.3.10. Reduction of the amplitude $\omega_i(x,y)a_i(x,\xi)$ uses finitely many derivatives, independently of i. Theorem 15.6.2, applied componentwise, therefore provides $N_0 = N_0(n,m,s)$ such that, for every $\mathbf{u} \in \Gamma_c(M,\mathbf{E})$,

$$\|A_{\mathrm{near}}\mathbf{u}\|_{H^{s}(M,\mathbf{F})} \leq C_s q_{N_0,m}^{\mathrm{u},\mathcal{Q}}((a_i))\|\mathbf{u}\|_{H^{s+m}(M,\mathbf{E})}. \tag{15.122}$$

For the remainder, localize $h_i R h_j$. This can be nonzero only if $j \in I_{\mathcal{Q}}(i,\rho)$. Apply Lemma 15.6.3, whose duality definition acts directly on restriction spaces, to the element $h_j\mathbf{u} \in H^{s+m}(U_j,\mathbf{E})$ on the fixed coordinate

rectangles, with

$$\sigma = s, \qquad \tau = s + m, \qquad N_1 := \max\{0, \lceil s \rceil\} + \max\{0, \lceil -s - m \rceil\}.$$

The supports of the localized kernels remain in a fixed compact subset of the rectangles, since the factors h_i, h_j are uniformly separated from the chart boundaries. Uniform comparison of partial and covariant derivatives up to order N_1 gives

$$\|h_i R h_j \mathbf{u}\|_{H^s(U_i,\mathbf{F})} \le C_s s^{\mathrm{u}}_{N_1}(R) \|h_j \mathbf{u}\|_{H^{s+m}(U_j,\mathbf{E})}.$$

Each sum has at most $N_Q(\rho)$ indices. Cauchy–Schwarz followed by reverse counting of the pairs (i, j) yields

$$\|R\mathbf{u}\|_{H^s(M,\mathbf{F})} \le C_s N_Q(\rho) s^{\mathrm{u}}_{N_1}(R) \|\mathbf{u}\|_{H^{s+m}(M,\mathbf{E})}. \tag{15.123}$$

With $N := \max\{N_0, N_1\}$, (15.122) and (15.123) first give

$$\|A\|_{\mathcal{L}(H^{s+m}(M,\mathbf{E}),H^s(M,\mathbf{F}))} \le C_s (1 + N_Q(\rho)) P^{\mathcal{Q}}_{m,N}(\mathfrak{r}).$$

Taking the infimum over all representations of radius ρ proves (15.120). If R is smoothing, use the representation $((0), R)$ and regard R as an operator of order $t - s$; this gives the mapping from $H^t(M,\mathbf{E})$ to $H^s(M,\mathbf{F})$. Density of $\Gamma_c(M,\mathbf{E})$, proved using the same analysis–synthesis pair in the preceding chapter, completes the extensions. □

Corollary 15.6.15. (Uniform elliptic estimate and regularity) Let $(M, \mathbf{g})$ be a complete Riemannian manifold without boundary of bounded geometry, and let $\mathbf{E}, \mathbf{F} \to M$ be complex vector bundles of the same finite rank equipped with Hermitian bundle metrics and compatible connections of bounded geometry. Let $m \in \mathbb{R}$ and let $A \in \Psi^m_{\mathrm{u,cl}}(M; \mathbf{E}, \mathbf{F})$ be uniformly elliptic. For $s, t \in \mathbb{R}$, there exists $C_{s,t} > 0$ such that

$$\|\mathbf{u}\|_{H^{s+m}(M,\mathbf{E})} \le C_{s,t} \big(\|A\mathbf{u}\|_{H^s(M,\mathbf{F})} + \|\mathbf{u}\|_{H^t(M,\mathbf{E})} \big), \qquad \mathbf{u} \in \Gamma_c(M,\mathbf{E}). \tag{15.124}$$

If $\mathbf{u} \in H^t(M,\mathbf{E})$ and $A\mathbf{u} \in H^s(M,\mathbf{F})$ in the distributional sense, then $\mathbf{u} \in H^{s+m}(M,\mathbf{E})$.

Proof. Let B be the parametrix of (15.119) and write $BA = I - R_{\mathbf{E}}$. The preceding theorem gives

$$\|\mathbf{u}\|_{H^{s+m}(M,\mathbf{E})} \le \|BA\mathbf{u}\|_{H^{s+m}(M,\mathbf{E})} + \|R_{\mathbf{E}}\mathbf{u}\|_{H^{s+m}(M,\mathbf{E})} \le C_{s,t} \big(\|A\mathbf{u}\|_{H^s(M,\mathbf{F})} + \|\mathbf{u}\|_{H^t(M,\mathbf{E})} \big),$$

since B has order $-m$ and $R_{\mathbf{E}}$ is uniformly smoothing. The same distributional identity proves the last assertion. □

Remark 15.6.16. (Uniform does not mean compact) On a noncompact manifold without boundary, an element of $\Psi^{-\infty}_{\mathrm{u}}(M; \mathbf{E}, \mathbf{F})$ need not be compact or Fredholm. On $\mathbb{R}^n$, let $0 \neq \kappa \in C^\infty_c(\mathbb{R}^n)$ and

$$Ru := \kappa * u.$$

Its symbol $(2\pi)^{\frac{n}{2}} \widehat{\kappa}(\xi)$ belongs to $S^{-\infty}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{C}, \mathbb{C}))$ with the unitary normalization of Definition 4.11.1, and its kernel has finite propagation and translation-invariant bounds. Thus $R \in \Psi^{-\infty}_{\mathrm{u}}(\mathbb{R}^n; \mathbb{C}, \mathbb{C})$. Choose $u \in C^\infty_c(\mathbb{R}^n)$ with $Ru \neq 0$ and set $u_j(x) = u(x - je_1)$. Then Ru_j is the corresponding translate of Ru; a subsequence

with disjoint supports cannot converge. Thus R is not compact. Moreover, $\widehat{\kappa}(\xi) \to 0$ as $\|\xi\| \to \infty$; normalized functions with Fourier supports in disjoint balls escaping to infinity form an orthonormal sequence (v_j) with $Rv_j \to 0$. Since v_j converges weakly to zero, its distance to any finite-dimensional subspace tends to one. This contradicts the lower bound for a Fredholm operator on the orthogonal complement of its kernel; thus R is not Fredholm either.

For the same reason, existence of a parametrix with *uniformly* smoothing remainders does not prove the Fredholm property. For example, $D_1 = \partial_1 \colon H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is uniformly elliptic but its range is not closed. On open manifolds, additional hypotheses at infinity are needed; they will not be introduced here.

15.6.4 Closed manifolds: compactness and Schatten classes

On a noncompact manifold, a uniformly smooth kernel can translate to infinity without producing a compact operator. On a closed manifold this possibility disappears: smooth kernels are compact and, in fact, belong to Schatten classes. This difference will allow passage from a smoothing parametrix to the Fredholm property.

First fix the functional-analytic terminology used in this subsection. Let H_1, H_2 be separable Hilbert spaces and let $T \colon H_1 \to H_2$ be compact. Its **singular values** are the eigenvalues, repeated according to multiplicity, arranged in nonincreasing order and padded with zeros when the rank is finite,

$$s_j(T) := \lambda_j\big((T^{\dagger}T)^{\frac{1}{2}}\big), \qquad j \geq 1.$$

For $1 \leq p < \infty$, the **Schatten class** $\mathfrak{S}_p(H_1, H_2)$ consists of compact operators such that

$$\|T\|_{\mathfrak{S}_p} := \left(\sum_{j=1}^{\infty} s_j(T)^p \right)^{\frac{1}{p}} < \infty.$$

In particular, $\mathfrak{S}_2$ is the Hilbert–Schmidt class and $\mathfrak{S}_1$ the trace class. If (e_j) is an orthonormal basis of H_1, then

$$\|T\|_{\mathfrak{S}_2}^2 = \sum_{j=1}^{\infty} \|Te_j\|_{H_2}^2; \tag{15.125}$$

this quantity is independent of the basis and is called the **Hilbert–Schmidt norm**. If $H_1 = H_2 = H$ and $T \in \mathfrak{S}_1(H)$, its **trace** is

$$\operatorname{Tr}(T) := \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle_H. \tag{15.126}$$

The series converges absolutely, and its sum is independent of the orthonormal basis. These assertions follow first for finite-rank operators by singular value decomposition and then by completeness in the norms $\mathfrak{S}_2$ and $\mathfrak{S}_1$.

Proposition 15.6.17. (Square-integrable kernels and products of Schatten classes) Let $(M, \mathbf{g})$ be a smooth closed Riemannian manifold of dimension n, and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics. An integral operator

$$(T_K u)(x) = \int_{M_y} K(x,y) u(y)\, d\lambda_{\mathbf{g}}(y)$$

is Hilbert–Schmidt if and only if, up to almost-everywhere equality, its kernel belongs to

$$L^2\big(M\times M;\operatorname{Hom}(\mathbf{E}_y,\mathbf{F}_x)\big).$$

Here $\|K(x,y)\|_{\mathrm{HS}}$ is the Euclidean norm of the matrix of $K(x,y)\colon \mathbf{E}_y\to\mathbf{F}_x$ in orthonormal bases, independent of those bases. In this case,

$$\|T_K\|_{\mathfrak{S}_2}^2=\int_{M_x\times M_y}\|K(x,y)\|_{\mathrm{HS}}^2\,d\lambda_{\mathbf{g}}(x)\,d\lambda_{\mathbf{g}}(y). \tag{15.127}$$

Moreover, if $C\in\mathfrak{S}_2(H_1,H_2)$ and $B\in\mathfrak{S}_2(H_2,H_3)$, then

$$BC\in\mathfrak{S}_1(H_1,H_3),\qquad \|BC\|_{\mathfrak{S}_1}\le\|B\|_{\mathfrak{S}_2}\|C\|_{\mathfrak{S}_2}. \tag{15.128}$$

Proof. The isometric identification between the Hilbert tensor product $L^2(M,\mathbf{F})\widehat{\otimes}\overline{L^2(M,\mathbf{E})}$ and Hilbert–Schmidt operators sends an elementary tensor $f\otimes\overline{e}$ to the operator $u\mapsto\langle u,e\rangle f$. In local orthonormal frames, this identification is precisely $K\mapsto T_K$; Parseval and Tonelli give (15.127). Surjectivity of the identification also proves the converse and uniqueness of the kernel in $L^2(M\times M;\operatorname{Hom}(\mathbf{E}_y,\mathbf{F}_x))$.
For the last assertion, use a singular value decomposition

$$Cu=\sum_{j=1}^{\infty}s_j(C)\langle u,e_j\rangle f_j,$$

where (e_j) is orthonormal in H_1 and (f_j) is an orthonormal system in H_2. Then

$$BCu=\sum_{j=1}^{\infty}s_j(C)\langle u,e_j\rangle Bf_j$$

is a nuclear representation. Indeed, a rank-one operator $u\mapsto\langle u,e\rangle h$ has $\mathfrak{S}_1$ norm equal to $\|e\|\|h\|$; by the triangle inequality, a series of such operators for which these quantities have finite sum converges in $\mathfrak{S}_1$. Now, by Cauchy–Schwarz and (15.125),

$$\sum_{j=1}^{\infty}s_j(C)\|Bf_j\|\le\left(\sum_{j=1}^{\infty}s_j(C)^2\right)^{\frac12}\left(\sum_{j=1}^{\infty}\|Bf_j\|^2\right)^{\frac12}\le\|C\|_{\mathfrak{S}_2}\|B\|_{\mathfrak{S}_2}.$$

In the last inequality, (f_j) was extended to an orthonormal basis of H_2. The partial sums therefore converge in trace-class norm and prove (15.128). □

Proposition 15.6.18. (Smoothing operators on a closed manifold) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold, and let $\mathbf{E},\mathbf{F}\longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and Hermitian connections. If $R\in\Psi^{-\infty}(M;\mathbf{E},\mathbf{F})$, then, for all $s,t\in\mathbb{R}$,

$$R\colon H^s(M,\mathbf{E})\longrightarrow H^t(M,\mathbf{F})$$

is continuous and compact. In fact, as an operator from $L^2(M,\mathbf{E})$ to $L^2(M,\mathbf{F})$, R is trace class.

Proof. After localizing the kernel on coordinate rectangles, the duality definition of Lemma 15.6.3 applies directly to every element of $H^s(M,\mathbf{E})$ and allows differentiation within the pairing with the distribution. Thus R maps $H^s(M,\mathbf{E})$ continuously into $H^N(M,\mathbf{F})$ for every N. Choose $N > t$. The inclusion $H^N(M,\mathbf{F}) \hookrightarrow H^t(M,\mathbf{F})$ is compact by the Rellich–Kondrashov theorem, obtained from 6.6.4 using a finite atlas; factorization proves compactness.

For the last assertion, localize the kernel on finitely many coordinate rectangles and extend each piece smoothly to a torus. Its double Fourier coefficients decay faster than any power. Thus the piece is a sum of rank-one operators with absolutely summable coefficients and is trace class. A finite sum preserves this property. □

Lemma 15.6.19. (Diagonal formula for a continuous kernel) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold, and let $\mathbf{E} \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric. If $T \in \mathfrak{S}_1(L^2(M,\mathbf{E}))$ has a continuous integral kernel $\mathbf{K}_T(x,y)$, then

$$\operatorname{Tr}(T) = \int_{M_x} \operatorname{tr}_{\mathbf{E}_x} \mathbf{K}_T(x,x)\, d\lambda_{\mathbf{g}}(x).$$

Proof. Fix a finite cover (V_i) with $\overline{V}_i$ contained in an open set U_i equipped with a unitary frame $(\mathbf{e}_{i,a})$. Take finite measurable subdivisions $\mathcal{P}_\nu$ whose positive-measure cells lie in some V_i and whose maximum diameter δ_ν tends to zero; discard null cells. For each cell $Q \subset V_i$ and each vector of the local frame, let

$$\mathbf{v}_{Q,a} := \lambda_{\mathbf{g}}(Q)^{-\frac{1}{2}} \mathbf{1}_Q \mathbf{e}_{i,a},$$

, and denote by P_ν the orthogonal projection onto the space spanned by these vectors. If $\mathbf{u} \in \Gamma(M,\mathbf{E})$ and $\mathbf{u} = \sum_{a=1}^{r_{\mathbf{E}}} u_{i,a}\mathbf{e}_{i,a}$ in U_i, then, on $Q \subset V_i$,

$$P_\nu \mathbf{u} = \sum_{a=1}^{r_{\mathbf{E}}} \left(\frac{1}{\lambda_{\mathbf{g}}(Q)} \int_{Q_y} u_{i,a}(y)\, d\lambda_{\mathbf{g}}(y) \right) \mathbf{e}_{i,a}.$$

Uniformity of the moduli of continuity of the coefficients over this finite cover gives $\|P_\nu \mathbf{u} - \mathbf{u}\|_{L^2(M,\mathbf{E})} \leq C\omega_{\mathbf{u}}(\delta_\nu)$. Density of smooth sections and $\|P_\nu\| \leq 1$ imply that $P_\nu \to I$ strongly.

Since T is trace class,

$$\|T - P_\nu T P_\nu\|_{\mathfrak{S}_1} \longrightarrow 0;$$

indeed, $T - P_\nu T P_\nu = (I - P_\nu)T + P_\nu T(I - P_\nu)$, and both terms converge to zero in trace norm, first for T of finite rank and then by approximating T and $T^\dagger$ in $\mathfrak{S}_1$. Consequently,

$$\operatorname{Tr}(T) = \lim_{\nu \to \infty} \sum_{Q \in \mathcal{P}_\nu} \sum_{a=1}^{r_{\mathbf{E}}} \frac{1}{\lambda_{\mathbf{g}}(Q)} \int_{Q_x \times Q_y} \left\langle \mathbf{K}_T(x,y)\mathbf{e}_{i,a}(y), \mathbf{e}_{i,a}(x) \right\rangle d\lambda_{\mathbf{g}}(y)\, d\lambda_{\mathbf{g}}(x).$$

On $\overline{V}_i \times \overline{V}_i$, set

$$k_{i,a}(x,y) := \left\langle \mathbf{K}_T(x,y)\mathbf{e}_{i,a}(y), \mathbf{e}_{i,a}(x) \right\rangle.$$

These functions are uniformly continuous. If ω is the maximum of their moduli of continuity, the difference

between the last sum and the diagonal integral is bounded by

$$\operatorname{rk}(\mathbf{E})\lambda_{\mathbf{g}}(M)\omega(\delta_\nu),$$

which tends to zero. Finally, $\sum_{a=1}^{r_{\mathbf{E}}} k_{i,a}(x,x) = \operatorname{tr}_{\mathbf{E}_x} \mathbf{K}_T(x,x)$, so the limit is independent of the frame and proves the formula. □

Theorem 15.6.20. (Hilbert–Schmidt and trace-class thresholds) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold of dimension n, and let $\mathbf{E},\mathbf{F} \longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and Hermitian connections. Let $m \in \mathbb{R}$.

(i) If $A \in \Psi^m(M;\mathbf{E},\mathbf{F})$ and $m < -\frac{n}{2}$, then

$$A \in \mathfrak{S}_2\big(L^2(M,\mathbf{E}), L^2(M,\mathbf{F})\big),$$

that is, A is Hilbert–Schmidt.

(ii) If $m < -n$, then

$$A \in \mathfrak{S}_1\big(L^2(M,\mathbf{E}), L^2(M,\mathbf{F})\big).$$

If $\mathbf{E} = \mathbf{F}$, its trace is given by the diagonal formula

$$\operatorname{Tr}(A) = \int_{M_x} \operatorname{tr}_{\mathbf{E}_x} \mathbf{K}_A(x,x)\, d\lambda_{\mathbf{g}}(x). \tag{15.129}$$

If, with the convention of Definition 15.4.7,

$$A = \operatorname{Op}_{\mathbf{g},\nabla,\chi}(a) + R, \qquad R \in \Psi^{-\infty}(M;\mathbf{E},\mathbf{E}),$$

and $m < -n$, then

$$\operatorname{Tr}(A) = (2\pi)^{-n} \int_{M_x} \int_{T_x^*M} \operatorname{tr}_{\mathbf{E}_x} a(x,\xi)\, d\xi_x\, d\lambda_{\mathbf{g}}(x) + \int_{M_x} \operatorname{tr}_{\mathbf{E}_x} \mathbf{K}_R(x,x)\, d\lambda_{\mathbf{g}}(x). \tag{15.130}$$

Proof. First localize a piece whose symbol $a(x,\xi)$ has compact support in x. Its kernel is

$$K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi.$$

Plancherel in the variable y gives

$$\int_{\mathbb{R}^n_y} \|K(x,y)\|_{\mathrm{HS}}^2\, dy = (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} \|a(x,\xi)\|_{\mathrm{HS}}^2\, d\xi.$$

The right-hand side is integrable in x when $2m < -n$. The square of the Hilbert–Schmidt norm of an integral operator is the $L^2(M\times M;\operatorname{Hom}(\mathbf{E}_y,\mathbf{F}_x))$ norm of its kernel; a finite atlas and Proposition 15.6.18 for the remainders prove (i).

Suppose $m < -n$. Construct a scalar elliptic operator $Q_{\mathbf{E}} \in \Psi^{\frac{m}{2}}(M;\mathbf{E},\mathbf{E})$ with local full symbol $\langle\xi\rangle^{\frac{m}{2}} I_{\mathbf{E}}$ modulo lower-order terms and hence principal symbol $\|\xi\|^{\frac{m}{2}} I_{\mathbf{E}}$, and a parametrix $C_{\mathbf{E}} \in \Psi^{-\frac{m}{2}}(M;\mathbf{E},\mathbf{E})$. If $C_{\mathbf{E}} Q_{\mathbf{E}} = I - R_{\mathbf{E}}$, then

$$A = (AC_{\mathbf{E}})Q_{\mathbf{E}} + AR_{\mathbf{E}}.$$

The first two factors have order $\frac{m}{2} < -\frac{n}{2}$ and are Hilbert–Schmidt by (i); their product is trace class. The last term is smoothing and trace class by the preceding proposition. This proves (ii).

When $m < -n$, in each localized piece the integral defining the kernel has the uniform majorant $C\langle\xi\rangle^m \in L^1(\mathbb{R}^n)$. Dominated convergence shows that the kernel is jointly continuous in (x,y); a finite sum of pieces and the smooth remainder preserve this property. Lemma 15.6.19, applied to the trace-class operator already constructed, gives (15.129). In geometric quantization, $j_x(0) = 1$ and parallel transport on the diagonal is the identity; thus

$$\mathbf{K}_{\mathrm{Op}_{\mathbf{g},\nabla,\chi}(a)}(x,x) = (2\pi)^{-n} \int_{T_x^*M} a(x,\xi)\, d\xi_x.$$

Adding the term R proves (15.130). □

15.6.5 Elliptic operators and Fredholm theory

An elliptic parametrix gives inverse identities modulo smoothing operators. On a closed manifold these remainders are compact, so Atkinson's theory immediately turns symbolic inversion into a Fredholm statement. We begin with the form of this criterion suited to the two identities of a parametrix.

Lemma 15.6.21. (Compact parametrix criterion) Let X, Y be Hilbert spaces and let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,X)$. If

$$ST - I_X \in \mathcal{K}(X), \qquad TS - I_Y \in \mathcal{K}(Y),$$

, then T is Fredholm in the sense of Definition B.2.40.

Proof. Let $T^\dagger\colon Y \to X$ and $S^\dagger\colon X \to Y$ denote the bounded Hilbert space adjoints. These are not formal adjoints of differential operators: they depend on the specific inner products of X and Y. If $Tu = 0$, then $u = (I_X - ST)u$; the unit ball of $\ker T$ is therefore relatively compact. A normed space with relatively compact unit ball is finite-dimensional, so $\dim\ker T < \infty$. Applying the same argument to $T^\dagger$ and $S^\dagger$ gives $\dim\ker T^\dagger < \infty$.

Let us prove that the range is closed. On $(\ker T)^\perp$, if there were no $c > 0$ with $\|Tu\|_Y \geq c\|u\|_X$, there would be a sequence $u_j \perp \ker T$, $\|u_j\|_X = 1$, and $Tu_j \to 0$. From

$$u_j = STu_j + (I_X - ST)u_j$$

and compactness of $I_X - ST$, we could extract a subsequence converging to u. Then $\|u\| = 1$, $Tu = 0$, and $u \perp \ker T$, a contradiction. The lower bound on $(\ker T)^\perp$ implies that $\mathrm{Ran}\, T$ is closed. Finally,

$$\mathrm{coker}\, T \cong (\mathrm{Ran}\, T)^\perp = \ker T^\dagger$$

is finite-dimensional. □

Theorem 15.6.22. (Elliptic Fredholm theorem on a closed manifold) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold, and let $\mathbf{E},\mathbf{F}\longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and Hermitian connections. Let $m,s\in\mathbb{R}$ and let $A\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic. Then

$$A_s\colon H^{s+m}(M,\mathbf{E})\longrightarrow H^s(M,\mathbf{F})$$

is Fredholm. Its kernel and cokernel have smooth representatives:

$$\ker A_s=\{\mathbf{u}\in\Gamma(M,\mathbf{E})\mid A\mathbf{u}=0\},\tag{15.131}$$

$$\big(\operatorname{coker}A_s\big)'\cong\{\mathbf{w}\in\Gamma(M,\overline{\mathbf{F}})\mid\overline{A_h^*}\mathbf{w}=0\}\cong\overline{\{\mathbf{v}\in\Gamma(M,\mathbf{F})\mid A_h^*\mathbf{v}=0\}}.\tag{15.132}$$

In particular,

$$\operatorname{ind}(A_s)=\dim\ker\big(A\colon\Gamma(M,\mathbf{E})\longrightarrow\Gamma(M,\mathbf{F})\big)-\dim\ker\big(A_h^*\colon\Gamma(M,\mathbf{F})\longrightarrow\Gamma(M,\mathbf{E})\big).\tag{15.133}$$

Proof. The elliptic construction provides a parametrix $B\in\Psi^{-m}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{E})$ and remainders

$$BA=I-R_{\mathbf{E}},\qquad AB=I-R_{\mathbf{F}},\qquad R_{\mathbf{E}}\in\Psi^{-\infty}(M;\mathbf{E},\mathbf{E}),\qquad R_{\mathbf{F}}\in\Psi^{-\infty}(M;\mathbf{F},\mathbf{F}).$$

By Corollary 15.6.6, $B\colon H^s(M,\mathbf{F})\longrightarrow H^{s+m}(M,\mathbf{E})$ is continuous, and by Proposition 15.6.18 both remainders are compact on the corresponding spaces. Lemma 15.6.21 proves that A_s is Fredholm.
If $\mathbf{u}$ is a distribution and $A\mathbf{u}=0$, the first identity gives $\mathbf{u}=R_E\mathbf{u}$. Since $R_{\mathbf{E}}$ is smoothing, $\mathbf{u}$ is smooth. This proves (15.131) and shows that the kernel is independent of the Sobolev order. By Proposition 14.5.5, applied on the closed manifold, the complex-linear dual of $H^s(M,\mathbf{F})$ is $H^{-s}(M,\overline{\mathbf{F}})$. Define the conjugate of the Hermitian adjoint by

$$\overline{A_h^*}\colon\Gamma(M,\overline{\mathbf{F}})\longrightarrow\Gamma(M,\overline{\mathbf{E}}),\qquad\overline{A_h^*}(\overline{\mathbf{v}}):=\overline{A_h^*\mathbf{v}}.$$

It is complex-linear. In charts it is obtained by conjugating the coefficients and kernel of A_h^*; hence it belongs to $\Psi^m_{\mathrm{cl}}(M;\overline{\mathbf{F}},\overline{\mathbf{E}})$ and is elliptic. The defining identity of the Hermitian adjoint becomes the bilinear identity

$$\langle A\mathbf{u},\mathbf{w}\rangle_{\mathbf{F},\overline{\mathbf{F}}}=\langle\mathbf{u},\overline{A_h^*}\mathbf{w}\rangle_{\mathbf{E},\overline{\mathbf{E}}}.$$

Consequently, the annihilator of the range of A_s is identified with

$$\ker\left(\overline{A_h^*}\colon H^{-s}(M,\overline{\mathbf{F}})\longrightarrow H^{-s-m}(M,\overline{\mathbf{E}})\right).$$

The conjugate operator is elliptic, and the same smoothing argument makes its solutions smooth. Pointwise conjugation identifies its kernel with the conjugate space of $\ker A_h^*$. This gives (15.132) and, together with the definition of the index, yields (15.133). □

Corollary 15.6.23. (Independence of the Sobolev exponent) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E},\mathbf{F}\longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and Hermitian connections; and let $m\in\mathbb{R}$ and let $A\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic. Then

$$\operatorname{ind}\left(A\colon H^{s+m}(M,\mathbf{E})\longrightarrow H^s(M,\mathbf{F})\right)$$

is independent of $s \in \mathbb{R}$.

Proof. The two spaces of smooth solutions appearing in (15.133) are independent of s. □

15.6.6 Analytic stability of the index

Once the Fredholm property is established, the index can change only if the family leaves the set of Fredholm operators. Compact perturbations and continuous deformations preserve this set; applied to the pseudodifferential calculus, they show that lower-order terms do not change the index.

Theorem 15.6.24. (Compact perturbations and continuous families) Let X, Y be Hilbert spaces.

(i) If $T \in \mathcal{L}(X,Y)$ is Fredholm and $K \in \mathcal{K}(X,Y)$, then $T+K$ is Fredholm and

$$\operatorname{ind}(T+K) = \operatorname{ind}(T).$$

(ii) The set of Fredholm operators is open in $\mathcal{L}(X,Y)$, and the index is locally constant.

(iii) If Λ is connected and $\lambda \mapsto T_\lambda \in \mathcal{L}(X,Y)$ is a continuous family of Fredholm operators, then $\operatorname{ind}(T_\lambda)$ is constant on Λ.

Proof. Since T is Fredholm, there are orthogonal decompositions

$$X = \ker T \oplus X_0, \qquad Y = \operatorname{Ran} T \oplus Y_0,$$

with Y_0 finite-dimensional. The restriction $T\colon X_0 \to \operatorname{Ran} T$ is a continuous isomorphism. Define $S\colon Y \to X$ as its inverse on $\operatorname{Ran} T$ and zero on Y_0. Then $ST - I_X$ and $TS - I_Y$ have finite rank. For $T+K$, the products with S differ from the identity by compact operators, so Lemma 15.6.21 proves that $T+K$ is Fredholm.
For operators sufficiently close to T, the components between X_0 and $\operatorname{Ran} T$ remain invertible by stability of invertibility. Eliminating this block by invertible operations reduces the nearby operator to a map between $\ker T$ and Y_0. Its index is

$$\dim \ker T - \dim Y_0 = \operatorname{ind} T,$$

regardless of the rank of this finite block. This proves openness and local constancy simultaneously. Applied to the path $T+tK$, it proves equality of indices in (i). Finally, a locally constant integer-valued function is constant on a connected space, giving (iii). □

Corollary 15.6.25. (Pseudodifferential stability) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and Hermitian connections; let $m, s \in \mathbb{R}$; and let $A \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic.

(a) If $K\colon H^{s+m}(M,\mathbf{E}) \longrightarrow H^s(M,\mathbf{F})$ is compact, then $A+K$ is Fredholm with the same index as A.

(b) In particular, this holds if K is smoothing. It also holds for $K \in \Psi^{m-\varepsilon}(M;\mathbf{E},\mathbf{F})$ with $\varepsilon > 0$, since

$$H^{s+m}(M,\mathbf{E}) \xrightarrow{K} H^{s+\varepsilon}(M,\mathbf{F}) \hookrightarrow H^s(M,\mathbf{F})$$

and the second arrow is compact.

(c) If $(A_\lambda)_{\lambda\in\Lambda}$ is an elliptic family of order m, continuous in the pseudodifferential topology of order m, and Λ is connected, then $\operatorname{ind} A_\lambda$ is independent of λ.

Proof. The first two parts follow from Proposition 15.6.18, the preceding theorem, and Rellich. For (c), the mapping theorem shows that continuity in sufficiently many symbol seminorms implies continuity in

$$\mathcal{L}\big(H^{s+m}(M,\mathbf{E}),H^s(M,\mathbf{F})\big).$$

Each A_λ is Fredholm by Theorem 15.6.22; part (iii) of the preceding theorem then applies. □

The stability obtained here is entirely analytic: it uses parametrices, regularity, compactness, and norm continuity. On a closed manifold, these are exactly the arguments that turn ellipticity of the symbol into a well-defined stable integer.

15.7 Complex powers, heat, and the analytic index

The parametrix of an elliptic operator inverts its symbol with respect to composition. To construct noninteger powers, we must uniformly invert an entire family: the resolvent $\lambda I - A$. The variable λ does not have the same weight as the covariable ξ. If A has order $m > 0$, the natural scaling is

$$(\xi,\lambda)\longmapsto(t\xi,t^m\lambda).$$

This scaling governs the resolvent parametrix, complex powers, and their comparison with the spectral calculus.

Throughout this section, $(M,\mathbf{g})$ is a smooth closed Riemannian manifold of dimension n, and $\mathbf{E}\longrightarrow M$ is a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. These data remain fixed, and within this section we abbreviate

$$\Gamma(\mathbf{E}):=\Gamma(M,\mathbf{E}),\qquad \Psi^\mu(M;\mathbf{E}):=\Psi^\mu(M;\mathbf{E},\mathbf{E}).$$

Let

$$A\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E}),\qquad m>0,$$

be a classical elliptic operator. Write $a\sim\sum_{j=0}^{\infty}a_{m-j}$ for its Fourier symbol in coordinates and local frames. If A is differential, recall the conversion fixed at the beginning of the chapter:

$$a_m(x,\xi)=i^m\sigma_m(A)(x,\xi).$$

The sectorial condition is formulated for a_m, since this is the symbol entering quantization. By $\sigma(A)$ and $\rho(A)$ we mean the spectrum and resolvent of the closed realization on $L^2(M,\mathbf{E})$ with domain $H^m(M,\mathbf{E})$.

Lemma 15.7.1. (Graph norm and elliptic realization) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E}\longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$; and let $A\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$ be elliptic, with $m>0$. Then there are constants $c,C>0$

such that

$$c\|\mathbf{u}\|_{H^m(M,\mathbf{E})} \le \|\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|A\mathbf{u}\|_{L^2(M,\mathbf{E})} \le C\|\mathbf{u}\|_{H^m(M,\mathbf{E})}, \qquad \mathbf{u} \in \Gamma(M,\mathbf{E}). \tag{15.134}$$

The operator $A\colon H^m(M,\mathbf{E}) \subset L^2(M,\mathbf{E}) \longrightarrow L^2(M,\mathbf{E})$ is closed, $\Gamma(M,\mathbf{E})$ is a core, and its graph norm is equivalent to the $H^m(M,\mathbf{E})$ norm.

Proof. The mapping property $A : H^m(M,\mathbf{E}) \to L^2(M,\mathbf{E})$ gives $\|\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|A\mathbf{u}\|_{L^2(M,\mathbf{E})} \le C\|\mathbf{u}\|_{H^m(M,\mathbf{E})}$. Let $B \in \Psi^{-m}_{\mathrm{cl}}(M;\mathbf{E})$ and $R \in \Psi^{-\infty}(M;\mathbf{E},\mathbf{E})$ satisfy $BA = I - R$. Then

$$\|\mathbf{u}\|_{H^m(M,\mathbf{E})} \le \|BA\mathbf{u}\|_{H^m(M,\mathbf{E})} + \|R\mathbf{u}\|_{H^m(M,\mathbf{E})} \le C_1\|A\mathbf{u}\|_{L^2(M,\mathbf{E})} + C_2\|\mathbf{u}\|_{L^2(M,\mathbf{E})},$$

since $B : L^2(M,\mathbf{E}) \to H^m(M,\mathbf{E})$ and $R : L^2(M,\mathbf{E}) \to H^m(M,\mathbf{E})$ are continuous. This proves (15.134) on $\Gamma(\mathbf{E})$ and, by density, on $H^m(M,\mathbf{E})$.

If $\mathbf{u}_j \in H^m(M,\mathbf{E})$, $\mathbf{u}_j \to \mathbf{u}$ in $L^2(M,\mathbf{E})$, and $A\mathbf{u}_j \to \mathbf{f}$ in $L^2(M,\mathbf{E})$, the first inequality applied to $\mathbf{u}_j - \mathbf{u}_k$ shows that $(\mathbf{u}_j)$ is Cauchy in $H^m(M,\mathbf{E})$. Its limit in $H^m(M,\mathbf{E})$ agrees with $\mathbf{u}$, and continuity $A : H^m(M,\mathbf{E}) \to L^2(M,\mathbf{E})$ gives $A\mathbf{u} = \mathbf{f}$; hence the realization is closed. Finally, for $\mathbf{u} \in H^m(M,\mathbf{E})$ choose $\mathbf{v}_j \in \Gamma(\mathbf{E})$ with $\mathbf{v}_j \to \mathbf{u}$ in $H^m(M,\mathbf{E})$. The second inequality implies $\mathbf{v}_j \to \mathbf{u}$ in graph norm. This is exactly the assertion that $\Gamma(\mathbf{E})$ is a core. □

Proposition 15.7.2. (Compact resolvent and discrete spectrum) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E} \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$; and let $A \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$ be elliptic, with $m > 0$, realized on $L^2(M,\mathbf{E})$ with domain $H^m(M,\mathbf{E})$. If $\rho(A) \neq \varnothing$, every resolvent of this realization is compact: for each $\lambda_0 \in \rho(A)$,

$$(\lambda_0 I - A)^{-1} : L^2(M,\mathbf{E}) \longrightarrow L^2(M,\mathbf{E})$$

is compact. Under this hypothesis, $\sigma(A)$ consists of isolated eigenvalues, each of finite algebraic multiplicity, and has no finite accumulation points. If A is self-adjoint and bounded below, then $\rho(A) \neq \varnothing$ and there is an orthonormal basis $(\boldsymbol{\varphi}_j)$ of $L^2(M,\mathbf{E})$ consisting of smooth eigensections. Repeating eigenvalues according to multiplicity,

$$A\boldsymbol{\varphi}_j = \lambda_j \boldsymbol{\varphi}_j, \qquad \lambda_j \longrightarrow +\infty.$$

Proof. A parametrix $B \in \Psi^{-m}_{\mathrm{cl}}(M;\mathbf{E})$ and a remainder $R \in \Psi^{-\infty}(M;\mathbf{E},\mathbf{E})$ give $\mathbf{u} = BAu + R\mathbf{u}$ for $\mathbf{u} \in \Gamma(M,\mathbf{E})$. The mapping properties and extension by density yield the elliptic estimate

$$\|\mathbf{u}\|_{H^m(M,\mathbf{E})} \le C\big(\|A\mathbf{u}\|_{L^2(M,\mathbf{E})} + \|\mathbf{u}\|_{L^2(M,\mathbf{E})}\big), \qquad \mathbf{u} \in H^m(M,\mathbf{E}). \tag{15.135}$$

If $\mathbf{u} = (\lambda_0 I - A)^{-1} f$, continuity of the resolvent on $L^2(M,\mathbf{E})$ and $A\mathbf{u} = \lambda_0 \mathbf{u} - f$ show, through (15.135), that $(\lambda_0 I - A)^{-1}$ is continuous from $L^2(M,\mathbf{E})$ to $H^m(M,\mathbf{E})$. The inclusion $H^m(M,\mathbf{E}) \hookrightarrow L^2(M,\mathbf{E})$ is compact by Rellich, so the resolvent viewed on $L^2(M,\mathbf{E})$ is compact.

The spectral correspondence

$$\lambda \in \sigma(A) \quad \Longleftrightarrow \quad (\lambda_0 - \lambda)^{-1} \in \sigma\big((\lambda_0 I - A)^{-1}\big), \qquad \lambda \neq \lambda_0,$$

reduces the assertions on isolation and algebraic multiplicity to the spectral theorem for compact operators.

In the self-adjoint bounded-below case, choose a real λ_0 below the spectrum. The resolvent is then compact and self-adjoint, and its spectral theorem provides an orthonormal basis of eigenvectors. The preceding correspondence and the lower bound force $\lambda_j \to +\infty$. Finally, $A\varphi_j = \lambda_j \varphi_j$ and elliptic regularity imply $\varphi_j \in \Gamma(\mathbf{E})$. □

15.7.1 Spectral cut and a minimal parameter-dependent calculus

Constructing A^z requires choosing a branch of the logarithm and controlling the resolvent along a sector avoiding both the principal spectrum and the spectrum of the closed realization. We first fix the cut, then construct the uniform parametrix of $\lambda I - A$.

For $\theta \in \mathbb{R}$, set

$$L_\theta := \{re^{i\theta} \mid r > 0\}.$$

On $\mathbb{C} \setminus (L_\theta \cup \{0\})$, fix the branch

$$\log_\theta \lambda = \log|\lambda| + i \arg_\theta \lambda, \qquad \theta - 2\pi < \arg_\theta \lambda < \theta,$$

and write $\lambda^z := e^{z \log_\theta \lambda}$.

Definition 15.7.3. (Agmon angle and spectral cut) Let $(M, \mathbf{g})$ be a smooth closed Riemannian manifold, let $(\mathbf{E}, \mathbf{h}_\mathbf{E}) \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric, and let $A \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{E})$ be elliptic, with $m > 0$ and principal Fourier symbol a_m. Let $\theta \in \mathbb{R}$. We say that θ is an **Agmon angle for the principal symbol** of A if

$$\operatorname{spec} a_m(x, \xi) \cap L_\theta = \varnothing$$

for every $(x, \xi) \in T^*M \setminus 0_M$. It is an **admissible spectral cut for** A if, in addition,

$$\sigma(A) \cap L_\theta = \varnothing.$$

To define all powers, positive and negative, we also assume $0 \in \rho(A)$, that is, invertibility of A. If 0 is an eigenvalue, reduced powers require separating the generalized eigenspace, as specified next.

If $0 \in \sigma(A)$ and A is not necessarily self-adjoint, the correct separation uses more than $\ker A$ alone. Existence of the admissible spectral cut implies $\rho(A) \neq \varnothing$; Proposition 15.7.2 then shows that 0 is isolated. Let

$$\Pi_0 := \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} (\lambda I - A)^{-1} \, d\lambda \tag{15.136}$$

be its Riesz projection, with the circle positively oriented and containing no other spectral point. Its range is the finite-dimensional generalized eigenspace of 0, and $L^2(M, \mathbf{E}) = \operatorname{ran} \Pi_0 \oplus \ker \Pi_0$ is a A-invariant decomposition. The restriction of A to $\ker \Pi_0$ is invertible; by **reduced power** we mean exclusively the power of this restriction, extended by zero on $\operatorname{ran} \Pi_0$. This convention asserts no group law on all of $L^2(M, \mathbf{E})$. If A is self-adjoint, Π_0 is the orthogonal projection onto $\ker A$, and no nilpotent part occurs.

In a cotangent chart, with the Euclidean norm fixed by this chapter's local convention, the quantity

$$\rho_m(\xi,\lambda) := \|\xi\|_{\mathbb{R}^n} + |\lambda|^{\frac{1}{m}}$$

is homogeneous of degree one under the weighted scaling fixed above. For smooth excisions, use the equivalent weight

$$\widehat{\rho}_m(\xi,\lambda) := \left(\langle\xi\rangle^{2m} + |\lambda|^2\right)^{\frac{1}{2m}}. \tag{15.137}$$

Unlike ρ_m, this weight is smooth also on the axes; the two are comparable in the region of high frequencies and large parameters.

Definition 15.7.4. (Parameter-dependent symbol and ellipticity) Let $m > 0$, let $U \subseteq \mathbb{R}^n$ be open, let $r \in \mathbb{N}$, and let $\Lambda \subseteq \mathbb{C}$ be a closed sector. Consider families with values in $\mathrm{End}(\mathbb{C}^r)$ and set

$$\Omega_m(\xi,\lambda) := |\lambda| + \langle\xi\rangle^m.$$

If

$$c \in C^\infty(U \times \mathbb{R}^n \times \Lambda; \mathrm{End}(\mathbb{C}^r))$$

is smooth up to the faces of the sector in the real variables $(x,\xi,\operatorname{Re}\lambda,\operatorname{Im}\lambda)$, for $K \Subset U$, $j \in \mathbb{N}_0$, $\alpha,\beta \in \mathbb{N}_0^n$, and $k,\ell \in \mathbb{N}_0$ define the following seminorms, where $\|\cdot\|_{\mathrm{op},\mathbb{C}^r}$ is the operator norm induced by the standard Hermitian norm:

$$\mathfrak{p}^{\mathrm{R},j}_{K,\alpha,\beta,k,\ell}(c) := \sup_{\substack{x\in K,\ \xi\in\mathbb{R}^n\\ \lambda\in\Lambda}} \langle\xi\rangle^{j+|\beta|}\Omega_m(\xi,\lambda)^{1+k+\ell}\left\|D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell c(x,\xi,\lambda)\right\|_{\mathrm{op},\mathbb{C}^r}, \tag{15.138}$$

$$\mathfrak{p}^{\mathrm{D},j}_{K,\alpha,\beta,k,\ell}(c) := \sup_{\substack{x\in K,\ \xi\in\mathbb{R}^n\\ \lambda\in\Lambda}} \langle\xi\rangle^{j+|\beta|}\Omega_m(\xi,\lambda)^{k+\ell}\left\|D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell c(x,\xi,\lambda)\right\|_{\mathrm{op},\mathbb{C}^r}. \tag{15.139}$$

Here $D_x^\alpha = \partial_x^\alpha$ and $D_\xi^\beta = \partial_\xi^\beta$, in accordance with this chapter's convention, while $\partial_\lambda = \frac{1}{2}(\partial_{\operatorname{Re}\lambda} - i\partial_{\operatorname{Im}\lambda})$ and $\partial_{\bar\lambda} = \frac{1}{2}(\partial_{\operatorname{Re}\lambda} + i\partial_{\operatorname{Im}\lambda})$ are the Wirtinger derivatives.

A family $q^{(j)}$, with $j \geq 0$, has **resolvent type** j if all seminorms (15.138) are finite. It has **defect type** j if all seminorms (15.139) are finite. In particular, for the first type,

$$\|D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell q^{(j)}(x,\xi,\lambda)\|_{\mathrm{op},\mathbb{C}^r} \leq C_{K,\alpha,\beta,k,\ell,j}\langle\xi\rangle^{-j-|\beta|}\Omega_m(\xi,\lambda)^{-1-k-\ell},$$

and for the second the last factor is replaced by $\Omega_m^{-k-\ell}$. Families of resolvent type j and defect type j behave as orders $-m-j$ and $-j$, respectively, in the cone $|\lambda| \leq C\langle\xi\rangle^m$, while retaining the correct estimate when $|\lambda|$ dominates.

A family is **holomorphic in the parameter** if $\partial_{\bar\lambda} c = 0$ in the interior of Λ. It is **simultaneously smoothing in ξ and λ** if, for all $P,Q \in \mathbb{N}_0$, every compact set K, and all α,β,k,ℓ, there exists a constant $C_{K,P,Q,\alpha,\beta,k,\ell} > 0$ such that

$$\left\|D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell c(x,\xi,\lambda)\right\|_{\mathrm{op},\mathbb{C}^r} \leq C_{K,P,Q,\alpha,\beta,k,\ell}\langle\xi\rangle^{-P}(1+|\lambda|)^{-Q}. \tag{15.140}$$

We abbreviate this condition by saying that c is smoothing with parameter. Membership in every resolvent type j imposes only arbitrary decay in ξ and the resolvent factor in λ; condition (15.140) additionally requires every power of $1+|\lambda|$ and does not follow from the former condition.

This is the weak version of the parameter-dependent calculus. The strong bound condensed into a single power of ρ_m fails for a general classical pseudodifferential operator when $|\lambda|$ dominates $\langle\xi\rangle^m$; the preceding separate bounds remain valid. At high frequencies, a family of resolvent type 0 is **classical with parameter** if its components are homogeneous under $(\xi,\lambda)\mapsto(t\xi,t^m\lambda)$ and the remainder after N components has resolvent type N for $\|\xi\|\geq 1$. The complementary bounded-frequency part is required only to be smoothing in ξ, retaining the decay factor $\Omega_m^{-1-k-\ell}$. If c_j has resolvent type j, write

$$c\sim\sum_{j=0}^{\infty}c_j\quad\Longleftrightarrow\quad c-\sum_{j=0}^{N-1}c_j\text{ has resolvent type }N\text{ for every }N\in\mathbb{N}_0. \tag{15.141}$$

The same convention, replacing *resolvent* by *defect*, is used for defect-type families.

The homogeneous principal symbol a_m is **elliptic with parameter** in Λ if $\lambda I-a_m(x,\xi)$ is invertible for $\xi\neq 0$ and $\lambda\in\Lambda$ and, for every $K\Subset U$, there exists $C_K>0$ such that its inverse satisfies

$$\|\big(\lambda I-a_m(x,\xi)\big)^{-1}\|\leq C_K\big(|\lambda|+\|\xi\|^m\big)^{-1},\qquad x\in K.$$

This condition is stronger than ellipticity of A without a parameter.

The smooth weight in (15.137) satisfies

$$2^{-\frac12}\Omega_m(\xi,\lambda)\leq\widehat{\rho}_m(\xi,\lambda)^m\leq\Omega_m(\xi,\lambda). \tag{15.142}$$

Indeed, for $a,b\geq 0$ we have $\frac{a+b}{\sqrt2}\leq(a^2+b^2)^{\frac12}\leq a+b$; it suffices to take $a=\langle\xi\rangle^m$ and $b=|\lambda|$. If $\chi\in C^\infty(\mathbb{R})$ is constant outside a compact subset of $(0,\infty)$, fix $0<c_\chi<C_\chi$ such that $\operatorname{supp}\chi'\subseteq[c_\chi,C_\chi]$. The chain rule gives, when $|\gamma|+r+s\geq 1$,

$$\big|D_\xi^\gamma\partial_\lambda^r\partial_{\bar\lambda}^s\chi(\varepsilon\widehat{\rho}_m)\big|\leq C_{\gamma,r,s}\langle\xi\rangle^{-|\gamma|}\Omega_m^{-r-s}\mathbf{1}_{\{c_\chi\varepsilon^{-1}\leq\widehat{\rho}_m\leq C_\chi\varepsilon^{-1}\}}. \tag{15.143}$$

To justify every derivative order, first differentiate $\widehat{\rho}_m^{2m}=\langle\xi\rangle^{2m}+|\lambda|^2$. A derivative in ξ of the resulting quotient is bounded by $C\langle\xi\rangle^{-1}$, and a Wirtinger derivative by $C\widehat{\rho}_m^{-m}$. Repeated use of the Leibniz rule gives terms containing $|\gamma|$ factors of the first type and $r+s$ of the second, possibly grouped into higher-order derivatives; their total exponents remain the same. Comparison (15.142) proves (15.143).

Lemma 15.7.5. (Asymptotic summation in weak parameter-dependent classes) Fix $m>0$, an open set $U\subseteq\mathbb{R}^n$, a rank $r\in\mathbb{N}$, and a closed sector $\Lambda\subseteq\mathbb{C}$ as in Definition 15.7.4. Let $(c_j)_{j\geq0}$ be a sequence of $\operatorname{End}(\mathbb{C}^r)$-valued families such that c_j has resolvent type j. There is a family c of resolvent type 0 satisfying $c\sim\sum_{j=0}^{\infty}c_j$ in the sense of (15.141). The assertion also holds for defect-type families. If all c_j are holomorphic in λ, the sum can be chosen holomorphic. Any two sums differ by a family belonging to every resolvent type N; the difference is smoothing with parameter precisely when it additionally satisfies the powers of $1+|\lambda|$ in (15.140). Finally, suppose that, for each P,Q and each finite family of derivatives, there exists J such that the terms c_j, $j\geq J$, satisfy an inequality of the form (15.140) with constants that may depend on j. Then one may use the joint cutoff $\chi(\varepsilon_j\widehat{\rho}_m)$ and choose ε_j with the following quantified property. For each

simultaneous seminorm $\mathfrak{s}_{K,P,Q,\alpha,\beta,k,\ell}$, there exists J such that, for every $N \geq J$,

$$\mathfrak{s}_{K,P,Q,\alpha,\beta,k,\ell}\left(\sum_{j=N}^{\infty} \chi(\varepsilon_j \widehat{\rho}_m) c_j\right) \leq \sum_{j=N}^{\infty} 2^{-j}.$$

This second auxiliary sum is not generally holomorphic in λ.

Proof. Choose $\chi \in C^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, $\chi(t) = 0$ for $t \leq 1$, and $\chi(t) = 1$ for $t \geq 2$. Exhaust each chart by compact sets $K_1 \Subset K_2 \Subset \cdots$ and enumerate in a sequence $(\mathfrak{p}_\nu)_{\nu \geq 1}$ the seminorms (15.138) with $K = K_d$ and all indices $d, \alpha, \beta, k, \ell$. For $0 < \varepsilon \leq 1$, set

$$\chi_{j,\varepsilon}(\xi) := \chi(\varepsilon \langle \xi \rangle).$$

Successive applications of the chain rule give

$$|D_\xi^\gamma \chi_{j,\varepsilon}(\xi)| \leq C_\gamma \langle \xi \rangle^{-|\gamma|} \mathbf{1}_{\{\langle \xi \rangle \geq \varepsilon^{-1}\}}. \tag{15.144}$$

For $\gamma = 0$, the indicator merely expresses the cutoff support. Where a derivative of the cutoff is nonzero, $\varepsilon \langle \xi \rangle \in [1, 2]$; each derivative of $\langle \xi \rangle$ contributes a power bounded by that on the right-hand side.
The Leibniz rule, (15.144), and the defining inequality for c_j show that each term of $D_\xi^\beta(\chi_{j,\varepsilon} c_j)$ is bounded by

$$C \langle \xi \rangle^{-j-|\beta|} \Omega_m^{-1-k-\ell} \mathbf{1}_{\{\langle \xi \rangle \geq \varepsilon^{-1}\}}.$$

Thus, for each seminorm $\mathfrak{p}_\nu$ of resolvent type 0, there exists $C_{j,\nu} > 0$ such that

$$\mathfrak{p}_\nu(\chi_{j,\varepsilon} c_j) \leq C_{j,\nu} \varepsilon^j. \tag{15.145}$$

The derivatives $\partial_\lambda^k \partial_{\bar{\lambda}}^\ell$ fall only on c_j, since this cutoff depends on ξ and not on λ; hence the exponent $-1-k-\ell$ of Ω_m is preserved exactly.
Set $\varepsilon_0 = 1$ and recursively choose $\varepsilon_j \in (0, 2^{-j}]$ for $j \geq 1$. At step j, require, for the first j seminorms and every $N = 0, 1, \ldots, j-1$, that the resolvent-type N seminorm of the cutoff term not exceed 2^{-j}; for $N = 0$ this is equivalent to requiring

$$C_{j,\nu} \varepsilon_j^j \leq 2^{-j}, \qquad 1 \leq \nu \leq j. \tag{15.146}$$

There are only finitely many conditions at each step, all containing a positive power ε_j^{j-N}, so the choice is possible. Define

$$c := \sum_{j=0}^{\infty} \chi(\varepsilon_j \langle \xi \rangle) c_j. \tag{15.147}$$

The series is locally finite: for fixed ξ, $\varepsilon_j \langle \xi \rangle \leq 2^{-j} \langle \xi \rangle < 1$ holds from some index onward. For a fixed seminorm $\mathfrak{p}_\nu$, its tail from $j \geq \max\{\nu, 1\}$ is dominated by $\sum_{j=0}^{\infty} 2^{-j}$; hence it converges in every seminorm of resolvent type 0.
Fix N. For $j > N$, the preceding calculation, normalized with $\langle \xi \rangle^{N+|\beta|}$, contains $\langle \xi \rangle^{N-j} \leq \varepsilon_j^{j-N}$ on the cutoff support. The conditions imposed when choosing ε_j prove convergence of $\sum_{j=N}^{\infty} \chi(\varepsilon_j \langle \xi \rangle) c_j$ in every seminorm

of resolvent type N. Moreover,

$$c - \sum_{j=0}^{N-1} c_j = \sum_{j=N}^{\infty} \chi(\varepsilon_j\langle\xi\rangle)c_j - \sum_{j=0}^{N-1}\big(1-\chi(\varepsilon_j\langle\xi\rangle)\big)c_j. \tag{15.148}$$

The second sum is finite, and each summand has ξ in a compact set, so it belongs to every resolvent type N. This proves the remainder estimate for each N.

For defect type, repeat the same inequalities with $\Omega_m^{-k-\ell}$; no step changes that power. If the c_j are holomorphic, the locally finite sum (15.147) is also holomorphic. Subtracting two sums and using (15.148) for every N shows that their difference belongs to all the indicated types. If it additionally has the factor $(1+|\lambda|)^{-Q}$ for each Q, one obtains exactly (15.140); this is the quantified formulation of simultaneous residual decay.

To prove the last assertion, now use $\chi(\varepsilon_j\widehat{\rho}_m)$ and also enumerate the seminorms

$$\mathfrak{s}_{K,P,Q,\alpha,\beta,k,\ell}(d) := \sup_{\substack{x\in K,\,\xi\in\mathbb{R}^n\\ \lambda\in\Lambda}} \langle\xi\rangle^{P}(1+|\lambda|)^{Q}\|D_x^{\alpha}D_\xi^{\beta}\partial_\lambda^{k}\partial_{\bar\lambda}^{\ell}d\|.$$

Fix one of them and apply the hypothesis to exponents $P+L$ and $Q+L$. The Leibniz rule and (15.143) give, on the cutoff support, the bound

$$\mathfrak{s}_{K,P,Q,\alpha,\beta,k,\ell}\big(\chi(\varepsilon_j\widehat{\rho}_m)c_j\big) \le C_{j,L} \sup_{\substack{x\in K,\ \xi\in\mathbb{R}^n,\ \lambda\in\Lambda\\ \widehat{\rho}_m(\xi,\lambda)\ge\varepsilon_j^{-1}}} \langle\xi\rangle^{-L}(1+|\lambda|)^{-L}.$$

By (15.142), every point in the last supremum satisfies $\langle\xi\rangle \ge c\widehat{\rho}_m$ or $1+|\lambda| \ge c\widehat{\rho}_m^{m}$. Thus the right-hand side does not exceed $C_{j,L}\varepsilon_j^{\min\{1,m\}L}$. For a weak seminorm of resolvent or defect type, first absorb the positive powers of $\langle\xi\rangle$ and Ω_m by choosing $P+L$ and $Q+L$ still larger, obtaining the same positive power of ε_j. Recursively decrease ε_j so that the first j applicable seminorms do not exceed 2^{-j}. For a fixed seminorm, increase the index J in the hypothesis if necessary until that seminorm is among the first j for every $j \ge J$. The condition is then imposed on each summand of index $j \ge J$, and for every $N \ge J$ its tail is dominated by $\sum_{j=N}^{\infty} 2^{-j}$. Moreover, $1-\chi(\varepsilon_j\widehat{\rho}_m)$ has jointly compact support in (ξ,λ) for each fixed j. Thus the finite corrections are smoothing with parameter, and the tails satisfy (15.140). This proves the assertion without identifying simultaneous smoothing with mere membership in every resolvent type. □

The cutoff in this proof depends only on ξ and preserves holomorphy. In contrast, the global cutoffs $\chi(\varepsilon\widehat{\rho}_m)$ needed to excise the origin $(\xi,\lambda)=(0,0)$ jointly contain $|\lambda|^2$ and, unless constant, are not holomorphic. We always distinguish the auxiliary symbol family containing these cutoffs, the holomorphic homogeneous components, and the holomorphic exact resolvent.

The first condition is not a statement about the global spectrum of A. It is a uniform condition on a finite-dimensional family of matrices. Homogeneity of a_m and compactness of the cosphere bundle allow the ray to be widened: there exists $\varepsilon_{\mathrm{p}} > 0$ such that

$$\lambda I - a_m(x,\xi) \quad \text{is invertible if} \quad \xi \neq 0, \quad \lambda \in \Lambda^{\mathrm{p}}_{\theta,\varepsilon_{\mathrm{p}}},$$

where

$$\Lambda^{\mathrm{p}}_{\theta,\varepsilon_{\mathrm{p}}} := \{re^{i\varphi} \mid r>0,\ |\varphi-\theta| \le \varepsilon_{\mathrm{p}}\}$$

is understood modulo 2π. We do not yet assert that this principal sector is contained in $\rho(A)$. That conclusion is obtained first for large $|\lambda|$ using the parametrix and Neumann series. Only afterward is the sector narrowed to avoid the remaining compact spectral set; this logical order avoids using resolvent existence to construct the resolvent. Until then, all subsequent parameter seminorms and asymptotic sums are taken with $\Lambda = \Lambda^{\mathrm{p}}_{\theta,\varepsilon_{\mathrm{p}}}$.

Inversion on the normalized compact set followed by dilation gives

$$\left\|(\lambda I - a_m(x,\xi))^{-1}\right\|_{\mathcal{L}((\mathbf{E}_x,\mathbf{h}_{\mathbf{E}}(x)))} \le C\left(|\lambda| + |\xi|_{\mathbf{g}}^{m}\right)^{-1}.$$

Differentiating the identity $(\lambda I - a_m)(\lambda I - a_m)^{-1} = I$ gives, for the homogeneous terms introduced below and, in a local unitary frame, for $\|\xi\|_{\mathbb{R}^n} \ge 1$, weak estimates of the form

$$\left\|D_x^{\alpha} D_{\xi}^{\beta} \partial_{\lambda}^{k} q_{-m-j}(x,\xi,\lambda)\right\|_{\mathrm{op},\mathbb{C}^r} \le C_{\alpha,\beta,k} \langle \xi \rangle^{-j-|\beta|} \Omega_m(\xi,\lambda)^{-1-k}.$$

These bounds, together with holomorphy in λ, are the portion of the parameter-dependent calculus we use.

Proposition 15.7.6. (Resolvent parametrix) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $(\mathbf{E},\mathbf{h}_{\mathbf{E}}) \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$; and let $A \in \Psi^{m}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$ be elliptic, with $m > 0$ and local full symbol a. Suppose θ is an admissible spectral cut for A and $0 \in \rho(A)$. There exists a global auxiliary symbol family $\widetilde{q}$ of resolvent type 0, not necessarily holomorphic, which is a two-sided inverse of $\lambda I - a$ modulo families smoothing with parameter in the principal sector $\Lambda^{\mathrm{p}}_{\theta,\varepsilon_{\mathrm{p}}}$. At high frequencies its classical expansion is

$$\widetilde{q}(x,\xi,\lambda) \underset{\|\xi\|_{\mathbb{R}^n}\to\infty}{\sim} \sum_{j=0}^{\infty} q_{-m-j}(x,\xi,\lambda), \qquad \lambda \in \Lambda^{\mathrm{p}}_{\theta,\varepsilon_{\mathrm{p}}},$$

where the components are considered only for $\xi \neq 0$, are holomorphic in λ, satisfy

$$q_{-m-j}(x,t\xi,t^{m}\lambda) = t^{-m-j} q_{-m-j}(x,\xi,\lambda), \qquad t > 0,$$

and are determined recursively as follows. If

$$p_m := \lambda I - a_m, \qquad p_{m-k} := -a_{m-k} \quad (k \ge 1),$$

, then

$$q_{-m} = p_m^{-1}$$

and, for $j \ge 1$,

$$q_{-m-j} = -q_{-m} \sum_{\substack{k+\ell+|\alpha|=j \\ 0\le\ell<j}} \frac{i^{-|\alpha|}}{\alpha!} (D_{\xi}^{\alpha} p_{m-k})(D_x^{\alpha} q_{-m-\ell}). \tag{15.149}$$

The same component also satisfies the right recurrence

$$q_{-m-j} = -\sum_{\substack{k+\ell+|\alpha|=j \\ 0\le\ell<j}} \frac{i^{-|\alpha|}}{\alpha!} (D_{\xi}^{\alpha} q_{-m-\ell})(D_x^{\alpha} p_{m-k}) q_{-m}. \tag{15.150}$$

For every compact set K, all $\alpha, \beta \in \mathbb{N}_0^n$, and $k \in \mathbb{N}_0$, there exists $C_{K,j,\alpha,\beta,k} > 0$ such that, if $\|\xi\| \geq 1$,

$$\left\| D_x^\alpha D_\xi^\beta \partial_\lambda^k q_{-m-j}(x,\xi,\lambda) \right\| \leq C_{K,j,\alpha,\beta,k} \langle \xi \rangle^{-j-|\beta|} \Omega_m(\xi,\lambda)^{-1-k}, \qquad \partial_{\bar{\lambda}} q_{-m-j} = 0. \tag{15.151}$$

There exist $0 < \varepsilon \leq \varepsilon_{\mathrm{p}}$ and $R_0 > 0$ such that

$$\Lambda_{\theta,\varepsilon} := \{ re^{i\varphi} \mid r > 0,\ |\varphi - \theta| \leq \varepsilon \} \subseteq \rho(A).$$

The full symbol of the resolvent in this sector, $r(x,\xi,\lambda)$, taken in a fixed local quantization, is holomorphic in λ; for $|\lambda| \geq R_0$ it has the same high-frequency expansion

$$r(x,\xi,\lambda) \underset{\|\xi\| \to \infty}{\sim} \sum_{j=0}^{\infty} q_{-m-j}(x,\xi,\lambda).$$

For every $s \in \mathbb{R}$, there is a constant $C_s > 0$ such that, for $\lambda \in \Lambda_{\theta,\varepsilon}$ with $|\lambda| \geq R_0$, the resolvent satisfies

$$\|(\lambda I - A)^{-1}\mathbf{u}\|_{H^{s+m}(M,\mathbf{E})} + |\lambda| \|(\lambda I - A)^{-1}\mathbf{u}\|_{H^s(M,\mathbf{E})} \leq C_s \|\mathbf{u}\|_{H^s(M,\mathbf{E})}. \tag{15.152}$$

The constant C_s depends on s, the sector, finitely many parameter seminorms of the symbol, and the choices of a finite atlas, but not on $\mathbf{u}$ or λ in the indicated region; the estimate holds for every $\mathbf{u} \in H^s(M,\mathbf{E})$.

Proof. The symbolic composition formula gives

$$(p \# q)(x,\xi,\lambda) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!} (D_\xi^\alpha p)(D_x^\alpha q).$$

Transfer of Euclidean weights in the composition estimates is governed by Peetre's inequality in Lemma 6.1.1; the weight $\Omega_m(\xi,\lambda)$ also preserves the resolvent factor recorded separately below. The term of degree zero in the total weight is $p_m q_{-m}$; imposing $p_m q_{-m} = I$ gives $q_{-m} = p_m^{-1}$. The parameter-ellipticity estimate proves (15.151) for $j = 0$, $\alpha = \beta = 0$, and $k = 0$. If $\mathscr{D}$ is one of the derivatives D_{x^r}, D_{ξ_r}, or ∂_λ, differentiating $p_m q_{-m} = I$ gives the exact identity

$$\mathscr{D} q_{-m} = -q_{-m} (\mathscr{D} p_m) q_{-m}. \tag{15.153}$$

For $\mathscr{D} = \partial_{\bar{\lambda}}$, the right-hand side is zero. To order the derivatives together, write $\mathscr{D}^\nu = D_x^\alpha D_\xi^\beta \partial_\lambda^h$ and use the componentwise order on triple multi-indices. Applying the Leibniz rule to $\mathscr{D}^\nu(p_m q_{-m}) = 0$ and then multiplying on the left by q_{-m} gives the exact recurrence

$$\mathscr{D}^\nu q_{-m} = -q_{-m} \sum_{0 < \delta \leq \nu} \binom{\nu}{\delta} (\mathscr{D}^\delta p_m)(\mathscr{D}^{\nu-\delta} q_{-m}). \tag{15.154}$$

This formula closes an induction on $|\nu|$: in each summand, the derivative of q_{-m} in the last factor has strictly smaller order. If the λ component of δ is zero, the symbol bounds on p_m and the induction hypothesis bound this summand by

$$C \Omega_m^{-1} \langle \xi \rangle^{m-|\delta_\xi|} \langle \xi \rangle^{-|\beta|+|\delta_\xi|} \Omega_m^{-1-h} \leq C \langle \xi \rangle^{-|\beta|} \Omega_m^{-1-h}.$$

If that component is one, the summand can be nonzero only when $\mathscr{D}^{\delta} p_m = I$; the induction hypothesis then gives $C\langle\xi\rangle^{-|\beta|}\Omega_m^{-1-h}$ directly. For two or more derivatives in λ, the differentiated factor of p_m is zero. The sum in (15.154) is finite, so

$$\|D_x^{\alpha} D_{\xi}^{\beta} \partial_{\lambda}^{k} q_{-m}\| \le C_{K,\alpha,\beta,k}\langle\xi\rangle^{-|\beta|}\Omega_m^{-1-k}.$$

This establishes the initial case, including holomorphy.

Now suppose the components $q_{-m},\dots,q_{-m-j+1}$ have been constructed and their homogeneity, holomorphy, and (15.151) proved. The coefficient of degree $-j$ in $p\#q$ is

$$p_m q_{-m-j} + \sum_{\substack{k+\ell+|\gamma|=j \\ 0\le\ell<j}} \frac{i^{-|\gamma|}}{\gamma!}(D_{\xi}^{\gamma} p_{m-k})(D_x^{\gamma} q_{-m-\ell}).$$

Setting it equal to zero and multiplying on the left by q_{-m} gives exactly (15.149); thus the new component is unique. Under dilation $(\xi,\lambda)\mapsto(t\xi,t^m\lambda)$, the product inside the sum has degree

$$(m-k-|\gamma|)+(-m-\ell) = -(k+\ell+|\gamma|) = -j.$$

The outer factor q_{-m} lowers the degree to $-m-j$. Every factor in the recurrence is holomorphic in λ, so the new component is also holomorphic.

To prove its estimates, now write the symbol index as κ and the number of derivatives in λ as h. Distributing $D_x^{\alpha} D_{\xi}^{\beta} \partial_{\lambda}^{h}$ among the three factors of a summand with $\kappa+\ell+|\gamma| = j$ gives Leibniz terms of the form, apart from their multinomial coefficient,

$$\left(D_x^{\alpha_0} D_{\xi}^{\beta_0} \partial_{\lambda}^{h_0} q_{-m}\right)\left(D_x^{\alpha_1} D_{\xi}^{\beta_1+\gamma} \partial_{\lambda}^{h_1} p_{m-\kappa}\right)\left(D_x^{\alpha_2+\gamma} D_{\xi}^{\beta_2} \partial_{\lambda}^{h_2} q_{-m-\ell}\right),$$

where $\alpha_0+\alpha_1+\alpha_2 = \alpha$, $\beta_0+\beta_1+\beta_2 = \beta$, and $h_0+h_1+h_2 = h$. If $h_1 = 0$, the induction hypotheses bound its norm by

$$C\langle\xi\rangle^{-j-|\beta|+m}\Omega_m^{-2-h} \le C\langle\xi\rangle^{-j-|\beta|}\Omega_m^{-1-h}.$$

If $h_1 = 1$, the term is nonzero only for $\kappa = 0$, $\gamma = 0$, and $\alpha_1 = \beta_1 = 0$; then $j = \ell$ and the two resolvent factors give exactly $C\langle\xi\rangle^{-j-|\beta|}\Omega_m^{-1-h}$. If $h_1 \ge 2$, the term is zero. There are finitely many Leibniz distributions and finitely many triples (κ,ℓ,γ) for each j, so summing proves (15.151) for the new component. This completes the induction.

For two-sided compatibility, the equation $q\#p = I$ gives, at degree $-j$, the equation

$$q_{-m-j} p_m + \sum_{\substack{k+\ell+|\alpha|=j \\ 0\le\ell<j}} \frac{i^{-|\alpha|}}{\alpha!}(D_{\xi}^{\alpha} q_{-m-\ell})(D_x^{\alpha} p_{m-k}) = 0,$$

, and multiplying on the right by q_{-m} gives (15.150). To check that this does not define different components, let q^{d} be the formal series determined by $p\#q^{\mathrm{d}} = I$ and q^{i} the one determined by $q^{\mathrm{i}}\#p = I$. Formal associativity, whose coefficient at each degree contains only finite sums, gives

$$q^{\mathrm{i}} = q^{\mathrm{i}}\#(p\#q^{\mathrm{d}}) = (q^{\mathrm{i}}\#p)\#q^{\mathrm{d}} = q^{\mathrm{d}}.$$

Thus the two recurrences determine the same series term by term. This entire construction is at high

frequency: a_{m-k} and q_{-m-j} are evaluated only for $\xi \neq 0$.
To complete the construction on the axis $\xi = 0$, use the full symbol

$$p(x,\xi,\lambda) := \lambda I - a(x,\xi),$$

, which is smooth for every ξ. There exists $R > 0$ such that p is pointwise invertible when $\widehat{\rho}_m \geq R$. To see this quantitatively, fix $C_0 > 0$. In the cone $|\lambda| \leq C_0\langle\xi\rangle^m$, the principal estimate and the bound $\|a - a_m\| \leq C_1\langle\xi\rangle^{m-1}$ imply

$$\|p_m^{-1}(a - a_m)\| \leq C_2\langle\xi\rangle^{-1}.$$

If $\langle\xi\rangle \geq 2C_2$, the matrix geometric series inverts $p = p_m - (a - a_m) = p_m[I - p_m^{-1}(a - a_m)]$. In the complementary cone, choose $C_0 \geq 2C_3$, where $\|a(x,\xi)\| \leq C_3\langle\xi\rangle^m$; then

$$\|\lambda^{-1}a(x,\xi)\| \leq \tfrac{1}{2} \quad \text{if} \quad |\lambda| \geq C_0\langle\xi\rangle^m,$$

and $p = \lambda[I - \lambda^{-1}a]$ is invertible. Comparison (15.142) allows a single R to be chosen for both cones.
Let $\chi \in C^\infty(\mathbb{R})$ vanish on $(-\infty, 1]$ and equal one on $[2,\infty)$. Extending by zero where the inverse is not used, define

$$b_0(x,\xi,\lambda) := \chi\left(\frac{\widehat{\rho}_m(\xi,\lambda)}{R}\right) p(x,\xi,\lambda)^{-1}.$$

The inverse-derivative formula (15.153), now applied to p^{-1}, the symbol bounds on a, and (15.143) show, derivative by derivative, that b_0 is a global family of resolvent type 0. It is not holomorphic, since the cutoff depends on $|\lambda|^2$. For

$$e := I - p\#b_0$$

, we have e of defect type 1: the contribution $I - \chi$ has jointly compact support in (ξ,λ), and every remaining term in the symbolic calculus contains at least one derivative in ξ. Outside that compact support, we therefore have the sharper bound

$$\left\| D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell e \right\| \leq C_{\alpha,\beta,k,\ell}\langle\xi\rangle^{m-1-|\beta|}\Omega_m(\xi,\lambda)^{-1-k-\ell}. \tag{15.155}$$

That is, besides losing one power of $\langle\xi\rangle$, the defect contains the factor $\langle\xi\rangle^{m-1}\Omega_m^{-1}$. To obtain powers of the defect, start from $e^{\#(N+1)} = e^{\#N}\#e$. If the following bound holds for N, the composition formula distributes derivatives between $e^{\#N}$ and e; the product of the weight factors is

$$\langle\xi\rangle^{N(m-1)}\Omega_m^{-N}\langle\xi\rangle^{m-1}\Omega_m^{-1} = \langle\xi\rangle^{(N+1)(m-1)}\Omega_m^{-(N+1)}.$$

Each derivative in ξ lowers the power of $\langle\xi\rangle$, and each Wirtinger derivative lowers that of Ω_m; the Leibniz rule produces a finite sum. The case $N = 1$ is (15.155), so induction gives, modulo a family of jointly compact support,

$$\left\| D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell e^{\#N} \right\| \leq C_{N,\alpha,\beta,k,\ell}\langle\xi\rangle^{N(m-1)-|\beta|}\Omega_m^{-N-k-\ell}.$$

Given $P, Q \in \mathbb{N}_0$, choose an integer $N \geq \max\{Q, P + mQ + 1\}$. Since $\Omega_m \geq 1 + |\lambda|$, $\Omega_m \geq \langle\xi\rangle^m$, and $N - Q \geq 0$, we have

$$\Omega_m^{-N-k-\ell} \leq (1 + |\lambda|)^{-Q-k-\ell}\langle\xi\rangle^{-m(N-Q)}.$$

Thus the preceding inequality implies

$$\left\| D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell e^{\#N} \right\| \le C_{P,Q,N,\alpha,\beta,k,\ell} \langle \xi \rangle^{-P-|\beta|} (1+|\lambda|)^{-Q-k-\ell}. \tag{15.156}$$

Indeed, after extracting $(1+|\lambda|)^{-Q-k-\ell}$, the remaining power is $\langle \xi \rangle^{N(m-1)-m(N-Q)} = \langle \xi \rangle^{-N+mQ}$, which does not exceed $\langle \xi \rangle^{-P}$. This is quantified simultaneous smoothing. In particular, $c_k := b_0 \# e^{\#k}$ has resolvent type k, and for every P, Q and each finite family of derivatives there exists an integer $J = J(P, Q, \alpha, \beta, k, \ell)$ such that terms of index $j \ge J$ satisfy the filtered simultaneous-smoothing hypothesis. Apply the second part of Lemma 15.7.5, with the joint cutoffs $\chi(\varepsilon_k \widehat{\rho}_m)$, to sum the formal series

$$\widetilde{q} \sim b_0 \# \sum_{k=0}^{\infty} e^{\#k}$$

and obtain the nonholomorphic auxiliary family $\widetilde{q}$. If

$$T_N := b_0 \# \sum_{k=0}^{N-1} e^{\#k},$$

, the finite geometric identity gives

$$p \# T_N = (I - e) \# \sum_{k=0}^{N-1} e^{\#k} = I - e^{\#N}. \tag{15.157}$$

In the algebra of full symbols, possible corrections from the composition formula already belong to the ideal of families smoothing with parameter. Thus

$$s_1 := I - p \# \widetilde{q} = e^{\#N} - p \# (\widetilde{q} - T_N).$$

Given P, Q and a finite family of derivatives, choose N as in (15.156). The first part of the last expression satisfies the required bound there, and the diagonal choice of cutoffs in the lemma gives the same bound for the second. Thus

$$\| D_x^\alpha D_\xi^\beta \partial_\lambda^k \partial_{\bar\lambda}^\ell s_1 \| \le C_{P,Q,\alpha,\beta,k,\ell} \langle \xi \rangle^{-P} (1+|\lambda|)^{-Q}.$$

The left construction provides $\widetilde{q}_{\mathrm{i}} \# p = I - s_2$ with the same estimate. The identity

$$\widetilde{q} - \widetilde{q}_{\mathrm{i}} = s_2 \# \widetilde{q} - \widetilde{q}_{\mathrm{i}} \# s_1$$

shows that both families differ by a family smoothing with parameter. Replacing one by the other preserves its expansion and yields a two-sided inverse modulo that ideal. Expanding p and $\widetilde{q}$ for large $\|\xi\|$, uniqueness at each degree recovers exactly the components q_{-m-j} of (15.149).

The cutoffs in this full sum depend on $\widehat{\rho}_m$ and therefore need not be holomorphic. We will not use $\widetilde{q}$ as though it were: contour deformations will be performed only on each recursive component; the exact resolvent, although holomorphic on its resolvent set, will always be integrated over the fixed global contour. Quantizing $\widetilde{q}$ in charts, inserting partitions of unity, and summing gives $Q(\lambda)$ with

$$(\lambda I - A) Q(\lambda) = I - S_1(\lambda), \qquad Q(\lambda)(\lambda I - A) = I - S_2(\lambda),$$

where S_1 and S_2 are smoothing. Relative to the measure $d\lambda_{\mathbf{g}}(y)$, their kernels satisfy

$$\mathbf{K}_{S_j}(\lambda;\cdot,\cdot) \in C^\infty(M\times M;\operatorname{Hom}(\operatorname{pr}_2^*\mathbf{E},\operatorname{pr}_1^*\mathbf{E})).$$

Equip this bundle with the connection induced by the Hermitian connection of $\mathbf{E}$ and the tensor norms induced by $\mathbf{g}$ and $\mathbf{h}_{\mathbf{E}}$. For all $Q,k,\ell,a,b\in\mathbb{N}_0$ there exists $C_{Q,k,\ell,a,b}>0$ such that

$$\sup_{x,y\in M}\left|\nabla_x^a\nabla_y^b\partial_\lambda^k\partial_{\bar\lambda}^\ell\mathbf{K}_{S_j}(\lambda;x,y)\right|_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\le C_{Q,k,\ell,a,b}(1+|\lambda|)^{-Q}. \tag{15.158}$$

Let $V=\operatorname{vol}_{\mathbf{g}}(M)$. Proposition B.8.3, applied fiberwise with $X=Y=M$ and $\mu=\nu=\lambda_{\mathbf{g}}$, together with (15.158) for $\alpha=\beta=k=\ell=0$, gives, for each $Q\in\mathbb{N}$,

$$\|S_j(\lambda)\|_{\mathcal{L}(L^2(M,\mathbf{E}))}\le V\sup_{x,y\in M}\|\mathbf{K}_{S_j}(\lambda;x,y)\|_{\mathcal{L}((\mathbf{E}_y,\mathbf{h}_{\mathbf{E}}(y)),(\mathbf{E}_x,\mathbf{h}_{\mathbf{E}}(x)))}\le VC_Q(1+|\lambda|)^{-Q}. \tag{15.159}$$

Choose R_1 so that the last expression is at most $\frac12$ for $j=1,2$, $\lambda\in\Lambda^{\mathrm{p}}_{\theta,\varepsilon_{\mathrm{p}}}$, and $|\lambda|\ge R_1$. The series $\displaystyle\sum_{\nu=0}^{\infty}S_j(\lambda)^\nu$ then converge in $\mathcal{L}(L^2(M,\mathbf{E}))$, their sums have norm at most two, and the two parametrix identities provide a right and a left inverse of $\lambda I-A$, respectively. They coincide and define the global resolvent. Thus

$$\{\lambda\in\Lambda^{\mathrm{p}}_{\theta,\varepsilon_{\mathrm{p}}}\mid|\lambda|\ge R_1\}\subseteq\rho(A). \tag{15.160}$$

Narrowing the sector is now justified. The set

$$\mathbf{F}:=\sigma(A)\cap\{\lambda\mid|\lambda|\le R_1\}$$

is finite: admissibility of the cut implies $L_\theta\subseteq\rho(A)$, so Proposition 15.7.2 applies. Moreover, $0\notin\mathbf{F}$ and $\mathbf{F}\cap L_\theta=\varnothing$ by the cut hypotheses. Finiteness of $\mathbf{F}$ gives $0<\varepsilon\le\varepsilon_{\mathrm{p}}$ such that $\mathbf{F}\cap\Lambda_{\theta,\varepsilon}=\varnothing$. Together with (15.160), this proves $\Lambda_{\theta,\varepsilon}\subseteq\rho(A)$. Fix $R_0\ge\max\{R_1,1\}$.
For $|\lambda|\ge R_0$, the algebraic identity

$$(\lambda I-A)^{-1}=Q(\lambda)(I-S_1(\lambda))^{-1}=(I-S_2(\lambda))^{-1}Q(\lambda)$$

shows that the result is the exact resolvent; its holomorphy follows from holomorphy of $\lambda\mapsto\lambda I-A$ and uniqueness of the inverse, not from the cutoffs used in summation. Explicitly, the resolvent identity gives

$$\frac{(\lambda+h-A)^{-1}-(\lambda-A)^{-1}}{h}=-(\lambda+h-A)^{-1}(\lambda-A)^{-1},$$

and the right-hand side converges in operator norm to $-(\lambda-A)^{-2}$ as $h\to0$. This proves operator holomorphy of the exact resolvent. The identity

$$(\lambda I-A)^{-1}-Q(\lambda)=Q(\lambda)\big[(I-S_1(\lambda))^{-1}-I\big]$$

and its left version, together with (15.158), show that the difference satisfies the same estimate for each Q and each finite family of kernel derivatives.

On the part $|\lambda| \le R_0$ of the sector, the resolvent exists by the choice just justified. Since $0 \in \rho(A)$, the closure of this part after adding the vertex is a compact subset of $\rho(A)$. The two parametrix identities give

$$(\lambda I - A)^{-1} - Q(\lambda) = (\lambda I - A)^{-1} S_1(\lambda) = S_2(\lambda)(\lambda I - A)^{-1}.$$

Elliptic regularity applied on both sides shows that this difference is smoothing; on each compact subset of the sector, all its kernel seminorms are uniform. The difference may fail to be holomorphic because Q contains cutoffs depending on $|\lambda|$.

In a fixed chart and quantization, extracting the full symbol from the kernel is a continuous linear operation. Applied directly to the exact resolvent, it produces a symbol $r(x, \xi, \lambda)$ holomorphic throughout the sector. For large parameters, $r - \widetilde{q}$ is smoothing with parameter, and on the compact part it is uniformly smoothing; thus r retains the high-frequency expansion formed by the q_{-m-j}. Bounds on its derivatives in λ follow from Cauchy inequalities in a slightly smaller sector. Finally, the weak estimates of resolvent type 0 give the first norm in (15.152); using $(\lambda I - A)(\lambda I - A)^{-1} = I$ gives the second. The construction is matrix-valued from the first step, so it is unchanged for bundle sections. □

15.7.2 Construction of complex powers

The parameter-dependent parametrix gives the high-frequency behavior of the resolvent, while the exact resolvent retains holomorphy and spectral information. Integrating them against λ^z over a contour around the cut combines both properties in an operator of complex order.

Let Γ_θ be a keyhole contour contained in $\rho(A)$, with two infinite branches in the spectrum-free sector and a small arc around zero. It is the limit, as the outer arc is removed, of positively oriented closed contours surrounding the spectrum of A in the cut plane. With this orientation and the resolvent $(\lambda I - A)^{-1}$, the scalar residue at an eigenvalue μ is $+\mu^z$. Invertibility prevents the small arc from meeting the spectrum, and estimate

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(L^2(M,\mathbf{E}))} \le C_0 |\lambda|^{-1}, \qquad \lambda \in \Lambda_{\theta,\varepsilon}, \quad |\lambda| \ge R_0,$$

makes the following integral converge in operator norm when $\operatorname{Re} z < 0$.

Definition 15.7.7. (Complex powers) Let $(M, \mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E} \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric; let $A \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{E})$ be elliptic and invertible, with $m > 0$; and let θ be an admissible spectral cut. Let $\Gamma_\theta \subset \rho(A)$ be the positively oriented keyhole contour described above, and use the branch $\log_\theta$ in λ^z. For $\operatorname{Re} z < 0$, define

$$A^z := \frac{1}{2\pi i} \int_{\Gamma_\theta} \lambda^z (\lambda I - A)^{-1} \, d\lambda. \tag{15.161}$$

For arbitrary $z \in \mathbb{C}$, choose $k \in \mathbb{N}$ with $\operatorname{Re}(z - k) < 0$ and set

$$A^z := A^k A^{z-k} \quad \text{on } \Gamma(M, \mathbf{E}). \tag{15.162}$$

The resolvent identity

$$(\lambda I - A)^{-1} - (\mu I - A)^{-1} = (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}$$

and two successive applications of Cauchy's formula first prove, for negative real parts, that

$$A^z A^w = A^{z+w}.$$

Let us check that the integral with $z=-1$ is the usual inverse of A. Indeed, the contour calculus for rational functions follows directly from the resolvent identity and Cauchy's formula; since $0\in\rho(A)$, applying it to $r(\lambda)=\lambda^{-1}$ gives

$$\frac{1}{2\pi i}\int_{\Gamma_\theta}\lambda^{-1}(\lambda I-A)^{-1}\,d\lambda = A^{-1}. \tag{15.163}$$

Let us give the analytic justification of this identity. Truncate the keyhole contour and apply Cauchy's formula before removing the truncation. On $\Gamma(M,\mathbf{E})$, use

$$A(\lambda I-A)^{-1}=\lambda(\lambda I-A)^{-1}-I, \qquad (\lambda I-A)^{-1}A=\lambda(\lambda I-A)^{-1}-I.$$

Moreover, for $\mathbf{u}\in\Gamma(M,\mathbf{E})$,

$$(\lambda I-A)^{-1}\mathbf{u}-\lambda^{-1}\mathbf{u}=\lambda^{-1}(\lambda I-A)^{-1}A\mathbf{u}.$$

The resolvent estimate makes the remaining boundary terms tend to zero as the outer arc is removed, while the inner arc remains inside $\rho(A)$ because $0\in\rho(A)$. The scalar Cauchy formula then gives, respectively,

$$AA^{-1}\mathbf{u}=\mathbf{u}, \qquad A^{-1}A\mathbf{u}=\mathbf{u}, \qquad \mathbf{u}\in\Gamma(M,\mathbf{E}),$$

proving (15.163) without identifying the integral with the inverse in advance.

We can now check that (15.162) is independent of k. If $\operatorname{Re}\zeta<-1$, the integrals converge after multiplication by A, and the preceding identities give

$$AA^{\zeta}=A^{\zeta}A=A^{\zeta+1};$$

the additional term $\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_\theta}\lambda^{\zeta}\,d\lambda$ is zero by Cauchy's formula and decay at infinity. If k is admissible and $\zeta:=z-k-1$, then $\operatorname{Re}\zeta<-1$ and

$$A^{k+1}A^{z-k-1}=A^k(AA^{\zeta})=A^kA^{z-k}.$$

By induction, any two admissible choices produce the same operator. The identity (15.163) gives $A^0=I$ and shows that integer powers are the usual ones.

Finally, for $z,w\in\mathbb{C}$ choose $k,\ell\in\mathbb{N}$ with $\operatorname{Re}(z-k)<0$ and $\operatorname{Re}(w-\ell)<0$. Resolvents commute with A on $\Gamma(M,\mathbf{E})$, so negative powers commute with integer powers. Using the law already proved in the negative half-plane,

$$A^zA^w=A^{k+\ell}A^{z-k}A^{w-\ell}=A^{k+\ell}A^{z+w-k-\ell}=A^{z+w}.$$

Invertibility is precisely what permits this group law for all $z\in\mathbb{C}$; without it, negative powers require a separate convention on the kernel.

For $\mu\in\mathbb{C}$, the notation $\Psi^{\mu}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$ means that the symbol has homogeneous components of degrees $\mu-j$ and satisfies estimates of order $\operatorname{Re}\mu$. For $t>0$, interpret $t^{\mu}=e^{\mu\log t}$ using the real logarithm.

Definition 15.7.8. (Holomorphy for a family of variable order) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold, let $\mathbf{E}\longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric, let $m>0$, and let $G\subseteq\mathbb{C}$ be open. A family $B(z)\in\Psi^{mz}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$ is **holomorphic of variable order** mz if the

following three levels hold.

Homogeneous components. In every chart and for $\xi \neq 0$, the components $b_{mz-j}(x,\xi;z)$ are holomorphic in G (entire when $G=\mathbb{C}$), and for each $K \Subset G$, compact subset K_x of the chart, $h \in \mathbb{N}_0$, and α, β, there exists $C>0$ such that

$$\|D_x^\alpha D_\xi^\beta \partial_z^h b_{mz-j}(x,\xi;z)\| \le C\|\xi\|^{m\operatorname{Re} z - j - |\beta|}(1+|\log\|\xi\||)^h, \quad \|\xi\| \ge 1, \quad z \in K. \tag{15.164}$$

Full symbol. With a fixed homogeneous cutoff $\chi(\xi)$, the full symbol $b(x,\xi;z)$ is holomorphic in z and, for each $N, h \in \mathbb{N}_0$, $\delta>0$, $K \Subset G$, and α, β, there exists $C>0$ such that

$$\left\| D_x^\alpha D_\xi^\beta \partial_z^h \left(b - \sum_{j=0}^{N-1} \chi\, b_{mz-j} \right)(x,\xi;z) \right\| \le C\langle\xi\rangle^{m\operatorname{Re} z - N - |\beta| + \delta}, \qquad z \in K. \tag{15.165}$$

The arbitrary loss δ absorbs the powers of $\log\langle\xi\rangle$ arising from differentiation in z. Relative to $d\lambda_{\mathbf{g}}(y)$, the smoothing remainder has a kernel

$$\mathbf{K}_R(z;\cdot,\cdot) \in C^\infty(M\times M; \operatorname{Hom}(\operatorname{pr}_2^*\mathbf{E}, \operatorname{pr}_1^*\mathbf{E}))$$

holomorphic in z. With the induced connection and tensor norm, for all $h, a, b \in \mathbb{N}_0$,

$$\sup_{\substack{z\in K\\ x,y\in M}} \left| \partial_z^h \nabla_x^a \nabla_y^b \mathbf{K}_R(z;x,y) \right|_{\mathbf{g},\mathbf{h}_\mathbf{E}} \le C_{K,h,a,b}.$$

Operator level. The map $z \mapsto B(z)$ is holomorphic from G into $\mathcal{L}_b(\Gamma(M,\mathbf{E}),\Gamma(M,\mathbf{E}))$, where $\Gamma(M,\mathbf{E})$ carries its Fréchet topology and the subscript b denotes uniform convergence on bounded subsets. In the presence of the two preceding symbol levels, this is equivalent to the following fixed-Sobolev formulation. If $\tau_K := m\max_{z\in K}\operatorname{Re} z$, then, for every $r \in \mathbb{R}$, $h \in \mathbb{N}_0$, and $\delta>0$,

$$\sup_{z\in K} \|\partial_z^h B(z)\|_{\mathcal{L}(H^r(M,\mathbf{E}),H^{r-\tau_K-\delta}(M,\mathbf{E}))} \le C_{K,r,h,\delta}, \tag{15.166}$$

and $z \mapsto B(z)$ is holomorphic on an open neighborhood of K contained in G, with values in that bounded-operator space. Choose this neighborhood so that $m\operatorname{Re} z < \tau_K + \delta/2$; the remaining margin absorbs the logarithmic powers of the derivatives. The entire family over G need not take values in a single Sobolev pair. The fixed order $\tau_K+\delta$ allows operators whose real orders vary with z to be compared in the same Banach space.

Theorem 15.7.9. (Complex powers of an elliptic operator) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E} \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h}_\mathbf{E}$ and a compatible connection $\nabla^\mathbf{E}$; let $A \in \Psi_{\mathrm{cl}}^m(M;\mathbf{E},\mathbf{E})$ be elliptic and invertible, with $m>0$ and principal Fourier symbol a_m; and fix an admissible spectral cut θ. For every $z \in \mathbb{C}$,

$$A^z \in \Psi_{\mathrm{cl}}^{mz}(M;\mathbf{E},\mathbf{E}).$$

The family is holomorphic in the sense of Definition 15.7.8 and satisfies

$$A^z A^w = A^{z+w}, \qquad \sigma_{mz}^\Psi(A^z)(x,\xi) = a_m(x,\xi)^z, \tag{15.167}$$

where the matrix power on the right uses the same spectral cut. Moreover,

$$A^z\colon H^r(M,\mathbf{E}) \longrightarrow H^{r-m\operatorname{Re} z}(M,\mathbf{E})$$

is continuous for every $r \in \mathbb{R}$ when initially considered on smooth sections.

Proof. Begin with $\operatorname{Re} z < 0$. By Proposition 15.7.6, the resolvent symbol is

$$r(x,\xi,\lambda) \underset{\|\xi\|\to\infty}{\sim} \sum_{j=0}^{\infty} q_{-m-j}(x,\xi,\lambda).$$

. Extraction of the full symbol is linear and continuous, so it commutes with the convergent integral defining A^z. Thus a symbol of A^z is

$$b(x,\xi;z) = \frac{1}{2\pi i}\int_{\Gamma_\theta} \lambda^z r(x,\xi,\lambda)\, d\lambda.$$

The following deformation is performed exclusively for homogeneous components, not for r. Indeed, compactness of the cosphere bundle, invertibility of a_m, and the principal angle give a normalized contour uniformly avoiding $\operatorname{spec} a_m(x,\omega)$ for every $\|\omega\| = 1$. Let $\Gamma_{\theta,\xi}$ denote its image under $\lambda = \|\xi\|^m\mu$. Its branches lie in the principal sector, and its curved part has radius $c\|\xi\|^m$, with $c > 0$ smaller than the modulus of every eigenvalue of $a_m(x,\omega)$. For large $\|\xi\|$, the fixed contour Γ_θ and $\Gamma_{\theta,\xi}$ are homotopic in the set where $\lambda I - a_m(x,\xi)$ is invertible. This homotopy may cross global eigenvalues of A and therefore does not apply to the exact symbol r; it does apply to each q_{-m-j}, whose only singularities in λ arise from the principal resolvent. For $\xi \neq 0$, define

$$b_{mz-j}(x,\xi;z) := \frac{1}{2\pi i}\int_{\Gamma_{\theta,\xi}} \lambda^z q_{-m-j}(x,\xi,\lambda)\, d\lambda. \tag{15.168}$$

Parameter estimates allow differentiation under the integral. Indeed, if $K \Subset \{z \mid \operatorname{Re} z < 0\}$ and $\eta := -\max_{z\in K} \operatorname{Re} z > 0$, then on the branches where $|\lambda| \geq 1$,

$$\left\|\partial_z^h\left(\lambda^z D_x^\alpha D_\xi^\beta q_{-m-j}\right)\right\| \leq C_{K,h,\alpha,\beta,j}(1+\log|\lambda|)^h|\lambda|^{-1-\eta}\langle\xi\rangle^{-j-|\beta|}. \tag{15.169}$$

The function $(1+\log r)^h r^{-1-\eta}$ is integrable in $[1,\infty)$; after normalization by $\lambda = \|\xi\|^m\mu$, the arc belongs to a compact set uniformly avoiding the principal spectrum. This justifies, in the symbol topology, both the integral and all its derivatives in z, x and ξ. Replacing ξ by $t\xi$ and making the change $\lambda = t^m\mu$, homogeneity of the symbolic resolvent gives

$$\begin{aligned} b_{mz-j}(x,t\xi;z) &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\xi}} (t^m\mu)^z t^{-m-j} q_{-m-j}(x,\xi,\mu) t^m\, d\mu \\ &= t^{mz-j} b_{mz-j}(x,\xi;z). \end{aligned}$$

Thus the terms in (15.168) have exactly the required degrees.
It remains to control the remainder without moving the global resolvent contour. Fix a cutoff $\chi_0(\xi)$ vanishing near zero and equal to one beyond a frequency for which the preceding principal homotopy is valid.

For $L \in \mathbb{N}$, set, on the fixed contour,

$$R_L(x,\xi,\lambda) := r(x,\xi,\lambda) - \sum_{j=0}^{L-1} \chi_0(\xi) q_{-m-j}(x,\xi,\lambda).$$

The parameter-dependent expansion of Proposition 15.7.6 and uniform smoothing of $r - \widetilde{q}$ give, on the two infinite branches,

$$\|D_x^\alpha D_\xi^\beta R_L(x,\xi,\lambda)\| \le C_{\alpha,\beta,L} \langle \xi \rangle^{-L-|\beta|} \Omega_m(\xi,\lambda)^{-1}. \tag{15.170}$$

On the fixed curved part of the contour, the same inequality for large $\|\xi\|$ is the ordinary estimate of order $-m-L$; the region of bounded ξ contributes only a smoothing symbol. Since each q_{-m-j} can be deformed by the principal homotopy while r remains on Γ_θ, for large $\|\xi\|$ we have exactly

$$b - \sum_{j=0}^{L-1} \chi_0 b_{mz-j} = \frac{1}{2\pi i} \int_{\Gamma_\theta} \lambda^z R_L(x,\xi,\lambda)\, d\lambda. \tag{15.171}$$

Write $X = \langle \xi \rangle^m$ and let $r_* > 0$ be the radius of the fixed curved part. If $K \Subset \{\operatorname{Re} z < 0\}$, then after ∂_z^h, (15.170) and splitting each branch into $r_* \le |\lambda| \le X$ and $|\lambda| \ge X$ give

$$\begin{aligned} &\left\| D_x^\alpha D_\xi^\beta \partial_z^h \left(b - \sum_{j=0}^{L-1} \chi_0 b_{mz-j} \right) \right\| \\ &\quad \le C_{K,\alpha,\beta,L,h} \langle \xi \rangle^{-L-|\beta|} \left[X^{-1} \int_{r_*}^{X} r^{\operatorname{Re} z} (1 + |\log r|)^h \, dr + \int_X^\infty r^{\operatorname{Re} z - 1} (1 + \log r)^h \, dr \right]. \end{aligned} \tag{15.172}$$

If $X < r_*$, absorb this region into the constant. For $X \ge r_*$, bound the first integral by separating the cases $\operatorname{Re} z > -1$, $= -1$, and < -1; integrate the second in $[X, \infty)$. Uniformly for $z \in K$ and every $\delta > 0$, the result is

$$\left\| D_x^\alpha D_\xi^\beta \partial_z^h \left(b - \sum_{j=0}^{L-1} \chi_0 b_{mz-j} \right) \right\| \le C_{K,\alpha,\beta,L,h,\delta} \left(\langle \xi \rangle^{m \operatorname{Re} z - L - |\beta| + \delta} + \langle \xi \rangle^{-m-L-|\beta|+\delta} \right). \tag{15.173}$$

Here we used $(1 + |\log \langle \xi \rangle|)^h \le C_{h,\delta} \langle \xi \rangle^\delta$; this inequality also covers the borderline case $\operatorname{Re} z = -1$.
Now given N, h, δ and K, choose $L \ge N$ so large that

$$-m - L + \frac{\delta}{2} \le m \min_{z \in K} \operatorname{Re} z - N.$$

Apply (15.173) with $\frac{\delta}{2}$. Both terms are bounded by $C \langle \xi \rangle^{m \operatorname{Re} z - N - |\beta| + \delta}$. The finite terms $\displaystyle\sum_{j=N}^{L-1} \chi_0 b_{mz-j}$ satisfy the same bound by (15.164). The identity

$$b - \sum_{j=0}^{N-1} \chi_0 b_{mz-j} = \left(b - \sum_{j=0}^{L-1} \chi_0 b_{mz-j} \right) + \sum_{j=N}^{L-1} \chi_0 b_{mz-j}$$

proves (15.165). The rescaling in (15.168) and the majorant (15.169) in turn prove (15.164). Thus

$$b(x,\xi;z)\sim\sum_{j=0}^{\infty}b_{mz-j}(x,\xi;z)$$

is a classical symbol of complex order mz and is holomorphic at the first two levels of Definition 15.7.8. The difference between the exact resolvent and the parametrix satisfies, for each Q and each finite number of kernel derivatives, the bound (15.158). Once $Q>\max_{z\in K}\operatorname{Re}z+1$ is chosen, its product with $|\lambda^z|(1+|\log\lambda|)^h$ is integrable on the infinite branches. The integral therefore has a smooth holomorphic kernel with all seminorms uniform on K and does not alter the classical symbol obtained. The symbol estimates, mapping theorem, and differentiation under the integral give the bound (15.166). To verify holomorphy in this norm, if the segment $[z,z+h]$ remains in a compact neighborhood, use Taylor's identity with integral remainder

$$b(z+h)-b(z)-h\partial_z b(z)=h^2\int_0^1(1-t)\partial_z^2 b(z+th)\,dt.$$

Each symbol seminorm of the right-hand side is at most $C|h|^2$ by the preceding bounds. Quantization is continuous from these seminorms into $\mathcal{L}(H^r(M,\mathbf{E}),H^{r-\tau_K-\delta}(M,\mathbf{E}))$, so the difference quotient converges there to $\partial_z A^z$. The same identity applied to each seminorm of $\Gamma(\mathbf{E})$ proves holomorphy on $\Gamma(\mathbf{E})$ itself. This establishes all three levels when $\operatorname{Re}z<0$.

For arbitrary z, choose k as in (15.162). The composition calculus gives

$$A^kA^{z-k}\in\Psi_{\mathrm{cl}}^{mk+m(z-k)}=\Psi_{\mathrm{cl}}^{mz}.$$

For a compact set $K\Subset\mathbb{C}$, choose a single integer k such that $\operatorname{Re}(z-k)\le-\eta<0$ for every $z\in K$. Composition with the fixed operator A^k is continuous in the symbol topologies and the operator spaces of (15.166); hence it transfers all uniform bounds already proved to A^z. The group law proves independence of k. If $w=z-k$ has negative real part, taking $j=0$ in (15.168), the holomorphic functional calculus of the matrix $a_m(x,\xi)$ gives

$$b_{mw}(x,\xi;w)=\frac{1}{2\pi i}\int_{\Gamma_{\theta,\xi}}\lambda^w(\lambda I-a_m(x,\xi))^{-1}\,d\lambda=a_m(x,\xi)^w.$$

Multiplicativity of the principal symbol under composition then yields $\sigma^{\Psi}_{mz}(A^z)=a_m^k a_m^w=a_m^z$ for every z. The mapping property is that of a pseudodifferential operator of real order $m\operatorname{Re}z$. □

Proposition 15.7.10. (Closed realizations of powers) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E}\longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric; let $A\in\Psi_{\mathrm{cl}}^m(M;\mathbf{E},\mathbf{E})$ be elliptic and invertible, with $m>0$ and a fixed admissible spectral cut; and let $z\in\mathbb{C}$. Let $d=m\operatorname{Re}z$. If $d>0$, the closure in $L^2(M,\mathbf{E})$ of $A^z|_{\Gamma(M,\mathbf{E})}$ has domain $H^d(M,\mathbf{E})$, and there exist $c_z,C_z>0$ such that

$$c_z\|\mathbf{u}\|_{H^d(M,\mathbf{E})}\le\|\mathbf{u}\|_{L^2(M,\mathbf{E})}+\|A^z\mathbf{u}\|_{L^2(M,\mathbf{E})}\le C_z\|\mathbf{u}\|_{H^d(M,\mathbf{E})},\qquad\mathbf{u}\in\Gamma(M,\mathbf{E}).\tag{15.174}$$

This realization is closed, and $\Gamma(M,\mathbf{E})$ is a core. If $d\le0$, $A^z|_{\Gamma(M,\mathbf{E})}$ extends uniquely to a bounded operator on $L^2(M,\mathbf{E})$; this is its realization, its domain is all of $L^2(M,\mathbf{E})$, and $\Gamma(M,\mathbf{E})$ is also a core.

Proof. Suppose $d > 0$. The preceding theorem gives continuous maps $A^z : H^d(M,\mathbf{E}) \to L^2(M,\mathbf{E})$ and $A^{-z} : L^2(M,\mathbf{E}) \to H^d(M,\mathbf{E})$. The group law, first on $\Gamma(\mathbf{E})$, provides the exact identity $\mathbf{u} = A^{-z}A^z\mathbf{u}$. Thus

$$\|\mathbf{u}\|_{H^d(M,\mathbf{E})} \le \|A^{-z}\|_{\mathcal{L}(L^2(M,\mathbf{E}),H^d(M,\mathbf{E}))}\|A^z\mathbf{u}\|_{L^2(M,\mathbf{E})}, \qquad \|A^z\mathbf{u}\|_{L^2(M,\mathbf{E})} \le \|A^z\|_{\mathcal{L}(H^d(M,\mathbf{E}),L^2(M,\mathbf{E}))}\|\mathbf{u}\|_{H^d(M,\mathbf{E})},$$

, giving (15.174). If $\mathbf{u}_j \to \mathbf{u}$ and $A^z\mathbf{u}_j \to \mathbf{f}$ in $L^2(M,\mathbf{E})$, the first inequality applied to $\mathbf{u}_j - \mathbf{u}_k$ proves that $(\mathbf{u}_j)$ converges in $H^d(M,\mathbf{E})$; continuity $A^z : H^d(M,\mathbf{E}) \to L^2(M,\mathbf{E})$ gives $A^z\mathbf{u} = \mathbf{f}$. Thus the realization with domain $H^d(M,\mathbf{E})$ is closed. For $\mathbf{u} \in H^d(M,\mathbf{E})$, a sequence of smooth sections converging to $\mathbf{u}$ in $H^d(M,\mathbf{E})$ also converges in graph norm by the second inequality. Consequently, the closure of the restriction to $\Gamma(\mathbf{E})$ is precisely this realization, and $\Gamma(\mathbf{E})$ is a core.
If $d \le 0$, the mapping property gives $A^z : L^2(M,\mathbf{E}) \to H^{-d}(M,\mathbf{E}) \hookrightarrow L^2(M,\mathbf{E})$ when $d < 0$, and $A^z : L^2(M,\mathbf{E}) \to L^2(M,\mathbf{E})$ when $d = 0$. Density of $\Gamma(\mathbf{E})$ in $L^2(M,\mathbf{E})$ proves uniqueness of the extension. If $\mathbf{u}_j \in \Gamma(\mathbf{E})$ converges to $\mathbf{u}$ in $L^2(M,\mathbf{E})$, continuity implies $A^z\mathbf{u}_j \to A^z\mathbf{u}$ in $L^2(M,\mathbf{E})$, so smooth sections are dense also in graph norm. □

This proof follows the parameter-dependent classes and sums, the resolvent, and the holomorphic families presented in [135, Chapter II, §§9.1, 9.4 and 11.1–11.4]. Historically, the result and the method of integrating the resolvent parametrix are due to Seeley [132].

15.7.3 Agreement with the spectral calculus

The contour construction also applies to non-self-adjoint operators. When A is self-adjoint and positive, each spectral projection reduces the contour to the scalar Cauchy formula. This identifies Seeley's powers with the spectral calculus used in Chapter 12.

Proposition 15.7.11. (Contour calculus, spectral calculus, and the semigroup) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E} \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric; and let $A \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$ be elliptic and self-adjoint on $L^2(M,\mathbf{E})$, with $m > 0$ and $A \ge cI$ for some $c > 0$. Take the negative real axis as the cut. Then the powers in Definition 15.7.7 agree with the powers from the spectral theorem. As closed operators,

$$A^z = \int_{[c,\infty)} \lambda^z \, dE_A(\lambda). \tag{15.175}$$

Moreover, for $\operatorname{Re} s > 0$,

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tA} \, dt, \tag{15.176}$$

where the Bochner integral converges in $\mathcal{L}(L^2(M,\mathbf{E}),L^2(M,\mathbf{E}))$.

Proof. Proposition 15.7.2 provides an orthonormal basis $(\boldsymbol{\varphi}_j)$ of smooth eigensections, with $A\boldsymbol{\varphi}_j = \lambda_j\boldsymbol{\varphi}_j$ and $\lambda_j \to \infty$. For $\operatorname{Re} z < 0$, Cauchy's formula applied to each eigenvalue gives

$$\frac{1}{2\pi i}\int_{\Gamma_\pi} \lambda^z (\lambda I - A)^{-1}\boldsymbol{\varphi}_j \, d\lambda = \left(\frac{1}{2\pi i}\int_{\Gamma_\pi} \frac{\lambda^z}{\lambda - \lambda_j}\, d\lambda\right)\boldsymbol{\varphi}_j = \lambda_j^z \boldsymbol{\varphi}_j.$$

For arbitrary z, choose $k \in \mathbb{N}$ with $\operatorname{Re}(z-k) < 0$ and use $A^z = A^k A^{z-k}$; since $A^k\boldsymbol{\varphi}_j = \lambda_j^k\boldsymbol{\varphi}_j$, this gives $A^z\boldsymbol{\varphi}_j =$

$\lambda_j^z \boldsymbol{\varphi}_j$. By linearity, both constructions agree on the space $\mathcal{F}$ of finite spectral sums.
The spectral power has domain

$$\mathcal{D}(A_{\mathrm{esp}}^z) = \left\{ \mathbf{u} = \sum_{j=1}^{\infty} u_j \boldsymbol{\varphi}_j \in L^2(M,\mathbf{E}) \;\middle|\; \sum_{j=1}^{\infty} \lambda_j^{2\operatorname{Re} z} |u_j|^2 < \infty \right\}. \tag{15.177}$$

If $\mathbf{u}$ belongs to this domain and

$$\mathbf{u}_N := \sum_{\lambda_j \le N} u_j \boldsymbol{\varphi}_j,$$

, then $\mathbf{u}_N \in \mathcal{F} \subseteq \Gamma(M,\mathbf{E})$ and

$$\|\mathbf{u} - \mathbf{u}_N\|_{L^2(M,\mathbf{E})}^2 + \|A_{\mathrm{esp}}^z(\mathbf{u} - \mathbf{u}_N)\|_{L^2(M,\mathbf{E})}^2 = \sum_{\lambda_j > N} (1 + |\lambda_j^z|^2)|u_j|^2 \longrightarrow 0.$$

Thus $\mathcal{F}$ is a core for the spectral power.
Let us prove that it is also a core for the pseudodifferential power. If $d = m \operatorname{Re} z \le 0$, this power is bounded on $L^2(M,\mathbf{E})$ by Proposition 15.7.10; density of $\mathcal{F}$ in $L^2(M,\mathbf{E})$ then implies density in graph norm. Suppose $d > 0$ and choose

$$N_0 := \max\left\{1, \left\lceil \frac{d}{m} \right\rceil \right\}, \qquad mN_0 \ge d.$$

If $\mathbf{v} = \sum_{j=1}^{\infty} v_j \boldsymbol{\varphi}_j \in \Gamma(M,\mathbf{E})$ and $\mathbf{v}_N = \sum_{\lambda_j \le N} v_j \boldsymbol{\varphi}_j$, then $A^{N_0}\mathbf{v} \in L^2(M,\mathbf{E})$. Since $\mathbf{v}$ belongs to the domain of A^{N_0} and A is self-adjoint, the coefficient of $A^{N_0}\mathbf{v}$ in $\boldsymbol{\varphi}_j$ is

$$(A^{N_0}\mathbf{v}, \boldsymbol{\varphi}_j)_{L^2(M,\mathbf{E})} = (\mathbf{v}, A^{N_0}\boldsymbol{\varphi}_j)_{L^2(M,\mathbf{E})} = \lambda_j^{N_0} v_j.$$

By Parseval,

$$\|\mathbf{v} - \mathbf{v}_N\|_{L^2(M,\mathbf{E})}^2 + \|A^{N_0}(\mathbf{v} - \mathbf{v}_N)\|_{L^2(M,\mathbf{E})}^2 = \sum_{\lambda_j > N} (1 + \lambda_j^{2N_0})|v_j|^2 \longrightarrow 0.$$

Graph-norm equivalence (15.174), applied to exponent N_0, gives $\mathbf{v}_N \to \mathbf{v}$ in $H^{mN_0}(M,\mathbf{E})$ and, by the continuous inclusion, in $H^d(M,\mathbf{E})$.
Now let $\mathbf{u} \in H^d(M,\mathbf{E})$ and $\varepsilon > 0$. By density of $\Gamma(M,\mathbf{E})$ in $H^d(M,\mathbf{E})$, choose $\mathbf{v} \in \Gamma(M,\mathbf{E})$ such that

$$\|\mathbf{u} - \mathbf{v}\|_{H^d(M,\mathbf{E})} < \frac{\varepsilon}{2}.$$

For this $\mathbf{v}$, the convergence just proved allows $N \in \mathbb{N}$ to be chosen so that

$$\|\mathbf{v} - \mathbf{v}_N\|_{H^d(M,\mathbf{E})} < \frac{\varepsilon}{2}.$$

Since $\mathbf{v}_N \in \mathcal{F}$, the triangle inequality gives

$$\|\mathbf{u} - \mathbf{v}_N\|_{H^d(M,\mathbf{E})} < \varepsilon.$$

Thus $\mathcal{F}$ is dense in $H^d(M,\mathbf{E})$. Equivalence (15.174) for z shows that this is also density in the graph norm of the pseudodifferential power. Hence $\mathcal{F}$ is a core for both closed realizations. Since they agree on $\mathcal{F}$, their

closures coincide, proving (15.175) without presupposing equality on all of $\Gamma(\mathbf{E})$.
The scalar identity

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda}\, dt, \qquad \lambda \geq c,$$

is transferred by the spectral theorem. Near zero, $t^{\operatorname{Re} s-1}$ is a majorant, and at infinity $t^{\operatorname{Re} s-1} e^{-ct}$ is a majorant. More precisely,

$$\|t^{s-1} e^{-tA}\|_{\mathcal{L}(L^2(M,\mathbf{E}))} \leq t^{\operatorname{Re} s-1} \quad (0 < t \leq 1), \qquad \|t^{s-1} e^{-tA}\|_{\mathcal{L}(L^2(M,\mathbf{E}))} \leq t^{\operatorname{Re} s-1} e^{-ct} \quad (t \geq 1).$$

The first function is integrable on $(0,1]$ because $\operatorname{Re} s > 0$, and the second on $[1,\infty)$. The Bochner integral therefore converges in $\mathcal{L}(L^2(M,\mathbf{E}))$. Applying both expressions to each spectral projection and using the same majorants allows the integral and spectral calculus to be interchanged, proving (15.176). □

15.7.4 Bessel potentials, the fractional Laplacian, and heat

The preceding identification returns us to the operators that motivated this chapter. Powers of the connection Laplacian, previously defined by the spectral theorem and the semigroup, now have classical symbols and can be studied within the same microlocal calculus.

Now let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold, and let $\mathbf{E} \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Denote its nonnegative self-adjoint realization by $L_{\mathbf{E}} = (\nabla^{\mathbf{E}})^*_h \nabla^{\mathbf{E}}$. Although the intrinsic differential symbol of $L_{\mathbf{E}}$ is $-\|\xi\|^2_{\mathbf{g}} I$, its principal Fourier symbol is $\|\xi\|^2_{\mathbf{g}} I$. Thus the operator

$$B_{\mathbf{E}} := I + L_{\mathbf{E}}$$

is elliptic, self-adjoint, satisfies $B_{\mathbf{E}} \geq I$, and admits the negative axis as a spectral cut.

Corollary 15.7.12. (Bessel powers and the fractional Laplacian) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold, and let $\mathbf{E} \longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $L_{\mathbf{E}} = (\nabla^{\mathbf{E}})^*_h \nabla^{\mathbf{E}}$ and $B_{\mathbf{E}} = I + L_{\mathbf{E}}$. For every $z \in \mathbb{C}$,

$$B_{\mathbf{E}}^z \in \Psi^{2z}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E}), \qquad \sigma^{\Psi}_{2z}(B_{\mathbf{E}}^z)(x,\xi) = \|\xi\|^{2z}_{\mathbf{g}} I_{\mathbf{E}_x}.$$

If $s > 0$, the Bessel potential constructed through the semigroup satisfies

$$J^{\mathbf{E}}_{s,2} = (I + L_{\mathbf{E}})^{-\frac{s}{2}} = B_{\mathbf{E}}^{-\frac{s}{2}} \in \Psi^{-s}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E}). \tag{15.178}$$

In particular, for every $r \in \mathbb{R}$,

$$B_{\mathbf{E}}^{-\frac{s}{2}} : H^r(M,\mathbf{E}) \longrightarrow H^{r+s}(M,\mathbf{E})$$

is an isomorphism with inverse $B_{\mathbf{E}}^{\frac{s}{2}}$. More precisely, for every $s \in \mathbb{R}$ there exist constants $c_s, C_s > 0$ such that, for $\mathbf{u} \in \Gamma(M,\mathbf{E})$,

$$c_s \|\mathbf{u}\|_{H^s(M,\mathbf{E})} \leq \|B_{\mathbf{E}}^{\frac{s}{2}} \mathbf{u}\|_{L^2(M,\mathbf{E})} \leq C_s \|\mathbf{u}\|_{H^s(M,\mathbf{E})}. \tag{15.179}$$

The constants depend on s, the metric $\mathbf{g}$, the Hermitian metric and connection of $\mathbf{E}$, and the finite cover, trivializations, and partition of unity chosen for the localized Sobolev norm; once these data are fixed, they are uniform in $\mathbf{u}$.

Let Π_0 be the orthogonal projection onto $\ker L_{\mathbf{E}}$. For $s > 0$,

$$L_{\mathbf{E}}^{\frac{s}{2}} = (L_{\mathbf{E}} + \Pi_0)^{\frac{s}{2}} - \Pi_0 \in \Psi_{\mathrm{cl}}^{s}(M; \mathbf{E}, \mathbf{E}), \tag{15.180}$$

and the reduced negative power on $(\ker L_{\mathbf{E}})^{\perp}$ is

$$L_{\mathbf{E},\perp}^{-\frac{s}{2}} = (L_{\mathbf{E}} + \Pi_0)^{-\frac{s}{2}} - \Pi_0 \in \Psi_{\mathrm{cl}}^{-s}(M; \mathbf{E}, \mathbf{E}). \tag{15.181}$$

Both have principal symbol $\|\xi\|_{\mathbf{g}}^{\pm s} I_{\mathbf{E}_x}$.

Proof. Theorem 15.7.9 applied to $B_{\mathbf{E}}$ gives the first assertion. Proposition 15.7.11 and the spectral characterization in Theorem 12.4.10 prove (15.178). The mapping properties of $B_{\mathbf{E}}^{\pm\frac{s}{2}}$ and the group law give the Sobolev isomorphism. The spectral identity

$$\|B_{\mathbf{E}}^{\frac{s}{2}}\mathbf{u}\|_{L^2(M,\mathbf{E})}^2 = \sum_{j=1}^{\infty} (1+\lambda_j)^s |u_j|^2$$

is the norm of the spectral scale in the preceding chapter. Equivalence of this scale with the localized scale, applied in both directions, gives exactly (15.179) and the stated dependencies. Theorem 15.7.9 adds the pseudodifferential description of the same scale.

The kernel of $L_{\mathbf{E}}$ is finite-dimensional and consists of smooth sections. Thus Π_0 has smooth kernel and belongs to $\Psi^{-\infty}(M; \mathbf{E}, \mathbf{E})$. The operator $L_{\mathbf{E}} + \Pi_0$ is strictly positive, invertible, and has the same principal symbol as $L_{\mathbf{E}}$. On $\ker L_{\mathbf{E}}$ its eigenvalue is one, while on the complement it agrees with $L_{\mathbf{E}}$. The spectral calculus then gives identities (15.180) and (15.181). Subtracting Π_0 does not change the principal symbol. □

On a complete manifold of bounded geometry, Theorems 13.3.25 and 13.3.28 already identify, under their stated hypotheses, the spectral scale of $I + L_{\mathbf{E}}$ with the scale defined by uniform localization. Both inequalities in (15.179) then retain the same form, with constants depending on s and quantitative bounded-geometry bounds, but not on the chart. Theorem 15.6.14 explains the mapping properties of the resulting uniform operators. We do not, however, assert that every complex power automatically belongs to $\Psi_{\mathrm{u}}^{2z}(M; \mathbf{E}, \mathbf{E})$: that conclusion requires a uniform parameter-dependent calculus and global resolvent estimates at infinity, hypotheses that do not follow from the local construction. On a closed manifold, finiteness of the atlas and compactness of the resolvent remove precisely this difficulty and yield the full result proved above.

For the trivial bundle over $\mathbb{R}^n$ and $s > 0$, the same identities can be read directly in Fourier variables. With the unitary normalization fixed in this chapter, the fractional Laplacian is defined exactly as the multiplier

$$\widehat{(-\Delta)^{\frac{s}{2}} u}(\xi) = \|\xi\|^s \widehat{u}(\xi), \qquad u \in \mathcal{S}(\mathbb{R}^n),$$

with closed domain given by

$$\left\{ u \in L^2(\mathbb{R}^n) \;\middle|\; \|\xi\|^s \widehat{u}(\xi) \in L^2(\mathbb{R}^n) \right\}.$$

In contrast,

$$(I - \Delta)^{\frac{s}{2}} = \mathrm{Op}\big((1 + \|\xi\|^2)^{\frac{s}{2}}\big)$$

belongs directly to the global inhomogeneous calculus. To express the pseudodifferential part of the first operator,

choose $\chi \in C^\infty(\mathbb{R}^n)$ vanishing near zero and equal to one for large $\|\xi\|$, and reserve quantization notation for the smooth symbol

$$\operatorname{Op}\big(\chi(\xi)\|\xi\|^s\big) \in \Psi^s_{\mathrm{cl,loc}}(\mathbb{R}^n;\mathbb{C},\mathbb{C}).$$

The symbol is classical, but the frequency cutoff does not imply proper support of the operator. The subscript loc is therefore retained. If a properly supported representative is required, multiply its kernel by a properly supported cutoff equal to one near the diagonal. The difference has smooth kernel because the original kernel is smooth off the diagonal; the resulting representative is not generally the exact Fourier multiplier. The difference between the exact fractional Laplacian and its high-frequency part is supported in a compact frequency set and maps $L^2(\mathbb{R}^n)$ continuously into $H^N(\mathbb{R}^n)$ for every N, without identifying $\|\xi\|^s$ with a symbol smooth at the origin. On a closed manifold, the frequency-zero difficulty is concentrated in the finite-dimensional space $\ker L_{\mathbf{E}}$, and the formulas with Π_0 isolate it exactly. At high frequencies, however, the principal symbols of $L_{\mathbf{E}}^{\frac{s}{2}}$ and $(I+L_{\mathbf{E}})^{\frac{s}{2}}$ coincide. Except for exponents producing a differential operator, these powers are nonlocal: their Schwartz kernels are not supported on the diagonal. For $0<s<2$, this nonlocality is also visible in the representation, valid for $\mathbf{u}\in\Gamma(\mathbf{E})$,

$$L_{\mathbf{E}}^{\frac{s}{2}}\mathbf{u} = \frac{s}{2\Gamma(1-\frac{s}{2})}\int_0^\infty \frac{\mathbf{u}-e^{-tL_{\mathbf{E}}}\mathbf{u}}{t^{1+\frac{s}{2}}}\,dt.$$

The integral converges as a Bochner integral in $L^2(M,\mathbf{E})$. For $\mathbf{u}\in\mathcal{D}(L_{\mathbf{E}})$, the semigroup equation and the fundamental theorem of calculus in $L^2(M,\mathbf{E})$ give the exact identity

$$\mathbf{u}-e^{-tL_{\mathbf{E}}}\mathbf{u} = \int_0^t L_{\mathbf{E}}e^{-\tau L_{\mathbf{E}}}\mathbf{u}\,d\tau. \tag{15.182}$$

Since the semigroup is contractive and commutes with $L_{\mathbf{E}}$ on $\mathcal{D}(L_{\mathbf{E}})$,

$$\|\mathbf{u}-e^{-tL_{\mathbf{E}}}\mathbf{u}\|_{L^2(M,\mathbf{E})} \le \int_0^t \|e^{-\tau L_{\mathbf{E}}}L_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})}\,d\tau \le t\|L_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})}. \tag{15.183}$$

Thus, for $0<t\le 1$, the norm of the integrand is at most $\|L_{\mathbf{E}}\mathbf{u}\|_{L^2(M,\mathbf{E})}t^{-\frac{s}{2}}$, which is integrable because $s<2$. For $t\ge 1$, contractivity gives

$$\left\|\frac{\mathbf{u}-e^{-tL_{\mathbf{E}}}\mathbf{u}}{t^{1+\frac{s}{2}}}\right\|_{L^2(M,\mathbf{E})} \le 2\|\mathbf{u}\|_{L^2(M,\mathbf{E})}t^{-1-\frac{s}{2}},$$

and this majorant is integrable because $s>0$. The $\ker L_{\mathbf{E}}$ component vanishes exactly in the numerator.

Negative powers and heat are two presentations of the same functional calculus. The first integrates the resolvent against λ^{-s}; the second integrates the spectrum against $e^{-t\lambda}$. The Mellin transform in (15.176) identifies them.

Proposition 15.7.13. (Heat smoothing and trace) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold, and let $\mathbf{E}\longrightarrow M$ be a complex vector bundle of finite rank equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Let $A\ge 0$ be a classical elliptic self-adjoint pseudodifferential operator of order $m>0$ on $\mathbf{E}$. For each $t>0$,

$$e^{-tA}\in\Psi^{-\infty}(M;\mathbf{E},\mathbf{E}).$$

Its kernel $\mathbf{K}_A(t,x,y)$ is smooth on $(0,\infty)\times M\times M$. Moreover, e^{-tA} is trace class and

$$\operatorname{Tr}(e^{-tA}) = \int_{M_x} \operatorname{tr}_{\mathbf{E}_x} \mathbf{K}_A(t,x,x)\, d\lambda_{\mathbf{g}}(x). \tag{15.184}$$

Proof. The spectral calculus gives, for each $N\in\mathbb{N}$,

$$\|A^N e^{-tA}\|_{\mathcal{L}(L^2(M,\mathbf{E}),L^2(M,\mathbf{E}))} \le \sup_{\lambda\ge 0} \lambda^N e^{-t\lambda} < \infty.$$

Thus e^{-tA} maps $L^2(M,\mathbf{E})$ into $\mathcal{D}(A^N)$. Elliptic regularity identifies this domain, locally and with equivalent norms, with $H^{mN}(M,\mathbf{E})$. Since N is arbitrary, the operator gains arbitrarily many derivatives. The same argument for the adjoint shows that its kernel is smooth; hence it belongs to $\Psi^{-\infty}(M;\mathbf{E},\mathbf{E})$. For t in a compact subset of $(0,\infty)$, the identities $\partial_t^k e^{-tA} = (-A)^k e^{-tA}$ and the same estimates, with arbitrary N, give joint smoothness in (t,x,y). On a compact manifold, a smooth kernel defines a trace-class operator and the diagonal formula (15.184) holds. □

For the McKean–Singer argument, smoothing, smoothness of the kernel, and the diagonal trace formula suffice; the fine heat asymptotics will be needed only later, in studying the local index density. For $B_{\mathbf{E}} = I + L_{\mathbf{E}}$, we obtain

$$e^{-tB_{\mathbf{E}}} = e^{-t} e^{-tL_{\mathbf{E}}}$$

and

$$B_{\mathbf{E}}^{-\frac{s}{2}} = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty t^{\frac{s}{2}-1} e^{-t} e^{-tL_{\mathbf{E}}}\, dt,$$

which is exactly the Bochner formula used to define Bessel potentials.

15.7.5 From heat to the index

The chapter's last step shows how the preceding theory enters the index problem. The parametrix gives the Fredholm property, and heat turns the difference between kernel and cokernel into a difference of traces. Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E}^+,\mathbf{E}^-\longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics and Hermitian connections; and let

$$P\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E}^+,\mathbf{E}^-), \qquad P\colon \Gamma(M,\mathbf{E}^+)\longrightarrow\Gamma(M,\mathbf{E}^-), \qquad m>0,$$

be a classical elliptic operator. Its parametrix implies that its closed realization

$$P\colon H^m(M,\mathbf{E}^+)\subset L^2(M,\mathbf{E}^+)\longrightarrow L^2(M,\mathbf{E}^-)$$

is Fredholm. Its Hilbert space adjoint $P^\dagger$ has domain $H^m(M,\mathbf{E}^-)$ and agrees with the closed realization of the formal adjoint P_h^*. Thus

$$\operatorname{ind} P = \dim\ker P - \dim\ker P^\dagger.$$

This functional-analytic part of the index is a consequence of Atkinson's theorem and compactness of remainders on a closed manifold; see also [21, Chapters 3 and 9].

Proposition 15.7.14. (McKean–Singer formula) Let $(M,\mathbf{g})$ be a smooth closed Riemannian manifold; let $\mathbf{E}^+,\mathbf{E}^- \longrightarrow M$ be complex vector bundles of finite rank equipped with Hermitian bundle metrics; and let $P \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E}^+,\mathbf{E}^-)$ be elliptic, with $m>0$. For every $t>0$,

$$\operatorname{ind} P = \operatorname{Tr}(e^{-tP^\dagger P}) - \operatorname{Tr}(e^{-tPP^\dagger}). \tag{15.185}$$

The spectral proof corresponds to [66, Lemma 1.6.5].

Proof. The operators $P^\dagger P$ and $PP^\dagger$ are nonnegative, self-adjoint, and elliptic. Their semigroups are smoothing and trace class. For $\lambda>0$, the map

$$P : \ker(P^\dagger P - \lambda I) \longrightarrow \ker(PP^\dagger - \lambda I)$$

is an isomorphism with inverse $\lambda^{-1}P^\dagger$. Thus the contributions $e^{-t\lambda}$ from all positive eigenvalues cancel in the difference of traces. At eigenvalue zero, what remains is

$$\ker(P^\dagger P) = \ker P, \qquad \ker(PP^\dagger) = \ker P^\dagger,$$

and the difference of multiplicities is precisely $\operatorname{ind} P$. This proves the spectral formula. □

The right-hand side of (15.185) is independent of t. The diagonal formula (15.184) interprets both traces and the spectral cancellation; its local limit is expressed through the fine asymptotics of the heat kernel.

The analytic conclusion depends only on the class of the principal symbol. To see this, let us specify which datum remains invariant under the preceding operations. If $\pi : T^*M \setminus 0_M \to M$ is the projection and P has order m, ellipticity says that

$$\sigma^\Psi_m(P) : \pi^*\mathbf{E}^+ \longrightarrow \pi^*\mathbf{E}^-$$

is a bundle isomorphism over $T^*M \setminus 0_M$. If $R \in \Psi^{m-1}_{\mathrm{cl}}(M;\mathbf{E}^+,\mathbf{E}^-)$, the family $P_t = P + tR$ preserves this principal symbol and is therefore elliptic for every $t \in [0,1]$. Viewed between the appropriate Sobolev spaces, it is a norm-continuous family of Fredholm operators; local constancy of the index on the Fredholm set gives $\operatorname{ind}(P+R) = \operatorname{ind} P$. If R is smoothing, the same conclusion follows directly from Atkinson's theorem, since R is compact. Finally, fixing bundle metrics h_+ and h_- on $\mathbf{E}^+$ and $\mathbf{E}^-$, compactness of S^*M gives

$$\delta := \inf_{(x,\xi)\in S^*M} \left\| \sigma^\Psi_m(P)(x,\xi)^{-1} \right\|^{-1}_{\mathcal{L}((\mathbf{E}^-_x,h_-(x)),(\mathbf{E}^+_x,h_+(x)))} > 0.$$

If another homogeneous principal symbol b_m satisfies

$$\sup_{(x,\xi)\in S^*M} \| b_m(x,\xi) - \sigma^\Psi_m(P)(x,\xi) \|_{\mathcal{L}((\mathbf{E}^+_x,h_+(x)),(\mathbf{E}^-_x,h_-(x)))} < \delta,$$

the straight-line segment between them remains invertible by the Neumann series. Quantizing it again gives a norm-continuous family of elliptic operators with unchanged index. The analytic problem is thus reduced to the topological class of the principal symbol. The following chapters construct that class and explain how the index is computed from it.

16

Fredholm operators and the analytic index

The pseudodifferential calculus of Chapter 15 assigns to every elliptic operator on a closed manifold a parametrix whose remainders are smoothing. Our task now is to isolate the functional-analytic consequences of this construction and establish the properties of the index that will be used in passing from the principal symbol to K–theory. The first part is formulated for bounded operators between Banach spaces. The second returns to Sobolev spaces of sections and shows that the resulting integer depends on the homotopy class of the elliptic symbol, independently of the chosen completion.

The functional-analytic exposition follows Chapters 1–3 of [21, pp. 3–88]; the application to elliptic operators corresponds to Theorem 9.10 and Corollary 9.18 of [21, pp. 240–242]. We give the proofs that make the passage between the two parts explicit.

Fredholm and the persistence of finite dimension

Ivar Fredholm discovered that certain infinite-dimensional integral equations retain an alternative governed by finite-dimensional kernels and obstructions. Modern theory abstracts this situation in the notion of a Fredholm operator. Its index does not count all solutions; it measures the stable imbalance between solutions and compatibility conditions. This stability explains why an analytic integer can survive deformations of the operator and ultimately depend only on its principal symbol.

16.1 Fredholm operators and parametrices modulo compact operators

Let X and Y be Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Recall Definition B.2.40: an operator $T \in \mathcal{L}(X, Y)$ is Fredholm when $\ker T$ is finite-dimensional, $\operatorname{Ran} T$ is closed, and the cokernel

$$\operatorname{coker} T := Y / \operatorname{Ran} T$$

is finite-dimensional. In this case,

$$\operatorname{ind} T := \dim \ker T - \dim \operatorname{coker} T.$$

We explicitly require the range to be closed because of its analytic role: it gives the cokernel a Banach space structure and ensures that the inverse induced by T on a closed complement of its kernel is continuous. For a

bounded operator between Banach spaces, however, closedness also follows from the finite dimensionality of the algebraic cokernel.

Indeed, suppose that $Y/\operatorname{Ran} T$ is finite-dimensional and set $K := \ker T$. The subspace K is closed by the continuity of T, so X/K is Banach by Proposition B.2.20. Choose representatives of a basis of $Y/\operatorname{Ran} T$ and denote their span by C. Then C is finite-dimensional and $Y = \operatorname{Ran} T \oplus C$ as an algebraic direct sum. Consider

$$S\colon (X/K)\times C \longrightarrow Y, \qquad S([x],c) := Tx + c.$$

This operator is well defined because T vanishes on K, and the algebraic direct sum shows that it is linear and bijective. Moreover, for every $k \in K$, $\|Tx\|_Y = \|T(x-k)\|_Y \le \|T\|\|x-k\|_X$; taking the infimum yields

$$\|S([x],c)\|_Y \le \|T\|\|[x]\|_{X/K} + \|c\|_Y.$$

Thus S is continuous when the product is equipped with the sum norm. Both spaces are Banach, so Corollary B.2.17 implies that S^{-1} is continuous. Consequently,

$$\operatorname{Ran} T = S\big((X/K)\times\{0\}\big)$$

is closed, since a homeomorphism takes closed sets to closed sets. This conclusion does not require $\ker T$ to be finite-dimensional.

Lemma 16.1.1. (Complements associated with a Fredholm operator) Let $T \in \mathcal{L}(X,Y)$ be Fredholm. There exist closed subspaces $X_0 \subseteq X$ and $C \subseteq Y$ such that

$$X = \ker T \oplus X_0, \qquad Y = \operatorname{Ran} T \oplus C,$$

and the restriction

$$T_0 := T\restriction_{X_0}\colon X_0 \longrightarrow \operatorname{Ran} T$$

is an isomorphism of Banach spaces. Moreover, $\dim C = \dim \operatorname{coker} T$.

Proof. Every finite-dimensional subspace of a normed space is complemented. Indeed, if $(e_1,\dots,e_r)$ is a basis of $\ker T$, the coordinate functionals on $\ker T$ are continuous, and Corollary B.2.29 allows us to extend them to $\ell_1,\dots,\ell_r \in X'$. The operator

$$P_N x := \sum_{j=1}^{r} \ell_j(x) e_j$$

is a continuous projection onto $\ker T$; therefore, $X_0 := \ker P_N$ is closed and $X = \ker T \oplus X_0$.

Since $\operatorname{Ran} T$ is closed and has finite codimension, choose $c_1,\dots,c_q \in Y$ whose classes form a basis of $Y/\operatorname{Ran} T$ and set $C := \operatorname{span}\{c_1,\dots,c_q\}$. Every $y \in Y$ has a unique expression $y = r + c$, with $r \in \operatorname{Ran} T$ and $c \in C$. The projection onto C has closed graph: if $y_\nu \to y$ and their components satisfy $c_\nu \to c$, then $y - c \in \operatorname{Ran} T$. Theorem B.2.22 shows that the projection is continuous. Thus $Y = \operatorname{Ran} T \oplus C$ is a topological direct sum and $\dim C = q = \dim \operatorname{coker} T$.

The restriction T_0 is linear, continuous, and injective. It is also surjective: if $Tx \in \operatorname{Ran} T$ and $x = n + x_0$ is the first decomposition, then $Tx = Tx_0$. Theorem B.2.16 shows that T_0^{-1} is continuous. □

The central characterization identifies the Fredholm property with invertibility up to compact errors. This equivalence is Atkinson's theorem.

Theorem 16.1.2. (Atkinson) For $T \in \mathcal{L}(X,Y)$, the following statements are equivalent.

(i) The operator T is Fredholm.

(ii) There exists $S \in \mathcal{L}(Y,X)$ such that

$$ST = I_X - K_X, \qquad TS = I_Y - K_Y, \tag{16.1}$$

with $K_X \in \mathcal{K}(X)$ and $K_Y \in \mathcal{K}(Y)$.

If $X = Y$, the equivalence states that the class of T in the Calkin algebra

$$\mathcal{C}(X) := \mathcal{L}(X)/\mathcal{K}(X)$$

is invertible.

Proof. First suppose that T is Fredholm and use the decompositions in Lemma 16.1.1. Define $S\colon Y \to X$ by

$$S(r+c) := T_0^{-1} r, \qquad r \in \operatorname{Ran} T, \quad c \in C.$$

The projections associated with the two direct sums and T_0^{-1} are continuous, so $S \in \mathcal{L}(Y,X)$. If P_N and P_C are the projections onto $\ker T$ and C, respectively, then

$$ST = I_X - P_N, \qquad TS = I_Y - P_C.$$

Both projections have finite rank and are therefore compact. This proves (ii).

Now assume (ii). For $x \in \ker T$, the first identity in (16.1) gives $x = K_X x$. The unit ball of $\ker T$ is therefore the image under K_X of a bounded set and is relatively compact. The unit ball of an infinite-dimensional normed space cannot be relatively compact; hence $\ker T$ is finite-dimensional.

Choose a closed complement X_0 of $\ker T$ as in the first part of Lemma 16.1.1. We claim that there exists $c > 0$ such that

$$\|Tx\|_Y \geq c\|x\|_X, \qquad x \in X_0. \tag{16.2}$$

If this estimate were false, there would be a sequence $(x_\nu) \subseteq X_0$ with $\|x_\nu\|_X = 1$ and $Tx_\nu \to 0$. By compactness, a subsequence of $(K_X x_\nu)$ would converge. The identity

$$x_\nu = STx_\nu + K_X x_\nu$$

would show that the same subsequence of (x_ν) converges to some $x \in X_0$. Then $\|x\|_X = 1$ and $Tx = 0$, contradicting $X_0 \cap \ker T = \{0\}$. This proves (16.2).

The estimate implies that the range is closed. Indeed, if $Tx_\nu \to y$, write $x_\nu = n_\nu + x_\nu^0$ with $n_\nu \in \ker T$ and $x_\nu^0 \in X_0$. Then $Tx_\nu^0 = Tx_\nu$ and

$$\|x_\nu^0 - x_\mu^0\|_X \leq \frac{1}{c}\|Tx_\nu - Tx_\mu\|_Y.$$

The sequence (x_ν^0) converges to an element $x^0 \in X_0$, and the continuity of T gives $y = Tx^0$.
Let $Q\colon Y \to Y/\operatorname{Ran} T$ be the quotient map. Since the range is closed, the codomain of Q is Banach. Set $K_Y = I_Y - TS$. If $y = Tx$, then

$$K_Y y = Tx - TSTx = T(I_X - ST)x = TK_X x \in \operatorname{Ran} T.$$

Consequently, K_Y induces a compact operator

$$\overline{K}_Y\colon Y/\operatorname{Ran} T \longrightarrow Y/\operatorname{Ran} T, \qquad \overline{K}_Y[y] := [K_Y y].$$

To verify compactness, observe that every class of norm less than one has a representative of norm less than two. The image of these representatives under K_Y is relatively compact in Y, and its projection under Q remains relatively compact in the quotient. The second identity in (16.1) implies

$$\overline{K}_Y[y] = [y - TSy] = [y],$$

so the identity on the cokernel is compact. Its unit ball is relatively compact, and the cokernel is finite-dimensional. Thus T is Fredholm.
When $X = Y$, the two identities in (16.1) are equivalent to $[S][T] = [I_X] = [T][S]$ in $\mathcal{C}(X)$, which is precisely the invertibility of $[T]$. □

The preceding theorem appears in its Calkin algebra formulation in [21, Theorem 3.2, pp. 63–65]. The proof reveals a point that will be decisive in geometry: there is no need to construct an exact inverse; a parametrix with compact remainders contains all the Fredholm information.

Example 16.1.3. (Unilateral shifts) Let $\ell^2(\mathbb{N}_0)$ be the space of square-summable sequences and define

$$S_+(a_0, a_1, a_2, \ldots) := (0, a_0, a_1, \ldots), \qquad S_-(a_0, a_1, a_2, \ldots) := (a_1, a_2, a_3, \ldots).$$

Both operators have norm one and

$$S_- S_+ = I, \qquad S_+ S_- = I - \Pi_0,$$

where $\Pi_0(a_0, a_1, \ldots) = (a_0, 0, 0, \ldots)$ has rank one. Thus each is a parametrix for the other modulo compact operators. Moreover,

$$\ker S_+ = \{0\}, \qquad \operatorname{Ran} S_+ = \{(a_j) \in \ell^2(\mathbb{N}_0) \mid a_0 = 0\},$$

whereas $\ker S_- = \operatorname{span}\{(1, 0, 0, \ldots)\}$ and S_- is surjective. Therefore,

$$\operatorname{ind} S_+ = -1, \qquad \operatorname{ind} S_- = 1.$$

This example shows that the index is not a formal difference between two infinite dimensions: it records the finite defect left after inverting the operator modulo compact operators.

16.2 Stability and index calculus

The definition of the index uses two dimensions that may change separately. Their difference, however, remains fixed under perturbations that preserve the Fredholm property.

Theorem 16.2.1. (Stability of the Fredholm index) Let X and Y be Banach spaces.

(a) The set
$$\operatorname{Fred}(X,Y) := \{T \in \mathcal{L}(X,Y) \mid T \text{ is Fredholm}\}$$
is open in the operator norm, and $T \mapsto \operatorname{ind} T$ is locally constant.

(b) If $T \in \operatorname{Fred}(X,Y)$ and $K \in \mathcal{K}(X,Y)$, then $T+K$ is Fredholm and
$$\operatorname{ind}(T+K) = \operatorname{ind} T.$$

(c) If Z is a connected topological space and $z \mapsto T_z \in \operatorname{Fred}(X,Y)$ is norm-continuous, then $\operatorname{ind} T_z$ is independent of z.

Proof. Fix $T \in \operatorname{Fred}(X,Y)$ and the decompositions
$$X = X_0 \oplus N, \qquad Y = R \oplus C, \qquad N = \ker T, \quad R = \operatorname{Ran} T,$$
of Lemma 16.1.1. With respect to these, T has matrix
$$T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_0 \colon X_0 \longrightarrow R \text{ invertible.}$$
For A sufficiently close to T, write
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
The block $a \colon X_0 \to R$ is invertible because $\|a - T_0\| < \|T_0^{-1}\|^{-1}$ implies that $a = T_0(I_{X_0} + T_0^{-1}(a - T_0))$ can be inverted using the Neumann series. Multiplying A on the left and right by invertible operators yields
$$\begin{pmatrix} I_R & 0 \\ -ca^{-1} & I_C \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} I_{X_0} & -a^{-1}b \\ 0 & I_N \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{pmatrix}.$$
The second block acts between the finite-dimensional spaces N and C. Composition on the left or right with an isomorphism identifies the corresponding kernels and cokernels and does not change the index. If $D := d - ca^{-1}b$, the rank–nullity theorem gives
$$\dim\ker D - \dim(C/\operatorname{Ran} D) = (\dim N - \dim \operatorname{Ran} D) - (\dim C - \dim \operatorname{Ran} D) = \dim N - \dim C.$$
Thus A is Fredholm and
$$\operatorname{ind} A = \dim N - \dim C = \operatorname{ind} T.$$
This proves both openness and local constancy.

If K is compact and S is a parametrix for T, then

$$S(T+K)-I_X=(ST-I_X)+SK,\qquad (T+K)S-I_Y=(TS-I_Y)+KS$$

are compact. Theorem 16.1.2 shows that $T+tK$ is Fredholm for every $t\in[0,1]$. Since $t\mapsto T+tK$ is continuous and $[0,1]$ is connected, the local constancy just proved gives $\operatorname{ind}(T+K)=\operatorname{ind}T$. The same argument, applied to the continuous image of any connected space Z, proves (c). □

For local constancy, see also [21, Theorem 3.11, pp. 68–71]. The block argument explains why the dimensions of the kernel and cokernel need not be constant: all their variation is confined to the finite-dimensional operator $d-ca^{-1}b$, whose index is always $\dim N-\dim C$.

Proposition 16.2.2. (Rules for index calculus) Let $T\colon X\to Y$ and $U\colon Y\to Z$ be Fredholm operators between Banach spaces, and let $V\colon X_1\to Y_1$ be another Fredholm operator. Then

$$\operatorname{ind}(T\oplus V)=\operatorname{ind}T+\operatorname{ind}V, \tag{16.3}$$

$$\operatorname{ind}(UT)=\operatorname{ind}T+\operatorname{ind}U. \tag{16.4}$$

If X and Y are Hilbert spaces and $T^*\colon Y\to X$ is the adjoint, then T^* is Fredholm and

$$\operatorname{ind}T^*=-\operatorname{ind}T. \tag{16.5}$$

Proof. The direct sum satisfies

$$\ker(T\oplus V)=\ker T\oplus\ker V,\qquad \operatorname{coker}(T\oplus V)\cong\operatorname{coker}T\oplus\operatorname{coker}V,$$

which proves (16.3).

Let $S_T\colon Y\to X$ and $S_U\colon Z\to Y$ be parametrices for T and U. Since compact operators form a two-sided ideal, S_TS_U is a parametrix for UT modulo compact operators. By Theorem 16.1.2, UT is Fredholm. Consider the sequence

$$\begin{aligned}0\longrightarrow\ker T\longrightarrow\ker(UT)\xrightarrow{T}\ker U\xrightarrow{q_T}\operatorname{coker}T\\ \xrightarrow{\overline{U}}\operatorname{coker}(UT)\xrightarrow{q_U}\operatorname{coker}U\longrightarrow 0,\end{aligned} \tag{16.6}$$

where $q_T(y)=[y]$, $\overline{U}([y])=[Uy]$, and $q_U([z])=[z]$. These maps are well defined: $U(\operatorname{Ran}T)=\operatorname{Ran}(UT)$, and the quotient by $\operatorname{Ran}(UT)$ projects naturally onto the quotient by $\operatorname{Ran}U$. Let us verify exactness at the points where it is not tautological. If $y\in\ker U$ and $q_T(y)=0$, then $y=Tx$ and $x\in\ker(UT)$, so y lies in the image of the third map. If $[y]\in\operatorname{coker}T$ and $[Uy]=0$ in $\operatorname{coker}(UT)$, there exists $x\in X$ such that $Uy=UTx$; therefore, $y-Tx\in\ker U$ and $[y]=q_T(y-Tx)$. Finally, if $[z]\in\operatorname{coker}(UT)$ vanishes in $\operatorname{coker}U$, then $z=Uy$ and $[z]=\overline{U}([y])$. The sequence is exact.

All its terms are finite-dimensional. The alternating sum of the dimensions in a finite exact sequence is zero; applying this to (16.6) gives

$$\dim\ker T-\dim\ker(UT)+\dim\ker U-\dim\operatorname{coker}T+\dim\operatorname{coker}(UT)-\dim\operatorname{coker}U=0,$$

which is equivalent to (16.4).

Now suppose that X and Y are Hilbert spaces. Their orthogonal decompositions are

$$X = \ker T \oplus (\ker T)^{\perp}, \qquad Y = \operatorname{Ran} T \oplus (\operatorname{Ran} T)^{\perp}.$$

The restriction $T_0 \colon (\ker T)^{\perp} \to \operatorname{Ran} T$ is an isomorphism. Its adjoint $T_0^* \colon \operatorname{Ran} T \to (\ker T)^{\perp}$ is also an isomorphism. Moreover,

$$\ker T^* = (\operatorname{Ran} T)^{\perp}, \qquad \operatorname{Ran} T^* = (\ker T)^{\perp}.$$

The first equality follows directly from the definition of the adjoint; the second follows from the invertibility of T_0^* and the fact that T^* vanishes on $(\operatorname{Ran} T)^{\perp}$. In particular, T^* has closed range,

$$\dim \ker T^* = \dim \operatorname{coker} T, \qquad \dim \operatorname{coker} T^* = \dim \ker T,$$

and we obtain (16.5). □

16.3 Continuous families and finite-dimensional stabilization

For a norm-continuous family of Fredholm operators, the numerical index is constant on each connected component of the parameter space. When the parameter ranges over a compact space, a single finite-dimensional correction can be chosen to make all the operators surjective simultaneously. This will be the appropriate form for constructing a K–theory class in the next chapter.

Proposition 16.3.1. (Stabilization of a Fredholm family) Let B be a compact Hausdorff space and let

$$T \colon B \longrightarrow \operatorname{Fred}(X, Y), \qquad b \longmapsto T_b,$$

be continuous in the operator norm. There exists a finite-dimensional subspace $V \subseteq Y$ such that

$$\operatorname{Ran} T_b + V = Y, \qquad b \in B. \tag{16.7}$$

The surjective operators

$$\widetilde{T}_b \colon X \oplus V \longrightarrow Y, \qquad \widetilde{T}_b(x, v) := T_b x + v,$$

have kernels forming a vector bundle of locally constant rank over B. If $V \subseteq V'$ are two stabilizations satisfying (16.7), then

$$[\ker \widetilde{T}^{V'}] - [B \times V'] = [\ker \widetilde{T}^{V}] - [B \times V] \tag{16.8}$$

as stable differences of vector bundles.

Proof. Fix $b_0 \in B$ and choose a finite-dimensional complement C_{b_0} of $\operatorname{Ran} T_{b_0}$. The operator

$$A_{b_0} \colon X \oplus C_{b_0} \longrightarrow Y, \qquad A_{b_0}(x, c) = T_{b_0} x + c,$$

is surjective. Since $C_{b_0} \cap \operatorname{Ran} T_{b_0} = \{0\}$, its kernel is $\ker T_{b_0} \oplus \{0\}$ and is therefore finite-dimensional. Hence the kernel of A_{b_0} admits a closed complement W_{b_0} and $A_{b_0} \restriction_{W_{b_0}} \colon W_{b_0} \to Y$ is an isomorphism. For b sufficiently close to b_0, the restriction $A_b \restriction_{W_{b_0}}$ remains invertible by the Neumann series; in particular, $\operatorname{Ran} T_b + C_{b_0} = Y$.

These neighborhoods cover B. Choose a finite subcover, associated with $b_1,\dots,b_N$, and set

$$V := C_{b_1} + \cdots + C_{b_N}.$$

The space V is finite-dimensional and satisfies (16.7).
To prove the bundle structure, fix b_0 again and choose a closed complement W of $N_0 := \ker \widetilde{T}_{b_0}$ in $X \oplus V$. The restriction

$$\widetilde{T}_{b_0} \restriction_W : W \longrightarrow Y$$

is an isomorphism and remains so for b near b_0. For $n \in N_0$ define

$$\Phi_b(n) := n - \big(\widetilde{T}_b \restriction_W\big)^{-1} \widetilde{T}_b n.$$

Then $\Phi_b(n) \in \ker \widetilde{T}_b$. If $z = n + w \in N_0 \oplus W$ belongs to $\ker \widetilde{T}_b$, the equation $\widetilde{T}_b z = 0$ forces $w = -(\widetilde{T}_b \restriction_W)^{-1} \widetilde{T}_b n$; therefore, $z = \Phi_b(n)$. Thus $\Phi_b : N_0 \to \ker \widetilde{T}_b$ is an isomorphism depending continuously on b. These maps are local trivializations of the set of kernels.
Finally, suppose that $V \subseteq V'$. The projection $V' \to V'/V$ induces an exact sequence of bundles

$$0 \longrightarrow \ker \widetilde{T}^{\,V} \longrightarrow \ker \widetilde{T}^{\,V'} \longrightarrow B \times (V'/V) \longrightarrow 0.$$

Surjectivity of the last arrow is checked fiber by fiber. Given $w \in V'$, surjectivity of $\widetilde{T}_b^{\,V}$ provides $(x, v) \in X \oplus V$ such that $T_b x + v = -w$; then $(x, v + w)$ belongs to $\ker \widetilde{T}_b^{\,V'}$ and projects to $[w] \in V'/V$. A bundle metric on the middle bundle, Euclidean if $\mathbb{K} = \mathbb{R}$ and Hermitian if $\mathbb{K} = \mathbb{C}$, splits the sequence orthogonally, so

$$\ker \widetilde{T}^{\,V'} \cong \ker \widetilde{T}^{\,V} \oplus B \times (V'/V).$$

Subtracting the trivial bundles $B \times V'$ and $B \times V$ yields (16.8).
It remains to compare two arbitrary stabilizations $V_0, V_1 \subseteq Y$. The subspace $W := V_0 + V_1$ is finite-dimensional and also satisfies (16.7). Applying the equality just proved to the inclusions $V_0 \subseteq W$ and $V_1 \subseteq W$ yields

$$[\ker \widetilde{T}^{\,V_0}] - [B \times V_0] = [\ker \widetilde{T}^{\,W}] - [B \times W] = [\ker \widetilde{T}^{\,V_1}] - [B \times V_1].$$

Thus the stable difference is independent of every choice used in its construction. □

Stabilization replaces a family whose kernels may vary in dimension with a well-defined difference of finite-dimensional bundles. The Grothendieck group in the next chapter will allow us to consider this difference without retaining the auxiliary subspace V. This is the index bundle construction developed in [21, Theorem 3.30, pp. 82–85].

16.4 Elliptic operators on closed manifolds

Let $(M, \mathbf{g})$ be a closed smooth Riemannian manifold and let $\mathbf{E}, \mathbf{F} \longrightarrow M$ be finite-rank complex vector bundles equipped with Hermitian bundle metrics and Hermitian connections. For $m, s \in \mathbb{R}$ and

$$P \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{F}),$$

, Corollary 15.6.6 gives a unique continuous extension

$$P_s\colon H^{s+m}(M,\mathbf{E}) \longrightarrow H^s(M,\mathbf{F}).$$

In what follows, P will be elliptic. Chapter 15 constructed a parametrix $Q \in \Psi^{-m}_{\mathrm{cl}}(M;\mathbf{F},\mathbf{E})$ with

$$QP = I_{\mathbf{E}} - R_{\mathbf{E}}, \qquad PQ = I_{\mathbf{F}} - R_{\mathbf{F}}, \qquad R_{\mathbf{E}}, R_{\mathbf{F}} \in \Psi^{-\infty}. \tag{16.9}$$

Theorem 16.4.1. (Elliptic Fredholm operators and the analytic index) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, let $\mathbf{E},\mathbf{F} \to M$ be finite-rank complex vector bundles equipped with Hermitian bundle metrics, and let $P \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic. For every $s \in \mathbb{R}$, the operator

$$P_s\colon H^{s+m}(M,\mathbf{E}) \longrightarrow H^s(M,\mathbf{F})$$

is Fredholm. Its kernel consists of the smooth solutions of $P\mathbf{u} = 0$, and its cokernel is identified, by means of the orthogonal projection extended by duality, with the space of smooth solutions of $P^*\mathbf{v} = 0$, where P^* is the Hermitian formal adjoint. In particular,

$$\operatorname{ind} P_s = \dim\ker\bigl(P\colon \Gamma(M,\mathbf{E}) \to \Gamma(M,\mathbf{F})\bigr) - \dim\ker\bigl(P^*\colon \Gamma(M,\mathbf{F}) \to \Gamma(M,\mathbf{E})\bigr). \tag{16.10}$$

The right-hand side is independent of s.

Proof. The parametrix in (16.9) induces

$$Q_s\colon H^s(M,\mathbf{F}) \longrightarrow H^{s+m}(M,\mathbf{E}).$$

By Proposition 15.6.18, $R_{\mathbf{E}}$ and $R_{\mathbf{F}}$ act compactly on the Sobolev spaces occurring in the two identities. Applying Theorem 16.1.2 to P_s and Q_s proves that P_s is Fredholm.

If $\mathbf{u} \in H^{s+m}(M,\mathbf{E})$ and $P_s\mathbf{u} = 0$, the first parametrix identity gives $\mathbf{u} = R_{\mathbf{E}}\mathbf{u}$. A smoothing operator takes every distribution to a smooth section, so $\mathbf{u} \in \Gamma(M,\mathbf{E})$. Conversely, every smooth solution belongs to $H^{s+m}(M,\mathbf{E})$ because M is compact. Thus

$$\ker P_s = \{\mathbf{u} \in \Gamma(M,\mathbf{E}) \mid P\mathbf{u} = 0\}, \tag{16.11}$$

independently of s.

The formal adjoint $P^* \in \Psi^m_{\mathrm{cl}}(M;\mathbf{F},\mathbf{E})$ is characterized by

$$\int_M \mathbf{h}_{\mathbf{F}}(P\mathbf{u},\mathbf{v})\,d\lambda_{\mathbf{g}} = \int_M \mathbf{h}_{\mathbf{E}}(\mathbf{u},P^*\mathbf{v})\,d\lambda_{\mathbf{g}}, \qquad \mathbf{u} \in \Gamma(M,\mathbf{E}), \quad \mathbf{v} \in \Gamma(M,\mathbf{F}). \tag{16.12}$$

The principal symbol of P^* is the adjoint of the principal symbol of P by Theorem 15.2.8; therefore, P^* is elliptic. Proposition 14.5.5, applied to the closed manifold, identifies the continuous annihilator of $\operatorname{Ran} P_s \subseteq H^s(M,\mathbf{F})$ with

$$\ker\bigl(P^*\colon H^{-s}(M,\mathbf{F}) \longrightarrow H^{-s-m}(M,\mathbf{E})\bigr),$$

by an antilinear identification, since the Hermitian inner product is linear in its first variable. The elliptic regularity of Corollary 15.6.5 shows that this kernel consists of smooth sections. Denote it by K and choose

an orthonormal basis $\mathbf{v}_1,\dots,\mathbf{v}_q$ of K for the $L^2(M,\mathbf{F})$ inner product. For $\mathbf{f}\in H^s(M,\mathbf{F})$ define

$$\Pi_K\mathbf{f} := \sum_{j=1}^{q}\langle \mathbf{f},\mathbf{v}_j\rangle \mathbf{v}_j.$$

Each pairing is continuous on H^s, because $\mathbf{v}_j$ is smooth and belongs to H^{-s}. Thus $\Pi_K : H^s(M,\mathbf{F})\to K$ is linear and continuous, agrees with the orthogonal projection when $\mathbf{f}\in L^2\cap H^s$, and is the identity on K. Its kernel is $\operatorname{Ran} P_s$: inclusion of the range follows from the formal adjoint; conversely, if all the pairings vanish, all continuous functionals annihilating $\operatorname{Ran} P_s$ vanish at $\mathbf{f}$. Hahn–Banach and closedness of the range imply $\mathbf{f}\in\operatorname{Ran} P_s$. Thus Π_K induces a complex-linear isomorphism $\operatorname{coker} P_s\cong K$, independent of the chosen orthonormal basis.

Since $\operatorname{Ran} P_s$ is closed, the dual of the cokernel is this annihilator. Both are finite-dimensional and therefore have the same dimension. Together with (16.11), this gives (16.10) and proves independence of s. □

Definition 16.4.2. (Analytic index) Let $(M,\mathbf{g})$ be a closed Riemannian manifold, let $\mathbf{E},\mathbf{F}\to M$ be finite-rank complex vector bundles equipped with Hermitian bundle metrics, and let $P\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic. The **analytic index** of a classical elliptic operator P is the integer

$$\operatorname{ind}_{\mathrm{a}}(P) := \operatorname{ind}\big(P_s\colon H^{s+m}(M,\mathbf{E})\longrightarrow H^s(M,\mathbf{F})\big),$$

where $s\in\mathbb{R}$ is arbitrary. Theorem 16.4.1 ensures that the definition is independent of s.

Formula (16.10) can be written as

$$\operatorname{ind}_{\mathrm{a}}(P) = \dim\ker P - \dim\ker P^*.$$

Here both kernels are taken among smooth sections. Using the Hilbert space adjoint of P_s for specific Sobolev inner products gives an operator different from P^*; the Riesz isomorphism identifies their kernels and yields the same dimension. The formal adjoint preserves the geometric content and the principal symbol.

Corollary 16.4.3. (Properties of the analytic index) Let M be a closed smooth manifold, let $\mathbf{E},\mathbf{F},\mathbf{E}_0,\mathbf{F}_0,\mathbf{G}\to M$ be finite-rank smooth complex vector bundles, and let $P\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$, $P_0\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E}_0,\mathbf{F}_0)$, and $Q\in\Psi^\ell_{\mathrm{cl}}(M;\mathbf{F},\mathbf{G})$ be elliptic. Then

$$\begin{aligned}\operatorname{ind}_{\mathrm{a}}(P\oplus P_0) &= \operatorname{ind}_{\mathrm{a}}(P)+\operatorname{ind}_{\mathrm{a}}(P_0),\\ \operatorname{ind}_{\mathrm{a}}(QP) &= \operatorname{ind}_{\mathrm{a}}(P)+\operatorname{ind}_{\mathrm{a}}(Q),\\ \operatorname{ind}_{\mathrm{a}}(P^*) &= -\operatorname{ind}_{\mathrm{a}}(P).\end{aligned}$$

If P is invertible, or if $P=P^*$, then $\operatorname{ind}_{\mathrm{a}}(P)=0$.

Proof. Choose Sobolev exponents making all compositions continuous. The first two identities are, respectively, (16.3) and (16.4); independence of the exponent allows us to change scales without changing the integer. An invertible operator has zero kernel and cokernel. For the third identity, apply (16.10) to P and P^*: since $(P^*)^*=P$, the two dimensions exchange places. If $P=P^*$, this identity gives $\operatorname{ind}_{\mathrm{a}}(P)=-\operatorname{ind}_{\mathrm{a}}(P)$, and since the index is an integer, it must be zero. □

16.5 Dependence on the principal symbol

Let $\pi\colon T^*M \to M$ be the projection. The principal symbol of P is a homogeneous homomorphism

$$\sigma_m^{\Psi}(P)\colon \pi^*\mathbf{E} \longrightarrow \pi^*\mathbf{F}$$

that is invertible over $T^*M \setminus 0_M$. Its restriction to the unit cosphere bundle S^*M contains the same information, since homogeneity recovers the symbol on every nonzero ray.

Proposition 16.5.1. (Continuous quantization of a homotopy of symbols) Let M be a closed smooth manifold, let $\mathbf{E},\mathbf{F} \to M$ be finite-rank smooth complex vector bundles, and let $a_t\colon \pi^*\mathbf{E} \to \pi^*\mathbf{F}$, $t \in [0,1]$, be classical principal symbols of order m, depending continuously on t in the C^∞ topology on S^*M. There exists a family

$$P_t \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$$

continuous as a map

$$[0,1] \longrightarrow \mathcal{L}\big(H^{s+m}(M,\mathbf{E}), H^s(M,\mathbf{F})\big)$$

for every $s \in \mathbb{R}$, such that $\sigma_m^{\Psi}(P_t) = a_t$. If each a_t is invertible, each P_t is elliptic.

Proof. Choose a finite atlas, bundle frames, a partition of unity, and cutoffs in the position and frequency variables as in the global construction of Chapter 15. Extend each restriction homogeneously from S^*M for $\|\xi\|_{\mathbf{g}} \geq 1$ and multiply it by a cutoff vanishing near the zero section. The localized sum of the quantizations has principal symbol a_t; terms arising from changes of chart and cutoffs have order at most $m-1$ and do not alter this symbol.

In a finite atlas, every symbol seminorm of the full symbol of $P_t - P_{t_0}$ is bounded by a finite sum of C^N seminorms of $a_t - a_{t_0}$ on S^*M. Theorem 15.6.2, followed by the finite localization used in Corollary 15.6.6, shows that

$$\|P_t - P_{t_0}\|_{\mathcal{L}(H^{s+m}(M,\mathbf{E}),H^s(M,\mathbf{F}))} \longrightarrow 0 \qquad (t \longrightarrow t_0).$$

The last assertion is the definition of ellipticity. □

Corollary 16.5.2. (The index depends only on the symbol) Let M be a closed smooth manifold and let $\mathbf{E},\mathbf{F} \to M$ be finite-rank smooth complex vector bundles. Let $P_0, P_1 \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ be elliptic.

(a) If $\sigma_m^{\Psi}(P_0) = \sigma_m^{\Psi}(P_1)$, then

$$\mathrm{ind_a}(P_0) = \mathrm{ind_a}(P_1).$$

(b) If the restrictions of $\sigma_m^{\Psi}(P_0)$ and $\sigma_m^{\Psi}(P_1)$ to S^*M are joined by a homotopy of bundle isomorphisms continuous in the C^∞ topology on sections, then the same equality holds.

Proof. In (a), the exact sequence in Theorem 15.3.7 shows that the difference $P_1 - P_0$ has order at most $m-1$. For any $s \in \mathbb{R}$, it factors as

$$H^{s+m}(M,\mathbf{E}) \xrightarrow{P_1-P_0} H^{s+1}(M,\mathbf{F}) \hookrightarrow H^s(M,\mathbf{F}).$$

The first arrow is continuous by the pseudodifferential mapping theorem and the second is compact by

Rellich–Kondrashov. Thus $P_1 - P_0$ is compact between the spaces defining the index, and Theorem 16.2.1 proves the equality.

In (b), Proposition 16.5.1 produces a norm-continuous family $(\widehat{P}_t)_{t\in[0,1]}$ of elliptic operators with the prescribed symbols. It is a Fredholm family by Theorem 16.4.1, so part (c) of Theorem 16.2.1 gives

$$\operatorname{ind}_{\mathrm{a}}(\widehat{P}_0) = \operatorname{ind}_{\mathrm{a}}(\widehat{P}_1).$$

Since $\widehat{P}_j$ and P_j have the same principal symbol for $j = 0, 1$, part (a) completes the proof. □

To see directly that small homotopies remain elliptic, fix metrics $\mathbf{h}_{\mathbf{E}}$ and $\mathbf{h}_{\mathbf{F}}$ on the bundles. Compactness of S^*M gives

$$\delta := \inf_{(x,\xi)\in S^*M} \left\| \boldsymbol{\sigma}_m^{\Psi}(P)(x,\xi)^{-1} \right\|^{-1}_{\mathcal{L}((\mathbf{F}_x,\mathbf{h}_{\mathbf{F}}),(\mathbf{E}_x,\mathbf{h}_{\mathbf{E}}))} > 0.$$

If another symbol b_m satisfies

$$\sup_{(x,\xi)\in S^*M} \| b_m(x,\xi) - \boldsymbol{\sigma}_m^{\Psi}(P)(x,\xi) \|_{\mathcal{L}((\mathbf{E}_x,\mathbf{h}_{\mathbf{E}}),(\mathbf{F}_x,\mathbf{h}_{\mathbf{F}}))} < \delta,$$

then

$$b_m(x,\xi) = \boldsymbol{\sigma}_m^{\Psi}(P)(x,\xi)\left[I + \boldsymbol{\sigma}_m^{\Psi}(P)(x,\xi)^{-1}\left(b_m(x,\xi) - \boldsymbol{\sigma}_m^{\Psi}(P)(x,\xi) \right) \right]$$

is invertible at every point, because the second factor can be inverted using the Neumann series. The same calculation applies to the segment joining the two symbols.

Corollary 16.5.2 thus reduces the analytic problem to a topological one: turning the isomorphism $\boldsymbol{\sigma}_m^{\Psi}(P)$ over $T^*M \setminus 0_M$ into a stable class and proving that the index defines a homomorphism on these classes. This will be the construction in Chapter 17.

17

K-theory of vector bundles and the symbol class

Let X be a compact Hausdorff space. Direct sum of complex vector bundles over X gives a commutative monoid, but does not allow bundles to be subtracted. The Grothendieck construction remedies precisely this deficiency: it adds formal inverses without identifying any more elements than direct sum requires. In this sense the construction is algebraic. The resulting group, however, depends on the base space and on homotopies of maps; it is therefore called *topological K–theory*. It is distinct from the algebraic K–theory of rings.

The usefulness of this construction for an elliptic operator is already apparent in its principal symbol. If M is closed and

$$P \in \Psi^{m}_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{F})$$

is elliptic, then

$$\sigma^{\Psi}_{m}(P)\colon \pi^{*}\mathbf{E} \longrightarrow \pi^{*}\mathbf{F}$$

is an isomorphism away from the zero section of $T^{*}M$. The pair of bundles and this isomorphism determine a compactly supported class on $T^{*}M$. In this chapter we construct this class and prove the properties of the operations used to define the topological and analytic indices. The construction follows [21, Chapter 10, in particular pp. 262–269].

All bundles considered will be complex, finite-rank, and locally trivial. Over a smooth manifold they will also be smooth. Real K–theory requires different groups and will not enter the proof of the complex index theorem.

Grothendieck and Atiyah: subtracting bundles

Grothendieck completion turns a monoid of geometric objects into a group of formal differences. Applied to vector bundles, this idea gives a stable expression for the difference between the domain and codomain of an elliptic symbol. Atiyah developed topological K–theory as the natural language for these classes and, together with Singer, made it the topological side of the index theorem. Stabilization does not erase the geometry: it retains precisely the information that persists when trivial bundles are added.

17.1 Grothendieck completion

Denote by $\mathrm{Vect}_{\mathbb{C}}(X)$ the set of isomorphism classes of complex vector bundles over X. The operation

$$[\mathbf{E}] + [\mathbf{F}] := [\mathbf{E} \oplus \mathbf{F}]$$

makes it a commutative monoid whose identity element is the class of the rank-zero bundle. Cancellation does not hold in general: $\mathbf{E} \oplus \mathbf{G} \cong \mathbf{F} \oplus \mathbf{G}$ does not necessarily imply $\mathbf{E} \cong \mathbf{F}$. This is why differences of bundles must not be defined by first imposing a cancellation property that does not hold.

Theorem 17.1.1. (Grothendieck group) Let $(A, +, 0)$ be a commutative monoid. On $A \times A$ define

$$(a, b) \sim (c, d) \quad \Longleftrightarrow \quad \text{there exists } e \in A \text{ such that } a + d + e = c + b + e. \tag{17.1}$$

Then

$$G(A) := (A \times A)/\!\sim$$

is an abelian group with

$$[(a, b)] + [(c, d)] = [(a + c, b + d)], \qquad -[(a, b)] = [(b, a)].$$

The map

$$\iota_A \colon A \longrightarrow G(A), \qquad \iota_A(a) = [(a, 0)],$$

is a monoid homomorphism and satisfies the following universal property: if H is an abelian group and $h \colon A \to H$ is a monoid homomorphism, there exists a unique group homomorphism $\widetilde{h} \colon G(A) \to H$ such that $\widetilde{h} \circ \iota_A = h$.

Proof. Let us prove that this is an equivalence relation. Reflexivity follows by taking $e = 0$, and symmetry by interchanging the two sides. If

$$a + d + e = c + b + e \quad \text{and} \quad c + f + e' = g + d + e',$$

then adding the equalities and regrouping yields

$$a + f + (c + d + e + e') = g + b + (c + d + e + e'),$$

so $(a, b) \sim (g, f)$. Thus the relation is transitive.
If (a, b) and (c, d) are replaced by equivalent representatives, adding the witnesses for the two equivalences proves that $(a + c, b + d)$ remains in the same class. Thus addition in $G(A)$ is well defined. Moreover,

$$[(a, b)] + [(b, a)] = [(a + b, a + b)] = [(0, 0)],$$

so each element has the stated inverse.
For the universal property, define

$$\widetilde{h}([(a, b)]) := h(a) - h(b).$$

If $(a,b) \sim (c,d)$, there exists $e \in A$ with $a+d+e = c+b+e$; applying h and canceling in the group H gives $h(a)-h(b) = h(c)-h(d)$. Consequently, $\widetilde{h}$ is well defined. It is a homomorphism, factors h, and is unique because every class can be written as

$$[(a,b)] = \iota_A(a) - \iota_A(b).$$

□

Remark 17.1.2. The relation is often defined by

$$(a,b) \sim (c,d) \quad \Longleftrightarrow \quad \text{there exist } u,v \in A \text{ with } (a+u, b+u) = (c+v, d+v).$$

This form and (17.1) are equivalent: in one direction take $e = u+v$, and in the other take $u = d+e$ and $v = b+e$. The second form is the one used in [21, Theorem 10.5, pp. 262–263].

Definition 17.1.3. (K^0 of a compact space) Let X be compact and Hausdorff. Its complex K–theory group is

$$K^0(X) := G\big(\mathrm{Vect}_{\mathbb{C}}(X)\big).$$

The class of the pair $(\mathbf{E},\mathbf{F})$ is denoted by

$$[\mathbf{E}] - [\mathbf{F}] \in K^0(X).$$

If $\underline{\mathbb{C}}^r_X = X \times \mathbb{C}^r$ is the trivial bundle, we abbreviate its class by r when the base space is unambiguous.

Equality between differences has a stable description that we will use repeatedly.

Proposition 17.1.4. (Criterion for stable equality) Let $\mathbf{E},\mathbf{F},\mathbf{E}',\mathbf{F}'$ be complex vector bundles over a compact Hausdorff space X. Then

$$[\mathbf{E}] - [\mathbf{F}] = [\mathbf{E}'] - [\mathbf{F}'] \tag{17.2}$$

if and only if there exists a bundle $\mathbf{G} \to X$ such that

$$\mathbf{E} \oplus \mathbf{F}' \oplus \mathbf{G} \cong \mathbf{E}' \oplus \mathbf{F} \oplus \mathbf{G}. \tag{17.3}$$

This is also equivalent to the existence of $N \in \mathbb{N}_0$ for which

$$\mathbf{E} \oplus \mathbf{F}' \oplus \underline{\mathbb{C}}^N_X \cong \mathbf{E}' \oplus \mathbf{F} \oplus \underline{\mathbb{C}}^N_X. \tag{17.4}$$

Moreover, every class in $K^0(X)$ can be written as

$$[\mathbf{V}] - [\underline{\mathbb{C}}^N_X] \tag{17.5}$$

for some bundle $\mathbf{V} \to X$ and some $N \in \mathbb{N}_0$.

Proof. The equivalence between (17.2) and (17.3) is precisely (17.1) applied to the monoid of bundle classes. By the complement theorem for bundles over compact bases, stated in Appendix A, there exist a bundle $\mathbf{G}^{\perp}$ and an integer N such that

$$\mathbf{G} \oplus \mathbf{G}^{\perp} \cong \underline{\mathbb{C}}^N_X.$$

Adding $\mathbf{G}^{\perp}$ to both sides of (17.3) yields (17.4). The reverse implication follows immediately by taking $\mathbf{G}$ trivial.

Finally, given $[\mathbf{E}]-[\mathbf{F}]$, choose a complement $\mathbf{F}^{\perp}$ with $\mathbf{F}\oplus\mathbf{F}^{\perp}\cong\underline{\mathbb{C}}^{N}_{X}$. Then

$$[\mathbf{E}]-[\mathbf{F}]=[\mathbf{E}\oplus\mathbf{F}^{\perp}]-[\underline{\mathbb{C}}^{N}_{X}],$$

which has the form (17.5). □

The complement theorem is not a formal property of monoids. Its proof uses a finite trivializing cover, a partition of unity, and an inclusion of the bundle into a trivial bundle; the necessary topological statements are collected in Appendix A. The algebraic consequence follows by applying that theorem in the preceding proposition.

17.1.1 Product, pullback, and homotopy

Tensor product distributes over direct sum. By the universal property of the Grothendieck group, it induces a bilinear product

$$\begin{aligned} K^{0}(X)\times K^{0}(X)&\longrightarrow K^{0}(X),\\ \bigl([\mathbf{E}]-[\mathbf{F}],[\mathbf{G}]-[\mathbf{H}]\bigr)&\longmapsto [\mathbf{E}\otimes\mathbf{G}]+[\mathbf{F}\otimes\mathbf{H}]-[\mathbf{E}\otimes\mathbf{H}]-[\mathbf{F}\otimes\mathbf{G}]. \end{aligned} \tag{17.6}$$

Associativity, commutativity, and distributivity follow from the corresponding canonical bundle isomorphisms. The class of the trivial rank-one bundle is the identity. Consequently, $K^{0}(X)$ is a commutative ring.

If $f\colon Y\to X$ is continuous between compact Hausdorff spaces, pullback satisfies

$$f^{*}(\mathbf{E}\oplus\mathbf{F})\cong f^{*}\mathbf{E}\oplus f^{*}\mathbf{F},\qquad f^{*}(\mathbf{E}\otimes\mathbf{F})\cong f^{*}\mathbf{E}\otimes f^{*}\mathbf{F}.$$

It therefore defines a ring homomorphism

$$f^{*}\colon K^{0}(X)\longrightarrow K^{0}(Y),\qquad f^{*}([\mathbf{E}]-[\mathbf{F}])=[f^{*}\mathbf{E}]-[f^{*}\mathbf{F}].$$

The identities

$$(f\circ g)^{*}=g^{*}\circ f^{*},\qquad \mathrm{Id}_{X}^{*}=\mathrm{Id}_{K^{0}(X)}$$

are verified first for bundles and then for differences. Thus K^{0} is a contravariant functor.

Proposition 17.1.5. (Homotopy invariance) If $f_{0},f_{1}\colon Y\to X$ are homotopic maps between compact Hausdorff spaces, then

$$f_{0}^{*}=f_{1}^{*}\colon K^{0}(X)\longrightarrow K^{0}(Y).$$

In particular, a homotopy equivalence induces an isomorphism in K^{0}.

Proof. Let $H\colon Y\times[0,1]\to X$ be a homotopy from f_{0} to f_{1}. The homotopy invariance theorem for vector bundles in Appendix A gives

$$(H^{*}\mathbf{E})|_{Y\times\{0\}}\cong(H^{*}\mathbf{E})|_{Y\times\{1\}}$$

for every bundle $\mathbf{E}\to X$. Consequently, $f_{0}^{*}[\mathbf{E}]=f_{1}^{*}[\mathbf{E}]$. Since bundle classes generate $K^{0}(X)$, the two homomorphisms agree on the whole group. □

17.2 Reduced groups, pairs, and compact support

Let (X, x_0) be a pointed compact Hausdorff space. The inclusion $i\colon \{x_0\} \hookrightarrow X$ induces the rank homomorphism at the base point

$$i^*\colon K^0(X) \longrightarrow K^0(\{x_0\}) \cong \mathbb{Z}.$$

Definition 17.2.1. (Reduced K–theory) Define

$$\widetilde{K}^0(X) := \ker\bigl(i^*\colon K^0(X) \to \mathbb{Z}\bigr).$$

Since the constant map $r\colon X \to \{x_0\}$ satisfies $i^* r^* = I$, we obtain the natural decomposition

$$K^0(X) \cong \widetilde{K}^0(X) \oplus \mathbb{Z}.$$

Now let $A \subseteq X$ be a nonempty closed subspace and let X/A be the pointed space obtained by collapsing A to a point.

Definition 17.2.2. (Relative K–theory) The relative group of the pair (X, A) is

$$K^0(X, A) := \widetilde{K}^0(X/A).$$

If $A = \varnothing$, we adopt the convention $K^0(X, \varnothing) = K^0(X)$.

The description by triples exhibits the symbol of an operator and is valid for a closed subset of a compact Hausdorff space. In particular, it covers pairs consisting of a disk bundle and its sphere bundle, as well as their products with compact Hausdorff parameter spaces.

Definition 17.2.3. (Relative triple) A triple over (X, A) consists of two complex bundles $\mathbf{E}, \mathbf{F} \to X$ and an isomorphism

$$\alpha\colon \mathbf{E}|_A \longrightarrow \mathbf{F}|_A.$$

We write $(\mathbf{E}, \mathbf{F}, \alpha)$. Direct sum is defined by

$$(\mathbf{E}, \mathbf{F}, \alpha) \oplus (\mathbf{E}', \mathbf{F}', \alpha') := (\mathbf{E} \oplus \mathbf{E}', \mathbf{F} \oplus \mathbf{F}', \alpha \oplus \alpha').$$

The group of triples is obtained by imposing the following relations:

(i) isomorphic triples represent the same class;

(ii) a homotopy from α to α' through isomorphisms over A does not change the class;

(iii) if α extends to an isomorphism $\mathbf{E} \to \mathbf{F}$ over X, then $(\mathbf{E}, \mathbf{F}, \alpha)$ represents zero;

(iv) if $\alpha\colon \mathbf{E}|_A \to \mathbf{F}|_A$ and $\beta\colon \mathbf{F}|_A \to \mathbf{G}|_A$, then

$$[(\mathbf{E}, \mathbf{F}, \alpha)] + [(\mathbf{F}, \mathbf{G}, \beta)] = [(\mathbf{E}, \mathbf{G}, \beta \circ \alpha)]. \tag{17.7}$$

The last relation contains both the existence of inverses and the correct cancellation of an intermediate bundle.

Indeed,

$$-[(\mathbf{E},\mathbf{F},\alpha)] = [(\mathbf{F},\mathbf{E},\alpha^{-1})]$$

because the sum of the two classes is $[(\mathbf{E},\mathbf{E},I_{\mathbf{E}|_A})] = 0$. Addition is commutative: the isomorphism interchanging the summands extends over all of X, so the interchange does not alter the class. Thus the preceding relations produce an abelian group.

Theorem 17.2.4. (The model by triples) Let X be a compact Hausdorff space and let $A \subseteq X$ be closed. The group of triples in Definition 17.2.3 is naturally isomorphic to $K^0(X,A)$. If $A \neq \varnothing$, addition of triples corresponds, under this isomorphism, to addition in $\widetilde{K}^0(X/A)$.

Proof. If $A = \varnothing$, a triple contains only the two bundles, and the relations are those of Grothendieck completion. Thus suppose that $A \neq \varnothing$. We will use stable complements, gluing, homotopy invariance, and exactness of K–theory from Appendix A, all in the category of compact Hausdorff spaces.

First recall why gluing along a closed subset is possible without assuming that the pair is a CW pair. An isomorphism $\alpha : \mathbf{E}|_A \to \mathbf{F}|_A$ extends to an isomorphism on some neighborhood of A. Indeed, the bundle $\mathrm{Hom}(\mathbf{E},\mathbf{F})$ is a summand of a finite-rank trivial bundle by Theorem A.3.1. Regard α as a section of this trivial bundle over A, extend its components to X by the Tietze theorem, and project onto the summand $\mathrm{Hom}(\mathbf{E},\mathbf{F})$. This gives a homomorphism agreeing with α on A. Equality of the ranks holds on a neighborhood of A, and invertibility is open among matrices of that rank, so the homomorphism is invertible after shrinking the neighborhood. Applied to a trivialization, this argument also allows a bundle to descend to the quotient; see [76, Proposition 2.9, pp. 51–52].

Form the double

$$D := X_+ \cup_A X_-,$$

where X_+ and X_- are two copies of X and their copies of A are identified pointwise. It is compact and Hausdorff: the equivalence relation is closed in the compact Hausdorff space $X_+ \coprod X_-$. Denote by $r : D \to X$ the map that is the identity on both copies, and by $i_- : X \to D$ the inclusion of the second copy. We have $r \circ i_- = \mathrm{Id}_X$ and a natural homeomorphism $D/X_- \cong X/A$. Let $q : D \to X/A$ be the corresponding quotient map.

The exact sequence in Theorem A.4.2, applied to (D, X_-), contains

$$K^1(D) \xrightarrow{i_-^*} K^1(X) \xrightarrow{\partial} \widetilde{K}^0(X/A) \xrightarrow{q^*} K^0(D) \xrightarrow{i_-^*} K^0(X).$$

The first map is surjective because $i_-^* r^* = I$ also in degree one. By exactness, $\partial = 0$, the map q^* is injective, and

$$q^* : \widetilde{K}^0(X/A) \xrightarrow{\cong} \ker\bigl(i_-^* : K^0(D) \to K^0(X)\bigr).$$

Thus it suffices to identify the group of triples with this last kernel.

Given $(\mathbf{E},\mathbf{F},\alpha)$, glue $\mathbf{E}$ over X_+ to $\mathbf{F}$ over X_- by means of α, and call $\mathbf{H}_\alpha$ the resulting bundle. Local triviality at the gluing points follows by extending α to a neighborhood of A, as just justified. Define

$$d(\mathbf{E},\mathbf{F},\alpha) := [\mathbf{H}_\alpha] - [r^*\mathbf{F}] \in K^0(D).$$

This class lies in the kernel of i_-^*, because both bundles restrict to $\mathbf{F}$ on X_-. The construction respects direct

sums and isomorphisms of triples. If α extends to an isomorphism $\mathbf{E} \to \mathbf{F}$ on X, this isomorphism on X_+ and the identity on X_- give $\mathbf{H}_\alpha \cong r^*\mathbf{F}$; thus an elementary triple has zero image.

A homotopy $(\alpha_t)_{t\in[0,1]}$ of isomorphisms over A allows us to glue the bundles over $D\times[0,1]$. The product is compact Hausdorff, and extension from $A\times[0,1]$ is justified by the same argument. Theorem A.3.2 identifies the restrictions of the glued bundle at the endpoints. Hence d is unchanged under the homotopy relation.

Let us also verify the composition relation. Let $\alpha : \mathbf{E}|_A \to \mathbf{F}|_A$ and $\beta : \mathbf{F}|_A \to \mathbf{G}|_A$. For $0 \le t \le \pi/2$ consider the isomorphisms

$$h_t(u,v) := \big(\cos t\,\alpha(u) + \sin t\,v, -\sin t\,\beta\alpha(u) + \cos t\,\beta(v)\big)$$

from $(\mathbf{E}\oplus\mathbf{F})|_A$ to $(\mathbf{F}\oplus\mathbf{G})|_A$. They are invertible because they are obtained by composing $\alpha\oplus I$, the rotation of $\mathbf{F}|_A\oplus\mathbf{F}|_A$ through angle t, and $I\oplus\beta$. At $t=0$ we have $h_0 = \alpha\oplus\beta$; at $t=\pi/2$, composition with the global isomorphism $(v,w)\mapsto(-w,v)$ from $\mathbf{F}\oplus\mathbf{G}$ to $\mathbf{G}\oplus\mathbf{F}$ turns $h_{\pi/2}$ into $(\beta\alpha)\oplus I_{\mathbf{F}}$. Gluing and homotopy give

$$\mathbf{H}_\alpha \oplus \mathbf{H}_\beta \cong \mathbf{H}_{\beta\alpha} \oplus r^*\mathbf{F}.$$

Subtracting the classes of $r^*\mathbf{F}$ and $r^*\mathbf{G}$ yields

$$d(\mathbf{E},\mathbf{F},\alpha) + d(\mathbf{F},\mathbf{G},\beta) = d(\mathbf{E},\mathbf{G},\beta\alpha).$$

Consequently, d is well defined on the group of triples.

Let us prove its surjectivity onto the kernel of i_-^*. Let $c \in K^0(D)$ with $i_-^*c = 0$. By Proposition 17.1.4, we may write $c = [\mathbf{V}] - [\underline{\mathbb{C}}^N_D]$. The equality $i_-^*c = 0$ allows us, after adding the same trivial bundle to both sides, to choose a trivialization

$$\tau : \mathbf{V}|_{X_-} \xrightarrow{\cong} \underline{\mathbb{C}}^N_{X_-},$$

where we have retained the letters $\mathbf{V}$ and N for the stabilized data. Take $\mathbf{E} := \mathbf{V}|_{X_+}$, $\mathbf{F} := \underline{\mathbb{C}}^N_X$, and $\alpha := \tau|_A$. The bundle $\mathbf{H}_\alpha$ is isomorphic to $\mathbf{V}$: use the identity on X_+ and τ^{-1} on X_-; these agree along the gluing locus. Thus $d(\mathbf{E},\mathbf{F},\alpha) = c$.

Finally, suppose that $d(\mathbf{E},\mathbf{F},\alpha) = 0$. The same proposition provides an integer N and an isomorphism

$$\mathbf{H}_\alpha \oplus \underline{\mathbb{C}}^N_D \cong r^*\mathbf{F} \oplus \underline{\mathbb{C}}^N_D.$$

Its restrictions to X_+ and X_- are isomorphisms

$$a : \mathbf{E}\oplus\underline{\mathbb{C}}^N_X \longrightarrow \mathbf{F}\oplus\underline{\mathbb{C}}^N_X, \qquad b : \mathbf{F}\oplus\underline{\mathbb{C}}^N_X \longrightarrow \mathbf{F}\oplus\underline{\mathbb{C}}^N_X$$

satisfying $a|_A = b|_A \circ (\alpha\oplus I)$. Thus $b^{-1}a$ extends $\alpha\oplus I$ to all of X. The original triple, stabilized by an identity triple, is elementary and represents zero. Every element of the group is represented by a triple, since addition is direct sum and the inverse interchanges the bundles. This proves injectivity of d.

The composition $(q^*)^{-1}d$ is the asserted isomorphism. The preceding verification also shows that the relations can be realized by stabilizations with elementary triples, isomorphisms, and homotopies: the composition relation is precisely the rotation just constructed. To compare two triples, apply the zero-kernel criterion to the sum of the first with the inverse of the second.

All constructions are compatible with pullbacks. Indeed, a map of pairs induces a map of their doubles commuting with the retractions and quotients, and pullback commutes with gluing. Uniqueness of the

preimage under q^* gives naturality. For a compact Hausdorff space Λ, the same argument applies to $(X \times \Lambda, A \times \Lambda)$ and its product with $[0,1]$; no CW structure on Λ is required. □

We turn to the noncompact case. Let Y be a locally compact, Hausdorff, noncompact space. Its one-point compactification is

$$Y^+ := Y \cup \{\infty\},$$

where neighborhoods of ∞ are complements of compact subsets of Y. If Y is already compact, take the disjoint union with a base point.

Definition 17.2.5. (Compactly supported K–theory) Define

$$K_c^0(Y) := \widetilde{K}^0(Y^+).$$

When Y is the interior of a compact CW pair, and in particular when Y is the total space of a vector bundle over a closed manifold, a class can equivalently be represented by a complex $(\mathbf{E}, \mathbf{F}, a)$ in which $\mathbf{E}$ and $\mathbf{F}$ are bundles over Y and

$$a\colon \mathbf{E} \longrightarrow \mathbf{F}$$

is a bundle homomorphism invertible outside some compact set $C \subseteq Y$. The same relations as in Definition 17.2.3 apply, after enlarging the compact set when necessary.

The word support refers to the region where the two-term complex

$$0 \longrightarrow \mathbf{E} \xrightarrow{a} \mathbf{F} \longrightarrow 0$$

fails to be exact. No compactly supported section is chosen. If Y is the interior of a compact space X with boundary $A = X \setminus Y$ and the collapse map $X/A \to Y^+$ is a homeomorphism, then

$$K_c^0(Y) \cong K^0(X, A). \tag{17.8}$$

If $f\colon Y \to Z$ is proper, it extends continuously to $f^+\colon Y^+ \to Z^+$ by sending ∞ to ∞. Pullback defines

$$f^*\colon K_c^0(Z) \longrightarrow K_c^0(Y).$$

Properness is necessary: without it, the preimage of the compact support of a representative need not be compact.

17.2.1 Suspension and the exact sequence

For a pointed space X, its reduced suspension is

$$\Sigma X := S^1 \wedge X,$$

where $\wedge$ denotes the smash product. We will need the odd group only to write the exact sequence of a pair.

Definition 17.2.6. (The group K^1) For a compact pointed space X, define

$$\widetilde{K}^1(X) := \widetilde{K}^0(\Sigma X).$$

For an unpointed compact space, adjoin a disjoint base point and set $K^1(X) := \widetilde{K}^1(X_+)$. For a locally compact space,

$$K^1_c(Y) := \widetilde{K}^1(Y^+).$$

For a compact pair, set

$$K^1(X,A) := \widetilde{K}^1(X/A).$$

The definition by suspension agrees with the description that we will use when applying periodicity. Indeed, Proposition A.4.1 provides a natural isomorphism

$$K^1_c(Y) \cong K^0_c(\mathbb{R} \times Y). \tag{17.9}$$

This is not a second definition: the pointed homeomorphism $(\mathbb{R}\times Y)^+ \cong S^1 \wedge Y^+$ identifies the right-hand side with the suspension appearing in Definition 17.2.6.

Theorem 17.2.7. (Exactness for a pair) Let (X,A) be a compact CW pair. There exist connecting homomorphisms for which the cyclic sequence

$$\begin{array}{ccccc} K^0(X,A) & \longrightarrow & K^0(X) & \longrightarrow & K^0(A) \\ \uparrow{\scriptstyle \partial} & & & & \downarrow{\scriptstyle \partial} \\ K^1(A) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,A) \end{array}$$

is exact. In particular,

$$\operatorname{im}\bigl(K^0(X,A) \to K^0(X)\bigr) = \ker\bigl(K^0(X) \to K^0(A)\bigr). \tag{17.10}$$

The proof belongs to algebraic topology: it identifies the connecting maps with suspensions of quotient maps and uses Bott periodicity to close the sequence. The statement, cofibration hypotheses, and references for the proof are included in Theorem A.4.2 of Appendix A. For the elementary construction of the segment (17.10), see [21, Exercise 10.12(c), pp. 264–265]; for the compactly supported formulation, see [21, §10.4, pp. 266–269].

17.3 External product

Let X and Y be compact, and let $p_X\colon X\times Y\to X$ and $p_Y\colon X\times Y\to Y$ be the projections. For bundles $\mathbf{E}\to X$ and $\mathbf{G}\to Y$ define

$$\mathbf{E}\boxtimes\mathbf{G} := p_X^*\mathbf{E}\otimes p_Y^*\mathbf{G}.$$

The fiber over (x,y) is $\mathbf{E}_x\otimes\mathbf{G}_y$. Distributivity of tensor product gives a bilinear map on the monoids of bundles and, by the universal property of Grothendieck completion, an external product

$$\boxtimes\colon K^0(X)\otimes_{\mathbb{Z}} K^0(Y)\longrightarrow K^0(X\times Y).$$

On representatives,

$$\begin{aligned}\big([\mathbf{E}]-[\mathbf{F}]\big)\boxtimes\big([\mathbf{G}]-[\mathbf{H}]\big) = {}& [\mathbf{E}\boxtimes\mathbf{G}]+[\mathbf{F}\boxtimes\mathbf{H}]\\ & -[\mathbf{E}\boxtimes\mathbf{H}]-[\mathbf{F}\boxtimes\mathbf{G}].\end{aligned} \tag{17.11}$$

The formula proves directly that replacing either representative by a stably equivalent one does not alter the result: a summand added to both the positive and negative parts appears twice with opposite signs. It also proves associativity. The interchange isomorphism $\mathbf{E}_x\otimes\mathbf{G}_y\to\mathbf{G}_y\otimes\mathbf{E}_x$ gives commutativity after interchanging the factors X and Y.

Proposition 17.3.1. (External product with compact support) Let X and Y be locally compact, Hausdorff, paracompact spaces. There exists a natural bilinear product

$$\boxtimes\colon K_c^0(X)\otimes_{\mathbb{Z}} K_c^0(Y)\longrightarrow K_c^0(X\times Y).$$

If a and b admit representatives whose supports are contained in the compact sets $C\subseteq X$ and $D\subseteq Y$, respectively, then $a\boxtimes b$ admits a representative supported in $C\times D$.

Proof. Existence, bilinearity, and naturality are the topological properties of the compactly supported product stated in Theorem A.4.2. In the compactification model, the reduced product on $X^+\times Y^+$ vanishes on both axes and descends to the smash product $X^+\wedge Y^+\cong (X\times Y)^+$.
In the complex model, each factor is exact outside C or D. The graded product of the two complexes is contractible if at least one of them is exact; hence it can fail to be exact only over $C\times D$. This proves the assertion about support. □

This construction agrees with that in [21, Theorem 10.16, pp. 267–268]. In particular, the Bott class on $\mathbb{R}^2$ can be multiplied externally by a class on any locally compact space.

17.4 The class of an elliptic symbol

Let $(M,\mathbf{g})$ be a closed smooth Riemannian manifold. Denote by $\pi\colon T^*M\to M$ the projection and by

$$B^*M:=\{(x,\xi)\in T^*M\mid \|\xi\|_{\mathbf{g}}\le 1\},\qquad S^*M:=\{(x,\xi)\in T^*M\mid \|\xi\|_{\mathbf{g}}=1\}$$

the cotangent disk and sphere bundles. The radial map

$$\begin{aligned}\rho\colon T^*M &\longrightarrow \operatorname{int}(B^*M),\\ (x,\xi)&\longmapsto\left(x,\frac{\xi}{\sqrt{1+\|\xi\|_{\mathbf{g}}^2}}\right)\end{aligned}$$

is a diffeomorphism. Consequently,

$$K_c^0(T^*M)\cong K^0(B^*M,S^*M). \tag{17.12}$$

Let $\mathbf{E},\mathbf{F}\to M$ be complex bundles and let

$$P\in\Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$$

be a classical elliptic operator. By Definition 15.3.4,

$$\sigma_m^{\Psi}(P)(x,t\xi) = t^m \sigma_m^{\Psi}(P)(x,\xi), \qquad t>0,$$

and ellipticity says that this bundle homomorphism is invertible when $\xi \neq 0$. In particular, its restriction to S^*M is an isomorphism.

Definition 17.4.1. (Symbol class) Let M be a closed smooth manifold, let $\mathbf{E},\mathbf{F}\to M$ be finite-rank smooth complex vector bundles, and let $P\in \Psi_{\mathrm{cl}}^m(M;\mathbf{E},\mathbf{F})$ be elliptic. The **K–theory class of the symbol** of P is

$$[\sigma(P)] := \left[\pi^*\mathbf{E}, \pi^*\mathbf{F}, \sigma_m^{\Psi}(P)|_{S^*M}\right] \in K^0(B^*M,S^*M) \cong K_c^0(T^*M). \tag{17.13}$$

The notation does not retain the order m: the class records only the isomorphism on nonzero cotangent directions. For a differential operator, the identity

$$\sigma_m^{\Psi}(P) = i^m \sigma_m(P)$$

from (15.8) does not change the class. Indeed, $i^m\in\mathbb{C}^\times$ can be joined to 1 by a path in $\mathbb{C}^\times$, and homotopy invariance of triples applies.

Proposition 17.4.2. (Independence of choices) Let M be a closed smooth manifold, let $\mathbf{E},\mathbf{F}\to M$ be finite-rank smooth complex vector bundles, and let $P\in\Psi_{\mathrm{cl}}^m(M;\mathbf{E},\mathbf{F})$ be a classical elliptic operator. The class (17.13) is independent of the metric used to define B^*M and S^*M, after the canonical identification with $K_c^0(T^*M)$. It is also unchanged when the symbol is replaced by another elliptic symbol homotopic to it through elliptic symbols.

Proof. Let $\mathbf{g}_0$ and $\mathbf{g}_1$ be two metrics. For every $(x,\xi)\neq 0$, the positive ray $\{(x,t\xi)\mid t>0\}$ meets each unit sphere exactly once. The radial map

$$(x,\xi)\in S^*_{\mathbf{g}_0}M \longmapsto \left(x, \frac{\xi}{\|\xi\|_{\mathbf{g}_1}}\right)\in S^*_{\mathbf{g}_1}M$$

extends to a homeomorphism of the disk bundles, preserves the zero section, and is isotopic, through positive radial rescalings, to the identification induced by the identity of T^*M. Homogeneity of the symbol makes the two boundary isomorphisms compatible. The resulting triples therefore determine the same class in $K_c^0(T^*M)$.

If a_t is a continuous family of invertible symbols over S^*M, the triple over $(B^*M\times[0,1], S^*M\times[0,1])$ restricts at the endpoints to the triples defined by a_0 and a_1. Homotopy invariance of relative K–theory identifies their classes. □

Compact support can also be described using a symbol on the entire cotangent bundle. The normalized symbol

$$a(x,\xi) := \|\xi\|_{\mathbf{g}}^{-m}\sigma_m^{\Psi}(P)(x,\xi), \qquad \xi\neq 0,$$

is homogeneous of degree zero, but does not yet define a bundle homomorphism on the zero section. Choose

$\chi \in C^\infty([0,\infty))$ with $\chi = 0$ for $r \le \varepsilon$ and $\chi = 1$ for $r \ge 2\varepsilon$, and set

$$\widetilde{a}(x,\xi) = \begin{cases} \chi(\|\xi\|_{\mathbf{g}})\|\xi\|_{\mathbf{g}}^{-m}\sigma_m^{\Psi}(P)(x,\xi), & \xi \neq 0, \\ 0, & \xi = 0. \end{cases}$$

Since $\widetilde{a}$ vanishes on a neighborhood of the zero section, it is smooth on all of T^*M; outside the disk of radius 2ε it agrees with a and is invertible. The complex $(\pi^*\mathbf{E}, \pi^*\mathbf{F}, \widetilde{a})$ represents the symbol class. Two cutoffs or two radii can be joined by a homotopy of cutoffs, retaining invertibility outside a common disk.

Theorem 17.4.3. (Realization by elliptic symbols) Let M be a closed smooth manifold. For every class

$$\kappa \in K_c^0(T^*M)$$

there exist smooth complex bundles $\mathbf{E}, \mathbf{F} \to M$ and an elliptic operator

$$P \in \Psi_{\mathrm{cl}}^0(M; \mathbf{E}, \mathbf{F})$$

such that $[\sigma(P)] = \kappa$.

Proof. Using (17.12), represent κ by a triple $(\mathbf{V}, \mathbf{W}, \alpha)$ over (B^*M, S^*M). The projection $\pi\colon B^*M \to M$ and the zero section $\mathbf{s}\colon M \to B^*M$ are homotopy inverses, since

$$H_t(x,\xi) = (x, t\xi), \qquad 0 \le t \le 1,$$

contracts every disk to its center. Homotopy invariance of bundles gives isomorphisms

$$\mathbf{V} \cong \pi^*\mathbf{E}, \qquad \mathbf{W} \cong \pi^*\mathbf{F}, \qquad \mathbf{E} := \mathbf{s}^*\mathbf{V}, \quad \mathbf{F} := \mathbf{s}^*\mathbf{W}.$$

The smoothing theorem for bundles over manifolds allows us to choose smooth structures on $\mathbf{E}$ and $\mathbf{F}$ and to take the two preceding isomorphisms smooth. This result is stated in Appendix A. Fix Hermitian metrics $\mathbf{h}_{\mathbf{E}}$ and $\mathbf{h}_{\mathbf{F}}$ and equip $\mathrm{Hom}(\pi^*\mathbf{E}, \pi^*\mathbf{F})$ with the induced operator norm. After transporting α, we obtain a continuous isomorphism

$$\alpha\colon \pi^*\mathbf{E}|_{S^*M} \longrightarrow \pi^*\mathbf{F}|_{S^*M}.$$

Since S^*M is compact,

$$\delta := \inf_{(x,\xi)\in S^*M} \|\alpha(x,\xi)^{-1}\|_{\mathrm{Hom}(\mathbf{F}_x,\mathbf{E}_x)}^{-1} > 0.$$

Choose a smooth approximation α_1 such that

$$\sup_{(x,\xi)\in S^*M} \|\alpha_1(x,\xi) - \alpha(x,\xi)\|_{\mathrm{Hom}(\mathbf{E}_x,\mathbf{F}_x)} < \delta.$$

For $\alpha_t := \alpha + t(\alpha_1 - \alpha)$ we have

$$\|\alpha(x,\xi)^{-1}(\alpha_t(x,\xi) - \alpha(x,\xi))\|_{\mathrm{End}(\mathbf{E}_x)} < 1.$$

The Neumann series shows that α_t is invertible. Thus α and α_1 define homotopic triples; replace α by α_1.

Extend α to $T^*M \setminus 0_M$ by homogeneity of degree zero:

$$a_0(x,\xi) := \alpha\left(x, \frac{\xi}{\|\xi\|_{\mathbf{g}}}\right).$$

This section is smooth and invertible. Surjectivity in the exact principal symbol sequence, Theorem 15.3.7, provides a classical operator $P \in \Psi^0_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$ with $\sigma^{\Psi}_0(P) = a_0$. Since a_0 is invertible in every nonzero direction, P is elliptic and its symbol triple is the triple representing κ. □

The smoothing used in the proof is an approximation result for continuous maps between bundles over a compact manifold. Its statement and topological reference are included in Appendix A. The analytic step, quantization of a_0, was already proved in Chapter 15. The topological reduction allowing the two bundles of the triple to be represented as pullbacks, and the subsequent realization, are also stated in [21, Exercise 11.4, pp. 279–280].

17.5 Operations on symbol classes

The rules of symbol calculus translate into exact identities in $K^0_c(T^*M)$. We will prove them in the model by triples, keeping track of the bundles and homomorphisms in each construction.

Proposition 17.5.1. (Sum and stabilization) Let M be a closed smooth manifold and let $\mathbf{E},\mathbf{F},\mathbf{E}',\mathbf{F}',\mathbf{G} \to M$ be finite-rank smooth complex vector bundles. Let

$$P \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F}), \qquad Q \in \Psi^m_{\mathrm{cl}}(M;\mathbf{E}',\mathbf{F}')$$

be elliptic operators of the same order. Then

$$[\sigma(P \oplus Q)] = [\sigma(P)] + [\sigma(Q)].$$

For an elliptic operator of arbitrary order and a bundle $\mathbf{G} \to M$, stabilization by the identity triple satisfies

$$[\sigma(P)] + [(\pi^*\mathbf{G}, \pi^*\mathbf{G}, I_{\pi^*\mathbf{G}})] = [\sigma(P)].$$

After reducing P to order zero, this equality is realized by direct sum with the operator $I_{\mathbf{G}}$.

Proof. The first equality is the definition of addition of triples:

$$(\pi^*\mathbf{E}, \pi^*\mathbf{F}, \sigma^{\Psi}_m(P)|_{S^*M}) \oplus (\pi^*\mathbf{E}', \pi^*\mathbf{F}', \sigma^{\Psi}_m(Q)|_{S^*M}) = (\pi^*(\mathbf{E}\oplus\mathbf{E}'), \pi^*(\mathbf{F}\oplus\mathbf{F}'), \sigma^{\Psi}_m(P)|_{S^*M} \oplus \sigma^{\Psi}_m(Q)|_{S^*M}).$$

The triple $(\pi^*\mathbf{G}, \pi^*\mathbf{G}, I)$ extends as an isomorphism over all of B^*M and therefore represents zero. To justify the last assertion at the operator level, fix a nonnegative covariant Laplacian $L_{\mathbf{E}}$. Theorem 15.7.9 shows that

$$\Lambda^{-m}_{\mathbf{E}} := (I + L_{\mathbf{E}})^{-\frac{m}{2}} \in \Psi^{-m}_{\mathrm{cl}}(M;\mathbf{E},\mathbf{E})$$

is elliptic and that its principal symbol on S^*M is the identity. Thus

$$P^{(0)} := P\Lambda^{-m}_{\mathbf{E}} \in \Psi^0_{\mathrm{cl}}(M;\mathbf{E},\mathbf{F})$$

has the same symbol class as P, and $P^{(0)} \oplus I_{\mathbf{G}}$ realizes the stated stabilization. □

Proposition 17.5.2. (Composition) Let M be a closed smooth manifold and let $\mathbf{E}, \mathbf{F}, \mathbf{G} \to M$ be finite-rank smooth complex vector bundles. Let

$$P \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{F}), \qquad Q \in \Psi^\ell_{\mathrm{cl}}(M; \mathbf{F}, \mathbf{G})$$

be elliptic operators. Then

$$[\sigma(QP)] = [\sigma(P)] + [\sigma(Q)]. \tag{17.14}$$

Proof. Theorem 15.3.7 gives

$$\sigma^\Psi_{m+\ell}(QP) = \sigma^\Psi_\ell(Q) \circ \sigma^\Psi_m(P).$$

Applying the composition relation (17.7) with

$$\alpha = \sigma^\Psi_m(P)|_{S^*M}, \qquad \beta = \sigma^\Psi_\ell(Q)|_{S^*M},$$

yields precisely (17.14). □

Proposition 17.5.3. (Adjoint) Let M be a closed smooth manifold and let $\mathbf{E}, \mathbf{F} \to M$ be finite-rank smooth complex vector bundles; fix Hermitian metrics on them and a positive density on M. If $P \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{F})$ is elliptic, then

$$[\sigma(P^*)] = -[\sigma(P)].$$

Proof. Write

$$a := \sigma^\Psi_m(P)|_{S^*M} \colon \pi^*\mathbf{E} \longrightarrow \pi^*\mathbf{F}.$$

By the symbolic adjoint rule, $\sigma^\Psi_m(P^*)|_{S^*M} = a^*$. The inverse of the class of the triple $(\pi^*\mathbf{E}, \pi^*\mathbf{F}, a)$ is represented by $(\pi^*\mathbf{F}, \pi^*\mathbf{E}, a^{-1})$. It suffices to compare a^* with a^{-1}.

The endomorphism a^*a of $\pi^*\mathbf{E}$ is positive definite. Matrix functional calculus depends smoothly on the point of S^*M, so

$$h_t := (a^*a)^{-t} a^*, \qquad 0 \le t \le 1,$$

is a smooth homotopy through isomorphisms $\pi^*\mathbf{F} \to \pi^*\mathbf{E}$. At the endpoints,

$$h_0 = a^*, \qquad h_1 = (a^*a)^{-1} a^* = a^{-1}.$$

Homotopy invariance of triples completes the proof. □

Proposition 17.5.4. (Homotopy and lower-order terms) Let M be a closed smooth manifold, let $\mathbf{E}, \mathbf{F} \to M$ be finite-rank smooth complex vector bundles, and let $P_t \in \Psi^m_{\mathrm{cl}}(M; \mathbf{E}, \mathbf{F})$, $0 \le t \le 1$, be a continuous family whose principal symbols form a family of isomorphisms over S^*M. Then

$$[\sigma(P_0)] = [\sigma(P_1)].$$

In particular, if $R \in \Psi_{\mathrm{cl}}^{m-1}(M;\mathbf{E},\mathbf{F})$ and P is elliptic, then

$$[\sigma(P+R)] = [\sigma(P)].$$

Proof. The first assertion is homotopy invariance of the symbol triple. For the second,

$$\sigma_m^{\Psi}(P+R) = \sigma_m^{\Psi}(P),$$

by exactness of the principal symbol sequence; in fact, the two triples are identical. □

Combining realization with these identities explains why $K_c^0(T^*M)$ is the natural domain of any additive invariant of elliptic operators that is stable under homotopy.

Lemma 17.5.5. (Relative smoothing of invertible homotopies) Let M be a closed smooth manifold, let $\mathbf{E},\mathbf{F} \longrightarrow M$ be finite-rank smooth bundles, and let

$$a\colon S^*M \times [0,1] \longrightarrow \mathrm{Iso}(\pi^*\mathbf{E},\pi^*\mathbf{F})$$

be a continuous homotopy with smooth endpoints. There exists a smooth homotopy $\widetilde{a}$ with the same endpoints, consisting entirely of isomorphisms. Moreover, a and $\widetilde{a}$ are joined, relative to the endpoints, by a homotopy of isomorphisms.

The relative version of this smoothing result also appears in [21, Exercise 12.13, pp. 291–292].

Proof. First reparametrize the interval by a function that is constant near 0 and 1. The resulting homotopy is smooth on a neighborhood of the endpoints. The relative Whitney approximation theorem, applied to the continuous section of the homomorphism bundle over the compact manifold $S^*M \times [0,1]$, produces a smooth map $\widetilde{a}$, equal to a near the endpoints and uniformly as close to it as desired.
After fixing bundle metrics, compactness gives

$$\mu := \min_{(x,\xi,t)\in S^*M\times[0,1]} \|a(x,\xi,t)^{-1}\|_{\mathrm{op}(\mathbf{F}_x,\mathbf{E}_x)}^{-1} > 0.$$

Choose the approximation with $\|\widetilde{a}-a\|_{C^0(S^*M\times[0,1],\mathrm{Hom}(\pi^*\mathbf{E},\pi^*\mathbf{F}))} < \frac{\mu}{2}$. Then

$$(1-\theta)a + \theta\widetilde{a} = a\left[I + \theta a^{-1}(\widetilde{a}-a)\right], \qquad 0 \le \theta \le 1,$$

is invertible by the Neumann series. This segment is fixed near the endpoints and proves all the assertions. □

Theorem 17.5.6. (The analytic index on the symbol class) Let M be a closed smooth manifold. There exists a unique homomorphism

$$\mathrm{ind_a}\colon K_c^0(T^*M) \longrightarrow \mathbb{Z} \tag{17.15}$$

such that, for every classical elliptic pseudodifferential operator P,

$$\mathrm{ind_a}([\sigma(P)]) = \mathrm{ind}\, P.$$

Proof. By Theorem 17.4.3, every class is the symbol of some elliptic operator. For a smooth triple $(\pi^*\mathbf{E}, \pi^*\mathbf{F}, a)$ over (B^*M, S^*M) define

$$I(\mathbf{E}, \mathbf{F}, a) := \operatorname{ind} \operatorname{Op}(a),$$

where $\operatorname{Op}(a)$ is any elliptic quantization of order zero with principal symbol a. Two quantizations differ by an operator of order -1, compact between the corresponding L^2 spaces by Rellich; Corollary 16.5.2 shows that I is independent of the quantization. We must verify that this integer respects the relations in the model by triples.

First consider two isomorphic triples, after any stabilizations forming part of the isomorphism. Write them as

$$(\pi^*\mathbf{E}, \pi^*\mathbf{F}, a), \qquad (\pi^*\mathbf{E}', \pi^*\mathbf{F}', a')$$

over (B^*M, S^*M). There exist isomorphisms $U\colon \pi^*\mathbf{E} \to \pi^*\mathbf{E}'$ and $V\colon \pi^*\mathbf{F} \to \pi^*\mathbf{F}'$ over B^*M such that $a'U = Va$ on S^*M. Although U and V may depend on the covariable, radial contraction defines

$$U_t(x,\xi) = U(x, t\xi), \qquad V_t(x,\xi) = V(x,t\xi), \qquad 0 \le t \le 1.$$

Therefore,

$$a_t := V_t^{-1} a' U_t$$

is a homotopy of invertible symbols over S^*M joining a to $V_0^{-1}a'U_0$. At the second endpoint, U_0 and V_0 are pullbacks of continuous bundle isomorphisms over M. Theorem A.4.4 allows us to approximate them uniformly by smooth isomorphisms $\widehat{U_0}$ and $\widehat{V_0}$; an approximation smaller than the minimum of their singular values preserves invertibility, and the straight segments give homotopies of isomorphisms. Concatenate these segments with (a_t). Both endpoints of the resulting homotopy are now smooth, so Lemma 17.5.5 allows us to replace it by a smooth homotopy of symbols. A realization of the endpoint $\widehat{V_0}^{-1}a'\widehat{U_0}$ is obtained from a realization of a' by composition on the left and right with the bundle isomorphisms; these compositions are invertible operators of order zero and do not change the index. Thus an isomorphism of triples preserves the associated integer, even when its initial isomorphisms depend on ξ.

A continuous homotopy of the boundary isomorphism can be smoothed relative to its endpoints using Lemma 17.5.5 and quantized, by surjectivity of the principal symbol, as a family of elliptic operators: fix a single cover, a single partition of unity, and the same cutoffs in the construction of Theorem 15.3.7; the quantization formula is linear in the symbol and preserves continuity in the parameter. Corollary 16.5.2 gives constancy of its index. After reducing all operators to order zero as in Proposition 17.5.1, direct sum of triples corresponds to direct sum of operators. Proposition 16.2.2 proves additivity.

Suppose that the boundary isomorphism $\alpha\colon \pi^*\mathbf{E}|_{S^*M} \to \pi^*\mathbf{F}|_{S^*M}$ extends to an isomorphism $A\colon \pi^*\mathbf{E} \to \pi^*\mathbf{F}$ over B^*M. Radial contraction shows that $A|_{S^*M}$ is homotopic, through isomorphisms, to the pullback of $A|_{0_M}\colon \mathbf{E} \to \mathbf{F}$. This isomorphism may only be continuous. Approximate it uniformly by a smooth isomorphism $\widehat{A_0}\colon \mathbf{E} \to \mathbf{F}$; if the error is smaller than the minimum of the singular values of $A|_{0_M}$, the segment between them remains invertible. Concatenate this segment with radial contraction and apply Lemma 17.5.5. The smooth endpoint $\pi^*\widehat{A_0}$ is the constant symbol of multiplication by $\widehat{A_0}$, whose index is zero. By homotopy invariance, any realization of this triple has index zero. Thus the third relation in Definition 17.2.3 is respected.

Finally, the composition relation is respected because, for elliptic $P\colon \mathbf{E} \to \mathbf{F}$ and $Q\colon \mathbf{F} \to \mathbf{G}$,

$$\operatorname{ind}(QP) = \operatorname{ind} Q + \operatorname{ind} P$$

by Proposition 16.2.2, while their symbols satisfy (17.14). Consequently, the index descends to the quotient defined by triples and determines the homomorphism (17.15). Realization of all classes also proves uniqueness. □

Bott periodicity and the Thom isomorphism will allow us to construct the topological homomorphism on the same group.

18

Bott periodicity, the Thom isomorphism, and the Euclidean index

The symbol class constructed in Chapter 17 belongs to a compactly supported group,

$$[\sigma(P)] \in K_c^0(T^*M).$$

Two operations are needed to turn it into an integer. Bott periodicity identifies the K–theory of a space with that of its product with $\mathbb{R}^2$, and the Thom isomorphism allows this identification to be made fiber by fiber in a complex vector bundle. Both operations are topological; the analytic work consists in proving that quantization of a class and the Fredholm index behave in the same way.

We begin with the orientation conventions that determine the signs. We then construct the Thom isomorphisms and the pushforward maps associated with embeddings, calculate the indices of two model operators, and prove the Euclidean formula. The final result will be

$$\operatorname{ind}_{\mathrm{a}}(P) = (-1)^n \alpha_n\big([\sigma(P)]\big), \qquad P \in \operatorname{Ell}_c^0(\mathbb{R}^n),$$

where $\alpha_n : K_c^0(\mathbb{R}^{2n}) \longrightarrow \mathbb{Z}$ is the iterated Bott isomorphism with the normalization specified below. This is the form of [21, Theorem 11.5, pp. 281–282] compatible with the Fourier conventions of Chapter 15.

Bott and Thom: transporting topological information

The periodicity discovered by Raoul Bott revealed a deep repetition in the homotopy groups of the classical groups and gave K–theory an extraordinarily rigid structure. The Thom isomorphism allows this periodicity to be applied fiber by fiber in a vector bundle oriented in the appropriate sense. In the proof of the index theorem, these ideas push a localized class from the cotangent bundle of a manifold into Euclidean space, where its index can be calculated using model operators.

18.1 The Bott class and orientation conventions

Identify $\mathbb{R}^2$ with $\mathbb{C}$ by

$$(x,\xi)\longmapsto x+i\xi$$

and orient $\mathbb{R}^2$ by the ordered basis $(\partial_x,\partial_\xi)$. Its one-point compactification is $(\mathbb{R}^2)^+\cong S^2$. Write

$$S^2=D_0\cup_{S^1}D_\infty$$

as the union of two closed disks. If $m\in\mathbb{Z}$, denote by $\mathbf{E}_m\longrightarrow S^2$ the line bundle obtained by identifying the trivial fibers over the equator through

$$(z,v)_0\sim(z,z^m v)_\infty,\qquad z\in S^1,\quad v\in\mathbb{C}.$$

The fiber of $\mathbf{E}_m$ over the point added at infinity is identified with $\mathbb{C}$. Thus the difference of two rank-one bundles defines a reduced class.

Definition 18.1.1. (Bott class) The **Bott class** used in this book is

$$b:=[\mathbf{E}_{-1}]-[\underline{\mathbb{C}}]\in\widetilde{K}^0(S^2)=K_c^0(\mathbb{R}^2).$$

If X is locally compact, write

$$b\boxtimes a\in K_c^0(\mathbb{R}^2\times X),\qquad a\in K_c^0(X),$$

for the external product of Proposition 17.3.1.

The choice of $\mathbf{E}_{-1}$ rather than $\mathbf{E}_1$ is essential to the sign of the index. Indeed, $\mathbf{E}_1\otimes\mathbf{E}_{-1}$ is trivial and the computation of the ring of the sphere gives $b^2=0$; see [76, Theorem 2.2, pp. 41–50]. Since $[\mathbf{E}_{-1}]=1+b$, we obtain

$$[\mathbf{E}_1]=[\mathbf{E}_{-1}]^{-1}=(1+b)^{-1}=1-b.$$

Thus the opposite class $-b=[\mathbf{E}_1]-[\underline{\mathbb{C}}]$ is also a generator of $K_c^0(\mathbb{R}^2)$, but interchanging these generators changes all the odd-degree signs that will appear later.

It is useful to collect the conventions maintained throughout the argument:

$$(x,\xi)\longmapsto x+i\xi,\qquad J(u,v)=(-v,u),\qquad b=[\mathbf{E}_{-1}]-[\underline{\mathbb{C}}],\qquad \lambda_{\mathbb{C}}=-b. \tag{18.1}$$

With this choice, the localized operator representing b will have index -1, whereas the normal oscillator representing $\lambda_{\mathbb{C}}$ will have index $+1$. This difference is not a subsequent sign correction; it is a consequence of the fact that the two operators represent opposite generators.

The deep topological result that we will use is stated, with its local compactness and naturality hypotheses, in Theorem A.5.1. A complete proof can be found in [21, Theorems 10.20 and 10.22, pp. 270–272].

Theorem 18.1.2. (Complex Bott periodicity) For every locally compact space X there exists a natural isomorphism

$$\alpha_X:K_c^0(\mathbb{R}^2\times X)\longrightarrow K_c^0(X)$$

whose inverse is

$$\beta_X : K_c^0(X) \longrightarrow K_c^0(\mathbb{R}^2 \times X), \qquad \beta_X(a) = b \boxtimes a.$$

In particular,

$$\alpha_{\{*\}}(b) = 1 \in K^0(\{*\}) = \mathbb{Z}.$$

These isomorphisms are compatible with external products: if Y is another locally compact space, $u \in K_c^0(\mathbb{R}^2 \times X)$ and $v \in K_c^0(Y)$, then

$$\alpha_{X\times Y}(u \boxtimes v) = \alpha_X(u) \boxtimes v. \tag{18.2}$$

We do not reproduce the topological proof of periodicity here. The consequences we need follow immediately, and it is useful to record them.

Corollary 18.1.3. (Iterated Bott isomorphism) For $n \in \mathbb{N}$ there exists an isomorphism

$$\alpha_{n,X} : K_c^0(\mathbb{R}^{2n} \times X) \longrightarrow K_c^0(X)$$

obtained by applying α successively to the n factors $\mathbb{R}^2$. Its inverse is

$$a \longmapsto b^{\boxtimes n} \boxtimes a, \qquad b^{\boxtimes n} := \underbrace{b \boxtimes \cdots \boxtimes b}_{n \text{ factors}}.$$

In particular,

$$K_c^0(\mathbb{R}^{2n}) = \mathbb{Z}\, b^{\boxtimes n}, \qquad \alpha_n(b^{\boxtimes n}) = 1,$$

where $\alpha_n := \alpha_{n,\{*\}}$.

Proof. The assertion follows by induction. To pass from n to $n+1$, use

$$\mathbb{R}^{2(n+1)} = \mathbb{R}^2 \times \mathbb{R}^{2n}$$

and apply first $\alpha_{\mathbb{R}^{2n}\times X}$ and then $\alpha_{n,X}$. The formula for the inverse follows from (18.2) and associativity of the external product. □

Periodicity also determines the compactly supported K–theory groups in odd dimensions. Using the natural isomorphism (17.9), the definition by suspension in the preceding chapter can be written as

$$K_c^1(X) \cong K_c^0(\mathbb{R} \times X).$$

Periodicity then states that $K_c^{j+2}(X) \cong K_c^j(X)$. In particular,

$$K_c^0(\mathbb{R}^{2n+1}) = 0, \qquad K_c^1(\mathbb{R}^{2n+1}) \cong \mathbb{Z}.$$

Only even degree will be used explicitly, because $T^*\mathbb{R}^n$ has real dimension $2n$.

18.2 The Thom class

Let $\pi : \mathbf{V} \longrightarrow X$ be a complex vector bundle of rank r over a compact Hausdorff space X, equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{V}}$. The following construction is performed fiber by fiber. If the base is merely locally compact, the same construction defines a Thom orientation acting on compactly supported classes; the orientation class itself may have noncompact support because its support is the entire zero section.

For $v \in \mathbf{V}_x$, let

$$\varepsilon_v : \Lambda^k \mathbf{V}_x \longrightarrow \Lambda^{k+1} \mathbf{V}_x, \qquad \varepsilon_v(\omega) = v \wedge \omega,$$

and let $\iota_v := \varepsilon_v^*$ be its adjoint with respect to the Hermitian inner product induced on the exterior algebra. The creation and annihilation identities are

$$\varepsilon_v^2 = 0, \qquad \iota_v^2 = 0, \qquad \varepsilon_v \iota_v + \iota_v \varepsilon_v = \|v\|^2 I. \tag{18.3}$$

Define

$$c_{\mathbf{V}}(v) := \varepsilon_v - \iota_v : \Lambda^{\mathrm{ev}} \mathbf{V}_x \longrightarrow \Lambda^{\mathrm{odd}} \mathbf{V}_x.$$

By (18.3),

$$c_{\mathbf{V}}(v)^* c_{\mathbf{V}}(v) = \|v\|^2 I, \qquad c_{\mathbf{V}}(v) c_{\mathbf{V}}(v)^* = \|v\|^2 I. \tag{18.4}$$

Consequently, $c_{\mathbf{V}}(v)$ is an isomorphism if $v \neq 0$, with inverse $\|v\|^{-2} c_{\mathbf{V}}(v)^*$.

Definition 18.2.1. (Complex Thom class) The **Thom class** of $\mathbf{V}$ is the class supported on the zero section represented by

$$\lambda_{\mathbf{V}} := [\pi^* \Lambda^{\mathrm{ev}} \mathbf{V}, \pi^* \Lambda^{\mathrm{odd}} \mathbf{V}, c_{\mathbf{V}}] \in K_c^0(\mathbf{V}).$$

Equivalently, if $B(\mathbf{V})$ and $S(\mathbf{V})$ are the unit ball and sphere bundles, then

$$\lambda_{\mathbf{V}} = \chi\big(\pi^* \Lambda^{\mathrm{ev}} \mathbf{V}, \pi^* \Lambda^{\mathrm{odd}} \mathbf{V}; c_{\mathbf{V}}|_{S(\mathbf{V})}\big) \in K^0(B(\mathbf{V}), S(\mathbf{V})).$$

Equality (18.4) proves all the analytic and algebraic facts needed for the class to be defined. The assertion that this class generates the K–theory of each fiber in a way compatible with variation of the base is the topological content of the Thom isomorphism. It is stated in Theorem A.5.3; see also [21, Theorem 12.8, pp. 289–290].

Theorem 18.2.2. (Thom isomorphism in K–theory) Let $\pi : \mathbf{V} \longrightarrow X$ be a finite-rank complex vector bundle over a locally compact space. The map

$$\mathrm{Th}_{\mathbf{V}} : K_c^0(X) \longrightarrow K_c^0(\mathbf{V}), \qquad \mathrm{Th}_{\mathbf{V}}(a) = \pi^* a \cdot \lambda_{\mathbf{V}},$$

is an isomorphism. If $f : Y \longrightarrow X$ is continuous and $f^{-1}(\operatorname{supp} a)$ is compact, then

$$(f_{\mathbf{V}})^* \mathrm{Th}_{\mathbf{V}}(a) = \mathrm{Th}_{f^*\mathbf{V}}(f^* a), \tag{18.5}$$

where $f_{\mathbf{V}} : f^*\mathbf{V} \longrightarrow \mathbf{V}$ is the induced map. If $\mathbf{V}$ and $\mathbf{W}$ are complex bundles over X, then, under the natural identification of the total bundle $\mathbf{V} \oplus \mathbf{W}$ with the bundle $\pi_{\mathbf{V}}^* \mathbf{W}$ over $\mathbf{V}$,

$$\lambda_{\mathbf{V}\oplus\mathbf{W}} = \lambda_{\mathbf{V}} \boxtimes_X \lambda_{\mathbf{W}}, \qquad \mathrm{Th}_{\mathbf{V}\oplus\mathbf{W}} = \mathrm{Th}_{\pi_{\mathbf{V}}^*\mathbf{W}} \circ \mathrm{Th}_{\mathbf{V}}. \tag{18.6}$$

For a noncompact base, the notation $\pi^*a \cdot \lambda_{\mathbf{V}}$ means that the Thom complex is restricted to the preimage of a compact set containing $\operatorname{supp} a$. Its support is contained in the zero section over that compact set and therefore defines an element of $K_c^0(\mathbf{V})$.

Multiplicativity of the classes can be seen directly before using the fact that the Thom maps are isomorphisms. Indeed,

$$\Lambda^*(\mathbf{V} \oplus \mathbf{W}) \cong \Lambda^*\mathbf{V} \widehat{\otimes} \Lambda^*\mathbf{W}$$

as $\mathbb{Z}_2$–graded bundles, and the corresponding odd operator is

$$c_{\mathbf{V}\oplus\mathbf{W}}(v,w) = c_{\mathbf{V}}(v)\widehat{\otimes} I + \Gamma_{\mathbf{V}} \widehat{\otimes} c_{\mathbf{W}}(w),$$

where $\Gamma_{\mathbf{V}}$ equals 1 in even degree and -1 in odd degree. The two summands anticommute; hence the product of the adjoint with the operator is

$$\big(\|v\|^2 + \|w\|^2\big) I.$$

This is precisely the product complex representing the right-hand side of (18.6).

Let us check the relationship between the Bott and Thom conventions. On a trivial complex line we use the complex structure

$$J(u_1, u_2) = (-u_2, u_1).$$

The map $c_{\mathbb{C}}(u_1 + iu_2)$ is multiplication by $u_1 + iu_2$ from even to odd degree. With the gluing convention of Definition 18.1.1, this class is

$$\lambda_{\mathbb{C}} = -b. \tag{18.7}$$

Indeed, in trivializations over the interior and exterior disks, the transition map of the preceding difference is z, whereas b was defined by z^{-1}. Consequently,

$$\lambda_{\mathbb{C}^r} = (-1)^r b^{\boxtimes r}. \tag{18.8}$$

This normalization will give the oscillator operator associated with the Thom class index 1 and the operator associated with b index -1.

Not every real vector bundle has a canonical Thom class in complex K–theory. A real bundle of even rank requires a Spin^c orientation; once it has been chosen, the spinor bundles $\mathbf{S}^+$ and $\mathbf{S}^-$ and Clifford multiplication produce a class

$$[\pi^*\mathbf{S}^+, \pi^*\mathbf{S}^-, c(v)] \in K_c^0(\mathbf{V}).$$

A complex bundle has a canonical Spin^c orientation, and the resulting class agrees with the exterior-algebra class above. We will not apply the Thom isomorphism directly to an arbitrary real bundle. The embedding used for the index theorem will instead involve the sum of two copies of a real bundle; this sum has a canonical complex structure and does not require an additional choice of Spin^c structure.

18.3 Embeddings and extension by zero

Let $U \subseteq Z$ be an open subset of a locally compact space. We will use the description by supports

$$K_c^0(U) = \varinjlim_{K \Subset U} \mathcal{K}_K^0(U), \tag{18.9}$$

with the notation by complexes invertible outside K introduced in Appendix A. For every compact set $K \Subset U$, excision states that restriction to U and extension to Z are inverse isomorphisms. We will use the second map, written

$$K^0_K(U) \xrightarrow{\cong} K^0_K(Z). \tag{18.10}$$

These identifications commute with the morphisms induced by inclusions of compact sets. Passing to the direct limit defines extension by zero

$$e_U : K^0_c(U) \longrightarrow K^0_c(Z). \tag{18.11}$$

If a class is represented by a complex elementary outside a compact set $K \Subset U$, its image is the relative class determined by the same support K through (18.10). This description avoids requiring the individual bundles of the representative to extend to all of Z. Compatibility of excision with inclusions gives

$$e_V \circ e_U = e_U$$

for open subsets $U \subseteq V \subseteq Z$, where on the left-hand side the first extension is from U to V and the second from V to Z.

Now consider a proper smooth embedding

$$j : X \hookrightarrow Y$$

between manifolds without boundary. Choose metrics and use their musical isomorphisms to identify cotangent bundles with tangent bundles. The differential induces an embedding

$$Tj : TX \hookrightarrow TY.$$

Let $\nu \longrightarrow X$ be the normal bundle of j, obtained from the chosen metric on Y. Identify X with its image and write $T_xY = T_xX \oplus \nu_x$. Fix a connection ∇^Y on TY. The tangential projection of ∇^Y induces a connection ∇^X on TX, and its normal component defines the tensor

$$B_x(w,v) := \big(\nabla^Y_w V\big)^{\perp}, \qquad w, v \in T_xX,$$

where V is any vector field tangent to X with $V(x) = v$. The expression is independent of V: writing a field vanishing at x in a tangent frame, the terms containing derivatives of its coefficients are tangent and disappear upon projection onto ν_x. If ∇^Y is the Levi–Civita connection, B is the second fundamental form.

Proposition 2.7.26, applied to these two connections, gives decompositions

$$T_v(TY) \cong T_xY \oplus T_xY, \qquad T_v(TX) \cong T_xX \oplus T_xX, \qquad (x,v) \in TX.$$

The tangent inclusion is generally not the direct inclusion of the two summands. To calculate it, represent a vector in $T_v(TX)$ by a curve $v(t) \in T_{x(t)}X$ and set $w = x'(0)$, $z = \nabla^X_{\partial t} v(0)$. Decomposition of the connection along the curve gives $\nabla^Y_{\partial t} v(0) = z + B_x(w,v)$. Thus, in horizontal–vertical coordinates, the differential of Tj is

$$T_v(Tj)(w,z) = \big(w, z + B_x(w,v)\big).$$

Now consider the linear map

$$F_v : T_xY \oplus T_xY \longrightarrow \nu_x \oplus \nu_x, \qquad F_v(a,b) := \big(a^{\perp}, b^{\perp} - B_x(a^{\top}, v)\big).$$

It is surjective, since $F_v(u_1,u_2)=(u_1,u_2)$ for $u_1,u_2\in\nu_x$. If $F_v(a,b)=0$, then $a\in T_xX$ and $b=b^{\top}+B_x(a,v)$, so $(a,b)=T_v(Tj)(a,b^{\top})$. Conversely, the formula for $T_v(Tj)$ shows that its entire image lies in the kernel of F_v. Thus F_v induces an isomorphism from the quotient by the image of $T_v(Tj)$ to $\nu_x\oplus\nu_x$.

These maps depend smoothly on (x,v), and their inverse in the quotient is the class of (u_1,u_2). Identifying the normal bundle with this quotient by means of an orthogonal complement yields

$$N(Tj)\cong\pi_{TX}^{*}(\nu\oplus\nu). \tag{18.12}$$

The identification depends on the connection. If two connections are joined by their affine segment, the induced connections, tensors B, and maps F_v depend smoothly on the parameter, and each F_v induces the preceding isomorphism. This gives a homotopy of identifications. On the right-hand side of (18.12), fix the complex structure

$$J(u_1,u_2)=(-u_2,u_1). \tag{18.13}$$

The first summand corresponds to the horizontal direction and the second to the vertical direction. Fix this structure J and a tubular neighborhood Φ_J of Tj; the complex orientation, Thom isomorphism, and extension by zero are always taken with respect to the same pair (J,Φ_J). Thus, even if ν is a real bundle without a Spin^c orientation, the normal bundle of TX in TY is a complex bundle.

A tubular neighborhood provides a diffeomorphism

$$\Phi_J:\mathcal{U}\subseteq N(Tj)\longrightarrow\mathcal{V}\subseteq TY$$

between neighborhoods of the zero sections. By excision, every compactly supported class in $N(Tj)$ can be represented inside $\mathcal{U}$ after a radial dilation. We then compose the Thom isomorphism with Φ_J and extension by zero from $\mathcal{V}$.

Definition 18.3.1. (Pushforward of an embedding) The **pushforward** associated with j is

$$j_!:K_c^0(T^*X)\longrightarrow K_c^0(T^*Y),\qquad j_!:=e_{\mathcal{V}}\circ(\Phi_J^{-1})^{*}\circ\mathrm{Th}_{N(Tj)}, \tag{18.14}$$

where the metrics have been used to identify tangent and cotangent bundles and $N(Tj)$ carries the complex structure (18.13).

Independence of the metrics, connection, and tubular neighborhood follows from homotopy invariance, naturality of Thom, and isotopy of tubular neighborhoods. Full functoriality is the topological result recorded in Theorem A.6.1; in the preceding notation it states

$$(k\circ j)_!=k_!\circ j_! \tag{18.15}$$

for composable proper embeddings $X\xrightarrow{j}Y\xrightarrow{k}Z$. The construction in the form used here is described in [21, pp. 289–294]; independence of choices and functoriality are proved in [92, Appendix C, Theorem C.8; Chapter III, §§12–13].

For the inclusion $i:\{0\}\hookrightarrow\mathbb{R}^N$, the normal bundle of $T\{0\}$ in $T\mathbb{R}^N$ is

$$\mathbb{R}^N\oplus\mathbb{R}^N\cong\mathbb{C}^N$$

with the complex structure of (18.13). By (18.8),

$$i_!(1) = (-1)^N b^{\boxtimes N}. \tag{18.16}$$

Since $\alpha_N(b^{\boxtimes N}) = 1$, we obtain

$$i_!^{-1} = (-1)^N \alpha_N : K_c^0(T\mathbb{R}^N) = K_c^0(\mathbb{R}^{2N}) \longrightarrow \mathbb{Z}. \tag{18.17}$$

This formula explains in advance the sign that will appear in the Euclidean index.

18.4 Euclidean operators equal to the identity at infinity

The phrase "at infinity" will refer to a strong support condition. After stabilizing the bundles by trivial summands, all operators will act on functions with values in a single space $\mathbb{C}^q$.

Definition 18.4.1. (The class $\mathrm{Ell}_c^0(\mathbb{R}^n)$) An operator P belongs to $\mathrm{Ell}_c^0(\mathbb{R}^n;\mathbb{C}^q)$ if it is a classical, properly supported, elliptic pseudodifferential operator of order zero, and there exists a compact set $K \subset \mathbb{R}^n$ such that

$$P = I + Q, \qquad \operatorname{supp}(\mathbf{K}_Q) \subseteq K \times K,$$

where $\mathbf{K}_Q$ is the Schwartz kernel of Q. Direct sum with identity operators is allowed, and we write simply $\mathrm{Ell}_c^0(\mathbb{R}^n)$ for the stable union of these classes.

The condition on the Schwartz kernel is equivalent to the two statements

$$(P - I)u = 0, \qquad (P^* - I)u = 0$$

for every $u \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^q)$ whose support does not meet K. One implication is immediate. For the converse, the first equality shows, in the distributional sense, that the kernel of $P - I$ vanishes when the second variable lies outside K; the equality for the adjoint shows that it vanishes when the first variable lies outside K. It also follows that the principal symbol is the identity if $x \notin K$.

Equality of the principal symbol with I outside a compact set alone is a weaker condition: it does not assert that the operator is exactly the identity there. To compare realizations of this symbol, first carry out the quantization on $S^n = \mathbb{R}^n \cup \{\infty\}$. A matrix-valued principal symbol a_0 equal to I outside a compact set $K \subset \mathbb{R}^n$ extends by I to the cotangent bundle of S^n. Let $I + A \in \Psi_{\mathrm{cl}}^0(S^n;\mathbb{C}^q)$ be a quantization of this extension. Choose $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on a neighborhood of K, extend it by zero to S^n, and consider

$$I + \chi A \chi.$$

Its principal symbol is $I + \chi^2(a_0 - I) = a_0$. Consequently, $A - \chi A \chi \in \Psi_{\mathrm{cl}}^{-1}(S^n)$ and it is compact on $L^2(S^n)$: it maps $L^2(S^n)$ continuously into $H^1(S^n)$, whose inclusion in $L^2(S^n)$ is compact. By Theorem 15.6.24, the two operators on S^n have the same index. Moreover, the kernel of $\chi A \chi$ is supported in $\operatorname{supp}\chi \times \operatorname{supp}\chi$, so its Euclidean restriction satisfies the strong condition of Definition 18.4.1. This construction allows realizations with strong support to be used in the stable theory described in [21, p. 275]. The compactness comparison has been made on S^n; equality of principal symbols alone does not allow it to be applied to an arbitrary quantization on $\mathbb{R}^n$.

In this compactification, if $P \in \mathrm{Ell}_c^0(\mathbb{R}^n)$, equality with the identity near infinity allows it to be extended by I to

an elliptic pseudodifferential operator

$$\overline{P} \in \Psi^0_{\mathrm{cl}}(S^n).$$

Equality holds exactly on a neighborhood of ∞, so restriction and extension identify the kernels of P and $\overline{P}$, as well as those of their adjoints. Theorem 15.6.22 then gives

$$\operatorname{ind} P = \operatorname{ind} \overline{P}. \tag{18.18}$$

In particular, P is Fredholm on $L^2(\mathbb{R}^n;\mathbb{C}^q)$.

The principal symbol of P gives, by Definition 17.4.1, a class

$$[\sigma(P)] \in K^0_c(T^*\mathbb{R}^n) = K^0_c(\mathbb{R}^{2n}).$$

Its support is compact because the symbol is the identity outside K in the base variable and is invertible in every nonzero cotangent direction. In the relative model it is represented by

$$[\underline{\mathbb{C}}^q, \underline{\mathbb{C}}^q, \sigma_0^{\Psi}(P)|_{S^*\mathbb{R}^n}].$$

Proposition 18.4.2. (Euclidean realization and descent of the index) Every class $a \in K^0_c(T^*\mathbb{R}^n)$ is the symbol class of some $P \in \mathrm{Ell}^0_c(\mathbb{R}^n)$ after stabilization. Moreover, there exists a unique homomorphism

$$I_n : K^0_c(T^*\mathbb{R}^n) \longrightarrow \mathbb{Z} \tag{18.19}$$

such that

$$I_n([\sigma(P)]) = \operatorname{ind} P.$$

Proof. Represent a by a compactly supported triple

$$(\mathbf{E}, \mathbf{F}, \theta)$$

over $T^*\mathbb{R}^n$. Since $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ is contractible, after adding an elementary triple we may assume that $\mathbf{E}$ and $\mathbf{F}$ are trivial bundles of the same rank.

Extend a by zero to the cotangent bundle of $S^n = \mathbb{R}^n \cup \{\infty\}$. In the relative model, choose a representative over (B^*S^n, S^*S^n) elementary over T^*U for a neighborhood U of ∞. The relative proof of Theorem 17.4.3 allows us to smooth it without changing that neighborhood. After stabilizing and fixing the elementary trivialization there, we obtain an elliptic symbol of order zero that is exactly the identity over T^*U.

Let us construct the operator representing this class. Let $a_0(x,\xi)$ be the resulting homogeneous symbol, equal to I when x is near ∞. Theorem 15.3.7 gives a quantization A with principal symbol a_0. Choose $\chi \in C^\infty_c(\mathbb{R}^n)$ equal to 1 on the projection to the base of $\operatorname{supp}(a_0 - I)$ and define

$$P := I + \chi(A - I)\chi.$$

Its principal symbol is

$$I + \chi(x)^2(a_0(x,\xi) - I) = a_0(x,\xi),$$

because $a_0 - I$ vanishes where $\chi \neq 1$. Without imposing an additional condition on quantization, the two

multiplications by χ show directly that the Schwartz kernel of $P - I$ is contained in $K \times K$, with $K = \operatorname{supp} \chi$. Therefore, $P \in \mathrm{Ell}_c^0(\mathbb{R}^n)$ and $[\sigma(P)] = a$.

To prove that the index descends to K–theory, all the relative information of the triple must be retained; an isomorphism of triples may depend on ξ and does not automatically become a conjugation of operators. Let a_0 and a_1 be two representatives of the same class. After adding elementary triples, the relations of the relative model provide a parametrized triple over

$$B^*S^n \times [0,1]$$

whose endpoint restrictions are the given representatives and which is elementary over $B^*U \times [0,1]$ for a common neighborhood U of ∞. Relative smoothing keeps this neighborhood fixed. Radial contraction in the fibers of B^*S^n turns the stabilized isomorphisms into bundle isomorphisms over the base, exactly as in the proof of Theorem 17.5.6. Homogeneous extension in the covariable produces a family of elliptic symbols of order zero equal to the identity over T^*U.

Using a single cover, a single partition of unity, and a single cutoff χ, quantization is continuous in the topology of symbols of order zero and, by Calderón–Vaillancourt, continuous in the norm of

$$\mathcal{L}\big(L^2(\mathbb{R}^n;\mathbb{C}^q)\big).$$

Each operator extends to an elliptic operator on S^n. Stability of the Fredholm index gives

$$\operatorname{ind} P_0 = \operatorname{ind} P_1.$$

If the triple is elementary, radial contraction takes it to a bundle isomorphism over the base; its quantization is an isomorphism of order zero plus an operator of order -1 and has index zero. Finally, direct sum of triples is quantized by direct sum of operators, and

$$\operatorname{ind}(P \oplus Q) = \operatorname{ind} P + \operatorname{ind} Q.$$

Thus the index respects the stabilization, homotopy, isomorphism, and addition relations defining $K_c^0(T^*\mathbb{R}^n)$. This determines I_n, and realization of all classes proves its uniqueness. □

The same construction works for families. If Λ is compact Hausdorff and P_λ is a family continuous in the topology of classical symbols of order zero, with a common compact support, then the family is norm-continuous and its principal symbols depend jointly on (x, ξ, λ). Under these hypotheses, the symbols define

$$[\sigma(P_\Lambda)] \in K_c^0(T^*\mathbb{R}^n \times \Lambda). \tag{18.20}$$

Realization of every class and descent of the index bundle will be proved in Proposition 18.7.2. The variable λ is not incorporated into the cotangent variables: it continuously parametrizes the bundles and the symbol isomorphism.

18.5 Two model operators

Periodicity determines the group $K_c^0(T^*\mathbb{R}^n)$, but does not determine the analytic homomorphism I_n until its value on a generator is fixed. We first calculate that value for the class b using a localized Toeplitz operator. We then calculate the oscillator operator representing the opposite class $-b$, which will be the analytic model for the Thom isomorphism.

18.5.1 The localized operator representing b

Let $S^1 \subset \mathbb{C}$ and let

$$e_k(z) = z^k, \qquad k \in \mathbb{Z},$$

be the orthonormal Fourier basis of $L^2(S^1)$. Define

$$H_+ := \overline{\operatorname{span}}\{e_k \mid k \geq 0\}, \qquad H_- := \overline{\operatorname{span}}\{e_k \mid k < 0\},$$

and denote by Π_+ and Π_- the corresponding orthogonal projections. The projection Π_+ is pseudodifferential of order zero: modulo the finite-dimensional projection onto the constants,

$$\Pi_+ = \frac{1}{2}\big(I + \operatorname{sgn} D\big), \qquad D = -i\frac{d}{d\theta},$$

and its principal symbol equals 1 on $\xi > 0$ and 0 on $\xi < 0$.

Let $g : S^1 \longrightarrow S^1$ be a smooth function of degree one equal to 1 on a neighborhood of a point $p \in S^1$. It can be obtained by deforming $z \mapsto z$ through functions of degree one and concentrating all angular variation in an arc disjoint from p. Let M_g be multiplication by g and consider

$$A_g := \Pi_+ M_g \Pi_+ + \Pi_- : L^2(S^1) = H_+ \oplus H_- \longrightarrow L^2(S^1). \tag{18.21}$$

Its principal symbol is

$$\sigma_0^{\Psi}(A_g)(z,\xi) = \begin{cases} g(z), & \xi > 0, \\ 1, & \xi < 0. \end{cases} \tag{18.22}$$

It is invertible and equals 1 over $T^*U \setminus 0$ for some neighborhood U of p. Thus A_g is elliptic.

Lemma 18.5.1. (Index and class of the Bott model) There exists $T_b \in \operatorname{Ell}_c^0(\mathbb{R})$ such that

$$[\sigma(T_b)] = b, \qquad \operatorname{ind} T_b = -1.$$

Proof. We begin with the index of (18.21). A smooth homotopy g_t between $g_0(z) = z$ and $g_1 = g$ gives, through (18.21), a norm-continuous family of elliptic operators. Its index is constant. For $g_0(z) = z$ we have

$$A_{g_0} e_k = \begin{cases} e_{k+1}, & k \geq 0, \\ e_k, & k < 0. \end{cases}$$

Thus

$$\ker A_{g_0} = \{0\}, \qquad \operatorname{Ran} A_{g_0} = \{e_0\}^{\perp},$$

and consequently

$$\operatorname{ind} A_g = \operatorname{ind} A_{g_0} = -1. \tag{18.23}$$

Choose $\chi \in C^\infty(S^1)$ with support disjoint from p, equal to 1 on the support of $g-1$, and define

$$\widetilde{T}_b := I + \chi(A_g - I)\chi.$$

By (18.22), $\widetilde{T}_b$ and A_g have the same principal symbol. Their difference has order -1 on the closed manifold S^1 and is therefore compact on $L^2(S^1)$. By Theorem 15.6.24,

$$\operatorname{ind} \widetilde{T}_b = -1.$$

Moreover, the kernel of $\widetilde{T}_b - I$ is contained in $\operatorname{supp}\chi \times \operatorname{supp}\chi$. Removing the point p and identifying $S^1 \setminus \{p\}$ with $\mathbb{R}$ gives an operator $T_b \in \mathrm{Ell}^0_c(\mathbb{R})$ with the same index.
It remains to identify its class. Compactify the phase plane $T^*\mathbb{R} \cong \mathbb{R}_x \times \mathbb{R}_\xi$ using a large rectangle. On the lower side, on the two vertical sides, and outside the support of $g-1$, the symbol is 1. On the upper side it is $g(x)$. The positive orientation of the boundary traverses the upper side from $x = +R$ to $x = -R$. Since g has degree one when x is traversed from $-R$ to $+R$, the gluing along the boundary has degree -1. By Definition 18.1.1, the glued bundle is $\mathbf{E}_{-1}$ and

$$[\sigma(T_b)] = [\mathbf{E}_{-1}] - [\underline{\mathbb{C}}] = b.$$

□

The preceding proof does not invoke a general Toeplitz index formula: the only index calculated reduces to that of the unilateral shift. With the conventions already specified, the argument agrees with the normalization of [21, proof of Theorem 11.5, pp. 281–282].

18.5.2 The oscillator and the Thom class

On $L^2(\mathbb{R})$ consider the closed operators

$$B^+ u = \frac{du}{dx} + xu, \qquad B^- v = -\frac{dv}{dx} + xv, \tag{18.24}$$

with common domain

$$\mathcal{D} := \{u \in H^1(\mathbb{R}) \mid xu \in L^2(\mathbb{R})\}.$$

The space $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{D}$ for the norm

$$\|u\|_{\mathcal{D}} := \|u\|_{L^2(\mathbb{R})} + \left\|\frac{du}{dx}\right\|_{L^2(\mathbb{R})} + \|xu\|_{L^2(\mathbb{R})}.$$

To justify density, fix $\chi \in C_c^\infty(\mathbb{R})$, $0 \le \chi \le 1$, equal to one on $[-1,1]$, and set $\chi_R(x) = \chi(x/R)$. For $v \in \mathcal{D}$, $\chi_R v \to v$ in $L^2(\mathbb{R})$ and $x\chi_R v \to xv$ in $L^2(\mathbb{R})$ by dominated convergence. Moreover,

$$(\chi_R v)' - v' = (\chi_R - 1)v' + \chi_R' v, \qquad \|\chi_R' v\|_{L^2(\mathbb{R})} \le R^{-1}\|\chi'\|_{L^\infty(\mathbb{R})}\|v\|_{L^2(\mathbb{R})},$$

so $\chi_R v \to v$ in $\mathcal{D}$. For fixed R, convolution regularizations of $\chi_R v$ converge in $H^1(\mathbb{R})$, and their supports remain in a fixed compact interval. On that interval, multiplication by x is bounded on $L^2(\mathbb{R})$, so they also converge in $\mathcal{D}$. Choosing successively $R \to \infty$ and convolution radii producing errors smaller than $1/R$ gives a sequence in $C_c^\infty(\mathbb{R})$ converging in $\mathcal{D}$.

Proposition 18.5.2. (Index of the Thom oscillator) The operator $B^+ : \mathcal{D} \longrightarrow L^2(\mathbb{R})$ is Fredholm and

$$\ker B^+ = \operatorname{span}\{e^{-\frac{x^2}{2}}\}, \qquad \ker B^- = \{0\}, \qquad \operatorname{ind} B^+ = 1.$$

Its elliptic total symbol in the phase plane is $x + i\xi$ and represents

$$[\sigma(B^+)] = -b = \lambda_{\mathbb{C}}.$$

Proof. First we prove that

$$\operatorname{Dom}\big((B^+)^*\big) = \mathcal{D}, \qquad (B^+)^* = B^-. \tag{18.25}$$

Let $v \in \operatorname{Dom}((B^+)^*)$. There then exists $f \in L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \left(\frac{du}{dx} + xu\right) \overline{v}\, dx = \int_{\mathbb{R}} u\overline{f}\, dx, \qquad u \in C_c^\infty(\mathbb{R}).$$

In the distributional sense,

$$-\frac{dv}{dx} + xv = f. \tag{18.26}$$

Since $xv \in L^2_{\mathrm{loc}}(\mathbb{R})$, this identity implies $v \in H^1_{\mathrm{loc}}(\mathbb{R})$.
Fix $\chi \in C_c^\infty(\mathbb{R})$ with $0 \le \chi \le 1$, $\chi = 1$ on $[-1,1]$, and $\operatorname{supp}\chi \subset [-2,2]$, and set $\chi_R(x) = \chi\left(\frac{x}{R}\right)$ for $R \ge 1$. The function $w_R := \chi_R v$ belongs to $\mathcal{D}$ and

$$B^- w_R = \chi_R f - \chi_R' v. \tag{18.27}$$

For every compactly supported $w \in \mathcal{D}$, integration by parts gives

$$\|B^- w\|^2_{L^2(\mathbb{R})} = \left\|\frac{dw}{dx}\right\|^2_{L^2(\mathbb{R})} + \|xw\|^2_{L^2(\mathbb{R})} + \|w\|^2_{L^2(\mathbb{R})}. \tag{18.28}$$

Applied to w_R and combined with (18.27), this equality yields the uniform bound

$$\|x\chi_R v\|^2_{L^2(\mathbb{R})} \le 2\|f\|^2_{L^2(\mathbb{R})} + \frac{2\|\chi'\|^2_{L^\infty(\mathbb{R})}}{R^2}\|v\|^2_{L^2(\mathbb{R})}. \tag{18.29}$$

Fatou's lemma, applied as $R \to \infty$, shows that $xv \in L^2(\mathbb{R})$. The equation (18.26) then gives

$$\frac{dv}{dx} = xv - f \in L^2(\mathbb{R}),$$

so $v \in \mathcal{D}$. The reverse inclusion follows by integrating by parts, now with both sides in $L^2(\mathbb{R})$, and proves (18.25).

The same argument shows that $(B^-)^* = B^+$ with domain $\mathcal{D}$. Indeed, if $B^+u = g$ distributionally, then

$$\left\|\frac{dw}{dx}\right\|^2_{L^2(\mathbb{R})} + \|xw\|^2_{L^2(\mathbb{R})} = \|B^+w\|^2_{L^2(\mathbb{R})} + \|w\|^2_{L^2(\mathbb{R})}$$

for $w = \chi_R u$; the same passage to the limit gives $xu \in L^2(\mathbb{R})$ and $\frac{du}{dx} = g - xu \in L^2(\mathbb{R})$. In particular, B^+ and B^- are closed operators and are adjoints of one another.
The homogeneous equations can be integrated directly:

$$B^+u = 0 \quad \Longleftrightarrow \quad u(x) = Ce^{-\frac{x^2}{2}},$$

whereas

$$B^-v = 0 \quad \Longleftrightarrow \quad v(x) = Ce^{\frac{x^2}{2}}.$$

Only the first solution belongs to $L^2(\mathbb{R})$.
For $v \in C^\infty_c(\mathbb{R})$, identity (18.28) reads

$$\|B^-v\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|-\frac{dv}{dx} + xv\right|^2 dx \tag{18.30}$$

$$= \left\|\frac{dv}{dx}\right\|^2_{L^2(\mathbb{R})} + \|xv\|^2_{L^2(\mathbb{R})} + \|v\|^2_{L^2(\mathbb{R})}. \tag{18.31}$$

The approximation in the norm of $\mathcal{D}$ constructed before the proposition preserves convergence of v, v', and xv simultaneously in $L^2(\mathbb{R})$. Since $B^-v = -v' + xv$, all terms in (18.31) converge and the identity holds on $\mathcal{D}$. In particular, B^- is bounded below by 1 and has closed range. The closed range theorem applied to the adjoint pair B^+, B^- implies that $\operatorname{Ran} B^+$ is closed. Moreover,

$$(\operatorname{Ran} B^+)^\perp = \ker B^- = \{0\},$$

so B^+ is surjective. Its kernel has dimension one and its cokernel is zero; this proves that its index is 1.
With the Fourier normalization $\widehat{\frac{du}{dx}}(\xi) = i\xi\,\widehat{u}(\xi)$, the total symbol is $x + i\xi$. Its restriction to a large circle has degree one. By the same gluing calculation as in the preceding proof, now with the usual orientation of the circle, it represents $[E_1] - [\underline{\mathbb{C}}] = -b$. Equality with $\lambda_{\mathbb{C}}$ is (18.7). □

The oscillator does not belong to $\mathrm{Ell}^0_c(\mathbb{R})$: it is an unbounded operator. It does not have compact resolvent; in fact, $e^{\lambda x - \frac{x^2}{2}} \in \mathcal{D}$ and, for every $\lambda \in \mathbb{C}$,

$$B^+ e^{\lambda x - \frac{x^2}{2}} = \lambda e^{\lambda x - \frac{x^2}{2}}.$$

The self-adjoint supersymmetric operator

$$\mathscr{B} = \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix}$$

does have compact resolvent, since $\mathscr{B}^2$ is the sum of the two scalar harmonic oscillators. The role of B^+ is to realize the Thom class analytically, not to replace the model T_b. The comparison

$$\operatorname{ind} T_b = -1, \qquad \operatorname{ind} B^+ = 1$$

confirms that T_b represents b and B^+ represents $-b$.

18.6 Products of operators and multiplicativity of the index

Reducing dimension n to the one-dimensional case requires checking that external product of classes corresponds to a product of operators and that their indices multiply. A subtlety cannot be omitted: although operators acting separately in each variable are well defined on the Hilbert tensor product, in general they are not classical pseudodifferential operators on the product manifold. This phenomenon is explained in [21, Exercise 9.20, pp. 243–244]. First we calculate the index of the matrix on Hilbert spaces and then regularize it using angular cutoffs.

Let

$$P : H_E^+ \longrightarrow H_E^-, \qquad Q : H_F^+ \longrightarrow H_F^-$$

be bounded Fredholm operators between complex Hilbert spaces. On the Hilbert tensor products define

$$P\#Q := \begin{pmatrix} P\otimes I & -I\otimes Q^* \\ I\otimes Q & P^*\otimes I \end{pmatrix}, \tag{18.32}$$

with domain and codomain

$$H^{\mathrm{ev}} := (H_E^+ \widehat{\otimes} H_F^+) \oplus (H_E^- \widehat{\otimes} H_F^-),$$
$$H^{\mathrm{odd}} := (H_E^- \widehat{\otimes} H_F^+) \oplus (H_E^+ \widehat{\otimes} H_F^-).$$

Proposition 18.6.1. (Product of Fredholm operators) The operator $P\#Q : H^{\mathrm{ev}} \longrightarrow H^{\mathrm{odd}}$ is Fredholm and

$$\operatorname{ind}(P\#Q) = \operatorname{ind} P \operatorname{ind} Q. \tag{18.33}$$

Proof. Multiplying the matrices in (18.32) and using the fact that operators acting on different factors commute yields

$$(P\#Q)^*(P\#Q) = \begin{pmatrix} P^*P\otimes I + I\otimes Q^*Q & 0 \\ 0 & PP^*\otimes I + I\otimes QQ^* \end{pmatrix}, \tag{18.34}$$

$$(P\#Q)(P\#Q)^* = \begin{pmatrix} PP^*\otimes I + I\otimes Q^*Q & 0 \\ 0 & P^*P\otimes I + I\otimes QQ^* \end{pmatrix}. \tag{18.35}$$

Each summand is nonnegative. If $\mathcal{H}_A, \mathcal{H}_B$ are Hilbert spaces and $A \in \mathcal{L}(\mathcal{H}_A)$, $B \in \mathcal{L}(\mathcal{H}_B)$ are nonnegative, then, on $\mathcal{H}_A \widehat{\otimes} \mathcal{H}_B$,

$$\ker(A\otimes I + I\otimes B) = \ker A \widehat{\otimes} \ker B;$$

indeed, for $u \in \mathcal{H}_A \widehat{\otimes} \mathcal{H}_B$,

$$\langle (A\otimes I + I\otimes B)u, u\rangle_{\mathcal{H}_A\widehat{\otimes}\mathcal{H}_B} = \|(A^{\frac{1}{2}}\otimes I)u\|^2_{\mathcal{H}_A\widehat{\otimes}\mathcal{H}_B} + \|(I\otimes B^{\frac{1}{2}})u\|^2_{\mathcal{H}_A\widehat{\otimes}\mathcal{H}_B}.$$

Since P and Q are Fredholm, their restrictions to the orthogonal complements of their kernels are bounded below. The sums in (18.34) and (18.35) are therefore bounded below on the complements of their finite-dimensional kernels. This proves that $P\#Q$ has closed range and finite-dimensional kernels.

The kernel formulas are

$$\ker(P\#Q) \cong (\ker P \widehat{\otimes} \ker Q) \oplus (\ker P^* \widehat{\otimes} \ker Q^*),$$
$$\ker(P\#Q)^* \cong (\ker P^* \widehat{\otimes} \ker Q) \oplus (\ker P \widehat{\otimes} \ker Q^*).$$

If

$$a = \dim\ker P, \quad b_0 = \dim\ker P^*, \quad c = \dim\ker Q, \quad d = \dim\ker Q^*,$$

then

$$\operatorname{ind}(P\#Q) = (ac + b_0 d) - (b_0 c + ad) = (a - b_0)(c - d),$$

which proves (18.33). □

The preceding matrix cannot simply be declared classical pseudodifferential on a product. To obtain a classical matrix while preserving its index, we first raise the order. Let (X, g_X) and (Y, g_Y) be closed Riemannian manifolds and let

$$P \in \Psi^0_{\mathrm{cl}}(X; \mathbb{C}^q), \qquad Q \in \Psi^0_{\mathrm{cl}}(Y; \mathbb{C}^r)$$

be elliptic. Choose nonnegative scalar Laplacians Δ_X and Δ_Y and set

$$\Lambda_{X,q} = (I + \Delta_X)^{\frac{1}{2}} \otimes I_{\mathbb{C}^q}, \qquad \Lambda_{Y,r} = (I + \Delta_Y)^{\frac{1}{2}} \otimes I_{\mathbb{C}^r}.$$

By Theorem 15.7.9, these are positive classical operators of order one and define isomorphisms

$$\Lambda_{X,q} : H^1(X; \mathbb{C}^q) \longrightarrow L^2(X; \mathbb{C}^q), \qquad \Lambda_{Y,r} : H^1(Y; \mathbb{C}^r) \longrightarrow L^2(Y; \mathbb{C}^r).$$

The operators

$$A := P\Lambda_{X,q} \in \Psi^1_{\mathrm{cl}}(X; \mathbb{C}^q), \qquad B := Q\Lambda_{Y,r} \in \Psi^1_{\mathrm{cl}}(Y; \mathbb{C}^r) \tag{18.36}$$

are elliptic of order one and

$$\operatorname{ind} A = \operatorname{ind} P, \qquad \operatorname{ind} B = \operatorname{ind} Q. \tag{18.37}$$

Define

$$\mathcal{D} \colon C^\infty(X \times Y; \mathbb{C}^{qr})^{\oplus 2} \longrightarrow C^\infty(X \times Y; \mathbb{C}^{qr})^{\oplus 2}, \qquad \mathcal{D} := A\#B = \begin{pmatrix} A \otimes I_{\mathbb{C}^r} & -I_{\mathbb{C}^q} \otimes B^* \\ I_{\mathbb{C}^q} \otimes B & A^* \otimes I_{\mathbb{C}^r} \end{pmatrix}. \tag{18.38}$$

Its closure, as an operator on $L^2(X \times Y; \mathbb{C}^{qr})^{\oplus 2}$, has domain $H^1(X \times Y; \mathbb{C}^{qr})^{\oplus 2}$. Indeed, if $\mathcal{L}_X := \Lambda_{X,q} \otimes I_{\mathbb{C}^r}$ and $\mathcal{L}_Y := I_{\mathbb{C}^q} \otimes \Lambda_{Y,r}$, then

$$\mathcal{L} := (\mathcal{L}_X^2 + \mathcal{L}_Y^2)^{\frac{1}{2}} : H^1(X \times Y; \mathbb{C}^{qr}) \longrightarrow L^2(X \times Y; \mathbb{C}^{qr}) \tag{18.39}$$

is an isomorphism, and the same holds for $\mathcal{L}^{\oplus 2}$. The elliptic estimates for A and B show that the graph norm of $\mathcal{D}$ is equivalent to that of $H^1(X \times Y; \mathbb{C}^{qr})^{\oplus 2}$.

The calculation in Proposition 18.6.1 remains valid for these closed operators. The squares A^*A, AA^*, B^*B, and BB^* now have compact resolvent. The spectral theorem applied to

$$\mathcal{D}^*\mathcal{D} = \begin{pmatrix} A^*A \otimes I + I \otimes B^*B & 0 \\ 0 & AA^* \otimes I + I \otimes BB^* \end{pmatrix}, \tag{18.40}$$

$$\mathcal{D}\mathcal{D}^* = \begin{pmatrix} AA^* \otimes I + I \otimes B^*B & 0 \\ 0 & A^*A \otimes I + I \otimes BB^* \end{pmatrix} \tag{18.41}$$

gives the same four kernel formulas as in the preceding proof. In particular,

$$\operatorname{ind} \mathcal{D} = \operatorname{ind} A \operatorname{ind} B = \operatorname{ind} P \operatorname{ind} Q. \tag{18.42}$$

The approximation we need is constructed within the ordinary pseudodifferential calculus on the product. In this way we do not assume that an operator defined by spectral calculus in the two variables is automatically an isotropic pseudodifferential operator.

Lemma 18.6.2. (Angular microlocal cutoffs) Fix finite atlases $\{U_i\}_{i\in I}$ of X and $\{V_j\}_{j\in J}$ of Y, together with real functions

$$\alpha_i \in C_c^\infty(U_i), \qquad \beta_j \in C_c^\infty(V_j), \qquad \sum_{i\in I} \alpha_i^2 = 1, \qquad \sum_{j\in J} \beta_j^2 = 1.$$

There is a choice, depending only on these data, of scalar operators

$$Q_R^X, Q_R^Y \in \Psi^0_{\mathrm{cl}}(X \times Y), \qquad R > 2,$$

with the following property. If $T \in \Psi^1_{\mathrm{cl}}(X; \mathbb{C}^d)$ and $S \in \Psi^1_{\mathrm{cl}}(Y; \mathbb{C}^e)$, extend Q_R^X, Q_R^Y diagonally to $\mathbb{C}^{de}$ and set

$$\mathfrak{K}_R^X(T) := (T \otimes I_{\mathbb{C}^e}) Q_R^X, \qquad \mathfrak{K}_R^Y(S) := (I_{\mathbb{C}^d} \otimes S) Q_R^Y \tag{18.43}$$

These operators belong to $\Psi^1_{\mathrm{cl}}(X \times Y; \mathbb{C}^{de})$ and satisfy

$$\left\| \mathfrak{K}_R^X(T) - T \otimes I_{\mathbb{C}^e} \right\|_{\mathcal{L}(H^1(X\times Y;\mathbb{C}^{de}), L^2(X\times Y;\mathbb{C}^{de}))} \leq \frac{C_T}{R}, \tag{18.44}$$

$$\left\| \mathfrak{K}_R^Y(S) - I_{\mathbb{C}^d} \otimes S \right\|_{\mathcal{L}(H^1(X\times Y;\mathbb{C}^{de}), L^2(X\times Y;\mathbb{C}^{de}))} \leq \frac{C_S}{R}. \tag{18.45}$$

Let $|\xi|_i$ be the Euclidean norm of the components of $\xi \in T_x^*X$ in the chart U_i, and define $|\eta|_j$ similarly. In each summand of the following formulas, these norms are used only on $\operatorname{supp} \alpha_i \times \operatorname{supp} \beta_j$. The principal symbols of T and S are denoted, respectively, by

$$t_1(x,\xi) := \sigma_1^\Psi(T)(x,\xi), \qquad s_1(y,\eta) := \sigma_1^\Psi(S)(y,\eta),$$

and are understood to be lifted to $T^*(X \times Y)$ in the region where their own covariable does not vanish. The principal symbols of the two cutoff operators are

$$\Phi_R(x,y,\xi,\eta) := \sum_{(i,j)\in I\times J} \alpha_i(x)^2 \beta_j(y)^2 \chi\left(\frac{|\eta|_j}{R|\xi|_i} \right), \tag{18.46}$$

$$\Psi_R(x,y,\xi,\eta) := \sum_{(i,j)\in I\times J} \alpha_i(x)^2 \beta_j(y)^2 \chi\left(\frac{|\xi|_i}{R|\eta|_j} \right). \tag{18.47}$$

Here $\chi \in C^\infty([0,\infty),[0,1])$ equals 1 on $[0,1]$, is positive on $[0,2)$, and equals 0 on $[2,\infty)$. On the axis $\xi = 0$, the summand of (18.46) is defined to be 0, and on the axis $\eta = 0$ to be 1; for (18.47) the opposite convention

is used. In particular,

$$\sigma_1^{\Psi}\big(\mathfrak{R}_R^X(T)\big) = \Phi_R t_1, \qquad \sigma_1^{\Psi}\big(\mathfrak{R}_R^Y(S)\big) = \Psi_R s_1. \tag{18.48}$$

The products in (18.48) extend smoothly by zero to the axis of the corresponding covariable: the factor Φ_R is identically zero on a conic neighborhood of $\xi = 0$ when $\eta \neq 0$, and Ψ_R has the symmetric property.

Proof. Set $\lambda_{ij}(x,y) := \alpha_i(x)\beta_j(y)$. Identify $U_i \times V_j$ with an open subset of $\mathbb{R}^{m+k}$ and write

$$\langle \xi \rangle_i = (1 + |\xi|_i^2)^{\frac{1}{2}}, \qquad \langle \eta \rangle_j = (1 + |\eta|_j^2)^{\frac{1}{2}}.$$

Let $M_{ij,R}^X$ and $M_{ij,R}^Y$ be the Fourier multipliers in this product chart with full symbols

$$\vartheta_{ij,R}^X(\xi,\eta) := \chi\left(\frac{\langle \eta \rangle_j}{R\langle \xi \rangle_i}\right), \tag{18.49}$$

$$\vartheta_{ij,R}^Y(\xi,\eta) := \chi\left(\frac{\langle \xi \rangle_i}{R\langle \eta \rangle_j}\right). \tag{18.50}$$

Multiplication by λ_{ij} before and after the multiplier makes its extension by zero outside the chart well defined. Take

$$Q_R^X := \sum_{(i,j)\in I\times J} \lambda_{ij} M_{ij,R}^X \lambda_{ij}, \qquad Q_R^Y := \sum_{(i,j)\in I\times J} \lambda_{ij} M_{ij,R}^Y \lambda_{ij}. \tag{18.51}$$

If all multipliers in (18.51) are replaced by the identity, the result is exactly the identity, since

$$\sum_{(i,j)\in I\times J} \lambda_{ij}^2 = \left(\sum_{i\in I} \alpha_i^2\right)\left(\sum_{j\in J} \beta_j^2\right) = 1. \tag{18.52}$$

For fixed R, the symbols of (18.49) and (18.50) are classical of order zero. We verify this for the first. If $\xi \neq 0$ and $\eta \neq 0$, then

$$\frac{\langle t\eta \rangle_j}{R\langle t\xi \rangle_i} \sim \frac{|\eta|_j}{R|\xi|_i}\left(1 + \sum_{\ell=1}^{\infty} c_{\ell,ij}(\xi,\eta) t^{-2\ell}\right), \qquad t \longrightarrow \infty. \tag{18.53}$$

Taylor's formula for χ gives a homogeneous expansion of degrees $0, -2, -4, \ldots$. Its coefficients extend smoothly to the axes: near $\{\xi = 0,\ \eta \neq 0\}$ the cutoff is identically zero, and near $\{\eta = 0,\ \xi \neq 0\}$ it is identically one. The homogeneous term of degree zero is

$$\chi\left(\frac{|\eta|_j}{R|\xi|_i}\right).$$

The same proof, with variables interchanged, applies to the second symbol. Thus Q_R^X and Q_R^Y are classical operators of order zero, with principal symbols Φ_R and Ψ_R, respectively.

We must check that the compositions in (18.43) belong to the isotropic calculus; this does not follow merely from Q_R^X and Q_R^Y being pseudodifferential operators. Consider Q_R^X. In a chart, the symbol of every summand of (18.51), modulo $S^{-\infty}$, has an expansion whose terms contain a derivative of $\vartheta_{ij,R}^X$. All these terms are supported in

$$\langle \eta \rangle_j \leq 2R\langle \xi \rangle_i, \qquad \langle (\xi,\eta) \rangle \leq C_R \langle \xi \rangle_i. \tag{18.54}$$

Asymptotic summation can be performed after multiplication by a cutoff equal to 1 on this cone; it therefore retains support in a slightly larger cone, up to a remainder in $S^{-\infty}$.

Use a finite partition to decompose the kernel of T into a part near the diagonal and a smooth part. For the first part, the composition formula contains terms

$$\frac{1}{\gamma!}\partial_\xi^\gamma t(x,\xi)D_x^\gamma q_{ij,R}(x,y,\xi,\eta), \tag{18.55}$$

where $q_{ij,R}$ is a full symbol of the corresponding summand of Q_R^X. On the cone (18.54), the estimates for t with respect to ξ become isotropic estimates of order one with respect to (ξ,η). The remainder formula, integrated by parts in the chart variables, gives the same estimates. The smooth part of the kernel of T has rapidly decreasing transform in ξ; since $|\eta|_j \le C_R\langle\xi\rangle_i$ on the support of the cutoff, its composition is rapidly decreasing in (ξ,η). This proves that $(T\otimes I_{\mathbb{C}^e})Q_R^X$ is classical of order one. Moreover, the first term in (18.55) proves the first equality in (18.48). The proof for $(I_{\mathbb{C}^d}\otimes S)Q_R^Y$ is symmetric.

The norm estimate remains. From (18.52) we obtain the exact identity

$$Q_R^X - I = \sum_{(i,j)\in I\times J} \lambda_{ij}(M_{ij,R}^X - I)\lambda_{ij}. \tag{18.56}$$

Let $u \in H^1(X\times Y;\mathbb{C}^{de})$ and let v be the coordinate expression, with values in $\mathbb{C}^{de}$, of $\lambda_{ij}u$. The factor $1-\vartheta_{ij,R}^X$ can be nonzero only when $\langle\eta\rangle_j > R\langle\xi\rangle_i$. By Plancherel,

$$\begin{aligned}
&\left\|(M_{ij,R}^X - I)v\right\|^2_{H^1(\mathbb{R}_x^m;L^2(\mathbb{R}_y^k;\mathbb{C}^{de}))}\\
&\quad= \int_{\mathbb{R}^k}\int_{\mathbb{R}^m} \langle\xi\rangle_i^2 |1-\vartheta_{ij,R}^X(\xi,\eta)|^2 |\widehat{v}(\xi,\eta)|^2_{\mathbb{C}^{de}}\,d\xi\,d\eta\\
&\quad\le \frac{1}{R^2}\int_{\mathbb{R}^k}\int_{\mathbb{R}^m} \langle\eta\rangle_j^2 |\widehat{v}(\xi,\eta)|^2_{\mathbb{C}^{de}}\,d\xi\,d\eta = \frac{1}{R^2}\|v\|^2_{L^2(\mathbb{R}_x^m;H^1(\mathbb{R}_y^k;\mathbb{C}^{de}))}.
\end{aligned} \tag{18.57}$$

Suppressing the chart identifications from the notation, we have

$$(T\otimes I_{\mathbb{C}^e})(Q_R^X - I)u = \sum_{(i,j)\in I\times J} (T\alpha_i\otimes\beta_j I_{\mathbb{C}^e})(M_{ij,R}^X - I)(\lambda_{ij}u). \tag{18.58}$$

The operator $T\alpha_i : H^1(X;\mathbb{C}^d)\to L^2(X;\mathbb{C}^d)$ is bounded. Multiplications by α_i and β_j, changes of coordinates, and their inverses are bounded on the Sobolev spaces appearing in (18.57). Applying $T\alpha_i$ for every value of y and using Fubini yields

$$\|(T\alpha_i\otimes\beta_j I_{\mathbb{C}^e})w\|_{L^2(X\times Y;\mathbb{C}^{de})} \le C_{ij}\|w\|_{H^1(\mathbb{R}_x^m;L^2(\mathbb{R}_y^k;\mathbb{C}^{de}))}.$$

Since the covers are finite, (18.56) and (18.57) give (18.44), with a constant independent of R. Interchanging X and Y gives (18.45). □

The two principal symbols constructed with the same atlas do not vanish simultaneously. Indeed, every summand of Φ_R and Ψ_R is nonnegative, and the weights $\alpha_i^2\beta_j^2$ sum to 1. First suppose that $\xi\neq 0$ and $\eta\neq 0$. If $\Phi_R = 0$, then for every active pair we have

$$\frac{|\eta|_j}{R|\xi|_i} \ge 2.$$

Therefore,

$$\frac{|\xi|_i}{R|\eta|_j} \le \frac{1}{2R^2} < 1$$

for every active pair. All cutoffs occurring in Ψ_R equal 1, whence $\Psi_R = 1$. The same reasoning with the variables interchanged shows that $\Psi_R = 0$ implies $\Phi_R = 1$. On the axes, the conclusion follows directly from the conventions in (18.46) and (18.47).

Apply the lemma to A, A^*, B, and B^* and define

$$\mathcal{D}_R := \begin{pmatrix} \mathfrak{K}_R^X(A) & -\mathfrak{K}_R^Y(B^*) \\ \mathfrak{K}_R^Y(B) & \mathfrak{K}_R^X(A^*) \end{pmatrix}. \tag{18.59}$$

This is an operator

$$\mathcal{D}_R \in \Psi^1_{\mathrm{cl}}(X \times Y; \mathbb{C}^{2qr}).$$

The four estimates in the lemma give

$$\|\mathcal{D}_R - \mathcal{D}\|_{\mathcal{L}(H^1(X\times Y;\mathbb{C}^{qr})^{\oplus 2}, L^2(X\times Y;\mathbb{C}^{qr})^{\oplus 2})} \le \frac{C}{R}. \tag{18.60}$$

Since $\mathcal{D}$ is Fredholm, openness of the set of Fredholm operators implies that, for R sufficiently large,

$$\operatorname{ind} \mathcal{D}_R = \operatorname{ind} \mathcal{D} = \operatorname{ind} P \operatorname{ind} Q. \tag{18.61}$$

It remains to check that the symbol of $\mathcal{D}_R$ is the product complex. Let $a_1(x,\xi)$ and $b_1(y,\eta)$ be the principal symbols of A and B. The full principal symbol is

$$c_R(x,y,\xi,\eta) = \begin{pmatrix} \Phi_R a_1 \otimes I & -\Psi_R I \otimes b_1^* \\ \Psi_R I \otimes b_1 & \Phi_R a_1^* \otimes I \end{pmatrix}. \tag{18.62}$$

Matrix multiplication gives

$$c_R^* c_R = \begin{pmatrix} \Phi_R^2 a_1^* a_1 \otimes I + \Psi_R^2 I \otimes b_1^* b_1 & 0 \\ 0 & \Psi_R^2 I \otimes b_1 b_1^* + \Phi_R^2 a_1 a_1^* \otimes I \end{pmatrix}. \tag{18.63}$$

Let $(\xi,\eta) \neq (0,0)$. If $\xi \neq 0$ and $\Phi_R > 0$, ellipticity of a_1 makes the contribution containing $a_1^* a_1$ in the first diagonal entry and that containing $a_1 a_1^*$ in the second positive definite. If $\xi = 0$ or $\Phi_R = 0$, then $\eta \neq 0$ and $\Psi_R = 1$ by the preceding property; ellipticity of b_1 makes the remaining contributions positive definite. Thus both diagonal entries of (18.63) are positive definite on the entire punctured cotangent bundle. Compactness of the cosphere bundle of $X \times Y$ gives a uniform lower bound, so c_R is elliptic.

To identify the class, a continuous homotopy of exact complexes over the cosphere bundle suffices, as this is one of the relations defining K_c^0; we do not assert that its intermediate terms are classical symbols. Extend a_1 and b_1 continuously by zero on the axes of their covariables. For $t < 1$ these extensions may be merely continuous on the axes, which suffices for this topological homotopy. Set

$$\Phi_{R,t} := (1-t) + t\Phi_R, \qquad \Psi_{R,t} := (1-t) + t\Psi_R, \qquad 0 \le t \le 1,$$

and replace Φ_R, Ψ_R by $\Phi_{R,t}, \Psi_{R,t}$ in (18.62). For $t < 1$ both coefficients are positive. For $t = 1$ they do not vanish

simultaneously. The calculation in (18.63) proves that the entire continuous family of complexes is invertible away from the zero section. At $t = 0$ we obtain

$$a_1 \# b_1 = \begin{pmatrix} a_1 \otimes I & -I \otimes b_1^* \\ I \otimes b_1 & a_1^* \otimes I \end{pmatrix}, \tag{18.64}$$

which is the tensor product complex of the two complexes of length one. This matrix expresses the sign rule for the product of complexes described in [21, Remark 11.3, pp. 278–279]. Proposition 17.3.1 and homotopy invariance therefore give

$$[c_R] = [\sigma(P)] \boxtimes [\sigma(Q)]. \tag{18.65}$$

Multiplication by the positive symbols of Λ_X and Λ_Y does not change the classes of P and Q. This argument is the Euclidean version of the cutoffs c_{r_0} used in the proof of the multiplicative property in [21, pp. 294–301]; raising the order to one beforehand also makes possible the norm comparison (18.60).

Finally,

$$C_R := \mathcal{D}_R(\mathcal{L}^{-1})^{\oplus 2} \in \Psi^0_{\mathrm{cl}}(X \times Y; \mathbb{C}^{2qr}) \tag{18.66}$$

has the same index as $\mathcal{D}_R$. Its symbol is obtained by dividing c_R by the positive function $\sqrt{|\xi|^2_{\mathbf{g}_X} + |\eta|^2_{\mathbf{g}_Y}}$; this division also leaves the class (18.65) unchanged.

Apply the construction to $P \in \mathrm{Ell}^0_c(\mathbb{R}^m)$ and $Q \in \mathrm{Ell}^0_c(\mathbb{R}^k)$. Extend them by the identity to $X = S^m$ and $Y = S^k$. The class on the right-hand side of (18.65) is compactly supported inside

$$T^*(\mathbb{R}^m \times \mathbb{R}^k) = T^*\mathbb{R}^{m+k}.$$

The relative realization of Proposition 18.4.2 allows us to stabilize the complex, deform it while keeping it fixed on a compact set to the identity near the divisor

$$(\{\infty\} \times S^k) \cup (S^m \times \{\infty\})$$

and quantize the homotopy. If ρ equals 1 on the projection of the symbol support, the final replacement

$$\widetilde{C} \longmapsto I + \rho(\widetilde{C} - I)\rho$$

produces an operator in $\mathrm{Ell}^0_c(\mathbb{R}^{m+k})$ with the same symbol class. The intermediate operators are elliptic on the closed manifold $S^m \times S^k$; by Theorem 17.5.6, their index depends only on this class. Combined with (18.61), this proves

$$I_{m+k}(a \boxtimes c) = I_m(a) I_k(c), \qquad a \in K^0_c(T^*\mathbb{R}^m), \quad c \in K^0_c(T^*\mathbb{R}^k). \tag{18.67}$$

18.7 The index theorem in Euclidean space

Define the Euclidean topological index by

$$\mathrm{ind}_{\mathrm{t}}^{\mathbb{R}^n} := (-1)^n \alpha_n : K^0_c(T^*\mathbb{R}^n) = K^0_c(\mathbb{R}^{2n}) \longrightarrow \mathbb{Z}. \tag{18.68}$$

Equality (18.17) shows that it can also be written as

$$\operatorname{ind}_{\mathrm{t}}^{\mathbb{R}^n} = i_!^{-1}, \qquad i : \{0\} \hookrightarrow \mathbb{R}^n.$$

Theorem 18.7.1. (Euclidean index) Let $P \in \mathrm{Ell}_c^0(\mathbb{R}^n)$. Then

$$\operatorname{ind}_{\mathrm{a}}(P) = \operatorname{ind}_{\mathrm{t}}^{\mathbb{R}^n}[\sigma(P)] = (-1)^n \alpha_n([\sigma(P)]). \tag{18.69}$$

Proof. By Corollary 18.1.3, every class in $K_c^0(T^*\mathbb{R}^n)$ has a unique expression

$$[\sigma(P)] = \ell\, b^{\boxtimes n}, \qquad \ell \in \mathbb{Z},$$

and

$$\ell = \alpha_n([\sigma(P)]).$$

Lemma 18.5.1 and multiplicativity (18.67) imply

$$I_n(b^{\boxtimes n}) = I_1(b)^n = (-1)^n.$$

Since I_n is a homomorphism,

$$\operatorname{ind}_{\mathrm{a}}(P) = I_n([\sigma(P)]) = \ell I_n(b^{\boxtimes n}) = (-1)^n \alpha_n([\sigma(P)]).$$

By (18.68), the last expression is the topological index. □

The proof uses Bott periodicity to identify the generator and calculates its index using the unilateral shift and angular regularization; it therefore does not presuppose the index theorem on closed manifolds.

18.7.1 An extension: the families index

Let Λ be a compact Hausdorff space and let $P = (P_\lambda)_{\lambda\in\Lambda}$ be a family of Euclidean elliptic operators of order zero, continuous in the topology of classical symbols and with a common compact support. In particular, both the operators and their principal symbols form continuous families. Their analytic index is a class in $K^0(\Lambda)$. Recall its construction. Choose a finite-dimensional subspace $V \subset L^2(\mathbb{R}^n;\mathbb{C}^q)$ such that

$$\operatorname{Ran} P_\lambda + V = L^2(\mathbb{R}^n;\mathbb{C}^q) \qquad \text{for every } \lambda \in \Lambda.$$

Existence of a single V for all parameters is obtained first locally from openness of surjectivity and then by taking the sum of the spaces corresponding to a finite subcover of Λ. The kernels of the surjective maps

$$\mathcal{P}_\lambda : L^2(\mathbb{R}^n;\mathbb{C}^q) \oplus V \longrightarrow L^2(\mathbb{R}^n;\mathbb{C}^q), \qquad \mathcal{P}_\lambda(u,v) = P_\lambda u + v,$$

form a finite-dimensional vector bundle $\mathbf{K} := \ker \mathcal{P} \longrightarrow \Lambda$. Indeed, fix λ_0 and let R_0 be a right inverse of $\mathcal{P}_{\lambda_0}$. For λ near λ_0, the operator $\mathcal{P}_\lambda R_0$ is invertible by the Neumann series, and

$$R_\lambda := R_0(\mathcal{P}_\lambda R_0)^{-1}$$

is a right inverse depending continuously on λ. The projections

$$\Pi_\lambda := I - R_\lambda \mathcal{P}_\lambda$$

have image $\ker \mathcal{P}_\lambda$, depend continuously on the parameter, and provide local trivializations of these kernels. Define

$$\mathrm{Ind}_{\mathrm{a}}(P) := [\mathbf{K}] - [\underline{V}] \in K^0(\Lambda). \tag{18.70}$$

The class is independent of V. Indeed, if $V \subseteq W$ and both spaces make the stabilized maps surjective, set $\mathbf{K}_V := \ker \mathcal{P}_V$ and $\mathbf{K}_W := \ker \mathcal{P}_W$. There is an exact sequence of bundles

$$0 \longrightarrow \mathbf{K}_V \longrightarrow \mathbf{K}_W \longrightarrow \underline{W/V} \longrightarrow 0.$$

The last arrow sends (u, w) to the class of w modulo V. It is surjective because $\operatorname{Ran} P_\lambda + V = L^2(\mathbb{R}^n; \mathbb{C}^q)$, and its kernel is $\mathbf{K}_V$. Thus

$$[\mathbf{K}_W] - [\underline{W}] = [\mathbf{K}_V] + [\underline{W/V}] - [\underline{V}] - [\underline{W/V}] = [\mathbf{K}_V] - [\underline{V}].$$

Two arbitrary choices are compared by passing to their sum.

Proposition 18.7.2. (Realization and descent for families) Let Λ be a compact Hausdorff space. Every class

$$\kappa \in K_c^0(T^*\mathbb{R}^n \times \Lambda)$$

is the symbol class of a family of operators in $\mathrm{Ell}_c^0(\mathbb{R}^n)$ continuous in the symbol topology, after a fixed stabilization. The index bundle of a family depends only on its symbol class. Consequently, there exists a well-defined homomorphism

$$I_{n,\Lambda} : K_c^0(\mathbb{R}^{2n} \times \Lambda) \longrightarrow K^0(\Lambda) \tag{18.71}$$

sending a class realized by P to $\mathrm{Ind}_{\mathrm{a}}(P)$.

Proof. Extend κ by zero to $T^*S^n \times \Lambda$. The projection of its compact support onto $T^*\mathbb{R}^n$ is compact. We may choose a common neighborhood U of ∞ whose closure does not meet the projection of this support onto S^n. Under the radial identification of T^*S^n with the interior of B^*S^n, the support lies in the complement of the closed set

$$\big(S^*S^n \times \Lambda\big) \cup \big(B^*\overline{U} \times \Lambda\big). \tag{18.72}$$

The description by supports and excision in Theorem A.4.2 allow the class to be represented in the relative group of this pair. Since the total space and the closed subspace are compact Hausdorff, Theorem 17.2.4 provides bundles and an isomorphism over this closed subset, elementary on the second set. No CW structure on Λ is needed. The relative isomorphism extends to a neighborhood of the closed subset by the extension argument in the proof of that theorem. In particular, the relative trivialization is available near $B^*\overline{U} \times \Lambda$. Radial contraction of B^*S^n reduces the bundles, after stabilization, to pullbacks of bundles over $S^n \times \Lambda$. On restriction to $\mathbb{R}^n \times \Lambda$, these bundles are pullbacks of bundles G and H over Λ. The elementary trivialization near ∞ identifies G with H. Choose a complement G' such that $G \oplus G' \cong \underline{\mathbb{C}}^N$ and add the elementary triple defined by G'. Thus, over $\mathbb{R}^n \times \Lambda$ the operators act on a trivial bundle of fixed rank and are the identity on the added summands.

Smoothing must be performed uniformly. Take a finite atlas of S^n and a partition of unity independent of λ, and regularize by convolution only in the variables of S^n and their covariables. Shrink U once and use a cutoff function vanishing near its closure to keep the relative trivialization intact; continue to denote the smaller neighborhood by U. The original isomorphism on the compact set $S^*S^n \times \Lambda$ satisfies

$$\delta := \inf_{(x,\xi,\lambda)\in S^*S^n\times\Lambda} \left\| \theta(x,\xi,\lambda)^{-1} \right\|^{-1}_{\mathrm{op}(\mathbb{C}^N)} > 0.$$

Denote the approximation by $\widetilde{\theta}$ and choose it so that

$$\|\widetilde{\theta} - \theta\|_{C^0(S^*S^n\times\Lambda,\mathrm{End}(\mathbb{C}^N))} < \frac{\delta}{2}.$$

The segment between the two maps remains invertible, so the relative class is unchanged. The smoothed symbol is C^∞ in (x,ξ), continuous in λ in every symbol seminorm, and exactly equal to the identity over $T^*U \times \Lambda$.

Homogeneous extension and quantization are performed with a single cover, a single partition, and the same cutoffs for all parameters. In each chart, Calderón–Vaillancourt estimates the difference of two quantized operators by finitely many seminorms of the difference of their symbols. Therefore,

$$\lambda \longmapsto P_\lambda \quad \text{is continuous in} \quad \mathcal{L}\big(L^2(\mathbb{R}^n;\mathbb{C}^N)\big). \tag{18.73}$$

Choose a single $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on the projection of the common support, and replace the quantized family A_λ by

$$P_\lambda := I + \chi(A_\lambda - I)\chi. \tag{18.74}$$

The Schwartz kernel of $P_\lambda - I$ is contained in $\operatorname{supp}\chi \times \operatorname{supp}\chi$, uniformly in λ. This proves realization.

We now prove descent. The same model by triples applies to the product of the preceding compact Hausdorff pair with $[0,1]$. Thus a relative homotopy of classes is represented, after stabilization, over

$$B^*S^n \times \Lambda \times [0,1]$$

and is elementary over $B^*U \times \Lambda \times [0,1]$. Repeat the preceding smoothing on the compact space $\Lambda \times [0,1]$ and quantize with the same cutoff χ. This gives a norm-continuous family of Fredholm operators. There exists a common finite-dimensional subspace V making all stabilizations surjective, since $\Lambda \times [0,1]$ is compact. Their kernels form a bundle $\mathcal{K} \to \Lambda \times [0,1]$, and the index of the family is

$$[\mathcal{K}] - [\underline{V}] \in K^0(\Lambda \times [0,1]).$$

The inclusions $i_0, i_1 : \Lambda \longrightarrow \Lambda \times [0,1]$ are homotopic; consequently,

$$i_0^*\big([\mathcal{K}] - [\underline{V}]\big) = i_1^*\big([\mathcal{K}] - [\underline{V}]\big).$$

Thus the index bundle is invariant under relative homotopies.

An isomorphism of triples depending on the covariable reduces, through $(x,\xi,\lambda,t) \mapsto (x,t\xi,\lambda)$, to bundle isomorphisms over the base; the procedure is uniform because Λ is compact. An elementary triple contracts in the same way to a family of isomorphisms and has zero index bundle. Direct sum of triples corresponds

to direct sum of stabilized families. Thus the index respects equivalences, homotopies, elementary triples, and sums, which are all the relations in the relative model. This proves that (18.71) is well defined. □

The Bott isomorphism with parameters is

$$\alpha_{n,\Lambda} : K_c^0(\mathbb{R}^{2n} \times \Lambda) \longrightarrow K^0(\Lambda).$$

Theorem 18.7.3. (Euclidean index for families) Under the preceding hypotheses,

$$\operatorname{Ind}_{\mathrm{a}}(P) = (-1)^n \alpha_{n,\Lambda}\big([\sigma(P_\Lambda)]\big) \in K^0(\Lambda). \tag{18.75}$$

Proof. Proposition 18.7.2 constructs the homomorphism $I_{n,\Lambda}$ and proves its independence of the relative representative and quantization. This homomorphism is $K^0(\Lambda)$–linear. Indeed, if $\mathbf{G} \longrightarrow \Lambda$ is a vector bundle, the stabilization defining the index of $P \otimes \mathbf{G}$ is the preceding stabilization tensored with $\mathbf{G}$; consequently,

$$\operatorname{Ind}_{\mathrm{a}}(P \otimes \mathbf{G}) = \operatorname{Ind}_{\mathrm{a}}(P)\,[\mathbf{G}].$$

By Theorem 18.1.2, every class in the domain has a unique expression

$$b^{\boxtimes n} \boxtimes a, \qquad a \in K^0(\Lambda).$$

Iterating the angular regularization of (18.59) and relative realization, we obtain an operator

$$R_n \in \mathrm{Ell}_c^0(\mathbb{R}^n)$$

such that

$$[\sigma(R_n)] = b^{\boxtimes n}, \qquad \operatorname{ind} R_n = (-1)^n.$$

Write $a = [G] - [H]$ and represent G and H, after adding trivial bundles if necessary, by continuous projections

$$p_G : \Lambda \longrightarrow M_N(\mathbb{C}), \qquad p_H : \Lambda \longrightarrow M_N(\mathbb{C}).$$

The family

$$\mathcal{R}_G(\lambda) := R_n \otimes p_G(\lambda) + I \otimes \big(I - p_G(\lambda)\big) \tag{18.76}$$

is the identity on the complement of G_λ. Its symbol class and index are

$$[\sigma(\mathcal{R}_G)] = b^{\boxtimes n} \boxtimes [G], \qquad \operatorname{Ind}_{\mathrm{a}}(\mathcal{R}_G) = (-1)^n [G].$$

For the negative term, use

$$\mathcal{R}_H^-(\lambda) := R_n^* \otimes p_H(\lambda) + I \otimes \big(I - p_H(\lambda)\big). \tag{18.77}$$

Proposition 17.5.3 and the equality $\operatorname{ind} R_n^* = -\operatorname{ind} R_n$ give

$$[\sigma(\mathcal{R}_H^-)] = -b^{\boxtimes n} \boxtimes [H], \qquad \operatorname{Ind}_{\mathrm{a}}(\mathcal{R}_H^-) = -(-1)^n [H].$$

The direct sum of (18.76) and (18.77) realizes $b^{\boxtimes n} \boxtimes a$ and has index $(-1)^n a$. Therefore,

$$I_{n,\Lambda}(b^{\boxtimes n} \boxtimes a) = (-1)^n a.$$

Since

$$\alpha_{n,\Lambda}(b^{\boxtimes n} \boxtimes a) = a,$$

we obtain (18.75). □

18.8 Compatibility with embeddings of closed manifolds

The preceding constructions supply the three maps in the embedding argument. If $j : X \hookrightarrow \mathbb{R}^N$ is an embedding of a closed manifold and $a \in K_c^0(T^*X)$, we have

$$K_c^0(T^*X) \xrightarrow{\mathrm{Th}_{N(Tj)}} K_c^0(N(Tj)) \xrightarrow{e} K_c^0(T^*\mathbb{R}^N) \xrightarrow{(-1)^N \alpha_N} \mathbb{Z}.$$

The composition of the first two arrows is $j_!$, and the last is the Euclidean index. Thus the topological expression associated with a is

$$\mathrm{ind}_{\mathrm{t}}(a) := (-1)^N \alpha_N(j_! a). \tag{18.78}$$

Naturality of Thom, functoriality (18.15), and (18.17) show that (18.78) is independent of the Euclidean space in which the embedding is stabilized. Full independence of the embedding and equality with the analytic index will be proved in Chapter 19. Analytically, the remaining step is to realize the Thom class using the normal oscillator and to prove that extension from a tubular neighborhood changes neither kernel nor cokernel. Topologically, all the ingredients are already contained in Theorems 18.1.2 and 18.2.2 and in functoriality of the pushforward.

19

The index theorem on closed manifolds

Let M be a closed smooth manifold of real dimension n, and let $\mathbf{E}^{+}, \mathbf{E}^{-} \longrightarrow M$ be complex vector bundles. If

$$P \colon \Gamma(M, \mathbf{E}^{+}) \longrightarrow \Gamma(M, \mathbf{E}^{-})$$

is a classical elliptic pseudodifferential operator, Theorem 15.6.22 and Corollary 15.6.23 show that

$$\operatorname{ind}_{\mathrm{a}}(P) = \dim_{\mathbb{C}} \ker P - \dim_{\mathbb{C}} \ker P^{*}$$

is well defined and independent of the Sobolev exponent used to realize P as a bounded operator. Definition 17.4.1 associates a class to its principal symbol,

$$[\sigma(P)] \in K_{c}^{0}(T^{*}M).$$

Theorem 17.5.6, using the stability of Theorem 15.6.24, proves that the index depends only on this class. It remains to identify the resulting homomorphism $K_{c}^{0}(T^{*}M) \longrightarrow \mathbb{Z}$.

The identification will be made using an embedding of M into a Euclidean space. The proof keeps two arguments separate. The first is topological: the Thom isomorphism and extension by zero produce an integer from a compactly supported class in $T^{*}M$. The second is analytic: a Bott operator in the normal directions allows an elliptic operator on M to be transferred to a tubular neighborhood without changing its index. The Euclidean formula of Chapter 18 then completes the comparison. This is the embedding argument of Atiyah and Singer; our reference exposition is [21, Chapter 12, pp. 284–301].

Atiyah and Singer: two ways to count

The theorem announced by Michael Atiyah and Isadore Singer in the 1960s states that two constructions of very different natures produce the same integer. The analytic index counts solutions and obstructions of an elliptic equation; the topological index combines the symbol class with operations in K–theory. The equality is more than a computational formula: it explains why global data of an equation remain invariant under deformation and why geometric operators recover invariants such as the Euler characteristic or characteristic genera [11].

19.1 The topological index

Fix a Riemannian metric on M. The musical isomorphism

$$\sharp_g \colon T^*M \longrightarrow TM, \qquad \xi \longmapsto \xi^{\sharp_g},$$

induces an isomorphism $K_c^0(T^*M) \cong K_c^0(TM)$. Although it will be used to write the construction, the final result is independent of the metric: two metrics are joined by the convex segment $g_t = (1-t)g_0 + tg_1$, and the corresponding musical isomorphisms form a proper homotopy of bundles.

By the Whitney embedding theorem in the form [21, Theorem 6.7(c), pp. 162–163], there exists a smooth embedding

$$j \colon M \hookrightarrow \mathbb{R}^N$$

for some N. Let $\nu \longrightarrow M$ be its normal bundle, of rank $r = N - n$. Theorem 2.12.3 provides open subsets $V \subseteq \nu$ and $U \subseteq \mathbb{R}^N$ containing, respectively, the zero section and $j(M)$, and a diffeomorphism

$$\Theta \colon V \longrightarrow U, \qquad \Theta(x, 0) = j(x).$$

The differential $Tj \colon TM \hookrightarrow T\mathbb{R}^N$ is also an embedding. The differential of the projection of ν defines a real vector bundle

$$T\pi_\nu \colon T\nu \longrightarrow TM$$

of rank $2r$: the fiber over $X \in T_xM$ consists of the vectors tangent to ν that project to X. A normal connection separates the horizontal and vertical directions and gives an identification

$$T\nu \cong \pi_{TM}^*(\nu \oplus \nu)$$

as bundles over TM. This is the normal bundle of $Tj(TM)$ in $T\mathbb{R}^N$ under the differential of a tubular identification; in the notation of Definition 18.3.1, $N(Tj) \cong (T\nu, T\pi_\nu)$. The identification is not canonical, but two connections give homotopic identifications. The complex structure

$$J(u, v) = (-v, u), \qquad (u, v) \in \nu_x \oplus \nu_x, \tag{19.1}$$

, however, determines a canonical complex orientation up to homotopy. By Theorem 18.2.2, we obtain the Thom isomorphism

$$\mathrm{Th}_{T\nu} \colon K_c^0(TM) \longrightarrow K_c^0(T\nu).$$

The tubular neighborhood identifies a neighborhood of TM in $T\mathbb{R}^N$ with a neighborhood of the zero section of $T\nu$. Extension by zero then defines

$$\mathrm{ext} \colon K_c^0(T\nu) \longrightarrow K_c^0(T\mathbb{R}^N).$$

For the structure (19.1), the complex coordinate is $u + iv$ and the Thom class of a line is $\lambda_{\mathbb{C}} = -b$ with the normalization of Definition 18.1.1. Thus, in rank s, we have simultaneously

$$\lambda_{\mathbb{C}^s} = (-1)^s b^{\boxtimes s}, \qquad \mathrm{ch}_c(\lambda_{\mathbf{W}}) = (-1)^s \Phi_{\mathbf{W}}\left(\mathrm{Td}(\overline{\mathbf{W}})^{-1}\right). \tag{19.2}$$

The complex orientation and Thom class are fixed by (19.2).

In accordance with Definition 18.3.1, and using the Euclidean metric to identify the last tangent bundle with the cotangent bundle, the *pushforward* of the embedding is

$$j_! := \mathrm{ext} \circ \mathrm{Th}_{T\nu} \circ (\sharp_g)_* \colon K_c^0(T^*M) \longrightarrow K_c^0(T^*\mathbb{R}^N). \tag{19.3}$$

Let $i_N \colon \{0\} \hookrightarrow \mathbb{R}^N$ be the inclusion. With the conventions of Chapter 18, if $b \in K_c^0(\mathbb{R}^2)$ is the Bott class and $\alpha_N \colon K_c^0(\mathbb{R}^{2N}) \longrightarrow \mathbb{Z}$ is the periodicity isomorphism, then

$$(i_N)_!(1) = (-1)^N b^{\boxtimes N}, \qquad (i_N)_!^{-1} = (-1)^N \alpha_N. \tag{19.4}$$

These are precisely identities (18.16) and (18.17). The sign comes from the complex structure (19.1): in rank one, the Thom bundle homomorphism is $z = v + i\eta$, whereas b was normalized using the gluing function z^{-1}.

Definition 19.1.1. (Topological index) Let M be a closed smooth manifold and fix a smooth embedding $j \colon M \hookrightarrow \mathbb{R}^N$. For $a \in K_c^0(T^*M)$ define

$$\mathrm{ind}_{\mathrm{t}}(a) := (i_N)_!^{-1}(j_! a) \in \mathbb{Z}. \tag{19.5}$$

The definition appears to depend on the metric, embedding, and tubular neighborhood. This dependence is only apparent.

Proposition 19.1.2. (Independence of choices) Let M be a closed smooth manifold. The integer in (19.5) is independent of the metric, normal connection, tubular neighborhood, and embedding $j \colon M \hookrightarrow \mathbb{R}^N$. Moreover,

$$\mathrm{ind}_{\mathrm{t}} \colon K_c^0(T^*M) \longrightarrow \mathbb{Z}$$

is a homomorphism.

Independence under stabilization is stated in [21, Definition 12.11 and Exercise 12.10, p. 290]; a detailed proof appears in [92, Chapter III, §13, Definition 13.1].

Proof. Independence of the metric was already obtained using the homotopy g_t. The space of connections on a fixed bundle is affine; thus two normal connections are joined by a segment and give homotopic Thom classes. Two tubular neighborhoods admit, after restricting their domains, an isotopy fixing the zero section. Homotopy invariance and excision in compactly supported K–theory show that the extensions by zero agree. These topological results are stated in Theorem A.6.1 of Appendix A.

It remains to compare two embeddings $j_0 \colon M \hookrightarrow \mathbb{R}^{N_0}$ and $j_1 \colon M \hookrightarrow \mathbb{R}^{N_1}$. Their stabilizations in $\mathbb{R}^{N_0+N_1}$ are joined by the family

$$j_\theta(x) = \big((\cos\theta) j_0(x), (\sin\theta) j_1(x)\big), \qquad 0 \le \theta \le \frac{\pi}{2}.$$

Each j_θ is an embedding. At the endpoints this is immediate; if both components are nonzero, injectivity of either component proves injectivity, and the same observation applied to the differentials proves that it is an immersion. Since M is compact, an injective immersion into a Hausdorff space is an embedding. Isotopy invariance gives equality of the stabilized pushforwards. By functoriality of the Thom isomorphism,

$$(\iota \circ j)_! = \iota_! \circ j_!$$

for a linear inclusion ι between Euclidean spaces. The same identity holds for inclusions of points; hence the Bott factors introduced by stabilization cancel when $(i_N)_!^{-1}$ is applied. This proves independence of the embedding. Additivity follows from additivity of the Thom isomorphism, extension by zero, and $(i_N)_!^{-1}$. □

19.2 Excision and products for the analytic index

Passing from M to a tubular neighborhood requires two analytic properties. The first allows a symbol supported in an open subset to be transferred without changing the index; the second calculates the index of an external product.

Proposition 19.2.1. (Analytic excision) Let X_0 and X_1 be manifolds without boundary and let U be an open set identified with open subsets of both. For $i = 0, 1$, let $\mathbf{E}_i, \mathbf{F}_i \to X_i$ be complex vector bundles equipped with Hermitian bundle metrics and identified over U, and let $R_i \colon \mathbf{E}_i \to \mathbf{F}_i$ be a bundle homomorphism inducing a bounded multiplication operator and corresponding, over U, to a common homomorphism R. Suppose that there exist $L \Subset U$ such that R_i is invertible outside L and a bounded operator

$$B \colon L^2(U, \mathbf{E}|_U) \longrightarrow L^2(U, \mathbf{F}|_U)$$

whose Schwartz kernel is supported in $L \times L$ and agrees under the identifications. If

$$P_i := R_i + \operatorname{ext}_i B \colon L^2(X_i, \mathbf{E}_i) \longrightarrow L^2(X_i, \mathbf{F}_i)$$

is Fredholm for $i = 0, 1$, then

$$\operatorname{ind} P_0 = \operatorname{ind} P_1. \tag{19.6}$$

In particular, the analytic index of a class in $K_c^0(T^*U)$ extended by zero depends only on a neighborhood of the projection of its support.

Proof. Choose on U a Riemannian metric, its associated measure, and Hermitian metrics on the bundles, and extend the metrics to X_0 and X_1. Thus the identifications over U preserve the L^2 inner products and adjoints. If $P_i u = 0$, then $R_i u = 0$ outside L; invertibility of R_i there implies $\operatorname{supp} u \subseteq L$. The same argument for P_i^* shows $\operatorname{supp} v \subseteq L$ for every $v \in \ker P_i^*$.
Over U, the operators P_0 and P_1 agree with $R + B$, and the same holds for their adjoints. Restriction therefore identifies $\ker P_0$ with $\ker P_1$ and $\ker P_0^*$ with $\ker P_1^*$. Subtracting their dimensions proves (19.6).
For a class $a \in K_c^0(T^*U)$, the extension-by-zero construction allows us, after stabilization, to choose a localized representative of this form: if A is a quantization and R its invertible part outside a compact set, take $Q := R + \chi(A - R)\chi$ with $\chi \in C_c^\infty(U)$ equal to one near the support. The operator $B := Q - R$ has kernel supported in a compact subset of $U \times U$, and the preceding argument applies to its two extensions. □

To state the second property, it is convenient to use graded operators. If $P \colon H_P^+ \longrightarrow H_P^-$ and $Q \colon H_Q^+ \longrightarrow H_Q^-$ are closed densely defined operators, write

$$D_P = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}, \qquad D_Q = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}, \qquad \Gamma_P = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The odd operator

$$D_P \widehat{\otimes} I + \Gamma_P \widehat{\otimes} D_Q \tag{19.7}$$

acts on the graded tensor product. Its component from even to odd degree is

$$P\#Q = \begin{pmatrix} P \otimes I & -I \otimes Q^* \\ I \otimes Q & P^* \otimes I \end{pmatrix}. \tag{19.8}$$

Theorem 19.2.2. (Analytic multiplicativity) Let $P \in \Psi^1_{\mathrm{cl}}(M;\mathbf{E}^+,\mathbf{E}^-)$ and $Q \in \Psi^1_{\mathrm{cl}}(N;\mathbf{F}^+,\mathbf{F}^-)$ be elliptic operators on closed smooth manifolds M and N, respectively. The closure of the operator $P\#Q$ in (19.8) is Fredholm and

$$\operatorname{ind_a}(P\#Q) = \operatorname{ind_a}(P)\operatorname{ind_a}(Q). \tag{19.9}$$

Proof. Set

$$\mathcal{H}_P := L^2(M,\mathbf{E}^+) \oplus L^2(M,\mathbf{E}^-), \qquad \mathcal{H}_Q := L^2(N,\mathbf{F}^+) \oplus L^2(N,\mathbf{F}^-), \qquad \mathcal{H} := \mathcal{H}_P \widehat{\otimes} \mathcal{H}_Q,$$

with their natural orthogonal sums and gradings. The anticommutation relation $D_P\Gamma_P + \Gamma_P D_P = 0$ implies

$$\left(D_P \widehat{\otimes} I + \Gamma_P \widehat{\otimes} D_Q\right)^2 = D_P^2 \widehat{\otimes} I + I \widehat{\otimes} D_Q^2. \tag{19.10}$$

For every vector u in the algebraic tensor product of smooth sections,

$$\begin{aligned} &\left\langle \left(D_P^2 \widehat{\otimes} I + I \widehat{\otimes} D_Q^2\right) u, u \right\rangle_{\mathcal{H}} \\ &\qquad = \left\| (D_P \widehat{\otimes} I) u \right\|_{\mathcal{H}}^2 + \left\| (I \widehat{\otimes} D_Q) u \right\|_{\mathcal{H}}^2. \end{aligned}$$

Let us specify the closure used in these identities. Elliptic regularity and Rellich imply that D_P and D_Q, with domain H^1 on their respective bases, are self-adjoint with compact resolvent. Let $(\lambda_j)_{j\in I}$ and $(\mu_k)_{k\in J}$ be the distinct eigenvalues of D_P^2 and D_Q^2, respectively, and let V_j and W_k be their eigenspaces. The index sets I, J are finite or countable. Every eigenspace is finite-dimensional and invariant under the corresponding operator and its grading. On $V_j \widehat{\otimes} W_k$, the operator in (19.7) is a self-adjoint matrix whose square is $(\lambda_j + \mu_k)I$. The orthogonal sum of these matrices is self-adjoint on the domain

$$\left\{ \mathbf{u} = \sum_{(j,k)\in I\times J} \mathbf{u}_{jk} \;\middle|\; \sum_{(j,k)\in I\times J} (\lambda_j + \mu_k)\|\mathbf{u}_{jk}\|^2 < \infty \right\},$$

where the first sum converges in $\mathcal{H}$. Vectors with finitely many components form a core in the graph norm, by truncation of the second sum. This proves that the orthogonal sum is the closure of the product initially defined on the algebraic tensor product of smooth sections. For every $R > 0$, there are only finitely many pairs (j,k) with $\lambda_j + \mu_k \le R$, since both spectral sequences have this property. Zero therefore has finite multiplicity and is separated from the rest of the spectrum. The product is Fredholm. The only blocks contributing to its kernel are those satisfying $\lambda_j = \mu_k = 0$. Hence

$$\ker\left(D_P \widehat{\otimes} I + \Gamma_P \widehat{\otimes} D_Q\right) = \ker D_P \widehat{\otimes} \ker D_Q \subset \mathcal{H}.$$

Separating even and odd components gives

$$\ker(P\#Q) \cong (\ker P \otimes \ker Q) \oplus (\ker P^* \otimes \ker Q^*),$$
$$\ker(P\#Q)^* \cong (\ker P^* \otimes \ker Q) \oplus (\ker P \otimes \ker Q^*).$$

Subtracting dimensions yields

$$\big(\dim\ker P - \dim\ker P^*\big)\big(\dim\ker Q - \dim\ker Q^*\big),$$

which is (19.9). □

19.3 The normal Bott operator

The preceding multiplicativity must be applied uniformly in the fibers of the normal bundle. We begin with the model on $\mathbb{R}^r$. On $\Lambda^*\mathbb{C}^r$ let ε_j be exterior multiplication by the jth basis vector and $\iota_j = \varepsilon_j^*$ contraction. On the Schwartz class define

$$\mathcal{B}_r = \sum_{j=1}^{r} \left(\varepsilon_j \left(\frac{\partial}{\partial v_j} + v_j \right) + \iota_j \left(-\frac{\partial}{\partial v_j} + v_j \right) \right). \tag{19.11}$$

This is an odd symmetric operator. Denote by $\mathcal{B}_r^+$ its component from even to odd forms.

Lemma 19.3.1. (Kernel and index of the Bott operator) The closure of $\mathcal{B}_r$ is self-adjoint, has compact resolvent, and

$$\ker \mathcal{B}_r = \mathbb{C} e^{-\frac{|v|^2}{2}} \otimes \Lambda^0 \mathbb{C}^r.$$

In particular,

$$\operatorname{ind}_a(\mathcal{B}_r^+) = 1.$$

Its total symbol in the variables $(v,\eta) \in \mathbb{R}^r \oplus \mathbb{R}^r$ is

$$\beta_r(v,\eta) = \varepsilon(v+i\eta) + \varepsilon(v+i\eta)^*, \tag{19.12}$$

where $\varepsilon(z) = \sum_{j=1}^{r} z_j \varepsilon_j$ and $\varepsilon(z)^* = \sum_{j=1}^{r} \overline{z_j} \iota_j$. Its component from even to odd degree represents the Thom class of $\mathbb{R}^r \oplus \mathbb{R}^r$ with the complex structure $J(v,\eta) = (-\eta, v)$.

Proof. The relations

$$\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0, \qquad \iota_i\iota_j + \iota_j\iota_i = 0, \qquad \varepsilon_i\iota_j + \iota_j\varepsilon_i = \delta_{ij} I$$

give, on $\mathcal{S}(\mathbb{R}^r, \Lambda^*\mathbb{C}^r)$,

$$\mathcal{B}_r^2 = -\Delta_v + |v|^2 - r + 2\mathcal{N}, \qquad \mathcal{N} = \sum_{j=1}^{r} \varepsilon_j \iota_j. \tag{19.13}$$

The operator $\mathcal{N}$ acts by multiplication by q on $\Lambda^q\mathbb{C}^r$. The scalar oscillator $-\Delta_v + |v|^2$ has eigenvalues $r + 2|\alpha|$, $\alpha \in \mathbb{N}_0^r$, and its ground state is $e^{-\frac{|v|^2}{2}}$. Thus the eigenvalues of (19.13) are $2|\alpha| + 2q$. Zero occurs exactly once, for $\alpha = 0$ and $q = 0$. The Hermite function basis proves both essential self-adjointness and compactness of

the resolvent.

The first-order part together with the linear potential gives (19.12). Since

$$\beta_r(v,\eta)^2 = (|v|^2 + |\eta|^2)I,$$

its restriction to the sphere is an isomorphism between the even and odd parts. To compare it with the sign in Definition 18.2.1, set $z = v + i\eta$ and consider $c_t(z) = \varepsilon(z) + e^{i\pi t}\varepsilon(z)^*$, $0 \le t \le 1$. The creation and annihilation relations give $c_t(z)^2 = e^{i\pi t}|z|^2 I$. Thus the even-to-odd components are invertible when $z \neq 0$ and join the preceding symbol to $\varepsilon(z) - \varepsilon(z)^*$, which defines the Thom class. The normalization agrees with (19.4): for $r = 1$, the bundle homomorphism is $v + i\eta$ and represents $-b$. □

Formula (19.11) is invariant under the natural action of $O(r)$. Indeed, differentiation, multiplication by v, and the exterior and interior operations transform simultaneously. This equivariance allows us to pass from the vector-space model to the normal bundle without choosing global frames.

Existence of a global representative on the fiber compactification uses the equivariant Bott normalization for the standard representation of $O(r)$, stated in Theorem A.5.2. The result provides the class and its index in $R(O(r))$; the next lemma develops the analytic normalization we need and adapts it to the normal bundle. Before stating it, let us specify the compactification used in this step.

Let $G = O(r)$ and let $S^r = \mathbb{R}^r \cup \{\infty\}$, with the action of G fixing ∞. Denote the orthonormal frame bundle of ν by $\mathcal{P}_\nu \longrightarrow M$. The fiberwise spherical compactification is the closed manifold

$$\widehat{\nu} := \mathcal{P}_\nu \times_G S^r, \qquad \widehat{\pi} : \widehat{\nu} \longrightarrow M. \tag{19.14}$$

The section $\mathbf{s}_\infty = \mathcal{P}_\nu \times_G \{\infty\}$ is the complement of the open copy of ν in $\widehat{\nu}$.

Lemma 19.3.2. (Equivariant normal representative) Let S^r be the closed smooth sphere and let $G = O(r)$ with the action described above. There exist complex bundles $\mathbf{W}^+, \mathbf{W}^- \longrightarrow S^r$ with an action of G and a classical elliptic G–equivariant pseudodifferential operator of order one

$$B \colon \Gamma(S^r, \mathbf{W}^+) \longrightarrow \Gamma(S^r, \mathbf{W}^-)$$

such that its symbol class is the extension by zero from $T^*\mathbb{R}^r$ of the class of (19.12). The equality holds in equivariant K–theory. Moreover, after modifying B by an G–equivariant smoothing operator, we may assume that

$$\ker B = \mathbb{C}, \qquad \ker B^* = 0, \tag{19.15}$$

where G acts trivially on $\mathbb{C}$.

Proof. The bounded transform

$$F_r^+ = \mathcal{B}_r^+ \left(I + \mathcal{B}_r^- \mathcal{B}_r^+\right)^{-\frac{1}{2}}$$

is Fredholm and G–equivariant. The calculation in Lemma 19.3.1 shows, as representations of G and not merely after taking dimensions, that

$$[\ker F_r^+] - [\ker (F_r^+)^*] = [\mathbb{C}] \quad \text{in } R(G). \tag{19.16}$$

To pass to the compactification, apply Theorem A.5.2 of Appendix A. If $i : \{0\} \hookrightarrow \mathbb{R}^r$, this result provides an elliptic G–equivariant representative of $\beta_r = i_!(1)$ on S^r, elementary near ∞, and establishes

$$\operatorname{Ind}_G(\beta_r) = [\mathbb{C}] \quad \text{in } R(G), \tag{19.17}$$

with the trivial representation on the right-hand side.

The normalization in (19.17) agrees with the one adopted here. Indeed, the local representative is (19.12), and (19.16) calculates its index as a representation, not just its dimension: the Gaussian is fixed by $O(r)$ and the cokernel is zero. This rules out the determinant representation and fixes the right-hand side of (19.17).

Let B_0 be an order-zero quantization of the symbol provided by that theorem. We can make it G–equivariant by averaging its conjugates with respect to normalized Haar measure on G: the principal symbol remains unchanged because it is equivariant. Then

$$[\ker B_0] - [\ker B_0^*] = [\mathbb{C}] \quad \text{in } R(G).$$

Since G is compact, its finite-dimensional representations are semisimple. The preceding equality provides a G–equivariant decomposition

$$\ker B_0 \cong \mathbb{C} \oplus \ker B_0^*.$$

Let K be the operator identifying the second summand with $\ker B_0^*$ and vanishing on its orthogonal complement. It has finite rank and is G–equivariant. Elliptic regularity of both kernels shows that its integral kernel is smooth. The operator $B_1 := B_0 + K$ is surjective: B_0 identifies $(\ker B_0)^\perp$ with its closed image, and K covers the orthogonal complement of that image. Its kernel is the first copy of $\mathbb{C}$. Finally, compose on the right with the positive, invertible, G–equivariant operator

$$\Lambda := \left(I + (\nabla^{\mathbf{W}^+})^* \nabla^{\mathbf{W}^+}\right)^{1/2},$$

of order one, using the round metric and a G–invariant Hermitian metric and connection on $\mathbf{W}^+$. The map $\Lambda : H^1(S^r, \mathbf{W}^+) \to L^2(S^r, \mathbf{W}^+)$ is an isomorphism by the calculus of powers in Chapter 15. Thus $B := B_1 \Lambda$ has the same image as B_1 and $\ker B = \Lambda^{-1}(\ker B_1)$. This identification is G–equivariant and preserves the one-dimensional trivial kernel. The positive principal symbol of Λ does not change the symbol class after normalizing the order. This gives the operator B in the statement. □

Fix a metric h_ν on ν. A metric connection on ν splits the cotangent bundle of $\widehat{\nu}$ into horizontal and vertical parts,

$$T^*\widehat{\nu} = \mathbf{H}^* \oplus \mathbf{V}^*, \qquad \mathbf{H}^* \cong \widehat{\pi}^* T^* M, \qquad \mathbf{V}^* \cong \boldsymbol{\mathcal{P}}_\nu \times_G T^* S^r. \tag{19.18}$$

Denote by $|\cdot|_{\mathbf{H}}$ and $|\cdot|_{\mathbf{V}}$ the norms induced on $\mathbf{H}^*$ and $\mathbf{V}^*$, and by $|\cdot|_{\widehat{g}}$ the norm of the orthogonal sum. The operator B induces a vertical operator B_ν on $\widehat{\nu}$, between the associated bundles

$$\mathbf{W}_\nu^\pm := \boldsymbol{\mathcal{P}}_\nu \times_G \mathbf{W}^\pm \longrightarrow \widehat{\nu}.$$

Its kernel is the trivial line over M and its cokernel is zero, by (19.15).

The product construction will use cutoffs depending on the vertical operator. These cutoffs have two properties that we will need simultaneously: they preserve its spectral subspaces and turn local products into classical operators on the total manifold. We begin by specifying the domain corresponding to the latter.

Lemma 19.3.3. (Domain of a classical elliptic operator) Let X be a closed smooth manifold and let $\mathbf{E},\mathbf{F}\to X$ be complex vector bundles equipped with Hermitian bundle metrics. If $A\in\Psi^{1}_{\mathrm{cl}}(X;\mathbf{E},\mathbf{F})$ is elliptic, its closure in L^2 has domain $H^1(X,\mathbf{E})$, and the domain of its Hilbert space adjoint is $H^1(X,\mathbf{F})$. There exist constants $C_0,C_1>0$ such that, for every $\mathbf{u}\in H^1(X,\mathbf{E})$,

$$\|A\mathbf{u}\|_{L^2(X,\mathbf{F})}\le C_0\|\mathbf{u}\|_{H^1(X,\mathbf{E})}, \tag{19.19}$$

$$\|\mathbf{u}\|_{H^1(X,\mathbf{E})}\le C_1\left(\|A\mathbf{u}\|_{L^2(X,\mathbf{F})}+\|\mathbf{u}\|_{L^2(X,\mathbf{E})}\right). \tag{19.20}$$

The corresponding assertions hold for the formal adjoint. In particular, a formally self-adjoint classical elliptic operator of order one is self-adjoint with domain H^1.

Proof. The first inequality is Corollary 15.6.6. A parametrix $Q\in\Psi^{-1}_{\mathrm{cl}}(X;\mathbf{F},\mathbf{E})$ satisfies $QA=I-K$, where K is smoothing. Applying the identity $\mathbf{u}=QA\mathbf{u}+K\mathbf{u}$ and the same corollary yields (19.20). If $\mathbf{u}_j\in\Gamma(\mathbf{E})$ converges to $\mathbf{u}$ in $H^1(X,\mathbf{E})$, the first bound gives $A\mathbf{u}_j\to A\mathbf{u}$ in $L^2(X,\mathbf{F})$; thus $\mathbf{u}$ belongs to the domain of the closure. Conversely, if $(\mathbf{u}_j,A\mathbf{u}_j)$ converges in $L^2(X,\mathbf{E})\times L^2(X,\mathbf{F})$ to $(\mathbf{u},\mathbf{f})$, the second bound applied to $\mathbf{u}_j-\mathbf{u}_k$ proves that $(\mathbf{u}_j)$ is Cauchy in $H^1(X,\mathbf{E})$. Its limit in that space agrees with $\mathbf{u}$ by the continuous inclusion into $L^2(X,\mathbf{E})$; the first bound identifies $A\mathbf{u}=\mathbf{f}$. This proves both inclusions for the domain.

If $\mathbf{v}$ belongs to the domain of the Hilbert space adjoint, the identity defining that adjoint says that $A_h^{*}\mathbf{v}$ belongs to L^2 in the distributional sense. The parametrix of the classical elliptic operator A_h^{*} then gives $\mathbf{v}\in H^1(X,\mathbf{F})$. Conversely, let $\mathbf{v}\in H^1(X,\mathbf{F})$ and take $\mathbf{v}_j\in\Gamma(\mathbf{F})$ with $\mathbf{v}_j\to\mathbf{v}$ in $H^1(X,\mathbf{F})$. The mapping bound for A_h^{*} gives $A_h^{*}\mathbf{v}_j\to A_h^{*}\mathbf{v}$ in $L^2(X,\mathbf{E})$. For each $\mathbf{u}\in\Gamma(\mathbf{E})$, Cauchy–Schwarz allows us to pass to the limit in

$$\langle A\mathbf{u},\mathbf{v}_j\rangle_{L^2(X,\mathbf{F})}=\langle\mathbf{u},A_h^{*}\mathbf{v}_j\rangle_{L^2(X,\mathbf{E})}.$$

The resulting identity and $|\langle\mathbf{u},A_h^{*}\mathbf{v}\rangle_{L^2(X,\mathbf{E})}|\le\|\mathbf{u}\|_{L^2(X,\mathbf{E})}\|A_h^{*}\mathbf{v}\|_{L^2(X,\mathbf{E})}$ are precisely the definition of the domain of the Hilbert space adjoint. The two operators agree there. If $A_h^{*}=A$, their domains and actions agree, which is the asserted self-adjointness. □

Lemma 19.3.4. (Compactness from the first-order domain) Let X be a closed smooth manifold and let $\mathbf{E},\mathbf{F}\to X$ be complex vector bundles equipped with Hermitian bundle metrics. If $T:L^2(X,\mathbf{E})\to L^2(X,\mathbf{F})$ is bounded, its restriction is compact as a map

$$T:H^1(X,\mathbf{E})\longrightarrow L^2(X,\mathbf{F}).$$

This applies, in particular, to classical operators of order zero.

Proof. The restriction is the composition of the compact inclusion $H^1(X,\mathbf{E})\hookrightarrow L^2(X,\mathbf{E})$, given by Theorem 9.2.2, with the bounded operator T. For classical operators of order zero, boundedness on L^2 is Corollary 15.6.6. □

Lemma 19.3.5. (Spectral cutoffs on a local product) Let $n\in\mathbb{N}$, let Z be a closed smooth manifold, and let $\mathbf{W}\to Z$ be a complex vector bundle equipped with a Hermitian bundle metric. Suppose that $D\in\Psi^{1}_{\mathrm{cl}}(Z;\mathbf{W},\mathbf{W})$ is elliptic and self-adjoint with domain $H^1(Z,\mathbf{W})$. Set $L=I+D^2$ and denote the principal

symbol of D by $b_1(z,\eta)$. Fix $h \in C_c^\infty(\mathbb{R})$ such that

$$0 \le h \le 1, \qquad h(t) = 1 \quad (-1 \le t \le 1), \qquad h(t) = 0 \quad (t \ge 4).$$

For $R \ge 1$ define Φ_R using the partial Fourier transform in $x \in \mathbb{R}^n$:

$$\widehat{\Phi_R \mathbf{u}}(\xi) = h\left(\frac{R^2 L}{1+|\xi|^2}\right)\widehat{\mathbf{u}}(\xi), \qquad \xi \in \mathbb{R}^n. \tag{19.21}$$

It is self-adjoint and $0 \le \Phi_R \le I$ on $L^2(\mathbb{R}^n \times Z, \mathbf{W})$. After multiplication on the left and right by smooth compactly supported functions in x, the operators

$$\Phi_R, \qquad (P \otimes I)\Phi_R, \qquad \Phi_R(P \otimes I), \qquad (I \otimes D)(I - \Phi_R)$$

are classical in the joint variables (x,z), of orders $0,1,1,1$, respectively, for every classical operator P of order one in $\mathbb{R}^n$ between trivial bundles of finite rank. Identities on the additional factors are understood in the formulas.

The principal symbol of the cutoff is

$$f_R(z;\xi,\eta) = h\left(\frac{R^2 b_1(z,\eta)^2}{|\xi|^2}\right) \quad (\xi \ne 0), \tag{19.22}$$

extended by zero when $\xi = 0$ and $\eta \ne 0$. Near the axis $\eta = 0$, with $\xi \ne 0$, it is the identity. The other three principal symbols are $p_1 f_R$, $f_R p_1$, and $b_1(I - f_R)$.

Every bounded operator in the vertical variable commuting with the resolvent of D commutes with Φ_R. Every grading anticommuting with D also commutes with Φ_R. If a compact group acts unitarily on $\mathbf{W}$ and D is equivariant, the cutoffs are equivariant.

Proof. By the spectral theorem, for every $\xi \in \mathbb{R}^n$ the operator $h(R^2 L/(1+|\xi|^2))$ is a positive self-adjoint contraction. Plancherel in the variable x proves the same properties for Φ_R. The commutation relations follow from functional calculus; if $\Gamma D = -D\Gamma$, then Γ commutes with L and all its spectral functions.

Let us prove the assertion about joint symbols. We use the construction of the resolvent parametrix in Proposition 15.7.6, retaining the dependence of the bounds on distance to the real axis. Write

$$\tau = R^{-1}(1+|\xi|^2)^{1/2}, \qquad h\left(\frac{R^2 L}{1+|\xi|^2}\right) = h(\tau^{-2} L).$$

In a chart of Z, the symbol of L has an expansion $\ell \sim \sum_{j=0}^{\infty} \ell_{2-j}$, whose first term is $\ell_2 = b_1^2$. It is a positive Hermitian matrix for $\eta \ne 0$. Compactness and ellipticity give constants $c_Z, C_Z > 0$ such that

$$c_Z |\eta|^2 I \le \ell_2(z,\eta) \le C_Z |\eta|^2 I. \tag{19.23}$$

To integrate the resolvent against a smooth function, fix $N \in \mathbb{N}$ and a cutoff $\vartheta \in C_c^\infty(\mathbb{R})$ equal to one near

zero. With $w = a + it$ set

$$\widetilde{h}_N(a+it) = \vartheta(t) \sum_{j=0}^{N} \frac{(it)^j}{j!} h^{(j)}(a).$$

The operator $\partial_{\overline{w}} = \frac{1}{2}(\partial_a + i\partial_t)$ satisfies $|\partial_{\overline{w}} \widetilde{h}_N(a+it)| \le C_N |t|^N$ for $|t|$ sufficiently small. The Cauchy–Green formula, first for a real number and then integrated against the spectral resolution of L, gives

$$h(\tau^{-2}L) = \frac{1}{\pi} \int_{\mathbb{R}^2} \partial_{\overline{w}} \widetilde{h}_N(w)\, \tau^2 (L - \tau^2 w)^{-1}\, da\, dt. \tag{19.24}$$

The integrand is considered for $t \neq 0$; the set $t = 0$ has measure zero. The spectral bound

$$\|\tau^2 (L - \tau^2 w)^{-1}\|_{\mathcal{L}(L^2(Z,\mathbf{W}))} \le |t|^{-1}$$

justifies the integral in operator norm when $N \ge 1$.

The symbol recurrence of Chapter 15 now applies to $L - \tau^2 w$. In the region $\eta \neq 0$ it starts with

$$r_{-2} = (\ell_2 - \tau^2 w)^{-1}$$

and, for $j \in \mathbb{N}$, continues by

$$r_{-2-j} = -r_{-2} \sum_{\substack{a,b \in \mathbb{N}_0,\ \gamma \in \mathbb{N}_0^{\dim Z} \\ a+b+|\gamma| = j,\ b<j}} \frac{i^{-|\gamma|}}{\gamma!} (\partial_\eta^\gamma \ell_{2-a})(\partial_z^\gamma r_{-2-b}). \tag{19.25}$$

On an annulus where $|\eta|$ and τ are comparable, all derivatives of r_{-2-j} satisfy the parameter-dependent estimates of order $-2-j$. Their constants are bounded by a power of $|t|^{-1}$: this is seen first for r_{-2} from (19.23), and when the inverse is differentiated, each new derivative adds finitely many factors r_{-2}. The finite sum in (19.25) retains this property. Specifically, for every finite set of derivatives, there exists an integer m_N bounding the power of $|t|^{-1}$ needed both in the finite parametrix and in its remainder.

In (19.24) choose an extension of order greater than $m_N + 2$. We can then integrate each term and its remainder. The left and right parametrix identities, followed by the bound for the exact resolvent, control the difference between the two resolvents: if $(L - \tau^2 w) Q_N = I - S_N$, then

$$(L - \tau^2 w)^{-1} - Q_N = (L - \tau^2 w)^{-1} S_N.$$

Choose the expansion length so that S_N has the required negative order before integration. This produces, on the annulus $|\eta| \asymp \tau$, a classical expansion whose first term is $h(\ell_2/\tau^2)$.

Let us also examine what happens outside this annulus. Every term in the recurrence is a rational function of w; its poles lie in the spectrum of ℓ_2/τ^2. If $|\eta|/\tau$ is sufficiently small, this spectrum lies in an interval where $h = 1$. The integral of the first term is then I. Those of the subsequent terms vanish: they are integrals of products of resolvents whose expansion in w starts with w^{-2}, and integrating a constant function around all its poles gives zero. This argument remains valid for noncommuting matrices, since it applies entry by entry to the complete rational function. If $|\eta|/\tau$ is sufficiently large, h vanishes near all the poles and all the integrals vanish. Introducing cutoffs in these three regions leaves the terms differentiating the cutoffs in the transition annuli.

To control the remainders near the axis $\eta = 0$ as well, separate the frequencies $|\eta| \le 2\tau^{1/2}$, assuming first that $\tau \ge 1$. Let Q_τ be a vertical localization whose symbol is one for $|\eta| \le \tau^{1/2}$ and zero for $|\eta| \ge 2\tau^{1/2}$, with the spatial cutoffs of the chart. For $N \in \mathbb{N}$ define

$$g_N(t) = \frac{h(t) - 1}{t^N} \quad (t > 0), \qquad g_N(0) = 0.$$

This function is smooth and bounded on $[0, +\infty)$, since its numerator vanishes near zero. Spectral calculus gives the exact identity

$$\big(h(\tau^{-2}L) - I\big)Q_\tau = \tau^{-2N} g_N(\tau^{-2}L) L^N Q_\tau.$$

Vertical Sobolev norms can be expressed using powers of L, by Proposition 15.7.10. Thus $g_N(\tau^{-2}L)$ is uniformly bounded on each of these spaces. For $s \in \mathbb{N}_0$, symbol calculus and the bound $\langle \eta \rangle \le C\tau^{1/2}$ on the cutoff support give

$$\|L^N Q_\tau\|_{\mathcal{L}(H^{-s}(Z,\mathbf{W}),H^s(Z,\mathbf{W}))} \le C_{N,s}\tau^{N+s}.$$

Indeed, conjugating by the powers of L defining the two norms produces a symbol of order $2N + 2s$ with this support; dividing it by τ^{N+s} leaves bounded the order-zero seminorms needed for the estimate on L^2. We thus obtain

$$\|\big(h(\tau^{-2}L) - I\big)Q_\tau\|_{\mathcal{L}(H^{-s}(Z,\mathbf{W}),H^s(Z,\mathbf{W}))} \le C_{N,s}\tau^{-N+s}.$$

The integer N is arbitrary. Differentiating in τ produces derivatives of the cutoff and functions obtained from g_N by applying $t\partial_t$; these functions remain bounded. The same choice of N therefore controls any prescribed number of parameter derivatives.

In the complementary region $|\eta| \ge \tau^{1/2}$, use the weak resolvent estimates from the chapter on pseudodifferential operators. For every prescribed set of derivatives, the remainders are bounded by a fixed power of $1 + \tau$ multiplied by a negative power of $\langle \eta \rangle$ whose exponent can be made as large in magnitude as desired by increasing the length of the parametrix. Conversion to a joint bound is seen in the inequality

$$(1+\tau)^a \langle \eta \rangle^{-J} \le C_{a,K} \langle (\tau, \eta) \rangle^{-K}, \qquad |\eta| \ge \tau^{1/2}, \quad \tau \ge 1, \quad J \ge 2(a+K),$$

where $a, K \ge 0$. If $|\eta| \le \tau$, use $\langle \eta \rangle^{-J} \le \tau^{-J/2}$; if $|\eta| \ge \tau$, absorb $(1+\tau)^a$ into $\langle \eta \rangle^a$. The order of the almost analytic extension is increased after fixing J, so that the powers of $|t|^{-1}$ that have appeared can be integrated. The derivatives of Q_τ lie in this same region, so the two localizations can be recombined with the stated bounds. Finally, for $R^{-1} \le \tau \le 1$ the horizontal parameter stays bounded; smoothing spectral calculus in the vertical variable and horizontal localization give jointly smooth remainders. This justifies, for the remainders as well, that in the extreme cones the symbol is respectively I and 0 modulo arbitrarily negative order.

More precisely, after localization in x and in a chart of Z, we obtain a symbol f with homogeneous components f_{-j}, extended using a common cutoff near the origin. For $J \in \mathbb{N}$ and multiindices $\alpha, \beta \in \mathbb{N}_0^{\dim Z}$ and $\delta \in \mathbb{N}_0^n$ we have

$$\left\| \partial_z^\alpha \partial_\eta^\beta \partial_\xi^\delta \left(f - \sum_{j=0}^{J-1} f_{-j} \right) \right\|_{\mathrm{op}} \le C_{J,\alpha,\beta,\delta,R} \langle (\xi, \eta) \rangle^{-J-|\beta|-|\delta|}. \tag{19.26}$$

The same estimate allows derivatives in x after inserting the cutoffs. The smooth-kernel remainders satisfy these bounds for every J: use the identity with S_N between vertical Sobolev spaces of arbitrary orders and apply the kernel regularity established in Chapter 15. Derivatives in ξ are obtained by differentiating the

resolvent and τ before integration. The principal term, upon replacing $1+|\xi|^2$ by its homogeneous part, is (19.22). Its extensions to the axes are smooth because h is constant on the specified intervals.

In Kohn–Nirenberg quantization, P acts only in x and the symbol of the cutoff is independent of x before localization. The symbol of $(P\otimes I)\Phi_R$ is therefore the product $p(x,\xi)f(z,\eta,\xi)$. In the cone where $|\eta|$ dominates $|\xi|$, the second factor has arbitrarily negative order; in its complement, $|\xi|$ controls the joint frequency. The symbol estimates for p and (19.26) prove that the product has joint order one and is classical. The assertion for $\Phi_R(P\otimes I)$ follows by taking adjoints.

For $D(I-\Phi_R)$ use vertical composition with a parameter. In the cone where $|\xi|$ dominates $|\eta|$, the symbol of $I-\Phi_R$ has arbitrarily negative order. In the remaining cone, $|\eta|$ controls the joint frequency, and the symbol estimates for D apply. This yields a classical joint symbol of order one with principal part $b_1(I-f_R)$. The localizations and changes of coordinates are those of the classical calculus already constructed, completing the proof. □

Lemma 19.3.6. (Product symbol and normal Thom class) Let M be a closed smooth manifold and let $a=[\mathbf{E}^+,\mathbf{E}^-,p]\in K_c^0(T^*M)$, where $\mathbf{E}^+,\mathbf{E}^-\to M$ are smooth complex vector bundles. Use the normal connection, metrics, and compactification $\nu\subset\widehat{\nu}$ fixed above. Denote by $q:T^*\nu\to M$ the projection and set

$$\mathbf{\Lambda}_\nu^+=q^*\left(\bigoplus_{\substack{j\in\{0,\dots,r\}\\ j\text{ even}}}\Lambda^j(\nu\otimes_{\mathbb{R}}\mathbb{C})\right),\qquad \mathbf{\Lambda}_\nu^-=q^*\left(\bigoplus_{\substack{j\in\{0,\dots,r\}\\ j\text{ odd}}}\Lambda^j(\nu\otimes_{\mathbb{R}}\mathbb{C})\right).$$

The component from even to odd degree of the normal Koszul homomorphism is

$$\beta^+(v,\eta)=\big(\varepsilon(v+i\eta^\sharp)+\iota(v-i\eta^\sharp)\big)\Big|_{\mathbf{\Lambda}_\nu^+}.$$

On $T^*\nu$, the graded matrix

$$\mathbf{c}(x,v;\xi,\eta)=\begin{pmatrix}p(x,\xi)\otimes I & -I\otimes\beta^+(v,\eta)^*\\ I\otimes\beta^+(v,\eta) & p(x,\xi)^*\otimes I\end{pmatrix},\tag{19.27}$$

is invertible outside a compact set. Its class, extended from this open subset, satisfies

$$\operatorname{ext}_{T^*\nu}^{T^*\widehat{\nu}}[\mathbf{c}]=\operatorname{ext}_{T^*\nu}^{T^*\widehat{\nu}}\operatorname{Th}_{T\nu}\big((\sharp_g)_*a\big)\quad\text{in } K_c^0(T^*\widehat{\nu}).\tag{19.28}$$

Proof. The connection identifies the normal bundle $T\nu$ over TM with $\pi_{TM}^*(\nu\oplus\nu)$. The metric identifies the second copy of ν with the vertical covariables, and the complex structure of (19.1) expresses its coordinate as

$$z=v+i\eta^\sharp,\qquad J(v,\eta^\sharp)=(-\eta^\sharp,v).$$

The normal Koszul homomorphism on the full exterior bundle is $\beta(v,\eta)=\varepsilon(z)+\iota(\overline{z})$. It is self-adjoint and odd and satisfies

$$\beta^2=(|v|^2+|\eta|^2)I,\qquad (\beta^+)^*\beta^+=(|v|^2+|\eta|^2)I\quad\text{on }\mathbf{\Lambda}_\nu^+.$$

The corresponding identity holds in odd degree. Thus β^+ is invertible away from the zero section. These formulas are preserved under changes of frame of $O(r)$ and define global homomorphisms of the stated bundles. The graded product of the horizontal complex $(\mathbf{E}^+,\mathbf{E}^-,p)$ with this normal complex has even and

odd parts, respectively,

$$(q^*\mathbf{E}^+ \otimes \mathbf{\Lambda}_\nu^+) \oplus (q^*\mathbf{E}^- \otimes \mathbf{\Lambda}_\nu^-), \qquad (q^*\mathbf{E}^- \otimes \mathbf{\Lambda}_\nu^+) \oplus (q^*\mathbf{E}^+ \otimes \mathbf{\Lambda}_\nu^-),$$

and its odd differential from even to odd degree is exactly the matrix (19.27). By the definition of the product of complexes and the Thom isomorphism, its class is $\mathrm{Th}_{T\nu}((\sharp_g)_* a)$, identifying $T\nu$ with $T^*\nu$ by the fixed metrics. To check the support, the product of the matrix with its adjoint has diagonal blocks

$$p^*p \otimes I + (|v|^2 + |\eta|^2)I, \qquad pp^* \otimes I + (|v|^2 + |\eta|^2)I.$$

Thus the matrix can fail to be invertible only when $v = \eta = 0$ and $p(x, \xi)$ is not invertible. The latter set is compact because a is a compactly supported class.

The factors are ordered with the horizontal factor first and the normal factor second; the sign of the upper-right block is the corresponding Koszul sign and does not introduce an additional sign into the class. In complex rank one, the preceding coordinate gives the convention $\lambda_{\mathbb{C}} = -b$ of (18.7); by multiplicativity, it fixes both the orientation and the sign in every rank. Finally, the matrix class has compact support in $T^*\nu$. The extension-by-zero map for the open subset $T^*\nu \subset T^*\widehat{\nu}$ gives, by definition, precisely the right-hand side of (19.28). □

Spectral cutoffs allow the product to be formed within the classical calculus on $\widehat{\nu}$. The vertical kernel provides a copy of the sections over M. On its complement, a symmetry interchanging the degrees will make the contributions to the index cancel.

Lemma 19.3.7. (Horizontal and vertical product) Let M be a closed smooth manifold and let $\mathbf{E}^+, \mathbf{E}^- \to M$ be smooth complex vector bundles. Let $P \in \Psi^1_{\mathrm{cl}}(M; \mathbf{E}^+, \mathbf{E}^-)$ be elliptic and let B be the operator of Lemma 19.3.2. There exist a classical elliptic operator of order one

$$C : \Gamma(\widehat{\nu}, \boldsymbol{\mathcal{E}}^+) \longrightarrow \Gamma(\widehat{\nu}, \boldsymbol{\mathcal{E}}^-)$$

and bundles

$$\boldsymbol{\mathcal{E}}^+ = (\widehat{\pi}^*\mathbf{E}^+ \otimes \mathbf{W}_\nu^+) \oplus (\widehat{\pi}^*\mathbf{E}^- \otimes \mathbf{W}_\nu^-),$$
$$\boldsymbol{\mathcal{E}}^- = (\widehat{\pi}^*\mathbf{E}^- \otimes \mathbf{W}_\nu^+) \oplus (\widehat{\pi}^*\mathbf{E}^+ \otimes \mathbf{W}_\nu^-)$$

such that, if $a = [\sigma(P)]$, then

$$[\sigma(C)] = \mathrm{ext}^{T^*\widehat{\nu}}_{T^*\nu} \mathrm{Th}_{T\nu}\left((\sharp_{\mathbf{g}})_* a\right) \quad \text{in } K^0_c(T^*\widehat{\nu}) \tag{19.29}$$

and

$$\mathrm{ind_a}(C) = \mathrm{ind_a}(P). \tag{19.30}$$

The domain of the closure of C is $H^1(\widehat{\nu}, \boldsymbol{\mathcal{E}}^+)$, and that of its Hilbert space adjoint is $H^1(\widehat{\nu}, \boldsymbol{\mathcal{E}}^-)$.

Proof. Set $\mathbf{W} = \mathbf{W}^+ \oplus \mathbf{W}^-$ and consider on the fiber the operator and grading

$$D_B = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}, \qquad \Gamma_B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

By Lemma 19.3.3, D_B is self-adjoint with domain $H^1(S^r,\mathbf{W})$. Denote by Π_0 the orthogonal projection onto its kernel. Elliptic regularity shows that Π_0 is smoothing. By (19.15), its image is a line in even degree, and the action of G on it is trivial. Choose a generator $\mathbf{e}_0$ with L^2 norm equal to one. Both D_B and Π_0 are G–equivariant and therefore induce vertical operators on $\widehat{\nu}$; we retain the same letters for them.

Equip $\widehat{\nu}$ with the connection metric making horizontal and vertical directions orthogonal, using the round metric on the fibers. In an associated trivialization, its measure is $d\lambda_{\mathbf{g}}(x)\,d\lambda_{S^r}(z)$. Changes of trivialization are fiber isometries. Thus

$$(\mathcal{I}\mathbf{u})(x,z)=\mathbf{u}(x)\otimes\mathbf{e}_0(z) \tag{19.31}$$

defines a global isometry from $L^2(M,\mathbf{E}^+\oplus\mathbf{E}^-)$ onto $\Pi_0L^2(\widehat{\nu},\boldsymbol{\mathcal{E}})$, where $\boldsymbol{\mathcal{E}}=\boldsymbol{\mathcal{E}}^+\oplus\boldsymbol{\mathcal{E}}^-$. It also identifies the corresponding H^1 spaces: in a product chart, derivatives in x act on $\mathbf{u}$ and vertical derivatives on the fixed smooth section $\mathbf{e}_0$; the norm identity in L^2 gives the reverse bound. Finitely many charts give the global equivalence.

First suppose that $n=\dim M\geq 1$. Choose a finite cover $(U_\ell)_{\ell=1}^J$ trivializing ν and $\mathbf{E}^\pm$, and real functions $\chi_\ell\in C_c^\infty(U_\ell)$ such that

$$\sum_{\ell=1}^J\chi_\ell^2=1\quad\text{on } M.$$

Use orthonormal frames for ν and unitary frames for the complex bundles. When passing to Euclidean coordinates in U_ℓ, incorporate the square root of the measure Jacobian into the isomorphism between the L^2 spaces. Thus the adjoints in coordinates are the Hilbert space adjoints of the transported operators.

Let P_ℓ be the local expression of $\zeta_\ell P\zeta_\ell$, extended to $\mathbb{R}^n$, where $\zeta_\ell\in C_c^\infty(U_\ell)$ equals one near $\operatorname{supp}\chi_\ell$. Set

$$A_\ell=\begin{pmatrix}0 & P_\ell^*\\ P_\ell & 0\end{pmatrix},\qquad \Gamma_P=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$$

on the factor $\mathbf{E}^+\oplus\mathbf{E}^-$. In the product chart $U_\ell\times S^r$ apply Lemma 19.3.5 to D_B and write $\Phi_{\ell,R}$ for the cutoff constructed there. Before multiplying by χ_ℓ, perform horizontal quantization on all of $\mathbb{R}^n$. Define the odd operator

$$\mathscr{C}_R=\sum_{\ell=1}^J\chi_\ell\left[\frac{1}{2}\left(A_\ell\Phi_{\ell,R}+\Phi_{\ell,R}A_\ell\right)+R\Gamma_PD_B(I-\Phi_{\ell,R})\right]\chi_\ell. \tag{19.32}$$

Transport each summand to $\widehat{\pi}^{-1}(U_\ell)$ and extend it by zero in the base variable. The outer functions make this extension globally defined. The cutoff lemma proves that $\mathscr{C}_R\in\Psi^1_{\mathrm{cl}}(\widehat{\nu};\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{E}})$. Symmetrization of the horizontal term and commutation of D_B with the cutoffs show that it is formally self-adjoint.

Let us verify its ellipticity and fix the size of R. Denote by $a_1(x,\xi)$ the odd principal matrix of P and by $b_1(y,\eta)$ that of D_B, where $y\in\widehat{\nu}$, $x=\widehat{\pi}(y)$, and $(\xi,\eta)\in\mathbf{H}^*_y\oplus\mathbf{V}^*_y$. Extend both symbols by zero at their respective origins. They are continuous and homogeneous of degree one. There exist constants $c_P,c_B,C_B>0$ such that

$$a_1(x,\xi)^2\geq c_P^2|\xi|^2I,\qquad c_B^2|\eta|^2I\leq b_1(y,\eta)^2\leq C_B^2|\eta|^2I. \tag{19.33}$$

Homogeneity also gives a constant $L_P>0$ for which

$$\|a_1(x,\xi')-a_1(x,\xi)\|_{\mathrm{op}}\leq L_P|\xi'-\xi|.$$

Indeed, first derivatives in the covariable are homogeneous of degree zero and bounded in a finite atlas.

Integrate this bound along the segment between ξ and ξ'; if it passes through the origin, apply it on the two subsegments and use continuity there.

The product coordinates of the trivialization and the fixed horizontal splitting need not give the same decomposition. In the trivialization ℓ, the coordinate horizontal covariable corresponds to

$$\xi_\ell = \xi + T_\ell(y)\eta \in T_x^*M, \qquad \|T_\ell(y)\|_{\mathrm{op}} \le A_0.$$

The constant A_0 exists because we use only finitely many trivializations over compact sets. If $|\cdot|_\ell$ is the Euclidean norm of the components in that chart, fix $c_x, C_x > 0$ such that

$$c_x|\omega| \le |\omega|_\ell \le C_x|\omega|, \qquad \omega \in T_x^*M, \quad x \in \operatorname{supp}\chi_\ell, \quad \ell \in \{1,\dots,J\}.$$

The symbol of the cutoff, transported to the corresponding fiber, is

$$f_{\ell,R}(y;\xi,\eta) = h\!\left(\frac{R^2 b_1(y,\eta)^2}{|\xi_\ell|_\ell^2}\right).$$

On the axes, use the extensions of the lemma. These endomorphisms are positive, bounded by I, and commute with one another and with b_1, since they are functions of the same matrix b_1^2. Set

$$F_R = \sum_{\ell=1}^{J} \chi_\ell(x)^2 f_{\ell,R},$$

extending each term by zero outside its trivialization. The principal symbol of (19.32) is

$$c_R = \sum_{\ell=1}^{J} \chi_\ell^2 a_1(x,\xi_\ell) f_{\ell,R} + R\Gamma_P b_1(y,\eta)(I - F_R). \tag{19.34}$$

The identity factors on $\mathbf{E}$ and $\mathbf{W}$ remain implicit in this formula; the two operators act on distinct tensor factors.

Compare c_R with

$$m_R = a_1(x,\xi)F_R + R\Gamma_P b_1(y,\eta)(I - F_R).$$

The preceding bounds imply

$$\|c_R - m_R\|_{\mathrm{op}} \le L_P A_0 |\eta|. \tag{19.35}$$

Since $a_1\Gamma_P = -\Gamma_P a_1$, and F_R commutes with the vertical factors, the cross terms in the square cancel exactly:

$$m_R^2 = a_1(x,\xi)^2 F_R^2 + R^2 b_1(y,\eta)^2 (I - F_R)^2. \tag{19.36}$$

Choose

$$\delta = \min\left\{\frac{1}{4}, \frac{c_x}{4C_B}, \frac{c_B}{8C_x}\right\}, \qquad R \ge \max\{1, A_0/\delta\}.$$

If $R|\eta| \le \delta|\xi|$, then $|\xi_\ell| \ge (1-\delta^2)|\xi| \ge \tfrac{3}{4}|\xi|$. All matrices $R^2 b_1^2/|\xi_\ell|_\ell^2$ have spectrum in $[0,1]$ for the indices $\ell \in \{1,\dots,J\}$ with $\chi_\ell(x) \neq 0$. Thus $f_{\ell,R} = I$ in the terms occurring in the sum, and $F_R = I$. If $|\xi| \le \delta R|\eta|$, then $|\xi_\ell|_\ell \le 2C_x\delta R|\eta|$. The corresponding spectra then lie in $[4,+\infty)$; hence $F_R = 0$. In the remaining region, we

have simultaneously

$$|\xi| > \delta R|\eta|, \qquad R|\eta| > \delta|\xi|.$$

Thus each of $|\xi|^2$ and $R^2|\eta|^2$ is at least $\frac{\delta^2}{2}(|\xi|^2 + R^2|\eta|^2)$. Moreover,

$$F_R^2 + (I - F_R)^2 = 2(F_R - \tfrac{1}{2}I)^2 + \tfrac{1}{2}I \geq \tfrac{1}{2}I.$$

Applying (19.36) in the three regions gives, with $c_* = \min\{c_P, c_B\}$,

$$\|m_R \mathbf{v}\| \geq \frac{\delta c_*}{2}(|\xi|^2 + R^2|\eta|^2)^{1/2}\|\mathbf{v}\|$$

for every fiber vector $\mathbf{v}$. Finally, fix

$$R \geq \max\left\{1, \frac{A_0}{\delta}, \frac{4L_P A_0}{\delta c_*}\right\}.$$

Inequality (19.35) implies

$$\|c_R \mathbf{v}\| \geq \frac{\delta c_*}{4}(|\xi|^2 + R^2|\eta|^2)^{1/2}\|\mathbf{v}\|. \tag{19.37}$$

This proves ellipticity of $\mathscr{C}_R$. Lemma 19.3.3 and Theorem 15.6.22 give

$$\mathcal{D}(\mathscr{C}_R) = H^1(\widehat{\nu}, \boldsymbol{\mathcal{E}}), \qquad \mathcal{D}(\mathscr{C}_R^{\pm}) = H^1(\widehat{\nu}, \boldsymbol{\mathcal{E}}^{\pm}). \tag{19.38}$$

The operator $\mathscr{C}_R$ is self-adjoint, and its component $C := \mathscr{C}_R^+$ is Fredholm, with adjoint $\mathscr{C}_R^-$ on the stated domain.

Let us calculate its index. The operator $D_B^2 + \Pi_0$ is positive, invertible, and classical elliptic of order two on the fiber. The complex powers of Theorem 15.7.9 allow us to define

$$J_B = D_B(D_B^2 + \Pi_0)^{-1/2} \in \Psi^0_{\mathrm{cl}}(S^r; \mathbf{W}, \mathbf{W}).$$

It is equivariant and satisfies

$$J_B^* = J_B, \qquad J_B^2 = I - \Pi_0, \qquad J_B\Gamma_B = -\Gamma_B J_B.$$

The vertical operators Π_0 and J_B are bounded on the spaces L^2 and H^1 of the total manifold. To check the second assertion, in a product chart they commute with derivatives in x and are bounded on vertical H^1 by the order-zero calculus; a finite partition of unity gives the global bound.

Every summand of (19.32) commutes with Π_0. On the complement of its image define

$$\mathcal{T} = \Gamma_P \Gamma_B J_B.$$

This operator is unitary, interchanges degrees, and satisfies $\mathcal{T}^2 = -I$. The cutoffs commute with J_B and Γ_B; the operators A_ℓ commute with them and anticommute with Γ_P. Thus

$$\mathcal{T}\mathscr{C}_R = -\mathscr{C}_R\mathcal{T} \quad \text{on } (I - \Pi_0)H^1(\widehat{\nu}, \boldsymbol{\mathcal{E}}). \tag{19.39}$$

First verify the identity on smooth sections. Boundedness of $\mathcal{T}$ on H^1 and (19.38) allows it to extend to the domain. Consequently, $\mathcal{T}$ identifies the even and odd parts of the kernel of $\mathscr{C}_R$ lying in $(I - \Pi_0)L^2$. Their

dimensions are finite and their contributions to the index cancel.

On $\Pi_0 L^2$, the vertical operator vanishes and the cutoff is the scalar horizontal multiplier

$$a_{0,R}(\xi) = h\left(\frac{R^2}{1+|\xi|^2}\right).$$

This function equals one outside a ball. Through (19.31), the even-to-odd component of the restriction of $\mathscr{C}_R$ is identified with the operator

$$P_R = \frac{1}{2}\sum_{\ell=1}^{J} \chi_\ell \big(P_\ell a_{0,R}(D_x) + a_{0,R}(D_x) P_\ell\big)\chi_\ell$$

on M, transporting each summand from its chart. Since $a_{0,R} - 1$ has compact support in the horizontal frequency,

$$P_R - P = \sum_{\ell=1}^{J} \chi_\ell [P, \chi_\ell] + S_R, \qquad S_R \in \Psi^{-\infty}(M; \mathbf{E}^+, \mathbf{E}^-). \tag{19.40}$$

The commutators are classical of order zero. By Lemma 19.3.4, $P_R - P$ is compact from H^1 to L^2, so $\operatorname{ind} P_R = \operatorname{ind} P$ by Theorem 16.2.1.

If $K^\pm$ denotes the part of the kernel of $\mathscr{C}_R$ lying in $(I - \Pi_0)L^2$ and of degree $\pm$, we have proved the decompositions

$$\begin{aligned} \ker C &= \mathcal{I}(\ker P_R) \oplus K^+, \\ \ker C^* &= \mathcal{I}(\ker P_R^*) \oplus K^-, \qquad \mathcal{T} : K^+ \xrightarrow{\cong} K^-. \end{aligned} \tag{19.41}$$

Subtracting dimensions yields

$$\operatorname{ind} C = \operatorname{ind} P_R = \operatorname{ind} P. \tag{19.42}$$

It remains to identify the symbol class. In (19.34) replace T_ℓ simultaneously by tT_ℓ, with $t \in [0,1]$, both in $a_1(x, \xi_\ell)$ and in $f_{\ell,R}$. The estimates leading to (19.37) remain valid, since $\|tT_\ell\|_{\mathrm{op}} \le A_0$. This gives a homotopy of invertible symbols on the cotangent bundle with its zero section removed. At $t = 0$ it takes the form

$$a_1(x,\xi)F + R\Gamma_P b_1(y,\eta)(I - F),$$

where F is a convex sum of functions of b_1^2 and $0 \le F \le I$. For $s \in [0,1]$ replace the two coefficients by

$$F_s = (1-s)F + sI, \qquad G_s = (1-s)(I-F) + sI.$$

The square of $a_1 F_s + R\Gamma_P b_1 G_s$ is $a_1^2 F_s^2 + R^2 b_1^2 G_s^2$. Since $F_s \ge F$ and $G_s \ge I - F$, the lower bound at the start of the homotopy is preserved. At the end we obtain $a_1 + R\Gamma_P b_1$. Deforming the positive factor R to one yields $a_1 + \Gamma_P b_1$.

These last homotopies are continuous homotopies of bundle homomorphisms on the cosphere bundle. They are not used to define operators at intermediate values. Continuity suffices for homotopy invariance of the K–theory class. The even-to-odd component of the final symbol is

$$\begin{pmatrix} p_1(x,\xi)\otimes I & -I \otimes b_1^+(y,\eta)^* \\ I \otimes b_1^+(y,\eta) & p_1(x,\xi)^* \otimes I \end{pmatrix},$$

where b_1^+ is the symbol of B. This is the graded product of the horizontal class with the vertical class. Lemma 19.3.2 identifies the latter with the extension of the normal Bott class. Lemma 19.3.6 identifies its product with the Thom class, with the order of factors and orientation already fixed. This gives (19.29).
If $\dim M = 0$, M is a finite set. Over each point, use the vertical operator $\Gamma_P D_B$ directly in the graded product with $\mathbf{E}^+ \oplus \mathbf{E}^-$. Its index is $\dim \mathbf{E}^+ - \dim \mathbf{E}^-$, which agrees with the index of any linear map from $\mathbf{E}^+$ to $\mathbf{E}^-$ over that point. Its symbol is the product of this virtual class with the Bott class. Summing over the points of M gives both identities in the statement and completes the proof. □

Theorem 19.3.8. (Analytic compatibility with the pushforward) Let M be a closed smooth manifold, let $j\colon M \hookrightarrow \mathbb{R}^N$ be a smooth embedding, and let $a \in K_c^0(T^*M)$. Then

$$\operatorname{ind}_{\mathrm{a}}^{\mathbb{R}^N}(j_! a) = \operatorname{ind}_{\mathrm{a}}^{M}(a). \tag{19.43}$$

Proof. Represent a by the symbol of an elliptic operator P on M. Theorem 17.4.3 allows this after stabilization, which adds an identity of index zero. If the resulting representative has order zero, fix a positive invertible operator

$$\Lambda_+ \in \Psi_{\mathrm{cl}}^1(M;\mathbf{E}^+), \qquad \Lambda_+^{-1} \in \Psi_{\mathrm{cl}}^{-1}(M;\mathbf{E}^+),$$

and replace P by $P\Lambda_+$. Composition with Λ_+ does not change the index. Its positive scalar symbol contracts to the identity symbol after normalizing the order, so it also leaves the class a unchanged. We may therefore apply Lemma 19.3.7. This gives an operator C on $\widehat{\nu}$ with

$$\operatorname{ind}_{\mathrm{a}}(C) = \operatorname{ind}_{\mathrm{a}}(P) \tag{19.44}$$

and symbol class

$$[\sigma(C)] = \operatorname{ext}_{T^*\nu}^{T^*\widehat{\nu}} \operatorname{Th}_{T\nu}\big((\sharp_g)_* a\big) \in K_c^0(T^*\widehat{\nu}), \tag{19.45}$$

where the metric and connection identify the Thom representative on $T\nu$ with the corresponding complex on $T^*\nu$, followed by extension from the open subset $T^*\nu \subset T^*\widehat{\nu}$. This equality is precisely (19.29).
To pass from $\widehat{\nu}$ to the tubular neighborhood, we reduce the order of the operator and localize its symbol. Let

$$\Lambda_{-1} \in \Psi_{\mathrm{cl}}^{-1}(\widehat{\nu};\boldsymbol{\mathcal{E}}^+)$$

be positive and elliptic, chosen so that $\Lambda_{-1} : L^2(\widehat{\nu},\boldsymbol{\mathcal{E}}^+) \longrightarrow H^1(\widehat{\nu},\boldsymbol{\mathcal{E}}^+)$ is an isomorphism; its inverse has order one. The operator

$$C\Lambda_{-1}\colon L^2(\widehat{\nu},\boldsymbol{\mathcal{E}}^+) \longrightarrow L^2(\widehat{\nu},\boldsymbol{\mathcal{E}}^-)$$

has order zero. Since Λ_{-1} is invertible, it preserves the kernel, cokernel, and index of C; its normalized positive symbol is homotopic to the identity and also leaves the symbol class unchanged. Denote by A an order-zero quantization obtained in this way. By (19.45), its symbol triple is elementary on a neighborhood of $\mathbf{s}_\infty$. After adding an elementary triple, the input and output bundles are identified on that neighborhood. The polar retraction

$$s \longmapsto s(s^*s)^{-\frac{1}{2}}$$

is applied only to the symbol isomorphism in that region and makes it unitary. The triple being elementary means precisely that, after the chosen stabilization, this unitary admits a homotopy there to the identity

through isomorphisms. With a single cutoff supported in the same region, quantization of this homotopy is continuous in

$$\mathcal{L}\big(L^2(\widehat{\nu},\boldsymbol{\mathcal{E}}^+),L^2(\widehat{\nu},\boldsymbol{\mathcal{E}}^-)\big);$$

and all its members are elliptic and Fredholm, so the index remains constant.

At the end of the homotopy, replacing A by the final quantization, the principal symbol is the identity on a neighborhood $\mathcal{U}_0$ of $\mathbf{s}_\infty$. Fix a smaller neighborhood $\mathcal{U}_\infty$ whose closure is contained in $\mathcal{U}_0$. The identification already chosen between $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ over $\mathcal{U}_0$ determines an identity morphism I_∞ there. Choose $\zeta\in C_c^\infty(\mathcal{U}_0)$ equal to one on a neighborhood of $\overline{\mathcal{U}_\infty}$ and extend the morphism ζI_∞ by zero. This gives a global smooth morphism

$$R:\boldsymbol{\mathcal{E}}^+\longrightarrow\boldsymbol{\mathcal{E}}^-$$

agreeing with I_∞ near $\overline{\mathcal{U}_\infty}$; we do not assert that R is invertible outside this region. If $r=\sigma_0^\Psi(R)$, let

$$K_A:=\operatorname{supp}\big(\sigma_0^\Psi(A)-r\big)\subset S^*\widehat{\nu}.$$

This actual support is compact, and its projection to $\widehat{\nu}$ does not meet $\mathcal{U}_\infty$. Choose $\chi\in C^\infty(\widehat{\nu})$ equal to one on a neighborhood of $\pi(K_A)$ and zero on $\mathcal{U}_\infty$, and define

$$A_0:=R+\chi(A-R)\chi. \tag{19.46}$$

The operators A and R act between the same pair $\boldsymbol{\mathcal{E}}^+,\boldsymbol{\mathcal{E}}^-$, and χ acts by scalar multiplication; therefore, A_0 is a global operator on this bundle. Moreover,

$$\sigma_0^\Psi(A_0)=r+\chi^2\big(\sigma_0^\Psi(A)-r\big)=\sigma_0^\Psi(A),$$

because $\chi=1$ on the support of the difference. By composition calculus, A_0-A has order -1 on the closed manifold $\widehat{\nu}$ and is compact as an operator

$$L^2(\widehat{\nu},\boldsymbol{\mathcal{E}}^+)\longrightarrow L^2(\widehat{\nu},\boldsymbol{\mathcal{E}}^-).$$

On $\mathcal{U}_\infty$ we have $\chi=0$ and $R=I_\infty$. Thus

$$\operatorname{ind_a}(A_0)=\operatorname{ind_a}(A)=\operatorname{ind_a}(P),\qquad A_0=I_\infty\text{ on }\mathcal{U}_\infty. \tag{19.47}$$

This formula supplies the exact equality near the boundary needed to extend the operator; it is not the unbounded oscillator that is being extended.

After shrinking the tubular domain, we may assume that $V\subset\nu$ is the open disk $|v|_{h_\nu}<\rho(x)$ for a positive smooth function ρ. The proper radial contraction

$$\kappa\colon\nu\longrightarrow V,\qquad \kappa(x,v)=\left(x,\frac{\rho(x)v}{\sqrt{1+|v|_{h_\nu}^2}}\right),$$

with a positive function ρ chosen within the tubular radius sends the regions $|v|_{h_\nu}\geq R$ to progressively thinner collars of the boundary of V. Conjugate A_0 by the unitary isomorphism incorporating the Jacobian

of κ and then by the tubular diffeomorphism $\Theta\colon V \longrightarrow U$. Since A_0 is exactly the identity near $\mathbf{s}_\infty$, the transported operator is the identity on a collar of the boundary of U, under the identification already chosen between the input and output bundles. Denote by $d \in K_c^0(T^*U)$ the class of this localized symbol.
Extension must be performed after stabilizing the bundles; identification on the collar alone does not suffice. Consider

$$e := \operatorname{ext}(d) \in K_c^0(T^*\mathbb{R}^N).$$

In the difference model, choose finite-rank Hermitian complements of the two bundles of the triple representing e over a pair (B^{2N}, S^{2N-1}) whose interior contains the support. The space B^{2N} is contractible, so these complements and both bundles are trivial. The isomorphism over S^{2N-1} forces the input and output bundles to have equal ranks. After adding the same trivial elementary triple of sufficiently large rank, both are identified with $\underline{\mathbb{C}}^q$. Stabilization adds the symbol of an identity, so it changes neither the K–theory class nor the index.
On the collar, the identification coming from $A_0 = I$ and the preceding trivializations make the input and output bundle the same. The polar retraction, followed by the elementary homotopy of the stabilized triple, allows us to choose a representative

$$s_e : T^*\mathbb{R}^N \longrightarrow \operatorname{End}(\mathbb{C}^q)$$

invertible outside a compact set in phase space, equal to I when the base variable lies outside a compact set, and whose restriction to T^*U represents d. Quantize this homotopy with a single cutoff and localize as in (19.46); this gives

$$\widetilde{A} \in \operatorname{Ell}_c^0(\mathbb{R}^N; \mathbb{C}^q), \qquad [\sigma(\widetilde{A})] = e.$$

All homotopies are norm-continuous on L^2, and stabilization only adds identities, so they do not alter the index. Now A_0, viewed on $\widehat{\nu}$, and $\widetilde{A}$, viewed on $\mathbb{R}^N$, are the two extensions of the same localized class d. Proposition 19.2.1 and (19.47) give

$$\operatorname{ind}_{\mathrm{a}}^{\mathbb{R}^N}(\widetilde{A}) = \operatorname{ind}_{\mathrm{a}}^{M}(P). \tag{19.48}$$

e is the extension by zero of (19.45); by (19.3), it is precisely $j_! a$. Since the analytic index depends only on the symbol class, (19.48) proves (19.43). □

The preceding compatibility is the property of the analytic index that allows passage from a closed manifold to its tubular neighborhood. The embedding construction and its relation to the product of symbols are also developed in [21, equations (12.8)–(12.16), pp. 293–301].

19.4 The Atiyah–Singer theorem

Before the main theorem, let us clarify why the analytic index can be evaluated directly on a K–theory class. Theorem 17.4.3 realizes every $a \in K_c^0(T^*M)$ as the symbol of an elliptic operator, and Theorem 17.5.6 proves that two realizations of the same class have the same index. Indeed, after adding identities, they are joined by a homotopy of elliptic symbols; this is quantized as a continuous family of Fredholm operators, and Theorem 15.6.24 keeps the index constant. This gives the homomorphism

$$\operatorname{ind}_{\mathrm{a}}^{M}\colon K_c^0(T^*M) \longrightarrow \mathbb{Z}.$$

Theorem 19.4.1. (Atiyah–Singer, K–theoretic version) Let M be a closed smooth manifold and let $P\colon \Gamma(M,\mathbf{E}^+) \longrightarrow \Gamma(M,\mathbf{E}^-)$ be a classical elliptic pseudodifferential operator between complex vector bundles. Then

$$\operatorname{ind}_{\mathrm{a}}(P) = \operatorname{ind}_{\mathrm{t}}([\sigma(P)]) \tag{19.49}$$

with the topological index of Definition 19.1.1.

Proof. Write $a = [\sigma(P)]$. By Theorem 19.3.8,

$$\operatorname{ind}_{\mathrm{a}}^{M}(a) = \operatorname{ind}_{\mathrm{a}}^{\mathbb{R}^N}(j_! a). \tag{19.50}$$

Theorem 18.7.1 and (19.4) state that, for every $c \in K_c^0(T^*\mathbb{R}^N)$,

$$\operatorname{ind}_{\mathrm{a}}^{\mathbb{R}^N}(c) = (i_N)_!^{-1}(c).$$

Applying this to $c = j_! a$ turns equality (19.50) into precisely

$$\operatorname{ind}_{\mathrm{a}}^{M}(a) = (i_N)_!^{-1}(j_! a) = \operatorname{ind}_{\mathrm{t}}(a).$$

□

The argument also proves two properties often included in the statement of the theorem. If P_0 and P_1 have symbols homotopic through elliptic symbols, their indices agree. If a symbol class is supported in T^*U for an open subset $U \subseteq M$, its index can be calculated in any manifold containing U. Neither assertion requires a local formula for the heat kernel.

19.5 Cohomological form

The K–theoretic version is integral and does not require an orientation of M. To obtain an integral formula, apply the compactly supported Chern character

$$\operatorname{ch}_c\colon K_c^0(TM) \longrightarrow H_c^{\mathrm{even}}(TM;\mathbb{Q}).$$

The total space TM has a canonical orientation, even if M is nonorientable: a connection identifies

$$T(TM) \cong \pi_{TM}^* TM \oplus \pi_{TM}^* TM,$$

and $J(X,Y) = (-Y,X)$ defines an almost complex structure. Denote by $[TM]_{\mathrm{BM}} \in H_{2n}^{\mathrm{BM}}(TM;\mathbb{Z})$ its Borel–Moore fundamental class.

For a complex bundle $\mathbf{W}$ with formal Chern roots $x_1,\dots,x_s$, fix the normalizations

$$\operatorname{ch}(\mathbf{W}) = \sum_{j=1}^{s} e^{x_j}, \qquad \operatorname{Td}(\mathbf{W}) = \prod_{j=1}^{s} \frac{x_j}{1-e^{-x_j}}. \tag{19.51}$$

In particular,

$$\operatorname{Td}(\mathbf{W}) = 1 + \frac{1}{2}c_1(\mathbf{W}) + \frac{1}{12}\bigl(c_1(\mathbf{W})^2 + c_2(\mathbf{W})\bigr) + \cdots.$$

Theorem 19.5.1. (Cohomological index formula) Let M be a closed smooth manifold of real dimension n, let $\mathbf{E}^+, \mathbf{E}^- \to M$ be smooth complex vector bundles, and let $P\colon \Gamma(M,\mathbf{E}^+) \to \Gamma(M,\mathbf{E}^-)$ be a classical elliptic pseudodifferential operator. Identify T^*M with TM using a metric and write $a = (\sharp_g)_*[\sigma(P)]$. Then

$$\operatorname{ind}_{\mathrm{a}}(P) = (-1)^n \left\langle \operatorname{ch}_c(a) \smile \pi_{TM}^* \operatorname{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}), [TM]_{\mathrm{BM}} \right\rangle. \tag{19.52}$$

If M is oriented and

$$\Phi_{TM}\colon H^*(M;\mathbb{Q}) \longrightarrow H_c^{*+n}(TM;\mathbb{Q})$$

is the cohomological Thom isomorphism for this orientation, then

$$\operatorname{ind}_{\mathrm{a}}(P) = (-1)^{\frac{n(n+1)}{2}} \int_M \left[\Phi_{TM}^{-1}(\operatorname{ch}_c(a)) \smile \operatorname{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}) \right]_n. \tag{19.53}$$

Here $[\cdot]_n$ denotes the component of degree n.

Compatibility between Thom and the Chern character is also proved in [21, Theorem 13.1, pp. 313–315].

Proof. Comparison of the Thom isomorphisms in K–theory and cohomology is a purely topological result. With the normalizations (19.51), if $\mathbf{W} \longrightarrow X$ is a complex bundle of rank s, $\lambda_{\mathbf{W}} \in K_c^0(\mathbf{W})$ is its Thom class, and $\Phi_{\mathbf{W}}$ is the cohomological Thom isomorphism, then

$$\Phi_{\mathbf{W}}^{-1}(\operatorname{ch}_c(\lambda_{\mathbf{W}})) = \prod_{j=1}^{s} \frac{1 - e^{x_j}}{x_j} = (-1)^s \operatorname{Td}(\overline{\mathbf{W}})^{-1}. \tag{19.54}$$

The proof using the splitting principle and naturality of the Chern character is given in Appendix A.
For the embedding $j\colon M \hookrightarrow \mathbb{R}^N$, the stable complex sum

$$\pi_{TM}^*(TM \otimes_{\mathbb{R}} \mathbb{C}) \oplus T\nu \cong TM \times \mathbb{C}^N$$

is trivial. By multiplicativity of the Todd class,

$$\operatorname{Td}(T\nu) = \pi_{TM}^* \operatorname{Td}(TM \otimes_{\mathbb{R}} \mathbb{C})^{-1}. \tag{19.55}$$

Here both $T\nu$, with the structure of (19.1), and $TM \otimes_{\mathbb{R}} \mathbb{C}$ are isomorphic to their conjugates; hence (19.54) produces the Todd class appearing in (19.55). Since $T\nu$ has complex rank $r = N - n$, the Thom formula gives explicitly

$$\operatorname{ch}_c(j_! a) = (-1)^r \operatorname{ext} \Phi_{T\nu}\left(\operatorname{ch}_c(a) \smile \pi_{TM}^* \operatorname{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}) \right). \tag{19.56}$$

On the other hand, $\alpha_N(b^{\boxtimes N}) = 1$ and the normalization $b = [\mathbf{E}_{-1}] - [1]$ satisfies

$$\left\langle \operatorname{ch}_c(b^{\boxtimes N}), [\mathbb{C}^N]_{\mathrm{BM}} \right\rangle = 1.$$

Thus (19.4) becomes

$$(i_N)_!^{-1}(c) = (-1)^N \left\langle \operatorname{ch}_c(c), [\mathbb{C}^N]_{\mathrm{BM}} \right\rangle. \tag{19.57}$$

Naturality of cohomological Thom and extension by zero identifies the pairing in (19.56) with the pairing

over TM. The two sign factors are

$$(-1)^r(-1)^N = (-1)^{N-n+N} = (-1)^n.$$

Consequently,

$$(i_N)_!^{-1}(j_!a) = (-1)^n \left\langle \mathrm{ch}_c(a) \smile \pi_{TM}^* \,\mathrm{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}), [TM]_{\mathrm{BM}} \right\rangle.$$

Theorem 19.4.1 proves (19.52).

If M is oriented, the complex orientation of TM orders each horizontal direction first and its vertical direction next. The tangent bundle orientation orders the n horizontal directions first, followed by the n vertical directions. Passing from one ordering to the other requires

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$$

transpositions. Thus, for $u \in H_c^{2n}(TM;\mathbb{Q})$,

$$\langle u, [TM]_{\mathrm{BM}} \rangle = (-1)^{\frac{n(n-1)}{2}} \int_M \Phi_{TM}^{-1}(u).$$

Multiplying this sign by $(-1)^n$ gives $(-1)^{\frac{n(n+1)}{2}}$ and proves (19.53). □

19.6 Three geometric operators

The following examples verify the analytic hypotheses and show how the symbol class recovers familiar invariants.

19.6.1 The Euler–de Rham operator

First suppose that $(M, \mathbf{g})$ is a closed Riemannian manifold. Define the total odd operator

$$D_{\mathrm{dR}} := d + d^* \quad \text{on } \Omega^\bullet(M;\mathbb{C}).$$

With the even and odd grading of complex forms, its restriction is

$$D_{\mathrm{dR}}^+ \colon \Omega^{\mathrm{even}}(M;\mathbb{C}) \longrightarrow \Omega^{\mathrm{odd}}(M;\mathbb{C}).$$

For $\xi \in T_x^*M \setminus \{0\}$, its two symbols are

$$\sigma_1(D_{\mathrm{dR}}^+)(x,\xi) = c(\xi), \qquad \sigma_1^\Psi(D_{\mathrm{dR}}^+)(x,\xi) = i\,c(\xi), \qquad c(\xi) := \varepsilon(\xi) - \iota(\xi^\sharp), \qquad c(\xi)^2 = -|\xi|_{\mathbf{g}}^2 I.$$

Consequently, D_{dR}^+ is elliptic.

Proposition 19.6.1. (Euler–de Rham index) Let $(M, \mathbf{g})$ be a closed Riemannian manifold of dimension n,

and let D^{+}_{dR} be the Euler–de Rham operator defined above. Then

$$\operatorname{ind}_{\mathrm{a}}(D^{+}_{\mathrm{dR}}) = \sum_{q=0}^{n} (-1)^{q} \dim_{\mathbb{C}} H^{q}_{\mathrm{dR}}(M;\mathbb{C}) = \chi(M). \tag{19.58}$$

If M is oriented, the cohomological formula reduces to

$$\operatorname{ind}_{\mathrm{a}}(D^{+}_{\mathrm{dR}}) = \int_{M} e(TM), \tag{19.59}$$

where $e(TM)$ is the Euler class represented in de Rham cohomology.

The identity of classes used in the proof can be found in [21, equation (17.148) and Theorem 17.68, pp. 611–612]; its integral is identified with $\chi(M)$ in [21, Theorem 13.6(d), pp. 320–321].

Proof. The square of $D_{\mathrm{dR}} = d + d^{*}$ is the Hodge Laplacian $\Delta = dd^{*} + d^{*}d$. The elliptic theory already developed gives

$$L^{2}\Omega^{q}(M;\mathbb{C}) = \ker \Delta_{q} \oplus \operatorname{Ran} \Delta_{q}$$

and a Green operator G_{q}, defined as Δ_{q}^{-1} on $(\ker \Delta_{q})^{\perp}$ and zero on $\ker \Delta_{q}$, which gains two derivatives. If $\omega \in \Omega^{q}(M;\mathbb{C})$ and $d\omega = 0$, write

$$\omega = h + \Delta G_{q}\omega, \qquad h \in \ker \Delta_{q}.$$

The identities $d\Delta = \Delta d$ and uniqueness of the inverse on $(\ker \Delta)^{\perp}$ imply that $dG_{q}\omega$ is harmonic. Since it is also exact,

$$\| dG_{q}\omega \|^{2}_{L^{2}\Omega^{q+1}(M;\mathbb{C})} = \langle G_{q}\omega, d^{*}dG_{q}\omega \rangle_{L^{2}\Omega^{q}(M;\mathbb{C})} = 0.$$

Therefore, $dG_{q}\omega = 0$ and

$$\omega = h + d(d^{*}G_{q}\omega).$$

Every de Rham cohomology class thus has a harmonic representative. If a harmonic form is exact, the same calculation shows that it is zero; the representative is unique. Consequently,

$$\ker \Delta_{q} \cong H^{q}_{\mathrm{dR}}(M;\mathbb{C}).$$

The kernels of D^{+}_{dR} and its adjoint are the even and odd harmonic forms, respectively, proving the first equality in (19.58). Theorem A.8.1 identifies the alternating sum of these dimensions with $\chi(M)$.
Finally, the preceding Clifford symbol defines the *Euler symbol class* in $K^{0}_{c}(T^{*}M)$; it is not identified here with a Thom class of a real bundle. Denote by $\sharp_{g}\colon T^{*}M \to TM$ the metric isomorphism and by $(\sharp_{g})_{*}$ the induced isomorphism in compactly supported K–theory. Its characteristic identity is

$$(-1)^{\frac{n(n+1)}{2}} \Phi^{-1}_{TM}\left(\operatorname{ch}_{c}\left((\sharp_{g})_{*}[\sigma(D^{+}_{\mathrm{dR}})]\right)\right) \operatorname{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}) = e(TM).$$

This is an identity for the full symbol class, including the Thom isomorphism and its orientations; it does not follow by identifying the exterior difference with a single difference of half-spinor bundles. Indeed, for a pair of roots $\pm x$ we have

$$\operatorname{ch}(\Lambda^{\mathrm{even}} - \Lambda^{\mathrm{odd}}) = 2 - e^{x} - e^{-x} = -\left(e^{x/2} - e^{-x/2}\right)^{2},$$

and the remaining factors come precisely from Thom. Substituting this identity into (19.53) yields (19.59). □

19.6.2 The Dolbeault operator

Now let M be a closed complex manifold of complex dimension m and let $\mathbf{E} \longrightarrow M$ be a holomorphic vector bundle equipped with a Hermitian metric. Also fix a Hermitian metric on M. The complex structure splits the complexified bundles as

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M, \qquad T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{*1,0}M \oplus T^{*0,1}M,$$

and we write

$$\Omega^{p,q}(M,\mathbf{E}) = \Gamma\left(M, \Lambda^p T^{*1,0}M \otimes \Lambda^q T^{*0,1}M \otimes \mathbf{E}\right).$$

In a holomorphic trivialization of $\mathbf{E}$ and holomorphic coordinates $z^1,\dots,z^m$, the twisted Dolbeault operator is

$$\bar\partial_{\mathbf{E}} = \sum_{j=1}^{m} d\bar z^j \wedge \frac{\partial}{\partial \bar z^j} \colon \Omega^{p,q}(M,\mathbf{E}) \longrightarrow \Omega^{p,q+1}(M,\mathbf{E}). \tag{19.60}$$

The transition functions of $\mathbf{E}$ are holomorphic, so (19.60) is independent of the trivialization. Commutation of the derivatives $\partial/\partial\bar z^j$ and anticommutation of exterior products prove $\bar\partial_{\mathbf{E}}^2 = 0$.

The Hermitian metrics on M and $\mathbf{E}$ define the formal adjoint $\bar\partial_{\mathbf{E}}^*$. The total odd operator

$$D_{\bar\partial,\mathbf{E}} := \sqrt{2}\,(\bar\partial_{\mathbf{E}} + \bar\partial_{\mathbf{E}}^*) \quad \text{on } \Omega^{0,\bullet}(M,\mathbf{E})$$

has as its component from even to odd degree the restriction

$$D^+_{\bar\partial,\mathbf{E}} := D_{\bar\partial,\mathbf{E}}\big|_{\Omega^{0,\mathrm{even}}} \colon \Omega^{0,\mathrm{even}}(M,\mathbf{E}) \longrightarrow \Omega^{0,\mathrm{odd}}(M,\mathbf{E})$$

This operator is elliptic. Indeed, if ξ is a real covariable, the preceding local formula and the adjoint symbol formula give, respectively,

$$\sigma_1(D^+_{\bar\partial,\mathbf{E}})(x,\xi) = \sqrt{2}\left(\varepsilon(\xi^{0,1}) - \varepsilon(\xi^{0,1})^*\right) \otimes I_{\mathbf{E}_x}, \qquad \sigma_1^{\Psi}(D^+_{\bar\partial,\mathbf{E}})(x,\xi) = i\sqrt{2}\left(\varepsilon(\xi^{0,1}) - \varepsilon(\xi^{0,1})^*\right) \otimes I_{\mathbf{E}_x}. \tag{19.61}$$

Since $2|\xi^{0,1}|_{\mathbf{g}}^2 = |\xi|_{\mathbf{g}}^2$ for a real covariable and $\varepsilon\varepsilon^* + \varepsilon^*\varepsilon = |\xi^{0,1}|_{\mathbf{g}}^2 I$, we obtain

$$\sigma_1^{\Psi}(D^+_{\bar\partial,\mathbf{E}})(x,\xi)^*\sigma_1^{\Psi}(D^+_{\bar\partial,\mathbf{E}})(x,\xi) = |\xi|_{\mathbf{g}}^2 I.$$

The factor $\sqrt{2}$ does not affect the symbol class. Write

$$H^{0,q}_{\bar\partial}(M,\mathbf{E}) := \frac{\ker(\bar\partial_{\mathbf{E}} \colon \Omega^{0,q}(M,\mathbf{E}) \longrightarrow \Omega^{0,q+1}(M,\mathbf{E}))}{\mathrm{Ran}(\bar\partial_{\mathbf{E}} \colon \Omega^{0,q-1}(M,\mathbf{E}) \longrightarrow \Omega^{0,q}(M,\mathbf{E}))}.$$

Proposition 19.6.2. (Analytic Hirzebruch–Riemann–Roch) Let M be a closed complex manifold of complex dimension m equipped with a Hermitian metric; let $\mathbf{E} \to M$ be a holomorphic vector bundle equipped

with a Hermitian bundle metric, and let $D^+_{\bar\partial,\mathbf{E}}$ be the twisted Dolbeault operator defined above. Then

$$\operatorname{ind}_a(D^+_{\bar\partial,\mathbf{E}}) = \sum_{q=0}^{m}(-1)^q \dim_{\mathbb{C}} H^{0,q}_{\bar\partial}(M,\mathbf{E}) = \int_M \left[\operatorname{ch}(\mathbf{E}) \smile \operatorname{Td}(T^{1,0}M)\right]_{2m}. \tag{19.62}$$

Proof. The square of $D_{\bar\partial,\mathbf{E}}$ is twice the Dolbeault Laplacian,

$$\Delta_{\bar\partial,\mathbf{E}} = \bar\partial_{\mathbf{E}}\bar\partial^*_{\mathbf{E}} + \bar\partial^*_{\mathbf{E}}\bar\partial_{\mathbf{E}}.$$

It is elliptic and self-adjoint, so its range is closed and it has a Green operator $G_{\bar\partial,\mathbf{E}}$ equal to the inverse on the orthogonal complement of its kernel and zero on the kernel. If $\boldsymbol{\alpha} \in \Omega^{0,q}(M,\mathbf{E})$ satisfies $\bar\partial_{\mathbf{E}}\boldsymbol{\alpha} = 0$, write

$$\boldsymbol{\alpha} = h + \Delta_{\bar\partial,\mathbf{E}} G_{\bar\partial,\mathbf{E}}\boldsymbol{\alpha}, \qquad h \in \ker \Delta_{\bar\partial,\mathbf{E}}.$$

The identity $\bar\partial_{\mathbf{E}}\Delta_{\bar\partial,\mathbf{E}} = \Delta_{\bar\partial,\mathbf{E}}\bar\partial_{\mathbf{E}}$ shows that $\bar\partial_E G_{\bar\partial,\mathbf{E}}\boldsymbol{\alpha}$ is harmonic. Since it is exact, its squared norm is

$$\left\|\bar\partial_E G_{\bar\partial,\mathbf{E}}\boldsymbol{\alpha}\right\|^2_{L^2\Omega^{0,q+1}(M,\mathbf{E})} = \left\langle G_{\bar\partial,\mathbf{E}}\boldsymbol{\alpha}, \bar\partial^*_{\mathbf{E}}\bar\partial_E G_{\bar\partial,\mathbf{E}}\boldsymbol{\alpha}\right\rangle_{L^2\Omega^{0,q}(M,\mathbf{E})} = 0.$$

Therefore,

$$\boldsymbol{\alpha} = h + \bar\partial_{\mathbf{E}}\left(\bar\partial^*_{\mathbf{E}} G_{\bar\partial,\mathbf{E}}\boldsymbol{\alpha}\right).$$

The same identity proves that an exact harmonic form is zero. We thus obtain, without losing or adding any classes,

$$\ker \Delta^{0,q}_{\bar\partial,\mathbf{E}} \cong H^{0,q}_{\bar\partial}(M,\mathbf{E}).$$

This gives the first equality in (19.62).
The Clifford module

$$\mathbf{S}_{\mathbb{C}} = \Lambda^{0,*}T^*M = \Lambda^{0,\mathrm{even}}T^*M \oplus \Lambda^{0,\mathrm{odd}}T^*M$$

defines the canonical Spin^c structure of a complex manifold and its corresponding orientation. Formula (19.61) shows that the symbol class is the Spin^c Thom class of this module multiplied by $\pi^*[\mathbf{E}]$; it does not depend on choosing the summand $T^{0,1}M$ in isolation. If $x_1,\dots,x_m$ are the formal Chern roots of $T^{1,0}M$, the identity (19.54) cancels, root by root, the factor coming from $TM\otimes_{\mathbb{R}}\mathbb{C}$ and leaves

$$\prod_{j=1}^{m}\frac{x_j}{1-e^{-x_j}} = \operatorname{Td}(T^{1,0}M).$$

The complex manifold carries its canonical orientation. Formula (19.53) gives the second equality in (19.62). □

19.6.3 The twisted Dirac operator

Let $(M^{2m},\mathbf{g})$ be a closed oriented Riemannian spin manifold, and let $\mathbf{E}\to M$ be a complex vector bundle equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. Clifford geometry, the spin structure, the spinor bundle, and its connection were constructed in Section 2.17. The twisted differential operator was defined

later, once the general theory of operators on bundles was available, in Definition 8.3.7. In particular,

$$\mathbf{S} := \Sigma M = \mathbf{S}^+ \oplus \mathbf{S}^-, \qquad D_{\mathbf{E}} = \begin{pmatrix} 0 & D_{\mathbf{E}}^- \\ D_{\mathbf{E}}^+ & 0 \end{pmatrix}, \qquad (D_{\mathbf{E}}^+)^* = D_{\mathbf{E}}^-.$$

$$D_{\mathbf{E}}^2 = \begin{pmatrix} D_{\mathbf{E}}^- D_{\mathbf{E}}^+ & 0 \\ 0 & D_{\mathbf{E}}^+ D_{\mathbf{E}}^- \end{pmatrix} \quad \text{with respect to } \mathbf{S}^+ \oplus \mathbf{S}^-.$$

The corresponding Weitzenböck formula is (8.35). To calculate the index, we also use the two symbol conventions introduced at different stages: by Proposition 11.2.4 and Example 15.3.5, the differential and Fourier symbols are, respectively,

$$\sigma_1(D_{\mathbf{E}}^+)(x,\xi) = c(\xi) \otimes I_{\mathbf{E}_x}, \qquad \sigma_1^{\Psi}(D_{\mathbf{E}}^+)(x,\xi) = i\, c(\xi) \otimes I_{\mathbf{E}_x}. \tag{19.63}$$

The Clifford relation proves their invertibility for $\xi \neq 0$.

The McKean–Singer identity expresses the Dirac index as the supertrace of the heat semigroup associated with the square of the operator.

Corollary 19.6.3. (McKean–Singer for the Dirac operator) Let $(M^{2m}, \mathbf{g})$ be a closed oriented Riemannian spin manifold, let $\mathbf{E} \to M$ be a complex vector bundle equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$, and let $D_{\mathbf{E}}^+$ be the twisted Dirac operator. For every $t > 0$,

$$\begin{aligned} \operatorname{ind}_{\mathrm{a}}(D_{\mathbf{E}}^+) &= \operatorname{Tr}\big(e^{-tD_{\mathbf{E}}^- D_{\mathbf{E}}^+}\big) - \operatorname{Tr}\big(e^{-tD_{\mathbf{E}}^+ D_{\mathbf{E}}^-}\big) \\ &= \operatorname{Str}\big(e^{-tD_{\mathbf{E}}^2}\big) = \int_M \operatorname{str}_{\mathbf{S}_x \otimes \mathbf{E}_x} \mathbf{K}_{D_{\mathbf{E}}^2}(t,x,x)\, d\lambda_{\mathbf{g}}(x), \end{aligned} \tag{19.64}$$

where $\mathbf{K}_{D_{\mathbf{E}}^2}$ is the heat kernel of $D_{\mathbf{E}}^2$ and $\operatorname{str}(A) = \operatorname{tr}(A|_{\mathbf{S}^+ \otimes \mathbf{E}}) - \operatorname{tr}(A|_{\mathbf{S}^- \otimes \mathbf{E}})$.

Proof. Apply Proposition 15.7.14 to $P = D_{\mathbf{E}}^+$. Formula (8.35) shows that $D_{\mathbf{E}}^2$ is a Laplace-type operator with a zeroth-order potential, but its heat kernel is not the kernel of the connection Laplacian constructed in Chapter 12. Regularity, the trace-class property, and the diagonal formula come from Proposition 15.7.13. No asymptotic expansion as $t \to 0^+$ is used. □

If the formal Chern roots of $TM \otimes_{\mathbb{R}} \mathbb{C}$ are ordered as $x_1, -x_1, \dots, x_m, -x_m$, define

$$\widehat{A}(TM) = \prod_{j=1}^{m} \frac{\frac{x_j}{2}}{\sinh\left(\frac{x_j}{2}\right)}. \tag{19.65}$$

Proposition 19.6.4. (Index of the Dirac operator) Let $(M^{2m}, \mathbf{g})$ be a closed oriented Riemannian spin manifold, let $\mathbf{E} \to M$ be a complex vector bundle equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$, and let $D_{\mathbf{E}}^+$ be the twisted Dirac operator. Then

$$\operatorname{ind}_{\mathrm{a}}(D_{\mathbf{E}}^+) = \int_M \big[\widehat{A}(TM) \smile \operatorname{ch}(\mathbf{E})\big]_{2m}. \tag{19.66}$$

In particular, for $\mathbf{E} = M \times \mathbb{C}$ we obtain the $\widehat{A}(M)$ genus.

The analytic construction and spinor Thom orientation are described in [21, Definitions 17.22–17.23, pp. 531–532] and [92, Appendix C, Theorem C.12; Chapter III, §12, Proposition 12.5].

Proof. The second formula in (19.63) shows that the symbol class is the complex

$$\pi^*(\mathbf{S}^+ \otimes \mathbf{E}) \xrightarrow{ic(\xi)\otimes I_{\mathbf{E}}} \pi^*(\mathbf{S}^- \otimes \mathbf{E}), \qquad \xi \in T^*M,$$

exact away from the zero section. The complex without $\mathbf{E}$ is the spinor Thom class, and the twisting factor is $\pi^*[\mathbf{E}]$; multiplying the morphism by i does not change its class. The Chern defect of the spinor Thom class, stated in Theorem A.8.5, is

$$\Phi_{TM}^{-1}\,\mathrm{ch}_c(s(TM)) = (-1)^m \widehat{A}(TM)^{-1}.$$

Moreover, the splitting principle applied to the pairs of roots $x_j, -x_j$ gives

$$\mathrm{Td}(TM \otimes_{\mathbb{R}} \mathbb{C}) = \widehat{A}(TM)^2.$$

The factor $\pi^*[\mathbf{E}]$ contributes $\mathrm{ch}(\mathbf{E})$. Since the overall sign in (19.53) in dimension $2m$ is $(-1)^m$, the two signs cancel, leaving $\widehat{A}(TM)\,\mathrm{ch}(\mathbf{E})$. This proves (19.66). □

Proposition 19.6.5. (Index of the Spin^c Dirac operator) Let M^{2m} be a closed oriented smooth manifold equipped with a Spin^c structure with determinant line $\mathbf{L}$. Let A be a unitary connection on $\mathbf{L}$ and let $\mathbf{E}$ be a complex vector bundle equipped with a Hermitian bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$. The operator $D_{A,\mathbf{E}}$ is obtained by twisting the operator of Proposition 8.3.9 by $\nabla^{\mathbf{E}}$, using the same product connection construction as in Definition 8.3.7. The geometric data of the Spin^c structure are those of Section 2.17. Its chiral restriction satisfies

$$D^+_{A,\mathbf{E}} \colon \Gamma(\Sigma^{c,+}M \otimes \mathbf{E}) \longrightarrow \Gamma(\Sigma^{c,-}M \otimes \mathbf{E}).$$

Then

$$\mathrm{ind}_{\mathrm{a}}(D^+_{A,\mathbf{E}}) = \int_M \left[\widehat{A}(TM) \smile e^{c_1(\mathbf{L})/2} \smile \mathrm{ch}(\mathbf{E})\right]_{2m}. \tag{19.67}$$

If the structure comes from an almost complex structure, then $\mathbf{L} = \det_{\mathbb{C}}(T^{1,0}M)$ and

$$\widehat{A}(TM)e^{c_1(\mathbf{L})/2} = \mathrm{Td}(T^{1,0}M).$$

If the almost complex structure is Kähler, the connection on $\mathbf{L} = K_M^{-1}$ is the one induced by Levi–Civita, and $\mathbf{E}$ is holomorphic with its Chern connection, the canonical operator is exactly $\sqrt{2}(\bar\partial_{\mathbf{E}} + \bar\partial^*_{\mathbf{E}})$. In the general almost complex case, the Spin^c Dirac operator and the associated Dolbeault-type operator have the same symbol and index, although they may differ by a zeroth-order term.

The canonical topological identifications are collected in [92, Appendix D, Theorem D.2 and Examples D.5–D.6; Appendix C, Theorem C.12; Appendix D, Theorem D.15].

Proof. The Spin^c Thom class is obtained from the spinor class formally twisted by the square root of $\mathbf{L}$; in the Chern character this introduces the well-defined factor $e^{c_1(\mathbf{L})/2}$. The same Thom calculation and (19.53) used in the preceding proposition give (19.67). For the almost complex structure, the determinant line and characteristic identity follow, under the splitting principle, from

$$\frac{x/2}{\sinh(x/2)}e^{x/2} = \frac{x}{1-e^{-x}}.$$

The Kähler identification follows from the Chern–Levi–Civita connection on $\Lambda^{0,\bullet}T^*M$. □

These cases illustrate different roles of the same theorem. For de Rham, the index is the Euler characteristic; for Dolbeault, the holomorphic Euler characteristic; in the spinor case, a characteristic number built from $\widehat{A}(TM)$. The equality is independent of the auxiliary metrics and connections: smooth variation preserves the symbol class and gives a continuous family of Fredholm operators.

Part V

Geometric evolution equations

Introduction to Part V

Until now, the Riemannian metric has been a fixed part of the geometry. In this part we allow it to change with time. The idea is simple to state: instead of seeking a metric with special properties directly, we start with an initial metric and let it evolve according to an equation determined by its curvature. The geometric problem thereby becomes a nonlinear partial differential equation.

Before studying Ricci flow, we need a parabolic theory that establishes existence, uniqueness, and regularity for systems on manifolds. The flow also presents a difficulty specific to geometry: the equation is invariant under diffeomorphisms, and this symmetry appears analytically as a degeneracy of the symbol. DeTurck's procedure removes this degeneracy through a choice of gauge and reduces the problem, for a short time, to a strictly parabolic system.

Once the solution has been constructed, the evolution equations for curvature and the maximum principles yield a priori estimates. When curvature grows and a singularity appears, rescaling allows the geometry to be observed at its natural scale. Compactness extracts limits of these magnified regions, while Perelman's monotone quantities and noncollapsing restrict the possible models. In dimension three, this analysis leads to surgery and shows particularly clearly how an evolution equation can reveal the topological structure of a manifold.

Hamilton, DeTurck, and Perelman

Hamilton introduced Ricci flow in 1982 with the idea of deforming a metric using its own curvature. It soon became clear that diffeomorphism invariance required careful treatment; DeTurck's procedure supplied the adjustment needed to apply parabolic theory. Two decades later, Perelman introduced new monotone quantities, the noncollapsing theorem, and a decisive analysis of singularities and surgery. This history illustrates a characteristic feature of geometric analysis: an intuition about the shape of space leads to an equation, the equation demands precise analytic tools, and those tools ultimately return to a geometric or topological problem.

20

Geometric parabolic equations on closed manifolds

A geometric evolution equation resembles the heat equation only after several difficulties have been resolved. The unknown is usually a section of a bundle, the coefficients of the principal part depend on that same section, and geometric symmetries may conceal the parabolicity of the system. Before studying Ricci flow, it is useful to separate these problems and develop a theory that identifies precisely which part of the argument comes from diffusion and which depends on the nonlinear structure.

We work on a closed Riemannian manifold $(M, \mathbf{g}_0)$ of dimension m. Let $\mathbf{E} \to M$ be a real vector bundle equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla^{\mathbf{E}}$, abbreviated to ∇ together with its induced connections. These structures are used to measure sections and write the equations; any other smooth choice gives equivalent norms. Denote by $H^r(M, \mathbf{E})$ the Sobolev space of integer order r constructed in Chapter 9. The operator $\Delta_\nabla = \nabla^*\nabla$ defining the spectral scale is nonnegative, whereas the covariant trace $\operatorname{tr}_{\mathbf{g}_0}(\nabla^2)$ used in the maximum principles has the opposite sign.

Throughout this chapter, fix an integer

$$s > \frac{m}{2} + 2. \tag{20.1}$$

Set $\Lambda = (I + \Delta_\nabla)^{1/2}$. By Theorem 12.4.10, the norm

$$\|\mathbf{u}\|_{H^r(M,\mathbf{E})} := \|\Lambda^r \mathbf{u}\|_{L^2(M,\mathbf{E})} \tag{20.2}$$

is equivalent to the covariant norm when $r \geq 0$ is an integer; for $r < 0$ it is interpreted by completion. We use this choice because it makes energy identities in the Sobolev scale exact. Estimates obtained with it remain valid for the covariant norms by equivalence. In particular, $H^{s-2}(M)$, $H^{s-1}(M)$, and $H^s(M)$ are algebras and the inclusion $H^{s-1}(M, \mathbf{E}) \hookrightarrow C^0(M, \mathbf{E})$ is continuous.

The linear estimate must control both the solution and its time derivative and retain its constants as the coefficients vary within fixed bounds. This uniformity allows us to solve the quasilinear equation: each approximation determines the coefficients of the next, while a common energy controls the entire sequence. Maximum principles will then provide pointwise information that integral norms alone do not supply.

From the heat equation to geometric flows

Parabolic theory grew out of the study of diffusion: heat dampens irregularities and produces smoothness at positive times. In a geometric flow the unknown may be a metric, a connection, or a section, and the coefficients of the equation change with it. Energy estimates, maximum principles, and product inequalities preserve the smoothing mechanism of heat in this nonlinear regime. The results of this chapter prepare this transition before applying it to Ricci flow.

20.1 Product estimates and energy spaces

The nonlinear estimates we need are precise consequences of the Leibniz rule and the Gagliardo–Nirenberg inequality in Theorem 9.3.1. We first collect them to make explicit where condition (20.1) is used.

Lemma 20.1.1. (Sobolev product with distributed orders) Let $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3 \to M$ be vector bundles with fixed metrics and connections, and let $\beta\colon \mathbf{V}_1 \otimes \mathbf{V}_2 \to \mathbf{V}_3$ be a smooth homomorphism. If $a, b \in \mathbb{N}_0$ and $a + b > m/2$, there exists a constant C such that

$$\|\beta(\mathbf{u}, \mathbf{v})\|_{L^2(M,\mathbf{V}_3)} \leq C\|\mathbf{u}\|_{H^a(M,\mathbf{V}_1)}\|\mathbf{v}\|_{H^b(M,\mathbf{V}_2)}. \tag{20.3}$$

The same estimate holds if one factor is a covariant derivative of a section and the orders a, b describe the regularity remaining after taking that derivative.

Proof. By a finite partition of unity and local equivalence of Sobolev norms, it suffices to prove the assertion for vector-valued functions supported in a relatively compact Euclidean open subset. Since $a + b > m/2$, we can choose $p, q \in [2, \infty]$ so that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \qquad H^a \hookrightarrow L^p, \qquad H^b \hookrightarrow L^q.$$

Let us specify the choice in the critical cases. If $a > m/2$, take $p = \infty$ and $q = 2$; if $b > m/2$, proceed symmetrically. Thus suppose that $a, b \leq m/2$. When both are less than $m/2$, the inequality $a + b > m/2$ allows us to choose a number θ such that

$$\frac{1}{2} - \frac{a}{m} < \theta < \frac{b}{m}, \qquad \frac{1}{p} = \theta, \qquad \frac{1}{q} = \frac{1}{2} - \theta.$$

The corresponding Sobolev inclusions are continuous. If $a = m/2$, choose $0 < \theta < b/m$; the inclusion $H^{m/2} \hookrightarrow L^p$ holds for every $p < \infty$. The case $b = m/2$ is analogous, taking $1/2 - a/m < \theta < 1/2$. This also covers the critical orders using finite exponents. The inclusion $H^{m/2} \hookrightarrow L^\infty$ does not hold in general. Hölder and the two inclusions give

$$\|\beta(\mathbf{u}, \mathbf{v})\|_{L^2(M,\mathbf{V}_3)} \leq C\|\mathbf{u}\|_{L^p(M,\mathbf{V}_1)}\|\mathbf{v}\|_{L^q(M,\mathbf{V}_2)} \leq C\|\mathbf{u}\|_{H^a(M,\mathbf{V}_1)}\|\mathbf{v}\|_{H^b(M,\mathbf{V}_2)}.$$

The coefficients of β, the connections, and the local frames are smooth on finitely many compact sets and can therefore be absorbed into the constant. □

Lemma 20.1.2. (Moser estimates) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold of dimension m. In the following estimates, $\|\cdot\|_\infty$ denotes the L^∞ norm on the corresponding bundle, with the fixed metrics. Let $\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3 \longrightarrow M$ be vector bundles with fixed metrics and connections, and let $\beta\colon \mathbf{V}_1\otimes\mathbf{V}_2 \longrightarrow \mathbf{V}_3$ be a smooth homomorphism. For every integer $r\geq 0$ there exists a constant C, depending on the fixed structures and r, such that, for $\mathbf{u}\in H^r(M,\mathbf{V}_1)\cap L^\infty(M,\mathbf{V}_1)$ and $\mathbf{v}\in H^r(M,\mathbf{V}_2)\cap L^\infty(M,\mathbf{V}_2)$,

$$\|\beta(\mathbf{u},\mathbf{v})\|_{H^r(M,\mathbf{V}_3)} \leq C\big(\|\mathbf{u}\|_\infty\|\mathbf{v}\|_{H^r(M,\mathbf{V}_2)} + \|\mathbf{u}\|_{H^r(M,\mathbf{V}_1)}\|\mathbf{v}\|_\infty\big), \tag{20.4}$$

$$\|[\nabla^r,\mathbf{a}]\mathbf{v}\|_{L^2(M,T^{(0,r)}(TM)\otimes\mathbf{V}_3)} \leq C\big(\|\nabla\mathbf{a}\|_\infty\|\mathbf{v}\|_{H^{r-1}(M,\mathbf{V}_2)} + \|\mathbf{a}\|_{H^r(M,\mathrm{Hom}(\mathbf{V}_2,\mathbf{V}_3))}\|\mathbf{v}\|_\infty\big) \tag{20.5}$$

In the commutator estimate, assume $r\geq 1$, $\mathbf{v}\in H^{r-1}(M,\mathbf{V}_2)\cap L^\infty(M,\mathbf{V}_2)$, and $\mathbf{a}\in H^r(M,\mathrm{Hom}(\mathbf{V}_2,\mathbf{V}_3))$ with $\nabla\mathbf{a}\in L^\infty(M,T^*M\otimes\mathrm{Hom}(\mathbf{V}_2,\mathbf{V}_3))$. The action of $\mathbf{a}$ on the tensor factors is the identity on $T^{(0,r)}(TM)$, and

$$[\nabla^r,\mathbf{a}]\mathbf{v} := \nabla^r(\mathbf{a}\mathbf{v}) - (I_{T^{(0,r)}(TM)}\otimes\mathbf{a})(\nabla^r\mathbf{v}) \in L^2\big(M,T^{(0,r)}(TM)\otimes\mathbf{V}_3\big).$$

For nonsmooth sections, the difference defining the commutator is taken first in distributions; the estimate states that this distribution is represented by an L^2 section. If $\mathbf{a}\in H^s(M,\mathrm{Hom}(T^*M\otimes\mathbf{V}_1,\mathbf{V}_3))$ and $\mathbf{u}\in H^r(M,\mathbf{V}_1)$, the same commutator applied to $\nabla\mathbf{u}$ satisfies

$$\|[\nabla^r,\mathbf{a}](\nabla\mathbf{u})\|_{L^2(M,T^{(0,r)}(TM)\otimes\mathbf{V}_3)} \leq C\|\mathbf{a}\|_{H^s(M,\mathrm{Hom}(T^*M\otimes\mathbf{V}_1,\mathbf{V}_3))}\|\mathbf{u}\|_{H^r(M,\mathbf{V}_1)}, \qquad 1\leq r\leq s. \tag{20.6}$$

Now let $\mathcal{U}\subseteq\mathbf{V}_1$ be an open subset of a bundle and let $G\colon\mathcal{U}\longrightarrow\mathbf{V}_2$ be a smooth bundle map. If $K\subseteq\mathcal{U}$ is compact, there exists a neighborhood $\mathcal{V}$ of K and, for each $R>0$, a constant $C_{K,R}$ with the following properties. If $\mathbf{u},\mathbf{v}\in H^r(M,\mathbf{V}_1)$, the images of $\mathbf{u}$ and $\mathbf{v}$ lie in $\mathcal{V}$, and $\|\mathbf{u}\|_{H^r(M,\mathbf{V}_1)}+\|\mathbf{v}\|_{H^r(M,\mathbf{V}_1)}\leq R$, then

$$\|G(\mathbf{u})\|_{H^r(M,\mathbf{V}_2)} \leq C_{K,R}, \tag{20.7}$$

$$\|G(\mathbf{u})-G(\mathbf{v})\|_{H^r(M,\mathbf{V}_2)} \leq C_{K,R}\|\mathbf{u}-\mathbf{v}\|_{H^r(M,\mathbf{V}_1)}. \tag{20.8}$$

These last two estimates hold whenever $r>\frac{m}{2}$. Moreover, for every integer $r\geq 0$ and every section $\mathbf{u}$ with image in $\mathcal{V}$, we have the higher-order estimate

$$\|G(\mathbf{u})\|_{H^r(M,\mathbf{V}_2)} \leq C_{K,r}\big(1+\|\mathbf{u}\|_{H^r(M,\mathbf{V}_1)}\big). \tag{20.9}$$

Here the constant depends on a uniform bound for the values of $\mathbf{u}$ and derivatives of G up to order r on a fixed neighborhood of those values; it does not depend on the order-r norm of $\mathbf{u}$. These estimates also hold for maps depending smoothly on several arguments, with the norm of the difference of all of them on the right-hand side.

Proof. Localize using a finite partition of unity and work in a chart and local frame. The covariant norms and Euclidean norms of the components are equivalent by Lemma 9.1.28. Its triangular formula shows which connection coefficients enter at each order; on the compact sets of this localization, all their required derivatives are bounded. It therefore suffices to justify the estimates for partial derivatives of vector-valued functions. Fix a regular coordinate ball (U,φ) and set $\Omega := \varphi(U)\subset\mathbb{R}^m$. In the following local calculations, u,v,a denote the scalar components of the sections or coefficients in the fixed frames; their Euclidean norms are taken over Ω.

Let $1 \le k \le r$ and $|\gamma| = k$. The Leibniz rule gives

$$\partial^\gamma (uv) = \sum_{\substack{\mu \in \mathbb{N}_0^m \\ \mu \le \gamma}} \binom{\gamma}{\mu} (\partial^\mu u)(\partial^{\gamma-\mu} v).$$

The terms $\mu = 0$ and $\mu = \gamma$ are bounded, respectively, by $\|u\|_{L^\infty(\Omega)} \|v\|_{H^k(\Omega)}$ and $\|u\|_{H^k(\Omega)} \|v\|_{L^\infty(\Omega)}$. If $0 < |\mu| < k$, set $\theta = |\mu|/k$ and choose $p = 2/\theta$, $q = 2/(1-\theta)$. The Gagliardo–Nirenberg inequality gives

$$\|\partial^\mu u\|_{L^p(\Omega)} \le C \|u\|_{L^\infty(\Omega)}^{1-\theta} \|u\|_{H^k(\Omega)}^{\theta},$$
$$\|\partial^{\gamma-\mu} v\|_{L^q(\Omega)} \le C \|v\|_{L^\infty(\Omega)}^{\theta} \|v\|_{H^k(\Omega)}^{1-\theta}.$$

Since $1/p + 1/q = 1/2$, Hölder and the weighted Young inequality give

$$\begin{aligned} \|(\partial^\mu u)(\partial^{\gamma-\mu} v)\|_{L^2(\Omega)} &\le C \left(\|u\|_{H^k(\Omega)} \|v\|_{L^\infty(\Omega)} \right)^{\theta} \left(\|u\|_{L^\infty(\Omega)} \|v\|_{H^k(\Omega)} \right)^{1-\theta} \\ &\le C \left(\|u\|_{H^k(\Omega)} \|v\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} \|v\|_{H^k(\Omega)} \right). \end{aligned}$$

Summing over $|\gamma| \le r$ proves (20.4).
For the commutator, observe that

$$[\partial^\gamma, a] v = \sum_{\substack{\mu \in \mathbb{N}_0^m \\ 0 < \mu \le \gamma}} \binom{\gamma}{\mu} (\partial^\mu a)(\partial^{\gamma-\mu} v).$$

Each summand distributes at most $k-1$ derivatives between ∂a and v. The product estimate just proved, applied at order $k-1$, gives

$$\|[\partial^\gamma, a] v\|_{L^2(\Omega)} \le C \left(\|\partial a\|_{L^\infty(\Omega)} \|v\|_{H^{k-1}(\Omega)} + \|\partial a\|_{H^{k-1}(\Omega)} \|v\|_{L^\infty(\Omega)} \right),$$

and summing over $k \le r$ yields (20.5).
Now consider the asymmetric commutator. For $|\gamma| = k \le r$, its components are sums of terms

$$(\partial^\mu a) \partial^{\gamma-\mu+1} u, \qquad 1 \le |\mu| \le k.$$

After taking the indicated derivatives, the first factor retains $s - |\mu|$ derivatives in L^2 and the second retains $|\mu| - 1$. The sum of these orders is $s - 1 > m/2$. Lemma 20.1.1 implies

$$\|(\partial^\mu a) \partial^{\gamma-\mu+1} u\|_{L^2(\Omega)} \le C \|a\|_{H^s(\Omega)} \|u\|_{H^k(\Omega)}.$$

Summing over all multiindices yields (20.6).
For the composition estimates, choose an open subset $\mathcal{V}$ whose closure is compact, contains K, and is contained in $\mathcal{U}$. A cutoff function on the total space, equal to one on $\mathcal{V}$ and supported in $\mathcal{U}$, allows us to extend G to a smooth bundle map $\widetilde{G}$ defined on all fibers. The Faà di Bruno formula of Theorem 1.1.19, including

derivatives in the base variable, expresses every derivative of order $k \le r$ of $\widetilde{G}(x,u(x))$ as terms of the form

$$(\partial_x^\alpha D_z^j \widetilde{G})(x,u(x))[\partial^{\mu_1}u,\dots,\partial^{\mu_j}u], \qquad \begin{aligned} &1 \le j \le k, \quad \alpha,\mu_1,\dots,\mu_j \in \mathbb{N}_0^m,\\ &|\mu_i| \ge 1 \ (1 \le i \le j), \qquad |\alpha| + \sum_{i=1}^{j} |\mu_i| = k, \end{aligned}$$

and uniformly bounded terms with no derivatives of u. For $r \ge 1$, choose $p_i = 2r/|\mu_i|$. The Gagliardo–Nirenberg inequality gives

$$\|\partial^{\mu_i} u\|_{L^{p_i}(\Omega)} \le C \|u\|_{L^\infty(\Omega)}^{1-|\mu_i|/r} \|u\|_{H^r(\Omega)}^{|\mu_i|/r}.$$

If $\theta := \sum_{i=1}^{j} |\mu_i|/r$, then $0 < \theta \le 1$ and $\sum_{i=1}^{j} 1/p_i = \theta/2 \le 1/2$. Hölder, followed by the inclusion $L^{2/\theta} \hookrightarrow L^2$ on the compact subset of the chart, bounds the product by $C_K \|u\|_{H^r(\Omega)}^{\theta}$. The inequality $x^\theta \le 1 + x$ for $x \ge 0$ proves (20.9). The case $r = 0$ follows by integrating the uniform bound for G on the image of u. The finite sum over charts preserves these estimates. In particular, a bound $\|u\|_{H^r(M,\mathbf{V}_1)} \le R$ gives (20.7). Finally,

$$\widetilde{G}(u) - \widetilde{G}(v) = \int_0^1 D\widetilde{G}\big(v + \theta(u-v)\big)[u-v]\,d\theta.$$

The global extension ensures that the segment always lies within the domain. Applying the preceding estimate to the integrand, with constants uniform in θ, yields (20.8). The same argument handles several arguments and completes the proof. □

For $T > 0$ and $r \ge 0$ define

$$\begin{aligned} \mathcal{X}_T^r(\mathbf{E}) &:= \big\{ \mathbf{u} \in L^2(0,T;H^{r+1}(M,\mathbf{E})) \;\big|\; \partial_t \mathbf{u} \in L^2(0,T;H^{r-1}(M,\mathbf{E})) \big\},\\ \mathcal{Y}_T^r(\mathbf{E}) &:= L^2(0,T;H^{r-1}(M,\mathbf{E})), \end{aligned}$$

where the time derivative is understood in the sense of distributions with values in $H^{r-1}(M,\mathbf{E})$. Initially equip $\mathcal{X}_T^r(\mathbf{E})$ with the graph norm

$$\|\mathbf{u}\|_{\mathcal{X}_T^r(\mathbf{E}),0}^2 := \int_0^T \|\mathbf{u}(t)\|_{H^{r+1}(M,\mathbf{E})}^2\,dt + \int_0^T \|\partial_t \mathbf{u}(t)\|_{H^{r-1}(M,\mathbf{E})}^2\,dt. \tag{20.10}$$

This norm is complete. Indeed, if $(\mathbf{u}_j)$ is Cauchy, there exist $\mathbf{u} \in L^2(0,T;H^{r+1})$ and $\mathbf{v} \in L^2(0,T;H^{r-1})$ such that $\mathbf{u}_j \to \mathbf{u}$ and $\partial_t \mathbf{u}_j \to \mathbf{v}$ in the corresponding spaces. Passing to the limit in the identity defining the distributional derivative gives $\partial_t \mathbf{u} = \mathbf{v}$. The next lemma allows the supremum in time to be incorporated into the norm.

We use the pairing of the Hilbert triple $H^{r+1}(M,\mathbf{E}) \subseteq H^r(M,\mathbf{E}) \subseteq H^{r-1}(M,\mathbf{E})$ given, on smooth sections, by

$$\langle \mathbf{f}, \mathbf{v} \rangle_{H^{r-1},H^{r+1}} := (\Lambda^{r-1}\mathbf{f}, \Lambda^{r+1}\mathbf{v})_{L^2(M,\mathbf{E})}. \tag{20.11}$$

Thus H^{r-1} plays the role of the dual of H^{r+1} when H^r is the pivot space.

Lemma 20.1.3. (Time trace) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold and let $\mathbf{E} \to M$ be a vector bundle with a bundle metric and compatible connection. Every element of $\mathcal{X}_T^r(\mathbf{E})$ has a unique representative in

$C([0,T];H^r(M,\mathbf{E}))$. For this representative,

$$\|\mathbf{u}(t)\|^2_{H^r(M,\mathbf{E})} - \|\mathbf{u}(\tau)\|^2_{H^r(M,\mathbf{E})} = 2\int_\tau^t \langle \partial_\rho \mathbf{u}(\rho), \mathbf{u}(\rho)\rangle_{H^{r-1},H^{r+1}}\, d\rho \tag{20.12}$$

for $0 \le \tau \le t \le T$. In particular, evaluation $\mathbf{u} \mapsto \mathbf{u}(0)$ is continuous from $\mathcal{X}^r_T(\mathbf{E})$ to $H^r(M,\mathbf{E})$.

Proof. Let $\Delta_\nabla = \nabla^*\nabla$ be the connection Laplacian and set $\Lambda = (I+\Delta_\nabla)^{1/2}$. By Theorem 12.4.10, there exists an orthonormal basis $(\mathbf{e}_j)_{j\ge1}$ of $L^2(M,\mathbf{E})$ consisting of smooth eigensections of Δ_∇. Denote by P_N the orthogonal projection onto the subspace spanned by $\mathbf{e}_1,\dots,\mathbf{e}_N$. The projections P_N commute with all powers of Λ, are contractions on each $H^\sigma(M,\mathbf{E})$, and converge strongly to the identity on these spaces.

Let $\mathbf{u} \in \mathcal{X}^r_T(\mathbf{E})$ and set $\mathbf{u}_N = P_N\mathbf{u}$. Since $\partial_t \mathbf{u}_N = P_N \partial_t \mathbf{u}$ in the distributional sense, each scalar coefficient of $\mathbf{u}_N$ belongs to $H^1(0,T)$. Thus $\mathbf{u}_N \in C([0,T];H^r(M,\mathbf{E}))$, and the differentiation rule on the finite-dimensional subspace gives

$$\|\mathbf{u}_N(t)\|^2_{H^r(M,\mathbf{E})} - \|\mathbf{u}_N(\tau)\|^2_{H^r(M,\mathbf{E})} = 2\int_\tau^t \langle \partial_\rho \mathbf{u}_N(\rho), \mathbf{u}_N(\rho)\rangle_{H^{r-1},H^{r+1}}\, d\rho. \tag{20.13}$$

Let us prove that $(\mathbf{u}_N)$ converges in $C([0,T];H^r(M,\mathbf{E}))$. If $\mathbf{w} = \mathbf{u}_N - \mathbf{u}_M$, there exists $\tau \in [0,T]$ such that

$$\|\mathbf{w}(\tau)\|^2_{H^r(M,\mathbf{E})} \le \frac{1}{T}\int_0^T \|\mathbf{w}(\rho)\|^2_{H^r(M,\mathbf{E})}\, d\rho \le \frac{1}{T}\|\mathbf{w}\|^2_{L^2(0,T;H^{r+1}(M,\mathbf{E}))}.$$

Applying (20.13) to $\mathbf{w}$ and using Cauchy–Schwarz in the pairing of (20.11) yields, for every $t \in [0,T]$,

$$\begin{aligned}\|\mathbf{w}(t)\|^2_{H^r(M,\mathbf{E})} &\le \frac{1}{T}\|\mathbf{w}\|^2_{L^2(0,T;H^{r+1}(M,\mathbf{E}))} + 2\|\partial_t\mathbf{w}\|_{L^2(0,T;H^{r-1}(M,\mathbf{E}))}\|\mathbf{w}\|_{L^2(0,T;H^{r+1}(M,\mathbf{E}))}\\ &\le \left(1+\frac{1}{T}\right)\|\mathbf{w}\|^2_{L^2(0,T;H^{r+1}(M,\mathbf{E}))} + \|\partial_t\mathbf{w}\|^2_{L^2(0,T;H^{r-1}(M,\mathbf{E}))}.\end{aligned}$$

Since $P_N \to I$ strongly in both Bochner spaces occurring here, the right-hand side tends to zero as $M,N \to \infty$. Consequently, there exists a section $\widetilde{\mathbf{u}} \in C([0,T];H^r(M,\mathbf{E}))$ such that $\mathbf{u}_N \to \widetilde{\mathbf{u}}$ uniformly on H^r. On the other hand, $\mathbf{u}_N \to \mathbf{u}$ in $L^2(0,T;H^{r+1})$; hence $\widetilde{\mathbf{u}} = \mathbf{u}$ for almost every time. Thus $\widetilde{\mathbf{u}}$ is a representative of $\mathbf{u}$. Two continuous representatives agreeing almost everywhere agree on all of $[0,T]$, proving uniqueness.

In (20.13), the endpoint terms converge uniformly. Moreover,

$$\begin{aligned}&\left|\int_\tau^t \left(\langle \partial_\rho \mathbf{u}_N, \mathbf{u}_N\rangle_{H^{r-1},H^{r+1}} - \langle \partial_\rho \mathbf{u}, \mathbf{u}\rangle_{H^{r-1},H^{r+1}}\right) d\rho\right|\\ &\quad\le \|\partial_t\mathbf{u}_N - \partial_t\mathbf{u}\|_{L^2(0,T;H^{r-1}(M,\mathbf{E}))}\|\mathbf{u}_N\|_{L^2(0,T;H^{r+1}(M,\mathbf{E}))}\\ &\qquad + \|\partial_t\mathbf{u}\|_{L^2(0,T;H^{r-1}(M,\mathbf{E}))}\|\mathbf{u}_N - \mathbf{u}\|_{L^2(0,T;H^{r+1}(M,\mathbf{E}))},\end{aligned}$$

where the norms are taken over $(0,T)$. The right-hand side tends to zero. Letting $N \to \infty$ gives (20.12) for all $0 \le \tau \le t \le T$.

Finally, choose $\tau \in [0,T]$ such that

$$\|\mathbf{u}(\tau)\|^2_{H^r(M,\mathbf{E})} \le \frac{1}{T}\|\mathbf{u}\|^2_{L^2(0,T;H^r)} \le \frac{1}{T}\|\mathbf{u}\|^2_{L^2(0,T;H^{r+1}(M,\mathbf{E}))}.$$

The energy identity and the same preceding argument give

$$\|\mathbf{u}\|^2_{L^\infty(0,T;H^r(M,\mathbf{E}))} \le \left(1+\frac{1}{T}\right)\|\mathbf{u}\|^2_{L^2(0,T;H^{r+1}(M,\mathbf{E}))} + \|\partial_t \mathbf{u}\|^2_{L^2(0,T;H^{r-1}(M,\mathbf{E}))}.$$

This proves continuity of all time evaluations with respect to the graph norm, in particular of the trace at $t=0$. □

From now on, equip $\mathcal{X}^r_T(\mathbf{E})$ with the norm

$$\|\mathbf{u}\|^2_{\mathcal{X}^r_T(\mathbf{E})} := \|\mathbf{u}\|^2_{L^\infty(0,T;H^r(M,\mathbf{E}))} + \int_0^T \|\mathbf{u}(t)\|^2_{H^{r+1}(M,\mathbf{E})}\,dt + \int_0^T \|\partial_t\mathbf{u}(t)\|^2_{H^{r-1}(M,\mathbf{E})}\,dt. \tag{20.14}$$

The preceding estimate shows that this norm is equivalent to (20.10).

20.2 Strongly parabolic linear systems

The preceding estimates control the products arising when an equation is differentiated. We now turn to the linear problem used at each step of the quasilinear iteration. Set $\mathcal{H} := T^*M \otimes \mathbf{E}$, with its induced metric and connection. Let

$$\mathbf{A}(t) \in \Gamma(\operatorname{End}\mathcal{H}), \qquad \mathbf{B}(t) \in \Gamma(\operatorname{Hom}(\mathcal{H},\mathbf{E})), \qquad \mathbf{C}(t) \in \Gamma(\operatorname{End}\mathbf{E})$$

be time-dependent sections. Consider the intrinsic operator

$$P_t\mathbf{u} := \nabla^*\big(\mathbf{A}(t)\nabla\mathbf{u}\big) + \mathbf{B}(t)(\nabla\mathbf{u}) + \mathbf{C}(t)\mathbf{u} \tag{20.15}$$

and the equation

$$\partial_t\mathbf{u} + P_t\mathbf{u} = \mathbf{f}, \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{20.16}$$

After fixing a chart and a local frame of $\mathbf{E}$, the indices i,j,k range over $\{1,\dots,m\}$, and we write $(\mathbf{A}\nabla\mathbf{u})_i = \sum_{j=1}^m A_i{}^j \nabla_j u$ and $\mathbf{B}(\nabla\mathbf{u}) = \sum_{j=1}^m B^j \nabla_j u$. With $A^{ij} := \sum_{k=1}^m g_0^{ik} A_k{}^j$, the local formula for (20.15) is

$$P_t u = -\sum_{i,j=1}^m \nabla_i\big(A^{ij}(t)\nabla_j u\big) + \sum_{j=1}^m B^j(t)\nabla_j u + C(t)u.$$

Definition 20.2.1. (Strong parabolicity) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold and let $\mathbf{E}\to M$ be a vector bundle with a bundle metric and compatible connection. We say that the family of operators (20.15) is **strongly parabolic** with constant $\lambda>0$ if $\mathbf{A}(t)$ is self-adjoint on $\mathcal{H}$ and

$$\langle \mathbf{A}(x,t)\zeta,\zeta\rangle_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}} \ge \lambda|\zeta|^2_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}} \tag{20.17}$$

for every $(x,t)\in M\times[0,T]$ and $\zeta\in T^*_xM\otimes\mathbf{E}_x$. Taking $\zeta=\xi\otimes v$ shows that this condition implies uniform positivity of the symbol. For operators with scalar symbol $a^{ij}\operatorname{Id}_\mathbf{E}$, the two conditions are equivalent.

The systems we use admit a divergence-form formulation. In a fixed chart and frame, an operator with prin-

cipal part $-\sum_{i,j=1}^{m} A^{ij}(\nabla^2\cdot)_{ij}$ can be written as $-\sum_{i,j=1}^{m}\nabla_i(A^{ij}\nabla_j)+\sum_{j=1}^{m}\widetilde{B}^j\nabla_j$, with $\widetilde{B}^j = B^j + \sum_{i=1}^{m}\nabla_i A^{ij}$ after incorporating the original term $\sum_{j=1}^{m} B^j\nabla_j$.

For integrations by parts, use the covariant energy

$$\mathscr{E}_r(\mathbf{u}) := \sum_{q=0}^{r} \|\nabla^q \mathbf{u}\|^2_{L^2(M,T^{(0,q)}(TM)\otimes\mathbf{E})}. \tag{20.18}$$

There exist constants $c_r, C_r > 0$ such that

$$c_r\|\mathbf{u}\|^2_{H^r(M,\mathbf{E})} \le \mathscr{E}_r(\mathbf{u}) \le C_r\|\mathbf{u}\|^2_{H^r(M,\mathbf{E})}.$$

The trace identity refers to the spectral norm, whereas the following differential identities are written for $\mathscr{E}_r$; integrated estimates transfer between them through this equivalence.

Lemma 20.2.2. (Higher-order energy estimate) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold and let $\mathbf{E}\to M$ be a vector bundle with a bundle metric and compatible connection. Suppose that (20.17) holds and that

$$\sup_{t\in[0,T]}\Big(\|\mathbf{A}(t)\|_{H^s(M,\operatorname{End}\mathcal{H})} + \|\mathbf{B}(t)\|_{H^s(M,\operatorname{Hom}(\mathcal{H},\mathbf{E}))} + \|\mathbf{C}(t)\|_{H^s(M,\operatorname{End}\mathbf{E})}\Big) \le K. \tag{20.19}$$

For $r\in\{0,1,\dots,s\}$ there exist constants $c>0$ and $C>0$, depending only on r,s,λ,K and the fixed metrics and connections, such that every smooth solution of (20.16) satisfies

$$\frac{d}{dt}\mathscr{E}_r(\mathbf{u}(t)) + c\|\mathbf{u}(t)\|^2_{H^{r+1}(M,\mathbf{E})} \le C\mathscr{E}_r(\mathbf{u}(t)) + C\|\mathbf{f}(t)\|^2_{H^{r-1}(M,\mathbf{E})}. \tag{20.20}$$

Consequently,

$$\|\mathbf{u}\|^2_{L^\infty(0,T;H^r(M,\mathbf{E}))} + c\int_0^T \|\mathbf{u}(t)\|^2_{H^{r+1}(M,\mathbf{E})}\,dt \le Ce^{CT}\left(\|\mathbf{u}_0\|^2_{H^r(M,\mathbf{E})} + \int_0^T \|\mathbf{f}(t)\|^2_{H^{r-1}(M,\mathbf{E})}\,dt\right). \tag{20.21}$$

Proof. For $q\in\{0,1,\dots,r\}$, apply ∇^q to the equation and pair with $\nabla^q\mathbf{u}$. The new derivative occupies the first slot, according to Definition 9.1.1. Let $\mathbf{V}_q$ be the tensor obtained from $\nabla(\nabla^q\mathbf{u})$ by moving this first slot to the last covariant slot:

$$\mathbf{V}_q(X_1,\dots,X_q,Y) := (\nabla(\nabla^q\mathbf{u}))(Y,X_1,\dots,X_q).$$

Thus $\mathbf{A}$ acts on the pair consisting of Y and the value in $\mathbf{E}$, and $|\mathbf{V}_q| = |\nabla^{q+1}\mathbf{u}|$. Green's formula in Theorem 9.1.29 gives the principal contribution

$$\int_M \langle (I_{T^{(0,q)}(TM)}\otimes\mathbf{A})\mathbf{V}_q,\mathbf{V}_q\rangle\,d\lambda_{\mathbf{g}_0} + \mathcal{R}_q.$$

The integral is at least $\lambda\|\nabla^{q+1}\mathbf{u}\|^2_{L^2(M,T^{(0,q+1)}(TM)\otimes\mathbf{E})}$.

Let us describe the contributions of $\mathcal{R}_q$. When the q derivatives are distributed between $\mathbf{A}$ and $\nabla\mathbf{u}$, the terms with one derivative on $\mathbf{A}$ form $[\nabla^q,\mathbf{A}](\nabla\mathbf{u})$. Reordering the remaining covariant slots introduces commutators of two derivatives. For $q=1$ use the curvature identity of the induced connection. If the

interchanges have been carried out up to order q, applying one more derivative produces two types of terms: a derivative of a curvature coefficient already present and a derivative of the section to which that coefficient is applied. The new interchange contributes another curvature multiplied by a lower-order derivative of the section. The Leibniz rule in Proposition 9.1.40 controls the first two types. This is the same commutation step developed in Lemma 9.1.41; here it also applies to $T^{(0,q)}(TM)\otimes\mathbf{E}$. At order q, only curvatures of the fixed structures and their derivatives up to order $q-1$ occur. These terms therefore have bounded coefficients and contain at most q derivatives of $\mathbf{u}$ after integration by parts.
For $q=0$ there is no commutator. For $1\le q\le r$, estimate (20.6) gives, for each $\varepsilon>0$,

$$|\mathcal{R}_q|\le CK\|\mathbf{u}\|_{H^r(M,\mathbf{E})}\|\mathbf{u}\|_{H^{r+1}(M,\mathbf{E})}\le \varepsilon\|\mathbf{u}\|^2_{H^{r+1}(M,\mathbf{E})}+C_\varepsilon\|\mathbf{u}\|^2_{H^r(M,\mathbf{E})}.$$

The lower-order terms obey

$$\|\mathbf{B}(\nabla\mathbf{u})+\mathbf{C}\mathbf{u}\|_{H^q(M,\mathbf{E})}\le CK\|\mathbf{u}\|_{H^{q+1}(M,\mathbf{E})},\qquad q\in\{0,\dots,r\}.$$

Indeed, in every Leibniz product containing $|\mu|$ derivatives of a coefficient, the available orders for applying Lemma 20.1.1 are $s-|\mu|$ and $|\mu|$; their sum is $s>m/2$. This observation includes $q=0$ and avoids requiring an L^∞ norm of the solution. After pairing with $\nabla^q\mathbf{u}$, Young gives the same bound with ε for each of these terms.
If $r=0$, interpret the source term directly by duality:

$$|\langle\mathbf{f},\mathbf{u}\rangle_{H^{-1}(M,\mathbf{E}),H^1(M,\mathbf{E})}|\le\|\mathbf{f}\|_{H^{-1}(M,\mathbf{E})}\|\mathbf{u}\|_{H^1(M,\mathbf{E})},$$

and Young gives the required bound. Now suppose that $r\ge1$. For smooth $\mathbf{f}$, the contribution obtained by summing the covariant energies is

$$\mathcal{B}_r(\mathbf{f},\mathbf{u}):=\sum_{q=0}^{r}\int_M\langle\nabla^q\mathbf{f},\nabla^q\mathbf{u}\rangle\,d\lambda_{\mathbf{g}_0}.$$

For $0\le q\le r-1$, Cauchy–Schwarz gives directly

$$\left|\int_M\langle\nabla^q\mathbf{f},\nabla^q\mathbf{u}\rangle\,d\lambda_{\mathbf{g}_0}\right|\le C\|\mathbf{f}\|_{H^{r-1}(M,\mathbf{E})}\|\mathbf{u}\|_{H^{r-1}(M,\mathbf{E})}.$$

In the highest-order term, write $\nabla^r\mathbf{f}=\nabla(\nabla^{r-1}\mathbf{f})$ and apply the definition of the covariant adjoint:

$$\int_M\langle\nabla^r\mathbf{f},\nabla^r\mathbf{u}\rangle\,d\lambda_{\mathbf{g}_0}=\int_M\langle\nabla^{r-1}\mathbf{f},\nabla^*(\nabla^r\mathbf{u})\rangle\,d\lambda_{\mathbf{g}_0}.$$

The operator $\mathbf{u}\mapsto\nabla^*(\nabla^r\mathbf{u})$ has order $r+1$, so its L^2 norm is bounded by $C\|\mathbf{u}\|_{H^{r+1}(M,\mathbf{E})}$. Summing the preceding terms yields

$$|\mathcal{B}_r(\mathbf{f},\mathbf{u})|\le C\|\mathbf{f}\|_{H^{r-1}(M,\mathbf{E})}\|\mathbf{u}\|_{H^{r+1}(M,\mathbf{E})}.$$

For $\mathbf{f}\in H^{r-1}(M,\mathbf{E})$ and $\mathbf{u}\in H^{r+1}(M,\mathbf{E})$, the expression defining the extension is

$$\mathcal{B}_r(\mathbf{f},\mathbf{u})=\sum_{q=0}^{r-1}\int_M\langle\nabla_w^q\mathbf{f},\nabla_w^q\mathbf{u}\rangle\,d\lambda_{\mathbf{g}_0}+\int_M\langle\nabla_w^{r-1}\mathbf{f},\nabla^*(\nabla_w^r\mathbf{u})\rangle\,d\lambda_{\mathbf{g}_0}.$$

Each pairing is between two L^2 sections of the same bundle $T^{(0,q)}(TM)\otimes\mathbf{E}$; in the last term, take $q=r-1$. Thus the integrals are defined. If $\mathbf{f}_j\to\mathbf{f}$ in $H^{r-1}(M,\mathbf{E})$ and $\mathbf{u}_j\to\mathbf{u}$ in $H^{r+1}(M,\mathbf{E})$ are smooth Meyers–Serrin approximations, bilinearity and the preceding bound give

$$\begin{aligned}|\mathcal{B}_r(\mathbf{f}_j,\mathbf{u}_j)-\mathcal{B}_r(\mathbf{f},\mathbf{u})| &\le C\|\mathbf{f}_j-\mathbf{f}\|_{H^{r-1}(M,\mathbf{E})}\|\mathbf{u}_j\|_{H^{r+1}(M,\mathbf{E})}\\ &\quad + C\|\mathbf{f}\|_{H^{r-1}(M,\mathbf{E})}\|\mathbf{u}_j-\mathbf{u}\|_{H^{r+1}(M,\mathbf{E})}\longrightarrow 0.\end{aligned}$$

Thus the extension agrees with the smooth expression, is independent of both approximations, and satisfies the same bound. Young then gives

$$|\mathcal{B}_r(\mathbf{f},\mathbf{u})|\le \varepsilon\|\mathbf{u}\|^2_{H^{r+1}(M,\mathbf{E})}+C\|\mathbf{f}\|^2_{H^{r-1}(M,\mathbf{E})}.$$

Now sum all contributions. The sum of the coercive terms controls $\|\mathbf{u}\|^2_{H^{r+1}(M,\mathbf{E})}$ up to a multiple of $\|\mathbf{u}\|^2_{L^2(M,\mathbf{E})}$, which is incorporated into the right-hand side. Choose ε after this summation, smaller than the coercivity constant divided by twice the number of terms to be absorbed. A positive constant c remains, and we obtain (20.20). Integration over $[0,t]$ and Grönwall's inequality in Lemma 1.8.19 prove (20.21). □

Theorem 20.2.3. (Linear existence and parabolic regularity) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold and let $\mathbf{E}\to M$ be a vector bundle with a bundle metric and compatible connection. Let $r\in\{0,1,\dots,s\}$. Suppose that

$$\begin{aligned}&\mathbf{A}\in C([0,T];H^s(M,\operatorname{End}\mathcal{H})),\\ &\mathbf{B}\in C([0,T];H^s(M,\operatorname{Hom}(\mathcal{H},\mathbf{E}))),\\ &\mathbf{C}\in C([0,T];H^s(M,\operatorname{End}\mathbf{E})),\end{aligned}$$

satisfy (20.17) with a constant independent of t and obey (20.19). For every

$$\mathbf{u}_0\in H^r(M,\mathbf{E}),\qquad \mathbf{f}\in\mathcal{Y}^r_T(\mathbf{E}),$$

there exists a unique solution $\mathbf{u}\in\mathcal{X}^r_T(\mathbf{E})$ of (20.16). Moreover,

$$\|\mathbf{u}\|_{\mathcal{X}^r_T(\mathbf{E})}\le C_T\big(\|\mathbf{u}_0\|_{H^r(M,\mathbf{E})}+\|\mathbf{f}\|_{\mathcal{Y}^r_T(\mathbf{E})}\big),\tag{20.22}$$

where C_T can be chosen uniformly when T, K, and λ^{-1} remain bounded. If the coefficients, $\mathbf{f}$, and $\mathbf{u}_0$ are smooth, then $\mathbf{u}$ is smooth on $M\times[0,T]$.

Proof. Let $(\mathbf{e}_k)_{k\ge1}$ be an orthonormal basis of $L^2(M,\mathbf{E})$ consisting of eigensections of the connection Laplacian, ordered by eigenvalues $0\le\mu_1\le\mu_2\le\cdots$ counted with multiplicity. Let V_N be the span of $\mathbf{e}_1,\dots,\mathbf{e}_N$. Seek $\mathbf{u}_N(t)=\sum_{k=1}^{N}d_k(t)\mathbf{e}_k$ and require that, for every $\mathbf{v}\in V_N$,

$$(\partial_t\mathbf{u}_N,\mathbf{v})_{L^2(M,\mathbf{E})}+\int_M\langle\mathbf{A}\nabla\mathbf{u}_N,\nabla\mathbf{v}\rangle_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}}\,d\lambda_{\mathbf{g}_0}+\int_M\langle\mathbf{B}(\nabla\mathbf{u}_N)+\mathbf{C}\mathbf{u}_N,\mathbf{v}\rangle_{\mathbf{h}_\mathbf{E}}\,d\lambda_{\mathbf{g}_0}=\langle\mathbf{f},\mathbf{v}\rangle_{H^{-1}(M,\mathbf{E}),H^1(M,\mathbf{E})}.\tag{20.23}$$

For smooth data, this is a finite-dimensional linear differential equation with continuous coefficients and

therefore has a solution on all of $[0,T]$, with $\mathbf{u}_N(0)$ equal to the orthogonal projection of $\mathbf{u}_0$ onto V_N. Taking $\mathbf{v} = \mathbf{u}_N$ and using (20.17) gives, with constants independent of N,

$$\|\mathbf{u}_N\|^2_{L^\infty(0,T;L^2(M,\mathbf{E}))} + \int_0^T \|\mathbf{u}_N(t)\|^2_{H^1(M,\mathbf{E})}\,dt \le C_T\left(\|\mathbf{u}_0\|^2_{L^2(M,\mathbf{E})} + \int_0^T \|\mathbf{f}(t)\|^2_{H^{-1}(M,\mathbf{E})}\,dt\right). \tag{20.24}$$

Let Π_N be the spectral projection onto V_N. It is a contraction on H^1 and H^{-1}, because it retains the first coefficients in the spectral norms. The projected equation reads $\partial_t \mathbf{u}_N = \Pi_N(\mathbf{f} - P_t\mathbf{u}_N)$. The operator $P_t\colon H^1 \to H^{-1}$ is uniformly bounded by the coefficient bounds and the weak formula. The order-zero energy then gives a uniform bound for $\partial_t\mathbf{u}_N$ in $L^2(0,T;H^{-1}(M,\mathbf{E}))$. Reflexivity allows us to extract a subsequence converging weakly in $L^2(0,T;H^1(M,\mathbf{E}))$ whose time derivatives converge weakly in $L^2(0,T;H^{-1}(M,\mathbf{E}))$. Strong convergence in $L^2(0,T;L^2(M,\mathbf{E}))$ can be seen directly. If Q_J is the projection onto the first J eigensections, the bound H^1 gives

$$\int_0^T \|(I-Q_J)\mathbf{u}_N(t)\|^2_{L^2(M,\mathbf{E})}\,dt \le \frac{C}{1+\mu_{J+1}},$$

where $\mu_{J+1} \to \infty$ is the next eigenvalue. For fixed J, the coordinates of $Q_J\mathbf{u}_N$ are uniformly bounded in $H^1(0,T)$ by the bound for $\partial_t\mathbf{u}_N$ in $H^{-1}(M,\mathbf{E})$; a subsequence converges uniformly in the finite-dimensional space $Q_JL^2(M,\mathbf{E})$. First choose J large and then a diagonal subsequence. Passing to the limit in (20.23) gives a weak solution $\mathbf{u}$. Moreover, the two preceding weak convergences show that $\mathbf{u}_N \rightharpoonup \mathbf{u}$ in the energy space $\mathcal{X}^0_T(\mathbf{E})$. The trace $\mathcal{X}^0_T(\mathbf{E}) \to L^2(M,\mathbf{E})$ is linear and continuous by Lemma 20.1.3 and is therefore weakly continuous. Since $\mathbf{u}_N(0) = \Pi_N\mathbf{u}_0 \to \mathbf{u}_0$ strongly in L^2, we conclude that $\mathbf{u}(0) = \mathbf{u}_0$.

We now show that data of order r give exactly the asserted regularity. For $r = 0$, the preceding construction and the equation already give $\mathbf{u} \in \mathcal{X}^0_T(\mathbf{E})$. Thus suppose that $r \ge 1$. We obtain each spatial derivative using difference quotients. Begin with the first derivative.

Let us justify the test sections depending on the solution. First fix the spatial step $h \ne 0$ and a time $T' < T$. For $0 < \varepsilon < T - T'$ use the Steklov averages

$$\mathbf{u}^\varepsilon(t) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbf{u}(\rho)\,d\rho, \qquad t \in [0,T'].$$

Spatial difference quotients of $\mathbf{u}^\varepsilon$, localized in a chart, belong to $L^2(0,T';H^1)$ and are admissible in the averaged weak equation. Integrate between 0 and a time in $[0,T']$, using time cutoffs approximating its indicator function and including the initial term. For fixed h, first let $\varepsilon \to 0^+$. For example, $(\mathbf{A}\nabla\mathbf{u})^\varepsilon$ and $\mathbf{A}\nabla\mathbf{u}^\varepsilon$ both converge to $\mathbf{A}\nabla\mathbf{u}$ in L^2; continuity in time and the uniform bound for $\mathbf{A}$ justify this assertion. Quotients with a fixed step are bounded operators on L^2 and therefore preserve this convergence. Moreover, $\mathbf{u}^\varepsilon(0) \to \mathbf{u}_0$ in L^2 by the order-zero trace already obtained. This recovers the initial term for the difference quotients. Only afterward do we let $h \to 0$, using the uniform estimates below. At the end we may take $T' \to T^-$.

In a chart and a trivialization of rank $d := \operatorname{rank}\mathbf{E}$, let δ^ℓ_h be the difference quotient in the coordinate direction ℓ, and let η be a smooth function compactly supported in the chart. Choose an open subset U' relatively compact in the same chart and containing a neighborhood of $\operatorname{supp}\eta$; the steps h are chosen so that all translations used remain in U'. If $d\lambda_{\mathbf{g}_0} = a(x)\,dx$ and $H(x)$ is the matrix of the bundle metric, set $\mathsf{G}(x) := a(x)H(x)$. In the weak formulation, use as test section the weighted discrete adjoint

$$-\mathsf{G}^{-1}\delta^\ell_{-h}\big(\mathsf{G}\eta^2\delta^\ell_h\mathbf{u}\big).$$

The exact discrete identity in the chart is

$$\int \langle \mathsf{G}\delta_h^\ell v, w\rangle_{\mathbb{R}^d}\, dx = -\int \langle \mathsf{G}v, \mathsf{G}^{-1}\delta_{-h}^\ell(\mathsf{G}w)\rangle_{\mathbb{R}^d}\, dx. \tag{20.25}$$

The weighted operator differs from δ_{-h}^ℓ by an operator uniformly bounded on L^2. Indeed, if $\tau_h^\ell w(x) = w(x + he_\ell)$, the discrete product rule is

$$\delta_h^\ell(fw) = (\tau_h^\ell f)\delta_h^\ell w + (\delta_h^\ell f)w.$$

In these estimates, compactly supported functions are understood to be extended by zero to $\mathbb{R}^m$. Applying it to $\mathsf{G}w$, the first additional coefficient is $O(|h|)$; its product with $\delta_{-h}^\ell w$ is bounded because $\|h\delta_{-h}^\ell w\|_{L^2(\mathbb{R}^m,\mathbb{R}^d)} \le 2\|w\|_{L^2(\mathbb{R}^m,\mathbb{R}^d)}$. The second coefficient is bounded by the first derivatives of G.

The principal part contains the translated matrix $A(x + he_\ell, t)$, together with the translated metric weights. Comparison with the weights at x introduces an error $O(|h|)$, uniform by the C^1 bounds for the fixed coefficients. Thus, for $|h|$ sufficiently small, the principal integral controls

$$\frac{\lambda}{2}\int_M \eta^2 |\nabla \delta_h^\ell \mathbf{u}|^2_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}}\, d\lambda_{\mathbf{g}_0}.$$

The time term is

$$\frac{1}{2}\frac{d}{dt}\int_M \eta^2 |\delta_h^\ell \mathbf{u}|^2_{\mathbf{h}_\mathbf{E}}\, d\lambda_{\mathbf{g}_0}.$$

Terms in which the difference quotient falls on the local representation of $\mathbf{A}$, on η, or on the connection coefficients are bounded by

$$\frac{\lambda}{4}\int_M \eta^2 |\nabla \delta_h^\ell \mathbf{u}|^2_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}}\, d\lambda_{\mathbf{g}_0} + C\int_{U'} \left(|\nabla \mathbf{u}|^2_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}} + |\mathbf{u}|^2_{\mathbf{h}_\mathbf{E}} + |\mathbf{f}|^2_{\mathbf{h}_\mathbf{E}}\right) d\lambda_{\mathbf{g}_0}.$$

The bound is uniform for $|h|$ smaller than the distance between $\operatorname{supp}\eta$ and the edge of the chart. Integrating over $[0,T]$ and letting $h \to 0$ gives one additional spatial derivative in a smaller chart.

To iterate, we do not assume in advance that $\nabla\mathbf{u}$ is bounded. Apply difference quotients of order q, $1 \le q \le r$, and use as test sections the backward quotients of order q of the same localized expression. The commutator with the principal part follows from the discrete product rule. For $\gamma \in \mathbb{N}_0^m$ and a common step in each direction, writing τ_h^γ for translation by $h\gamma$, we have

$$\delta_h^\gamma(fw) = \sum_{\substack{\mu\in\mathbb{N}_0^m \\ \mu\le\gamma}} \binom{\gamma}{\mu} (\tau_h^{\gamma-\mu}\delta_h^\mu f)(\delta_h^{\gamma-\mu} w).$$

This identity follows by applying the first-order rule once for each direction in the multiindex. Translations are bounded in the localized norms, and quotients of order $|\mu|$ are controlled by derivatives of that order in U'. Thus the commutator terms satisfy the same bounds as the differential expressions to which they converge:

$$(\partial^\mu A)\,\partial^{\gamma-\mu+1}\mathbf{u}, \qquad \mu,\gamma \in \mathbb{N}_0^m, \quad 0 < \mu \le \gamma, \quad |\gamma| = q, \tag{20.26}$$

and the connection terms have strictly lower order. The asymmetric estimate (20.6) shows that the sum of

all terms (20.26) is bounded by $C\|\mathbf{A}\|_{H^s(M,\operatorname{End}\mathcal{H})}\|\mathbf{u}\|_{H^{|\gamma|}(M,\mathbf{E})}$. The first-order and zeroth-order terms obey

$$\|\mathbf{B}(\nabla\mathbf{u})+\mathbf{C}\mathbf{u}\|_{H^{q-1}(M,\mathbf{E})}\le CK\|\mathbf{u}\|_{H^q(M,\mathbf{E})},$$

while the sum of the source terms is the preceding covariant form $\mathcal{B}_q(\mathbf{f},\mathbf{u})$. Moving one derivative in the highest-order term and then using spectral duality (20.11) gives

$$|\mathcal{B}_q(\mathbf{f},\mathbf{u})|\le\frac{c}{4}\|\mathbf{u}\|^2_{H^{q+1}(M,\mathbf{E})}+C\|\mathbf{f}\|^2_{H^{q-1}(M,\mathbf{E})}.$$

Let us specify the induction that passes from these local estimates to global regularity. For $0\le q\le r$, let $\mathcal{P}_q$ be the assertion

$$\mathbf{u}\in C([0,T];H^q(M,\mathbf{E}))\cap L^2(0,T;H^{q+1}(M,\mathbf{E}))$$

together with the energy estimate of order q written below. The assertion $\mathcal{P}_0$ is (20.24) after passage to the limit. The difference-quotient calculation just carried out proves $\mathcal{P}_1$: the uniform bound for $\delta_h^\ell\mathbf{u}$ allows a weak limit to be extracted, and the difference-quotient characterization identifies that limit with $\partial_\ell\mathbf{u}$.

Now suppose that $\mathcal{P}_0,\dots,\mathcal{P}_{q-1}$ have been proved, where $2\le q\le r$. Apply a difference quotient of order q to the equation and test against its localized backward quotient. When all quotients fall on $\mathbf{u}$, parabolicity gives the coercive term of order $q+1$. If at least one falls on $\mathbf{A}$, we obtain a difference-quotient and translation version of one of the terms in (20.26); the preceding Moser estimate and Young allow the part containing $\|\mathbf{u}\|^2_{H^{q+1}(M,\mathbf{E})}$ to be absorbed. Quotients falling on $\mathbf{B},\mathbf{C}$, connection coefficients, or cutoffs contain at most q derivatives of $\mathbf{u}$ and are controlled by $\mathcal{P}_{q-1}$ and the uniform coefficient bound. The source term is controlled by $\mathcal{B}_q$, as shown above. Finally, quotients of the initial data are uniformly bounded by $\|\mathbf{u}_0\|_{H^q(M,\mathbf{E})}$.

Choose a finite cover by charts and a subordinate partition of unity. Summing the local estimates, terms in which derivatives reach the cutoffs have lower order and are already covered by $\mathcal{P}_{q-1}$. The resulting bounds are uniform in the quotient step; letting $h\to0$, weak lower semicontinuity of the norm gives $\mathbf{u}\in L^2(0,T;H^{q+1}(M,\mathbf{E}))$. The equation itself, viewed as an identity in $H^{q-1}(M,\mathbf{E})$, then gives $\partial_t\mathbf{u}\in L^2(0,T;H^{q-1}(M,\mathbf{E}))$. Applying Lemma 20.1.3 to the Hilbert triple

$$H^{q+1}(M,\mathbf{E})\hookrightarrow H^q(M,\mathbf{E})\hookrightarrow H^{q-1}(M,\mathbf{E})$$

yields $\mathbf{u}\in C([0,T];H^q(M,\mathbf{E}))$. This proves $\mathcal{P}_q$. The inductive step is complete, and for each $1\le q\le r$ we obtain

$$\frac{d}{dt}\mathscr{E}_q(\mathbf{u}(t))+c\|\mathbf{u}(t)\|^2_{H^{q+1}(M,\mathbf{E})}\le C\mathscr{E}_q(\mathbf{u}(t))+C\|\mathbf{f}(t)\|^2_{H^{q-1}(M,\mathbf{E})}. \tag{20.27}$$

Each step uses only the coefficient bound and regularity from the preceding step. Integrating the last inequality gives

$$\|\mathbf{u}\|^2_{L^\infty(0,T;H^r(M,\mathbf{E}))}+\int_0^T\|\mathbf{u}(t)\|^2_{H^{r+1}(M,\mathbf{E})}\,dt\le C_T\left(\|\mathbf{u}_0\|^2_{H^r(M,\mathbf{E})}+\int_0^T\|\mathbf{f}(t)\|^2_{H^{r-1}(M,\mathbf{E})}\,dt\right). \tag{20.28}$$

Control up to $t=0$ follows by applying the same calculation to approximations with smooth initial data and using the term $\|\mathbf{u}_0\|_{H^r(M,\mathbf{E})}$ when integrating the energy identity. No additional compatibility condition arises, since M is closed.

The equation and continuity of multiplication imply

$$\|\partial_t \mathbf{u}\|_{L^2(0,T;H^{r-1}(M,\mathbf{E}))} \le C\big(\|\mathbf{u}\|_{L^2(0,T;H^{r+1}(M,\mathbf{E}))} + \|\mathbf{u}\|_{L^2(0,T;H^{r}(M,\mathbf{E}))} + \|\mathbf{f}\|_{L^2(0,T;H^{r-1}(M,\mathbf{E}))}\big).$$

This proves (20.22) for smooth data and coefficients.

We turn to coefficients and data with the regularity in the statement. Choose smooth approximations $\mathbf{A}_j, \mathbf{B}_j, \mathbf{C}_j, \mathbf{f}_j$ and $\mathbf{u}_{0,j}$ converging in the stated spaces. Replace $\mathbf{A}_j$ by its self-adjoint part. Since $H^s(M) \hookrightarrow C^1(M)$, convergence of $\mathbf{A}_j$ is uniform; thus, from some index onward,

$$\langle \mathbf{A}_j(x,t)\zeta, \zeta\rangle \ge \frac{\lambda}{2}|\zeta|^2$$

for every (x,t) and every $\zeta \in T^*_xM \otimes \mathbf{E}_x$. The smooth solutions $\mathbf{u}_j$ satisfy a uniform bound in $\mathcal{X}^r_T(\mathbf{E})$ by (20.21) and the equation itself.

Reflexivity of the Bochner spaces allows us to extract a subsequence such that

$$\begin{aligned} \mathbf{u}_j &\rightharpoonup \mathbf{u} && \text{in } L^2(0,T;H^{r+1}(M,\mathbf{E})),\\ \partial_t \mathbf{u}_j &\rightharpoonup \partial_t \mathbf{u} && \text{in } L^2(0,T;H^{r-1}(M,\mathbf{E})). \end{aligned}$$

To pass to the limit in the principal term, write

$$\mathbf{A}_j \nabla \mathbf{u}_j - \mathbf{A}\nabla \mathbf{u} = (\mathbf{A}_j - \mathbf{A})\nabla \mathbf{u}_j + \mathbf{A}\nabla(\mathbf{u}_j - \mathbf{u}).$$

The first part converges strongly to zero in $L^2(0,T;H^{r-1})$ by the product estimates, and the second converges weakly. The terms containing $\mathbf{B}_j$ and $\mathbf{C}_j$ are handled in the same way. Testing the equation against time-dependent smooth sections therefore gives the limiting problem.

The trace map $\mathcal{X}^r_T(\mathbf{E}) \to H^r(M,\mathbf{E})$ is linear and continuous by Lemma 20.1.3 and is therefore weakly continuous. Since $\mathbf{u}_j(0) = \mathbf{u}_{0,j} \to \mathbf{u}_0$, we conclude that $\mathbf{u}(0) = \mathbf{u}_0$. Weak lower semicontinuity preserves estimate (20.22). If two solutions have the same data, the order-zero energy estimate applied to their difference gives uniqueness. Finally, when all data are smooth, repeat the estimate at every spatial order; the equation expresses time derivatives in terms of the spatial derivatives already controlled and gives joint smoothness. □

Proposition 20.2.4. (Stability with respect to coefficients) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold and let $\mathbf{E} \to M$ be a vector bundle with a bundle metric and compatible connection. Let P_t and $\widetilde{P}_t$ be two families satisfying the hypotheses of Theorem 20.2.3 with the same bounds K and λ. Let $\mathbf{u}, \widetilde{\mathbf{u}} \in \mathcal{X}^s_T(\mathbf{E})$ be the solutions corresponding to $(\mathbf{u}_0, \mathbf{f})$ and $(\widetilde{\mathbf{u}}_0, \widetilde{\mathbf{f}})$. Then

$$\begin{aligned} &\|\mathbf{u} - \widetilde{\mathbf{u}}\|_{\mathcal{X}^{s-1}_T(\mathbf{E})}\\ &\quad\le C_T\Big(\|\mathbf{u}_0 - \widetilde{\mathbf{u}}_0\|_{H^{s-1}(M,\mathbf{E})} + \|\mathbf{f} - \widetilde{\mathbf{f}}\|_{\mathcal{Y}^{s-1}_T(\mathbf{E})}\\ &\qquad + \|\mathbf{A} - \widetilde{\mathbf{A}}\|_{C([0,T];H^{s-1}(M,\operatorname{End}\mathcal{H}))}\|\widetilde{\mathbf{u}}\|_{L^2(0,T;H^{s+1}(M,\mathbf{E}))}\\ &\qquad + \|\mathbf{B} - \widetilde{\mathbf{B}}\|_{C([0,T];H^{s-1}(M,\operatorname{Hom}(\mathcal{H},\mathbf{E})))}\|\widetilde{\mathbf{u}}\|_{L^2(0,T;H^{s}(M,\mathbf{E}))}\\ &\qquad + \|\mathbf{C} - \widetilde{\mathbf{C}}\|_{C([0,T];H^{s-1}(M,\operatorname{End}\mathbf{E}))}\|\widetilde{\mathbf{u}}\|_{L^2(0,T;H^{s}(M,\mathbf{E}))}\Big). \end{aligned} \tag{20.29}$$

Proof. Set $\mathbf{w} = \mathbf{u} - \widetilde{\mathbf{u}}$. Subtracting the equations and retaining P_t on the left-hand side gives

$$\partial_t \mathbf{w} + P_t \mathbf{w} = \mathbf{f} - \widetilde{\mathbf{f}} - \nabla^*\big((\mathbf{A} - \widetilde{\mathbf{A}})\nabla\widetilde{\mathbf{u}}\big) - (\mathbf{B} - \widetilde{\mathbf{B}})(\nabla\widetilde{\mathbf{u}}) - (\mathbf{C} - \widetilde{\mathbf{C}})\widetilde{\mathbf{u}}.$$

Since $s - 1 > \frac{m}{2}$, product estimate (20.4) shows that the right-hand side belongs to $L^2(0,T;H^{s-2}(M,\mathbf{E}))$ and is bounded by the terms appearing in (20.29). Apply Theorem 20.2.3 with $r = s - 1$. □

20.3 Quasilinear equations

The linear theory can be applied with coefficients frozen at a given section. The nonlinear solution will be constructed by repeating this procedure and showing that, on a sufficiently short interval, the iteration stays in a uniformly parabolic region and becomes contractive in a weaker norm. Let $\mathcal{U} \subseteq \mathbf{E}$ be an open subset of a bundle and consider two smooth bundle maps

$$A\colon \mathcal{U} \longrightarrow \operatorname{End}(T^*M \otimes \mathbf{E}),$$
$$F\colon J^1(\mathcal{U}) \longrightarrow \mathbf{E}.$$

Here $J^1(\mathcal{U})$ denotes the open subset of the bundle $\mathbf{E} \oplus (T^*M \otimes \mathbf{E})$ consisting of pairs (z,p) with $z \in \mathcal{U}$. We study

$$\partial_t \mathbf{u} + \nabla^*\big(A(\mathbf{u})\nabla\mathbf{u}\big) = F(j^1\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{20.30}$$

For $\xi \in T^*_xM$, let $\iota_\xi\colon \mathbf{E}_x \to T^*_xM \otimes \mathbf{E}_x$ be the map $v \mapsto \xi \otimes v$. The principal symbol is $\iota_\xi^* A(\mathbf{u}(x))\iota_\xi$. In a chart and local frame, the equation takes the form

$$\partial_t u - \sum_{i,j=1}^{m} \nabla_i\big(A^{ij}(x,u)\nabla_j u\big) = F(x,u,\nabla u).$$

Expanding the divergence, the horizontal derivative $\nabla_x A$ and vertical derivative $D_z A(u)[\nabla u]$ contribute only lower-order terms.

Theorem 20.3.1. (Short-time quasilinear existence) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold, let $\mathbf{E} \to M$ be a vector bundle with a bundle metric and compatible connection, and let $\mathcal{U} \subseteq \mathbf{E}$ be an open subset of the bundle. Let $\mathbf{u}_0 \in H^s(M,\mathbf{E})$ be a section whose image is contained in $\mathcal{U}$. Suppose that there exists $\lambda_0 > 0$ such that

$$\langle A(\mathbf{u}_0(x))\zeta,\zeta\rangle_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}} \geq \lambda_0 |\zeta|^2_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}} \tag{20.31}$$

for every $x \in M$ and $\zeta \in T^*_xM \otimes \mathbf{E}_x$, and that $A(z)$ is self-adjoint for every $z \in \mathcal{U}$. Then there exists $T > 0$ for which (20.30) has a unique solution

$$\mathbf{u} \in \mathcal{X}^s_T(\mathbf{E}) \quad \text{con} \quad \mathbf{u}(x,t) \in \mathcal{U}.$$

The time T can be chosen uniformly as $\mathbf{u}_0$ ranges over a bounded subset of $H^s(M,\mathbf{E})$ whose image stays a positive distance from the complement of $\mathcal{U}$ and for which the constant in (20.31) has a positive lower bound.

Proof. The image of $\mathbf{u}_0$ is compact because $s > \frac{m}{2}$ and M is compact. By continuity of A, choose an open subset $\mathcal{V} \subseteq \mathbf{E}$ such that

$$\mathbf{u}_0(M) \subseteq \mathcal{V}, \qquad \overline{\mathcal{V}} \text{ is compact and contained in } \mathcal{U},$$

and on whose closure we have

$$\langle A(z)\zeta, \zeta\rangle_{\mathbf{g}_0, \mathbf{h}_\mathbf{E}} \geq \frac{\lambda_0}{2} |\zeta|^2_{\mathbf{g}_0, \mathbf{h}_\mathbf{E}}. \tag{20.32}$$

Define $\mathbf{u}^{(0)}(t) = \mathbf{u}_0$ and, once $\mathbf{u}^{(k)}$ is constructed, solve the linear problem

$$\partial_t \mathbf{u}^{(k+1)} + \nabla^*\big(A(\mathbf{u}^{(k)})\nabla \mathbf{u}^{(k+1)}\big) = F(j^1\mathbf{u}^{(k)}), \qquad \mathbf{u}^{(k+1)}(0) = \mathbf{u}_0. \tag{20.33}$$

We show simultaneously that the iteration is well defined and contractive.
Let $D_s = c_s^{-1}$, with c_s as in the equivalence following (20.18). Fix R large enough that $R^2 > 2D_s\mathscr{E}_s(\mathbf{u}_0) + 1$. Suppose inductively that

$$\|\mathbf{u}^{(k)}\|_{L^\infty(0,T;H^s(M,\mathbf{E}))} \leq R, \qquad \mathbf{u}^{(k)}(M \times [0,T]) \subseteq \mathcal{V}. \tag{20.34}$$

Lemma 20.1.2 gives bounds independent of k for $A(\mathbf{u}^{(k)})$ in $C([0,T]; H^s(M, \operatorname{End}\mathcal{H}))$ and for $F(j^1\mathbf{u}^{(k)})$ in $C([0,T]; H^{s-1}(M,\mathbf{E}))$. Together with (20.32), Theorem 20.2.3 gives $\mathbf{u}^{(k+1)} \in \mathcal{X}^s_T(\mathbf{E})$. Differential inequality (20.20), before replacement by the coarser bound (20.22), gives a constant C_R independent of k and $T \leq 1$ such that

$$\|\mathbf{u}^{(k+1)}\|^2_{L^\infty(0,T;H^s(M,\mathbf{E}))} \leq D_s e^{C_R T}\big(\mathscr{E}_s(\mathbf{u}_0) + C_R T\big), \tag{20.35}$$

$$\|\mathbf{u}^{(k+1)}\|_{\mathcal{X}^s_T(\mathbf{E})} \leq C_R. \tag{20.36}$$

The first bound tends to $D_s\mathscr{E}_s(\mathbf{u}_0)$ as $T \to 0$. Thus, after reducing T, its right-hand side is less than R^2. Moreover, by Lemma 20.1.3,

$$\begin{aligned}\|\mathbf{u}^{(k+1)}(t) - \mathbf{u}_0\|_{H^{s-1}(M,\mathbf{E})} &\leq \int_0^t \|\partial_\rho \mathbf{u}^{(k+1)}(\rho)\|_{H^{s-1}(M,\mathbf{E})}\, d\rho \\ &\leq T^{\frac{1}{2}} \|\partial_t \mathbf{u}^{(k+1)}\|_{L^2(0,T;H^{s-1}(M,\mathbf{E}))}.\end{aligned} \tag{20.37}$$

The embedding $H^{s-1}(M,\mathbf{E}) \hookrightarrow C^0(M,\mathbf{E})$ and (20.36) show that, for T still smaller, the image of $\mathbf{u}^{(k+1)}$ stays in $\mathcal{V}$. This closes the induction.
Set $\mathbf{w}^{(k+1)} = \mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}$ and $\mathbf{A}_k = A(\mathbf{u}^{(k)})$, $\mathbf{F}_k = F(j^1\mathbf{u}^{(k)})$. The difference of two consecutive equations is

$$\begin{aligned}\partial_t \mathbf{w}^{(k+1)} + \nabla^*(\mathbf{A}_k \nabla \mathbf{w}^{(k+1)}) = &-\nabla^*\big((\mathbf{A}_k - \mathbf{A}_{k-1})\nabla \mathbf{u}^{(k)}\big) \\ &+ \mathbf{F}_k - \mathbf{F}_{k-1}, \qquad \mathbf{w}^{(k+1)}(0) = 0.\end{aligned} \tag{20.38}$$

Contraction is obtained in the lower-order energy space

$$\mathcal{Z}_T(\mathbf{E}) := C([0,T]; L^2(M,\mathbf{E})) \cap L^2(0,T; H^1(M,\mathbf{E})),$$

with

$$\|\mathbf{w}\|^2_{\mathcal{Z}_T(\mathbf{E})} := \|\mathbf{w}\|^2_{L^\infty(0,T;L^2(M,\mathbf{E}))} + \int_0^T \|\mathbf{w}(t)\|^2_{H^1(M,\mathbf{E})}\, dt. \tag{20.39}$$

The preceding bounds and the embedding $H^s(M,\mathbf{E}) \hookrightarrow C^1(M,\mathbf{E})$ confine the first jets of the iteration to a compact subset of $J^1(\mathcal{U})$. Using cutoff functions as in the proof of Lemma 20.1.2, extend A and F to smooth maps defined on the full vector fibers of a neighborhood of this compact set. The mean value theorem applied to these extensions allows us to write

$$\mathbf{A}_k - \mathbf{A}_{k-1} = \boldsymbol{\mathcal{A}}_k[\mathbf{w}^{(k)}], \tag{20.40}$$

$$\mathbf{F}_k - \mathbf{F}_{k-1} = \boldsymbol{\mathcal{B}}_k[\mathbf{w}^{(k)}] + \boldsymbol{\mathcal{C}}_k[\nabla \mathbf{w}^{(k)}]. \tag{20.41}$$

For every $t \in [0,T]$, these coefficients belong to

$$\begin{aligned}
\boldsymbol{\mathcal{A}}_k(t) &\in H^s\big(M, \operatorname{Hom}(\mathbf{E}, \operatorname{End}\boldsymbol{\mathcal{H}})\big),\\
\boldsymbol{\mathcal{B}}_k(t) &\in H^{s-1}(M, \operatorname{End}\mathbf{E}),\\
\boldsymbol{\mathcal{C}}_k(t) &\in H^{s-1}(M, \operatorname{Hom}(\boldsymbol{\mathcal{H}}, \mathbf{E})),
\end{aligned}$$

and depend continuously on t in these spaces. The bounds (20.34) and the embedding $H^s(M,\mathbf{E}) \hookrightarrow C^2(M,\mathbf{E})$ uniformly bound, in the corresponding L^∞ spaces, $\boldsymbol{\mathcal{A}}_k, \boldsymbol{\mathcal{B}}_k, \boldsymbol{\mathcal{C}}_k$, $\nabla\boldsymbol{\mathcal{C}}_k$, and $\nabla \mathbf{u}^{(k)}$. Here $\nabla\boldsymbol{\mathcal{C}}_k$ is a section of $T^*M \otimes \operatorname{Hom}(\boldsymbol{\mathcal{H}}, \mathbf{E})$ with the induced connection.

Take the $L^2(M,\mathbf{E})$ inner product of (20.38) with $\mathbf{w}^{(k+1)}$. Integrate by parts the term containing $\mathbf{A}_k - \mathbf{A}_{k-1}$ and also the term with $\nabla \mathbf{w}^{(k)}$. Since $\boldsymbol{\mathcal{C}}_k$ and $\nabla\boldsymbol{\mathcal{C}}_k$ are uniformly bounded, the latter satisfies

$$\left| \int_M \left\langle \boldsymbol{\mathcal{C}}_k[\nabla \mathbf{w}^{(k)}], \mathbf{w}^{(k+1)} \right\rangle_{\mathbf{h}_\mathbf{E}} d\lambda_{\mathbf{g}_0} \right| \le C_R \|\mathbf{w}^{(k)}\|_{L^2(M,\mathbf{E})} \big(\|\nabla \mathbf{w}^{(k+1)}\|_{L^2(M,\boldsymbol{\mathcal{H}})} + \|\mathbf{w}^{(k+1)}\|_{L^2(M,\mathbf{E})} \big).$$

Parabolicity and Young's inequality give

$$\frac{d}{dt}\|\mathbf{w}^{(k+1)}(t)\|^2_{L^2(M,\mathbf{E})} + \frac{\lambda_0}{2}\|\nabla \mathbf{w}^{(k+1)}(t)\|^2_{L^2(M,\boldsymbol{\mathcal{H}})} \le C_R \|\mathbf{w}^{(k+1)}(t)\|^2_{L^2(M,\mathbf{E})} + C_R \|\mathbf{w}^{(k)}(t)\|^2_{L^2(M,\mathbf{E})}. \tag{20.42}$$

Since $\mathbf{w}^{(k+1)}(0) = 0$, Grönwall's lemma 1.8.19 implies

$$\|\mathbf{w}^{(k+1)}\|^2_{\mathcal{Z}_T(\mathbf{E})} \le C_R T e^{C_R T} \|\mathbf{w}^{(k)}\|^2_{C([0,T];L^2(M,\mathbf{E}))} \le C_R T e^{C_R T} \|\mathbf{w}^{(k)}\|^2_{\mathcal{Z}_T(\mathbf{E})}. \tag{20.43}$$

Reduce T until $q := C_R T e^{C_R T} < 1$. Then

$$\|\mathbf{w}^{(k+1)}\|_{\mathcal{Z}_T(\mathbf{E})} \le q^{\frac{1}{2}} \|\mathbf{w}^{(k)}\|_{\mathcal{Z}_T(\mathbf{E})},$$

For $k = 1$ the bound $\|\mathbf{w}^{(k)}\|_{\mathcal{Z}_T} \le q^{(k-1)/2}\|\mathbf{w}^{(1)}\|_{\mathcal{Z}_T}$ is an equality. If it holds for k, the preceding inequality gives it for $k+1$ after multiplication by $q^{1/2}$. Thus, summing the geometric series,

$$\sum_{k=1}^{\infty} \|\mathbf{w}^{(k)}\|_{\mathcal{Z}_T(\mathbf{E})} \le \frac{\|\mathbf{w}^{(1)}\|_{\mathcal{Z}_T(\mathbf{E})}}{1 - q^{\frac{1}{2}}} < \infty.$$

The space $\mathcal{Z}_T(\mathbf{E})$ is complete as the intersection of the two Banach spaces defining it, with the sum norm. Therefore, $(\mathbf{u}^{(k)})$ is Cauchy in $\mathcal{Z}_T(\mathbf{E})$.

The uniform bounds for $(\mathbf{u}^{(k)})_{k\ge1}$ in $\mathcal{X}^s_T(\mathbf{E})$ improve this convergence. The section $\mathbf{u}^{(0)}(t) = \mathbf{u}_0$ is used only to start the iteration and need not belong to $L^2(0,T;H^{s+1}(M,\mathbf{E}))$. At each time, interpolation of the Sobolev

scale gives

$$\|\mathbf{w}\|_{H^{s-1}(M,\mathbf{E})} \le C\|\mathbf{w}\|^{\frac{1}{s}}_{L^2(M,\mathbf{E})}\|\mathbf{w}\|^{\frac{s-1}{s}}_{H^s(M,\mathbf{E})}, \tag{20.44}$$

and, after integration in time,

$$\|\mathbf{w}\|_{L^2(0,T;H^s(M,\mathbf{E}))} \le C\|\mathbf{w}\|^{\frac{1}{s}}_{L^2(0,T;H^1(M,\mathbf{E}))}\|\mathbf{w}\|^{\frac{s-1}{s}}_{L^2(0,T;H^{s+1}(M,\mathbf{E}))}. \tag{20.45}$$

Thus $(\mathbf{u}^{(k)})$ converges strongly in $C([0,T];H^{s-1}(M,\mathbf{E}))\cap L^2(0,T;H^s(M,\mathbf{E}))$. Weak compactness in the L^2 factors, weak-* compactness in $L^\infty(0,T;H^s(M,\mathbf{E}))$, and the uniform bounds produce a limit $\mathbf{u}\in\mathcal{X}^s_T(\mathbf{E})$. The composition estimates allow passage to the limit in $A(\mathbf{u}^{(k)})$ and $F(j^1\mathbf{u}^{(k)})$, so $\mathbf{u}$ satisfies (20.30). Continuity in H^{s-1} and the embedding into C^0 ensure that its image stays in $\mathcal{U}$.

If $\mathbf{u}$ and $\mathbf{v}$ are two solutions, repeat the calculation (20.40)– (20.42) for $\mathbf{w}=\mathbf{u}-\mathbf{v}$. Now $\mathbf{w}(0)=0$, and the right-hand side contains $C_R\|\mathbf{w}(t)\|^2_{L^2(M,\mathbf{E})}$ in place of the difference from the preceding step. Grönwall's lemma 1.8.19 gives $\mathbf{w}=0$ on all of $[0,T]$. Uniformity of the existence time follows because all constants used depend only on the bound H^s, distance from the complement of $\mathcal{U}$, and the lower parabolicity bound. □

The autonomous form of the equation suffices for most of the chapter. Geometric applications will also involve coefficients with explicit time dependence; the same argument gives the following version.

Corollary 20.3.2. (Explicit time dependence) Let $T_*>0$ and let

$$A\colon [0,T_*]\times\mathcal{U}\longrightarrow \operatorname{End}(T^*M\otimes\mathbf{E}),$$
$$F\colon [0,T_*]\times J^1(\mathcal{U})\longrightarrow \mathbf{E}$$

be smooth bundle maps. Suppose that $A(t,z)$ is self-adjoint and the strong parabolicity condition holds on $(0,\mathbf{u}_0)$ with a positive constant. Then there exist $T\in(0,T_*]$ and a unique solution $\mathbf{u}\in\mathcal{X}^s_T(\mathbf{E})$ of

$$\partial_t\mathbf{u}+\nabla^*\big(A(t,\mathbf{u})\nabla\mathbf{u}\big)=F(t,j^1\mathbf{u}),\qquad \mathbf{u}(0)=\mathbf{u}_0.$$

The assertions on regularity, dependence on data, and uniformity of the time remain valid when the bounds for derivatives of A and F are uniform for $t\in[0,T_*]$ on the compact sets considered.

Proof. In the iteration of (20.33), replace the coefficients by $A(t,\mathbf{u}^{(k)})$ and $F(t,j^1\mathbf{u}^{(k)})$. The composition estimates of Lemma 20.1.2 are uniform in t because the time interval is compact. The energy estimate, remaining inside $\mathcal{V}$, and contraction in $\mathcal{Z}_T(\mathbf{E})$ are unchanged. The proofs of regularity and parameter dependence add only derivatives of the coefficients with respect to t, which are uniformly bounded lower-order terms. □

Corollary 20.3.3. (Regularity and dependence on data) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold, let $\mathbf{E}\to M$ be a vector bundle with a bundle metric and compatible connection, and consider the quasilinear equation (20.30). Under the hypotheses of Theorem 20.3.1, the following assertions hold.

(a) On bounded subsets of initial data sharing bounds for parabolicity and distance from $\mathbf{E}\setminus\mathcal{U}$, the solutions satisfy

$$\|\mathbf{u}-\mathbf{v}\|_{\mathcal{X}^{s-1}_T(\mathbf{E})}\le C\|\mathbf{u}_0-\mathbf{v}_0\|_{H^{s-1}(M,\mathbf{E})}. \tag{20.46}$$

(b) If $\mathbf{u}_0 \in H^{s+q}(M,\mathbf{E})$ for some integer $q \geq 0$, then, after reducing T using only a bound for $\|\mathbf{u}_0\|_{H^s(M,\mathbf{E})}$, we have $\mathbf{u} \in \mathcal{X}_T^{s+q}(\mathbf{E})$.

(c) If $\mathbf{u}_0$ is smooth, the solution is smooth on $M \times [0,T]$. For merely H^s initial data, the solution is smooth on $M \times (0,T]$.

(d) A smooth family of smooth initial data gives a smooth family of solutions, as long as they all remain in a set on which the preceding bounds are uniform.

Proof. The equation for $\mathbf{u}-\mathbf{v}$ is

$$\begin{aligned}\partial_t(\mathbf{u}-\mathbf{v}) + \nabla^*(A(\mathbf{u})\nabla(\mathbf{u}-\mathbf{v})) = &-\nabla^*\big((A(\mathbf{u})-A(\mathbf{v}))\nabla\mathbf{v}\big)\\ &+F(j^1\mathbf{u})-F(j^1\mathbf{v}).\end{aligned}$$

Apply ∇^q, $0 \leq q \leq s-1$, and repeat the integration by parts in Lemma 20.2.2. For the term containing the coefficient difference, use

$$\|(A(\mathbf{u})-A(\mathbf{v}))\nabla\mathbf{v}\|_{H^{s-1}(M,\boldsymbol{\mathcal{H}})} \leq C_R\|\mathbf{u}-\mathbf{v}\|_{H^{s-1}(M,\mathbf{E})}\|\mathbf{v}\|_{H^s(M,\mathbf{E})}. \tag{20.47}$$

This estimate is valid because $H^{s-1}(M)$ is an algebra. The difference of the F terms is bounded in $H^{s-2}(M,\mathbf{E})$ by $C_R\|\mathbf{u}-\mathbf{v}\|_{H^{s-1}(M,\mathbf{E})}$. After absorbing the derivative of order s of $\mathbf{u}-\mathbf{v}$, we obtain

$$\frac{d}{dt}\mathscr{E}_{s-1}(\mathbf{u}-\mathbf{v}) + c\|\mathbf{u}-\mathbf{v}\|^2_{H^s(M,\mathbf{E})} \leq C_R\mathscr{E}_{s-1}(\mathbf{u}-\mathbf{v}). \tag{20.48}$$

Grönwall's lemma 1.8.19 controls the $C([0,T];H^{s-1}(M,\mathbf{E}))\cap L^2(0,T;H^s(M,\mathbf{E}))$ part. The difference equation also controls $\partial_t(\mathbf{u}-\mathbf{v})$ in $L^2(0,T;H^{s-2}(M,\mathbf{E}))$, giving (20.46).

We now prove persistence of regularity using an energy of arbitrary order. Let $N \geq s$ and first suppose that $\mathbf{u}$ is smooth. On an interval $[0,T]$ where $\|\mathbf{u}\|_{L^\infty(0,T;H^s(M,\mathbf{E}))} \leq R$, the Moser estimates with higher-order control give

$$\|[\nabla^N, A(\mathbf{u})](\nabla\mathbf{u})\|_{L^2(M,T^{(0,N)}(TM)\otimes\boldsymbol{\mathcal{H}})} + \|F(j^1\mathbf{u})\|_{H^{N-1}(M,\mathbf{E})} \leq C_{N,R}\big(1+\|\mathbf{u}\|_{H^N(M,\mathbf{E})}\big). \tag{20.49}$$

Indeed, (20.5) gives

$$\begin{aligned}&\|[\nabla^N, A(\mathbf{u})]\nabla\mathbf{u}\|_{L^2(M,T^{(0,N)}(TM)\otimes\boldsymbol{\mathcal{H}})}\\ &\quad\leq C_N\|\nabla A(\mathbf{u})\|_{L^\infty(M,T^*M\otimes\mathrm{End}(\boldsymbol{\mathcal{H}}))}\|\nabla\mathbf{u}\|_{H^{N-1}(M,\boldsymbol{\mathcal{H}})}\\ &\qquad+C_N\|A(\mathbf{u})\|_{H^N(M,\mathrm{End}(\boldsymbol{\mathcal{H}}))}\|\nabla\mathbf{u}\|_{L^\infty(M,\boldsymbol{\mathcal{H}})}.\end{aligned}$$

The bound $H^s \hookrightarrow C^2$ controls both factors with infinity norms. Estimate (20.9) controls $A(\mathbf{u})$ in H^N by $C_{N,R}(1+\|\mathbf{u}\|_{H^N(M,\mathbf{E})})$. Applied to $(\mathbf{u},\nabla\mathbf{u}) \mapsto F(\mathbf{u},\nabla\mathbf{u})$ at order $N-1$, the same estimate controls the remaining term in (20.49). All constants use the bounds for the first jets and the fixed structures. Applying ∇^N to the equation, integrating by parts, and using (20.49) yields

$$\frac{d}{dt}\mathscr{E}_N(\mathbf{u}(t)) + c\|\mathbf{u}(t)\|^2_{H^{N+1}(M,\mathbf{E})} \leq C_{N,R}\big(1+\mathscr{E}_N(\mathbf{u}(t))\big). \tag{20.50}$$

For a solution $\mathbf{u} \in \mathcal{X}_T^N(\mathbf{E})$, the identity allowing this energy to be used is

$$\frac{d}{dt}\mathscr{E}_N(\mathbf{u}(t)) = 2\mathcal{B}_N(\partial_t \mathbf{u}(t), \mathbf{u}(t)) \quad \text{for almost every } t \in (0,T), \tag{20.51}$$

where $\mathcal{B}_N$ is the form extended in the proof of Lemma 20.2.2. To verify it, fix a compact interval $J \subset (0,T)$ and regularize only in time: $\mathbf{u}_\varepsilon = \rho_\varepsilon *_t \mathbf{u}$, with ε smaller than the distance from J to the endpoints. Continuity of translations in Bochner spaces gives

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{in } L^2(J; H^{N+1}(M,\mathbf{E})), \qquad \partial_t \mathbf{u}_\varepsilon \to \partial_t \mathbf{u} \quad \text{in } L^2(J; H^{N-1}(M,\mathbf{E})).$$

The first convergence implies $\mathscr{E}_N(\mathbf{u}_\varepsilon) \to \mathscr{E}_N(\mathbf{u})$ in $L^1(J)$. By the bound for $\mathcal{B}_N$, the error on the right-hand side, integrated over J, is at most

$$\begin{aligned} &C\|\partial_t \mathbf{u}_\varepsilon - \partial_t \mathbf{u}\|_{L^2(J;H^{N-1}(M,\mathbf{E}))}\|\mathbf{u}_\varepsilon\|_{L^2(J;H^{N+1}(M,\mathbf{E}))} \\ &\quad + C\|\partial_t \mathbf{u}\|_{L^2(J;H^{N-1}(M,\mathbf{E}))}\|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^2(J;H^{N+1}(M,\mathbf{E}))}, \end{aligned}$$

which tends to zero. For $\mathbf{u}_\varepsilon$, differentiation of the covariant norm and the formula for $\mathcal{B}_N$ give the classical identity; testing against a function in $C_c^\infty(J)$ and taking limits gives (20.51) in distributions. Its right-hand side belongs to $L^1(0,T)$. Lemma 20.1.3 provides the continuous representative in $H^N(M,\mathbf{E})$, so the energy has the absolutely continuous representative determined by this identity.

Finally, for almost every t, substitute the equation into $\mathcal{B}_N(\partial_t \mathbf{u}, \mathbf{u})$. Spatial integrations by parts are valid for $\mathbf{u}(t) \in H^{N+1}(M,\mathbf{E})$ by the explicit definition of that form. The product and Moser estimates in (20.49) control the resulting terms in the same spaces; they are also obtained as limits of smooth approximations in $H^{N+1}(M,\mathbf{E})$. Thus the energy estimate holds for almost every time and can be integrated on any subinterval of $[0,T]$.

Let us justify application of this estimate on the existence interval determined by the norm of order s. If $\mathbf{u}_0 \in H^N$, the quasilinear theorem applied initially at order N gives a solution on a possibly smaller interval. By uniqueness, it agrees there with the solution of order s. While both are defined, Grönwall applied to (20.50) bounds its H^N norm in terms of N, R, T and $\|\mathbf{u}_0\|_{H^N(M,\mathbf{E})}$. Distance to the boundary of $\mathcal{U}$ and parabolicity retain the bounds from the order-s construction. Restarting the existence theorem at any time in this interval, these bounds and the bound H^N give a positive additional time independent of the restart time. If the interval of H^N regularity ended before T, restarting sufficiently near its endpoint would extend beyond it. By uniqueness, the extension would remain the same solution. This gives H^N regularity up to T.

To justify the identities for H^N data, approximate the data in that norm by smooth sections. The order-s estimate gives a common interval, and the preceding argument and (20.50) give uniform bounds in $\mathcal{X}_T^N$. Weak passage to the limit and identification of the trace are as in the linear theorem; uniqueness identifies the limit with $\mathbf{u}$. We conclude that $\mathbf{u} \in \mathcal{X}_T^N$. The interval T is fixed using the order-s bounds, although the order-N norm of the solution also depends on that of the initial data.

The same energy proves smoothing without attributing to the data derivatives it does not have. Fix $0 < \tau < T$ and an integer $q \geq 1$. Choose numbers

$$0 = \tau_0 < \tau_1 < \cdots < \tau_q = \tau$$

and smooth functions χ_j vanishing on a neighborhood of $[0,\tau_{j-1}]$ and equal to one on $[\tau_j,T]$. Suppose inductively that, after shifting the time origin to τ_{j-1}, the restriction of the solution to $[\tau_{j-1},T]$ belongs to $\mathcal{X}^{s+j-1}_{T-\tau_{j-1}}(\mathbf{E})$; for $j=1$ this is precisely the existence regularity. Apply the calculation in (20.50) with $N=s+j$ and weight χ_j^2 to smooth approximations of $\mathbf{u}$. The only new term is

$$2\chi_j\chi_j'\mathscr{E}_{s+j}(\mathbf{u}),$$

whose integral is finite because the induction hypothesis includes $\mathbf{u}\in L^2(\tau_{j-1},T;H^{s+j}(M,\mathbf{E}))$. Passing to the limit gives

$$\begin{aligned}\|\mathbf{u}\|^2_{L^\infty(\tau_j,T;H^{s+j}(M,\mathbf{E}))}+\int_{\tau_j}^{T}\|\mathbf{u}(t)\|^2_{H^{s+j+1}(M,\mathbf{E})}\,dt\\ \le C_{j,R,\tau}\left(1+\int_{\tau_{j-1}}^{\tau_j}\|\mathbf{u}(t)\|^2_{H^{s+j}(M,\mathbf{E})}\,dt\right).\end{aligned}\tag{20.52}$$

The induction reaches every q. Sobolev embedding gives spatial smoothness on $M\times(0,T]$, and the equation expresses every time derivative in terms of spatial derivatives already controlled; this gives joint smoothness. If $\mathbf{u}_0$ is smooth, estimate (20.50) starts at $t=0$ for every N, and the solution is smooth up to the initial time.

Finally, we justify smooth dependence without assuming in advance that differentiation with respect to the parameter is possible. Let $a\mapsto\mathbf{u}_{0,a}$ be a smooth family of smooth data and let $\mathbf{u}_a$ be the family of solutions obtained on a common interval. For a fixed parameter direction define the quotient

$$\mathbf{h}_\varepsilon:=\frac{\mathbf{u}_{a+\varepsilon}-\mathbf{u}_a}{\varepsilon}.$$

The integral mean value formula applied to the extensions of A and F constructed in the proof of the theorem shows that $\mathbf{h}_\varepsilon$ satisfies a linear system whose principal term is $\nabla^*(A(\mathbf{u}_a)\nabla\mathbf{h}_\varepsilon)$ and whose other coefficients are averages of D_zA, D_zF, and D_pF along the segment between the two solutions. The energies (20.48) and (20.50) uniformly bound $\mathbf{h}_\varepsilon$ at every order, because the initial-data quotients are uniformly bounded. Subtracting the equations for $\mathbf{h}_\varepsilon$ and $\mathbf{h}_\delta$, the averaged coefficients converge by the Moser composition estimates. The energy of the difference shows that $(\mathbf{h}_\varepsilon)$ is Cauchy in every space of one lower order. Its limit $\mathbf{h}=\partial_a\mathbf{u}_a$ satisfies

$$\begin{aligned}\partial_t\mathbf{h}+\nabla^*\big(A(\mathbf{u}_a)\nabla\mathbf{h}+D_zA(\mathbf{u}_a)[\mathbf{h}]\nabla\mathbf{u}_a\big)\\ =D_zF(\mathbf{u}_a,\nabla\mathbf{u}_a)[\mathbf{h}]+D_pF(\mathbf{u}_a,\nabla\mathbf{u}_a)[\nabla\mathbf{h}].\end{aligned}\tag{20.53}$$

Let us make the passage to higher parameter derivatives explicit. For a real variable a write $\mathbf{u}^{[k]}=\partial_a^k\mathbf{u}_a$ and define the nonlinear differential operator

$$\mathcal{H}(\mathbf{v}):=-\nabla^*(A(\mathbf{v})\nabla\mathbf{v})+F(\mathbf{v},\nabla\mathbf{v}).$$

Suppose that the derivatives $\mathbf{u}^{[1]},\dots,\mathbf{u}^{[q-1]}$ have been constructed and are continuous at all spatial orders,

with $q \geq 2$. The Faà di Bruno formula gives the candidate equation

$$(\partial_t - D\mathcal{H}(\mathbf{u}_a))\mathbf{u}^{[q]} = \sum_{j=2}^{q} \sum_{\substack{k_1,\dots,k_j \in \mathbb{N} \\ k_1+\cdots+k_j=q}} \frac{q!}{j!\,k_1!\cdots k_j!} D^j\mathcal{H}(\mathbf{u}_a)[\mathbf{u}^{[k_1]},\dots,\mathbf{u}^{[k_j]}].$$

In this sum, every k_i is less than q; by the induction hypothesis, the right-hand side is known. The operator on the left is precisely that of (20.53) and retains the principal part $\nabla^*(A(\mathbf{u}_a)\nabla\cdot)$. The initial data are $\partial_a^q \mathbf{u}_{0,a}$. Linear theory gives a unique solution at every spatial order.

To identify it as a derivative, take the incremental quotient of the equation for $\mathbf{u}^{[q-1]}$. The integral product and composition rules express its coefficients as averages of derivatives of A and F. The induction hypothesis makes the sources converge to the preceding right-hand side. Subtracting two quotient equations, the linear estimate controls their difference by differences of the data, coefficients, and sources, which tend to zero. The quotients are therefore Cauchy, and their limit is the linear solution just constructed. The same estimate, applied to two nearby values of a, proves continuity. This closes the step from $q-1$ to q, with the case $q = 1$ already proved. For parameters in an open subset of $\mathbb{R}^d$, apply this argument to successive directional derivatives; the multilinear Faà di Bruno formulas give the mixed derivatives with the same control. The bounds are taken locally in the parameter, on compact sets where the interval and existence hypotheses are uniform. This proves the last assertion. □

Theorem 20.3.4. (Continuation criterion) Let $(M, \mathbf{g}_0)$ be a closed Riemannian manifold, let $\mathbf{E} \to M$ be a vector bundle with a bundle metric and compatible connection, and let $\mathbf{u}$ be a maximal solution of (20.30) defined on $[0, T_{\max})$, with $T_{\max} < \infty$, and suppose that $\mathbf{u} \in \mathcal{X}_T^s(\mathbf{E})$ for every $T < T_{\max}$. If there exist $R, \delta > 0$ such that, for every $0 \leq t < T_{\max}$,

$$\|\mathbf{u}(t)\|_{H^s(M,\mathbf{E})} \leq R, \tag{20.54}$$

$$\inf_{x \in M} \operatorname{dist}_{\mathbf{E}_x}\big(\mathbf{u}(x,t), \mathbf{E}_x \setminus \mathcal{U}_x\big) \geq \delta, \tag{20.55}$$

$$\langle A(\mathbf{u}(x,t))\zeta, \zeta\rangle_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}} \geq \delta |\zeta|^2_{\mathbf{g}_0,\mathbf{h}_\mathbf{E}}. \tag{20.56}$$

The last inequality is required for every $x \in M$ and $\zeta \in T_x^*M \otimes \mathbf{E}_x$. Then $\mathbf{u}$ extends to an interval $[0, T_{\max} + \varepsilon)$, contradicting maximality. Thus, if the maximal time is finite, at least one quantity controlled in (20.54)–(20.56) must degenerate.

Proof. The last assertion of Theorem 20.3.1 gives a time $\varepsilon > 0$ depending only on R, δ, and bounds for A, F on the compact set traversed by the first jets of $\mathbf{u}$. These latter bounds follow from $H^s(M,\mathbf{E}) \hookrightarrow C^1(M,\mathbf{E})$. For any $t_0 \in (T_{\max} - \frac{\varepsilon}{2}, T_{\max})$, apply the existence theorem with initial data $\mathbf{u}(t_0)$. This gives a solution on $[t_0, t_0 + \varepsilon]$. Uniqueness makes it agree with $\mathbf{u}$ on $[t_0, T_{\max})$, thus extending it beyond $T_{\max}$. □

20.4 Maximum principles

Energy estimates control the solution in global norms, but do not by themselves record the sign of a function or whether a tensor belongs to a convex cone. Maximum principles provide this pointwise information and will be the basic tool for studying curvature under Ricci flow.

Theorem 20.4.1. (Scalar maximum principle) Let $(M, \mathbf{g}_0)$ be a closed Riemannian manifold. Let

$$L_t\varphi = \sum_{i,j=1}^m a^{ij}(x,t)(\nabla^2\varphi)_{ij} + \sum_{i=1}^m b^i(x,t)\nabla_i\varphi + c(x,t)\varphi,$$

where a^{ij}, b^i, and c are continuous on $M\times[0,T]$, (a^{ij}) is symmetric, and there exists $\lambda > 0$ such that $\sum_{i,j=1}^m a^{ij}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2_{\mathbf{g}_0}$ for every $(x,t)\in M\times[0,T]$ and every $\xi\in T^*_xM$. Suppose that $f\in C^0(M\times[0,T])$ and that $u\in C^0(M\times[0,T])\cap C^{2,1}(M\times(0,T])$ satisfies

$$\partial_t u - L_t u \leq f \tag{20.57}$$

and let $\gamma\in\mathbb{R}$ be an upper bound for c on $M\times[0,T]$. Then, for $0\leq t\leq T$,

$$\sup_{x\in M} u(x,t) \leq e^{\gamma t}\left(\max\left\{0, \sup_{x\in M} u(x,0)\right\} + \int_0^t e^{-\gamma\rho}\max\left\{0, \sup_{x\in M} f(x,\rho)\right\} d\rho\right). \tag{20.58}$$

In particular, if $c\leq 0$, $f\leq 0$, and $u(0)\leq 0$, then $u\leq 0$ on $M\times[0,T]$.

Proof. Set $v(x,t) = e^{-\gamma t}u(x,t)$ and

$$F_+(t) := \max\left\{0, \sup_{x\in M} e^{-\gamma t}f(x,t)\right\}, \qquad z(t) := \max\left\{0, \sup_{x\in M} u(x,0)\right\} + \int_0^t F_+(\rho)\,d\rho.$$

The function $w = v - z$ satisfies

$$\partial_t w - \sum_{i,j=1}^m a^{ij}(\nabla^2 w)_{ij} - \sum_{i=1}^m b^i\nabla_i w - (c-\gamma)w \leq 0, \qquad w(0)\leq 0, \tag{20.59}$$

because $c-\gamma\leq 0$ and $z\geq 0$.

If w attained a positive value, then for small $\varepsilon > 0$ the function $w_\varepsilon = w - \varepsilon t - \varepsilon$ would have a first nonnegative maximum at some (x_0,t_0) with $t_0 > 0$. There,

$$\partial_t w_\varepsilon \geq 0, \qquad \nabla w_\varepsilon = 0, \qquad \sum_{i,j=1}^m a^{ij}(\nabla^2 w_\varepsilon)_{ij} \leq 0.$$

If $D = \partial_t - \sum_{i,j=1}^m a^{ij}(\nabla^2\cdot)_{ij} - \sum_{i=1}^m b^i\nabla_i - (c-\gamma)$, the three preceding relations and $c-\gamma\leq 0$ imply $Dw_\varepsilon(x_0,t_0)\geq 0$. On the other hand,

$$Dw_\varepsilon = Dw - \varepsilon + (c-\gamma)\varepsilon(t+1) \leq -\varepsilon,$$

a contradiction. Therefore, $w\leq 0$. Multiplying by $e^{\gamma t}$ gives (20.58). □

Corollary 20.4.2. (Scalar comparison) Let $(M,\mathbf{g}_0)$ be a closed Riemannian manifold and let L_t be the

operator of Theorem 20.4.1. Let $u, v \in C^0(M\times[0,T]) \cap C^{2,1}(M\times(0,T])$. If

$$\partial_t u - L_t u \le \partial_t v - L_t v \quad \text{y} \quad u(0) \le v(0),$$

then $u \le v$ on $M\times[0,T]$ when $c \le 0$. For bounded c, the same conclusion follows after multiplying both functions by $e^{-\gamma t}$ with $\gamma \ge \sup_{(x,t)\in M\times[0,T]} c(x,t)$.

Proof. Apply Theorem 20.4.1 to $u - v$. □

To formulate the tensor version, a family $\mathcal{K} = (K_x)_{x\in M}$ of subsets of $\mathbf{E}_x$ is called *invariant under parallel transport* if transport along any curve takes K_x into the set corresponding to its endpoint. In particular, this condition identifies all the K_x locally using the connection. If $q \in \partial K_x$, a vector $\nu \in \mathbf{E}_x$ is an outward supporting normal if

$$\langle \nu, z - q\rangle_{\mathbf{E}_x} \le 0 \qquad \text{for every } z \in K_x.$$

Theorem 20.4.3. (Tensor maximum principle) Let $(M, \mathbf{g}_0)$ be a closed Riemannian manifold and let $\mathbf{E} \to M$ be a real vector bundle with a bundle metric and compatible connection. Let $\mathcal{K} = (K_x)_{x\in M}$ be a family of nonempty closed convex sets invariant under parallel transport. Let $\mathbf{S} \in C^0(M\times[0,T],\mathbf{E}) \cap C^{2,1}(M\times(0,T],\mathbf{E})$ be a solution of

$$\partial_t \mathbf{S} = \sum_{i,j=1}^{m} a^{ij}(x,t)(\nabla^2 \mathbf{S})_{ij} + \sum_{i=1}^{m} b^i(x,t)\nabla_i \mathbf{S} + \Phi(x,t,\mathbf{S}), \tag{20.60}$$

where $\mathbf{a} \in C^0(M\times[0,T], S^2TM)$ and $\mathbf{b} \in C^0(M\times[0,T], TM)$ are fields whose components a^{ij} and b^i multiply the identity of $\mathbf{E}$, and (a^{ij}) is symmetric and uniformly positive. Suppose that Φ is continuous and locally Lipschitz in its last variable, uniformly in (x,t) on compact sets, and that for every $q \in \partial K_x$ and every outward supporting normal ν at q we have

$$\langle \nu, \Phi(x,t,q)\rangle_{\mathbf{h}_\mathbf{E}} \le 0. \tag{20.61}$$

If $\mathbf{S}(x,0) \in K_x$ for every $x \in M$, then $\mathbf{S}(x,t) \in K_x$ for every $(x,t) \in M\times[0,T]$.

Proof. Let $\Pi_x \colon \mathbf{E}_x \longrightarrow K_x$ be the metric projection, unique by convexity. Define

$$\rho(x,t) := \frac{1}{2}\,|\mathbf{S}(x,t) - \Pi_x(\mathbf{S}(x,t))|^2_{\mathbf{E}_x}.$$

The squared-distance function to a closed convex set in a Euclidean space is of class C^1 and has gradient $z \mapsto z - \Pi_x z$. It is also convex and of class $C^{1,1}$. Invariance of $\mathcal{K}$ under parallel transport identifies the distance functions on neighboring fibers. In particular, ρ is continuous, is of class C^1 in positive time, and

$$\partial_t \rho(x,t) = \langle \mathbf{S}(x,t) - \Pi_x \mathbf{S}(x,t), \partial_t \mathbf{S}(x,t)\rangle_{\mathbf{h}_\mathbf{E}}. \tag{20.62}$$

Fix $t > 0$ and a point x where $\rho(\cdot,t)$ attains its maximum. Set $q = \Pi_x \mathbf{S}(x,t)$ and $\nu = \mathbf{S}(x,t) - q$. Let $X \in T_xM$, let $\gamma(h) = \exp_x(hX)$, and parallel-transport $\mathbf{S}(\gamma(h),t)$ back to $\mathbf{E}_x$; denote the transported curve by $z(h)$. Transport

invariance identifies $K_{\gamma(h)}$ with K_x, and maximality of x gives

$$\frac{1}{2}\operatorname{dist}_{\mathbf{E}_x}(z(h),K_x)^2 \le \frac{1}{2}\operatorname{dist}_{\mathbf{E}_x}(z(0),K_x)^2.$$

The subgradient inequality for the convex squared-distance function is

$$\frac{1}{2}\operatorname{dist}_{\mathbf{E}_x}(z,K_x)^2 \ge \frac{1}{2}\operatorname{dist}_{\mathbf{E}_x}(z(0),K_x)^2 + \langle \nu, z - z(0)\rangle_{\mathbf{h_E}}.$$

Apply it to $z(h)$ and $z(-h)$, add, and divide by h^2. Letting $h \to 0$ gives

$$\langle \nu, (\nabla^2 \mathbf{S})(X,X)(x,t)\rangle_{\mathbf{h_E}} \le 0. \tag{20.63}$$

The first derivative at $h = 0$ also gives

$$\langle \nu, \nabla_X \mathbf{S}(x,t)\rangle_{\mathbf{h_E}} = 0. \tag{20.64}$$

Diagonalize $(a^{ij}(x,t))$ in an orthonormal basis of T_xM. Summing (20.63) with the positive eigenvalues as weights and using (20.64) yields

$$\left\langle \nu, \sum_{i,j=1}^{m} a^{ij}(\nabla^2\mathbf{S})_{ij} + \sum_{i=1}^{m} b^i \nabla_i \mathbf{S} \right\rangle_{\mathbf{h_E}} \le 0. \tag{20.65}$$

If $\nu = 0$, all expressions to be estimated are zero. If $\nu \neq 0$, the projection property implies $q \in \partial K_x$, and ν is an outward supporting normal at q. If L is a Lipschitz constant for Φ on the compact set traversed by $\mathbf{S}$ and its projection, then (20.61) implies

$$\begin{aligned}\langle \nu, \Phi(x,t,\mathbf{S})\rangle_{\mathbf{h_E}} &= \langle \nu, \Phi(x,t,q)\rangle_{\mathbf{h_E}} + \langle \nu, \Phi(x,t,\mathbf{S}) - \Phi(x,t,q)\rangle_{\mathbf{h_E}} \\ &\le L|\nu|^2_{\mathbf{h_E}} = 2L\rho(x,t).\end{aligned} \tag{20.66}$$

Identities (20.62) and (20.60), together with (20.65) and (20.66), show that

$$\partial_t \rho(x,t) \le 2L\rho(x,t) \tag{20.67}$$

at every spatial maximum of $\rho(\cdot,t)$.

It remains to pass from the spatial maximum to a function of one variable. Define $Q(t) = \max_{x\in M} \rho(x,t)$. Its upper Dini derivative satisfies

$$D^+Q(t) \le \max_{\substack{x\in M \\ \rho(x,t)=Q(t)}} \partial_t \rho(x,t) \le 2LQ(t). \tag{20.68}$$

For the first inequality, choose $h_j \to 0^+$ and points x_j where $\rho(\cdot, t+h_j)$ attains its maximum. A subsequence converges to a maximum of $\rho(\cdot,t)$, and

$$\frac{Q(t+h_j)-Q(t)}{h_j} \le \frac{\rho(x_j,t+h_j)-\rho(x_j,t)}{h_j};$$

uniform continuity of $\partial_t\rho$ near the positive time t allows passage to the limit. The following Grönwall-type

estimate for upper Dini derivatives is proved by applying the first-maximum argument to $e^{-2Lt}Q(t)$; thus,

$$Q(t) \le e^{2L(t-\tau)}Q(\tau), \qquad 0 < \tau < t \le T.$$

Since Q is continuous and $Q(0) = 0$, let $\tau \to 0^+$ to obtain $Q(t) = 0$. Therefore, $\mathbf{S}(x,t) \in K_x$ for every (x,t). □

Remark 20.4.4. The proof remains valid if the connection depends smoothly on t, provided that it is compatible with the same fixed bundle metric and each connection preserves $\mathcal{K}$ under parallel transport. Indeed, the spatial argument is performed at a fixed time, and the function $\frac{1}{2}\operatorname{dist}(\mathbf{S}, K_x)^2$ does not introduce a varying metric upon time differentiation. If the bundle metric also varies, one must first transport the problem to a bundle with a fixed metric or include its derivative in the time identity.

Corollary 20.4.5. (Cones of nonnegative forms) Let $(M, \mathbf{g}_0)$ be a closed Riemannian manifold, let $\mathbf{V} \longrightarrow M$ be a real vector bundle with a bundle metric and compatible connection, and let $\mathbf{S}(t)$ be a section of $S^2\mathbf{V}^*$ satisfying

$$\partial_t \mathbf{S} = \sum_{i,j=1}^{m} a^{ij}(\nabla^2 \mathbf{S})_{ij} + \Phi(\mathbf{S}).$$

Suppose that, for every form $Q \ge 0$ and every $v \in \mathbf{V}_x$ with $Q(v,v) = 0$, we have

$$\Phi(Q)(v,v) \ge 0.$$

Also assume the regularity and uniform positivity of the coefficients and the local Lipschitz condition on Φ from Theorem 20.4.3. If $\mathbf{S}(0) \ge 0$, then $\mathbf{S}(t) \ge 0$ for as long as the solution exists.

Proof. The cone of positive semidefinite forms is closed, convex, and invariant under parallel transport. Its outward normal functionals at a boundary point are nonnegative combinations of the functionals $Q \mapsto -Q(v,v)$ with v in the kernel of Q. The hypothesis is precisely (20.61); apply Theorem 20.4.3. □

For classical treatments of parabolic theory with less regular coefficients, see [61, Chapter 7] and [35, Chapter 3].

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Ricci flow

A Riemannian metric determines how lengths, angles, and curvature are measured. Ricci flow allows this rule of measurement to change with time according to an equation built from its own Ricci tensor. Rather than searching directly for a special metric, one starts from an initial metric and studies which geometric features improve, which are preserved, and how curvature may concentrate.

The analogy with heat is useful, but should not be taken literally. The unknown is the metric itself and the equation is invariant under diffeomorphisms. This symmetry makes its symbol degenerate in the directions tangent to the orbits of the diffeomorphism action. The DeTurck procedure fixes a gauge using a vector field and turns the problem into a strictly parabolic system; a family of diffeomorphisms then recovers the solution of the original flow [75, 43].

We begin with variation formulas for the metric, connection, and curvature. We then identify the symbol degeneracy, construct the Ricci–DeTurck system, and prove short-time existence and uniqueness. Curvature evolution equations and maximum principles lead to the first a priori estimates. Finally, we study rescalings, normalized flow, Einstein metrics, and solitons. These last solutions provide self-similar models that will reappear when regions in which singularities form are magnified.

Richard Hamilton and a new strategy for geometry

Hamilton introduced Ricci flow in 1982 and proposed studying the topology of a manifold by letting its geometry evolve. The strategy required combining three levels: a parabolic equation to construct and regularize the metric, Riemannian estimates to control curvature, and topological arguments to interpret possible limits. The compactness and singularity analysis of the next chapter continues this program.

Let M^m be a closed smooth manifold, with $m \geq 2$. Whenever an assertion requires connectedness, this will be stated explicitly. For a tensor $\mathbf{T}$ write $\Delta \mathbf{T} = \operatorname{tr}_{\mathbf{g}}(\nabla^2 \mathbf{T}) = -\nabla^* \nabla \mathbf{T}$. All time-dependent norms, contractions, and covariant derivatives are calculated with $\mathbf{g}(t)$ unless otherwise stated. Coordinate indices of M belong to $\{1, \dots, m\}$. For higher-order derivatives, retain the order in Definition 9.1.1:

$$(\nabla^2 \mathbf{T})(X, Y, \dots) = \big(\nabla_X(\nabla_Y \mathbf{T}) - \nabla_{\nabla_X Y} \mathbf{T}\big)(\dots).$$

Thus, in $(\nabla^2 \mathbf{T})_{abi_1 \dots i_q}$, the index a corresponds to the outer derivative and b to the inner one. Permutations of a

and b will be performed using the curvature identity of the cited chapter. We also retain $R_{ijk}{}^{\ell} = (R(\partial_i,\partial_j)\partial_k)^{\ell}$, $\mathrm{Ric}_{ij} = \sum_{k=1}^{m} R_{kij}{}^{k}$, and $R_{ijk\ell} = \sum_{q=1}^{m} g_{\ell q} R_{ijk}{}^{q}$. For a symmetric covariant tensor $\mathbf{h}$ set

$$(\operatorname{div}\mathbf{h})_i := \sum_{j=1}^{m} \nabla^{j} h_{ij}, \qquad \operatorname{div}^2\mathbf{h} := \sum_{i=1}^{m} \nabla^{i} (\operatorname{div}\mathbf{h})_i .$$

Denote by $\lambda_{\mathbf{g}}$ the Riemann–Lebesgue measure determined by $\mathbf{g}$. In a chart, its weight relative to Lebesgue measure is $\sqrt{\det(\mathbf{g})}$; thus no orientation of M is required. When M is orientable and an orientation is fixed, integration with respect to $\lambda_{\mathbf{g}}$ agrees with integration of the Riemannian volume form. Identities containing $\partial_t d\lambda_{\mathbf{g}(t)}$ are understood as identities of measures depending smoothly on t, verified through these coordinate weights.

21.1 Variations of the metric and the Einstein–Hilbert functional

The following variation identities fix the signs that will appear later. Let $\mathbf{g}(t)$ be a smooth family of metrics, $\mathbf{g}(0) = \mathbf{g}$, and $\mathbf{h} = \partial_t \mathbf{g}(t)|_{t=0}$.

Lemma 21.1.1. (Metric variation identities) Let M^m be a smooth manifold with or without boundary and let $\mathbf{g}(t)$ be a smooth family of metrics on M, with $\mathbf{g}(0) = \mathbf{g}$ and $\mathbf{h} = \partial_t \mathbf{g}(t)|_{t=0}$. Then

$$\partial_t g^{ij}(t)\Big|_{t=0} = -h^{ij}, \tag{21.1}$$

$$\partial_t d\lambda_{\mathbf{g}(t)}\Big|_{t=0} = \frac{1}{2}\operatorname{tr}_{\mathbf{g}}(\mathbf{h})\, d\lambda_{\mathbf{g}}, \tag{21.2}$$

$$\partial_t \Gamma(\mathbf{g}(t))^{k}_{ij}\Big|_{t=0} = \frac{1}{2}\sum_{\ell=1}^{m} g^{k\ell}\big(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}\big), \tag{21.3}$$

$$\partial_t R(\mathbf{g}(t))_{ijr}{}^{s}\Big|_{t=0} = \nabla_i \dot\Gamma^{s}_{jr} - \nabla_j \dot\Gamma^{s}_{ir}, \tag{21.4}$$

$$\partial_t \operatorname{Ric}(\mathbf{g}(t))_{ij}\Big|_{t=0} = \sum_{k=1}^{m}\big(\nabla_k \dot\Gamma^{k}_{ij} - \nabla_i \dot\Gamma^{k}_{kj}\big), \tag{21.5}$$

$$DR_{\mathbf{g}}[\mathbf{h}] = -\langle \operatorname{Ric}_{\mathbf{g}}, \mathbf{h}\rangle_{\mathbf{g}} + \operatorname{div}^2\mathbf{h} - \Delta_{\mathbf{g}} \operatorname{tr}_{\mathbf{g}}\mathbf{h}. \tag{21.6}$$

Here $\dot\Gamma^{k}_{ij}$ denotes the expression in the third line.

Proof. Differentiating $\sum_{k=1}^{m} g^{ik}(t) g_{kj}(t) = \delta^{i}_{j}$ gives

$$\sum_{k=1}^{m}\Big(\partial_t g^{ik}(t)\Big|_{t=0} g_{kj} + g^{ik} h_{kj}\Big) = 0,$$

and multiplication by $g^{j\ell}$ proves (21.1). In coordinates, Jacobi's identity for the determinant gives

$$\partial_t \sqrt{\det \mathbf{g}(t)}\Big|_{t=0} = \frac{1}{2}\sqrt{\det \mathbf{g}}\sum_{i,j=1}^{m} g^{ij} h_{ij},$$

whence (21.2).
If ∇^t is the Levi–Civita connection of $\mathbf{g}(t)$ and

$$\mathbf{A}(\mathbf{X},\mathbf{Y}) = \frac{d}{dt}\Big|_{t=0} \nabla^t_{\mathbf{X}}\mathbf{Y},$$

, differentiating the Koszul formula and canceling the terms containing $\nabla_X Y$ gives

$$2\mathbf{g}(\mathbf{A}(\mathbf{X},\mathbf{Y}),\mathbf{Z}) = (\nabla_{\mathbf{X}}\mathbf{h})(\mathbf{Y},\mathbf{Z}) + (\nabla_{\mathbf{Y}}\mathbf{h})(\mathbf{X},\mathbf{Z}) - (\nabla_{\mathbf{Z}}\mathbf{h})(\mathbf{X},\mathbf{Y}).$$

In a coordinate frame, this is (21.3). Differentiating the definition of curvature and using that ∇ is torsion-free yields

$$\partial_t \mathbf{R}^t(\mathbf{X},\mathbf{Y})\mathbf{Z}\big|_{t=0} = (\nabla_{\mathbf{X}}\mathbf{A})(\mathbf{Y},\mathbf{Z}) - (\nabla_{\mathbf{Y}}\mathbf{A})(\mathbf{X},\mathbf{Z}),$$

which proves (21.4); contraction of the first covariant index with the contravariant index gives (21.5).
The contractions of (21.3) are

$$\sum_{k=1}^{m} \dot\Gamma^k_{kj} = \frac{1}{2}\nabla_j \operatorname{tr}_{\mathbf{g}} \mathbf{h}, \qquad \sum_{i,j=1}^{m} g^{ij}\dot\Gamma^k_{ij} = \sum_{i=1}^{m} \nabla^i h_i{}^k - \frac{1}{2}\nabla^k \operatorname{tr}_{\mathbf{g}} \mathbf{h}.$$

Therefore,

$$\sum_{i,j=1}^{m} g^{ij}\, \partial_t \operatorname{Ric}(\mathbf{g}(t))_{ij}\Big|_{t=0} = \operatorname{div}^2 \mathbf{h} - \Delta_{\mathbf{g}} \operatorname{tr}_{\mathbf{g}} \mathbf{h}.$$

Finally, differentiate $R(\mathbf{g}(t)) = \sum_{i,j=1}^{m} g^{ij}(t)\operatorname{Ric}(\mathbf{g}(t))_{ij}$ and use (21.1). This gives (21.6). □

For $s > \frac{m}{2} + 2$, the set $\operatorname{Met}^s(M)$ of H^s metrics is open in $H^s(S^2T^*M)$. Indeed, fixing a smooth metric $\mathbf{g}_*$, the embedding $H^s \hookrightarrow C^0$ makes the function

$$\mathbf{g} \longmapsto \min_{x\in M} \min_{\substack{v\in T_xM \\ |v|_{\mathbf{g}_*}=1}} \frac{\mathbf{g}_x(v,v)}{\mathbf{g}_{*,x}(v,v)}.$$

continuous. Denote this minimum by $m(\mathbf{g})$. Its existence is justified in local frames of $\mathbf{g}_*$ over a finite cover by compact sets: the unit sphere is identified with a fixed Euclidean sphere and evaluation of $\mathbf{g}$ is continuous. This is the fiberwise extrema argument of Lemma 9.1.43. Moreover, if $\mathbf{g}_1, \mathbf{g}_2$ are two symmetric fields of class H^s and $|v|_{\mathbf{g}_*} = 1$, Cauchy–Schwarz gives

$$|(\mathbf{g}_1 - \mathbf{g}_2)_x(v,v)| \le \|\mathbf{g}_1 - \mathbf{g}_2\|_{\infty;\mathbf{g}_*}.$$

Taking minima and then interchanging $\mathbf{g}_1$ and $\mathbf{g}_2$ yields

$$|m(\mathbf{g}_1) - m(\mathbf{g}_2)| \le \|\mathbf{g}_1 - \mathbf{g}_2\|_{\infty;\mathbf{g}_*} \le C\|\mathbf{g}_1 - \mathbf{g}_2\|_{H^s(M,S^2T^*M)}.$$

This proves continuity with respect to the Sobolev norm. Since M is compact, $m(\mathbf{g}) > 0$ precisely when $\mathbf{g}$ is positive definite in every fiber. The condition $m(\mathbf{g}) > 0$ therefore defines an open set. Inversion of positive matrices, Sobolev

products, and the local curvature formulas show that

$$\mathbf{g} \longmapsto \int_M R_{\mathbf{g}}\, d\lambda_{\mathbf{g}}$$

is a smooth map from $\mathrm{Met}^s(M)$ to $\mathbb{R}$.

Proposition 21.1.2. (First variation of Einstein–Hilbert) Let M^m be a closed connected smooth manifold and let $\mathbf{g}$ be a Riemannian metric on M. The functional

$$\mathcal{S}(\mathbf{g}) = \int_M R_{\mathbf{g}}\, d\lambda_{\mathbf{g}} \tag{21.7}$$

satisfies

$$D\mathcal{S}(\mathbf{g})[\mathbf{h}] = -\int_M \left\langle \mathrm{Ric}_{\mathbf{g}} - \frac{1}{2} R_{\mathbf{g}}\mathbf{g}, \mathbf{h} \right\rangle_{\mathbf{g}} d\lambda_{\mathbf{g}}. \tag{21.8}$$

If $m \geq 3$, the critical points of $\mathcal{S}$ restricted to metrics of fixed volume are precisely the Einstein metrics.

Proof. Equations (21.2) and (21.6) give

$$D\mathcal{S}(\mathbf{g})[\mathbf{h}] = \int_M \left(-\langle \mathrm{Ric}, \mathbf{h} \rangle_{\mathbf{g}} + \frac{1}{2} R \,\mathrm{tr}_{\mathbf{g}} \mathbf{h} + \sum_{i=1}^m \nabla_i \left(\sum_{j=1}^m \nabla_j h^{ij} - \nabla^i \,\mathrm{tr}_{\mathbf{g}} \mathbf{h} \right) \right) d\lambda_{\mathbf{g}}.$$

The integral of the divergence over the closed manifold M is zero, proving (21.8).
On the other hand,

$$D\,\mathrm{Vol}_{\mathbf{g}}[\mathbf{h}] = \frac{1}{2} \int_M \mathrm{tr}_{\mathbf{g}} \mathbf{h}\, d\lambda_{\mathbf{g}}$$

is not the zero functional, since its value at $\mathbf{h} = \mathbf{g}$ is $\frac{m}{2}\mathrm{Vol}(M, \mathbf{g})$. If $\mathbf{g}$ is critical on the volume level set, choose $\mathbf{h}_0$ with $D\,\mathrm{Vol}_{\mathbf{g}}[\mathbf{h}_0] = 1$. Every tensor $\mathbf{h}$ decomposes as

$$\mathbf{h} = \left(\mathbf{h} - D\,\mathrm{Vol}_{\mathbf{g}}[\mathbf{h}]\mathbf{h}_0\right) + D\,\mathrm{Vol}_{\mathbf{g}}[\mathbf{h}]\mathbf{h}_0.$$

The first summand belongs to the kernel of $D\,\mathrm{Vol}_{\mathbf{g}}$, where $D\mathcal{S}_{\mathbf{g}}$ vanishes. Therefore, $D\mathcal{S}_{\mathbf{g}} = cD\,\mathrm{Vol}_{\mathbf{g}}$, with $c = D\mathcal{S}_{\mathbf{g}}[\mathbf{h}_0]$. The formulas for the two differentials imply

$$\mathrm{Ric} - \frac{1}{2} R\mathbf{g} = c\mathbf{g}$$

after changing the value of the constant c if necessary. Taking the trace of this equality when $m \geq 3$ shows that R is constant; substituting it back gives $\mathrm{Ric} = \lambda\mathbf{g}$ with constant λ. The reverse implication follows directly from (21.8), because vectors tangent to the volume level set satisfy $\int_M \mathrm{tr}_{\mathbf{g}} \mathbf{h}\, d\lambda_{\mathbf{g}} = 0$. □

In dimension two, Theorem A.8.7 makes $\mathcal{S}$ constant on each topological component of the space of metrics; this is why the last characterization requires $m \geq 3$. The first variation formula, however, remains valid in every dimension.

21.2 Diffeomorphism invariance and symbol degeneracy

Definition 21.2.1. (Ricci flow) Let M be a smooth manifold with or without boundary. A *Ricci flow* on M is a family of metrics $\mathbf{g}(t)$ satisfying

$$\partial_t \mathbf{g}(t) = -2\,\mathrm{Ric}_{\mathbf{g}(t)}. \tag{21.9}$$

The Levi–Civita connection is natural: if φ is a diffeomorphism, $\nabla^{\varphi^*\mathbf{g}} = \varphi^*\nabla^{\mathbf{g}}$. Taking curvature and contracting gives

$$\mathrm{Ric}_{\varphi^*\mathbf{g}} = \varphi^*\,\mathrm{Ric}_{\mathbf{g}}. \tag{21.10}$$

Thus pullback by a time-independent diffeomorphism takes solutions to solutions. For a family φ_t, if

$$\mathbf{X}_t = (\partial_t \varphi_t) \circ \varphi_t^{-1},$$

, the pullback differentiation rule is

$$\frac{d}{dt}(\varphi_t^*\mathbf{g}(t)) = \varphi_t^*(\partial_t \mathbf{g}(t) + \mathcal{L}_{\mathbf{X}_t}\mathbf{g}(t)). \tag{21.11}$$

It is proved first for functions and one-forms, then using the Leibniz rule for tensor products.

Naturality explains why the right-hand side of (21.9) is not strictly parabolic. With $(\delta_{\mathbf{g}}\mathbf{h})_j = -\sum_{i=1}^{m} \nabla^i h_{ij}$, define the formal adjoint of $\delta_{\mathbf{g}}$ by

$$(\delta_{\mathbf{g}}^*\boldsymbol{\omega})_{ij} = \frac{1}{2}\big(\nabla_i \omega_j + \nabla_j \omega_i\big). \tag{21.12}$$

If $\nabla^*\nabla = -\mathrm{tr}_{\mathbf{g}}(\nabla^2)$, Lemma 21.1.1, followed by commutation of covariant derivatives, gives

$$2D\,\mathrm{Ric}_{\mathbf{g}}[\mathbf{h}] = \nabla^*\nabla\mathbf{h} - \nabla^2\,\mathrm{tr}_{\mathbf{g}}\,\mathbf{h} - 2\delta_{\mathbf{g}}^*\delta_{\mathbf{g}}\mathbf{h} + \mathcal{Q}_{\mathbf{g}}(\mathbf{h}), \tag{21.13}$$

where the zeroth-order term is

$$\mathcal{Q}_{\mathbf{g}}(\mathbf{h})_{ij} = \sum_{k=1}^{m} \mathrm{Ric}_i{}^k h_{kj} + \sum_{k=1}^{m} \mathrm{Ric}_j{}^k h_{ki} - 2\sum_{k,\ell=1}^{m} R_{kij\ell} h^{k\ell}. \tag{21.14}$$

Before commuting, substitution of (21.3) into (21.5) gives the tensor identity

$$2D\,\mathrm{Ric}_{\mathbf{g}}[\mathbf{h}]_{ij} = \sum_{k,a=1}^{m} g^{ka}\big((\nabla^2\mathbf{h})_{aijk} + (\nabla^2\mathbf{h})_{ajik}\big) + (\nabla^*\nabla\mathbf{h})_{ij} - (\nabla^2\,\mathrm{tr}_{\mathbf{g}}\,\mathbf{h})_{ij}. \tag{21.15}$$

To see where $\mathcal{Q}_{\mathbf{g}}$ comes from, apply the commutation identity to the two slots of $\mathbf{h}$:

$$\begin{aligned}
&\sum_{a,k=1}^{m} g^{ak}\big((\nabla^2\mathbf{h})_{aijk} - (\nabla^2\mathbf{h})_{iajk}\big) \\
&\quad = -\sum_{a,k,p=1}^{m} g^{ak}\big(R_{aij}{}^p h_{pk} + R_{aik}{}^p h_{jp}\big) \\
&\quad = \sum_{p=1}^{m} \mathrm{Ric}_i{}^p h_{pj} - \sum_{k,\ell=1}^{m} R_{kij\ell} h^{k\ell}.
\end{aligned}$$

The same equality with i and j interchanged produces the other Ricci term. The Riemann symmetries and symmetry of $\mathbf{h}$ make the two remaining contractions equal. The derivatives already in the order i,a and j,a form $-2\delta_{\mathbf{g}}^{*}\delta_{\mathbf{g}}\mathbf{h}$. This proves (21.13) and (21.14) with the chosen differentiation order.

If $\mathbf{h} = \mathcal{L}_{X}\mathbf{g}$, its symbol part is $h_{ij} = \xi_i X_j + \xi_j X_i$, and substitution into (21.15) annihilates the symbol of $D\,\mathrm{Ric}$. This is precisely the infinitesimal direction of the diffeomorphism orbit. Thus strongly parabolic theory cannot be applied directly to (21.9).

21.3 The Ricci–DeTurck system

The preceding degeneracy reflects the freedom to change coordinates during evolution rather than a loss of diffusion. The DeTurck procedure removes this freedom by comparing the connection of the unknown metric with a fixed connection. Fix a smooth metric $\bar{\mathbf{g}}$, denote its Levi–Civita connection by $\bar{\nabla}$, and write

$$\mathbf{A}(\mathbf{g})_{ij}^{k} = \Gamma(\mathbf{g})_{ij}^{k} - \Gamma(\bar{\mathbf{g}})_{ij}^{k}, \qquad \mathbf{W}_{\bar{\mathbf{g}}}(\mathbf{g})^{k} = \sum_{i,j=1}^{m} g^{ij}\mathbf{A}(\mathbf{g})_{ij}^{k}.$$

The difference of connections is tensorial, so $\mathbf{W}_{\bar{\mathbf{g}}}(\mathbf{g})$ is a globally defined vector field. The modified system is

$$\partial_t \mathbf{g} = -2\,\mathrm{Ric}_{\mathbf{g}} + \mathcal{L}_{\mathbf{W}_{\bar{\mathbf{g}}}(\mathbf{g})}\mathbf{g}. \tag{21.16}$$

Proposition 21.3.1. (Parabolicity of the DeTurck system) Let M^m be a smooth manifold with or without boundary, let $\bar{\mathbf{g}}$ be a fixed Riemannian metric, and consider the Ricci–DeTurck system (21.16). The right-hand side of (21.16) is a quasilinear operator whose principal part is locally strongly parabolic. More precisely, relative to $\bar{\nabla}$ it can be written as

$$-2\,\mathrm{Ric}(\mathbf{g})_{ij} + (\mathcal{L}_{\mathbf{W}_{\bar{\mathbf{g}}}(\mathbf{g})}\mathbf{g})_{ij} = \sum_{a,b=1}^{m} g^{ab}(\bar{\nabla}^2\mathbf{g})_{abij} + P_{ij}(\mathbf{g}, \mathbf{g}^{-1}, \bar{\nabla}\mathbf{g}, \overline{\mathrm{Rm}}), \tag{21.17}$$

where P is smooth, contains no derivatives of $\mathbf{g}$ of order greater than one, and is polynomial in $\bar{\nabla}\mathbf{g}$. The Fourier symbol of the linearization at a covariable $\xi \neq 0$ is

$$-\sum_{a,b=1}^{m} g^{ab}\xi_a\xi_b\,\mathrm{Id}_{S^2T^*M}. \tag{21.18}$$

On each compact subset of M, the parabolicity bound can be chosen uniformly for metrics sufficiently close in C^0 to a fixed metric. In particular, if M is compact, this bound is global.

Proof. The difference-of-connections formula is

$$\mathbf{A}(\mathbf{g})_{ij}^{k} = \frac{1}{2}\sum_{\ell=1}^{m} g^{k\ell}\left(\bar{\nabla}_i g_{j\ell} + \bar{\nabla}_j g_{i\ell} - \bar{\nabla}_\ell g_{ij}\right).$$

Expressing the curvature of $\mathbf{g}$ using $\bar{\nabla}$ gives

$$\mathbf{R}(\mathbf{g})_{ijk}{}^{\ell} = \overline{R}_{ijk}{}^{\ell} + \bar{\nabla}_i \mathbf{A}(\mathbf{g})^{\ell}_{jk} - \bar{\nabla}_j \mathbf{A}(\mathbf{g})^{\ell}_{ik} + \sum_{p=1}^{m} \big(\mathbf{A}(\mathbf{g})^{p}_{jk} \mathbf{A}(\mathbf{g})^{\ell}_{ip} - \mathbf{A}(\mathbf{g})^{p}_{ik} \mathbf{A}(\mathbf{g})^{\ell}_{jp} \big).$$

The contraction defining Ricci contains second derivatives of $\mathbf{g}$. On the other hand,

$$(\mathcal{L}_{\mathbf{W}}\mathbf{g})_{ij} = \nabla_i W_j + \nabla_j W_i.$$

At a point where normal coordinates for $\bar{\mathbf{g}}$ are chosen, the parts containing two derivatives of $\mathbf{g}$ are

$$(-2\,\mathrm{Ric}(\mathbf{g}))^{(2)}_{ij} = \sum_{a,b=1}^{m} g^{ab} \big((\bar{\nabla}^2\mathbf{g})_{abij} + (\bar{\nabla}^2\mathbf{g})_{ijab} - (\bar{\nabla}^2\mathbf{g})_{aibj} - (\bar{\nabla}^2\mathbf{g})_{ajbi} \big),$$

$$(\mathcal{L}_{\mathbf{W}}\mathbf{g})^{(2)}_{ij} = \sum_{a,b=1}^{m} g^{ab} \big((\bar{\nabla}^2\mathbf{g})_{iabj} + (\bar{\nabla}^2\mathbf{g})_{jabi} - (\bar{\nabla}^2\mathbf{g})_{ijab} \big).$$

The three gauge terms cancel in pairs, leaving as the only second-order term $\sum_{a,b=1}^{m} g^{ab}(\bar{\nabla}^2\mathbf{g})_{abij}$. Commutations of $\bar{\nabla}$ produce $\overline{\mathrm{Rm}} * \mathbf{g}$, and all other terms are contractions of $\mathbf{g}$, $\mathbf{g}^{-1}$, $\bar{\nabla}\mathbf{g}$, and $\overline{\mathrm{Rm}}$. This proves (21.17). To apply the divergence-form theory, observe the exact identity

$$\sum_{a,b=1}^{m} g^{ab}(\bar{\nabla}^2\mathbf{g})_{ab} = \sum_{a,b=1}^{m} \bar{\nabla}_a \big(g^{ab} \bar{\nabla}_b \mathbf{g} \big) - \sum_{a,b=1}^{m} (\bar{\nabla}_a g^{ab}) \bar{\nabla}_b \mathbf{g}. \tag{21.19}$$

The last summand contains only first derivatives of $\mathbf{g}$. The principal coefficient is therefore

$$A^{ab}(\mathbf{g}) = g^{ab}\,\mathrm{Id}_{S^2T^*M}.$$

As an endomorphism of $T^*M \otimes S^2T^*M$ measured with $\bar{\mathbf{g}}$, it reads $A_a{}^b = \sum_{c=1}^{m} \bar{g}_{ac} g^{cb}\,\mathrm{Id}_{S^2T^*M}$; raising the first index with $\bar{\mathbf{g}}$ recovers A^{ab}. This coefficient satisfies $A^{ab*} = A^{ba}$. To specify uniformity, fix a compact set $K \subseteq M$ and a continuous metric $\mathbf{g}_*$. The comparison on compact sets proved after Lemma 7.1.8, whose argument also applies to continuous metrics, gives $0 < c_* \le C_* < \infty$ such that

$$c_* \bar{\mathbf{g}}_x(v,v) \le (\mathbf{g}_*)_x(v,v) \le C_* \bar{\mathbf{g}}_x(v,v) \qquad (x \in K,\ v \in T_xM).$$

If $\|\mathbf{g} - \mathbf{g}_*\|_{C^0(K, S^2T^*M; \bar{\mathbf{g}})} < c_*/2$, Cauchy–Schwarz for the tensor inner product gives, for $x \in K$ and $v \in T_xM$,

$$|(\mathbf{g} - \mathbf{g}_*)_x(v,v)| \le |\mathbf{g} - \mathbf{g}_*|_{\bar{\mathbf{g}}}(x)\, |v|^2_{\bar{\mathbf{g}}} < \frac{c_*}{2} \bar{\mathbf{g}}_x(v,v) \qquad (v \neq 0).$$

Indeed, in a $\bar{\mathbf{g}}$–orthonormal basis, apply Cauchy–Schwarz to $\sum_{a,b=1}^{m} (g - g_*)_{ab} v^a v^b$ and use $\sum_{a,b=1}^{m} (v^a v^b)^2 =$

$(\sum_{a=1}^{m}(v^a)^2)^2$. Adding this bound to the comparison for $\mathbf{g}_*$ gives

$$\frac{c_*}{2}\bar{\mathbf{g}} \le \mathbf{g} \le \Big(C_* + \frac{c_*}{2}\Big)\bar{\mathbf{g}} \qquad \text{on } K.$$

The dual comparison in Proposition 7.1.9 gives, with $\lambda = (C_* + c_*/2)^{-1} > 0$,

$$\mathbf{g}_x^{-1}(\xi,\xi) \ge \lambda \bar{\mathbf{g}}_x^{-1}(\xi,\xi) \qquad (x \in K,\ \xi \in T_x^*M).$$

To apply this inequality to a covector with values in $S^2T_x^*M$, choose a basis $F_1,\dots,F_d$ of that space orthonormal for the metric induced by $\bar{\mathbf{g}}$, where $d = m(m+1)/2$. Write $\zeta_a = \sum_{\alpha=1}^{d} z_a^\alpha F_\alpha$ in a $\bar{\mathbf{g}}$–orthonormal tangent frame. For each α, the numbers z_a^α are the components of a covector to which the preceding bound applies. Thus

$$\begin{aligned}\sum_{a,b=1}^{m}\big\langle A^{ab}(\mathbf{g})\zeta_b,\zeta_a\big\rangle_{\bar{\mathbf{g}}} &= \sum_{\alpha=1}^{d}\sum_{a,b=1}^{m} g^{ab} z_b^\alpha z_a^\alpha \\ &\ge \lambda\sum_{\alpha=1}^{d}\sum_{a=1}^{m}|z_a^\alpha|^2 = \lambda|\zeta|_{\bar{\mathbf{g}}}^2.\end{aligned}$$

If M is compact, take $K = M$. Sobolev embedding into C^0 allows the neighborhood in $\mathrm{Met}^s(M)$ to be chosen so that the preceding bound holds; in particular, this applies for $s > m/2 + 2$, as in this chapter. For closed manifolds, this gives the uniform Legendre condition of Theorem 20.3.1; the permitted open subset of bundle values is the cone of positive definite bilinear forms. If M is noncompact, a comparison $\mathbf{g} \le C\bar{\mathbf{g}}$ on all of M, with C independent of the point, likewise gives a lower bound for the principal part $\lambda = C^{-1}$. Without uniform comparison, pointwise positivity alone does not imply a global constant.

Linearization of the principal term in the direction $\mathbf{h}$ is $\sum_{a,b=1}^{m} g^{ab}(\bar\nabla^2\mathbf{h})_{abij}$ plus terms of order at most one in $\mathbf{h}$. The substitution $\bar\nabla_a \mapsto i\xi_a$ gives (21.18). Since $\sum_{a,b=1}^{m} g^{ab}\xi_a\xi_b > 0$, the system is locally strongly parabolic, and the preceding bounds provide uniformity in the compact case. The presence of a boundary does not alter this symbol calculation; the theory of Chapter 20 is applied here with M closed. □

The uniqueness argument will use harmonic map heat flow. To construct it through an equation on a trivial bundle, we must check that the ambient solution remains in the embedded submanifold. The following estimate makes this invariance explicit.

Lemma 21.3.2. (Remaining in a submanifold through tubular distance) Let N be a compact embedded submanifold without boundary of a Riemannian manifold $(\mathcal{N},\mathbf{h})$, and let $\mathcal{T}$ be a tubular neighborhood with retraction $\pi\colon \mathcal{T} \to N$. Set $\rho(y) = \frac{1}{2}\,\mathrm{dist}_{\mathbf{h}}(y,N)^2$. Let $\widetilde{\boldsymbol{\mathcal{A}}}$ be a smooth extension to $\mathcal{T}$ of the second fundamental form $\boldsymbol{\mathcal{A}}$ of N, defined as follows. For $y = \exp_z^{\perp} \nu$, with $z = \pi(y)$, let $\mathsf{P}_{z,y}\colon T_z\mathcal{N} \to T_y\mathcal{N}$ be parallel transport along the normal geodesic. Set

$$\widetilde{\boldsymbol{\mathcal{A}}}_y(V,W) := \mathsf{P}_{z,y}\big(\boldsymbol{\mathcal{A}}_z(d\pi_y V, d\pi_y W)\big), \qquad V,W \in T_y\mathcal{N}.$$

If a smooth map $U\colon M\times[0,T]\to\mathcal{T}$ satisfies

$$\partial_t U=\tau_{\mathbf{g}(t),\mathbf{h}}(U)-\sum_{i,j=1}^{m}g^{ij}(t)\widetilde{\boldsymbol{\mathcal{A}}}_U(\partial_i U,\partial_j U),\tag{21.20}$$

, there exists a constant C, uniform while U ranges over a compact subset of $\mathcal{T}$ and $\mathbf{g}(t)$ stays in a uniformly equivalent family, such that

$$(\partial_t-\Delta_{\mathbf{g}(t)})(\rho\circ U)\le C\bigl(1+|dU|^2_{\mathbf{g}(t),\mathbf{h}}\bigr)(\rho\circ U).\tag{21.21}$$

In particular, if M is closed and $U(\cdot,0)$ takes values in N, then $U(M\times[0,T])\subseteq N$.

Proof. The chain rule for the tension field of a map gives

$$\begin{aligned}(\partial_t-\Delta_{\mathbf{g}(t)})(\rho\circ U)=&-d\rho_U\left(\sum_{i,j=1}^{m}g^{ij}\widetilde{\boldsymbol{\mathcal{A}}}_U(\partial_i U,\partial_j U)\right)\\&-\sum_{i,j=1}^{m}g^{ij}\operatorname{Hess}_{\mathbf{h}}\rho_U(\partial_i U,\partial_j U).\end{aligned}$$

To estimate the right-hand side, use Fermi coordinates $y=(z^a,\nu^\alpha)$, where $z=\pi(y)$ and ν is normal to N at z. Write $d=\dim N$ and $c=\dim\mathcal{N}-d$. Choose a local orthonormal normal frame $(\mathbf{n}_\alpha)_{\alpha=1}^{c}$ and set $\nu=\sum_{\alpha=1}^{c}\nu^\alpha\mathbf{n}_\alpha(z)$. In these coordinates, $a,b\in\{1,\dots,d\}$ and $\alpha,\beta\in\{1,\dots,c\}$. On a compact subset of the tubular neighborhood, the Taylor expansions are uniform and give

$$\rho(z,\nu)=\frac12\sum_{\alpha=1}^{c}(\nu^\alpha)^2,\qquad\partial_a\rho=0,\qquad\partial_\alpha\rho=\nu^\alpha,\tag{21.22}$$

$$(\operatorname{Hess}\rho)_{ab}=-\langle\nu,\boldsymbol{\mathcal{A}}_{ab}\rangle+O(|\nu|^2),\qquad(\operatorname{Hess}\rho)_{a\beta}=O(|\nu|),\qquad(\operatorname{Hess}\rho)_{\alpha\beta}=\delta_{\alpha\beta}+O(|\nu|).\tag{21.23}$$

The first line is exact: normal geodesics minimize distance to N in the tubular neighborhood. For the second line, use $(\operatorname{Hess}\rho)_{AB}=\partial_A\partial_B\rho-\sum_{C=1}^{d+c}\Gamma^C_{AB}\partial_C\rho$. The indices A,B range over $\{1,\dots,d+c\}$. On the zero section, Γ^α_{ab} are the normal components of $\boldsymbol{\mathcal{A}}_{ab}$; multiplying them by ν^α gives the linear term of the tangential block. The normal block starts with the identity and the mixed block vanishes on N, justifying the remainder orders by Taylor's formula. Here $\boldsymbol{\mathcal{A}}_{ab}$ is the second fundamental form of N on the zero section. By the way this form was extended, transport $V\in T_y\mathcal{N}$ back to $T_z\mathcal{N}$ and decompose it as $V^\top+V^\perp$ according to $T_zN\oplus(T_zN)^\perp$. Smoothness of $d\pi$ and its value on the zero section give $d\pi_y V=V^\top+O(|\nu|)|V|$. Moreover, $d\rho_y(\mathsf{P}_{z,y}Z)=\langle\nu,Z\rangle$ for $Z\in T_z\mathcal{N}$ by the first variation of distance. Thus

$$d\rho_y\bigl(\widetilde{\boldsymbol{\mathcal{A}}}_y(V,V)\bigr)=\langle\nu,\boldsymbol{\mathcal{A}}_z(V^\top,V^\top)\rangle+O(|\nu|^2)|V|^2_{\mathbf{h}}.\tag{21.24}$$

Fix a point of the domain and choose there an orthonormal frame $(\mathbf{e}_i)_{i=1}^{m}$ for $\mathbf{g}(t)$. If $V_i=dU(\mathbf{e}_i)=V_i^\top+V_i^\perp$,

the two preceding formulas show, for each i, that

$$\begin{aligned} &-d\rho_U\big(\widetilde{\boldsymbol{\mathcal{A}}}_U(V_i,V_i)\big)-\operatorname{Hess}_{\mathbf{h}}\rho_U(V_i,V_i)\\ &\qquad=-|V_i^{\perp}|_{\mathbf{h}}^2+\mathcal{E}_i, \end{aligned}$$

with

$$|\mathcal{E}_i|\le C|\nu||V_i^{\perp}|_{\mathbf{h}}|V_i|_{\mathbf{h}}+C|\nu|^2\big(1+|V_i|_{\mathbf{h}}^2\big).$$

Indeed, the term $\langle \nu,\boldsymbol{\mathcal{A}}(V_i^{\top},V_i^{\top})\rangle$ coming from $d\rho(\widetilde{\boldsymbol{\mathcal{A}}})$ cancels with the linear term of opposite sign in $\operatorname{Hess}\rho(V_i^{\top},V_i^{\top})$. The normal block of the Hessian leaves $-|V_i^{\perp}|^2$, while the mixed block and Taylor remainders produce the stated errors. Summing over i gives

$$\begin{aligned} &-d\rho_U\left(\sum_{i,j=1}^{m}g^{ij}\widetilde{\boldsymbol{\mathcal{A}}}_U(\partial_iU,\partial_jU)\right)-\sum_{i,j=1}^{m}g^{ij}\operatorname{Hess}_{\mathbf{h}}\rho_U(\partial_iU,\partial_jU)\\ &\qquad=-|(dU)^{\perp}|_{\mathbf{g}(t),\mathbf{h}}^2+\mathcal{E}, \end{aligned}\tag{21.25}$$

where

$$|\mathcal{E}|\le C|\nu||(dU)^{\perp}||dU|+C|\nu|^2\big(1+|dU|^2\big).\tag{21.26}$$

Young's inequality absorbs the cross term:

$$C|\nu||(dU)^{\perp}||dU|\le\frac{1}{2}|(dU)^{\perp}|^2+C|\nu|^2|dU|^2.$$

In these coordinates, $|\nu|^2=2\rho(z,\nu)$. The three preceding estimates imply (21.21). On an interval where U is smooth, $|dU|$ is bounded because M is compact. The maximum principle applied to $e^{-Ct}(\rho\circ U)$ with zero initial data gives $\rho\circ U=0$. □

Corollary 21.3.3. (Harmonic map heat flow with varying domain metric) Let M be a closed smooth manifold. Let $\mathbf{g}(t)$ be a smooth family of metrics on M, let $\bar{\mathbf{g}}$ be a fixed smooth metric, and let $F_0\colon M\to M$ be a smooth map. There exists $T_0>0$ for which the problem

$$\partial_tF=\tau_{\mathbf{g}(t),\bar{\mathbf{g}}}(F),\qquad F(0)=F_0,\tag{21.27}$$

has a unique smooth solution. The time can be chosen uniformly when the domain metrics and their inverses satisfy uniform bounds for ellipticity and regularity, and the data $\iota\circ F_0$ are bounded in an H^s norm, with $s>m/2+2$, for a fixed embedding ι of the target. The smooth geometry of the target remains fixed.

Proof. Theorem A.6.2 gives an embedding $\iota\colon M\hookrightarrow\mathbb{R}^N$. On a neighborhood of $\iota(M)$, choose an ambient metric $\mathbf{h}$ making the inclusion $\iota\colon(M,\bar{\mathbf{g}})\to(\mathbb{R}^N,\mathbf{h})$ isometric. To construct it, on the normal bundle of a Euclidean embedding take the orthogonal sum of $\bar{\mathbf{g}}$ in the horizontal directions and a smooth metric in the normal directions. A fixed connection defines this splitting; the tubular diffeomorphism transports the metric to a neighborhood of $\iota(M)$. Its restriction to the zero section is the prescribed one. Let $\mathcal{T}$ be a tubular neighborhood, let $\pi\colon\mathcal{T}\to\iota(M)$ be its retraction, and let $\boldsymbol{\mathcal{A}}$ be the second fundamental form of

$\iota(M)$. Shrinking $\mathcal{T}$, smoothly extend the map

$$(y, V, W) \longmapsto \mathrm{P}_{\pi(y),y}\big(\boldsymbol{\mathcal{A}}_{\pi(y)}(d\pi_y V, d\pi_y W)\big)$$

to an open subset of $\mathbb{R}^N$ and denote the extension by $\widetilde{\boldsymbol{\mathcal{A}}}$. Existence of the tubular neighborhood and retraction was proved in Theorem 2.12.3.

For $U_0 = \iota \circ F_0$, solve on the trivial bundle $M \times \mathbb{R}^N$ the system

$$\partial_t U = \tau_{\mathbf{g}(t),\mathbf{h}}(U) - \sum_{i,j=1}^{m} g^{ij}(t)\widetilde{\boldsymbol{\mathcal{A}}}_U(\partial_i U, \partial_j U), \qquad U(0) = U_0. \tag{21.28}$$

In ambient-space coordinates, the principal part is $\sum_{i,j=1}^{m} g^{ij}(t)\partial_i\partial_j U^\alpha$ and the remaining terms depend smoothly on (x, t, U, dU). Corollary 20.3.2 gives a unique smooth solution on a short interval. Shrinking this interval, its image stays in $\mathcal{T}$.

Apply Lemma 21.3.2 with $N = \iota(M)$. Since U_0 takes values in N, we have $U(M \times [0, T_0]) \subseteq \iota(M)$. Thus there exists a smooth map F such that $U = \iota \circ F$. The Gauss formula states that the normal component of the tension field of $\iota \circ F$ is $\sum_{i,j=1}^{m} g^{ij}\boldsymbol{\mathcal{A}}(dF(\partial_i), dF(\partial_j))$; restricting (21.28) to the submanifold cancels this term, leaving exactly (21.27).

Composing two intrinsic solutions with ι gives two solutions of the same ambient system, so they agree. Finally, the existence constants depend only on the uniform bounds for $\mathbf{g}$ and $\mathbf{g}^{-1}$ and finitely many of their derivatives, the fixed geometry of the tubular neighborhood, and an H^s bound for $U_0 = \iota \circ F_0$. The embedding $H^s \hookrightarrow C^1$ also controls the values and first derivatives occurring in the nonlinearity. This proves the uniform assertion. □

Theorem 21.3.4. (Short-time existence and uniqueness) For every smooth metric $\mathbf{g}_0$ on a closed smooth manifold M, there exist $T > 0$ and a unique smooth solution

$$\mathbf{g}\colon M \times [0, T] \longrightarrow S^2 T^*M$$

of Ricci flow with $\mathbf{g}(0) = \mathbf{g}_0$. A smooth family of smooth initial metrics gives, locally in the parameter, a smooth family of solutions on a common interval. The interval can be chosen uniformly over compact subsets of the parameter space. In particular, for $\tau > 0$, dependence is smooth in every C^k norm on $M \times [\tau, T]$, as long as the existence interval is common.

Proof. Take $\bar{\mathbf{g}} = \mathbf{g}_0$. Proposition 21.3.1 and Theorem 20.3.1 give a unique solution $\widetilde{\mathbf{g}}(t)$ of

$$\partial_t \widetilde{\mathbf{g}} = -2\,\mathrm{Ric}_{\widetilde{\mathbf{g}}} + \mathcal{L}_{\mathbf{W}_{\bar{\mathbf{g}}}(\widetilde{\mathbf{g}})}\widetilde{\mathbf{g}}, \qquad \widetilde{\mathbf{g}}(0) = \mathbf{g}_0. \tag{21.29}$$

Positive definiteness is an open condition; continuity in time allows T to be reduced so that $\widetilde{\mathbf{g}}(t)$ remains a metric. Let φ_t be the nonautonomous flow determined by

$$\partial_t \varphi_t = -\mathbf{W}_{\bar{\mathbf{g}}}(\widetilde{\mathbf{g}}(t)) \circ \varphi_t, \qquad \varphi_0 = \mathrm{id}_M. \tag{21.30}$$

The field is smooth on $M\times[0,T]$. Compactness of M ensures that the flow exists on the whole interval, and each φ_t is a diffeomorphism. With $\mathbf{g}(t)=\varphi_t^*\widetilde{\mathbf{g}}(t)$, equations (21.11), (21.29), and (21.10) give

$$\begin{aligned}\partial_t\mathbf{g} &= \varphi_t^*\left(-2\operatorname{Ric}_{\widetilde{\mathbf{g}}}+\mathcal{L}_{\mathbf{W}_{\widetilde{\mathbf{g}}}(\widetilde{\mathbf{g}})}\widetilde{\mathbf{g}}-\mathcal{L}_{\mathbf{W}_{\widetilde{\mathbf{g}}}(\widetilde{\mathbf{g}})}\widetilde{\mathbf{g}}\right)\\ &= -2\operatorname{Ric}_{\varphi_t^*\widetilde{\mathbf{g}}} = -2\operatorname{Ric}_{\mathbf{g}}.\end{aligned}$$

This proves existence.

Uniqueness does not follow by applying uniqueness of the DeTurck system to two Ricci flows: the gauge-fixing diffeomorphisms depend on the solution. To address this difficulty, consider two solutions $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$ with the same initial data and solve, for $a=1,2$, the harmonic map heat flow

$$\partial_t F_a = \tau_{\mathbf{g}_a(t),\bar{\mathbf{g}}}(F_a),\qquad F_a(0)=\mathrm{id}_M. \tag{21.31}$$

Corollary 21.3.3 gives a smooth solution on a common interval. Since $dF_a(0)=\mathrm{Id}$, uniform continuity of $dF_a(t)$ and compactness of M allow the interval to be reduced so that every $dF_a(t)$ is a linear isomorphism. Thus $F_a(t)$ is a local diffeomorphism.

Let us also verify global injectivity. Choose a finite family of charts with three concentric coordinate balls $V_\alpha\Subset W_\alpha\Subset U_\alpha$, such that the V_α cover M and each W_α is convex in coordinates. Convergence $F_a(t)\to\mathrm{id}_M$ in C^1 allows time to be reduced so that $F_a(t)(W_\alpha)\subseteq U_\alpha$ and the coordinate representation $f_{a,t,\alpha}$ satisfies

$$\|D(f_{a,t,\alpha}-\mathrm{id})\|_\infty<\frac{1}{2}$$

on W_α, simultaneously for all indices of the finite family. Integration along the segment between two points x,y of this ball gives

$$|f_{a,t,\alpha}(x)-f_{a,t,\alpha}(y)|\geq\frac{1}{2}|x-y|.$$

Thus each restriction to W_α is injective.

Let $\delta>0$ be a Lebesgue number of the cover (V_α) for a fixed Riemannian distance. Reduce time again so that $d(F_a(t)(x),x)<\delta/3$ for every $x\in M$. If $F_a(t)(x)=F_a(t)(y)$, the triangle inequality gives $d(x,y)<2\delta/3$, so both points belong to some V_α. The local injectivity just proved implies $x=y$. Finally, $F_a(t)$ preserves each connected component, since $s\mapsto F_a(st)(x)$ is a path starting at x. On each component, its image is nonempty, open because it is a local diffeomorphism, and closed by compactness. Connectedness makes it the whole component. Thus $F_a(t)$ is bijective, and its inverse is smooth by the inverse function theorem.

Define

$$\widehat{\mathbf{g}}_a(t)=(F_a(t)^{-1})^*\mathbf{g}_a(t).$$

The tension field of the identity from $(M,\widehat{\mathbf{g}}_a)$ to $(M,\bar{\mathbf{g}})$ is

$$\tau_{\widehat{\mathbf{g}}_a,\bar{\mathbf{g}}}(\mathrm{id})^k=\sum_{i,j=1}^m\widehat{g}_a^{ij}\left(\overline{\Gamma}_{ij}^k-\Gamma(\widehat{\mathbf{g}}_a)_{ij}^k\right)=-\mathbf{W}_{\bar{\mathbf{g}}}(\widehat{\mathbf{g}}_a)^k. \tag{21.32}$$

Set

$$\mathbf{V}_a=(\partial_tF_a)\circ F_a^{-1}.$$

Precomposing the equation for F_a with F_a^{-1}, naturality of the tension field under changes of domain gives

the exact identity

$$\mathbf{V}_a = \tau_{\widehat{\mathbf{g}}_a,\bar{\mathbf{g}}}(\mathrm{id}) = -\mathbf{W}_{\bar{\mathbf{g}}}(\widehat{\mathbf{g}}_a). \tag{21.33}$$

Differentiation of pullback by the inverse family gives

$$\partial_t \widehat{\mathbf{g}}_a = (F_a^{-1})^*(\partial_t \mathbf{g}_a) - \mathcal{L}_{\mathbf{V}_a}\widehat{\mathbf{g}}_a. \tag{21.34}$$

Indeed, differentiating $F_a^{-1} \circ F_a = \mathrm{id}$ identifies the Eulerian velocity of the inverse with the pullback of $-\mathbf{V}_a$, and rule (21.11) gives (21.34). By (21.10) and (21.33), this formula becomes

$$\partial_t \widehat{\mathbf{g}}_a = -2\,\mathrm{Ric}_{\widehat{\mathbf{g}}_a} + \mathcal{L}_{\mathbf{W}_{\bar{\mathbf{g}}}(\widehat{\mathbf{g}}_a)}\widehat{\mathbf{g}}_a, \qquad \widehat{\mathbf{g}}_a(0) = \mathbf{g}_0.$$

Uniqueness of the quasilinear parabolic problem gives $\widehat{\mathbf{g}}_1 = \widehat{\mathbf{g}}_2 =: \widehat{\mathbf{g}}$. Moreover,

$$\partial_t F_a = -\mathbf{W}_{\bar{\mathbf{g}}}(\widehat{\mathbf{g}}) \circ F_a, \qquad F_a(0) = \mathrm{id}_M .$$

Uniqueness for this nonautonomous ordinary differential equation implies $F_1 = F_2$. Since $\mathbf{g}_a = F_a^*\widehat{\mathbf{g}}$, we conclude that $\mathbf{g}_1 = \mathbf{g}_2$ on an initial interval. The set of times up to which they agree is closed by continuity and open by the same argument restarted at any time; connectedness of the time interval shows that they agree on every common interval of existence.

For smooth families of smooth data, keep a reference metric fixed and reduce time uniformly. The fourth assertion of Corollary 20.3.3, parameter dependence of equation (21.30), and the pullback rule give joint smoothness. Parabolic smoothing allows it to be measured in every C^k norm on $M \times [\tau, T]$ when $\tau > 0$. □

Remark 21.3.5. The second half of the proof also gives the reverse direction of the DeTurck procedure. Harmonic map heat flow takes any Ricci solution to a solution of the same strictly parabolic system; the ordinary differential equation for the diffeomorphisms recovers the geometric solution unambiguously.

21.4 Evolution equations

Let $\mathbf{g}(t)$ be a smooth solution of (21.9). Substituting $\mathbf{h} = -2\,\mathrm{Ric}$ into Lemma 21.1.1 gives the identities

$$\partial_t g^{ij} = 2\,\mathrm{Ric}^{ij}, \tag{21.35}$$

$$\partial_t d\lambda_{\mathbf{g}} = -R\, d\lambda_{\mathbf{g}}, \tag{21.36}$$

$$\partial_t \Gamma_{ij}^k = -\sum_{\ell=1}^{m} g^{k\ell}\left(\nabla_i \mathrm{Ric}_{j\ell} + \nabla_j \mathrm{Ric}_{i\ell} - \nabla_\ell \mathrm{Ric}_{ij}\right). \tag{21.37}$$

In particular,

$$\frac{d}{dt}\mathrm{Vol}(M, \mathbf{g}(t)) = -\int_M R_{\mathbf{g}(t)}\, d\lambda_{\mathbf{g}(t)}. \tag{21.38}$$

Proposition 21.4.1. (Variation of the Laplacian) Let M be a smooth manifold with or without boundary,

let $\mathbf{g}(t)$ be a smooth family of metrics on M, let $\mathbf{h} = \partial_t \mathbf{g}$, and let $f = f(x,t)$ be a smooth function. Then

$$\partial_t(\Delta f) = \Delta(\partial_t f) - \sum_{i,j=1}^{m} h^{ij}(\nabla^2 f)_{ij} - \sum_{k=1}^{m}\left(\sum_{i=1}^{m} \nabla^i h_i{}^k - \frac{1}{2}\nabla^k \operatorname{tr}_{\mathbf{g}} \mathbf{h}\right)\nabla_k f. \tag{21.39}$$

Under Ricci flow, the contracted Bianchi identity gives

$$\partial_t(\Delta f) = \Delta(\partial_t f) + 2\sum_{i,j=1}^{m} \operatorname{Ric}^{ij}(\nabla^2 f)_{ij}. \tag{21.40}$$

Under a flow of the form $\partial_t \mathbf{g} = -2\operatorname{Ric}_{\mathbf{g}} + \frac{2}{m} r(t)\mathbf{g}$, with $r(t)$ constant in space, there is the additional term $-\frac{2}{m} r(t)\Delta_{\mathbf{g}} f$.

Proof. In coordinates,

$$\Delta f = \sum_{i,j=1}^{m} g^{ij}\left(\partial_i\partial_j f - \sum_{k=1}^{m} \Gamma_{ij}^{k}\partial_k f\right).$$

Differentiate, use $\partial_t g^{ij} = -h^{ij}$, and contract the Christoffel variation formula. This gives

$$\sum_{i,j=1}^{m} g^{ij}\partial_t \Gamma_{ij}^{k} = \sum_{i=1}^{m} \nabla^i h_i{}^k - \frac{1}{2}\nabla^k \operatorname{tr}_{\mathbf{g}} \mathbf{h},$$

and hence (21.39). For $\mathbf{h} = -2\operatorname{Ric}$, the equality $\sum_{i=1}^{m} \nabla^i \operatorname{Ric}_{ik} = \frac{1}{2}\nabla_k R$ cancels the first-order term. Adding $\frac{2}{m} r\mathbf{g}$, its spatial derivative is zero and its contraction with $\nabla^2 f$ is $\frac{2}{m} r\Delta f$. □

Curvature evolution starts with the uncommuted formula obtained directly from variation of the connection.

Lemma 21.4.2. (Uncommuted curvature evolution) Let M^m be a smooth manifold with or without boundary and let $\mathbf{g}(t)$ be a smooth solution of Ricci flow on M. The tensor of type $(1,3)$ satisfies

$$\partial_t R_{ijk}{}^{\ell} = \sum_{q=1}^{m} g^{\ell q}\big(-(\nabla^2 \operatorname{Ric})_{ijkq} - (\nabla^2 \operatorname{Ric})_{ikjq} + (\nabla^2 \operatorname{Ric})_{iqjk} \tag{21.41}$$

$$+ (\nabla^2 \operatorname{Ric})_{jikq} + (\nabla^2 \operatorname{Ric})_{jkiq} - (\nabla^2 \operatorname{Ric})_{jqik}\big). \tag{21.42}$$

For the fully covariant tensor, add the term coming from the metric:

$$\partial_t R_{ijk\ell} = -2\sum_{q=1}^{m} \operatorname{Ric}_{\ell q} R_{ijk}{}^{q} + \sum_{q=1}^{m} g_{\ell q}\partial_t R_{ijk}{}^{q}. \tag{21.43}$$

Proof. Equation (21.4) and (21.37) give

$$\partial_t R_{ijk}{}^{\ell} = \nabla_i(\partial_t \Gamma_{jk}^{\ell}) - \nabla_j(\partial_t \Gamma_{ik}^{\ell}).$$

Substitute (21.37) and use $\nabla \mathbf{g} = 0$; expanding the two parentheses produces exactly the six terms of (21.42).

Finally, differentiate $R_{ijk\ell} = \sum_{q=1}^{m} g_{\ell q} R_{ijk}{}^{q}$ and use $\partial_t g_{\ell q} = -2\,\mathrm{Ric}_{\ell q}$. □

The commutations needed to recognize the Laplacian follow from

$$([\nabla_a, \nabla_b]\mathbf{T})_{i_1 \dots i_r} = -\sum_{\alpha=1}^{r} \sum_{p=1}^{m} R_{abi_\alpha}{}^{p}\, T_{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_r} \tag{21.44}$$

and the second Bianchi identity

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0. \tag{21.45}$$

Contracting the latter gives

$$\sum_{i=1}^{m} \nabla^i \mathrm{Ric}_{ij} = \frac{1}{2} \nabla_j R. \tag{21.46}$$

In estimate formulas, use the notation $\mathbf{S} * \mathbf{T}$ for a universal finite sum of contractions of $\mathbf{S} \otimes \mathbf{T}$ using $\mathbf{g}$ and $\mathbf{g}^{-1}$. This is Notation 9.1.37; Lemma 9.1.39 controls each contraction. The number of summands and their coefficients depend only on tensor type and m. For covariant tensors of orders $r, q \in \mathbb{N}_0$, there exists $C = C(m, r, q)$ such that

$$|\mathbf{S} * \mathbf{T}|_{\mathbf{g}} \le C |\mathbf{S}|_{\mathbf{g}} |\mathbf{T}|_{\mathbf{g}}$$

for fixed tensor types. This notation does not suppress derivatives: it only collects algebraic contractions completely determined by applying (21.44).

Proposition 21.4.3. (Evolution of Ricci, scalar, and Riemann curvature) Let M^m be a smooth manifold with or without boundary and let $\mathbf{g}(t)$ be a smooth solution of Ricci flow on M. Define

$$B_{ijk\ell} = \sum_{p,q,r,s=1}^{m} g^{pr} g^{qs} R_{pijq} R_{rk\ell s}. \tag{21.47}$$

Then

$$\partial_t \mathrm{Ric}_{ij} = \Delta \mathrm{Ric}_{ij} + 2 \sum_{k,\ell=1}^{m} R_{kij\ell}\, \mathrm{Ric}^{k\ell} - 2 \sum_{k=1}^{m} \mathrm{Ric}_i{}^{k}\, \mathrm{Ric}_{kj}, \tag{21.48}$$

$$\partial_t R_{\mathbf{g}(t)} = \Delta_{\mathbf{g}(t)} R_{\mathbf{g}(t)} + 2 |\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)}, \tag{21.49}$$

Moreover, with all indices covariant,

$$\begin{aligned} \partial_t R_{ijk\ell} = {} & \Delta R_{ijk\ell} + 2\big(B_{ij\ell k} - B_{ijk\ell} - B_{ikj\ell} + B_{i\ell jk}\big) \\ & - \sum_{p,q=1}^{m} g^{pq} \big(R_{pjk\ell}\, \mathrm{Ric}_{qi} + R_{ipk\ell}\, \mathrm{Ric}_{qj} + R_{ijkp}\, \mathrm{Ric}_{q\ell} + R_{ijp\ell}\, \mathrm{Ric}_{qk}\big). \end{aligned} \tag{21.50}$$

In particular,

$$(\partial_t - \Delta)\,\mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}. \tag{21.51}$$

The schematic equality is exactly (21.50): the right-hand side collects only algebraic contractions of two

curvature tensors and contains no additional derivatives.

Proof. First derive the four-index formula. Commuting in (21.42) the two derivatives falling on Ric_{kq} and using (21.43) gives

$$\partial_t R_{ijk\ell} = (\nabla^2 \mathrm{Ric})_{i\ell jk} - (\nabla^2 \mathrm{Ric})_{ikj\ell} - (\nabla^2 \mathrm{Ric})_{j\ell ik} + (\nabla^2 \mathrm{Ric})_{jki\ell} - \sum_{p,q=1}^{m} g^{pq}\left(R_{ijkp}\,\mathrm{Ric}_{q\ell} + R_{ijp\ell}\,\mathrm{Ric}_{qk}\right). \tag{21.52}$$

Indeed, the four terms written with two derivatives are the four uncommuted terms of (21.42). The two remaining terms form

$$([\nabla_j, \nabla_i]\mathrm{Ric})_{k\ell} = \sum_{p=1}^{m} R_{ijk}{}^{p}\,\mathrm{Ric}_{p\ell} + \sum_{p=1}^{m} R_{ij\ell}{}^{p}\,\mathrm{Ric}_{kp},$$

by (21.44). Adding the term coming from lowering the index gives

$$\sum_{p=1}^{m} R_{ijk}{}^{p}\,\mathrm{Ric}_{p\ell} + \sum_{p=1}^{m} R_{ij\ell}{}^{p}\,\mathrm{Ric}_{kp} - 2\sum_{q=1}^{m} \mathrm{Ric}_{\ell q}\,R_{ijk}{}^{q} = -\sum_{p,q=1}^{m} g^{pq}\left(R_{ijkp}\,\mathrm{Ric}_{q\ell} + R_{ijp\ell}\,\mathrm{Ric}_{qk}\right),$$

where $R_{ij\ell p} = -R_{ijp\ell}$ was used. This proves (21.52).

We now prove the Simons identity transforming the second derivatives. The second Bianchi identity and its contraction give, with the index order used here,

$$\Delta R_{ijk\ell} = \sum_{p,q=1}^{m} g^{pq}(\nabla^2 \mathrm{Rm})_{piqjk\ell} - \sum_{p,q=1}^{m} g^{pq}(\nabla^2 \mathrm{Rm})_{pjqik\ell}, \tag{21.53}$$

$$\sum_{p,q=1}^{m} g^{pq}\nabla_p R_{qjk\ell} = \nabla_\ell \mathrm{Ric}_{jk} - \nabla_k \mathrm{Ric}_{j\ell}. \tag{21.54}$$

To carry out the commutation, fix a point and an orthonormal frame there. Identity (21.44) gives

$$\sum_{p=1}^{m} ([\nabla_p, \nabla_i]\mathrm{Rm})_{pjk\ell} = \sum_{r=1}^{m} \mathrm{Ric}_{ir}\,R_{rjk\ell} - \sum_{p,r=1}^{m}\left(R_{pijr}R_{prk\ell} + R_{pikr}R_{pjr\ell} + R_{pi\ell r}R_{pjkr}\right).$$

The first term comes from $\sum_{p=1}^{m} R_{pipr} = -\mathrm{Ric}_{ir}$. The first Bianchi identity and antisymmetry in the last slots transform the three remaining sums into

$$-\sum_{p,r=1}^{m} R_{pijr}R_{prk\ell} = B_{ijk\ell} - B_{ij\ell k},$$

$$-\sum_{p,r=1}^{m} R_{pikr}R_{pjr\ell} = B_{ikj\ell},$$
$$-\sum_{p,r=1}^{m} R_{pi\ell r}R_{pjkr} = -B_{i\ell jk}.$$

For example, the first equality follows by substituting $R_{prk\ell} = -R_{pk\ell r} + R_{p\ell kr}$, which is Bianchi with indices permuted. The other two equalities use directly $R_{pjr\ell} = -R_{pj\ell r}$ and the definition of B. Returning to arbitrary coordinates gives

$$\sum_{p,q=1}^{m} g^{pq}\big((\nabla^2 \mathrm{Rm})_{piqjk\ell} - (\nabla^2 \mathrm{Rm})_{ipqjk\ell}\big)$$
$$= \sum_{r=1}^{m} \mathrm{Ric}_i{}^r R_{rjk\ell} - B_{ij\ell k} + B_{ijk\ell} - B_{i\ell jk} + B_{ikj\ell}.$$

Substitute this equality and its analogue with i and j interchanged into (21.53), then use (21.54). Grouping the four terms B yields

$$\begin{aligned}\Delta R_{ijk\ell} &+ 2\big(B_{ij\ell k} - B_{ijk\ell} - B_{ikj\ell} + B_{i\ell jk}\big)\\ &= (\nabla^2 \mathrm{Ric})_{i\ell jk} - (\nabla^2 \mathrm{Ric})_{ikj\ell} - (\nabla^2 \mathrm{Ric})_{j\ell ik} + (\nabla^2 \mathrm{Ric})_{jki\ell}\\ &\quad + \sum_{p,q=1}^{m} g^{pq}\big(R_{pjk\ell}\,\mathrm{Ric}_{qi} + R_{ipk\ell}\,\mathrm{Ric}_{qj}\big).\end{aligned} \tag{21.55}$$

Substituting (21.55) into (21.52) proves (21.50) term by term.

To obtain the Ricci equation, substitute $\mathbf{h} = -2\,\mathrm{Ric}$ into (21.13). The contracted Bianchi identity gives

$$\delta_{\mathbf{g}}(-2\,\mathrm{Ric}) = dR, \qquad \mathrm{tr}_{\mathbf{g}}(-2\,\mathrm{Ric}) = -2R.$$

Thus the two Hessian terms cancel. The explicit formula (21.14) leaves

$$\partial_t \mathrm{Ric}_{ij} = \Delta \mathrm{Ric}_{ij} + 2\sum_{k,\ell=1}^{m} R_{kij\ell}\,\mathrm{Ric}^{k\ell} - 2\sum_{k=1}^{m} \mathrm{Ric}_i{}^k\,\mathrm{Ric}_{kj},$$

which is (21.48).

The scalar equation can also be obtained without contracting the preceding one. Indeed, (21.6), with $\mathbf{h} = -2\,\mathrm{Ric}$, gives

$$\begin{aligned}\partial_t R_{\mathbf{g}(t)} &= 2|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} - 2\,\mathrm{div}^2_{\mathbf{g}(t)}\mathrm{Ric}_{\mathbf{g}(t)} + 2\Delta_{\mathbf{g}(t)}R_{\mathbf{g}(t)}\\ &= 2|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} + \Delta_{\mathbf{g}(t)}R_{\mathbf{g}(t)},\end{aligned}$$

where (21.46) was used. This proves (21.49).

Finally, every additional term in (21.50) is a contraction of two copies of Rm, because Ric is a contraction of Rm. This proves (21.51). □

Corollary 21.4.4. (Evolution of curvature norms) Let M^m be a smooth manifold with or without boundary and let $\mathbf{g}(t)$ be a smooth solution of Ricci flow on M. There exist constants $C_m, C_m' > 0$, depending only on dimension, such that

$$(\partial_t - \Delta_{\mathbf{g}(t)})|\mathrm{Rm}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} = -2|\nabla_{\mathbf{g}(t)}\mathrm{Rm}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} - 8\sum_{i,j,k,\ell=1}^{m}\big(B_{ijk\ell} + B_{ikj\ell}\big)R^{ijk\ell}, \tag{21.56}$$

$$(\partial_t - \Delta_{\mathbf{g}(t)})|\mathrm{Rm}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} \le -2|\nabla_{\mathbf{g}(t)}\mathrm{Rm}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} + C_m|\mathrm{Rm}_{\mathbf{g}(t)}|^3_{\mathbf{g}(t)}, \tag{21.57}$$

$$(\partial_t - \Delta_{\mathbf{g}(t)})|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} = -2|\nabla_{\mathbf{g}(t)}\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} + 4\sum_{i,j,k,\ell=1}^{m} R_{kij\ell}\,\mathrm{Ric}^{ij}\,\mathrm{Ric}^{k\ell}, \tag{21.58}$$

$$(\partial_t - \Delta_{\mathbf{g}(t)})R^2_{\mathbf{g}(t)} = -2|dR_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} + 4R_{\mathbf{g}(t)}|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)}. \tag{21.59}$$

In particular,

$$(\partial_t - \Delta_{\mathbf{g}(t)})|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} \le -2|\nabla_{\mathbf{g}(t)}\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} + C_m'|\mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)}|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)}. \tag{21.60}$$

Proof. For any time-dependent covariant tensor $\mathbf{T}$,

$$\Delta_{\mathbf{g}(t)}|\mathbf{T}|^2_{\mathbf{g}(t)} = 2\langle\Delta_{\mathbf{g}(t)}\mathbf{T},\mathbf{T}\rangle_{\mathbf{g}(t)} + 2|\nabla_{\mathbf{g}(t)}\mathbf{T}|^2_{\mathbf{g}(t)}.$$

Time differentiation of the norm also differentiates each copy of $\mathbf{g}^{-1}$. By (21.35), these terms are contractions of $\mathrm{Ric}*\mathbf{T}*\mathbf{T}$. With $\mathbf{T} = \mathrm{Rm}$, contract (21.50) with $2R^{ijk\ell}$ and add the four terms obtained by differentiating the inverse metrics. The terms $\mathrm{Ric}*\mathrm{Rm}*\mathrm{Rm}$ cancel and, for every covariant slot, the Ricci term in the tensor evolution cancels the term differentiating the inverse metric in that same slot. For the summands B, renaming k and ℓ and using $R^{ij\ell k} = -R^{ijk\ell}$ gives

$$\sum_{i,j,k,\ell=1}^{m} B_{ij\ell k}R^{ijk\ell} = -\sum_{i,j,k,\ell=1}^{m} B_{ijk\ell}R^{ijk\ell},$$
$$\sum_{i,j,k,\ell=1}^{m} B_{i\ell jk}R^{ijk\ell} = -\sum_{i,j,k,\ell=1}^{m} B_{ikj\ell}R^{ijk\ell}.$$

Consequently, the remainder is

$$-8\sum_{i,j,k,\ell=1}^{m}(B_{ijk\ell} + B_{ikj\ell})R^{ijk\ell}.$$

This proves (21.56). The algebraic estimate for B gives (21.57).

For the Ricci tensor, directly differentiate the two inverse metrics occurring in $|\mathrm{Ric}|^2 = \sum_{i,j,a,b=1}^{m} g^{ia}g^{jb}\,\mathrm{Ric}_{ij}\,\mathrm{Ric}_{ab}$ and use (21.35). This yields

$$\partial_t|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} = 2\langle\partial_t\mathrm{Ric}_{\mathbf{g}(t)},\mathrm{Ric}_{\mathbf{g}(t)}\rangle_{\mathbf{g}(t)} + 4\sum_{i,j,k=1}^{m}\mathrm{Ric}_i{}^j\,\mathrm{Ric}_j{}^k\,\mathrm{Ric}_k{}^i.$$

The cubic Ricci terms coming from (21.48) cancel the second summand exactly. Subtracting the formula

for $\Delta_{\mathbf{g}(t)}|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)}$ leaves (21.58). The following inequality is Cauchy–Schwarz in each fiber. Finally,

$$(\partial_t - \Delta_{\mathbf{g}(t)})R^2_{\mathbf{g}(t)} = 2R_{\mathbf{g}(t)}(\partial_t - \Delta_{\mathbf{g}(t)})R_{\mathbf{g}(t)} - 2|dR_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)},$$

and (21.49) proves (21.59). □

21.5 The maximum principle and a priori estimates

The evolution equations become useful when combined with the maximum principle. Scalar curvature satisfies a closed inequality, while the norm of the Riemann tensor obeys an equation with a cubic term. These two observations give the first bounds and, through a Bernstein induction, estimates for all derivatives of curvature.

To apply the maximum principle from the preceding chapter, fix $\nabla^0 = \nabla^{\mathbf{g}(0)}$. At every time, on functions,

$$\Delta_{\mathbf{g}(t)} f = \sum_{i,j=1}^{m} g^{ij}(t)((\nabla^0)^2 f)_{ij} + \sum_{k=1}^{m} b^k(x,t)\nabla^0_k f,$$

where

$$b^k(x,t) = \sum_{i,j=1}^{m} g^{ij}(t)\bigl(\Gamma(\mathbf{g}(0))^k_{ij} - \Gamma(\mathbf{g}(t))^k_{ij}\bigr), \qquad k \in \{1,\dots,m\}.$$

On a compact subinterval of existence, smoothness and positivity of $\mathbf{g}(t)$ make ellipticity and the coefficient bounds uniform. Thus the operator satisfies the hypotheses of Theorem 20.4.1, even though the metric defining the Laplacian varies with time.

Proposition 21.5.1. (Lower bound for scalar curvature) Let M^m be a closed smooth manifold and let $\mathbf{g}(t)$ be a smooth solution of Ricci flow on M. Let

$$a_0 = \min_{x \in M} R_{\mathbf{g}(0)}(x).$$

As long as the denominator is positive,

$$R_{\mathbf{g}(t)}(x) \geq \frac{a_0}{1 - \frac{2}{m}a_0 t} \qquad (x \in M). \tag{21.61}$$

In particular, the condition $R_{\mathbf{g}(0)} \geq 0$ is preserved. If $R_{\mathbf{g}(0)} > 0$, then $R_{\mathbf{g}(t)} > 0$ throughout the existence interval.

Proof. Cauchy–Schwarz for the Ricci eigenvalues gives

$$|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} \geq \frac{1}{m} R^2_{\mathbf{g}(t)}.$$

Let $a(t)$ be the solution of

$$a'(t) = \frac{2}{m}a(t)^2, \qquad a(0) = a_0.$$

If $a_0 \neq 0$, then

$$a(t) = \frac{a_0}{1 - \frac{2}{m}a_0 t};$$

if $a_0 = 0$, we have $a(t) = 0$. For $u = R_{\mathbf{g}(t)} - a$,

$$\begin{aligned}(\partial_t - \Delta)u &= 2|\mathrm{Ric}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} - \frac{2}{m}a^2 \\ &\geq \frac{2}{m}(R^2_{\mathbf{g}(t)} - a^2) = \frac{2}{m}(R_{\mathbf{g}(t)} + a)u.\end{aligned}$$

Set $b = \frac{2}{m}(R_{\mathbf{g}(t)} + a)$ and $v = -u$. On every compact subinterval of existence, b has an upper bound B. The preceding inequality reads

$$(\partial_t - \Delta)v \leq bv.$$

For $\widetilde{v} = e^{-Bt}v$ we have

$$(\partial_t - \Delta)\widetilde{v} \leq (b - B)\widetilde{v}.$$

Since $b - B \leq 0$ and $\widetilde{v}(0) \leq 0$, Theorem 20.4.1 applied to the positive maximum of $\widetilde{v}$ gives $\widetilde{v} \leq 0$. Thus $u \geq 0$. Strict positivity also follows from the principle already proved: if $a_0 = \min_{x\in M} R_{\mathbf{g}(0)}(x) > 0$, then $(\partial_t - \Delta)(a_0 - R) = -2|\mathrm{Ric}|^2 \leq 0$. The maximum principle implies $R(x,t) \geq a_0 > 0$ on every compact subinterval of existence. The same argument with $a_0 = 0$ preserves nonnegativity. □

Proposition 21.5.2. (Initial curvature control) Let M^m be a closed smooth manifold and let $\mathbf{g}(t)$ be a smooth solution of Ricci flow on M. If

$$K_0 = \max_{x\in M} |\mathrm{Rm}_{\mathbf{g}(0)}(x)|_{\mathbf{g}(0)},$$

then there exists $c_m > 0$ such that

$$\max_{x\in M} |\mathrm{Rm}_{\mathbf{g}(t)}(x)|_{\mathbf{g}(t)} \leq \frac{K_0}{1 - c_m K_0 t} \tag{21.62}$$

if $1 - c_m K_0 t > 0$. When $K_0 = 0$, the flow remains flat.

Proof. Set $q(x,t) = |\mathrm{Rm}_{\mathbf{g}(t)}(x)|^2_{\mathbf{g}(t)}$. Equation (21.57) and the scalar maximum principle show that the upper right derivative of the maximum $Q(t) = \max_{x\in M} q(x,t)$ satisfies

$$D^+Q(t) \leq C_m Q(t)^{\frac{3}{2}}.$$

The solution of $y' = C_m y^{\frac{3}{2}}$, $y(0) = K_0^2$, is

$$y(t) = \frac{K_0^2}{\left(1 - \frac{C_m}{2}K_0 t\right)^2}.$$

Comparison for ordinary differential equations gives the conclusion with $c_m = \frac{C_m}{2}$. If $K_0 = 0$, comparison gives $q = 0$. □

Curvature derivative estimates are needed to rule out loss of regularity while curvature remains bounded.

Proposition 21.5.3. (Global derivative estimates) Let M^m be a closed smooth manifold and let $\mathbf{g}(t)$ be a smooth solution of Ricci flow on M. Suppose that $\mathbf{g}(t)$ is defined on $M \times [0,T]$ and

$$|\mathrm{Rm}_{\mathbf{g}(t)}(x)|_{\mathbf{g}(t)} \leq K \qquad ((x,t) \in M \times [0,T]).$$

For every integer $r \geq 1$ there exists a constant $C_r = C_r(m, r, KT)$ such that

$$|\nabla^r_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)}(x)|_{\mathbf{g}(t)} \leq C_r K t^{-\frac{r}{2}} \qquad ((x,t) \in M \times (0,T]). \tag{21.63}$$

If $K = 0$, all derivatives vanish and the assertion is interpreted accordingly.

Proof. Write $\mathcal{D} = \partial_t - \Delta$ and $\mathbf{T}_r = \nabla^r \operatorname{Rm}$ for $r \in \mathbb{N}_0$. If $\mathbf{T}$ is covariant of order q, differentiating its components gives

$$([\partial_t, \nabla]\mathbf{T})_{ij_1\dots j_q} = -\sum_{\alpha=1}^{q} \sum_{p=1}^{m} (\partial_t \Gamma^p_{ij_\alpha}) T_{j_1\dots j_{\alpha-1} p j_{\alpha+1}\dots j_q}.$$

By (21.37), this is a sum of contractions of $\nabla \operatorname{Rm}$ with $\mathbf{T}$. On the other hand, Lemma 9.1.41, applied to the tensor connection with $\Delta = -\nabla^*\nabla$, gives

$$[\Delta, \nabla]\mathbf{T} = \operatorname{Rm} * \nabla\mathbf{T} + \nabla \operatorname{Rm} * \mathbf{T}.$$

Here the bundle in the lemma is the bundle of tensors of the type of $\mathbf{T}$; its curvature is the sum of the Riemann actions on each slot, as expressed by (21.44).
For every $r \in \mathbb{N}_0$, we prove the identity

$$(\partial_t - \Delta_{\mathbf{g}(t)}) \nabla^r_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)} = \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b=r}} \nabla^a_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)} * \nabla^b_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)}. \tag{21.64}$$

For $r = 0$, this is (21.51). Suppose that it holds for some $r \geq 0$. Since $\mathbf{T}_{r+1} = \nabla \mathbf{T}_r$,

$$\mathcal{D}\mathbf{T}_{r+1} = \nabla(\mathcal{D}\mathbf{T}_r) + [\partial_t, \nabla]\mathbf{T}_r - [\Delta, \nabla]\mathbf{T}_r.$$

Proposition 9.1.40 turns the first term into the sum, for $a + b = r$, of $\mathbf{T}_{a+1} * \mathbf{T}_b$ and $\mathbf{T}_a * \mathbf{T}_{b+1}$. The two commutators contribute only $\mathbf{T}_1 * \mathbf{T}_r$ and $\mathbf{T}_0 * \mathbf{T}_{r+1}$. All these products have total order $r+1$; grouping those sharing the same pair of orders gives (21.64) for $r+1$. The number of terms and their coefficients depend only on m and r.
Passing to the norm of $\mathbf{T}_r$ differentiates each of the $r+4$ copies of the inverse metric. Identity $\partial_t \mathbf{g}^{-1} = 2\operatorname{Ric}^{\sharp\sharp}$ bounds their contribution by $C_r|\operatorname{Rm}||\mathbf{T}_r|^2$, which is the type of term corresponding to $a = 0$ or $b = 0$ in the following sum. By the Leibniz rule,

$$(\partial_t - \Delta_{\mathbf{g}(t)})|\nabla^r_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} \leq -2|\nabla^{r+1}_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)}|^2_{\mathbf{g}(t)} \tag{21.65}$$

$$+ C_r \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b=r}} |\nabla^r_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} |\nabla^a_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} |\nabla^b_{\mathbf{g}(t)} \operatorname{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)}. \tag{21.66}$$

Scale normalization avoids mixing different powers of K. If $K > 0$, define

$$\widetilde{\mathbf{g}}(\tau) = K\mathbf{g}\left(\frac{\tau}{K}\right), \qquad 0 \leq \tau \leq KT. \tag{21.67}$$

Then $\widetilde{\mathbf{g}}$ satisfies Ricci flow and $|\operatorname{Rm}_{\widetilde{\mathbf{g}}(\tau)}|_{\widetilde{\mathbf{g}}(\tau)} \leq 1$. In the original variables, the term with no derivatives in (21.57) has size K^3, not K^2; transformation (21.67) turns it into a dimensionless constant. It therefore

suffices to prove the estimate when $K = 1$ on an interval $[0, S]$, with $S = KT$.
Write

$$u_j(x,\tau) = |\nabla^j_{\widetilde{\mathbf{g}}(\tau)} \mathrm{Rm}_{\widetilde{\mathbf{g}}(\tau)}(x)|^2_{\widetilde{\mathbf{g}}(\tau)}.$$

Proceed by induction on r. Suppose the bounds $u_j \le C_{j,S}\tau^{-j}$ are known for $j < r$; for $r = 1$ this hypothesis uses only $u_0 \le 1$. In (21.66), summands with $a = 0$ or $b = 0$ are bounded by $C_r u_r$. If $1 \le a, b \le r-1$, the induction hypothesis gives $\tau^r\sqrt{u_r u_a u_b} \le C_{r,S}\tau^{r/2}\sqrt{u_r}$. Write the last product as $C_{r,S}(\tau^{(r-1)/2}\sqrt{u_r})\tau^{1/2}$ and apply Young. Since $\tau \le S$, we obtain

$$\tau^r u_r^{\frac12} u_a^{\frac12} u_b^{\frac12} \le \varepsilon\tau^{r-1}u_r + C_{r,S,\varepsilon}, \qquad a + b = r.$$

Differentiating the power of τ and also absorbing $\tau^r u_r \le S\tau^{r-1}u_r$ gives

$$(\partial_\tau - \Delta_{\widetilde{\mathbf{g}}(\tau)})(\tau^r u_r) \le -2\tau^r u_{r+1} + C_{r,S}\tau^{r-1}u_r + C_{r,S}. \tag{21.68}$$

The same calculation, already justified by the induction hypothesis, gives for $1 \le j < r$

$$(\partial_\tau - \Delta_{\widetilde{\mathbf{g}}(\tau)})(\tau^j u_j) \le -2\tau^j u_{j+1} + C_{j,S}\tau^{j-1}u_j + C_{j,S}, \tag{21.69}$$

and for $j = 0$ gives

$$(\partial_\tau - \Delta_{\widetilde{\mathbf{g}}(\tau)})u_0 \le -2u_1 + C_m. \tag{21.70}$$

Set $A_r = 1$ and, for $j = r, r-1, \dots, 1$, define $A_{j-1} = 1 + C_{j,S}A_j$. This choice makes the term $C_{j,S}A_j\tau^{j-1}u_j$ in (21.68) or (21.69) absorbable by $-2A_{j-1}\tau^{j-1}u_j$. For

$$F_r = \tau^r u_r + \sum_{j=0}^{r-1} A_j \tau^j u_j \tag{21.71}$$

we therefore obtain

$$(\partial_\tau - \Delta_{\widetilde{\mathbf{g}}(\tau)})F_r \le C_{r,S}. \tag{21.72}$$

Since $F_r(0) = A_0 u_0(0) \le A_0$, the maximum principle implies $F_r \le A_0 + C_{r,S}S$. In particular,

$$|\nabla^r_{\widetilde{\mathbf{g}}(\tau)} \mathrm{Rm}_{\widetilde{\mathbf{g}}(\tau)}|_{\widetilde{\mathbf{g}}(\tau)} \le C_{r,S}\tau^{-\frac{r}{2}}.$$

This closes the induction.
Finally, under the constant rescaling in (21.67),

$$|\nabla^r_{\widetilde{\mathbf{g}}(\tau)} \mathrm{Rm}_{\widetilde{\mathbf{g}}(\tau)}|_{\widetilde{\mathbf{g}}(\tau)} = K^{-1-\frac{r}{2}}|\nabla^r_{\mathbf{g}(\tau/K)} \mathrm{Rm}_{\mathbf{g}(\tau/K)}|_{\mathbf{g}(\tau/K)}.$$

Taking $\tau = Kt$ gives

$$|\nabla^r_{\mathbf{g}(t)} \mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} \le C_r(m, r, KT)Kt^{-\frac{r}{2}},$$

which is (21.63). The case $K = 0$ already follows from Proposition 21.5.2. □

Theorem 21.5.4. (Continuation criterion) Let $\mathbf{g}(t)$ be the maximal solution of Ricci flow on a closed smooth manifold, defined on $[0, T_{\max})$. If $T_{\max} < \infty$, then

$$\lim_{t \nearrow T_{\max}} \max_{x \in M} |\mathrm{Rm}_{\mathbf{g}(t)}(x)|_{\mathbf{g}(t)} = \infty. \tag{21.73}$$

Proof. Suppose, toward a contradiction, that $|\mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} \le K$ on $M \times [0, T_{\max})$. Since $|\mathrm{Ric}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} \le \sqrt{m}|\mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)}$, for every vector $v \neq 0$ fixed in TM we have

$$|\partial_t \log \mathbf{g}(t)(v,v)| = 2\frac{|\mathrm{Ric}_{\mathbf{g}(t)}(v,v)|}{\mathbf{g}(t)(v,v)} \le 2\sqrt{m}K.$$

Integration in time gives

$$e^{-2\sqrt{m}KT_{\max}}\mathbf{g}(0) \le \mathbf{g}(t) \le e^{2\sqrt{m}KT_{\max}}\mathbf{g}(0). \tag{21.74}$$

Inequality (21.74) states that, for every $x \in M$ and every $v \in T_xM$,

$$e^{-2\sqrt{m}KT_{\max}}\mathbf{g}(0)_x(v,v) \le \mathbf{g}(t)_x(v,v) \le e^{2\sqrt{m}KT_{\max}}\mathbf{g}(0)_x(v,v).$$

The constants are independent of x and $t < T_{\max}$. In particular, any possible limit cannot lose positivity. Proposition 7.1.9 allows every tensor of fixed type to be measured with $\mathbf{g}(0)$ or $\mathbf{g}(t)$ using common constants. To replace also the t-dependent Levi–Civita derivatives by derivatives with respect to a fixed connection, we need the following change-of-connection argument.

Apply Proposition 21.5.3 to the shifted flow starting at $\frac{T_{\max}}{4}$. All covariant derivatives of Riemann are uniformly bounded on $M \times [\frac{T_{\max}}{2}, T_{\max})$.

Let us explain the passage to a fixed connection. Let $\nabla^0 = \nabla^{\mathbf{g}(0)}$ and

$$\mathbf{A}(t) = \nabla^{\mathbf{g}(t)} - \nabla^0.$$

The change-of-connection identities have the form

$$\nabla^0\mathbf{T} = \nabla^{\mathbf{g}(t)}\mathbf{T} + \mathbf{A} * \mathbf{T}, \qquad \nabla^0\mathbf{g} = \mathbf{A} * \mathbf{g}, \tag{21.75}$$

where, in the second equality, the two summands corresponding to the two indices of $\mathbf{g}$ are included in the notation $*$. Since ∇^0 is fixed, (21.37) reads

$$\partial_t\mathbf{A} = -\mathbf{g}^{-1} * \nabla^{\mathbf{g}(t)}\mathrm{Ric}. \tag{21.76}$$

The first-order estimate in Proposition 21.5.3 bounds the right-hand side and, after integration from $\frac{T_{\max}}{2}$, bounds $\mathbf{A}$.

For higher orders, fix $t_0 = T_{\max}/2$ and measure all infinity norms with $\mathbf{g}(0)$. First specify the conversion of derivatives. The triangular formula of Lemma 9.1.28, or its application to the two connections, takes the form

$$(\nabla^0)^r\mathbf{T} = (\nabla^{\mathbf{g}(t)})^r\mathbf{T} + \sum_{j=0}^{r-1} \mathbf{C}_{r,j} * (\nabla^{\mathbf{g}(t)})^j\mathbf{T}, \qquad r \ge 1. \tag{21.77}$$

Each $\mathbf{C}_{r,j}$ is a finite sum of products of $\mathbf{A}, \nabla^0\mathbf{A}, \dots, (\nabla^0)^{r-j-1}\mathbf{A}$. For $r = 1$ the formula is (21.75). If it holds for

r, apply ∇^0: when the derivative falls on $\mathbf{C}_{r,j}$, it raises by at most one the order of a derivative of $\mathbf{A}$; when it falls on $(\nabla^{\mathbf{g}})^j\mathbf{T}$, the first identity in (21.75) gives $(\nabla^{\mathbf{g}})^{j+1}\mathbf{T}$ and $\mathbf{A}*(\nabla^{\mathbf{g}})^j\mathbf{T}$. These are precisely the terms allowed for $r+1$. Contractions are understood as the natural actions of a difference of connections on every tensor slot; they introduce no new metric derivatives.

Now suppose that $(\nabla^0)^j\mathbf{A}$ are bounded for $j \in \{0,\dots,r-1\}$, with $r \geq 1$. Identity $\nabla^0\mathbf{g} = \mathbf{A}*\mathbf{g}$ shows that derivatives of $\mathbf{g}$ up to order r are bounded: in the step from order k to $k+1$, distribute k derivatives between $\mathbf{A}$ and $\mathbf{g}$, so at most $(\nabla^0)^k\mathbf{A}$ and the already controlled derivatives of $\mathbf{g}$ occur. For $0 \leq k \leq r-1$ all these factors are available. Differentiating $\mathbf{g}^{-1}\mathbf{g} = I$ in the same way controls derivatives of $\mathbf{g}^{-1}$ up to order r, also using uniform equivalence of metrics.

Apply $(\nabla^0)^r$ to (21.76). The Leibniz rule distributes derivatives between $\mathbf{g}^{-1}$ and $\nabla^{\mathbf{g}(t)}\mathrm{Ric}$. For the second factor, (21.77) uses only derivatives of $\mathbf{A}$ up to order $r-1$ and derivatives of Riemann with respect to $\nabla^{\mathbf{g}(t)}$ up to order $r+1$. By the induction hypothesis and curvature estimates, all are bounded. Consequently,

$$\|\partial_t(\nabla^0)^r\mathbf{A}(t)\|_{L^\infty(M,T^{(1,r+2)}(TM);\mathbf{g}(0))} \leq C_r, \qquad t \in [t_0, T_{\max}).$$

If $a_r(t) := \|(\nabla^0)^r\mathbf{A}(t)\|_{L^\infty(M,T^{(1,r+2)}(TM);\mathbf{g}(0))}$, integration even gives $a_r(t) \leq a_r(t_0) + C_r(t-t_0)$ and, in particular,

$$a_r(t) \leq a_r(t_0) + C_r\int_{t_0}^{t}(1+a_r(\rho))\,d\rho. \tag{21.78}$$

This proves the inductive step. The case $r = 0$ was already obtained by integrating the evolution of $\mathbf{A}$. All derivatives of $\mathbf{A}$ with the fixed connection are bounded; the preceding identities then give those of $\mathbf{g}$, $\mathbf{g}^{-1}$, and Rm.

In particular, conversion of connections gives, for every r, a constant C_r such that

$$\|(\nabla^0)^r\partial_t\mathbf{g}(t)\|_{C^0(M,(T^*M)^{\otimes(r+2)};\mathbf{g}(0))} = 2\|(\nabla^0)^r\,\mathrm{Ric}_{\mathbf{g}(t)}\|_{C^0(M,(T^*M)^{\otimes(r+2)};\mathbf{g}(0))} \leq C_r.$$

Consequently, if $t_0 \leq t_1 < t_2 < T_{\max}$,

$$\|\mathbf{g}(t_2) - \mathbf{g}(t_1)\|_{C^r(M,S^2T^*M;\mathbf{g}(0))} \leq C_r|t_2 - t_1|. \tag{21.79}$$

Completeness of each C^r space and compatibility of the limits produce a smooth metric $\mathbf{g}_*$ such that

$$\mathbf{g}(t) \longrightarrow \mathbf{g}_* \quad \text{in } C^\infty(M) \quad \text{as } t \nearrow T_{\max}.$$

Inequality (21.74) ensures that $\mathbf{g}_*$ is positive definite.

Theorem 21.3.4, applied to $\mathbf{g}_*$, gives a solution on $[T_{\max}, T_{\max}+\varepsilon)$ with initial data $\mathbf{g}_*$. The piecewise definition is continuous, and its first time derivative agrees on both sides, since it equals $-2\,\mathrm{Ric}_{\mathbf{g}_*}$. To check all derivatives, define the differential operators $\mathcal{P}_1(\mathbf{g}) = -2\,\mathrm{Ric}_{\mathbf{g}}$ and

$$\mathcal{P}_{k+1}(\mathbf{g}) := D\mathcal{P}_k(\mathbf{g})[-2\,\mathrm{Ric}_{\mathbf{g}}], \qquad k \in \mathbb{N}.$$

If $\partial_t^k\mathbf{g} = \mathcal{P}_k(\mathbf{g})$, the chain rule gives $\partial_t^{k+1}\mathbf{g} = \mathcal{P}_{k+1}(\mathbf{g})$. Thus every time derivative is a smooth expression in the inverse metric and finitely many spatial derivatives of the metric. Convergence in all C^r spaces makes these expressions and their spatial derivatives agree with those of the new solution at $T_{\max}$. This gives a smooth

extension, contradicting maximality. This proof is also the geometric application of Theorem 20.3.4: the derivative estimates verify that no coefficient norm of the DeTurck system diverges while curvature remains bounded. This first proves that the limit superior is infinite.
If the full limit were not infinite, there would exist $K < \infty$ and $t_j \nearrow T_{\max}$ such that

$$\max_{x\in M} |\operatorname{Rm}_{\mathbf{g}(t_j)}(x)|_{\mathbf{g}(t_j)} \le K.$$

Replacing K by $\max\{K,1\}$, we may assume $K > 0$. Apply Proposition 21.5.2 to the shifted flow starting at t_j. For j large enough that $T_{\max} - t_j < (2c_m K)^{-1}$, we obtain

$$\max_{x\in M} |\operatorname{Rm}_{\mathbf{g}(t)}(x)|_{\mathbf{g}(t)} \le 2K, \qquad t_j \le t < T_{\max}.$$

Curvature is also bounded on the compact interval $[0, t_j]$. Thus there is a uniform bound on all of $[0, T_{\max})$, contradicting the first part of the proof. The conclusion is the stated limit. □

21.6 Rescalings and normalized flow

If $c > 0$ is constant and $\widehat{\mathbf{g}} = c\mathbf{g}$, the Levi–Civita connections of $\mathbf{g}$ and $\widehat{\mathbf{g}}$ agree. Consequently,

$$\operatorname{Rm}^{(1,3)}_{c\mathbf{g}} = \operatorname{Rm}^{(1,3)}_{\mathbf{g}}, \qquad \operatorname{Rm}^{(0,4)}_{c\mathbf{g}} = c\operatorname{Rm}^{(0,4)}_{\mathbf{g}}, \qquad \operatorname{Ric}_{c\mathbf{g}} = \operatorname{Ric}_{\mathbf{g}}, \qquad R_{c\mathbf{g}} = c^{-1}R_{\mathbf{g}}, \qquad d\lambda_{c\mathbf{g}} = c^{\frac{m}{2}} d\lambda_{\mathbf{g}}, \tag{21.80}$$

as well as

$$d_{c\mathbf{g}} = c^{1/2} d_{\mathbf{g}}, \qquad \operatorname{inj}_{c\mathbf{g}} = c^{1/2}\operatorname{inj}_{\mathbf{g}}, \qquad |\nabla^q_{c\mathbf{g}} \operatorname{Rm}_{c\mathbf{g}}|_{c\mathbf{g}} = c^{-1-q/2}|\nabla^q_{\mathbf{g}} \operatorname{Rm}_{\mathbf{g}}|_{\mathbf{g}}. \tag{21.81}$$

The equality $\operatorname{Ric}_{c\mathbf{g}} = \operatorname{Ric}_{\mathbf{g}}$ is an equality of covariant tensors of order two; it is important not to replace it by $c^{-1}\operatorname{Ric}_{\mathbf{g}}$.

Denote by

$$r(\mathbf{g}) = \frac{1}{\operatorname{Vol}(M,\mathbf{g})}\int_M R_{\mathbf{g}}\, d\lambda_{\mathbf{g}} \tag{21.82}$$

the average scalar curvature. The volume-normalized Ricci flow is

$$\partial_t \mathbf{g} = -2\operatorname{Ric}_{\mathbf{g}} + \frac{2}{m} r(\mathbf{g})\mathbf{g}. \tag{21.83}$$

Proposition 21.6.1. (Volume and curvature of normalized flow) Let M^m be a closed smooth manifold and let $\mathbf{g}(t)$ be a smooth solution of normalized Ricci flow (21.83). Every solution of (21.83) satisfies

$$\partial_t g_{ij} = -2\operatorname{Ric}_{ij} + \frac{2}{m} r g_{ij}, \tag{21.84}$$

$$\partial_t d\lambda_{\mathbf{g}} = (-R_{\mathbf{g}} + r) d\lambda_{\mathbf{g}}, \tag{21.85}$$

$$\frac{d}{dt}\operatorname{Vol}(M,\mathbf{g}(t)) = 0, \tag{21.86}$$

$$\partial_t R_{\mathbf{g}} = \Delta_{\mathbf{g}} R_{\mathbf{g}} + 2|\operatorname{Ric}_{\mathbf{g}}|^2_{\mathbf{g}} - \frac{2}{m} r R_{\mathbf{g}}. \tag{21.87}$$

Proof. The first identity is the component expression of (21.83). The trace of $\mathbf{h} = -2\operatorname{Ric}_{\mathbf{g}} + \frac{2}{m} r\mathbf{g}$ is $-2R_{\mathbf{g}} + 2r$. By (21.2) we obtain (21.85). Integrating this identity,

$$\frac{d}{dt}\operatorname{Vol}(M, \mathbf{g}(t)) = -\int_M R_{\mathbf{g}}\, d\lambda_{\mathbf{g}} + r\int_M d\lambda_{\mathbf{g}} = 0,$$

where definition (21.82) was used. The part $-2\operatorname{Ric}$ of $\mathbf{h}$ gives $\Delta_{\mathbf{g}} R_{\mathbf{g}} + 2|\operatorname{Ric}_{\mathbf{g}}|^2_{\mathbf{g}}$ by (21.49). For the part $\frac{2}{m} r\mathbf{g}$, the number $r(t)$ is spatially constant at each time, so both derivative terms in (21.6) vanish, while

$$-\left\langle \operatorname{Ric}_{\mathbf{g}}, \frac{2}{m} r\mathbf{g} \right\rangle_{\mathbf{g}} = -\frac{2}{m} r R_{\mathbf{g}}.$$

Their sum proves (21.87). □

Normalized flow is not a different equation from the local viewpoint. A spatial rescaling by a positive function of time, together with a change of the time variable, turns an unnormalized solution into a normalized one. Indeed, if $\mathbf{g}(t)$ solves (21.9), set

$$c(t) := \left(\frac{\operatorname{Vol}(M, \mathbf{g}(0))}{\operatorname{Vol}(M, \mathbf{g}(t))}\right)^{2/m}, \qquad \tau(t) := \int_0^t c(s)\, ds, \qquad \widehat{\mathbf{g}}(\tau(t)) := c(t)\mathbf{g}(t).$$

Then $\operatorname{Vol}(\widehat{\mathbf{g}}) = \operatorname{Vol}(\mathbf{g}(0))$. Moreover, viewing c as a function of τ and $t = t(\tau)$, we have

$$\frac{dt}{d\tau} = \frac{1}{c}, \qquad \frac{1}{c}\frac{dc}{d\tau} = \frac{2}{m} r(\widehat{\mathbf{g}}),$$

since $\partial_t \log \operatorname{Vol}(\mathbf{g}) = -r(\mathbf{g})$ and $r(c\mathbf{g}) = c^{-1} r(\mathbf{g})$. Then, using (21.80),

$$\begin{aligned} \partial_\tau \widehat{\mathbf{g}} &= \frac{dc}{d\tau}\mathbf{g} + c\frac{dt}{d\tau}\partial_t \mathbf{g} \\ &= \frac{1}{c}\frac{dc}{d\tau}\widehat{\mathbf{g}} - 2\operatorname{Ric}_{\mathbf{g}} \\ &= \frac{2}{m} r(\widehat{\mathbf{g}})\widehat{\mathbf{g}} - 2\operatorname{Ric}_{\widehat{\mathbf{g}}}. \end{aligned}$$

This gives precisely (21.83).

21.7 Einstein metrics, surfaces, and solitons

The simplest solutions combine rescalings, diffeomorphisms, and special curvature structures.

Example 21.7.1. (Einstein metrics) Let M be a closed smooth manifold and let $\mathbf{g}_0$ be a Riemannian metric on M. Suppose that $\operatorname{Ric}_{\mathbf{g}_0} = \lambda \mathbf{g}_0$, with constant λ. On the interval where $1 - 2\lambda t > 0$,

$$\mathbf{g}(t) = (1 - 2\lambda t)\mathbf{g}_0 \tag{21.88}$$

solves Ricci flow. Indeed,

$$\partial_t \mathbf{g} = -2\lambda \mathbf{g}_0, \qquad \operatorname{Ric}_{\mathbf{g}(t)} = \operatorname{Ric}_{\mathbf{g}_0} = \lambda \mathbf{g}_0.$$

If $\lambda > 0$, the homothetic solution becomes extinct at $t = \frac{1}{2\lambda}$; if $\lambda = 0$, it is stationary; if $\lambda < 0$, it exists for all $t \geq 0$ and expands. The same metric $\mathbf{g}_0$ is a fixed point of normalized flow, since $r(\mathbf{g}_0) = m\lambda$.

In particular, a metric of constant sectional curvature K is Einstein with $\lambda = (m-1)K$. Formula (21.88) describes its evolution completely.

Proposition 21.7.2. (Ricci flow in dimension two) On a closed surface,

$$\partial_t \mathbf{g} = -R\mathbf{g}, \qquad \partial_t R = \Delta R + R^2 \tag{21.89}$$

for unnormalized flow. For normalized flow,

$$\partial_t \mathbf{g} = (r - R)\mathbf{g}, \qquad \partial_t R = \Delta R + R(R - r) \tag{21.90}$$

In the latter case,

$$r = \frac{4\pi\chi(M)}{\operatorname{Vol}(M, \mathbf{g}(0))} \tag{21.91}$$

is constant in time.

Proof. In dimension two, the algebraic Riemann symmetries imply

$$\operatorname{Ric} = K\mathbf{g}, \qquad R = 2K,$$

and hence $\operatorname{Ric} = \frac{1}{2}R\mathbf{g}$. Substitution into (21.9) and (21.49) gives

$$\partial_t \mathbf{g} = -R\mathbf{g}, \qquad \partial_t R = \Delta R + 2\left|\frac{1}{2}R\mathbf{g}(t)\right|^2_{\mathbf{g}(t)} = \Delta R + R^2,$$

because $|\mathbf{g}(t)|^2_{\mathbf{g}(t)} = 2$. The normalized formulas follow in the same way from (21.83) and (21.87). If M is orientable, Theorem A.8.7 states

$$\int_M K \, d\lambda_{\mathbf{g}} = 2\pi\chi(M).$$

If M is nonorientable, apply this equality to the orientable two-sheeted covering space $\pi\colon \widetilde{M} \to M$ with the lifted metric. Since

$$\int_{\widetilde{M}} K_{\pi^*\mathbf{g}} \, d\lambda_{\pi^*\mathbf{g}} = 2\int_M K_{\mathbf{g}} \, d\lambda_{\mathbf{g}}, \qquad \chi(\widetilde{M}) = 2\chi(M),$$

the same formula holds on M. Consequently, $\displaystyle\int_M R \, d\lambda_{\mathbf{g}} = 4\pi\chi(M)$. Volume is constant by (21.86), proving (21.91). □

Definition 21.7.3. (Ricci soliton) Let M be a smooth manifold with or without boundary. A *Ricci soliton* on M is a metric $\mathbf{g}_0$ for which there exist a smooth vector field $\mathbf{X}$ and a constant λ such that

$$\operatorname{Ric}_{\mathbf{g}_0} + \frac{1}{2}\mathcal{L}_{\mathbf{X}}\mathbf{g}_0 = \lambda\mathbf{g}_0. \tag{21.92}$$

It is called shrinking, steady, or expanding according as $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$. If $\mathbf{X} = \nabla^{\mathbf{g}_0} f$, the equation is

equivalent to

$$\mathrm{Ric}_{\mathbf{g}_0} + \nabla^2 f = \lambda \mathbf{g}_0 \tag{21.93}$$

and it is called a gradient soliton.

Proposition 21.7.4. (Self-similar solutions) Let M be a closed smooth manifold and let $(\mathbf{g}_0, \mathbf{X}, \lambda)$ be a Ricci soliton on M. Set

$$\sigma(t) = 1 - 2\lambda t$$

and, while $\sigma(t) > 0$, let φ_t be the family of diffeomorphisms generated by the time-dependent field

$$\mathbf{Y}_t = \frac{1}{\sigma(t)} \mathbf{X}, \qquad \varphi_0 = \mathrm{id}_M . \tag{21.94}$$

Then

$$\mathbf{g}(t) = \sigma(t) \varphi_t^* \mathbf{g}_0 \tag{21.95}$$

is a solution of Ricci flow.

Proof. The pullback differentiation rule and $\sigma'(t) = -2\lambda$ give

$$\begin{aligned} \partial_t \mathbf{g}(t) &= -2\lambda \varphi_t^* \mathbf{g}_0 + \sigma(t) \varphi_t^* (\mathcal{L}_{\mathbf{Y}_t} \mathbf{g}_0) \\ &= \varphi_t^* (-2\lambda \mathbf{g}_0 + \mathcal{L}_{\mathbf{X}} \mathbf{g}_0) \\ &= -2 \varphi_t^* \mathrm{Ric}_{\mathbf{g}_0}, \end{aligned}$$

where the last equality is (21.92). By naturality of Ricci and its invariance under constant rescaling,

$$\mathrm{Ric}_{\sigma(t)\varphi_t^* \mathbf{g}_0} = \mathrm{Ric}_{\varphi_t^* \mathbf{g}_0} = \varphi_t^* \mathrm{Ric}_{\mathbf{g}_0} .$$

Therefore, $\partial_t \mathbf{g} = -2 \mathrm{Ric}_{\mathbf{g}}$. □

Einstein metrics are the solitons with $\mathbf{X} = 0$. Solitons with $\mathbf{X} \neq 0$ also incorporate the motion by diffeomorphisms that the geometric equation does not detect in its principal symbol. Proposition 21.7.4 describes the combination of this motion with rescaling.

22

Compactness, singularities, and Perelman's theory

When curvature in a Ricci flow grows without bound, the relevant geometry concentrates in regions whose natural size tends to zero. To observe them, one chooses a point of large curvature, multiplies the metric by that scale, and reparametrizes time. If volume does not collapse and curvature derivatives remain controlled, a subsequence of these rescalings converges to a complete flow retaining the geometry of the singularity at unit scale.

This is the first chain of ideas in the chapter: Shi estimates, injectivity radius, and Hamilton compactness. The second begins with conjugate heat and Perelman's monotone quantities. The functionals $\mathcal{F}$ and $\mathcal{W}$, the quantity μ, and reduced volume prevent regions of controlled curvature from losing all their volume. In dimension three, Hamilton–Ivey pinching turns rescaling limits into ancient solutions of nonnegative curvature, the κ–solutions.

Surgery theory additionally requires a uniform description of regions of large curvature, the standard solution inserted after cutting a neck, and an inductive argument preserving pinching, noncollapsing, and canonical neighborhoods. We first study the analytic part leading to singularity limits. In the final section, we state with precise hypotheses the structural surgery and extinction theorems used to conclude the Poincaré Conjecture; for these results we follow Chapters 12–19 of [104].

Perelman introduced these tools in his 2002 and 2003 papers [114, 116, 115]. The presentation also draws on the compactness and singularity estimates in [36] and the entropy approach of [154].

Grisha Perelman and the geometry of singularities

Perelman's contribution went beyond continuing the flow through a singularity. He first found quantities distinguishing an irreversible direction of evolution; he then used noncollapsing and compactness to prove that magnifying a region of large curvature can produce only very rigid models. Surgery relies on this rigidity: one cuts where the geometry is nearly cylindrical and replaces the singular region by a cap whose behavior can be controlled. In this way, a singularity ceases to be the point where the argument ends and becomes an organized part of the evolution.

Throughout this chapter, dimension is denoted by m.

22.1 Pointed smooth convergence

A rescaled sequence rarely converges on the entire original manifold. The appropriate notion fixes a base point and requires smooth convergence only on increasingly large compact subsets of the limit space. For flows, spatial identifications must be time-independent; otherwise the limiting equation would acquire additional Lie derivative terms.

Definition 22.1.1. (Pointed smooth convergence) A sequence of pointed complete Riemannian manifolds without boundary $(M_i, \mathbf{g}_i, p_i)$ *converges smoothly in the pointed sense* to $(M_\infty, \mathbf{g}_\infty, p_\infty)$ if there exist an exhaustion

$$p_\infty \in U_1 \Subset U_2 \Subset \cdots \Subset M_\infty, \qquad \bigcup_{a=1}^{\infty} U_a = M_\infty,$$

, a sequence of integers $a_i \to \infty$, and embeddings $\Phi_i \colon U_{a_i} \to M_i$. For every $a \in \mathbb{N}$ and every i with $a \le a_i$, denote their restrictions by

$$\Phi_{i,a} \colon U_a \longrightarrow M_i, \qquad \Phi_{i,a}(p_\infty) = p_i,$$

such that

$$\Phi_{i,a}^{*}\mathbf{g}_i \longrightarrow \mathbf{g}_\infty \quad \text{in } C^\infty(U_a).$$

We also require that the images exhaust the pointed balls of the approximating spaces: for every $R > 0$ there exists $a(R)$ such that, for all sufficiently large i,

$$\overline{B_{\mathbf{g}_\infty}(p_\infty, R)} \subseteq U_{a(R)}, \qquad B_{\mathbf{g}_i}(p_i, R) \subseteq \Phi_{i,a(R)}(U_{a(R)}). \tag{22.1}$$

The notation $C^\infty(U_a)$ means uniform convergence of every derivative on each compact subset of U_a. Since $\Phi_{i,a} = \Phi_i|_{U_a}$, the identifications are compatible under restriction of domains. We work in the connected component of each base point, and the limit manifold is also taken to be connected and without boundary.

Definition 22.1.2. (Pointed convergence of flows) Let M_i and M_∞ be smooth manifolds without boundary and let $\mathbf{g}_i(t)$ be complete Ricci flows on M_i, defined on a common interval I. We say that

$$(M_i, \mathbf{g}_i(t), p_i) \longrightarrow (M_\infty, \mathbf{g}_\infty(t), p_\infty)$$

smoothly on compact spacetime subsets if the embeddings in the preceding definition are independent of t and, for every $J \Subset I$,

$$\Phi_{i,a}^{*}\mathbf{g}_i(t) \longrightarrow \mathbf{g}_\infty(t) \quad \text{in } C^\infty(U_a \times J).$$

Exhaustion of balls is also required uniformly on compact time intervals: given $J \Subset I$ and $R > 0$, one can choose $a = a(J, R)$ so that, for sufficiently large i and every $t \in J$,

$$\overline{B_{\mathbf{g}_\infty(t)}(p_\infty, R)} \subseteq U_a, \qquad B_{\mathbf{g}_i(t)}(p_i, R) \subseteq \Phi_{i,a}(U_a). \tag{22.2}$$

Lemma 22.1.3. (Passing geometry to the limit) Let M_i and M_∞ be smooth manifolds without boundary and suppose that $(M_i, \mathbf{g}_i(t), p_i)$ converges smoothly in the pointed sense to $(M_\infty, \mathbf{g}_\infty(t), p_\infty)$. Then, on every identified compact subset, the Levi–Civita connections, the Rm and Ric tensors, scalar curvatures, and all their covariant derivatives converge. Convergence also commutes with tensor products, contractions, and

traces. If the $\mathbf{g}_i(t)$ are Ricci flows and convergence is in spacetime, the limit satisfies $\partial_t \mathbf{g}_\infty = -2\operatorname{Ric}_{\mathbf{g}_\infty}$.

Proof. In a fixed chart, Christoffel symbols are smooth rational functions of g_{ij}, g^{ij}, and ∂g_{ij}. Curvature and its derivatives are obtained by iterating derivatives and products of these coefficients. C^∞ convergence allows passage to the limit in all these expressions. Tensor operations are continuous in each C^k space on compact subsets. Finally, C^∞ convergence in the time variable allows passage to the limit in the flow equation. □

The injectivity radius prevents a curvature bound from concealing collapse.

Lemma 22.1.4. (Volume, curvature, and injectivity radius) For every $m \in \mathbb{N}$ and $v > 0$ there exists $c = c(m, v) > 0$ with the following property. If $(N^m, \mathbf{h})$ is a complete Riemannian manifold without boundary,

$$|\operatorname{Rm}_{\mathbf{h}}|_{\mathbf{h}} \le r^{-2} \quad \text{en } B_{\mathbf{h}}(x, r), \qquad \operatorname{Vol}_{\mathbf{h}} B_{\mathbf{h}}(x, r) \ge v r^m,$$

then $\operatorname{inj}_{\mathbf{h}}(x) \ge cr$.

The preceding estimate is local: it involves only the geometry of the ball $B_{\mathbf{h}}(x, r)$ and completeness of the metric. Its proof, based on the short-loop lemma and comparison of the exponential Jacobian, can be found in Chapter 1, in the section *Local volume and the injectivity radius*, of [104].

Lemma 22.1.5. (Local propagation of injectivity radius) Given $m \in \mathbb{N}$, $R > 0$, $\iota_0 > 0$, and $0 \le K < \infty$, there exists $\iota = \iota(m, R, \iota_0, K) > 0$ with the following property. If $(N^m, \mathbf{h}, p)$ is a complete Riemannian manifold without boundary,

$$\operatorname{inj}_{\mathbf{h}}(p) \ge \iota_0, \qquad |\operatorname{Rm}_{\mathbf{h}}|_{\mathbf{h}} \le K \quad \text{on } B_{\mathbf{h}}(p, 2R + 2),$$

then

$$\operatorname{inj}_{\mathbf{h}}(x) \ge \iota \qquad \text{for every } x \in B_{\mathbf{h}}(p, R).$$

Proof. After increasing K by a constant depending only on dimension, we may assume that

$$\operatorname{Sec}_{\mathbf{h}} \le K, \qquad \operatorname{Ric}_{\mathbf{h}} \ge -(m-1)K\mathbf{h}$$

on the indicated ball. Choose

$$0 < r_0 < \min\left\{\frac{\iota_0}{2}, 1, \frac{\pi}{4\sqrt{K+1}}\right\}.$$

The exponential map at p is a diffeomorphism on the tangent ball of radius $2r_0$. The local Jacobi-field argument used in the proof of Theorem 13.1.8, applied on this ball, gives a constant $c_0 = c_0(m, K, r_0) > 0$ such that

$$\operatorname{Vol}_{\mathbf{h}} B_{\mathbf{h}}(p, r_0) \ge c_0 r_0^m.$$

Let $x \in B_{\mathbf{h}}(p, R)$. Since $B_{\mathbf{h}}(p, r_0) \subseteq B_{\mathbf{h}}(x, R + r_0)$, we have

$$\operatorname{Vol}_{\mathbf{h}} B_{\mathbf{h}}(x, R + r_0) \ge c_0 r_0^m.$$

Denote by $V_{-K}(s)$ the volume of a ball of radius s in the simply connected space of constant sectional

curvature $-K$. Relative Bishop–Gromov comparison, valid because $\mathrm{Ric}_{\mathbf{h}} \geq -(m-1)K\mathbf{h}$, gives

$$\frac{\mathrm{Vol}_{\mathbf{h}} B_{\mathbf{h}}(x,\rho)}{V_{-K}(\rho)} \geq \frac{\mathrm{Vol}_{\mathbf{h}} B_{\mathbf{h}}(x,R+r_0)}{V_{-K}(R+r_0)}$$

for $0 < \rho \leq R + r_0$. For this form of comparison, see [104, Chapter 1]. Take

$$\rho := \min\{1, (K+1)^{-1/2}, R+r_0\}.$$

The preceding estimate implies

$$\mathrm{Vol}_{\mathbf{h}} B_{\mathbf{h}}(x,\rho) \geq v\rho^m$$

with $v = v(m,R,\iota_0,K) > 0$. Moreover, $B_{\mathbf{h}}(x,\rho) \subseteq B_{\mathbf{h}}(p,R+1)$ and $|\mathrm{Rm}_{\mathbf{h}}| \leq K \leq \rho^{-2}$ on this ball. Lemma 22.1.4 gives

$$\mathrm{inj}_{\mathbf{h}}(x) \geq c(m,v)\rho.$$

The constant on the right is independent of x and the manifold. □

The version of Shi's estimates proved in the preceding chapter was formulated for closed manifolds. For compactness we will need the complete noncompact version.

Theorem 22.1.6. (Shi estimates for complete flows) Let $(M^m, \mathbf{g}(t))$, $t \in [s_0, s_1]$, be a complete Ricci flow and suppose that

$$\sup_{t\in[s_0,s_1]} \|\mathrm{Rm}_{\mathbf{g}(t)}\|_{L^\infty(M,\mathbf{g}(t))} \leq K < \infty.$$

For every $q \in \mathbb{N}$ and every $\delta \in (0, s_1 - s_0]$ there exists a constant $C_q = C_q(m,q,K,s_1-s_0,\delta) > 0$ such that

$$\sup_{t\in[s_0+\delta,s_1]} \|\nabla^q_{\mathbf{g}(t)} \mathrm{Rm}_{\mathbf{g}(t)}\|_{L^\infty(M,\mathbf{g}(t))} \leq C_q.$$

In particular, if $J \Subset I$ and there exists $s < \min J$ such that $[s, \max J] \subseteq I$, a uniform curvature bound on this interval gives uniform bounds for all its covariant derivatives on $M \times J$.

We will use this theorem in its complete version, proved through local estimates in [104, Chapter 3, Section 7]. The Bernstein quantities are those of Proposition 21.5.3; localization allows the maximum principle to be applied on compact domains before their radius is sent to infinity. The time margin $\delta > 0$ is essential: the constant may diverge as $\delta \to 0^+$. In particular, a curvature bound with no initial bounds for its derivatives only gives smooth convergence at times later than the initial time.

We separate from the parabolic argument the compactness theorem for a single metric.

Theorem 22.1.7. (Cheeger–Gromov pointed smooth compactness) Let $(M_i^m, \mathbf{h}_i, p_i)$ be a sequence of complete Riemannian manifolds without boundary. Suppose that there exists $\iota_0 > 0$ such that

$$\mathrm{inj}_{\mathbf{h}_i}(p_i) \geq \iota_0 \qquad \text{for every } i,$$

and that, for every $R > 0$ and $q \in \mathbb{N}_0$, there exists $C_{q,R} < \infty$ such that

$$\sup_{i\in\mathbb{N}} \|\nabla^q_{\mathbf{h}_i} \operatorname{Rm}_{\mathbf{h}_i}\|_{L^\infty(B_{\mathbf{h}_i}(p_i,R),\mathbf{h}_i)} \le C_{q,R}.$$

Then there exist a complete Riemannian manifold without boundary $(M_\infty, \mathbf{h}_\infty, p_\infty)$ and a subsequence converging smoothly to it in the pointed sense. The convergence embeddings can be chosen compatibly on an exhaustion

$$p_\infty \in U_1 \Subset U_2 \Subset \cdots \Subset M_\infty$$

satisfying the ball-exhaustion property (22.1).

Proof. Replace each M_i by the connected component of p_i. Fix $R > 0$. The hypothesis with radius $2R+4$ and Lemma 22.1.5 give $\iota_R > 0$ such that

$$\operatorname{inj}_{\mathbf{h}_i}(x) \ge \iota_R \qquad \big(x \in B_{\mathbf{h}_i}(p_i, R+1)\big)$$

for every i. Bounds for curvature derivatives on the ball of radius $R+2$, together with the local part of the proof of Theorem 13.1.8, allow a radius $\rho_R > 0$ to be chosen independently of i and the center $x \in B_{\mathbf{h}_i}(p_i, R+1)$, such that normal coordinates of radius $4\rho_R$ are defined and the coefficients of $\mathbf{h}_i$, its inverse, and all their jets are uniformly bounded in them. The same bounds hold for transitions between two such charts.

Choose in $B_{\mathbf{h}_i}(p_i, R)$ a maximal net of points separated by ρ_R. The balls of radius ρ_R centered at net points cover the pointed ball, while the concentric balls of radius $\rho_R/2$ are disjoint. The Bishop–Gromov argument used in the preceding lemma gives a uniform lower bound for the volume of each small ball. The Ricci lower bound gives, by the same comparison, a uniform upper bound for the volume of $B_{\mathbf{h}_i}(p_i, R+1)$. Consequently, the number of net points is bounded by a constant N_R independent of i. Thus, for every R, we obtain a finite atlas with uniformly bounded number of charts, common radius, and uniform bounds for all jets of the metrics and transitions.

To turn these uniform atlases into a limit manifold, apply the geometric convergence theorem of [104, Theorem 5.6]. In the complete case under consideration, its hypotheses are: a nondegenerate base point, upper volume bounds on fixed-radius balls, bounds for all curvature derivatives on these balls, and lower volume bounds at small radii. The injectivity bound verifies the first; Bishop–Gromov comparison and the preceding paragraph's construction verify the second and fourth; the third is our hypothesis. Completeness also ensures that closed balls in every approximating manifold are compact.

The theorem gives a smooth Hausdorff manifold M_∞, a metric $\mathbf{h}_\infty$, relatively compact open subsets $U_1 \Subset U_2 \Subset \cdots$, and, after choosing a subsequence, embeddings $\Phi_i \colon U_{a_i} \to M_i$, with $a_i \to \infty$. For every $a \in \mathbb{N}$, the restrictions $\Phi_{i,a} = \Phi_i|_{U_a}$ satisfy

$$\Phi_{i,a}^* \mathbf{h}_i \longrightarrow \mathbf{h}_\infty \quad \text{in } C^\infty(U_a).$$

The topological part of that theorem constructs the transition domains and embeddings simultaneously; it therefore also preserves exhaustion of balls in the approximating manifolds. This is the complete form of Cheeger–Gromov compactness stated in [104, Theorem 5.9].

In the cited construction, the atlases of radius $R+1$ cover the ball $B_{\mathbf{h}_i}(p_i, R+1)$ before the charts are restricted.

Reindexing the exhaustion gives, for every $R \in \mathbb{N}$ and all sufficiently large i,

$$\overline{B_{\mathbf{h}_\infty}(p_\infty, R)} \subseteq U_{R+1}, \qquad B_{\mathbf{h}_i}(p_i, R) \subseteq \Phi_{i,R+1}(U_{R+1}). \tag{22.3}$$

The second inclusion follows by retaining all net charts meeting the ball of radius R; the unit margin allows their domains to be restricted without leaving points of that ball outside the image. This verifies (22.1). The first inclusion also shows that every closed bounded ball in $(M_\infty, \mathbf{h}_\infty)$ is compact. The Hopf–Rinow theorem implies that $\mathbf{h}_\infty$ is complete. □

The next estimate compares different time slices of a flow.

Lemma 22.1.8. (Metric and distance distortion) Let $(M, \mathbf{g}(t))$, $t \in J$, be a Ricci flow and suppose that

$$|\mathrm{Ric}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} \le K \qquad \text{on } M \times J.$$

For $s, t \in J$ we have

$$e^{-2K|t-s|}\mathbf{g}(s) \le \mathbf{g}(t) \le e^{2K|t-s|}\mathbf{g}(s), \tag{22.4}$$

and, if the distances are finite,

$$e^{-K|t-s|}d_{\mathbf{g}(s)}(x,y) \le d_{\mathbf{g}(t)}(x,y) \le e^{K|t-s|}d_{\mathbf{g}(s)}(x,y). \tag{22.5}$$

In particular,

$$B_{\mathbf{g}(s)}\big(x, e^{-K|t-s|}r\big) \subseteq B_{\mathbf{g}(t)}(x,r) \subseteq B_{\mathbf{g}(s)}\big(x, e^{K|t-s|}r\big). \tag{22.6}$$

If one metric $\mathbf{g}(s)$ is complete, every metric in the family is complete.

Proof. Fix $v \in T_xM \setminus \{0\}$. Since $\partial_t \mathbf{g} = -2\,\mathrm{Ric}_{\mathbf{g}(t)}$,

$$\left|\frac{d}{dt}\log \mathbf{g}(t)(v,v)\right| = 2\frac{|\mathrm{Ric}_{\mathbf{g}(t)}(v,v)|}{\mathbf{g}(t)(v,v)} \le 2K.$$

Integration between s and t gives

$$-2K|t-s| \le \log \frac{\mathbf{g}(t)_x(v,v)}{\mathbf{g}(s)_x(v,v)} \le 2K|t-s|.$$

The exponential function is increasing. Applying it and multiplying by $\mathbf{g}(s)_x(v,v) > 0$ yields

$$e^{-2K|t-s|}\mathbf{g}(s)_x(v,v) \le \mathbf{g}(t)_x(v,v) \le e^{2K|t-s|}\mathbf{g}(s)_x(v,v).$$

The inequality also holds for $v = 0$. Since the constants are independent of x and v, this is precisely (22.4) with the interpretation of Definition 2.1.5.
Taking square roots gives

$$e^{-K|t-s|}|v|_{\mathbf{g}(s)} \le |v|_{\mathbf{g}(t)} \le e^{K|t-s|}|v|_{\mathbf{g}(s)}.$$

For a piecewise smooth curve $\gamma\colon [a,b] \to M$, apply this bound to $v = \dot\gamma(r)$ on each smooth segment and

integrate with respect to r. The identity $L_{\mathbf{g}(t)}(\gamma) = \int_a^b |\dot\gamma(r)|_{\mathbf{g}(t)}\,dr$ gives

$$e^{-K|t-s|}L_{\mathbf{g}(s)}(\gamma) \le L_{\mathbf{g}(t)}(\gamma) \le e^{K|t-s|}L_{\mathbf{g}(s)}(\gamma).$$

Taking the infimum over curves joining x to y yields (22.5); the ball inclusions are an immediate consequence.

Suppose that $\mathbf{g}(s)$ is complete. A Cauchy sequence for $d_{\mathbf{g}(t)}$ is Cauchy for $d_{\mathbf{g}(s)}$ by (22.5) and therefore converges with respect to $d_{\mathbf{g}(s)}$. The two distances induce the same topology, and the second inequality in (22.5) gives convergence with respect to $d_{\mathbf{g}(t)}$. Thus $\mathbf{g}(t)$ is complete. □

We also need to turn Shi's covariant bounds into coefficient bounds in time-independent coordinates.

Lemma 22.1.9. (Control with respect to a reference connection) Let $J \subseteq \mathbb{R}$ be a compact interval containing 0 and let $\mathbf{g}_i(t)$ be Ricci flows defined on open subsets V_i, with $\mathbf{h}_i := \mathbf{g}_i(0)$. Suppose that the metrics $\mathbf{h}_i$ have uniform bounds for all jets of their coefficients and inverses in fixed charts over V_i. Also suppose that, for every $q \in \mathbb{N}_0$, there exists C_q such that

$$\sup_{i\in\mathbb{N}}\sup_{t\in J}\|\nabla^q_{\mathbf{g}_i(t)}\operatorname{Rm}_{\mathbf{g}_i(t)}\|_{L^\infty(V_i,\mathbf{g}_i(t))} \le C_q.$$

Then, in each coordinate subdomain whose closure lies within the fixed charts, all coefficients of $\mathbf{g}_i(t)$ and $\mathbf{g}_i(t)^{-1}$ and all their spatial and time derivatives are uniformly bounded on J.

Proof. The order-zero curvature bound gives $|\operatorname{Ric}_{\mathbf{g}_i(t)}|_{\mathbf{g}_i(t)} \le K := \sqrt{m}\,C_0$ for all indices and times. Since $0 \in J$, Lemma 22.1.8 implies, with $Q = e^{2K\operatorname{diam} J}$,

$$Q^{-1}(\mathbf{h}_i)_x(v,v) \le \mathbf{g}_i(t)_x(v,v) \le Q(\mathbf{h}_i)_x(v,v) \qquad (i\in\mathbb{N},\ t\in J,\ x\in V_i,\ v\in T_xV_i).$$

By Proposition 7.1.9, norms of the same tensor of fixed type are comparable with constants independent of i,x,t. This allows the metric used in a tensor bound to be changed; to change covariant derivatives, we study the difference of connections. Let $\nabla^i(t)$ and $\nabla^{i,0}$ be their Levi–Civita connections and set

$$\mathbf{A}_i(t) := \nabla^i(t) - \nabla^{i,0}.$$

Then $\mathbf{A}_i(0) = 0$, and identity (21.37) takes the form

$$\partial_t\mathbf{A}_i = -\mathbf{g}_i(t)^{-1} * \nabla^i(t)\operatorname{Ric}_{\mathbf{g}_i(t)}. \tag{22.7}$$

Moreover,

$$\nabla^{i,0}\mathbf{T} = \nabla^i(t)\mathbf{T} + \mathbf{A}_i * \mathbf{T}, \qquad \nabla^{i,0}\mathbf{g}_i(t) = \mathbf{A}_i * \mathbf{g}_i(t). \tag{22.8}$$

Write $D_i = \nabla^{i,0}$ and $\nabla_i = \nabla^i(t)$. The difference of connections acts on every tensor slot as in Chapter 9. Apply the formula already proved in (21.77): for every integer $r \ge 1$,

$$D_i^r\mathbf{T} = \nabla_i^r\mathbf{T} + \sum_{j=0}^{r-1}\mathbf{C}_{i,r,j} * \nabla_i^j\mathbf{T}. \tag{22.9}$$

Each coefficient $\mathbf{C}_{i,r,j}$ is a finite sum of products of $D_i^b \mathbf{A}_i$ with $b \in \{0,\dots,r-j-1\}$; the number of terms depends only on r, dimension, and the type of $\mathbf{T}$. The Leibniz rule of Proposition 9.1.40 and the contraction estimates in the chapter on bundles bound these products by the norms of their factors.

The induction starts at $r = 0$. The evolution of $\mathbf{A}_i$ and the bound for $\nabla_i \operatorname{Ric}$ give $\|\partial_t \mathbf{A}_i(t)\|_{L^\infty(V_i, T^{(1,2)}(TV_i); \mathbf{h}_i)} \le C$. Integrating from 0 and using $\mathbf{A}_i(0) = 0$ yields $\|\mathbf{A}_i(t)\|_{L^\infty(V_i, T^{(1,2)}(TV_i); \mathbf{h}_i)} \le C|J|$.

Now let $r \ge 1$ and suppose that the derivatives $D_i^b \mathbf{A}_i$ are bounded uniformly in i and $t \in J$ for $b \in \{0,\dots,r-1\}$. Define $\mathbf{F}_i = \partial_t \mathbf{A}_i$. Equation (22.7) expresses $\mathbf{F}_i$ as contractions of $\mathbf{g}_i^{-1}$ and $\nabla_i \operatorname{Ric}$. Since $\nabla_i \mathbf{g}_i = 0$, for each $j \in \{0,\dots,r\}$, $\nabla_i^j \mathbf{F}_i$ contains only Ricci derivatives of order $j+1$ contracted with the inverse metric. All are bounded by the curvature hypothesis. Applying (22.9) to $\mathbf{T} = \mathbf{F}_i$, the additional coefficients contain only derivatives of $\mathbf{A}_i$ of order at most $r-1$. By the induction hypothesis,

$$\|\partial_t D_i^r \mathbf{A}_i(t)\|_{L^\infty(V_i, T^{(1,r+2)}(TV_i); \mathbf{h}_i)} = \|D_i^r \mathbf{F}_i(t)\|_{L^\infty(V_i, T^{(1,r+2)}(TV_i); \mathbf{h}_i)} \le C_r, \qquad i \in \mathbb{N}, \quad t \in J.$$

Since $D_i^r \mathbf{A}_i(0) = 0$, integration toward either endpoint of J gives $\|D_i^r \mathbf{A}_i(t)\|_{L^\infty(V_i, T^{(1,r+2)}(TV_i); \mathbf{h}_i)} \le C_r|J|$. This proves the step from $r-1$ to r without introducing a derivative of $\mathbf{A}_i$ of higher order than the one controlled.

Now apply (22.9) to $\mathbf{T} = \mathbf{g}_i$ and $\mathbf{T} = \mathbf{g}_i^{-1}$. All their positive-order derivatives with respect to ∇_i vanish. Thus $D_i^r \mathbf{g}_i$ and $D_i^r \mathbf{g}_i^{-1}$ are expressed in terms of $\mathbf{C}_{i,r,0}$ and already bounded order-zero tensors. The local expression for higher covariant derivatives in Lemma 9.1.28, applied to the fixed connection D_i, converts these bounds into coordinate derivative bounds. Its coefficients are uniform because the jets of $\mathbf{h}_i$ and $\mathbf{h}_i^{-1}$ are uniform by hypothesis.

For time derivatives, use the coordinate expression for Ricci: it is a finite sum of products of coefficients of $\mathbf{g}_i^{-1}$, first derivatives of $\mathbf{g}_i$, and second derivatives of $\mathbf{g}_i$. Suppose that all spatial derivatives of $\partial_t^b \mathbf{g}_i$ and $\partial_t^b \mathbf{g}_i^{-1}$ are controlled for $b \in \{0,\dots,a\}$. Applying ∂_t^a to $\partial_t \mathbf{g}_i = -2\operatorname{Ric}_{\mathbf{g}_i}$, each product of d factors is differentiated by

$$\partial_t^a (F_1 \cdots F_d) = \sum_{\substack{(b_1,\dots,b_d) \in \mathbb{N}_0^d \\ b_1+\cdots+b_d = a}} \frac{a!}{b_1! \cdots b_d!} (\partial_t^{b_1} F_1) \cdots (\partial_t^{b_d} F_d).$$

All time indices on the right-hand side are at most a. The induction hypothesis therefore bounds all spatial derivatives of $\partial_t^{a+1} \mathbf{g}_i$. Differentiating the matrix identity $\mathbf{g}_i^{-1} \mathbf{g}_i = I$ then gives

$$\partial_t^{a+1} \mathbf{g}_i^{-1} = -\sum_{b=0}^{a} \binom{a+1}{b} (\partial_t^b \mathbf{g}_i^{-1})(\partial_t^{a+1-b} \mathbf{g}_i) \mathbf{g}_i^{-1}.$$

All factors here are also controlled, even after spatial derivatives are applied. The initial case $a = 0$ is the spatial control already proved. Bounds for all mixed orders follow. □

Theorem 22.1.10. (Hamilton compactness) Let M_i be smooth manifolds without boundary of dimension m and let $(M_i, \mathbf{g}_i(t), p_i)$ be complete Ricci flows defined on a common interval I, open on the left, with $0 \in I$ and $\inf I < 0$. The interval I may contain its right endpoint, in particular $I = (a, 0]$ with $-\infty \le a < 0$. In this statement, $J \Subset I$ denotes a compact interval contained in I; it may include the right endpoint if this belongs to I. Suppose that

$$\operatorname{inj}_{\mathbf{g}_i(0)}(p_i) \ge \iota_0 > 0.$$

Also suppose that, for every $J \Subset I$, there exists $C(J) < \infty$ such that

$$\sup_{i\in\mathbb{N}} \sup_{t\in J} \|\operatorname{Rm}_{\mathbf{g}_i(t)}\|_{L^\infty(M_i,\mathbf{g}_i(t))} \le C(J).$$

Then a subsequence converges smoothly, in the pointed sense and on compact spacetime subsets, to a complete Ricci flow

$$(M_\infty, \mathbf{g}_\infty(t), p_\infty), \qquad t \in I,$$

where M_∞ is a smooth manifold without boundary and $\mathbf{g}_\infty(t)$ is complete for every t.

Proof. Set $\mathbf{h}_i := \mathbf{g}_i(0)$. First extract a limit of the time slices at $t = 0$. Fix $q \in \mathbb{N}_0$ and choose $s < 0$ with $[s, 0] \subseteq I$. The hypothesis gives a uniform curvature bound on $M_i \times [s, 0]$. If $q \ge 1$, apply Theorem 22.1.6 to the shifted flow starting at s and evaluate at $t = 0$; for $q = 0$, use the hypothesis directly. We obtain a constant C_q independent of i such that

$$\|\nabla^q_{\mathbf{h}_i} \operatorname{Rm}_{\mathbf{h}_i}\|_{L^\infty(M_i,\mathbf{h}_i)} \le C_q, \qquad i \in \mathbb{N}.$$

Together with the injectivity radius bound at p_i, these estimates verify the hypotheses of Theorem 22.1.7. After passing to a subsequence, there exist a complete Riemannian manifold $(M_\infty, \mathbf{h}_\infty, p_\infty)$, an exhaustion

$$p_\infty \in U_1 \Subset U_2 \Subset \cdots \Subset M_\infty$$

, and compatible embeddings

$$\Phi_{i,a} \colon U_a \longrightarrow M_i, \qquad \Phi_{i,a}(p_\infty) = p_i,$$

such that

$$\Phi^*_{i,a}\mathbf{h}_i \longrightarrow \mathbf{h}_\infty \quad \text{in } C^\infty(U_a). \tag{22.10}$$

Now fix $a \in \mathbb{N}$, a compact set $K \Subset U_a$, and a compact interval $J \Subset I$. Let $\widetilde{J} \Subset I$ be a compact interval containing $J \cup \{0\}$. Choose $s < \min \widetilde{J}$ so that $[s, \max \widetilde{J}] \subseteq I$. The curvature bound on this interval and Theorem 22.1.6 give, for every $q \in \mathbb{N}_0$, constants $C_{q,\widetilde{J}}$ such that

$$\sup_{i\in\mathbb{N}} \sup_{t\in\widetilde{J}} \|\nabla^q_{\mathbf{g}_i(t)} \operatorname{Rm}_{\mathbf{g}_i(t)}\|_{L^\infty(M_i,\mathbf{g}_i(t))} \le C_{q,\widetilde{J}}. \tag{22.11}$$

Consider on U_a the metrics

$$\widehat{\mathbf{g}}_{i,a}(t) := \Phi^*_{i,a}\mathbf{g}_i(t), \qquad \widehat{\mathbf{h}}_{i,a} := \widehat{\mathbf{g}}_{i,a}(0).$$

By (22.10), in any finite atlas covering K, the coefficients of $\widehat{\mathbf{h}}_{i,a}$ and their inverses have uniform bounds of all orders. The estimates (22.11) are preserved under pullback. Lemma 22.1.9, applied on $\widetilde{J}$, then shows that on $K \times J$ all coefficients of $\widehat{\mathbf{g}}_{i,a}(t)$ and all their spatial and time derivatives are uniformly bounded. Metric distortion also ensures a uniform lower bound for these positive definite tensors.
Exhaust each U_a by compact sets and I by compact intervals. Arzelà–Ascoli, applied successively to all coefficient derivatives in finitely many charts, and a diagonal argument give a subsequence for which

$$\Phi^*_{i,a}\mathbf{g}_i(t) \longrightarrow \mathbf{g}_\infty(t) \quad \text{in } C^\infty(K \times J) \tag{22.12}$$

for every a, every $K \Subset U_a$, and every $J \Subset I$. Compatibility of the embeddings $\Phi_{i,a}$ makes the limits obtained on U_a and U_{a+1} agree on U_a; they therefore define a global family of smooth metrics $\mathbf{g}_\infty(t)$ on M_∞. At $t = 0$, identity (22.10) gives $\mathbf{g}_\infty(0) = \mathbf{h}_\infty$. Lemma 22.1.3, applied to (22.12), shows that

$$\partial_t \mathbf{g}_\infty(t) = -2\,\mathrm{Ric}_{\mathbf{g}_\infty(t)}.$$

Let us also verify uniform exhaustion of balls. Fix $J \Subset I$ and $R > 0$. Let $\widehat{J}$ be the compact interval joining J to 0. The curvature bound on $M_i \times \widehat{J}$ bounds $|\mathrm{Ric}|$ by a constant K_J independent of i. By (22.5), for every $t \in J$,

$$B_{\mathbf{g}_i(t)}(p_i, R) \subseteq B_{\mathbf{h}_i}(p_i, e^{K_J|t|}R).$$

The exhaustion property of the static limit allows a common U_a to be chosen whose image contains the balls on the right for all $t \in J$. Passing to the limit, the analogous inequality between $\mathbf{g}_\infty(t)$ and $\mathbf{h}_\infty$ shows that the same U_a contains $\overline{B_{\mathbf{g}_\infty(t)}(p_\infty, R)}$. Thus (22.2) holds, establishing pointed smooth convergence on compact spacetime subsets.

It remains to prove completeness of each slice. Fix $t \in I$ and choose a compact interval $J \Subset I$ containing 0 and t. The curvature bound gives $K_J < \infty$ such that

$$|\mathrm{Ric}_{\mathbf{g}_i(\rho)}|_{\mathbf{g}_i(\rho)} \le K_J \qquad ((x,\rho) \in M_i \times J)$$

for every i. By Lemma 22.1.8,

$$e^{-2K_J|t|}\mathbf{h}_i \le \mathbf{g}_i(t) \le e^{2K_J|t|}\mathbf{h}_i.$$

Pull back by $\Phi_{i,a}$ and pass to the limit. Since the U_a cover M_∞, we obtain

$$e^{-2K_J|t|}\mathbf{h}_\infty \le \mathbf{g}_\infty(t) \le e^{2K_J|t|}\mathbf{h}_\infty \quad \text{on } M_\infty.$$

The metric $\mathbf{h}_\infty$ is complete by the static compactness theorem; the last assertion of Lemma 22.1.8 implies that $\mathbf{g}_\infty(t)$ is also complete. □

22.2 Rescalings and singularity models

The preceding compactness theorem becomes useful once the correct scale is chosen. At a point where $|\mathrm{Rm}| = Q$, the metric $Q\mathbf{g}$ makes the curvature have size one. Time must be multiplied by the same factor to preserve the Ricci equation. The following formulas fix all conventions used for singularity limits.

Definition 22.2.1. (Temporal types of solutions) Let M be a smooth manifold without boundary and let $\mathbf{g}(t)$ be a solution of Ricci flow on M. For a problem with initial data at $t = 0$, the solution is *forward maximal* if it admits no smooth extension beyond its right time endpoint. This is the meaning of maximality used for flows on $[0, T)$. Independently of this property, a solution is called *ancient* if its time interval has left endpoint $-\infty$, for example $(-\infty, T]$; *eternal* if defined on $\mathbb{R}$, and *immortal* if defined on $[0, \infty)$. A *rescaling limit model* is a nonflat pointed smooth limit of parabolic rescalings about points whose curvature tends to infinity.

Suppose the original flow is defined on $[0,T)$. Let $(x_i,t_i)\in M\times[0,T)$, with $i\in\mathbb{N}$, and set

$$Q_i := |\mathrm{Rm}_{\mathbf{g}(t_i)}|_{\mathbf{g}(t_i)}(x_i) > 0.$$

The parabolic rescaling centered at (x_i,t_i) is

$$\mathbf{g}_i(s) := Q_i\,\mathbf{g}\Big(t_i+\frac{s}{Q_i}\Big), \qquad s\in\big[-Q_i t_i, Q_i(T-t_i)\big). \tag{22.13}$$

For each $i\in\mathbb{N}$ and $q\in\mathbb{N}_0$, the scaling laws are

$$\begin{aligned}
s &= Q_i(t-t_i), & \mathbf{g}_i &= Q_i\mathbf{g}, & d_{\mathbf{g}_i} &= Q_i^{1/2} d_{\mathbf{g}},\\
d\lambda_{\mathbf{g}_i} &= Q_i^{m/2} d\lambda_{\mathbf{g}}, & \mathrm{inj}_{\mathbf{g}_i} &= Q_i^{1/2}\,\mathrm{inj}_{\mathbf{g}}, & R_{\mathbf{g}_i} &= Q_i^{-1} R_{\mathbf{g}},\\
\mathrm{Rm}^{(1,3)}_{\mathbf{g}_i} &= \mathrm{Rm}^{(1,3)}_{\mathbf{g}}, & \mathrm{Rm}^{(0,4)}_{\mathbf{g}_i} &= Q_i\,\mathrm{Rm}^{(0,4)}_{\mathbf{g}}, & \mathrm{Ric}_{\mathbf{g}_i} &= \mathrm{Ric}_{\mathbf{g}},\\
|\nabla^q_{\mathbf{g}_i(s)}\mathrm{Rm}_{\mathbf{g}_i(s)}|_{\mathbf{g}_i(s)} &= Q_i^{-1-q/2}|\nabla^q_{\mathbf{g}(t_i+s/Q_i)}\mathrm{Rm}_{\mathbf{g}(t_i+s/Q_i)}|_{\mathbf{g}(t_i+s/Q_i)}.
\end{aligned} \tag{22.14}$$

In particular, $|\mathrm{Rm}_{\mathbf{g}_i(0)}|_{\mathbf{g}_i(0)}(x_i)=1$. Moreover,

$$\partial_s\mathbf{g}_i = Q_i\frac{1}{Q_i}\partial_t\mathbf{g}\Big(t_i+\frac{s}{Q_i}\Big) = -2\,\mathrm{Ric}_{\mathbf{g}(t_i+s/Q_i)} = -2\,\mathrm{Ric}_{\mathbf{g}_i},$$

so the flow is invariant.

The choice of points cannot be arbitrary. If

$$Q_i = |\mathrm{Rm}_{\mathbf{g}(t_i)}|_{\mathbf{g}(t_i)}(x_i) = \max_{(x,t)\in M\times[0,t_i]}|\mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)}(x), \qquad t_i\nearrow T, \tag{22.15}$$

then

$$|\mathrm{Rm}_{\mathbf{g}_i(s)}|_{\mathbf{g}_i(s)}\le 1 \quad \text{for } -Q_i t_i\le s\le 0, \qquad |\mathrm{Rm}_{\mathbf{g}_i(0)}|_{\mathbf{g}_i(0)}(x_i)=1.$$

Shi estimates control all derivatives on compact subintervals of $(-\infty,0]$. A lower injectivity radius bound is still needed; we will obtain it from Perelman's noncollapsing. Hamilton compactness will then produce a complete nonflat ancient solution.

22.3 Heat with a time-dependent metric

Perelman's monotone quantities are constructed by coupling Ricci flow with a backward heat equation. Since Riemannian measure is also time-dependent, the spacetime adjoint is not $-\partial_t-\Delta$; a scalar curvature term appears to compensate for volume variation.

For a Ricci flow define

$$\Box := \partial_t - \Delta_{\mathbf{g}(t)}.$$

The measure evolution, proved in (21.36), is $\partial_t d\lambda_{\mathbf{g}(t)} = -R_{\mathbf{g}(t)}\,d\lambda_{\mathbf{g}(t)}$.

Proposition 22.3.1. (Spacetime adjoint) If u, v are smooth and M is closed, then

$$\int_{t_1}^{t_2}\int_M v\,\Box u\,d\lambda_{\mathbf{g}(t)}dt = \Big[\int_M uv\,d\lambda_{\mathbf{g}(t)}\Big]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_M u\,\Box^* v\,d\lambda_{\mathbf{g}(t)}dt, \tag{22.16}$$

where

$$\Box^* := -\partial_t - \Delta_{\mathbf{g}(t)} + R_{\mathbf{g}(t)}. \tag{22.17}$$

Thus the conjugate equation is

$$\Box^* u = 0, \qquad \partial_t u = -\Delta_{\mathbf{g}(t)}u + R_{\mathbf{g}(t)}u.$$

Every positive solution of this equation preserves $\int_M u\,d\lambda_{\mathbf{g}(t)}$.

Proof. Differentiating the time pairing gives

$$\frac{d}{dt}\int_M uv\,d\lambda_{\mathbf{g}(t)} = \int_M (u_t v + u v_t - R_{\mathbf{g}(t)}uv)\,d\lambda_{\mathbf{g}(t)}.$$

Spatial self-adjointness of $\Delta_{\mathbf{g}(t)}$ and integration in time give (22.16). If $\Box^* u = 0$, then

$$\frac{d}{dt}\int_M u\,d\lambda_{\mathbf{g}(t)} = \int_M \big(-\Delta_{\mathbf{g}(t)}u + R_{\mathbf{g}(t)}u - R_{\mathbf{g}(t)}u\big)\,d\lambda_{\mathbf{g}(t)} = 0.$$

□

Time dependence prevents use of an autonomous semigroup. For $s \le t$ write $U(t,s)$ for the propagator of $\partial_t u = \Delta_{\mathbf{g}(t)}u$. It satisfies

$$U(t,r)U(r,s) = U(t,s), \qquad s \le r \le t,$$

but does not depend only on $t-s$. Its two-time kernel $H(x,t;y,s)$ satisfies the same composition law, integrating over the intermediate slice. The adjoint of $U(t,s)$ propagates backward through $\Box^*$; in general there is no symmetry under simultaneous interchange of (x,t) and (y,s). The construction follows by applying the parabolic theory of Chapter 20 on short intervals and composing the evolution operators.

22.4 The functional $\mathcal{F}$

Perelman's variational description considers a metric and a density simultaneously. The auxiliary function f allows the weighted measure to be fixed and introduces an energy whose derivative along the flow is a square.

Let M be closed. For $f \in C^\infty(M)$ define

$$\mathcal{F}(\mathbf{g}, f) := \int_M \big(R_{\mathbf{g}} + |\nabla f|_{\mathbf{g}}^2\big)e^{-f}\,d\lambda_{\mathbf{g}}. \tag{22.18}$$

The natural normalization is $\int_M e^{-f}\,d\lambda_{\mathbf{g}} = 1$.

Before differentiating $\mathcal{F}$, collect the integrations by parts explaining the Hessian terms that will appear.

Lemma 22.4.1. (Integration by parts with the weighted measure) Let M be a closed smooth manifold, let $f \in C^\infty(M)$, and let $\mathbf{h}$ be a symmetric covariant tensor of order two. Then

$$\int_M \operatorname{div}_{\mathbf{g}}^2 \mathbf{h}\, e^{-f}\, d\lambda_{\mathbf{g}} = \int_M \left[-\langle \mathbf{h}, \nabla_{\mathbf{g}}^2 f\rangle_{\mathbf{g}} + \mathbf{h}(\nabla^{\mathbf{g}} f, \nabla^{\mathbf{g}} f)\right] e^{-f}\, d\lambda_{\mathbf{g}}, \tag{22.19}$$

$$-\int_M \Delta_{\mathbf{g}}(\operatorname{tr}_{\mathbf{g}} \mathbf{h}) e^{-f}\, d\lambda_{\mathbf{g}} = \int_M \operatorname{tr}_{\mathbf{g}} \mathbf{h}\left(\Delta_{\mathbf{g}} f - |\nabla f|_{\mathbf{g}}^2\right) e^{-f}\, d\lambda_{\mathbf{g}}, \tag{22.20}$$

$$2\int_M \langle \nabla^{\mathbf{g}} f, \nabla^{\mathbf{g}} \varphi\rangle_{\mathbf{g}} e^{-f}\, d\lambda_{\mathbf{g}} = -2\int_M \varphi\left(\Delta_{\mathbf{g}} f - |\nabla f|_{\mathbf{g}}^2\right) e^{-f}\, d\lambda_{\mathbf{g}}. \tag{22.21}$$

Proof. For the first identity, integrate the two divergences successively:

$$\begin{aligned}\int_M \sum_{i,j=1}^m \nabla^i \nabla^j h_{ij}\, e^{-f}\, d\lambda_{\mathbf{g}} &= \int_M \sum_{i,j=1}^m (\nabla^j h_{ij}) \nabla^i f\, e^{-f}\, d\lambda_{\mathbf{g}} \\ &= -\int_M \sum_{i,j=1}^m h_{ij} \nabla^j (\nabla^i f\, e^{-f})\, d\lambda_{\mathbf{g}}.\end{aligned}$$

Since $\nabla^j(\nabla^i f\, e^{-f}) = (\nabla^2 f)^{ij} e^{-f} - \nabla^i f \nabla^j f\, e^{-f}$, we obtain (22.19). If $q = \operatorname{tr}_{\mathbf{g}} \mathbf{h}$, then

$$-\int_M (\Delta_{\mathbf{g}} q) e^{-f}\, d\lambda_{\mathbf{g}} = -\int_M \langle \nabla q, \nabla f\rangle_{\mathbf{g}} e^{-f}\, d\lambda_{\mathbf{g}}.$$

Integrating the divergence of $q e^{-f} \nabla^{\mathbf{g}} f$ gives (22.20). The same calculation with $q = 2\varphi$ proves (22.21). □

Proposition 22.4.2. (First variation of $\mathcal{F}$) Let M be a closed smooth manifold. For a metric $\mathbf{g}$, a function $f \in C^\infty(M)$, and variations $\mathbf{h} = \delta\mathbf{g}$, $\varphi = \delta f$, we have

$$D\mathcal{F}_{(\mathbf{g},f)}[\mathbf{h}, \varphi] = \int_M \left[-\langle \mathbf{h}, \operatorname{Ric}_{\mathbf{g}} + \nabla_{\mathbf{g}}^2 f\rangle_{\mathbf{g}} + \left(\frac{1}{2}\operatorname{tr}_{\mathbf{g}} \mathbf{h} - \varphi\right)\left(2\Delta_{\mathbf{g}} f - |\nabla f|_{\mathbf{g}}^2 + R_{\mathbf{g}}\right)\right] e^{-f}\, d\lambda_{\mathbf{g}}. \tag{22.22}$$

Proof. The variation formulas of the preceding chapter give

$$\begin{aligned}\delta R_{\mathbf{g}} &= -\langle \mathbf{h}, \operatorname{Ric}_{\mathbf{g}}\rangle_{\mathbf{g}} + \operatorname{div}_{\mathbf{g}}^2 \mathbf{h} - \Delta_{\mathbf{g}} \operatorname{tr}_{\mathbf{g}} \mathbf{h}, \\ \delta|\nabla f|_{\mathbf{g}}^2 &= -\mathbf{h}(\nabla^{\mathbf{g}} f, \nabla^{\mathbf{g}} f) + 2\langle \nabla^{\mathbf{g}} f, \nabla^{\mathbf{g}} \varphi\rangle_{\mathbf{g}}, \\ \delta(e^{-f}\, d\lambda_{\mathbf{g}}) &= \left(\frac{1}{2}\operatorname{tr}_{\mathbf{g}} \mathbf{h} - \varphi\right) e^{-f}\, d\lambda_{\mathbf{g}}.\end{aligned}$$

Substituting into (22.18), the sum of terms containing derivatives of $\mathbf{h}$ becomes, by (22.19) and (22.20),

$$-\langle \mathbf{h}, \nabla^2 f\rangle + \mathbf{h}(\nabla f, \nabla f) + \operatorname{tr}_{\mathbf{g}} \mathbf{h}(\Delta f - |\nabla f|^2).$$

The second summand cancels the metric variation in $|\nabla f|^2$ exactly. By (22.21), the term containing $\nabla\varphi$ becomes $-2\varphi(\Delta f - |\nabla f|^2)$. Adding the measure variation, the coefficients of $\operatorname{tr}_{\mathbf{g}} \mathbf{h}$ and φ are, respectively,

$$\frac{1}{2}(2\Delta f - |\nabla f|^2 + R), \qquad -(2\Delta f - |\nabla f|^2 + R).$$

This proves (22.22). □

Theorem 22.4.3. (Monotonicity of $\mathcal{F}$) Let M be a closed smooth manifold and suppose that

$$\partial_t \mathbf{g} = -2\,\mathrm{Ric}_{\mathbf{g}(t)}, \qquad \partial_t f = -\Delta_{\mathbf{g}(t)} f + |\nabla f|^2_{\mathbf{g}(t)} - R_{\mathbf{g}(t)}. \tag{22.23}$$

Then $u = e^{-f}$ satisfies $\Box^* u = 0$ and

$$\frac{d}{dt}\mathcal{F}(\mathbf{g}(t), f(t)) = 2\int_M |\mathrm{Ric}_{\mathbf{g}(t)} + \nabla^2_{\mathbf{g}(t)} f|^2_{\mathbf{g}(t)} e^{-f}\, d\lambda_{\mathbf{g}(t)} \geq 0. \tag{22.24}$$

Equality on an interval characterizes, modulo diffeomorphisms, a steady gradient soliton. On a closed manifold, such a soliton is Ricci-flat and f is constant on every connected component.

Proof. The equation for f implies

$$\partial_t(e^{-f}) = (\Delta f - |\nabla f|^2 + R)e^{-f} = -\Delta(e^{-f}) + Re^{-f},$$

so $\Box^* u = 0$. Let φ_t be the diffeomorphisms generated by $-\nabla^{\mathbf{g}(t)} f$. By naturality of Ricci and the Hessian, the pullback of $(\mathbf{g}, f)$ satisfies

$$\partial_t \mathbf{g} = -2(\mathrm{Ric}_{\mathbf{g}} + \nabla^2_{\mathbf{g}} f), \qquad \partial_t f = -\Delta_{\mathbf{g}} f - R_{\mathbf{g}}.$$

In this gauge, $\frac{1}{2}\,\mathrm{tr}_{\mathbf{g}}(\partial_t \mathbf{g}) - \partial_t f = 0$; thus the measure $e^{-f} d\lambda_{\mathbf{g}}$ remains fixed pointwise. Apply (22.22) with $\mathbf{h} = -2(\mathrm{Ric} + \nabla^2 f)$ and $\varphi = -\Delta f - R$. The second variation term vanishes, and the first is the right-hand side of (22.24). Diffeomorphism invariance of $\mathcal{F}$ returns the formula to the original flow.

If equality holds, then $\mathrm{Ric} + \nabla^2 f = 0$. Taking the trace and integrating gives $\int_M R\, d\lambda_{\mathbf{g}} = 0$. Naturality of scalar curvature and its evolution equation give

$$\Delta_f R := \Delta R - \langle \nabla f, \nabla R \rangle = -2|\mathrm{Ric}|^2.$$

The measure $e^{-f} d\lambda_{\mathbf{g}}$ makes Δ_f symmetric, so the integral of the left-hand side is zero. We conclude that $\mathrm{Ric} = 0$ and then $\nabla^2 f = 0$. A function with zero Hessian on a closed manifold is constant on every connected component. □

22.5 The entropy $\mathcal{W}$ and the quantity μ

The functional $\mathcal{F}$ is not invariant under parabolic rescaling. The entropy $\mathcal{W}$ incorporates a time scale $\tau > 0$ and a Boltzmann entropy term so that the entire expression remains invariant when $\mathbf{g}$ and τ are multiplied by the same constant.

Let

$$u = (4\pi\tau)^{-m/2} e^{-f}, \qquad \int_M u\, d\lambda_{\mathbf{g}} = 1,$$

and define

$$\mathcal{W}(\mathbf{g}, f, \tau) := \int_M \left[\tau(R_{\mathbf{g}} + |\nabla f|^2_{\mathbf{g}}) + f - m\right] u\, d\lambda_{\mathbf{g}}. \tag{22.25}$$

Lemma 22.5.1. (Boltzmann entropy and $\mathcal{F}$) Suppose that $\mathbf{g}(t)$ solves Ricci flow and a positive function u satisfies $\Box^* u = 0$ and $\int_M u \, d\lambda_{\mathbf{g}(t)} = 1$. If

$$\mathcal{B}(t) := \int_M u \log u \, d\lambda_{\mathbf{g}(t)}, \qquad \mathcal{F}(t) := \int_M \left(\frac{|\nabla u|^2}{u} + Ru \right) d\lambda_{\mathbf{g}(t)},$$

then $\mathcal{B}'(t) = \mathcal{F}(t)$.

Proof. Use $u_t = -\Delta u + Ru$ and $\partial_t d\lambda_{\mathbf{g}} = -R \, d\lambda_{\mathbf{g}}$. Differentiation under the integral gives

$$\mathcal{B}'(t) = \int_M \left[u_t (1 + \log u) - Ru \log u \right] d\lambda_{\mathbf{g}} = \int_M \left[-\Delta u (1 + \log u) + Ru \right] d\lambda_{\mathbf{g}}.$$

The integral of Δu is zero and $-\int_M (\Delta u) \log u = \int_M |\nabla u|^2 / u$. This proves the identity. □

Theorem 22.5.2. (Monotonicity of $\mathcal{W}$) Let M^m be a closed smooth manifold. Suppose that

$$\partial_t \mathbf{g} = -2 \operatorname{Ric}_{\mathbf{g}(t)}, \qquad \tau' = -1, \qquad \partial_t f = -\Delta_{\mathbf{g}(t)} f + |\nabla f|^2_{\mathbf{g}(t)} - R_{\mathbf{g}(t)} + \frac{m}{2\tau}, \tag{22.26}$$

and $\tau(t) > 0$. Also suppose that the density $u = (4\pi\tau)^{-m/2} e^{-f}$ has mass one at some time in the interval. Then u solves $\Box^* u = 0$, preserves this normalization, and

$$\frac{d}{dt} \mathcal{W}(\mathbf{g}(t), f(t), \tau(t)) = 2\tau \int_M \left| \operatorname{Ric}_{\mathbf{g}(t)} + \nabla^2_{\mathbf{g}(t)} f - \frac{1}{2\tau} \mathbf{g}(t) \right|^2_{\mathbf{g}(t)} u \, d\lambda_{\mathbf{g}(t)} \geq 0. \tag{22.27}$$

Equality on an interval is equivalent to the equation of a shrinking gradient soliton.

Proof. From $\log u = -\frac{m}{2} \log(4\pi\tau) - f$ and (22.26) we obtain $u_t = -\Delta u + Ru$. Mass is preserved by Proposition 22.3.1.

Write

$$\mathcal{F}_u(t) := \int_M (R + |\nabla f|^2) u \, d\lambda_{\mathbf{g}}, \qquad \mathcal{B}(t) := \int_M u \log u \, d\lambda_{\mathbf{g}}.$$

Since $u = (4\pi\tau)^{-m/2} e^{-f}$ and its mass is one,

$$\mathcal{W} = \tau \mathcal{F}_u - \mathcal{B} - \frac{m}{2} \log(4\pi\tau) - m. \tag{22.28}$$

Lemma 22.5.1 gives $\mathcal{B}' = \mathcal{F}_u$. To calculate the derivative of the first term, we do not apply Theorem 22.4.3 directly to f, since its equation contains the summand $m/(2\tau)$. Define

$$\widetilde{f} := f + \frac{m}{2} \log(4\pi\tau) = -\log u.$$

Since $\tau' = -1$, we have

$$\partial_t \widetilde{f} = -\Delta \widetilde{f} + |\nabla \widetilde{f}|^2 - R, \qquad e^{-\widetilde{f}} = u.$$

Moreover, $d\widetilde{f} = df$ and $\nabla^2\widetilde{f} = \nabla^2 f$. Thus,

$$\mathcal{F}_u = \mathcal{F}(\mathbf{g}, \widetilde{f}), \qquad \mathcal{F}_u' = 2\int_M |\mathrm{Ric} + \nabla^2 f|^2 u \, d\lambda_{\mathbf{g}}$$

by Theorem 22.4.3. Differentiating (22.28) and using $\tau' = -1$ gives

$$\mathcal{W}' = 2\tau \int_M |\mathrm{Ric} + \nabla^2 f|^2 u \, d\lambda_{\mathbf{g}} - 2\mathcal{F}_u + \frac{m}{2\tau}. \tag{22.29}$$

Weighted integration by parts gives

$$\int_M (\Delta f) u \, d\lambda_{\mathbf{g}} = \int_M |\nabla f|^2 u \, d\lambda_{\mathbf{g}};$$

consequently, $\mathcal{F}_u = \displaystyle\int_M (R + \Delta f) u = \int_M \mathrm{tr}_{\mathbf{g}}(\mathrm{Ric} + \nabla^2 f) u$. If $\mathbf{A} = \mathrm{Ric} + \nabla^2 f$, then

$$2\tau \left| \mathbf{A} - \frac{1}{2\tau}\mathbf{g} \right|^2 = 2\tau |\mathbf{A}|^2 - 2\,\mathrm{tr}_{\mathbf{g}} \mathbf{A} + \frac{m}{2\tau}.$$

Integrating this identity and comparing it with (22.29) proves (22.27). Equality is equivalent to $\mathrm{Ric} + \nabla^2 f = \mathbf{g}/(2\tau)$. □

Definition 22.5.3. (The quantity μ) Let M^m be a closed smooth manifold, let $\mathbf{g}$ be a Riemannian metric, and let $\tau > 0$. Set

$$\mu(\mathbf{g}, \tau) := \inf \left\{ \mathcal{W}(\mathbf{g}, f, \tau) \,\middle|\, f \in C^\infty(M), \quad (4\pi\tau)^{-m/2} \int_M e^{-f} \, d\lambda_{\mathbf{g}} = 1 \right\}. \tag{22.30}$$

Lemma 22.5.4. (Strong principle for the entropy nonlinearity) Let Ω be a connected domain in a Riemannian manifold without boundary and let $w \in C^2(\Omega)$ be nonnegative. Suppose that there exists $s_0 > 0$ such that, when $0 \le w \le s_0$,

$$\Delta w \le \beta(w), \qquad \beta(s) = C_0 s - C_1 s \log s^2, \quad C_1 > 0.$$

Then w is identically zero or strictly positive. The same alternative holds for a nonnegative function $w \in W^{2,p_0}_{\mathrm{loc}}(\Omega)$ with $p_0 > \max\{2, \dim \Omega\}$, taking its C^1 representative, if the inequality holds almost everywhere on $\{0 \le w \le s_0\}$.

Proof. First prove the assertion for $w \in C^2(\Omega)$. Extend β by $\beta(0) = 0$ and reduce $s_0 > 0$ if necessary. Then β is continuous, nonnegative, and increasing on $[0, s_0]$, because $\beta'(s) = C_0 - C_1(\log s^2 + 2) > 0$ for sufficiently small s. We may also assume that $s_0 < 1$ and β is strictly increasing.

Suppose w has both a zero and a positive point. By connectedness, the open set $\{w > 0\}$ has a boundary point in Ω. Take a point p in this open set sufficiently close to its boundary and set $R = d(p, \{w = 0\}) > 0$. It can be chosen so that the closed ball $\overline{B(p, R)}$ lies in a relatively compact normal neighborhood of Ω and R is smaller than the injectivity radius at p. There then exists $q \in \partial B(p, R)$ with $w(q) = 0$, while $w > 0$ on $B(p, R)$. Existence of q follows by minimizing distance on a compact subset of that neighborhood; choosing p sufficiently close to the boundary makes points outside the neighborhood farther away than R.

The function $r(x) = d(p,x)$ is smooth on the annulus $\mathcal{A} = \{R/2 < r < R\}$. Fix $C \geq 0$ such that $\Delta r \leq C$ on its closure and set $L = R/2$. For $0 < \epsilon < 1$, define

$$k_\epsilon = C + \sqrt{|C_0| + 2C_1|\log\epsilon| + 2\frac{C_1}{e} + 1}, \qquad z_\epsilon(t) = \epsilon(e^{k_\epsilon t} - 1), \quad 0 \leq t \leq L.$$

Since $k_\epsilon = O(\sqrt{|\log\epsilon|})$, $z_\epsilon(L) \to 0$ as $\epsilon \to 0^+$. Choose ϵ small enough that

$$0 < z_\epsilon(L) < \min\left\{s_0, \min_{r=R/2} w\right\}.$$

Let $\varphi(x) = z_\epsilon(R - r(x))$. On both components of $\partial\mathcal{A}$ we have $\varphi \leq w$.

Let us verify the differential inequality for the barrier. For $t > 0$, write $y = e^{k_\epsilon t} - 1 > 0$. Since $\dfrac{y}{1+y}\max\{-\log y, 0\} \leq 1/e$, we obtain

$$\beta(z_\epsilon(t)) \leq \epsilon e^{k_\epsilon t}\left(|C_0| + 2C_1|\log\epsilon| + 2C_1/e\right).$$

On the other hand, $|dr| = 1$ and the choice of k_ϵ give

$$\Delta\varphi = z_\epsilon''(R-r) - z_\epsilon'(R-r)\Delta r \geq \epsilon e^{k_\epsilon(R-r)}(k_\epsilon^2 - Ck_\epsilon) \geq \beta(\varphi).$$

If $\varphi - w$ had a positive maximum in $\mathcal{A}$, at that point we would have $0 \leq w < \varphi < s_0$ and

$$0 \geq \Delta(\varphi - w) \geq \beta(\varphi) - \beta(w) > 0,$$

a contradiction. Thus $w \geq \varphi$ in the annulus. Let $\gamma(t)$ be the radial geodesic from q toward p, parametrized by arc length. For $0 < t < L$ we have $w(\gamma(t)) \geq z_\epsilon(t)$, and both functions vanish at $t = 0$. Their right derivatives satisfy

$$dw_q(\dot\gamma(0)) \geq z_\epsilon'(0) = \epsilon k_\epsilon > 0.$$

This contradicts $dw_q = 0$, since $w \geq 0$ and $w(q) = 0$ at an interior point of Ω. This proves the alternative.

Now consider $w \in W^{2,p_0}_{\mathrm{loc}}(\Omega)$ with $p_0 > \max\{2, \dim\Omega\}$. The local Sobolev–Morrey embedding, obtained by applying Theorem 3.5.7 in charts after multiplication by cutoff functions, provides a representative of class C^1. Suppose again that w has a zero and a positive point. The choice of the ball, the annulus, and the barrier φ uses only the continuity of w and remains valid. To compare w with φ, set $h := \varphi - w$ and, given $\delta > 0$,

$$\psi_\delta := (h - \delta)_+.$$

Since $h \leq 0$ on $\partial\mathcal{A}$, the support of ψ_δ is contained in the compact set $\{x \in \overline{\mathcal{A}} \mid h(x) \geq \delta\} \subseteq \mathcal{A}$. The chain rule in Theorem 9.1.49 gives $\psi_\delta \in W^{1,2}(\mathcal{A})$ and

$$d\psi_\delta = \mathbf{1}_{\{h>\delta\}}\, dh \qquad \text{almost everywhere.}$$

We can approximate ψ_δ in $W^{1,2}$ by smooth functions with compact support in $\mathcal{A}$: apply Theorem 9.1.48 to the trivial bundle and multiply the approximations by a fixed cutoff function with compact support in $\mathcal{A}$ that equals one near the support of ψ_δ. Moreover, $\Delta h \in L^{p_0}(\mathcal{A}) \subseteq L^2(\mathcal{A})$, because $p_0 > 2$ and the annulus has finite volume. Thus the weak integration by parts identity admits ψ_δ as a test function. Where $\psi_\delta > 0$, we

have $0 \le w < \varphi < s_0$, and we obtain

$$-\int_{\mathcal{A}} |d\psi_\delta|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} = \int_{\mathcal{A}} (\Delta h)\psi_\delta \, d\lambda_{\mathbf{g}} \ge \int_{\mathcal{A}} \big(\beta(\varphi) - \beta(w)\big)\psi_\delta \, d\lambda_{\mathbf{g}} \ge 0.$$

The last integrand is strictly positive wherever $\psi_\delta > 0$. Thus this set has measure zero and, being open, is empty. It follows that $h \le \delta$ for every $\delta > 0$, and hence $w \ge \varphi$ in $\mathcal{A}$. The contradiction with $dw_q = 0$ now follows along the same radial geodesic, since the representative of w is of class C^1. This proves the weak assertion.

For this nonlinearity, the barrier above implements the criterion in Vázquez's strong principle [151]. Indeed, with $B(s) = \displaystyle\int_0^s \beta(r)\,dr$,

$$\int_0^{s_0} \frac{ds}{\sqrt{B(s)}} = \infty.$$

In our case, integration gives $B(s) = (C_0 + C_1)\frac{s^2}{2} - C_1 s^2 \log s$. Consequently, $B(s) \le C s^2(1 + |\log s|)$ near zero. Under the substitution $z = -\log s$, the integral providing the lower bound is a multiple of $\displaystyle\int_{-\log s_0}^{\infty} (1+z)^{-1/2}\,dz$, which diverges. This divergence explains why the logarithmic term still permits the strong principle, even though $\beta(s)/s$ is unbounded near zero. □

Proposition 22.5.5. (Minimizers and invariances of μ) On a smooth closed connected manifold, the infimum of (22.30) is attained by a smooth function. If

$$v = (4\pi\tau)^{-m/4} e^{-f/2}, \qquad \int_M v^2 \, d\lambda_{\mathbf{g}} = 1,$$

the minimizer can be chosen positive and satisfies

$$-4\tau\Delta_{\mathbf{g}} v + \tau R_{\mathbf{g}} v - v\log v^2 - \frac{m}{2}\log(4\pi\tau)v - mv = \mu(\mathbf{g},\tau)v. \tag{22.31}$$

Equivalently,

$$2\tau\Delta_{\mathbf{g}} f - \tau|\nabla f|_{\mathbf{g}}^2 + \tau R_{\mathbf{g}} + f - m = \mu(\mathbf{g},\tau). \tag{22.32}$$

Moreover, for every diffeomorphism φ and every $c > 0$,

$$\mu(\varphi^*\mathbf{g},\tau) = \mu(\mathbf{g},\tau), \qquad \mu(c\mathbf{g}, c\tau) = \mu(\mathbf{g},\tau). \tag{22.33}$$

Proof. In the variable v, the problem is to minimize

$$\mathcal{J}_{\mathbf{g},\tau}(v) := \int_M \Big[4\tau|\nabla v|_{\mathbf{g}}^2 + \tau R_{\mathbf{g}} v^2 - v^2\log v^2 - \frac{m}{2}\log(4\pi\tau)v^2 - mv^2\Big]\, d\lambda_{\mathbf{g}} \tag{22.34}$$

over the sphere in $H^1(M)$ given by $\|v\|_{L^2(M)} = 1$, with the convention $0\log 0 = 0$. Enlarging the admissible set in this way does not change the infimum. To see this, approximate $|v|$ in H^1 by nonnegative smooth functions using the heat flow for the fixed metric; add a positive constant tending to zero and normalize in L^2. This gives strictly positive smooth functions converging to $|v|$ in H^1. By the Sobolev embeddings, they also converge in some L^q with $q > 2$. The power estimate detailed below and Vitali's theorem give convergence of their entropy terms. The quadratic terms converge directly. Finally, $\mathcal{J}(|v|) = \mathcal{J}(v)$, since

$|d|v|| = |dv|$ almost everywhere for real functions in H^1.
Fix $q > 2$ for which $H^1(M) \hookrightarrow L^q(M)$ is continuous. Jensen's inequality, applied to the probability measure $v^2 d\lambda_{\mathbf{g}}$, gives

$$\int_M v^2 \log v^2 \, d\lambda_{\mathbf{g}} \le \frac{2q}{q-2} \log \|v\|_{L^q(M)}.$$

The Sobolev embedding and the inequality $\log(1+a) \le \varepsilon a^2 + C_\varepsilon$ imply, for every $\varepsilon > 0$,

$$\int_M v^2 \log v^2 \, d\lambda_{\mathbf{g}} \le \varepsilon \|dv\|^2_{L^2(M,T^*M)} + C_\varepsilon \qquad (\|v\|_{L^2(M)} = 1). \tag{22.35}$$

Taking $0 < \varepsilon < 4\tau$ and using the boundedness of $R_{\mathbf{g}}$, we obtain a lower bound for $\mathcal{J}_{\mathbf{g},\tau}$ and a uniform bound in H^1 for every minimizing sequence (v_j).
We may replace v_j by $|v_j|$. After passing to a subsequence,

$$v_j \rightharpoonup v \quad \text{in } H^1(M), \qquad v_j \to v \quad \text{in } L^{q_0}(M)$$

for some $2 < q_0 < q$, and also almost everywhere. In particular, $\|v\|_{L^2(M)} = 1$. The Dirichlet energy is lower semicontinuous, and the curvature term converges by strong convergence in L^2.
For the entropy term, choose $\eta \in (0,1)$ with $2(1+\eta) < q_0$. The inequality

$$a|\log a| \le C_\eta (a^{1-\eta} + a^{1+\eta}), \qquad a \ge 0,$$

applied to $a = v_j^2$ shows that $(v_j^2 \log v_j^2)$ is uniformly integrable: the higher power is bounded in a space L^p with $p > 1$ by convergence in L^{q_0}, and the lower power is bounded in $L^{1/(1-\eta)}$ by normalization in L^2. Vitali's theorem gives

$$\int_M v_j^2 \log v_j^2 \, d\lambda_{\mathbf{g}} \longrightarrow \int_M v^2 \log v^2 \, d\lambda_{\mathbf{g}}.$$

Therefore, v is a nonnegative minimizer.
The real function $s \mapsto s^2 \log s^2$, extended by zero at $s = 0$, is C^1, and its derivative grows more slowly than any power $|s|^{1+\varepsilon}$ at infinity. The preceding embeddings allow us to differentiate (22.34) in directions in H^1. The method of Lagrange multipliers yields

$$-4\tau \Delta v + \tau R v - v \log v^2 - v - \frac{m}{2} \log(4\pi\tau) v - m v = \lambda v$$

in the weak sense. Multiplying by v and integrating gives $\lambda = \mu(\mathbf{g}, \tau) - 1$, and hence (22.31).
Let us establish the regularity before using positivity. The weak equation has the form

$$-\Delta v = F(x, v), \qquad |F(x,a)| \le C_\varepsilon (1 + |a|^{1+\varepsilon}), \quad x \in M, \quad a \in \mathbb{R},$$

for each $\varepsilon > 0$. The term $a \log a^2$, extended by zero, is bounded near $a = 0$, which explains the constant summand.
If $m \ge 3$, we start with $v \in L^{q_0}(M)$ with $q_0 = 2m/(m-2)$. Fix $0 < \varepsilon < \min\{q_0 - 1, q_0/m\}$. If $v \in L^{q_j}$, then $F(\cdot, v) \in L^{p_j}$, where $p_j = q_j/(1+\varepsilon) > 1$, and elliptic regularity gives $v \in W^{2,p_j}$. As long as $p_j < m/2$, the Sobolev

embedding yields the next exponent:

$$\frac{1}{q_{j+1}} = \frac{1+\varepsilon}{q_j} - \frac{2}{m}, \qquad j \in \mathbb{N}_0 \text{ with } p_j < m/2.$$

Since $q_j \geq q_0$, the choice of ε implies $\varepsilon/q_j < 1/m$ and therefore $1/q_{j+1} < 1/q_j - 1/m$. The reciprocals decrease by at least $1/m$ at each step, so this alternative can occur only finitely many times. If $p_j = m/2$ is reached, the critical embedding gives $v \in L^q$ for every $q < \infty$; we then choose $q > (1+\varepsilon)m/2$ and apply the equation again. In either case, we reach $W^{2,p}$ with $p > m/2$, and hence $C^{0,\alpha}$ for some $\alpha \in (0,1)$.

If $m = 2$, the embedding of H^1 gives $v \in L^q$ for each $q < \infty$. Choose $q > 2(1+\varepsilon)$; the equation gives $v \in W^{2,q/(1+\varepsilon)} \hookrightarrow C^{1,\alpha}$ for some $\alpha > 0$. If $m = 1$, the embedding of H^1 already gives a continuous bounded function. Thus v is bounded in every dimension.

The function $F(\cdot, v)$ now belongs to L^p for each $1 < p < \infty$. Another application of elliptic regularity gives $v \in W^{2,p}$ for all these exponents and, taking $p > m$, $v \in C^{1,\alpha}$. In particular, v is Lipschitz. Take $p > \max\{2, m\}$ to apply the weak version of the strong principle just proved.

From (22.31) we obtain, almost everywhere, $\Delta v \leq C_0 v - C_1 v \log v^2$, where we may take

$$C_1 = \frac{1}{4\tau}, \qquad C_0 = \frac{1}{4\tau}\left\| \tau R_{\mathbf{g}} - \frac{m}{2}\log(4\pi\tau) - m - \mu(\mathbf{g},\tau)\right\|_\infty.$$

The weak version of Lemma 22.5.4 and the normalization exclude the zero solution; since M is connected, it follows that $v > 0$. Compactness of M gives $\min_{x\in M} v(x) > 0$. Thus F is smooth on a neighborhood of the compact set $\{(x, v(x)) \mid x \in M\}$. We start with $v \in H^2(M)$. If $v \in H^r(M)$ for an integer $r \geq 2$, the composition estimate (20.9) implies $F(\cdot, v) \in H^r(M)$; the equation and Theorem 11.4.2 then give $v \in H^{r+2}(M)$, using a finite cover of M. Induction yields all even orders and, by the Sobolev–Morrey embeddings, $v \in C^\infty(M)$. The substitution $v = (4\pi\tau)^{-m/4} e^{-f/2}$ gives (22.32).

Diffeomorphism invariance follows from the change of variables formula. Under $\mathbf{g} \mapsto c\mathbf{g}$, we have $R \mapsto c^{-1}R$, $|\nabla f|^2 \mapsto c^{-1}|\nabla f|^2$, and $d\lambda \mapsto c^{m/2} d\lambda$; the factor $(4\pi c\tau)^{-m/2}$ compensates for the change in volume. This proves (22.33). □

If M has connected components $M_1, \ldots, M_N$, set $a_\alpha = \int_{M_\alpha} v^2 \, d\lambda_{\mathbf{g}}$ for $\alpha \in \{1, \ldots, N\}$. On the components with $a_\alpha > 0$, write $v = \sqrt{a_\alpha}\, w_\alpha$, with $\|w_\alpha\|_{L^2(M_\alpha)} = 1$. Then

$$\mathcal{J}_{\mathbf{g},\tau}(v) = \sum_{\substack{\alpha\in\{1,\ldots,N\} \\ a_\alpha > 0}} a_\alpha \mathcal{J}_{\mathbf{g}|_{M_\alpha},\tau}(w_\alpha) - \sum_{\alpha=1}^N a_\alpha \log a_\alpha, \qquad \sum_{\alpha=1}^N a_\alpha = 1.$$

The last term is nonnegative. Concentrating the mass on a component that minimizes its quantity μ, and approximating if positivity on all components is required, gives $\mu(\mathbf{g},\tau) = \min_{\alpha\in\{1,\ldots,N\}} \mu(\mathbf{g}|_{M_\alpha}, \tau)$. Thus the following arguments apply component by component.

Corollary 22.5.6. (Monotonicity of μ) Let M be a smooth closed connected manifold. If $\mathbf{g}(t)$ solves the Ricci flow, $\tau' = -1$, and $\tau(t) > 0$, then $t \mapsto \mu(\mathbf{g}(t), \tau(t))$ is nondecreasing. If it remains constant on an interval, the flow is, modulo diffeomorphisms and rescaling, a gradient shrinking soliton.

Proof. Let $t_1 < t_2$ and choose a positive minimizer for $(\mathbf{g}(t_2), \tau(t_2))$. Its terminal density determines a positive solution of the conjugate equation backward to t_1; by conservation of mass, it can be written as $(4\pi\tau)^{-m/2} e^{-f}$. Theorem 22.5.2 gives

$$\mu(\mathbf{g}(t_2), \tau(t_2)) = \mathcal{W}(t_2) \geq \mathcal{W}(t_1) \geq \mu(\mathbf{g}(t_1), \tau(t_1)).$$

If the endpoints are equal, both inequalities are equalities, and the square in (22.27) vanishes throughout the interval. This is the gradient shrinking soliton equation. □

22.6 Reduced length, distance, and volume

The entropy $\mathcal{W}$ yields noncollapsing for smooth flows. To study the spacetime geometry more precisely, Perelman introduced a second monotone quantity based on a variational problem for curves. Its integrand combines kinetic energy with scalar curvature and uses backward time as its parameter.

Throughout this section, M is closed and connected. Fix $p \in M$, a time t_0, and $\tau_{\max} > 0$ such that $[t_0 - \tau_{\max}, t_0]$ is contained in the smooth time interval of the flow. For $0 \leq s \leq \tau_{\max}$, write

$$\mathbf{g}_s := \mathbf{g}(t_0 - s), \qquad R_s := R_{\mathbf{g}_s}, \qquad \mathrm{Ric}_s := \mathrm{Ric}_{\mathbf{g}_s}.$$

We denote the Levi–Civita connection and Laplacian of $\mathbf{g}_s$ by $\nabla^{(s)}$ and Δ_s. Since s is backward time, $\partial_s \mathbf{g}_s = 2\,\mathrm{Ric}_s$.

To differentiate a field along a curve, we use at each instant the connection corresponding to that time. If $\mathbf{X}(s) = \dot\gamma(s)$ and $\mathbf{Z}(s) \in T_{\gamma(s)}M$, we write $\nabla_{\mathbf{X}}\mathbf{Z} = D_s\mathbf{Z}$ for the total covariant derivative. In coordinates,

$$(D_s\mathbf{Z})^k = \frac{dZ^k}{ds} + \sum_{i,j=1}^{m} \Gamma_{ij}^k(s, \gamma(s)) X^i Z^j, \qquad k \in \{1, \dots, m\}. \tag{22.36}$$

In particular, $D_s^2\mathbf{Z}$ includes the time derivative of the Christoffel symbols appearing in $D_s\mathbf{Z}$. For two fields $\mathbf{Z}, \mathbf{W}$ along the curve, metric compatibility on each slice gives

$$\frac{d}{ds}\langle \mathbf{Z}, \mathbf{W} \rangle_{\mathbf{g}_s} = \langle D_s\mathbf{Z}, \mathbf{W} \rangle_{\mathbf{g}_s} + \langle \mathbf{Z}, D_s\mathbf{W} \rangle_{\mathbf{g}_s} + 2\,\mathrm{Ric}_s(\mathbf{Z}, \mathbf{W}). \tag{22.37}$$

Tensor derivatives involving only a spatial direction, such as $\nabla_{\mathbf{Y}} \mathrm{Ric}_s$, are taken with the connection $\nabla^{(s)}$ at fixed time s.

For an absolutely continuous curve $\gamma\colon [0, \bar\tau] \to M$ with $\gamma(0) = p$ and $\gamma(\bar\tau) = q$, define

$$\mathcal{L}(\gamma) := \int_0^{\bar\tau} \sqrt{s}\, \left(R_s(\gamma(s)) + |\dot\gamma(s)|_{\mathbf{g}_s}^2 \right) ds. \tag{22.38}$$

The $\mathcal{L}$-distance and the reduced distance based at (p, t_0) are

$$L(q, \bar\tau) := \inf_{\gamma \in \mathcal{A}_{p,q}(\bar\tau)} \mathcal{L}(\gamma), \qquad l(q, \bar\tau) := \frac{L(q, \bar\tau)}{2\sqrt{\bar\tau}}. \tag{22.39}$$

Here $\mathcal{A}_{p,q}(\bar\tau)$ is the set of absolutely continuous curves from $[0, \bar\tau]$ into M with these endpoints and finite weighted

energy, that is, $\int_0^{\bar\tau} \sqrt{s}\,|\dot\gamma(s)|^2_{\mathbf{g}_s}\,ds < \infty$.

The apparent singularity of the integrand at $s = 0$ disappears after reparametrization by the square root of time. This observation allows us to apply the variational method and the theory of ordinary differential equations directly.

Lemma 22.6.1. (Regular reparametrization and existence of minimizers) Let M be closed. If $\alpha(r) = \gamma(r^2)$ and $\mathbf{A}(r) = \alpha'(r)$, then

$$\mathcal{L}(\gamma) = \int_0^{\sqrt{\bar\tau}} \left(\frac{1}{2}|\mathbf{A}(r)|^2_{\mathbf{g}_{r^2}} + 2r^2 R_{r^2}(\alpha(r)) \right) dr. \tag{22.40}$$

For every $q \in M$ and every $0 < \bar\tau \le \tau_{\max}$, there exists an $\mathcal{L}$-minimizing curve from $(p, 0)$ to $(q, \bar\tau)$. Every minimizer is smooth in the variable r and satisfies

$$\nabla^{(r^2)}_{\mathbf{A}}\mathbf{A} - 2r^2 \nabla^{\mathbf{g}_{r^2}} R_{r^2} + 4r\,\mathrm{Ric}_{r^2}(\mathbf{A}, \cdot)^{\sharp_{\mathbf{g}_{r^2}}} = 0. \tag{22.41}$$

In the original variable, with $\mathbf{X} = \dot\gamma$, this equation becomes

$$\nabla^{(s)}_{\mathbf{X}}\mathbf{X} - \frac{1}{2}\nabla^{\mathbf{g}_s} R_s + \frac{1}{2s}\mathbf{X} + 2\,\mathrm{Ric}_s(\mathbf{X}, \cdot)^{\sharp_{\mathbf{g}_s}} = 0. \tag{22.42}$$

Proof. The change of variables $s = r^2$ gives $ds = 2r\,dr$ and $\mathbf{X}(r^2) = \mathbf{A}(r)/(2r)$. Direct substitution into (22.38) yields (22.40).

Fix a reference metric $\mathbf{h}$ on M. On the compact interval under consideration, the metrics $\mathbf{g}_{r^2}$ are uniformly equivalent to $\mathbf{h}$, and R_{r^2} is uniformly bounded. Thus there exist $c, C > 0$ such that

$$\mathcal{L}(\alpha) \ge c \int_0^{\sqrt{\bar\tau}} |\alpha'(r)|^2_{\mathbf{h}}\,dr - C. \tag{22.43}$$

Choose a smooth embedding $\iota\colon M \hookrightarrow \mathbb{R}^N$. The preceding estimate and the uniform equivalence of the metric induced by ι and $\mathbf{h}$ bound $\iota\circ\alpha_j$ in $H^1([0,\sqrt{\bar\tau}],\mathbb{R}^N)$ for every minimizing sequence. After passing to a subsequence, these curves converge weakly in H^1 and uniformly to a curve a. Since $\iota(M)$ is closed and each $\iota\circ\alpha_j$ takes values in $\iota(M)$, so does a; write $a = \iota\circ\alpha$, with values in $\iota(M)$. The endpoints are preserved by uniform convergence.

In charts around the compact image of α, the coefficients of $\mathbf{g}_{r^2}$ evaluated at α_j converge uniformly to those evaluated at α. The kinetic integrand is convex in the velocity; by lower semicontinuity of convex integrals,

$$\int_0^{\sqrt{\bar\tau}} |\alpha'|^2_{\mathbf{g}_{r^2}}\,dr \le \liminf_{j\to\infty} \int_0^{\sqrt{\bar\tau}} |\alpha_j'|^2_{\mathbf{g}_{r^2}}\,dr.$$

The curvature term converges by uniform convergence and the dominated convergence theorem. Thus α attains the infimum.

The first variation of (22.40) gives the Euler–Lagrange equation in the weak sense. To justify its regularity, consider a subinterval whose image lies in a chart and define the momenta

$$P_i(r) = \sum_{j=1}^{m} g_{ij}(r^2, \alpha(r))\alpha'^j(r), \qquad i \in \{1,\dots,m\}.$$

The weak equation is

$$P_i'(r) = \frac{1}{2}\sum_{j,k=1}^{m} \partial_i g_{jk}(r^2, \alpha(r))\alpha'^j(r)\alpha'^k(r) + 2r^2\partial_i R_{r^2}(\alpha(r)).$$

The right-hand side belongs to L^1 because $\alpha' \in L^2$. Thus each P_i has an absolutely continuous, and hence continuous, representative. The identity $\alpha'^j = \sum_{i=1}^{m} g^{ji}(r^2, \alpha)P_i$ then shows that α is C^1. The right-hand side of the equation for P_i' is now continuous, so P_i is C^1 and α is C^2. The resulting ordinary differential system has smooth coefficients, even at $r = 0$, and ordinary differential equation regularity gives smoothness up to the endpoints. Writing it with the connection of $\mathbf{g}_{r^2}$ gives (22.41). Finally, $\mathbf{A} = 2\sqrt{s}\,\mathbf{X}$ and

$$\nabla^{(r^2)}_{\mathbf{A}}\mathbf{A} = 2\mathbf{X} + 4r^2\nabla^{(s)}_{\mathbf{X}}\mathbf{X}.$$

Substituting this identity into (22.41) proves (22.42). □

Lemma 22.6.2. (The $\mathcal{L}$-exponential map) For each $v \in T_pM$, there exists a unique solution of (22.41) with

$$\alpha_v(0) = p, \qquad \alpha_v'(0) = 2v.$$

It is defined on all of $[0, \sqrt{\tau_{\max}}]$ and depends smoothly on (v, r). If $\gamma_v(s) = \alpha_v(\sqrt{s})$, then

$$\lim_{s\to 0^+} \sqrt{s}\,\dot\gamma_v(s) = v.$$

Define

$$\mathcal{L}\exp_p^\tau(v) := \gamma_v(\tau). \tag{22.44}$$

Proof. Existence, uniqueness, and smooth dependence near $r = 0$ follow from the usual theorem for ordinary differential systems applied to (22.41). To exclude blowup of the velocity on a finite interval, take the inner product of that equation with $\mathbf{A}$. The uniform bounds on ∇R and Ric give

$$\frac{d}{dr}|\mathbf{A}|^2_{\mathbf{g}_{r^2}} \le C\big(1 + |\mathbf{A}|^2_{\mathbf{g}_{r^2}}\big).$$

Grönwall's inequality keeps the velocity bounded. Since M is compact, the solution extends to $\sqrt{\tau_{\max}}$. The identity $\mathbf{A}(r) = 2r\mathbf{X}(r^2)$ gives the last assertion. □

Definition 22.6.3. ($\mathcal{L}$-injectivity domain and cut locus) For $\tau > 0$, let $\widetilde{U}_p(\tau) \subseteq T_pM$ be the set of vectors v satisfying the following two conditions:

(a) $d(\mathcal{L}\exp_p^\tau)_v$ is an isomorphism;

(b) there exists a neighborhood $\mathcal{V}$ of v in T_pM such that, for every $w \in \mathcal{V}$, the curve $\gamma_w|_{[0,\tau]}$ is the unique $\mathcal{L}$-minimizer between its endpoints.

Set

$$U_p(\tau) := \mathcal{L}\exp_p^\tau(\widetilde{U}_p(\tau)), \qquad \mathrm{Cut}_{\mathcal{L}}(p, \tau) := M \setminus U_p(\tau).$$

The second set is called the *$\mathcal{L}$-cut locus* based at (p, t_0) at backward time τ.

By definition, $\widetilde{U}_p(\tau)$ is open. The restriction of $\mathcal{L}\exp_p^\tau$ to this set is injective: two preimages of the same point would determine two minimizers, but each is unique. The inverse function theorem therefore shows that this restriction is a diffeomorphism onto the open set $U_p(\tau)$. In particular, $L(\cdot,\tau)$ and $l(\cdot,\tau)$ are smooth on $U_p(\tau)$.

Lemma 22.6.4. (First variation) Let γ_u be a variation of an $\mathcal{L}$–geodesic $\gamma=\gamma_0$, smooth in (u,r) with $r=\sqrt{s}$, that fixes the initial endpoint $\gamma_u(0)=p$, and let $\mathbf{Y}=\partial_u\gamma_u\big|_{u=0}$. Then $\mathbf{Y}(0)=0$ and

$$\delta\mathcal{L}(\gamma)=2\sqrt{s}\langle\mathbf{Y},\mathbf{X}\rangle_{\mathbf{g}_s}\Big|_0^{\bar\tau}+\int_0^{\bar\tau}\sqrt{s}\left\langle\mathbf{Y},\nabla R_s-\frac{1}{s}\mathbf{X}-2\nabla_{\mathbf{X}}\mathbf{X}-4\operatorname{Ric}_s(\mathbf{X},\cdot)^\sharp\right\rangle_{\mathbf{g}_s}ds. \tag{22.45}$$

In particular, if $q\in U_p(\tau)$ and γ is the regular minimizer ending at q, then

$$\nabla^{\mathbf{g}_\tau}L(q,\tau)=2\sqrt{\tau}\,\mathbf{X}(\tau),\qquad \partial_\tau L(q,\tau)=\sqrt{\tau}\big(R_\tau(q)-|\mathbf{X}(\tau)|^2_{\mathbf{g}_\tau}\big). \tag{22.46}$$

Consequently,

$$\nabla^{\mathbf{g}_\tau}l=\mathbf{X}(\tau),\qquad \partial_\tau l=-\frac{l}{2\tau}+\frac{1}{2}\big(R_\tau-|\nabla l|^2_{\mathbf{g}_\tau}\big) \tag{22.47}$$

on the regular set.

Proof. Let γ_u be a variation and let $\mathbf{Y}=\partial_u\gamma_u|_{u=0}$. Since $\partial_s\mathbf{g}_s=2\operatorname{Ric}_s$ and the coordinate fields of the variation commute,

$$\partial_u|\mathbf{X}|^2_{\mathbf{g}_s}=2\langle\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{X}\rangle_{\mathbf{g}_s}.$$

When integrating by parts, we must also differentiate the s-dependent metric:

$$\frac{d}{ds}\big(2\sqrt{s}\langle\mathbf{Y},\mathbf{X}\rangle\big)=\frac{1}{\sqrt{s}}\langle\mathbf{Y},\mathbf{X}\rangle+2\sqrt{s}\langle\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{X}\rangle+2\sqrt{s}\langle\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{X}\rangle+4\sqrt{s}\operatorname{Ric}_s(\mathbf{Y},\mathbf{X}).$$

Substituting this identity into the derivative of (22.38) gives (22.45). The endpoint term at $s=0$ vanishes because $\mathbf{Y}(0)=0$ and $\sqrt{s}\,\mathbf{X}(s)$ has a limit.

On the regular set, the inverse function theorem applied to $\mathcal{L}\exp_p^\tau$ produces a smooth family of minimizers as the endpoint varies. The boundary part of (22.45) gives the first identity in (22.46). Along the minimizer itself,

$$\frac{d}{d\tau}L(\gamma(\tau),\tau)=\sqrt{\tau}(R_\tau+|\mathbf{X}(\tau)|^2).$$

By the chain rule, the left-hand side is also $\partial_\tau L+\langle\nabla L,\mathbf{X}(\tau)\rangle$. The gradient identity gives the second formula. Dividing by $2\sqrt{\tau}$ proves (22.47). □

Lemma 22.6.5. ($\mathcal{L}$-Jacobi fields and index) Let $\mathbf{C}_s=\partial_s\nabla^{(s)}$ be the variation tensor of the connection. Along an $\mathcal{L}$–geodesic, the linearization of (22.42), with the total derivative in (22.36), is

$$\begin{aligned}\mathcal{J}_{\mathcal{L}}\mathbf{Y}:=D_s^2\mathbf{Y}+\operatorname{Rm}_{\mathbf{g}_s}(\mathbf{Y},\mathbf{X})\mathbf{X}-\mathbf{C}_s(\mathbf{Y},\mathbf{X})-\frac{1}{2}\nabla_{\mathbf{Y}}(\nabla R_s)+\frac{1}{2s}D_s\mathbf{Y}\\+2(\nabla_{\mathbf{Y}}\operatorname{Ric}_s)(\mathbf{X},\cdot)^\sharp+2\operatorname{Ric}_s(D_s\mathbf{Y},\cdot)^\sharp.\end{aligned} \tag{22.48}$$

The fields satisfying $\mathcal{J}_{\mathcal{L}}\mathbf{Y}=0$ that are smooth in $r=\sqrt{s}$ and vanish at $s=0$ are precisely the derivatives of families of $\mathcal{L}$–geodesics with fixed initial point p.

For fields $\mathbf{Y},\mathbf{Z}$ smooth in $r=\sqrt{s}$ and vanishing at $s=0$, define

$$\begin{aligned} I_{\mathcal{L}}(\mathbf{Y},\mathbf{Z}) := \int_0^\tau \sqrt{s}\Big\{ &\operatorname{Hess}_{\mathbf{g}_s} R_s(\mathbf{Y},\mathbf{Z}) + 2\langle \operatorname{Rm}_{\mathbf{g}_s}(\mathbf{Y},\mathbf{X})\mathbf{Z},\mathbf{X}\rangle_{\mathbf{g}_s} \\ &+ 2\langle D_s\mathbf{Y}, D_s\mathbf{Z}\rangle_{\mathbf{g}_s} + 2(\nabla_{\mathbf{X}}\operatorname{Ric}_s)(\mathbf{Y},\mathbf{Z}) - 2(\nabla_{\mathbf{Y}}\operatorname{Ric}_s)(\mathbf{X},\mathbf{Z}) \\ &- 2(\nabla_{\mathbf{Z}}\operatorname{Ric}_s)(\mathbf{X},\mathbf{Y})\Big\}\, ds. \end{aligned} \tag{22.49}$$

If γ_u is a variation of a minimizer, smooth in $(u,\sqrt{s})$, with fixed initial point and variation field $\mathbf{Y}$, then

$$\left.\frac{d^2}{du^2}\right|_{u=0} \mathcal{L}(\gamma_u) = 2\sqrt{\tau}\,\langle \nabla_{\mathbf{Y}}\mathbf{Y},\mathbf{X}\rangle_{\mathbf{g}_\tau}\Big|_{s=\tau} + I_{\mathcal{L}}(\mathbf{Y},\mathbf{Y}). \tag{22.50}$$

At a regular point,

$$\operatorname{Hess}_{\mathbf{g}_\tau} L(\mathbf{Y}(\tau),\mathbf{Y}(\tau)) \le I_{\mathcal{L}}(\mathbf{Y},\mathbf{Y}). \tag{22.51}$$

Moreover, $d(\mathcal{L}\exp_p^\tau)_v$ is singular if and only if there exists a nonzero Jacobi field vanishing at both endpoints.

Proof. The connection variation formula from the previous chapter, applied to $\partial_s \mathbf{g}_s = 2\operatorname{Ric}_s$, gives

$$\langle \mathbf{C}_s(\mathbf{U},\mathbf{V}),\mathbf{W}\rangle_{\mathbf{g}_s} = (\nabla_{\mathbf{U}}\operatorname{Ric}_s)(\mathbf{V},\mathbf{W}) + (\nabla_{\mathbf{V}}\operatorname{Ric}_s)(\mathbf{U},\mathbf{W}) - (\nabla_{\mathbf{W}}\operatorname{Ric}_s)(\mathbf{U},\mathbf{V}). \tag{22.52}$$

Here $\mathbf{U},\mathbf{V},\mathbf{W}$ are tangent vectors at the same point, and all derivatives on the right-hand side are spatial. On a variation $\gamma(u,s)$, write D_u for the covariant derivative in the $\mathbf{Y}=\partial_u\gamma$ direction at fixed s. The absence of torsion gives $D_u\mathbf{X} = D_s\mathbf{Y}$. For a field $\mathbf{Z}$ on the variation, the commutator is

$$D_u D_s\mathbf{Z} - D_s D_u\mathbf{Z} = \operatorname{Rm}_{\mathbf{g}_s}(\mathbf{Y},\mathbf{X})\mathbf{Z} - \mathbf{C}_s(\mathbf{Y},\mathbf{Z}). \tag{22.53}$$

To check the time-dependent term, write each derivative as in (22.36). The terms containing spatial derivatives of Γ_{ij}^k and products of two Christoffel symbols combine into the curvature. Differentiating $D_u\mathbf{Z}$ with respect to s also produces $\displaystyle\sum_{i,j=1}^m (\partial_s\Gamma_{ij}^k)Y^iZ^j$, for each $k\in\{1,\dots,m\}$, with a negative sign in the commutator. This is exactly the last summand in (22.53).

In particular,

$$D_u(D_s\mathbf{X}) = D_s^2\mathbf{Y} + \operatorname{Rm}(\mathbf{Y},\mathbf{X})\mathbf{X} - \mathbf{C}_s(\mathbf{Y},\mathbf{X}).$$

The derivative of $\operatorname{Ric}_s(\mathbf{X},\cdot)^\sharp$ in the u direction is $(\nabla_{\mathbf{Y}}\operatorname{Ric}_s)(\mathbf{X},\cdot)^\sharp + \operatorname{Ric}_s(D_s\mathbf{Y},\cdot)^\sharp$. Differentiating the remaining terms of (22.42) yields (22.48).

The equation in $r=\sqrt{s}$ is a regular second-order linear ordinary differential system. If $\mathbf{Y}(0)=0$, it is determined by $\lim_{s\to0^+}\sqrt{s}\,D_s\mathbf{Y}$. Indeed, for a family of initial data $v(u)$, $D_r\mathbf{Y}(0) = 2v'(0)$, and this limit is $v'(0)$. Smooth dependence and uniqueness of the initial value problem establish the correspondence with variations of $\mathcal{L}$–geodesics.

Now differentiate (22.45) along the central curve. Its Euler–Lagrange equation makes the integrand of the

first variation vanish. We obtain

$$\delta^2\mathcal{L} = 2\sqrt{\tau}\big(\langle\nabla_{\mathbf{Y}}\mathbf{Y},\mathbf{X}\rangle + \langle D_s\mathbf{Y},\mathbf{Y}\rangle\big)_{s=\tau} - 2\int_0^\tau \sqrt{s}\,\langle \mathcal{J}_{\mathcal{L}}\mathbf{Y},\mathbf{Y}\rangle\,ds.$$

To integrate the last term, use (22.37) with $\mathbf{Z} = D_s\mathbf{Y}$ and $\mathbf{W} = \mathbf{Y}$:

$$\frac{d}{ds}\langle D_s\mathbf{Y},\mathbf{Y}\rangle_{\mathbf{g}_s} = \langle D_s^2\mathbf{Y},\mathbf{Y}\rangle_{\mathbf{g}_s} + |D_s\mathbf{Y}|^2_{\mathbf{g}_s} + 2\,\mathrm{Ric}_s(D_s\mathbf{Y},\mathbf{Y}). \tag{22.54}$$

On the other hand, setting $\mathbf{U} = \mathbf{W} = \mathbf{Y}$ and $\mathbf{V} = \mathbf{X}$ in (22.52), the two terms involving $\nabla_{\mathbf{Y}}\mathrm{Ric}_s$ cancel:

$$\langle \mathbf{C}_s(\mathbf{Y},\mathbf{X}),\mathbf{Y}\rangle = (\nabla_{\mathbf{X}}\mathrm{Ric}_s)(\mathbf{Y},\mathbf{Y}).$$

Consequently,

$$\begin{aligned}-2\sqrt{s}\,\langle \mathcal{J}_{\mathcal{L}}\mathbf{Y},\mathbf{Y}\rangle = &-\frac{d}{ds}\big(2\sqrt{s}\langle D_s\mathbf{Y},\mathbf{Y}\rangle\big)\\ &+\sqrt{s}\Big\{\mathrm{Hess}\,R_s(\mathbf{Y},\mathbf{Y}) + 2\langle \mathrm{Rm}(\mathbf{Y},\mathbf{X})\mathbf{Y},\mathbf{X}\rangle + 2|D_s\mathbf{Y}|^2\\ &\qquad + 2(\nabla_{\mathbf{X}}\mathrm{Ric}_s)(\mathbf{Y},\mathbf{Y}) - 4(\nabla_{\mathbf{Y}}\mathrm{Ric}_s)(\mathbf{X},\mathbf{Y})\Big\}.\end{aligned}$$

The total derivative term cancels the summand $2\sqrt{\tau}\langle D_s\mathbf{Y},\mathbf{Y}\rangle$ at the final endpoint. At the initial endpoint, smoothness in r and $\mathbf{Y}(0) = 0$ imply $\mathbf{Y} = O(\sqrt{s})$ and $D_s\mathbf{Y} = O(s^{-1/2})$; hence $2\sqrt{s}\langle D_s\mathbf{Y},\mathbf{Y}\rangle \to 0$. These same bounds justify convergence of the improper integrals. This proves (22.50), and polarization gives (22.49). If both fields also vanish at $s = \tau$, bilinear integration by parts gives

$$I_{\mathcal{L}}(\mathbf{Y},\mathbf{Z}) = -2\int_0^\tau \sqrt{s}\,\langle \mathcal{J}_{\mathcal{L}}\mathbf{Y},\mathbf{Z}\rangle\,ds.$$

The symmetry of this expression also follows directly from (22.49) and the curvature symmetries.

To estimate the Hessian, realize $\mathbf{Y}$ by a variation with fixed initial point and final endpoint $q(u)$. We have $L(q(u),\tau) \le \mathcal{L}(\gamma_u)$, with equality at $u = 0$. The second derivative of the difference is nonnegative. When we use $\nabla L = 2\sqrt{\tau}\,\mathbf{X}(\tau)$, the term containing the acceleration of $q(u)$ is the same on both sides and cancels. This proves (22.51). Finally, $d(\mathcal{L}\exp_p^\tau)_v(w) = \mathbf{Y}_w(\tau)$, where $\lim_{s\to 0^+}\sqrt{s}D_s\mathbf{Y}_w = w$. Uniqueness for the regular system implies that $w \neq 0$ corresponds to a nonzero field, which proves the assertion about the differential of the exponential map. □

Lemma 22.6.6. (Identities for K and a Hessian estimate) Let $q \in U_p(\tau)$ and let γ be the regular minimizer ending at q. Along γ, define

$$\mathcal{H}(\mathbf{X}) := -\partial_s R_s - \frac{R_s}{s} - 2\langle\nabla R_s,\mathbf{X}\rangle + 2\,\mathrm{Ric}_s(\mathbf{X},\mathbf{X}), \tag{22.55}$$

$$K(\tau) := \int_0^\tau s^{3/2}\mathcal{H}(\mathbf{X}(s))\,ds. \tag{22.56}$$

Then

$$\partial_\tau l = R_\tau - \frac{l}{\tau} + \frac{K(\tau)}{2\tau^{3/2}}, \tag{22.57}$$

$$|\nabla^{\mathbf{g}_\tau} l|^2_{\mathbf{g}_\tau} = -R_\tau + \frac{l}{\tau} - \frac{K(\tau)}{\tau^{3/2}}, \tag{22.58}$$

$$\Delta_\tau l \le \frac{m}{2\tau} - R_\tau - \frac{K(\tau)}{2\tau^{3/2}}. \tag{22.59}$$

Consequently, on the regular set,

$$\partial_\tau l - \Delta_\tau l + |\nabla l|^2_{\mathbf{g}_\tau} - R_\tau + \frac{m}{2\tau} \ge 0. \tag{22.60}$$

Proof. Differentiate $R_s(\gamma(s)) + |\mathbf{X}(s)|^2$. The time variation of the metric contributes $2\,\mathrm{Ric}_s(\mathbf{X},\mathbf{X})$, and equation (22.42) gives

$$\frac{d}{ds}\left(R_s + |\mathbf{X}|^2\right) = -\mathcal{H}(\mathbf{X}) - \frac{1}{s}(R_s + |\mathbf{X}|^2). \tag{22.61}$$

Thus

$$\frac{d}{ds}\left[s^{3/2}(R_s + |\mathbf{X}|^2)\right] = \frac{1}{2}\sqrt{s}(R_s + |\mathbf{X}|^2) - s^{3/2}\mathcal{H}(\mathbf{X}).$$

Integration between 0 and τ, together with $\lim_{s\to 0^+} s^{3/2}(R_s + |\mathbf{X}|^2) = 0$, yields

$$\tau^{3/2}(R_\tau + |\mathbf{X}(\tau)|^2) = \frac{1}{2}L(q,\tau) - K(\tau). \tag{22.62}$$

The second formula in (22.46) and the identity $\nabla l = \mathbf{X}(\tau)$ transform (22.62) into (22.57) and (22.58).
It remains to estimate the trace of the Hessian. Let $(\mathbf{e}_a)_{a=1}^m$ be an orthonormal basis of T_qM, and let $\mathbf{Y}_a$ be the comparison field determined by

$$\mathbf{Y}_a(\tau) = \mathbf{e}_a, \qquad \nabla_{\mathbf{X}}\mathbf{Y}_a = -\mathrm{Ric}_s(\mathbf{Y}_a,\cdot)^\sharp + \frac{1}{2s}\mathbf{Y}_a. \tag{22.63}$$

Differentiating $\langle \mathbf{Y}_a, \mathbf{Y}_b\rangle_{\mathbf{g}_s}$ and using $\partial_s \mathbf{g}_s = 2\,\mathrm{Ric}_s$ gives

$$\langle \mathbf{Y}_a(s), \mathbf{Y}_b(s)\rangle_{\mathbf{g}_s} = \frac{s}{\tau}\delta_{ab}.$$

Thus the fields vanish at $s = 0$, and $\mathbf{W}_a(s) = \sqrt{\tau/s}\,\mathbf{Y}_a(s)$ form an orthonormal frame satisfying $\nabla_{\mathbf{X}}\mathbf{W}_a = -\mathrm{Ric}_s(\mathbf{W}_a,\cdot)^\sharp$ for $a \in \{1,\dots,m\}$. In the variable $r = \sqrt{s}$, this equation is $D_r\mathbf{W}_a = -2r\,\mathrm{Ric}_{r^2}(\mathbf{W}_a,\cdot)^\sharp$; its coefficients are smooth up to $r = 0$. Hence the fields $\mathbf{W}_a$ extend smoothly, and $\mathbf{Y}_a = (r/\sqrt{\tau})\mathbf{W}_a$ are admissible variations. Inequality (22.51) implies

$$\Delta_\tau L(q,\tau) \le \sum_{a=1}^m I_{\mathcal{L}}(\mathbf{Y}_a, \mathbf{Y}_a).$$

Substitute $\mathbf{Y}_a = \sqrt{s/\tau}\,\mathbf{W}_a$ into (22.49). The required contractions are

$$\sum_{a=1}^m \mathrm{Hess}\, R_s(\mathbf{W}_a,\mathbf{W}_a) = \Delta_s R_s,$$

$$\sum_{a=1}^m \langle \mathrm{Rm}(\mathbf{W}_a,\mathbf{X})\mathbf{W}_a, \mathbf{X}\rangle = -\mathrm{Ric}_s(\mathbf{X},\mathbf{X}),$$

$$\sum_{a=1}^m \left|\frac{1}{2s}\mathbf{W}_a - \mathrm{Ric}_s(\mathbf{W}_a,\cdot)^\sharp\right|^2 = \frac{m}{4s^2} - \frac{R_s}{s} + |\mathrm{Ric}_s|^2,$$

$$\sum_{a=1}^{m} (\nabla_{\mathbf{W}_a} \mathrm{Ric}_s)(\mathbf{X}, \mathbf{W}_a) = \frac{1}{2} \langle \nabla R_s, \mathbf{X} \rangle.$$

The last equality is the contracted Bianchi identity. Together with $\partial_s R_s = -\Delta_s R_s - 2|\mathrm{Ric}_s|^2$, these identities give, term by term,

$$\sum_{a=1}^{m} I_{\mathcal{L}}(\mathbf{Y}_a, \mathbf{Y}_a) = \frac{1}{\tau} \int_0^\tau \left[\frac{m}{2\sqrt{s}} - 2\sqrt{s} R_s - s^{3/2} \partial_s R_s - 2 s^{3/2} \mathrm{Ric}_s(\mathbf{X}, \mathbf{X}) \right] ds. \tag{22.64}$$

Here all quantities are evaluated at $\gamma(s)$. Now

$$\frac{d}{ds} \left(2 s^{3/2} R_s(\gamma(s)) \right) = 3\sqrt{s} R_s + 2 s^{3/2} (\partial_s R_s + \langle \nabla R_s, \mathbf{X} \rangle).$$

The definition of $\mathcal{H}$ allows us to rewrite the part of the preceding integral that does not involve the dimension as

$$-2\tau^{3/2} R_\tau(q) - K(\tau).$$

Since $\displaystyle\int_0^\tau m/(2\sqrt{s})\, ds = m\sqrt{\tau}$, we conclude that

$$\Delta_\tau L(q, \tau) \le \frac{m}{\sqrt{\tau}} - 2\sqrt{\tau} R_\tau(q) - \frac{K(\tau)}{\tau}.$$

Dividing by $2\sqrt{\tau}$ proves (22.59). Combining the three formulas gives (22.60). □

The preceding inequality was obtained where the minimizer is unique and the exponential map has no conjugate points. To integrate it over the entire manifold, we need to control the reduced function on its cut locus.

Definition 22.6.7. (Local semiconcavity) Let $\Omega \subseteq \mathbb{R}^m$ be open. A continuous function $u : \Omega \longrightarrow \mathbb{R}$ is *locally semiconcave* if each point has a convex neighborhood $V \subseteq \Omega$ on which, for some constant $C \ge 0$, the function

$$x \longmapsto u(x) - \frac{C}{2}|x|^2$$

is concave. On a smooth manifold, this condition is imposed on the coordinate representations of u. The constants may depend on the chosen chart and neighborhood; no global constant is required.

Concavity of a function v on V means that

$$v((1-t)x + ty) \ge (1-t)v(x) + tv(y), \qquad x, y \in V, \quad 0 \le t \le 1.$$

For a continuous function, it is enough to check this inequality at midpoints: repeated application of that inequality gives all dyadic coefficients, and continuity allows passage to any $t \in [0,1]$. We will use this observation both to recognize semiconcavity and to check that the definition is independent of coordinates.

Lemma 22.6.8. (Upper supports and changes of coordinates) Let $V \subseteq \mathbb{R}^m$ be open and convex, and let

$u \in C(V)$. Suppose that for each $x \in V$ there exists $\varphi_x \in C^2(V)$ such that $u \le \varphi_x$ on V, $u(x) = \varphi_x(x)$, and

$$D^2\varphi_x(y)[e,e] \le C|e|^2, \qquad y \in V, \quad e \in \mathbb{R}^m,$$

with the same constant $C \ge 0$. Then $u - C|x|^2/2$ is concave on V. Moreover, every locally semiconcave function is locally Lipschitz, and local semiconcavity is preserved under composition with a C^2 change of coordinates.

Proof. Let $x-h, x+h \in V$. Since V is convex, the segment joining these points lies in V. Taylor's formula with integral remainder, applied to $t \mapsto \varphi_x(x+th)$, gives

$$\varphi_x(x+h) + \varphi_x(x-h) - 2\varphi_x(x) = \int_{-1}^{1} (1-|t|) D^2\varphi_x(x+th)[h,h]\,dt \le C|h|^2.$$

The properties of the upper support imply

$$u(x+h) + u(x-h) - 2u(x) \le C|h|^2.$$

After subtracting $C|x|^2/2$, this inequality becomes midpoint concavity. The preceding observation proves the first assertion.

Let us prove local Lipschitz regularity. It suffices to prove it for a continuous concave function v, since the quadratic term is smooth. Choose $\overline{B(x_0,2r)} \subset V$ and set $M_r = \max_{\overline{B(x_0,2r)}} |v|$. If $x, y \in B(x_0,r)$ and $x \ne y$, define $d = |y-x|$ and $e = (y-x)/d$. The points $x-re$ and $y+re$ belong to $B(x_0,2r)$. Concavity along the line containing these four points gives

$$\frac{v(x) - v(x-re)}{r} \ge \frac{v(y)-v(x)}{d} \ge \frac{v(y+re)-v(y)}{r}.$$

Both outer quotients have absolute value at most $2M_r/r$. Therefore, $|v(y)-v(x)| \le (2M_r/r)|y-x|$.

Now let κ be a C^2 change of coordinates. Shrink its domains so that u is L–Lipschitz on a convex open set containing the image under consideration, $u - C|x|^2/2$ is concave there, and $\|D\kappa\| \le K_0$ and $\|D^2\kappa\| \le K_1$. This is possible by working in neighborhoods with compact closure contained in the domains. For $x-h, x+h$ in a convex ball in the smaller domain, set

$$z_\pm = \kappa(x \pm h), \qquad z_0 = \kappa(x), \qquad z_* = (z_+ + z_-)/2.$$

Semiconcavity at the points z_-, z_+ and the Lipschitz bound give

$$u(z_+) + u(z_-) - 2u(z_0) \le \frac{C}{4}|z_+ - z_-|^2 + 2L|z_* - z_0|.$$

The mean value theorem and the same Taylor formula imply

$$|z_+ - z_-| \le 2K_0|h|, \qquad |z_* - z_0| \le \frac{K_1}{2}|h|^2.$$

Consequently, the second increment of $u \circ \kappa$ is bounded above by $(CK_0^2 + LK_1)|h|^2$. The midpoint criterion proves its semiconcavity. Applying the same argument to κ^{-1} establishes the chart independence asserted in the definition. □

Lemma 22.6.9. (Regularity and cut locus of the reduced distance) For each compact interval $[\tau_0, \tau_1] \subset (0, \tau_{\max}]$, the functions L and l are locally Lipschitz on $M \times [\tau_0, \tau_1]$. For each $\tau > 0$, $U_p(\tau)$ is open, L and l are smooth there, and $\mathrm{Cut}_{\mathcal{L}}(p, \tau)$ has measure zero. Moreover, the restriction $l(\cdot, \tau)$ is locally semiconcave.

Proof. Fix a compact spacetime set $K \times [\tau_0, \tau_1] \subset M \times (0, \tau_{\max}]$. A comparison curve obtained by traversing a geodesic of a fixed metric, reparametrized by $r = \sqrt{s}$, gives a uniform upper bound for L on this compact set. The coercivity estimate (22.43) then implies

$$\int_0^{\sqrt{\tau}} |\mathbf{A}(r)|^2_{\mathbf{g}_{r^2}} \, dr \le C_K \tag{22.65}$$

for every minimizer with endpoint in the compact set. Since $\sqrt{\tau} \ge \sqrt{\tau_0}$, there exists $r_* \in [\sqrt{\tau}/2, 3\sqrt{\tau}/4]$ such that $|\mathbf{A}(r_*)|^2 \le C_K$. Equation (22.41) and the identity $\partial_r \mathbf{g}_{r^2} = 4r \, \mathrm{Ric}_{r^2}$ give, for $E(r) = |\mathbf{A}(r)|^2_{\mathbf{g}_{r^2}}$,

$$E'(r) = 4r^2 \langle \nabla R_{r^2}, \mathbf{A} \rangle - 4r \, \mathrm{Ric}_{r^2}(\mathbf{A}, \mathbf{A}), \qquad |E'(r)| \le C_K (1 + E(r)).$$

Integrate from r_* in both directions. Grönwall's inequality bounds E on all of $[0, \sqrt{\tau}]$, including the initial datum and the final velocity. Since $\tau \ge \tau_0$, $|\mathbf{X}(\tau)| = |\mathbf{A}(\sqrt{\tau})|/(2\sqrt{\tau})$ is also uniformly bounded. The initial data $\mathbf{A}(0)/2 \in T_pM$ lie in a compact ball independent of the minimizer.

Now fix a minimizer γ up to (q, τ). Choose a number $d > 0$ smaller than $\tau_0/4$ and set $s_0 = \tau - d$. We will shrink d once, depending on the preceding uniform bounds. On the interval $[s_0, \tau]$, the Lagrangian

$$(x, V, s) \longmapsto \sqrt{s}\big(R_s(x) + |V|^2_{\mathbf{g}_s}\big)$$

is smooth and uniformly convex in V. In normal coordinates for a fixed metric around q, the Euler–Lagrange equation is a regular ordinary differential system. By smooth dependence of its solutions and the inverse function theorem, if d is sufficiently small, the map sending the initial velocity at s_0 to the endpoint at τ is a diffeomorphism between neighborhoods. In coordinates, its differential is $dI + O(d^2)$, uniformly over the compact set of positions, times, and velocities under consideration. Thus the choice of d and the bounds on the derivatives of the inverse can be made uniform. In particular, the segment $\gamma|_{[s_0,\tau]}$ is the unique short extremal joining its endpoints, has no conjugate point, and depends smoothly on the final endpoint and final time.

Let us make the control of the domain of the inverse precise. In a fixed chart, denote by $E_\sigma(w)$ the endpoint at time σ of the solution starting at $\gamma(s_0)$ with velocity w, and let w_0 be the velocity of the chosen segment. The velocities w_0 remain in a bounded set. Choose a radius $b > 0$ and shrink d so that, on $\overline{B(w_0, b)}$, all these solutions remain in the chart and

$$\big\| d^{-1} D_w E_\tau(w) - I \big\| \le \frac{1}{4}.$$

This choice can be made uniform: the positions, times, and velocities range over compact sets in a finite atlas, and the derivatives of the ordinary differential system are bounded on a neighborhood of these compact sets. By uniform continuity, for $|\sigma - \tau|$ sufficiently small, the preceding bound remains at most $1/2$ and $|E_\sigma(w_0) - E_\tau(w_0)| < db/8$. If, in addition, $|y - E_\tau(w_0)| < db/8$, the map

$$w \longmapsto w + d^{-1}(y - E_\sigma(w))$$

is a contraction with constant $1/2$ that maps $\overline{B(w_0, b)}$ into itself: its distance from w_0 is at most $b/2 + b/4$. Its fixed point solves $E_\sigma(w) = y$. We thus obtain a spacetime neighborhood of uniform radius on which the inverse is defined. Its first two derivatives are uniformly bounded after shrinking this neighborhood once more, by the formula for the derivative of the inverse and the bounds on the first two derivatives of E_σ. In particular, after fixing a spacetime point and shrinking a convex coordinate neighborhood around it, all supports constructed from points of this neighborhood will be defined throughout it. Changes between the charts of the finite atlas preserve the bounds, since their first two derivatives are bounded on the compact sets under consideration.

After shrinking a neighborhood of (q, τ), denote the action of this short extremal by $L_{(\gamma(s_0),s_0)}(y, \sigma)$ and set

$$\Phi_{s_0}(y, \sigma) := \mathcal{L}(\gamma|_{[0,s_0]}) + L_{(\gamma(s_0),s_0)}(y, \sigma).$$

Concatenating the first portion of γ with the short extremal ending at (y, σ) gives an admissible curve; hence $L(y, \sigma) \le \Phi_{s_0}(y, \sigma)$, with equality at (q, τ). Smooth dependence of this extremal on the endpoints and the uniform bounds for the ordinary differential system and its inverse give bounds on its first two derivatives with respect to (y, σ). Differentiating the action integral over the short interval gives a uniform bound on the spatial Hessian of Φ_{s_0} and on its first spatial and time derivatives. The interval is bounded away from $s = 0$, and its length has a positive lower bound in the chosen neighborhood, so these constants remain finite.

Work in one of the common convex neighborhoods obtained above, and write $z = (x, \tau)$ for its spacetime coordinates. Denote by Φ_z the support constructed from z. If z' lies in this same neighborhood, the uniform bound $|D\Phi_z| \le A$ implies

$$L(z') - L(z) \le \Phi_z(z') - \Phi_z(z) \le A|z' - z|.$$

Interchanging z and z' gives the reverse inequality. This proves local Lipschitz regularity of L and, in particular, its continuity. On each time slice, the supports have a common Hessian bound and are defined on a common domain. Lemma 22.6.8 then gives semiconcavity of $L(\cdot, \tau)$. Since $\tau \ge \tau_0 > 0$, multiplication by $1/(2\sqrt{\tau})$ preserves local spacetime Lipschitz regularity and spatial semiconcavity. The constants for both properties of l can be chosen uniformly on compact sets bounded away from $\tau = 0$.

It remains to determine the size of the cut locus. We already observed after Definition 22.6.3 that $U_p(\tau)$ is open and the functions are smooth there. Now let $q \notin U_p(\tau)$ and suppose that q is not a critical value of $\mathcal{L}\exp_p^\tau$. If $L(\cdot, \tau)$ were differentiable at q, the first variation would force all minimizers ending at q to have the same final velocity. Backward uniqueness for (22.42) would show that there is only one minimizer, with regular initial datum v.

This uniqueness persists in a neighborhood. Indeed, otherwise there would exist $q_j \to q$ and minimizers with initial data outside a neighborhood of v, or two distinct data with the same endpoint. The bounds (22.65) and the preceding Grönwall argument make the family of these initial data compact. A subsequence would converge to a datum determining a minimizer up to q; by uniqueness, the limit would be v. For large j, all data would then lie in a neighborhood on which $\mathcal{L}\exp_p^\tau$ is a diffeomorphism, a contradiction. Thus q would belong to $U_p(\tau)$.

The same compactness argument for the initial data applies when both the endpoint and the final time vary. Therefore, uniqueness of a nonconjugate minimizer persists in a spacetime neighborhood. The inverse function theorem with time as a parameter shows that $\mathcal{U} = \{(q, \tau) \in M \times (0, \tau_{\max}] \mid q \in U_p(\tau)\}$ is open in that space and that L and l are smooth on $\mathcal{U}$.

We have proved that the cut locus is contained in the union of the critical values of $\mathcal{L}\exp_p^\tau$ and the set of points where $L(\cdot,\tau)$ is not differentiable. The first set has measure zero by Sard's theorem, and the second by Rademacher's theorem. This completes the proof. □

Lemma 22.6.10. (Semiconcavity and the distributional Hessian) Let $(N^m,\mathbf{h})$ be a smooth Riemannian manifold without boundary, and let u be locally semiconcave. Then $u \in W^{1,\infty}_{\mathrm{loc}}(N)$, its distributional Hessian is a symmetric tensor with values in Radon measures, and the singular part of $\mathrm{Hess}_{\mathbf{h}}\, u$ is negative semidefinite. In particular, the singular part of $\Delta_{\mathbf{h}} u$ is a nonpositive measure.

Proof. Fix a chart and write $\widehat{u} = u\circ\varphi^{-1}$. Shrink its image to a convex open set V on which $v(x) = \widehat{u}(x) - C|x|^2/2$ is concave. By Lemma 22.6.8, $\widehat{u}$ is locally Lipschitz. Theorem 7.2.1 gives its almost everywhere derivatives, which are locally bounded. They are also its weak derivatives: for a test function $\psi \in C_c^\infty(V)$, transfer the quotient $[\widehat{u}(x+te_j)-\widehat{u}(x)]/t$ to ψ by a change of variables. The Lipschitz bound and dominated convergence allow us to let $t \to 0$ and give

$$\int_V D_j\widehat{u}\,\psi\, d\lambda_m = -\int_V \widehat{u}\,\partial_j\psi\, d\lambda_m.$$

This proves that $u \in W^{1,\infty}_{\mathrm{loc}}(N)$.

Let us prove that the second derivatives of v are measures. For a constant vector $e \in \mathbb{R}^m$, concavity implies $v(x+te)+v(x-te)-2v(x) \le 0$ whenever the segment lies in V. If $\psi \in C_c^\infty(V)$ is nonnegative and $t \neq 0$ is sufficiently small, integrate this inequality after multiplying by ψ/t^2. Transferring the increments to the test function gives

$$\int_V v(x)\frac{\psi(x+te)+\psi(x-te)-2\psi(x)}{t^2}\, d\lambda_m(x) \le 0.$$

The quotient converges uniformly to $D_{ee}\psi$, with support in a fixed compact set. Since v is locally integrable, passage to the limit gives

$$\langle D_{ee}v, \psi\rangle \le 0.$$

Thus $-D_{ee}v$ is a positive distribution. Theorem 4.6.3 represents it by a positive Radon measure. In particular, if $K \subset V$ is compact and $0 \le \chi \in C_c^\infty(V)$ equals one near K, positivity gives the order-zero bound

$$|\langle D_{ee}v,\psi\rangle| \le \|\psi\|_\infty \langle -D_{ee}v, \chi\rangle, \qquad \mathrm{supp}(\psi) \subseteq K.$$

Here positivity is applied to $\|\psi\|_\infty\chi + \psi$ and $\|\psi\|_\infty\chi - \psi$.

Distributional derivatives commute. The polarization identity

$$2D_{ij}v = D_{e_i+e_j,e_i+e_j}v - D_{ii}v - D_{jj}v$$

shows that all $D_{ij}v$ are local signed Radon measures, in the sense of Definition B.5.22. Denote them by μ_{ij}. The matrix (μ_{ij}) is symmetric and, for every $e \in \mathbb{R}^m$,

$$\sum_{i,j=1}^m e_i e_j \mu_{ij} = D_{ee}v \le 0.$$

This inequality also determines the sign for directions varying with the point. To see this, on an open set

V_0 with compact closure contained in V, take the finite positive measure

$$\nu = \lambda_m \restriction_{V_0} + \sum_{i,j=1}^{m} |\mu_{ij}| \restriction_{V_0}.$$

Theorem B.6.29, applied to the positive and negative variations, gives densities $a_{ij} = d\mu_{ij}/d\nu$ and $r = d\lambda_m/d\nu$. For each vector $e \in \mathbb{Q}^m$, we have $\sum_{i,j=1}^{m} e_i e_j a_{ij} \le 0$ outside a ν-null set. Since $\mathbb{Q}^m$ is countable, we can exclude a single null set for all these directions. By density, the symmetric matrix $\mathbf{a} = (a_{ij})$ is negative semidefinite at ν-almost every point.

Set $S = \{x \in V_0 \mid r(x) = 0\}$. Then $\lambda_m(S) = 0$ and, on $V_0 \setminus S$, the preceding measures have densities a_{ij}/r with respect to λ_m. Thus their decomposition into absolutely continuous and singular parts is

$$\mu_{ij} = \mathbf{1}_{V_0 \setminus S} \frac{a_{ij}}{r}\, d\lambda_m + \mathbf{1}_S a_{ij}\, d\nu.$$

The measures in this equality are restricted to V_0, and the first coefficient is extended by zero on S. The second summand is the singular part, and its matrix has the sign already established. Since $D_{ij}\widehat{u} = \mu_{ij} + C\delta_{ij}\, d\lambda_m$, the quadratic term does not change this singular part.

We now pass to the Riemannian Hessian, taking care with the measure used to identify functions with distributions. In this chart, $D_{ij}\widehat{u}$ is a measure written with the Euclidean convention, whereas $d\lambda_{\mathbf{h}} = \sqrt{\det(\mathbf{h})}\, d\lambda_m$. If H_{ij} denotes the coefficient of the distributional Hessian when represented by measures, then

$$\mathsf{H}_{ij} = \sqrt{\det(\mathbf{h})}\, D_{ij}\widehat{u} - \left(\sum_{k=1}^{m} \Gamma_{ij}^{k} D_k \widehat{u} \right) d\lambda_{\mathbf{h}}.$$

Indeed, for a smooth function this is the coordinate formula $\operatorname{Hess}_{\mathbf{h}} u = \nabla du$ multiplied by the volume element. For the present function, regularize $\widehat{u}$ by convolution in a smaller chart. The approximations converge in $W^{1,1}$ on compact sets, and their second derivatives converge distributionally to $D_{ij}\widehat{u}$. Continuity of distributional differentiation and multiplication by smooth coefficients allows passage to the limit in the formula. On overlaps, we also pass to the limit in the tensor transformation law for the smooth Hessians, since the coordinate changes and their coefficients are smooth. The resulting measures therefore represent the intrinsic tensor ∇du and are independent of the chart.

The term containing the Christoffel symbols has locally bounded density. Moreover, λ_m and $\lambda_{\mathbf{h}}$ have the same null sets. Consequently,

$$\mathsf{H}^{\mathrm{s}}_{ij} = \sqrt{\det(\mathbf{h})}\, \mathbf{1}_S a_{ij}\, d\nu.$$

The square root of the determinant is positive, and $\mathbf{a}$ is negative semidefinite; this proves the assertion about the singular Hessian. Its contraction with the inverse metric is

$$(\Delta_{\mathbf{h}} u)^{\mathrm{s}} = \sqrt{\det(\mathbf{h})}\, \mathbf{1}_S \left(\sum_{i,j=1}^{m} h^{ij} a_{ij} \right) d\nu \le 0.$$

The sign follows because the trace of the product of a positive definite matrix and a negative semidefinite matrix is nonpositive.

We also record the local control that we will need as time varies. If a family of functions u_τ has uniform

bounds on $\|\widehat{u}_\tau\|_\infty$, its Lipschitz constant, and its semiconcavity constant in a common chart, the total variations of the Hessian measures are uniformly bounded on every compact subset of the interior. Indeed, for $v_\tau = \widehat{u}_\tau - C|x|^2/2$ and the preceding cutoff function χ,

$$(-D_{ee}v_\tau)(K) \le \langle -D_{ee}v_\tau, \chi\rangle \le \|v_\tau\|_{L^\infty(\operatorname{supp}\chi)}\|D_{ee}\chi\|_{L^1(\mathbb{R}^m)}.$$

Apply this bound to the finitely many directions e_i and $e_i + e_j$ and use polarization. The quadratic term and the Christoffel terms are controlled by the bounds already stated. This control remains uniform for a smooth family of metrics on a compact interval, since their coefficients, inverses, and first derivatives are bounded in the smaller chart. □

Proposition 22.6.11. (Variational geometry of $\mathcal{L}$) Let M be a smooth closed connected manifold, and let $\mathbf{g}(t)$ be a smooth Ricci flow on an interval containing $[t_0 - \tau_{\max}, t_0]$. For each $q \in M$ and $0 < \tau \le \tau_{\max}$, there exists a minimizer of $\mathcal{L}$. On the regular set, (22.46) and (22.57)–(22.59) hold. On the entire manifold, the reduced function satisfies, in the distributional sense,

$$\partial_\tau l - \Delta_\tau l + |\nabla^{\mathbf{g}_\tau} l|^2_{\mathbf{g}_\tau} - R_\tau + \frac{m}{2\tau} \ge 0. \tag{22.66}$$

Proof. Existence and the formulas on the regular set were proved in the preceding lemmas. By Lemma 22.6.9, l is locally Lipschitz in spacetime, and the cut locus has measure zero on each slice. Fubini's theorem then shows that (22.60) holds almost everywhere on $M \times (0, \tau_{\max}]$.
Fix a compact interval $J \subset (0, \tau_{\max})$. For almost every $\tau \in J$, write the Laplacian measure of $l(\cdot, \tau)$ as

$$\Delta_\tau l = (\Delta_\tau l)_{\mathrm{ac}}\, d\lambda_{\mathbf{g}_\tau} + \mu^{\mathrm{s}}_\tau.$$

Lemma 22.6.10 gives $\mu^{\mathrm{s}}_\tau \le 0$. On the regular open set, the density $(\Delta_\tau l)_{\mathrm{ac}}$ agrees with the classical Laplacian. Since the complement of that open set has volume measure zero, the almost everywhere inequality obtained above implies

$$\partial_\tau l - (\Delta_\tau l)_{\mathrm{ac}} + |\nabla^{\mathbf{g}_\tau} l|^2_{\mathbf{g}_\tau} - R_\tau + \frac{m}{2\tau} \ge 0$$

at almost every point of each of these slices.
Let $\psi \ge 0$ be a smooth function with compact support contained in $M \times \operatorname{Int}(J)$. Integrating on a slice, the sign of $-\mu^{\mathrm{s}}_\tau$ gives

$$\int_M \left(\partial_\tau l + |\nabla^{\mathbf{g}_\tau} l|^2_{\mathbf{g}_\tau} - R_\tau + \frac{m}{2\tau}\right)\psi\, d\lambda_{\mathbf{g}_\tau} - \langle \Delta_\tau l, \psi(\cdot,\tau)\rangle \ge 0.$$

The Laplacian pairing can be written without separating its two parts:

$$\langle \Delta_\tau l, \psi(\cdot,\tau)\rangle = -\int_M \langle \nabla^{\mathbf{g}_\tau} l, \nabla^{\mathbf{g}_\tau}\psi\rangle_{\mathbf{g}_\tau}\, d\lambda_{\mathbf{g}_\tau}.$$

This identity is the weak definition of the Laplacian and holds because $l(\cdot,\tau)$ is locally Lipschitz. Its right-hand side depends measurably on τ and is integrable on J: the weak gradient of l is jointly measurable and uniformly bounded on the support of ψ, and the metric coefficients are smooth. The same holds for the remaining terms. The uniform control of the measures on compact sets, proved at the end of the preceding lemma, also applies here, since the semiconcavity and Lipschitz bounds for l are uniform on J. Thus

time integration is justified directly by Fubini's theorem, without presupposing measurability of a separate choice of the singular measures. We obtain

$$\int_J \left\{ \int_M \left(\partial_\tau l + |\nabla^{\mathbf{g}_\tau} l|^2_{\mathbf{g}_\tau} - R_\tau + \frac{m}{2\tau} \right) \psi \, d\lambda_{\mathbf{g}_\tau} + \int_M \langle \nabla^{\mathbf{g}_\tau} l, \nabla^{\mathbf{g}_\tau} \psi \rangle_{\mathbf{g}_\tau} \, d\lambda_{\mathbf{g}_\tau} \right\} d\tau \geq 0.$$

Finally, the weak time derivative also agrees with the almost everywhere derivative of l. For the spacetime measure we use, $\partial_\tau d\lambda_{\mathbf{g}_\tau} = R_\tau d\lambda_{\mathbf{g}_\tau}$; hence integration by parts in time takes the form

$$\int_J \int_M (\partial_\tau l)\psi \, d\lambda_{\mathbf{g}_\tau} d\tau = -\int_J \int_M l(\partial_\tau \psi + R_\tau \psi) \, d\lambda_{\mathbf{g}_\tau} d\tau.$$

The preceding inequality is therefore exactly (22.66) in the distributional sense. Since the interval J and the test function were arbitrary, this justifies the formulation used below; it corresponds to the extension developed in [104, Theorem 7.13]. □

Definition 22.6.12. (Reduced volume) The reduced volume based at (p, t_0) is

$$\widetilde{V}(\tau) := \int_M (4\pi\tau)^{-m/2} e^{-l(q,\tau)} \, d\lambda_{\mathbf{g}(t_0-\tau)}(q). \tag{22.67}$$

To identify the initial value of the reduced volume, we need a quantitative comparison with the Euclidean model.

Lemma 22.6.13. (Gaussian behavior for small times) Fix $A > 0$. There exist $C_A > 0$ and $\tau_A > 0$ such that, in normal coordinates for $\mathbf{g}(t_0)$ centered at p,

$$\left| l(\exp_p(\sqrt{\tau}\, z), \tau) - \frac{|z|^2_{\mathbf{g}(t_0)}}{4} \right| \leq C_A \tau \tag{22.68}$$

for $|z| \leq A$ and $0 < \tau \leq \tau_A$. Moreover, there exist $c, C > 0$ such that

$$l(q, \tau) \geq c \frac{d_{\mathbf{g}(t_0)}(p, q)^2}{\tau} - C\tau \tag{22.69}$$

for every $q \in M$ and every sufficiently small τ.

Proof. The curvature bound on a short interval provides a uniform bound on $|\mathrm{Ric}_{\mathbf{g}_s}|_{\mathbf{g}_s}$ and $|R_s|$. Lemma 22.1.8 compares $\mathbf{g}_s$ with $\mathbf{g}(t_0)$ by factors $e^{\pm 2Ks}$. On a bounded interval, the inequalities $e^{-2Ks} \geq 1 - 2Ks$ and $e^{2Ks} \leq 1 + 2Ke^{2K\tau_0} s$ allow us to choose $C, \tau_0 > 0$ such that

$$(1 - C\tau)\mathbf{g}(t_0) \leq \mathbf{g}_s \leq (1 + C\tau)\mathbf{g}(t_0), \qquad |R_s| \leq C, \quad 0 \leq s \leq \tau \leq \tau_0.$$

Shrink τ_0 so that $C\tau_0 \leq 1/2$. The first inequality means, at each point and for each vector,

$$(1 - C\tau)|v|^2_{\mathbf{g}(t_0)} \leq |v|^2_{\mathbf{g}_s}.$$

Thus $|v|_{\mathbf{g}(t_0)} \le \sqrt{2}|v|_{\mathbf{g}_s}$, with a constant independent of s and τ. Let $d_0 = d_{\mathbf{g}(t_0)}$. For any admissible curve,

$$d_0(p,q) \le \int_0^\tau |\dot\gamma(s)|_{\mathbf{g}(t_0)}\, ds \le \sqrt{2}\int_0^\tau s^{1/4}|\dot\gamma(s)|_{\mathbf{g}_s} s^{-1/4}\, ds.$$

Cauchy–Schwarz with the weights $s^{1/4}$ and $s^{-1/4}$ gives

$$d_0(p,q)^2 \le C\left(\int_0^\tau \sqrt{s}|\dot\gamma|^2_{\mathbf{g}_s} ds\right)\left(\int_0^\tau s^{-1/2} ds\right).$$

Since the second integral is $2\sqrt{\tau}$ and the curvature term is bounded by $C\tau^{3/2}$, we obtain (22.69).
For the local estimate, use the variable $r = \sqrt{s}$. If $q = \exp_p(\sqrt{\tau}z)$, the radial curve $\alpha(r) = \exp_p(rz)$ gives, by (22.40),

$$L(q,\tau) \le \frac{\sqrt{\tau}}{2}|z|^2 + C_A\tau^{3/2}.$$

Now let α be a minimizer. Since the scalar term is bounded below and the preceding upper bound is $O_A(\sqrt{\tau})$, we obtain the sharper estimate

$$\int_0^{\sqrt{\tau}} |\alpha'(r)|^2_{\mathbf{g}(t_0)}\, dr \le C_A\sqrt{\tau}. \tag{22.70}$$

By Cauchy–Schwarz, for $0 \le r \le \sqrt{\tau}$,

$$d_{\mathbf{g}(t_0)}(p,\alpha(r)) \le \int_0^r |\alpha'(\rho)|_{\mathbf{g}(t_0)}\, d\rho \le C_A\sqrt{\tau}.$$

The minimizer therefore remains in the normal ball of radius $C_A\sqrt{\tau}$. In these coordinates, the coefficient matrix of $\mathbf{g}(t_0)$ differs from its value at p by $O_A(\tau)$ along the curve, while the flow equation gives $\mathbf{g}_{r^2} = \mathbf{g}(t_0) + O(\tau)$. Consequently, the coefficients of $\mathbf{g}_{r^2}$ differ from those of the Euclidean form $\mathbf{g}(t_0)|_p$ by $O_A(\tau)$ uniformly along the minimizer. The Euclidean energy inequality then gives

$$\frac{1}{2}\int_0^{\sqrt{\tau}} |\alpha'(r)|^2_{\mathbf{g}_{r^2}}\, dr \ge \frac{\sqrt{\tau}}{2}|z|^2 - C_A\tau^{3/2}.$$

The curvature term has absolute value at most $C\tau^{3/2}$. Together with the radial upper bound, this gives

$$\left|L(\exp_p(\sqrt{\tau}z),\tau) - \frac{\sqrt{\tau}}{2}|z|^2\right| \le C_A\tau^{3/2},$$

, and division by $2\sqrt{\tau}$ proves (22.68). □

Theorem 22.6.14. (Monotonicity of the reduced volume) Let M be a smooth closed connected manifold, and let $\mathbf{g}(t)$ be a smooth Ricci flow on an interval containing $[t_0 - \tau_{\max}, t_0]$. The reduced volume is locally Lipschitz and nonincreasing on $(0, \tau_{\max}]$. In particular,

$$\frac{d}{d\tau}\widetilde{V}(\tau) \le 0 \quad \text{for almost every } \tau \in (0,\tau_{\max}), \qquad \lim_{\tau\to 0^+}\widetilde{V}(\tau) = 1.$$

Moreover, $\widetilde{V}$ is invariant under parabolic rescalings.

Proof. Set $v(q,\tau) = (4\pi\tau)^{-m/2} e^{-l(q,\tau)}$. Let us explain the passage to e^{-l} in the presence of the singular part of the Hessian. For almost every τ, write

$$\Delta_\tau l = (\Delta_\tau l)_{\mathrm{ac}}\, d\lambda_{\mathbf{g}_\tau} + \mu_\tau^{\mathrm{s}}, \qquad \mu_\tau^{\mathrm{s}} \le 0.$$

The weak chain rule gives $\nabla(e^{-l}) = -e^{-l}\nabla l$. For a smooth test function φ with compact support in a chart, apply the definition of the measure $\Delta_\tau l$ to φe^{-l}:

$$\int_M \varphi e^{-l}\, d(\Delta_\tau l) = -\int_M e^{-l}\langle \nabla\varphi, \nabla l\rangle\, d\lambda_{\mathbf{g}_\tau} + \int_M \varphi e^{-l}|\nabla l|^2\, d\lambda_{\mathbf{g}_\tau}.$$

The use of a Lipschitz test function here is justified by approximating it uniformly and in $W^{1,2}$ by smooth functions; uniform convergence controls the pairing with the measure. Comparison with the definition of $\Delta_\tau(e^{-l})$ gives

$$\Delta_\tau(e^{-l}) = e^{-l}\big(-(\Delta_\tau l)_{\mathrm{ac}} + |\nabla l|^2\big)\, d\lambda_{\mathbf{g}_\tau} - e^{-l}\mu_\tau^{\mathrm{s}}. \tag{22.71}$$

The absolutely continuous part of $(\partial_\tau - \Delta_{\mathbf{g}_\tau} + R_\tau)v$ is the product of $-v$ and the left-hand side of (22.66); its singular part is $v\mu_\tau^{\mathrm{s}}$, which is also nonpositive. Consequently,

$$(\partial_\tau - \Delta_{\mathbf{g}_\tau} + R_\tau)v \le 0 \tag{22.72}$$

in the sense of distributions. Since $\partial_\tau d\lambda_{\mathbf{g}_\tau} = R_\tau d\lambda_{\mathbf{g}_\tau}$ and M is closed, testing (22.72) against the spatially constant function one gives

$$\frac{d}{d\tau}\widetilde{V}(\tau) = \int_M (\partial_\tau v + R_\tau v)\, d\lambda_{\mathbf{g}_\tau} \le 0$$

in distributions. Since l is Lipschitz on each compact time interval bounded away from zero and M is compact, the integral defining $\widetilde{V}$ is locally Lipschitz. Its continuous representative is therefore nonincreasing, and its ordinary derivative is nonpositive almost everywhere.

For the initial limit, fix $A \ge 1$ and use $q = \exp_p(\sqrt{\tau}z)$. On each ball $|z| \le A$, (22.68) gives $l(q,\tau) \to |z|^2/4$ uniformly. Moreover,

$$d\lambda_{\mathbf{g}_\tau}(q) = \tau^{m/2} J_\tau(z)\, dz, \qquad J_\tau(z) \longrightarrow 1$$

uniformly on that ball. Compactness of M and smoothness of the initial metric give $\mathrm{Vol}_{\mathbf{g}(t_0)} B(p,r) \le Cr^m$ for every $r > 0$: for small radii we use normal charts, and for the remaining radii we use the total volume. The measures of $\mathbf{g}_\tau$ and $\mathbf{g}(t_0)$ are uniformly comparable for small τ. Divide the complement of $B(p, A\sqrt{\tau})$ into the annuli

$$\{q \in M \mid 2^k A\sqrt{\tau} \le d_0(p,q) < 2^{k+1}A\sqrt{\tau}\}, \qquad k \in \mathbb{N}_0.$$

The bound (22.69) then gives

$$\int_{M\setminus B_{\mathbf{g}(t_0)}(p, A\sqrt{\tau})} v\, d\lambda_{\mathbf{g}_\tau} \le C\sum_{k=0}^{\infty} (2^{k+1}A)^m e^{-c4^k A^2}.$$

This series tends to zero as $A \to \infty$, uniformly for small τ: the exponential absorbs the polynomial factor and bounds the series by a multiple of $\sum_{k=0}^{\infty} e^{-(c/2)4^k A^2}$. First let $\tau \to 0^+$ with A fixed, and then let $A \to \infty$.

Dominated convergence gives

$$\lim_{\tau\to 0^+}\widetilde{V}(\tau)=(4\pi)^{-m/2}\int_{\mathbb{R}^m}e^{-|z|^2/4}\,dz=1.$$

Finally, for $Q>0$, let $\widehat{\mathbf{g}}(\sigma)=Q\mathbf{g}(t_0+\sigma/Q)$. A change of variables in (22.38) shows that $l_{\widehat{\mathbf{g}}}(q,Q\tau)=l_{\mathbf{g}}(q,\tau)$, while $d\lambda_{\widehat{\mathbf{g}}(-Q\tau)}=Q^{m/2}d\lambda_{\mathbf{g}(t_0-\tau)}$. This factor cancels $(Q\tau)^{-m/2}$ and proves invariance. □

22.7 The noncollapsing theorem

A curvature bound alone does not suffice to obtain a smooth limit: a sequence may lose volume at the scale being observed. The entropy μ prevents this phenomenon during a finite time interval. The idea is to compare two estimates for the same quantity. Its monotonicity provides a lower bound determined by the initial metric; a test function concentrated on a ball, on the other hand, shows that a very small volume ratio would make μ arbitrarily negative.

Definition 22.7.1. (Noncollapsing at scale r) Let M be a smooth manifold without boundary, and let $\mathbf{g}(t)$ be a Ricci flow defined on an interval I. Whenever $[t_0-r^2,t_0]\subseteq I$, set

$$P^-(x_0,t_0,r):=B_{\mathbf{g}(t_0)}(x_0,r)\times[t_0-r^2,t_0]. \tag{22.73}$$

We say that the flow is κ–noncollapsed at scale r at (x_0,t_0) if the implication

$$|\mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)}\le r^{-2}\ \text{on}\ P^-(x_0,t_0,r)\quad\Longrightarrow\quad \mathrm{Vol}_{\mathbf{g}(t_0)}B_{\mathbf{g}(t_0)}(x_0,r)\ge\kappa r^m$$

holds.

We first isolate the two functional and geometric ingredients of the proof.

Lemma 22.7.2. (Lower bound for μ at bounded scales) Let $(M^m,\mathbf{h})$ be a closed Riemannian manifold. For each $A>0$, there exists $C=C(\mathbf{h},A)<\infty$ such that

$$\mu(\mathbf{h},\sigma)\ge -C\qquad(0<\sigma\le A).$$

Proof. To obtain a uniform bound as σ tends to zero, we need to retain the dimensional coefficient of the logarithmic term. Choose $q>2$ in the range of the embedding of H^1 into L^q: if $m\ge 3$, take $q<2m/(m-2)$; if $m\le 2$, any $q<\infty$ suffices. The Euclidean Gagliardo–Nirenberg inequality we need is

$$\|w\|_{L^q(\mathbb{R}^m)}\le C\|dw\|^{\theta}_{L^2(\mathbb{R}^m)}\|w\|^{1-\theta}_{L^2(\mathbb{R}^m)},\qquad \theta=m\left(\frac12-\frac1q\right),\quad w\in C_c^\infty(\mathbb{R}^m).$$

It follows from the Euclidean embedding $H^1\hookrightarrow L^q$ in the Sobolev chapter by applying it to $w(\lambda\cdot)$: for each $\lambda>0$,

$$\|w\|_{L^q(\mathbb{R}^m)}\le C\big(\lambda^{1-\theta}\|dw\|_{L^2(\mathbb{R}^m)}+\lambda^{-\theta}\|w\|_{L^2(\mathbb{R}^m)}\big).$$

If $w\neq 0$, choose $\lambda=\|w\|_{L^2(\mathbb{R}^m)}/\|dw\|_{L^2(\mathbb{R}^m)}$; the case $w=0$ is immediate. Applying this inequality to the

localizations of v in a finite atlas of $(M,\mathbf{h})$ gives

$$\|v\|_{L^q(M)} \le C\left(\|dv\|^{\theta}_{L^2(M,T^*M)}\|v\|^{1-\theta}_{L^2(M)} + \|v\|_{L^2(M)}\right), \qquad \theta = m\left(\frac{1}{2}-\frac{1}{q}\right) \in (0,1).$$

Derivatives of the partition of unity contribute the second summand; the finite number of charts allows us to use a common constant C. With $\|v\|_{L^2(M)} = 1$ and $E = \|dv\|^2_{L^2(M,T^*M)}$, the Jensen inequality used in (22.35) implies

$$\begin{aligned}\int_M v^2 \log v^2 \, d\lambda_{\mathbf{h}} &\le \frac{2q}{q-2}\log\|v\|_{L^q(M)} \\ &\le \frac{q\theta}{q-2}\log(1+E) + C = \frac{m}{2}\log(1+E) + C.\end{aligned}$$

For $0 < \eta \le 1$, set $z = \eta(1+E) > 0$. Then

$$\frac{m}{2}\log(1+E) - \eta E = -\frac{m}{2}\log\eta + \frac{m}{2}\log z - z + \eta \le -\frac{m}{2}\log\eta + C_m,$$

since the function $z \mapsto (m/2)\log z - z$ attains its maximum at $z = m/2$. Thus we obtain a constant $C_0 = C_0(M,\mathbf{h})$ such that

$$\int_M v^2 \log v^2 \, d\lambda_{\mathbf{h}} \le \eta \int_M |dv|^2_{\mathbf{h}} \, d\lambda_{\mathbf{h}} - \frac{m}{2}\log\eta + C_0. \tag{22.74}$$

The variational expression for μ is

$$\begin{aligned}\mathcal{J}_{\mathbf{h},\sigma}(v) = 4\sigma\int_M |dv|^2_{\mathbf{h}}\, d\lambda_{\mathbf{h}} + \sigma\int_M R_{\mathbf{h}} v^2 \, d\lambda_{\mathbf{h}} - \int_M v^2\log v^2 \, d\lambda_{\mathbf{h}} \\ - \frac{m}{2}\log(4\pi\sigma) - m.\end{aligned}$$

If $0 < \sigma \le 1/2$, take $\eta = 2\sigma$ in (22.74). The coefficient of the energy becomes $2\sigma \ge 0$, and the difference between $\frac{m}{2}\log(2\sigma)$ and $\frac{m}{2}\log(4\pi\sigma)$ is constant. Since $R_{\mathbf{h}}$ is bounded below, we obtain a bound independent of σ. For $1/2 \le \sigma \le A$, use $\eta = 1$; then $4\sigma - 1 \ge 1$ and the remaining terms are uniformly bounded below on this compact interval. Taking the infimum over $\|v\|_{L^2(M)} = 1$ proves the lemma. □

Lemma 22.7.3. (Cutoff function on a ball with bounded curvature) There exist constants $c_m, C_m > 0$ with the following property. If $(N^m,\mathbf{h})$ is complete and $|\mathrm{Rm}_{\mathbf{h}}|_{\mathbf{h}} \le r^{-2}$ on $B_{\mathbf{h}}(x,r)$, there exists a Lipschitz function $\zeta\colon N \to [0,1]$ such that

$$\zeta = 1 \text{ on } B_{\mathbf{h}}(x,r/2), \qquad \operatorname{supp}\zeta \subseteq \overline{B_{\mathbf{h}}(x,r)}, \qquad |d\zeta|_{\mathbf{h}} \le \frac{C_m}{r}$$

almost everywhere. If $V = \mathrm{Vol}_{\mathbf{h}} B_{\mathbf{h}}(x,r)$ and $Z = \displaystyle\int_N \zeta^2 \, d\lambda_{\mathbf{h}}$, then

$$c_m V \le Z \le V. \tag{22.75}$$

Proof. Let $\eta\colon [0,\infty) \to [0,1]$ be a Lipschitz function that equals one on $[0,1/2]$, vanishes on $[1,\infty)$, and satisfies $|\eta'| \le 3$. Take $\zeta(y) = \eta(d_{\mathbf{h}}(x,y)/r)$. The support and gradient properties follow because the distance

function is 1–Lipschitz. Moreover,

$$\operatorname{Vol}_{\mathbf{h}} B_{\mathbf{h}}(x, r/2) \le Z \le V.$$

The curvature assumption implies $\operatorname{Ric}_{\mathbf{h}} \ge -(m-1)r^{-2}\mathbf{h}$ on the ball. Relative Bishop–Gromov comparison between radii $r/2$ and r gives $\operatorname{Vol} B(x, r/2) \ge c_m \operatorname{Vol} B(x, r)$ after rescaling so that $r = 1$. This proves (22.75). □

Theorem 22.7.4. (κ–noncollapsing at bounded scales) Let $(M^m, \mathbf{g}(t))$, $t \in [0, T)$, be a Ricci flow on a smooth closed manifold, with $T < \infty$. For each $\rho > 0$, there exists $\kappa = \kappa(m, \mathbf{g}(0), T, \rho) > 0$ such that the flow is κ–noncollapsed at every (x_0, t_0) and every scale

$$0 < r \le \min\{\rho, \sqrt{t_0}\}$$

for which the curvature bound in Definition 22.7.1 holds.

Proof. Since a closed manifold has finitely many components, it suffices to work on the component of x_0 and finally take the minimum of the resulting constants. We therefore assume that M is connected.
Apply Corollary 22.5.6 on the interval $[0, t_0]$ with $\tau(s) = t_0 + r^2 - s$. Since $\tau(t_0) = r^2$ and $\tau(0) = t_0 + r^2 \le T + \rho^2$, Lemma 22.7.2 gives a constant $\mu_* > -\infty$, independent of (x_0, t_0, r), such that

$$\mu(\mathbf{g}(t_0), r^2) \ge \mu(\mathbf{g}(0), t_0 + r^2) \ge \mu_*. \tag{22.76}$$

Set $\mathbf{h} = \mathbf{g}(t_0)$, $B = B_{\mathbf{h}}(x_0, r)$, and $V = \operatorname{Vol}_{\mathbf{h}} B$. The parabolic assumption implies, in particular, $|\operatorname{Rm}_{\mathbf{h}}| \le r^{-2}$ on B. Let ζ be the function in Lemma 22.7.3, let $Z = \int_M \zeta^2 d\lambda_{\mathbf{h}}$, and define $v = Z^{-1/2}\zeta$. Then $v \in H^1(M)$, $v \ge 0$, and $\|v\|_{L^2(M)} = 1$. The extension of the variational formula to the sphere in H^1, justified in Proposition 22.5.5, allows us to use this test function.
We now estimate each term of $\mathcal{J}_{\mathbf{h}, r^2}(v)$. By (22.75),

$$4r^2 \int_M |dv|_{\mathbf{h}}^2 d\lambda_{\mathbf{h}} = \frac{4r^2}{Z} \int_M |d\zeta|_{\mathbf{h}}^2 d\lambda_{\mathbf{h}} \le C_m \frac{V}{Z} \le C_m.$$

Moreover, $|R_{\mathbf{h}}| \le C_m r^{-2}$ on the support of v, so

$$\left| r^2 \int_M R_{\mathbf{h}} v^2 d\lambda_{\mathbf{h}} \right| \le C_m.$$

For the entropy, use $v^2 = \zeta^2/Z$:

$$\begin{aligned} -\int_M v^2 \log v^2 d\lambda_{\mathbf{h}} &= \log Z - \frac{1}{Z} \int_M \zeta^2 \log \zeta^2 d\lambda_{\mathbf{h}} \\ &\le \log V + C_m, \end{aligned}$$

since $0 \le -\zeta^2 \log \zeta^2 \le e^{-1}$ and $Z \ge c_m V$. Thus

$$\mu(\mathbf{g}(t_0), r^2) \le \mathcal{J}_{\mathbf{h}, r^2}(v) \le \log \frac{V}{r^m} + C_m. \tag{22.77}$$

Combining (22.76) and (22.77) gives

$$V \geq e^{\mu_* - C_m} r^m.$$

We may take $\kappa = e^{\mu_* - C_m}$, decreased if necessary so that $0 < \kappa \leq 1$. □

Corollary 22.7.5. (Nondegenerate injectivity radius) Let $(M^m, \mathbf{g}(t))$, $t \in [0, T)$, be a Ricci flow on a smooth closed manifold. If it is κ–noncollapsed at scale r at (x_0, t_0) and $|\mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} \leq r^{-2}$ on $P^-(x_0, t_0, r)$, then there exists $c = c(m, \kappa) > 0$ such that

$$\mathrm{inj}_{\mathbf{g}(t_0)}(x_0) \geq cr.$$

Proof. On the final slice, we have $|\mathrm{Rm}| \leq r^{-2}$ on $B(x_0, r)$ and $\mathrm{Vol}\, B(x_0, r) \geq \kappa r^m$. Lemma 22.1.4 directly gives the bound $\mathrm{inj}(x_0) \geq c(m, \kappa) r$. □

Together with Shi's estimates, this corollary provides the two families of bounds needed to apply Hamilton compactness to curvature rescalings: control of all curvature jets and a lower bound on the injectivity radius at the basepoint.

22.8 Hamilton–Ivey pinching and κ–solutions

In dimension three, the curvature operator may initially have a negative part. Hamilton–Ivey pinching shows that this part becomes negligible relative to positive curvature as the curvature scale tends to infinity. This is why the limiting models of a three-dimensional singularity have nonnegative curvature operator.

Denote the smallest eigenvalue of $\mathcal{R}_{\mathbf{g}(t)} \colon \Lambda^2 T_x M \to \Lambda^2 T_x M$ by $\nu(x, t)$, with the normalization $\mathrm{tr}_{\Lambda^2} \mathcal{R} = R/2$.

Theorem 22.8.1. (Hamilton–Ivey pinching) Let $(M^3, \mathbf{g}(t))$, $0 \leq t < T < \infty$, be a Ricci flow on a smooth closed manifold. There exist constants $A, B > 0$, determined by $\mathbf{g}(0)$, such that at every point where $\nu < 0$ and $-\nu \geq B/(1 + Bt)$,

$$R \geq 2(-\nu)\left[\log\left(\frac{-\nu}{B}\right) + \log(1 + Bt) - A\right]. \tag{22.78}$$

In particular, for each $\varepsilon > 0$, there exists $C_\varepsilon = C_\varepsilon(\mathbf{g}(0), T)$ such that

$$\nu \geq -\varepsilon |\mathrm{Rm}_{\mathbf{g}(t)}|_{\mathbf{g}(t)} - C_\varepsilon \quad \text{on } M \times [0, T). \tag{22.79}$$

The logarithmic estimate is the Hamilton–Ivey theorem, in the normalization $\mathrm{tr}\mathcal{R} = R/2$ of [104, Theorem 4.26]. One may take $A = 3$ and $B \geq \max\{1, \|\max\{-\nu(\cdot, 0), 0\}\|_\infty\}$. Indeed, replacing $\mathbf{g}$ by $B\mathbf{g}$ and t by Bt makes the smallest initial eigenvalue at least -1; undoing this normalization gives (22.78). The maximum principle is applied through spectral support functions, so it does not require ν to be differentiable at points of multiplicity.

Let us derive the consequence we will use in the limits. Fix $\varepsilon > 0$ and a dimensional constant c_3 such that $|R| \leq c_3 |\mathrm{Rm}|$. If $a = -\nu > B \exp(A + c_3/(2\varepsilon))$, the threshold in the theorem is satisfied, and $\log(1 + Bt) \geq 0$ implies

$$\frac{c_3}{\varepsilon} a \leq 2a\big(\log(a/B) - A\big) \leq R \leq c_3 |\mathrm{Rm}|.$$

Thus $a \leq \varepsilon |\mathrm{Rm}|$. At the remaining points, the negative part of ν is bounded by $C_\varepsilon = B \exp(A + c_3/(2\varepsilon))$. Both alternatives give (22.79).

Definition 22.8.2. (Three-dimensional κ–solution) A κ–solution is a Ricci flow $(N^3, \mathbf{h}(s))$, $s \in (-\infty, 0]$, on a smooth, connected manifold without boundary, satisfying the following properties:

(a) each $(N, \mathbf{h}(s))$ is complete, and the curvature is uniformly bounded on $N \times [a, b]$ for each $-\infty < a \le b \le 0$;

(b) the flow is nonflat and has nonnegative curvature operator;

(c) the flow is κ–noncollapsed at all scales.

The last condition means that, for every (x, s) and every $r > 0$ such that the corresponding parabolic cylinder is contained in the time interval and $|\mathrm{Rm}| \le r^{-2}$ on it, we have $\mathrm{Vol}_{\mathbf{h}(s)} B_{\mathbf{h}(s)}(x, r) \ge \kappa r^3$.

Noncollapsing is preserved under smooth limits. We spell out this step because it will be needed to verify that the limit of the rescalings is still a κ–solution.

Lemma 22.8.3. (Stability of noncollapsing under smooth convergence) Let $(M_i^m, \mathbf{g}_i(t), p_i) \to (M_\infty^m, \mathbf{g}_\infty(t), p_\infty)$ be pointed smooth convergence on compact spacetime sets. Suppose there exist numbers $R_i \to \infty$ such that each flow $\mathbf{g}_i$ is κ–noncollapsed at all its points and at every scale $0 < r \le R_i$ whose backward parabolic cylinder is contained in its time interval. Then the limit flow is κ–noncollapsed at every finite scale for which the corresponding cylinder is defined.

Proof. Fix $x_\infty \in M_\infty$, a time t_0, and $r > 0$ for which $P^-(x_\infty, t_0, r)$ is defined and $|\mathrm{Rm}_{\mathbf{g}_\infty}| \le r^{-2}$ holds on it. Let $0 < \rho < \sigma < r$. By completeness, the closed ball of radius σ in the final slice is compact. Take a convergence embedding Φ_i defined on a neighborhood of this ball and on a larger open set containing the curves considered below; write $x_i = \Phi_i(x_\infty)$. The ball exhaustion property ensures that its image contains the pointed balls of fixed radius that we need.
For large i, we have

$$B_{\mathbf{g}_i(t_0)}(x_i, \rho) \subseteq \Phi_i\big(B_{\mathbf{g}_\infty(t_0)}(x_\infty, \sigma)\big). \tag{22.80}$$

Indeed, let γ_i be a minimizing geodesic of length less than ρ starting at x_i. It can be lifted through Φ_i as long as it remains in the image of the ball of radius σ. If the lifted curve first exited that ball, its length in the limit metric up to that instant would be at least σ. Uniform convergence of the metrics on the closed ball implies that the corresponding length of γ_i is at least $(1 - \varepsilon_i)\sigma$, with $\varepsilon_i \to 0$. For large i, this is greater than ρ, a contradiction. This proves the inclusion.
Convergence of the curvatures on the compact set $\overline{B_{\mathbf{g}_\infty(t_0)}(x_\infty, \sigma)} \times [t_0 - \rho^2, t_0]$ and the strict margin $r^{-2} < \rho^{-2}$ give

$$|\mathrm{Rm}_{\mathbf{g}_i}| \le \rho^{-2} \quad \text{on } P^-(x_i, t_0, \rho)$$

for large i, using (22.80). Moreover, $\rho < R_i$ for all these indices. Noncollapsing of the approximating flows yields

$$\mathrm{Vol}_{\mathbf{g}_i(t_0)} B_{\mathbf{g}_i(t_0)}(x_i, \rho) \ge \kappa \rho^m. \tag{22.81}$$

By the ball inclusion and convergence of the volume densities on a fixed domain,

$$\begin{aligned} \kappa \rho^m &\le \liminf_{i \to \infty} \mathrm{Vol}_{\mathbf{g}_i(t_0)} B_{\mathbf{g}_i(t_0)}(x_i, \rho) \\ &\le \mathrm{Vol}_{\mathbf{g}_\infty(t_0)} B_{\mathbf{g}_\infty(t_0)}(x_\infty, \sigma). \end{aligned}$$

First let $\rho \to \sigma^-$ and then $\sigma \to r^-$. Continuity of measure from below gives $\mathrm{Vol}_{\mathbf{g}_\infty(t_0)} B(x_\infty, r) \geq \kappa r^m$. □

Theorem 22.8.4. (Rescaling limit at a singular time) Let $(M^3, \mathbf{g}(t))$, $0 \leq t < T < \infty$, be a maximal solution on a smooth closed manifold. Let (x_i, t_i) be spacetime maxima as in (22.15), with $t_i \nearrow T$, and let $Q_i = |\mathrm{Rm}_{\mathbf{g}(t_i)}|_{\mathbf{g}(t_i)}(x_i)$. Then $Q_i \to \infty$, and a subsequence of the rescaled flows

$$\mathbf{g}_i(s) := Q_i \mathbf{g}(t_i + s/Q_i), \qquad -Q_i t_i \leq s \leq 0,$$

, pointed at x_i, converges smoothly on compact subsets of $M_\infty \times (-\infty, 0]$ to a κ–solution $(M_\infty, \mathbf{g}_\infty(s), x_\infty)$, for some $\kappa > 0$.

Proof. The continuation criterion in the previous chapter implies that $Q_i \to \infty$; since $t_i \to T > 0$, we also have $Q_i t_i \to \infty$. By the choice of the spacetime maxima,

$$|\mathrm{Rm}_{\mathbf{g}_i(s)}|_{\mathbf{g}_i(s)} \leq 1 \quad (-Q_i t_i \leq s \leq 0), \qquad |\mathrm{Rm}_{\mathbf{g}_i(0)}|_{\mathbf{g}_i(0)}(x_i) = 1. \tag{22.82}$$

Fix $A > 0$. For large i, the flow $\mathbf{g}_i$ is defined on $[-A-1, 0]$. Shi's estimates applied on this interval provide uniform bounds for all curvature derivatives on $[-A, 0]$. Now take unit scale at $(x_i, 0)$. In the original variables, this corresponds to radius $Q_i^{-1/2}$. For large i, this radius is smaller than ρ and $\sqrt{t_i}$, and (22.82) satisfies the parabolic assumption of Theorem 22.7.4. Invariance of the volume ratio under rescaling gives

$$\mathrm{Vol}_{\mathbf{g}_i(0)} B_{\mathbf{g}_i(0)}(x_i, 1) \geq \kappa$$

with a common constant $\kappa > 0$. Corollary 22.7.5 yields $\mathrm{inj}_{\mathbf{g}_i(0)}(x_i) \geq c(3, \kappa)$.

The first part of the proof of Theorem 22.1.10, applied to the metrics $\mathbf{g}_i(0)$, provides a single limit manifold M_∞ and spatial embeddings on an exhaustion. Fix these embeddings. For each $A \in \mathbb{N}$, the estimates on $[-A-1, 0]$ allow us to apply the second part of that proof on $[-A, 0]$. Choose nested subsequences as A increases and take their diagonal. The time-dependent limits agree on overlaps because they are computed using the same embeddings and on the same manifold. This gives a complete ancient flow $(M_\infty, \mathbf{g}_\infty(s), x_\infty)$ for every $s \leq 0$. The unit curvature value at the basepoint passes to the limit, so the flow is nonflat. The bound in (22.82) also passes to the limit and shows that each slice has bounded curvature.

Let us verify the two remaining properties. Fix, for example, the original range $\rho = 1$ in Theorem 22.7.4. On the rescaled interval $[-A, 0]$, a scale r corresponds to radius $rQ_i^{-1/2}$ and original time $t_i + s/Q_i$. Thus the theorem applies simultaneously at all points of this interval whenever

$$0 < r \leq R_{i,A} := \min\left\{\sqrt{Q_i}, \sqrt{Q_i t_i - A}\right\}.$$

Since $Q_i \to \infty$ and $Q_i t_i \to \infty$, we have $R_{i,A} \to \infty$. The same constant κ works for all rescaled flows and every bounded radius. Applying Lemma 22.8.3 on each interval $[-A, 0]$ and then letting $A \to \infty$, we conclude that the limit is κ–noncollapsed at all scales.

Finally, the eigenvalues of the curvature operator are multiplied by Q_i^{-1}. Inequality (22.79) becomes

$$\nu_{\mathbf{g}_i(s)} \geq -\varepsilon |\mathrm{Rm}_{\mathbf{g}_i(s)}|_{\mathbf{g}_i(s)} - \frac{C_\varepsilon}{Q_i}.$$

Letting $i \to \infty$ gives $\nu_{\mathbf{g}_\infty(s)} \geq -\varepsilon |\mathrm{Rm}_{\mathbf{g}_\infty(s)}|$. Since $\varepsilon > 0$ is arbitrary, the curvature operator of the limit is nonnegative. Thus all the conditions in Definition 22.8.2 hold. □

22.9 Structural theorems for surgery and the topological conclusion

The preceding results describe the limits that arise when a region of high curvature is magnified. To continue the flow beyond a singularity, we must identify these regions with canonical models, cut along nearly cylindrical necks, and attach caps whose geometry is compatible with pinching and noncollapsing.

The full construction includes the standard solution, persistence of caps, noncollapsing across surgeries, the canonical neighborhood theorem, and the inductive choice of parameters. These arguments occupy Chapters 12–17 of [104]. We will use their structure and existence theorems to determine the geometric and topological effect of each surgery. The final step will be to reconstruct the initial manifold from the components that disappear.

The model for a neck is the shrinking round cylinder $S^2 \times \mathbb{R}$. If $\mathbf{g}_{S^2(1)}$ is the metric of the unit sphere, set

$$\mathbf{g}_{\mathrm{cil}}(s) = (2 - 2s)\mathbf{g}_{S^2(1)} + (\mathbf{d}z)^2, \qquad -1 \leq s \leq 0.$$

The scalar curvature of the slice $s = 0$ equals one.

Definition 22.9.1. (Necks, caps, and canonical components) Fix $0 < \varepsilon < 1$ and $k = \lfloor \varepsilon^{-1} \rfloor$. Let $(M^3, \mathbf{g}(t))$ be a Ricci flow and let (x, t_0) be a point.

An open set $U \subseteq M$ is an *ε–neck of scale $h > 0$* centered at x if there exists a diffeomorphism

$$\Psi \colon S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \longrightarrow U, \qquad x \in \Psi(S^2 \times \{0\}),$$

such that

$$\left\| h^{-2}\Psi^*\mathbf{g}(t_0) - \mathbf{g}_{\mathrm{cil}}(0) \right\|_{C^k(\mathbf{g}_{\mathrm{cil}}(0))} < \varepsilon.$$

It is a *strong ε–neck* if Ψ extends to a spacetime identification with spatial map independent of s, in which, after setting $s = h^{-2}(t - t_0)$, we have

$$\left\| h^{-2}\Psi^*\mathbf{g}(t_0 + h^2 s) - \mathbf{g}_{\mathrm{cil}}(s) \right\|_{C^k} < \varepsilon.$$

This last norm is taken on $S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \times [-1, 0]$, using the reference metric $\mathbf{g}_{\mathrm{cil}}(0) + (\mathbf{d}s)^2$ to compute derivatives. The region under consideration must exist throughout this time interval.

An *(C, ε)–cap* is an open set U diffeomorphic to an open ball B^3 or to $\mathbb{RP}^3 \setminus \overline{B^3}$ that decomposes into a compact core K and an end that is an ε–neck. If $x \in K$, we require $R(x, t_0) > 0$ and, in the normalized metric $\widehat{\mathbf{g}} = R(x, t_0)\mathbf{g}(t_0)$,

$$\mathrm{diam}_{\widehat{\mathbf{g}}} K \leq C, \qquad \mathrm{Vol}_{\widehat{\mathbf{g}}} K \geq C^{-1}, \qquad C^{-1} \leq \frac{R(y, t_0)}{R(x, t_0)} \leq C \quad (y \in K).$$

A *C–component* is a compact component of positive curvature that, when normalized by the scalar curvature at any of its points, has diameter at most C, scalar curvature between C^{-1} and C, and volume between C^{-1} and C. An *ε–round component* is a component that, after this normalization, lies at distance less than ε in C^k from a round spherical space form of scalar curvature one.

Definition 22.9.2. (Strong canonical neighborhood) Given $C < \infty$ and $\varepsilon > 0$, a point (x, t) of positive scalar curvature has a strong canonical neighborhood (C, ε) if one of the following alternatives holds:

(a) x is the center of a strong ε–neck whose scale is comparable to $R(x,t)^{-1/2}$;

(b) x belongs to the core of an (C, ε)–cap;

(c) the component of x is a C–component;

(d) the component of x is ε–round.

In all cases, the quantitative estimates

$$|\nabla R|(x,t) \le C R(x,t)^{3/2}, \qquad |\partial_t R|(x,t) \le C R(x,t)^2, \tag{22.83}$$

and a volume lower bound at scale $R(x,t)^{-1/2}$ of the form $c(C,\varepsilon)R(x,t)^{-3/2}$ are included.

To close off a neck, fix a rotationally symmetric metric $\mathbf{g}_{\text{std}}$ on $\mathbb{R}^3$ with nonnegative curvature operator and positive scalar curvature that agrees outside a compact set with the normalized cylinder. Its flow is the model for the caps that are inserted.

Theorem 22.9.3. (Standard solution) For the fixed standard metric $\mathbf{g}_{\text{std}}$, there exists a unique complete solution of the Ricci flow starting from it, defined on $[0, 1)$ and with uniformly bounded curvature on each $\mathbb{R}^3 \times [0, T_0]$, $T_0 < 1$. On these intervals, it is asymptotic to all orders, away from the cap, to the round cylinder of scalar curvature $1/(1-t)$.
There exist $r_{\text{std}} > 0$ and $\kappa_{\text{std}} > 0$ such that it is κ_{std}–noncollapsed at scales smaller than r_{std} whose cylinders are contained in $[0, 1)$. For each sufficiently small $\varepsilon > 0$, there exist $C_\varepsilon < \infty$ and $R_\varepsilon < \infty$ such that the points with $R(x,t) \ge R_\varepsilon$ have strong canonical neighborhoods $(C_\varepsilon, \varepsilon)$. The constants are chosen after fixing ε and the standard metric.

Existence, rotational symmetry, behavior at infinity, and uniqueness are proved in [104, Theorem 12.5 and Theorems 12.28 and 12.32]. Uniqueness uses a harmonic map flow and is the noncompact version of the gauge mechanism studied in Chapter 21.

Definition 22.9.4. (Ricci flow with surgery) A Ricci flow with surgery, initially defined up to a time $T_* \le \infty$, consists of strictly increasing surgery times t_j, with j in an initial subset of $\mathbb{N}$, and of $t_0 = 0$. The list may be finite, empty, or infinite. In the last case, it may, a priori, accumulate at T_*. Between consecutive times there are closed manifolds M_j and smooth flows $\mathbf{g}_j(t)$ on $M_j \times [t_j, t_{j+1})$; if there is a last surgery time, the remaining interval ends at T_*. At a surgery time t_{j+1}, finite disjoint collections of strong necks are chosen; they are cut along their central spheres, the ends and components prescribed by the procedure are removed, and the new boundaries are closed off with caps obtained from the standard solution at the same scale. There is an open set, called the continuing region, that is identified isometrically before and after surgery, and the new metric agrees there with the limit of the previous metric. The flow becomes extinct at T_{ext} if all components have been discarded, and we set $M(t) = \varnothing$ for $t \ge T_{\text{ext}}$.

Cutting must be performed at scales sufficiently separated from the canonical neighborhood scale. The following parameters record this choice and the properties that must be preserved when it is repeated.

Definition 22.9.5. (Standard control of surgery parameters) Fix small $\varepsilon > 0$ and the canonical neighborhood constant C. Standard control consists of positive nonincreasing functions

$$r(t), \qquad \kappa(t), \qquad \delta(t)$$

and a surgery scale

$$\rho(t) = \delta(t) r(t), \qquad h(t) = h(\rho(t), \delta(t)) \le \delta(t)^2 r(t).$$

A flow with surgery is subject to this control if it satisfies:

(a) Hamilton–Ivey pinching;

(b) noncollapsing with constant $\kappa(t)$ at scales $0 < \rho \le \rho_*$, where $\rho_* > 0$ is fixed, on cylinders that survive without surgery throughout their time interval; in the normalization of the source, one may take $\rho_* = \varepsilon$;

(c) the strong canonical neighborhood assumption at every point with $R(x,t) \ge r(t)^{-2}$;

(d) each surgery is performed on strong necks of accuracy $\delta(t)$, centered at points with $R = h(t)^{-2}$, and the post-surgery metric is constructed using the standard cap of scale $h(t)$;

(e) the functions r, κ, and δ are positive and nonincreasing; in particular, for each $T < \infty$, their values on $[0,T]$ are bounded below by the positive numbers $r(T)$, $\kappa(T)$, and $\delta(T)$. The choice of scale h also satisfies $h_T := \inf_{t \in [0,T]} h(t) > 0$.

The full formulation includes the seven compatibility conditions for the spacetime, the continuing and disappearing regions, and maximality between consecutive surgery times in [104, Chapter 15, Sections 1 and 3].

The first structural result describes the ancient models that arise at the curvature scale.

Theorem 22.9.6. (Structure of three-dimensional κ–solutions) There exists $\overline{\varepsilon} > 0$ such that, for each $\kappa > 0$ and $0 < \varepsilon \le \overline{\varepsilon}$, one can choose $C = C(\kappa, \varepsilon)$ with the following property. In an orientable three-dimensional κ–solution, every point has, at its curvature scale, an ε–neck, an (C,ε)–cap, a C–component, or an ε–round component. The approximations include the curvature, derivative, and volume estimates of Definition 22.9.2. When the neck alternative holds, the neck can be chosen strong. The scalar curvature is positive at every point: if it vanished, the strong maximum principle for $\partial_s R = \Delta R + 2|\mathrm{Ric}|^2$, connectedness, and uniqueness of the complete flow with bounded curvature would force the flat solution, which is excluded by definition.

This is the qualitative description obtained in [104, Chapter 9, in particular Theorem 9.93 and Corollaries 9.94–9.95]. Orientability excludes the cylindrical quotient $\mathbb{RP}^2 \times \mathbb{R}$.

Theorem 22.9.7. (Canonical neighborhood theorem) Fix a smooth initial metric $\mathbf{g}_0$ on a closed, orientable three-dimensional manifold, a time horizon $T < \infty$, and sufficiently small $\varepsilon > 0$. There exist $r = r(\mathbf{g}_0, T, \varepsilon) > 0$ and $C = C(\mathbf{g}_0, T, \varepsilon) < \infty$ such that, in the smooth flow starting from $\mathbf{g}_0$, every point before T with $R(x,t) \ge r^{-2}$ has a strong canonical neighborhood (C,ε). The threshold is also chosen so that the strong necks have the

required time interval available.
For the flows with surgery of Theorem 22.9.8, this property is preserved with the time-dependent threshold $r(t)$ chosen together with the surgery parameters. After normalization by $R(x,t)$, the high-curvature regions are approximated by regions of κ–solutions or by regions of the standard solution modeling a recent surgery.

The proof is by contradiction. One chooses the first points where the conclusion fails, normalizes their curvature, and applies noncollapsing, curvature control at bounded distance, and Hamilton compactness. If the rescaled times extend to $-\infty$, the limit is a κ–solution, and the description of its neighborhoods gives a contradiction. If a surgery prevents this extension, the limit is modeled on the standard solution, yielding the same contradiction. The complete argument, including the curvature estimate at bounded distance and the case near a surgery cap, is developed in [104, Chapters 10–12 and 17].

Theorem 22.9.8. (Existence and control of the flow with surgery) Let M be a smooth closed connected orientable three-dimensional manifold, and let $\mathbf{g}_0$ be a smooth metric. There exist control parameters $r(t)$, $\kappa(t)$, $\delta(t)$, and $h(t)$ and a Ricci flow with surgery starting from $\mathbf{g}_0$, defined for every $t \geq 0$ up to possible extinction. The flow satisfies Hamilton–Ivey pinching, noncollapsing, and Theorem 22.9.7; each surgery is performed on a finite collection of strong necks of scale $h(t)$. The surgery times do not accumulate in any compact time interval.

To apply the formulation of [104, Theorem 15.9 and Corollary 15.10] to an arbitrary metric, first multiply $\mathbf{g}_0$ by a constant to normalize the curvature and the volume of unit balls. Parabolic invariance of the flow then allows this normalization to be undone. Persistence of noncollapsing across caps is proved in Chapter 16, and the strong canonical neighborhood assumption and absence of accumulation of surgeries are completed in Chapter 17 of the same source. Let us make the absence of accumulation precise, as used in [104, Lemma 17.12]. Fix $T < \infty$. Pinching gives a bound $R \geq -C_T$, so the total volume $V(t)$ satisfies $V'(t) \leq C_T V(t)$ during the smooth intervals. The cap construction and the bound $h(t) \geq h_T > 0$ give a number $v_T > 0$ such that each neck cut decreases the net volume by at least v_T after the cap volumes have been added. Upon multiplication by $e^{-C_T t}$, smooth evolution makes the weighted volume nonincreasing, and each cut produces a decrease of at least $e^{-C_T T} v_T$. Thus the total number $S(T)$ of cuts satisfies

$$S(T) \leq \frac{e^{C_T T} V(0)}{v_T}.$$

There may also be times when components are merely discarded, without any neck being cut. Each cut increases the number of components by at most one. If there were initially N_0 components, the number $D(T)$ of discarded components satisfies $D(T) \leq N_0 + S(T)$. Every surgery time contains a cut or a discard, so there are at most $S(T) + D(T)$ such times in $[0, T]$.

The topological effect of each surgery is also fully controlled: one cuts along spheres and discards only components covered by necks and caps.

Theorem 22.9.9. (Topological reconstruction from surgeries) Let M be a smooth closed connected orientable three-dimensional manifold. If a flow with surgery subject to the control of Theorem 22.9.8 becomes extinct, then M is diffeomorphic to a finite connected sum of spherical space forms and copies of $S^2 \times S^1$.

Proof. Cutting along an embedded sphere and capping the two resulting boundaries with balls gives two cases. If the sphere separates the component, one obtains two closed manifolds whose connected sum reconstructs the original component. If it does not separate it, the result is a closed manifold N, and the original component is $N\#(S^2\times S^1)$; orientability excludes the nonorientable sphere bundle over the circle. The classification of components discarded by the standard procedure gives spherical space forms, $S^2\times S^1$, and $\mathbb{RP}^3\#\mathbb{RP}^3$. The last case is already a connected sum of two spherical space forms. Control of these alternatives, including the regions that disappear upon reaching the singular time, is Proposition 15.3 and Corollary 15.4 of [104]. Thus, at each step, the preceding manifold is reconstructed from the subsequent components by adding only summands of the indicated types.
Let t_N be the extinction time. The existence and control theorem gives only finitely many surgery times in $[0,t_N]$. Start with the empty slice after t_N: the components discarded at that step have the decomposition described above. Suppose that all components immediately after t_j, with $j\in\{1,\dots,N\}$, have been reconstructed. Reverse the cuts at t_j using the two operations in the first paragraph and add the components discarded at that time. The result is again a finite connected sum of spherical space forms and copies of $S^2\times S^1$. This proves the step back to the preceding slice. Upon reaching $t_0=0$, we obtain the decomposition of M. □

For the Poincaré conjecture, it remains to know that, under the corresponding assumption on the fundamental group, the flow disappears in finite time.

Theorem 22.9.10. (Finite-time extinction) Let M be a smooth closed connected orientable three-dimensional manifold, and let $(M(t),\mathbf{g}(t))$ be a flow with surgery subject to the control of Theorem 22.9.8. If

$$\pi_1(M)\cong G_1*\cdots*G_a*\underbrace{\mathbb{Z}*\cdots*\mathbb{Z}}_{b\text{ factors}},\qquad a,b\in\mathbb{N}_0,\quad |G_j|<\infty\ (j\in\{1,\dots,a\}),\tag{22.84}$$

then the flow becomes extinct in finite time.

The complete proof is in Chapters 18–19 of [104], together with the correction to its Section 19.2 in [105]. First, the least area of spheres nontrivial in π_2 is controlled through a forward difference differential inequality. Then, when the remaining components have $\pi_2=0$ and finite fundamental group, one uses a nontrivial class in π_3 represented by families of loops and least disk areas. The same differential structure forces the minimax quantity to become negative in finite time if the component were to persist, which is impossible. The modifications at a surgery time are compatible with these quantities by semicontinuity.

Theorem 22.9.11. (Poincaré conjecture) Every smooth closed simply connected three-dimensional manifold is diffeomorphic to S^3.

$$M^3\text{ closed and simply connected}\quad\Longrightarrow\quad M^3\cong S^3.$$

Proof. Let M be a manifold as in the statement. It is orientable: otherwise, its orientation cover would be a connected two-sheeted cover and would give a nontrivial quotient of $\pi_1(M)$ isomorphic to $\mathbb{Z}_2$. Choose a smooth metric $\mathbf{g}_0$ and apply Theorem 22.9.8. Since $\pi_1(M)=0$, assumption (22.84) holds with the empty free product. Theorem 22.9.10 shows that the flow becomes extinct in finite time.

By Theorem 22.9.9, there exist spherical space forms $N_j = S^3/\Gamma_j$ and an integer $b \geq 0$ such that

$$M \cong N_1 \# \cdots \# N_a \# \underbrace{(S^2 \times S^1) \# \cdots \# (S^2 \times S^1)}_{b \text{ summands}}.$$

The Seifert–van Kampen theorem gives

$$\pi_1(M) \cong \Gamma_1 * \cdots * \Gamma_a * \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{b \text{ factors}}.$$

By the normal form theorem for free products, this group can be trivial only if $b = 0$ and each Γ_j is trivial. Consequently, each N_j is diffeomorphic to S^3. Since S^3 is the identity for the connected sum of closed orientable three-dimensional manifolds, we conclude that $M \cong S^3$. □

Part VI

Infinite-dimensional geometry

Introduction to Part VI

A change of perspective arises naturally in many problems of geometric analysis. So far, we have studied functions, sections, metrics, and connections as objects defined on a manifold. From this part onward, we will also regard them as points of spaces with their own geometric structure. A space of maps, the set of Riemannian metrics, or the space of connections can, under suitable hypotheses, behave as an infinite-dimensional manifold.

Working rigorously with these spaces requires a careful choice of regularity. Banach spaces allow the use of versions of the inverse function, rank, and regular preimage theorems. Hilbert spaces add an inner product structure and the Riesz identification. Banach–Finsler structures provide tangent norms and distances that will be useful in studying flows and deformations. When an operation loses derivatives, a single Sobolev completion is no longer sufficient, and Hilbert scales and ILH structures enter the picture.

Many identifications that are automatic in finite dimensions cease to be so here. The kinematic tangent space must be distinguished from the operational one, the continuous dual from the algebraic dual, and a strong metric from a weak metric. These differences are more than terminological details: they determine whether a Levi–Civita connection exists, whether the musical map is invertible, and whether a theorem of differential calculus can be applied. The final chapters show how this theory takes concrete form in spaces of maps and connections, gauge groups, diffeomorphisms, and problems such as harmonic maps, Yang–Mills theory, and the Euler equation.

Function spaces as manifolds

Fréchet opened the way to calculus on abstract spaces, and Banach provided a complete setting in which the fundamental theorems of analysis could be formulated precisely. Later, Eells and Palais developed global tools for infinite-dimensional manifolds. Ebin and Marsden demonstrated the usefulness of Sobolev completions in the study of diffeomorphism groups and fluid equations. With these ideas, many function spaces ceased to be viewed merely as sets of possible solutions and began to be studied as geometric objects in their own right.

23

Differential calculus in Banach spaces

The Sobolev spaces constructed in the preceding parts are themselves spaces in which we can do calculus. A function, a section, or a metric then becomes a point in an infinite-dimensional space, and an energy functional behaves like a function defined on that space. To study its variations, we need a notion of derivative that retains the linear approximation and continuity familiar from the Euclidean case.

The Fréchet derivative provides this language in Banach spaces. It will allow us to formulate chain rules, Taylor expansions, inverse and implicit function theorems, and Lagrange multipliers. In later chapters, these tools will describe manifolds of functions and variational constraints; at the end of the book, critical points of energies defined on Sobolev spaces will again be interpreted as weak solutions of differential equations.

Unless otherwise stated, all normed vector spaces in this chapter are taken over the field of real numbers.

A general reference for the results and exercises on differential calculus and nonlinear analysis in this chapter is [45, Chapters 3, 4, and 6].

Fréchet and Banach: expanding the domain of calculus

In the early twentieth century, Maurice Fréchet introduced structures capable of measuring convergence beyond Euclidean spaces. Stefan Banach later demonstrated the power of working in complete normed spaces, where approximation and fixed point arguments have limits within the space itself. Fréchet calculus combines these two ideas: the derivative is a continuous linear operator, and completeness allows local estimates to be converted into existence theorems [62, 18].

23.1 Fréchet differentiability

The Fréchet derivative generalizes the derivative of functions between Euclidean spaces to the setting of normed spaces. The fundamental idea is the same as in finite dimensions: a function is differentiable at a point if, in a neighborhood of that point, it can be approximated by a continuous linear map so that the approximation error divided by the norm of the increment tends to zero.

The following definition retains the idea of linear approximation but requires the error to be uniformly small in all directions measured by the norm. This uniformity distinguishes the Fréchet derivative from merely directional notions and underpins the fundamental theorems of calculus in infinite dimensions.

For a normed space $(X, \|\cdot\|_X)$, we write $d_X(x,y) := \|x-y\|_X$; thus $B_{d_X}(x,r)$ denotes the ball centered at x with radius r for the metric induced by the norm.

Definition 23.1.1. (Fréchet derivative) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces, let $U \subseteq X$ be open, and let $f\colon U \longrightarrow Y$. We say that f is **Fréchet differentiable** at $x_0 \in U$ if there exists a continuous linear operator $L \in \mathcal{L}(X,Y)$ such that $\lim_{h\to 0} \dfrac{\|f(x_0+h)-f(x_0)-L(h)\|_Y}{\|h\|_X} = 0$. The operator L is called the **Fréchet derivative** of f at x_0 and is denoted by $Df(x_0)$.

Equivalently, f is Fréchet differentiable at x_0 if there exist $\rho > 0$ and a function $r\colon B_{d_X}(0,\rho) \longrightarrow Y$ such that $B_{d_X}(x_0,\rho) \subseteq U$, $f(x_0+h) = f(x_0) + Df(x_0)(h) + r(h)$ for every $h \in B_{d_X}(0,\rho)$, and $\lim_{h\to 0} \dfrac{\|r(h)\|_Y}{\|h\|_X} = 0$.

Proposition 23.1.2. The Fréchet derivative of a function, when it exists, is unique.

Proof. Suppose $L_1, L_2 \in \mathcal{L}(X,Y)$ satisfy the condition in Definition 23.1.1. Since U is open, there exists $\rho > 0$ such that $B_{d_X}(x_0,\rho) \subseteq U$. Let $v \in X \setminus \{0\}$. For every $t \in \mathbb{R}$ such that $0 < |t| < \dfrac{\rho}{\|v\|_X}$, we have

$$\frac{\|(L_1-L_2)(v)\|_Y}{\|v\|_X} = \frac{\|(L_1-L_2)(tv)\|_Y}{\|tv\|_X} \leq \frac{\|f(x_0+tv)-f(x_0)-L_1(tv)\|_Y}{\|tv\|_X} + \frac{\|f(x_0+tv)-f(x_0)-L_2(tv)\|_Y}{\|tv\|_X}.$$

As $t \to 0$, the right-hand side tends to zero, so $(L_1-L_2)(v) = 0$. Equality is also immediate for $v = 0$, and therefore $L_1 = L_2$. □

Fréchet differentiability implies, in particular, continuity at the point under consideration.

Proposition 23.1.3. Let X and Y be normed spaces, let $U \subseteq X$ be open, and let $f\colon U \longrightarrow Y$. If f is Fréchet differentiable at $x_0 \in U$, then f is continuous at x_0.

Proof. By the preceding characterization, there exist $\rho > 0$ and a function $r\colon B_{d_X}(0,\rho) \longrightarrow Y$ such that $B_{d_X}(x_0,\rho) \subseteq U$, $f(x_0+h) - f(x_0) = Df(x_0)(h) + r(h)$ for every $h \in B_{d_X}(0,\rho)$, and

$$\lim_{h\to 0} \frac{\|r(h)\|_Y}{\|h\|_X} = 0.$$

Let $\varepsilon > 0$. There exists $\delta \in (0,\rho)$ such that $\|r(h)\|_Y < \varepsilon\|h\|_X$ whenever $0 < \|h\|_X < \delta$. For these increments,

$$\|f(x_0+h)-f(x_0)\|_Y \leq \big(\|Df(x_0)\|_{\mathcal{L}(X,Y)} + \varepsilon\big)\|h\|_X.$$

The right-hand side tends to zero as $h \to 0$. Thus $f(x_0+h) \to f(x_0)$, and f is continuous at x_0. □

Definition 23.1.4. Let X and Y be normed spaces, let $U \subseteq X$ be open, and let $f\colon U \longrightarrow Y$. We say that f is **Fréchet differentiable on** U if it is Fréchet differentiable at every point of U. In this case, the derivative map $Df\colon U \longrightarrow \mathcal{L}(X,Y)$ is defined. We say that f is of **class** C^1 on U if it is Fréchet differentiable on U and Df is continuous with respect to the operator norm on $\mathcal{L}(X,Y)$.

The directional derivative studies the behavior of a function only along a fixed direction. Existence of all directional derivatives is generally a weaker condition than Fréchet differentiability.

Definition 23.1.5. (Directional and Gâteaux derivatives) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces, let $U \subseteq X$ be open, and let $f\colon U \longrightarrow Y$. Given $x_0 \in U$ and $v \in X$, we say that f is **differentiable at x_0 in the direction v** if the limit $D_v f(x_0) := \lim_{t\to 0} \dfrac{f(x_0+tv)-f(x_0)}{t}$ exists. This value is called the **directional derivative** of f at x_0 in the direction v.
If $D_v f(x_0)$ exists for every $v \in X$ and the map $\delta f(x_0)\colon X \longrightarrow Y$, defined by $\delta f(x_0)(v) := D_v f(x_0)$, is linear and continuous, we say that f is **Gâteaux differentiable** at x_0. In this case, $\delta f(x_0)$ is called the **Gâteaux derivative** of f at x_0.

Mere existence of all directional derivatives does not guarantee Gâteaux differentiability, since dependence on the direction need be neither linear nor continuous.

Exercise 23.1.6. Consider the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f has a directional derivative at $(0,0)$ in every direction $v = (a,b) \in \mathbb{R}^2$ and that

$$D_v f(0,0) = \begin{cases} \dfrac{a^2}{b}, & \text{if } b \neq 0, \\ 0, & \text{if } b = 0. \end{cases}$$

Show that the map $v \longmapsto D_v f(0,0)$ is neither linear nor continuous, and conclude that f is not Gâteaux differentiable at $(0,0)$.

Fréchet differentiability does guarantee Gâteaux differentiability, and the two derivatives agree.

Proposition 23.1.7. Let X and Y be normed vector spaces, let $U \subseteq X$ be open, and let $f\colon U \longrightarrow Y$ be Fréchet differentiable at $x_0 \in U$. Then f is Gâteaux differentiable at x_0 and

$$\delta f(x_0) = Df(x_0).$$

Proof. Set $L := Df(x_0)$. By Fréchet differentiability, there exists a function r, defined for sufficiently small increments, such that

$$f(x_0+h) - f(x_0) = Lh + r(h), \qquad \frac{\|r(h)\|_Y}{\|h\|_X} \longrightarrow 0 \quad (h \longrightarrow 0).$$

Fix $v \in X$. For sufficiently small $t \neq 0$, we have

$$\frac{f(x_0+tv)-f(x_0)}{t} - Lv = \frac{r(tv)}{t}.$$

If $v = 0$, the left-hand side is identically zero. If $v \neq 0$, then

$$\left\| \frac{r(tv)}{t} \right\|_Y = \|v\|_X \frac{\|r(tv)\|_Y}{\|tv\|_X} \longrightarrow 0 \qquad (t \longrightarrow 0).$$

Consequently, $D_v f(x_0) = Lv$ for every $v \in X$. Since L is linear and continuous, f is Gâteaux differentiable at x_0 and $\delta f(x_0) = L = Df(x_0)$. □

The converse is false, even for scalar functions defined on finite-dimensional Euclidean spaces.

Exercise 23.1.8. Consider the function $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that $D_v g(0,0) = 0$ for every direction $v \in \mathbb{R}^2$. Conclude that g is Gâteaux differentiable at $(0,0)$ and that $\delta g(0,0) = 0$. Show, however, that $g(t,t^3) = \dfrac{1}{2}$ for every $t \neq 0$, so g is not continuous at $(0,0)$. Deduce from Proposition 23.1.3 that g is not Fréchet differentiable at that point.

In Hilbert spaces, continuous linear functionals can be identified with vectors by Theorem B.3.12. This allows the derivative of a scalar functional to be represented by its gradient.

Definition 23.1.9. (Gradient) Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space, let $U \subseteq H$ be open, and let $f\colon U \longrightarrow \mathbb{R}$ be Gâteaux differentiable at $x \in U$. By Theorem B.3.12, there exists a unique vector $\nabla f(x) \in H$ such that $\delta f(x)(v) = \langle v, \nabla f(x)\rangle_H$ for every $v \in H$. This vector is called the **gradient** of f at x. If f is Fréchet differentiable at x, then $Df(x)(v) = \langle v, \nabla f(x)\rangle_H$ for every $v \in H$.

Before studying mean value formulas, recall that, by Corollary B.2.30, for every real normed space Y and each $y \in Y \setminus \{0\}$, there exists $\varphi \in Y'$ such that $\|\varphi\|_{Y'} = 1$ and $\varphi(y) = \|y\|_Y$. In particular, the continuous dual of any normed space separates its points.

The following result estimates the increment of a function in terms of its directional derivatives along the segment joining the points under consideration. The second inequality also gives a quantitative estimate for the error in approximating that increment by the directional derivative at the initial point.

Theorem 23.1.10. (Mean value inequalities) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let $U \subseteq X$ be open, and let $f\colon U \longrightarrow Y$. Let $x, y \in U$ be such that the segment $[x,y] := \{x + t(y-x) \mid t \in [0,1]\}$ is contained in U. Suppose that, for each $t \in [0,1]$, the directional derivative $D_{y-x} f(x + t(y-x))$ exists. Then

$$\|f(y) - f(x)\|_Y \leq \sup_{t\in[0,1]} \|D_{y-x} f(x + t(y-x))\|_Y.$$

Moreover,

$$\|f(y) - f(x) - D_{y-x} f(x)\|_Y \leq \sup_{t\in[0,1]} \|D_{y-x} f(x + t(y-x)) - D_{y-x} f(x)\|_Y.$$

If, in addition, f is Fréchet differentiable at every point of $[x,y]$, then

$$\|f(y) - f(x)\|_Y \leq \sup_{t\in[0,1]} \big\|Df(x + t(y-x))(y-x)\big\|_Y \leq \left(\sup_{t\in[0,1]} \|Df(x + t(y-x))\|_{\mathcal{L}(X,Y)}\right) \|y - x\|_X,$$

and

$$\|f(y)-f(x)-Df(x)(y-x)\|_Y \le \sup_{t\in[0,1]} \Big\| \big(Df(x+t(y-x))-Df(x)\big)(y-x)\Big\|_Y$$

$$\le \left(\sup_{t\in[0,1]} \|Df(x+t(y-x))-Df(x)\|_{\mathcal{L}(X,Y)} \right) \|y-x\|_X.$$

In particular, if U is convex, f is Fréchet differentiable at every point of U, and there exists a constant $C \ge 0$ such that $\|Df(z)\|_{\mathcal{L}(X,Y)} \le C$ for every $z \in U$, then f is Lipschitz on U and

$$\|f(y)-f(x)\|_Y \le C\|y-x\|_X$$

for any $x, y \in U$.

Proof. If $f(y) = f(x)$, the first inequality is immediate. Suppose $f(y) \neq f(x)$. By Corollary B.2.30, there exists $\varphi \in Y'$ such that $\|\varphi\|_{Y'} = 1$ and $\varphi(f(y)-f(x)) = \|f(y)-f(x)\|_Y$.
Define $g\colon [0,1] \longrightarrow \mathbb{R}$ by $g(t) := \varphi(f(x+t(y-x)))$. Existence of $D_{y-x}f(x+t(y-x))$ for each $t \in [0,1]$ implies that g is continuous on $[0,1]$ and differentiable on $(0,1)$. For each $t \in (0,1)$, continuity of φ allows us to write

$$g'(t) = \lim_{s\to 0} \varphi\left(\frac{f(x+(t+s)(y-x))-f(x+t(y-x))}{s} \right) = \varphi\big(D_{y-x}f(x+t(y-x))\big).$$

By the classical mean value theorem, there exists $\theta \in (0,1)$ such that $g(1)-g(0) = g'(\theta)$. Consequently,

$$\|f(y)-f(x)\|_Y = \varphi(f(y)-f(x)) = \varphi\big(D_{y-x}f(x+\theta(y-x))\big)$$

$$\le \Big|\varphi\big(D_{y-x}f(x+\theta(y-x))\big)\Big| \le \|D_{y-x}f(x+\theta(y-x))\|_Y.$$

It follows that

$$\|f(y)-f(x)\|_Y \le \sup_{t\in[0,1]} \|D_{y-x}f(x+t(y-x))\|_Y.$$

To prove the second inequality, the result is immediate if $f(y)-f(x)-D_{y-x}f(x) = 0$. Suppose $f(y)-f(x)-D_{y-x}f(x) \neq 0$. By Corollary B.2.30, there exists $\psi \in Y'$ such that $\|\psi\|_{Y'} = 1$ and

$$\psi\big(f(y)-f(x)-D_{y-x}f(x)\big) = \|f(y)-f(x)-D_{y-x}f(x)\|_Y.$$

Define $q\colon [0,1] \longrightarrow \mathbb{R}$ by $q(t) := \psi(f(x+t(y-x))-f(x)-tD_{y-x}f(x))$. The function q is continuous on $[0,1]$ and differentiable on $(0,1)$, and, for each $t \in (0,1)$,

$$q'(t) = \psi\big(D_{y-x}f(x+t(y-x))-D_{y-x}f(x)\big).$$

By the classical mean value theorem, there exists $\vartheta \in (0,1)$ such that $q(1)-q(0) = q'(\vartheta)$. Since $q(0) = 0$, we obtain

$$\|f(y)-f(x)-D_{y-x}f(x)\|_Y = \psi\big(D_{y-x}f(x+\vartheta(y-x))-D_{y-x}f(x)\big)$$

$$\le \Big|\psi\big(D_{y-x}f(x+\vartheta(y-x))-D_{y-x}f(x)\big)\Big| \le \|D_{y-x}f(x+\vartheta(y-x))-D_{y-x}f(x)\|_Y.$$

Therefore,

$$\|f(y)-f(x)-D_{y-x}f(x)\|_Y \le \sup_{t\in[0,1]} \|D_{y-x}f(x+t(y-x))-D_{y-x}f(x)\|_Y.$$

Now suppose that f is Fréchet differentiable at every point of $[x,y]$. By Proposition 23.1.7, for each $t\in[0,1]$ we have $D_{y-x}f(x+t(y-x)) = Df(x+t(y-x))(y-x)$. The first inequality already proved then implies that

$$\|f(y)-f(x)\|_Y \le \sup_{t\in[0,1]} \big\|Df(x+t(y-x))(y-x)\big\|_Y.$$

On the other hand, for each $t\in[0,1]$,

$$\big\|Df(x+t(y-x))(y-x)\big\|_Y \le \|Df(x+t(y-x))\|_{\mathcal{L}(X,Y)}\|y-x\|_X.$$

Taking the supremum over $t\in[0,1]$ gives

$$\|f(y)-f(x)\|_Y \le \left(\sup_{t\in[0,1]} \|Df(x+t(y-x))\|_{\mathcal{L}(X,Y)}\right)\|y-x\|_X.$$

Likewise, $D_{y-x}f(x) = Df(x)(y-x)$ and, for each $t\in[0,1]$, we have

$$D_{y-x}f(x+t(y-x))-D_{y-x}f(x) = \big(Df(x+t(y-x))-Df(x)\big)(y-x).$$

The second inequality already proved implies that

$$\|f(y)-f(x)-Df(x)(y-x)\|_Y \le \sup_{t\in[0,1]} \big\|\big(Df(x+t(y-x))-Df(x)\big)(y-x)\big\|_Y.$$

For each $t\in[0,1]$,

$$\big\|\big(Df(x+t(y-x))-Df(x)\big)(y-x)\big\|_Y \le \|Df(x+t(y-x))-Df(x)\|_{\mathcal{L}(X,Y)}\|y-x\|_X.$$

Taking the supremum over $t\in[0,1]$ again gives

$$\|f(y)-f(x)-Df(x)(y-x)\|_Y \le \left(\sup_{t\in[0,1]} \|Df(x+t(y-x))-Df(x)\|_{\mathcal{L}(X,Y)}\right)\|y-x\|_X.$$

Finally, suppose that U is convex and $\|Df(z)\|_{\mathcal{L}(X,Y)}\le C$ for every $z\in U$. For any $x,y\in U$, the segment $[x,y]$ is contained in U and $\sup_{t\in[0,1]} \|Df(x+t(y-x))\|_{\mathcal{L}(X,Y)}\le C$. By the first estimate, $\|f(y)-f(x)\|_Y \le C\|y-x\|_X$, so f is Lipschitz on U. □

The preceding inequalities estimate the increment of the function without requiring continuity of the directional derivatives. Under the additional assumptions that these derivatives vary continuously along the segment and the target space is complete, one can obtain an integral identity for the increment.

Theorem 23.1.11. (Integral mean value formula) Let $(X,\|\cdot\|_X)$ be a real normed space, let $(Y,\|\cdot\|_Y)$ be a real Banach space, let $U\subseteq X$ be open, and let $f\colon U\longrightarrow Y$. Let $x,y\in U$ be such that the segment $[x,y] := \{x+t(y-x)\mid t\in[0,1]\}$ is contained in U. Suppose f has a directional derivative in the direction $y-x$ at every

point of $[x,y]$ and the map $F\colon [0,1] \longrightarrow Y$, defined by $F(t) := D_{y-x}f(x+t(y-x))$, is continuous. Then

$$f(y) - f(x) = \int_0^1 D_{y-x}f(x+t(y-x))\,dt.$$

Proof. By continuity of F and Theorem B.6.2, the Riemann integral $\displaystyle\int_0^1 F(t)\,dt$ is well defined. Let $\varphi \in Y'$ and define $g_\varphi\colon [0,1] \longrightarrow \mathbb{R}$ by $g_\varphi(t) := \varphi(f(x+t(y-x)))$. For each $t \in [0,1]$, interpreting derivatives at the endpoints as one-sided derivatives, continuity of φ allows us to write

$$g'_\varphi(t) = \lim_{s\to 0} \varphi\left(\frac{f(x+(t+s)(y-x)) - f(x+t(y-x))}{s}\right) = \varphi(F(t)).$$

Since F is continuous, so is g'_φ. By Theorem B.6.5 and Proposition B.6.4,

$$\varphi(f(y)-f(x)) = g_\varphi(1) - g_\varphi(0) = \int_0^1 \varphi(F(t))\,dt = \varphi\left(\int_0^1 F(t)\,dt\right).$$

Thus $\displaystyle\varphi\left(f(y)-f(x)-\int_0^1 F(t)\,dt\right) = 0$ for every $\varphi \in Y'$. Corollary B.2.30 implies that $\displaystyle f(y)-f(x)-\int_0^1 F(t)\,dt = 0$, proving the formula. □

The usual properties of the derivative also hold in this setting.

Theorem 23.1.12. (Properties of the directional derivative) Let X, Y, and Z be real normed spaces, let $U \subseteq X$ be an open set, and let $x_0 \in U$, $v \in X$, and $f, g\colon U \longrightarrow Y$. If $D_v f(x_0)$ and $D_v g(x_0)$ exist, then:

(a) $D_v(f+g)(x_0)$ exists and $D_v(f+g)(x_0) = D_v f(x_0) + D_v g(x_0)$.

(b) If $\lambda \in \mathbb{R}$, then $D_v(\lambda f)(x_0)$ exists and $D_v(\lambda f)(x_0) = \lambda D_v f(x_0)$.

(c) If $B\colon Y \times Y \longrightarrow Z$ is a continuous bilinear map, the function $B(f,g)\colon U \longrightarrow Z$, defined by $B(f,g)(x) = B(f(x),g(x))$, has a directional derivative at x_0 in the direction v, and

$$D_v\big(B(f,g)\big)(x_0) = B\big(D_v f(x_0), g(x_0)\big) + B\big(f(x_0), D_v g(x_0)\big).$$

(d) If $Y = \mathbb{R}$, g is continuous at x_0, and $g(x_0) \neq 0$, then $\displaystyle D_v\left(\frac{f}{g}\right)(x_0)$ exists and

$$D_v\left(\frac{f}{g}\right)(x_0) = \frac{D_v f(x_0) g(x_0) - f(x_0) D_v g(x_0)}{g(x_0)^2}.$$

Proof. For the first two items, simply apply linearity of limits in the normed space Y term by term. Let us prove the third. For brevity, write

$$f_t := f(x_0+tv), \qquad g_t := g(x_0+tv), \qquad f_0 := f(x_0), \qquad g_0 := g(x_0).$$

Existence of the directional derivatives implies $f_t \to f_0$ and $g_t \to g_0$ as $t \to 0$. By bilinearity of B,

$$\frac{B(f_t, g_t) - B(f_0, g_0)}{t} = B\left(\frac{f_t - f_0}{t}, g_t\right) + B\left(f_0, \frac{g_t - g_0}{t}\right).$$

Continuity of B allows passage to the limit and gives exactly the stated formula.
Finally, suppose $Y = \mathbb{R}$ and $g(x_0) \neq 0$. Continuity of g at x_0 ensures that $g_t \neq 0$ for sufficiently small t. Then

$$\frac{f_t/g_t - f_0/g_0}{t} = \frac{g_0(f_t - f_0) - f_0(g_t - g_0)}{t\, g_t g_0}.$$

Letting $t \to 0$ and using $g_t \to g_0$ and existence of both directional derivatives gives

$$D_v\left(\frac{f}{g}\right)(x_0) = \frac{D_v f(x_0) g(x_0) - f(x_0) D_v g(x_0)}{g(x_0)^2}.$$

□

Theorem 23.1.13. (Chain rule for directional derivatives) Let X, Y, and Z be real normed spaces, let $U \subseteq X$ and $V \subseteq Y$ be open sets, and let $f\colon U \longrightarrow Y$ and $g\colon V \longrightarrow Z$ be functions such that $f(U) \subseteq V$. Let $x_0 \in U$ and $v \in X$. If g is Fréchet differentiable at $f(x_0)$ and $D_v f(x_0)$ exists, then $D_v(g \circ f)(x_0)$ exists and

$$D_v(g \circ f)(x_0) = Dg(f(x_0))\big(D_v f(x_0)\big).$$

Proof. Since U and V are open, there exist $\delta_0 > 0$ and $\rho > 0$ such that $x_0 + tv \in U$ for every $t \in \mathbb{R}$ with $|t| < \delta_0$ and $B_{d_Y}(f(x_0), \rho) \subseteq V$. We proceed by cases.

(a) Case 1: $D_v f(x_0) \neq 0$.

Let $\varepsilon > 0$. Since g is Fréchet differentiable at $f(x_0)$, there exists $\delta_1 \in (0, \rho)$ such that if $0 < \|h\|_Y < \delta_1$, then

$$\frac{\|g(f(x_0) + h) - g(f(x_0)) - Dg(f(x_0))(h)\|_Z}{\|h\|_Y} < \frac{\varepsilon}{2\big(1 + \|D_v f(x_0)\|_Y\big)}.$$

On the other hand, since $D_v f(x_0)$ exists, there exists $\delta_2 > 0$ such that if $0 < |t| < \delta_2$, then

$$\left\|\frac{f(x_0 + tv) - f(x_0)}{t} - D_v f(x_0)\right\|_Y < \min\left\{1, \frac{\|D_v f(x_0)\|_Y}{2}, \frac{\varepsilon}{2\big(1 + \|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}\big)}\right\}.$$

Let $\delta = \min\left\{\delta_0, \delta_2, \dfrac{\delta_1}{1 + \|D_v f(x_0)\|_Y}\right\} > 0$ and let $t \in \mathbb{R}$ be such that $0 < |t| < \delta$. First note that

$$\frac{\|f(x_0 + tv) - f(x_0)\|_Y}{|t|} \leq \left\|\frac{f(x_0 + tv) - f(x_0)}{t} - D_v f(x_0)\right\|_Y + \|D_v f(x_0)\|_Y < 1 + \|D_v f(x_0)\|_Y.$$

On the other hand, by the reverse triangle inequality,

$$\frac{\|f(x_0 + tv) - f(x_0)\|_Y}{|t|} \geq \|D_v f(x_0)\|_Y - \left\|\frac{f(x_0 + tv) - f(x_0)}{t} - D_v f(x_0)\right\|_Y > \frac{\|D_v f(x_0)\|_Y}{2} > 0.$$

Consequently, $f(x_0 + tv) \neq f(x_0)$ and $0 < \|f(x_0 + tv) - f(x_0)\|_Y < |t|\big(1 + \|D_v f(x_0)\|_Y\big) < \delta_1$.

Adding and subtracting $\dfrac{Dg(f(x_0))\big(f(x_0+tv)-f(x_0)\big)}{t}$ and using the triangle inequality, the quantity

$$\left\|\frac{g(f(x_0+tv))-g(f(x_0))}{t}-Dg(f(x_0))\big(D_vf(x_0)\big)\right\|_Z$$

is bounded above by the sum of

$$\frac{\|g(f(x_0+tv))-g(f(x_0))-Dg(f(x_0))(f(x_0+tv)-f(x_0))\|_Z}{|t|}$$

and

$$\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}\left\|\frac{f(x_0+tv)-f(x_0)}{t}-D_vf(x_0)\right\|_Y.$$

The first summand equals

$$\frac{\|g(f(x_0+tv))-g(f(x_0))-Dg(f(x_0))(f(x_0+tv)-f(x_0))\|_Z}{\|f(x_0+tv)-f(x_0)\|_Y}\,\frac{\|f(x_0+tv)-f(x_0)\|_Y}{|t|},$$

, so

$$\frac{\|g(f(x_0+tv))-g(f(x_0))-Dg(f(x_0))(f(x_0+tv)-f(x_0))\|_Z}{|t|}<\frac{\varepsilon}{2}.$$

Likewise,

$$\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}\left\|\frac{f(x_0+tv)-f(x_0)}{t}-D_vf(x_0)\right\|_Y\le\frac{\varepsilon\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}}{2\big(1+\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}\big)}<\frac{\varepsilon}{2}.$$

Therefore,

$$\left\|\frac{g(f(x_0+tv))-g(f(x_0))}{t}-Dg(f(x_0))\big(D_vf(x_0)\big)\right\|_Z<\varepsilon.$$

Consequently, $D_v(g\circ f)(x_0)=Dg(f(x_0))\big(D_vf(x_0)\big)$.

(b) Case 2: $D_vf(x_0)=0$.

In this case, we must prove that $D_v(g\circ f)(x_0)=0$. Let $\varepsilon>0$. Since g is Fréchet differentiable at $f(x_0)$, there exists $\delta_1\in(0,\rho)$ such that if $0<\|h\|_Y<\delta_1$, then

$$\frac{\|g(f(x_0)+h)-g(f(x_0))-Dg(f(x_0))(h)\|_Z}{\|h\|_Y}<1.$$

On the other hand, since $D_vf(x_0)=0$, there exists $\delta_2>0$ such that if $0<|t|<\delta_2$, then

$$\frac{\|f(x_0+tv)-f(x_0)\|_Y}{|t|}<\min\left\{1,\frac{\varepsilon}{1+\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}}\right\}.$$

Let $\delta=\min\{\delta_0,\delta_1,\delta_2\}>0$ and let $t\in\mathbb{R}$ be such that $0<|t|<\delta$. If $f(x_0+tv)=f(x_0)$, then

$$\left\|\frac{g(f(x_0+tv))-g(f(x_0))}{t}\right\|_Z=0<\varepsilon.$$

Now suppose $f(x_0+tv)\neq f(x_0)$. In this case, $0<\|f(x_0+tv)-f(x_0)\|_Y<|t|<\delta\le\delta_1$.

Adding and subtracting $\dfrac{Dg(f(x_0))\big(f(x_0+tv)-f(x_0)\big)}{t}$ and using the triangle inequality, the quantity

$$\left\|\frac{g(f(x_0+tv))-g(f(x_0))}{t}\right\|_Z$$

is bounded above by the sum of

$$\frac{\|g(f(x_0+tv))-g(f(x_0))-Dg(f(x_0))(f(x_0+tv)-f(x_0))\|_Z}{|t|}$$

and

$$\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}\frac{\|f(x_0+tv)-f(x_0)\|_Y}{|t|}.$$

The first summand equals

$$\frac{\|g(f(x_0+tv))-g(f(x_0))-Dg(f(x_0))(f(x_0+tv)-f(x_0))\|_Z}{\|f(x_0+tv)-f(x_0)\|_Y}\,\frac{\|f(x_0+tv)-f(x_0)\|_Y}{|t|},$$

, so

$$\frac{\|g(f(x_0+tv))-g(f(x_0))-Dg(f(x_0))(f(x_0+tv)-f(x_0))\|_Z}{|t|}<\frac{\varepsilon}{1+\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}}.$$

Likewise,

$$\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}\frac{\|f(x_0+tv)-f(x_0)\|_Y}{|t|}\leq\frac{\varepsilon\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}}{1+\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)}}.$$

Therefore,

$$\left\|\frac{g(f(x_0+tv))-g(f(x_0))}{t}\right\|_Z<\varepsilon.$$

Consequently, $D_v(g\circ f)(x_0)=0=Dg(f(x_0))\big(D_vf(x_0)\big)$.

□

The chain rule for directional derivatives immediately gives the corresponding rule for the Gâteaux derivative. The version for the Fréchet derivative also requires checking that the linear approximation is uniform in the increments.

Corollary 23.1.14. (Chain rule for Gâteaux and Fréchet derivatives) Let X, Y, and Z be normed spaces, let $U\subseteq X$ and $V\subseteq Y$ be open sets, and let $f\colon U\longrightarrow Y$ and $g\colon V\longrightarrow Z$ be functions such that $f(U)\subseteq V$. Let $x_0\in U$ and suppose g is Fréchet differentiable at $f(x_0)$.
If f is Gâteaux differentiable at x_0, then $g\circ f$ is Gâteaux differentiable at x_0 and

$$\delta(g\circ f)(x_0)=Dg(f(x_0))\circ\delta f(x_0).$$

If f is Fréchet differentiable at x_0, then $g\circ f$ is Fréchet differentiable at x_0 and

$$D(g\circ f)(x_0)=Dg(f(x_0))\circ Df(x_0).$$

Proof. First suppose f is Gâteaux differentiable at x_0. For each $v \in X$, Theorem 23.1.13 implies that

$$D_v(g \circ f)(x_0) = Dg(f(x_0))\big(D_v f(x_0)\big) = Dg(f(x_0))\big(\delta f(x_0)(v)\big).$$

Since $Dg(f(x_0)) \circ \delta f(x_0) \in \mathcal{L}(X,Z)$, we conclude that $g \circ f$ is Gâteaux differentiable at x_0 and $\delta(g \circ f)(x_0) = Dg(f(x_0)) \circ \delta f(x_0)$.
Now suppose f is Fréchet differentiable at x_0. By Proposition 23.1.7, the first part shows that $g \circ f$ is Gâteaux differentiable at x_0 and the only candidate for its Fréchet derivative is $Dg(f(x_0)) \circ Df(x_0)$.
For each $h \in X \setminus \{0\}$ such that $x_0 + h \in U$, adding and subtracting $Dg(f(x_0))(f(x_0+h) - f(x_0))$ and using the triangle inequality, the quantity

$$\frac{\big\|g(f(x_0+h)) - g(f(x_0)) - Dg(f(x_0))\big(Df(x_0)(h)\big)\big\|_Z}{\|h\|_X}$$

is bounded above by the sum of

$$\frac{\big\|g(f(x_0+h)) - g(f(x_0)) - Dg(f(x_0))\big(f(x_0+h) - f(x_0)\big)\big\|_Z}{\|h\|_X}$$

and

$$\|Dg(f(x_0))\|_{\mathcal{L}(Y,Z)} \frac{\|f(x_0+h) - f(x_0) - Df(x_0)(h)\|_Y}{\|h\|_X}.$$

If $f(x_0+h) \neq f(x_0)$, the first quotient can be written as

$$\frac{\big\|g(f(x_0+h)) - g(f(x_0)) - Dg(f(x_0))\big(f(x_0+h) - f(x_0)\big)\big\|_Z}{\|f(x_0+h) - f(x_0)\|_Y} \frac{\|f(x_0+h) - f(x_0)\|_Y}{\|h\|_X}.$$

If $f(x_0+h) = f(x_0)$, that quotient equals zero. Moreover,

$$\frac{\|f(x_0+h) - f(x_0)\|_Y}{\|h\|_X} \leq \frac{\|f(x_0+h) - f(x_0) - Df(x_0)(h)\|_Y}{\|h\|_X} + \|Df(x_0)\|_{\mathcal{L}(X,Y)}.$$

By Fréchet differentiability of f at x_0, this last quotient remains bounded as $h \to 0$, while $f(x_0+h) \to f(x_0)$ by Proposition 23.1.3. Fréchet differentiability of g at $f(x_0)$ then implies that the first summand tends to zero, and differentiability of f implies the same for the second. Thus

$$\lim_{h \to 0} \frac{\big\|g(f(x_0+h)) - g(f(x_0)) - Dg(f(x_0))\big(Df(x_0)(h)\big)\big\|_Z}{\|h\|_X} = 0.$$

Consequently, $g \circ f$ is Fréchet differentiable at x_0 and $D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$. □

Corollary 23.1.15. (Composition with a continuous linear operator) Let X, Y, and Z be normed spaces, let $U \subseteq X$ be open, let $f : U \longrightarrow Y$, and let $A \in \mathcal{L}(Y,Z)$. Given $x_0 \in U$ and $v \in X$, the following assertions hold:

(a) If $D_v f(x_0)$ exists, then $D_v(A \circ f)(x_0)$ exists and $D_v(A \circ f)(x_0) = A\big(D_v f(x_0)\big)$.

(b) If f is Gâteaux differentiable at x_0, then $A \circ f$ is Gâteaux differentiable at x_0 and $\delta(A \circ f)(x_0) = A \circ \delta f(x_0)$.

(c) If f is Fréchet differentiable at x_0, then $A \circ f$ is Fréchet differentiable at x_0 and $D(A \circ f)(x_0) = A \circ Df(x_0)$.

Proof. For each $y \in Y$ and each $h \in Y \setminus \{0\}$, we have

$$\frac{\|A(y+h) - A(y) - A(h)\|_Z}{\|h\|_Y} = 0.$$

Thus A is Fréchet differentiable at every point of Y and $DA(y) = A$. The three assertions follow from Theorem 23.1.13 and Corollary 23.1.14. □

Existence of the Gâteaux derivative alone does not guarantee Fréchet differentiability. However, continuity of the Gâteaux derivative in the operator norm provides a sufficient condition.

Proposition 23.1.16. (Continuity of the Gâteaux derivative) Let X and Y be normed spaces, let $U \subseteq X$ be open, let $x_0 \in U$, and let $f : U \longrightarrow Y$. Suppose there exists $r > 0$ such that $B_{d_X}(x_0, r) \subseteq U$, f is Gâteaux differentiable at every point of $B_{d_X}(x_0, r)$, and the map $\delta f : B_{d_X}(x_0, r) \longrightarrow \mathcal{L}(X, Y)$ is continuous at x_0. Then f is Fréchet differentiable at x_0 and $Df(x_0) = \delta f(x_0)$.

Proof. Let $\varepsilon > 0$. By continuity of δf at x_0, there exists $\delta \in (0, r)$ such that $\|\delta f(z) - \delta f(x_0)\|_{\mathcal{L}(X,Y)} < \varepsilon$ for every $z \in B_{d_X}(x_0, \delta)$.
Let $h \in X$ be such that $0 < \|h\|_X < \delta$. For each $t \in [0,1]$, we have $\|x_0 + th - x_0\|_X = t\|h\|_X < \delta$, so $x_0 + th \in B_{d_X}(x_0, \delta)$.
Applying the second estimate in Theorem 23.1.10 to the points x_0 and $x_0 + h$ gives

$$\|f(x_0+h) - f(x_0) - D_h f(x_0)\|_Y \leq \sup_{t\in[0,1]} \|D_h f(x_0 + th) - D_h f(x_0)\|_Y.$$

Since f is Gâteaux differentiable at every point of the segment, for each $t \in [0,1]$ we have $D_h f(x_0 + th) = \delta f(x_0 + th)(h)$ and $D_h f(x_0) = \delta f(x_0)(h)$. Thus

$$\|f(x_0+h) - f(x_0) - \delta f(x_0)(h)\|_Y \leq \sup_{t\in[0,1]} \left\| \big(\delta f(x_0 + th) - \delta f(x_0)\big)(h) \right\|_Y.$$

By the definition of the operator norm and the choice of δ, for each $t \in [0,1]$ we have

$$\left\| \big(\delta f(x_0 + th) - \delta f(x_0)\big)(h) \right\|_Y \leq \|\delta f(x_0 + th) - \delta f(x_0)\|_{\mathcal{L}(X,Y)} \|h\|_X < \varepsilon \|h\|_X.$$

Consequently, $\|f(x_0 + h) - f(x_0) - \delta f(x_0)(h)\|_Y \leq \varepsilon \|h\|_X$ and therefore

$$\frac{\|f(x_0+h) - f(x_0) - \delta f(x_0)(h)\|_Y}{\|h\|_X} \leq \varepsilon.$$

This proves that f is Fréchet differentiable at x_0 and that $Df(x_0) = \delta f(x_0)$. □

The preceding proposition allows us to characterize functions of class C^1 using the Gâteaux derivative and the notion of differentiability already defined.

Corollary 23.1.17. (Characterization of functions of class C^1) Let X and Y be normed spaces, let $U \subseteq X$ be open, and let $f : U \longrightarrow Y$. Then $f \in C^1(U, Y)$ if and only if f is Gâteaux differentiable at every point of U and the map $\delta f : U \longrightarrow \mathcal{L}(X, Y)$ is continuous. In this case, $Df(x) = \delta f(x)$ for every $x \in U$.

Proof. If $f \in C^1(U,Y)$, Proposition 23.1.7 implies that f is Gâteaux differentiable at every point of U and that $\delta f = Df$. Continuity of δf follows from continuity of Df.
Conversely, suppose f is Gâteaux differentiable at every point of U and $\delta f\colon U \longrightarrow \mathcal{L}(X,Y)$ is continuous. Let $x \in U$. Since U is open, there exists $r > 0$ such that $B_{d_X}(x,r) \subseteq U$. Proposition 23.1.16 implies that f is Fréchet differentiable at x and that $Df(x) = \delta f(x)$. Since $x \in U$ was arbitrary, f is Fréchet differentiable on U and $Df = \delta f$. Continuity of δf then implies continuity of Df. By Definition 23.1.4, we conclude that $f \in C^1(U,Y)$. □

The notion of partial derivative also extends naturally to normed spaces. Here it is useful to distinguish Gâteaux and Fréchet partial derivatives according to the notion of differentiability used when the other variables are held fixed. On the product space $X \times Y$, we use the norm $\|(h,k)\|_{X\times Y} := \|h\|_X + \|k\|_Y$.

Definition 23.1.18. (Gâteaux and Fréchet partial derivatives) Let X, Y, and Z be normed spaces, let $U \subseteq X$ and $V \subseteq Y$ be open, let $f\colon U \times V \longrightarrow Z$, and let $(x,y) \in U \times V$.
If the function $f(\cdot,y)\colon U \longrightarrow Z$ is Gâteaux differentiable at x, the **Gâteaux partial derivative of f with respect to the first variable** is defined by $\delta_1 f(x,y) := \delta(f(\cdot,y))(x) \in \mathcal{L}(X,Z)$. If $f(x,\cdot)\colon V \longrightarrow Z$ is Gâteaux differentiable at y, the **Gâteaux partial derivative with respect to the second variable** is defined by $\delta_2 f(x,y) := \delta(f(x,\cdot))(y) \in \mathcal{L}(Y,Z)$.
If $f(\cdot,y)$ is Fréchet differentiable at x, the **Fréchet partial derivative of f with respect to the first variable** is defined by $D_1 f(x,y) := D(f(\cdot,y))(x) \in \mathcal{L}(X,Z)$. If $f(x,\cdot)$ is Fréchet differentiable at y, the **Fréchet partial derivative with respect to the second variable** is defined by $D_2 f(x,y) := D(f(x,\cdot))(y) \in \mathcal{L}(Y,Z)$.

Partial derivatives are obtained by restricting the total derivative to the directions corresponding to each factor of the product.

Proposition 23.1.19. Let X, Y, and Z be normed spaces, let $U \subseteq X$ and $V \subseteq Y$ be open, let $f\colon U \times V \longrightarrow Z$, and let $(x,y) \in U \times V$.
If f is Gâteaux differentiable at (x,y), then $\delta_1 f(x,y)$ and $\delta_2 f(x,y)$ exist and, for every $(h,k) \in X \times Y$,

$$\delta f(x,y)(h,k) = \delta_1 f(x,y)(h) + \delta_2 f(x,y)(k). \tag{23.1}$$

If f is Fréchet differentiable at (x,y), then $D_1 f(x,y)$ and $D_2 f(x,y)$ exist and, for every $(h,k) \in X \times Y$,

$$Df(x,y)(h,k) = D_1 f(x,y)(h) + D_2 f(x,y)(k). \tag{23.2}$$

Proof. First suppose f is Gâteaux differentiable at (x,y). Let $h \in X$. When only the first component varies, the increment th of x corresponds to the increment $(th,0) = t(h,0)$ of the point (x,y). Thus

$$\delta_1 f(x,y)(h) = \lim_{t\to 0} \frac{f(x+th,y) - f(x,y)}{t} = \lim_{t\to 0} \frac{f\big((x,y) + t(h,0)\big) - f(x,y)}{t} = \delta f(x,y)(h,0).$$

This proves that $\delta_1 f(x,y)$ exists and agrees with the restriction of $\delta f(x,y)$ to the subspace $X \times \{0\}$. For the second variable, varying only that component corresponds to the increment $(0,tk) = t(0,k)$. Hence, for each $k \in Y$,

$$\delta_2 f(x,y)(k) = \lim_{t\to 0} \frac{f(x,y+tk) - f(x,y)}{t} = \delta f(x,y)(0,k).$$

Since $(h,k) = (h,0) + (0,k)$ and $\delta f(x,y)$ is linear, it follows that

$$\delta f(x,y)(h,k) = \delta f(x,y)(h,0) + \delta f(x,y)(0,k) = \delta_1 f(x,y)(h) + \delta_2 f(x,y)(k),$$

, which proves (23.1).
Now suppose f is Fréchet differentiable at (x,y). For each $h \in X \setminus \{0\}$ such that $x+h \in U$, we have

$$\frac{\|f(x+h,y) - f(x,y) - Df(x,y)(h,0)\|_Z}{\|h\|_X} = \frac{\|f\big((x,y)+(h,0)\big) - f(x,y) - Df(x,y)(h,0)\|_Z}{\|(h,0)\|_{X\times Y}}.$$

The right-hand side tends to zero as $h \to 0$, since $(h,0) \to (0,0)$ in $X \times Y$. Consequently, $f(\cdot, y)$ is Fréchet differentiable at x and $D_1 f(x,y)(h) = Df(x,y)(h,0)$ for every $h \in X$.
For the second variable, if $k \in Y \setminus \{0\}$ and $y + k \in V$, then

$$\frac{\|f(x,y+k) - f(x,y) - Df(x,y)(0,k)\|_Z}{\|k\|_Y} = \frac{\|f\big((x,y)+(0,k)\big) - f(x,y) - Df(x,y)(0,k)\|_Z}{\|(0,k)\|_{X\times Y}}.$$

The last quotient tends to zero as $k \to 0$. Thus $D_2 f(x,y)(k) = Df(x,y)(0,k)$ for every $k \in Y$. Linearity of $Df(x,y)$ then implies

$$Df(x,y)(h,k) = Df(x,y)(h,0) + Df(x,y)(0,k) = D_1 f(x,y)(h) + D_2 f(x,y)(k),$$

, which proves (23.2). □

The converse generally does not follow from mere existence of the partial derivatives. However, continuity of one of them in the operator norm allows us to recover total differentiability.

Proposition 23.1.20. (Differentiability from partial derivatives) Let X, Y, and Z be normed spaces, let $U \subseteq X$ and $V \subseteq Y$ be open, let $f\colon U \times V \longrightarrow Z$, and let $(x_0,y_0) \in U \times V$. Suppose $\delta_1 f(x_0,y_0)$ exists, $\delta_2 f$ is defined on an open neighborhood $W \subseteq U \times V$ of (x_0,y_0), and the map

$$(x,y) \longmapsto \delta_2 f(x,y), \qquad W \longrightarrow \mathcal{L}(Y,Z),$$

is continuous at (x_0,y_0). Then f is Gâteaux differentiable at (x_0,y_0), and identity (23.1) holds with $(x,y) = (x_0,y_0)$.
If, in addition, $D_1 f(x_0,y_0)$ exists, then f is Fréchet differentiable at (x_0,y_0), and identity (23.2) holds with $(x,y) = (x_0,y_0)$. In this case,

$$D_2 f(x_0,y_0) = \delta_2 f(x_0,y_0).$$

Proof. Let $W \subseteq U \times V$ be the open neighborhood in the statement. Take $r > 0$ such that $B_{d_{X\times Y}}((x_0,y_0),r) \subseteq W$. Fix $(h,k) \in X \times Y$. The case $(h,k) = (0,0)$ is immediate. Suppose $(h,k) \neq (0,0)$ and $0 < |t| < \dfrac{r}{\|h\|_X + \|k\|_Y}$. For each $\tau \in [0,1]$, we have $\|(th, \tau tk)\|_{X\times Y} \leq |t|(\|h\|_X + \|k\|_Y) < r$; hence the segment joining (x_0+th, y_0) to (x_0+th, y_0+tk) is contained in W.
Adding and subtracting $f(x_0+th, y_0)$ gives

$$\frac{f(x_0+th, y_0+tk) - f(x_0,y_0)}{t} - \delta_1 f(x_0,y_0)(h) - \delta_2 f(x_0,y_0)(k)$$

$$= \frac{f(x_0+th, y_0+tk) - f(x_0+th, y_0) - \delta_2 f(x_0,y_0)(tk)}{t} + \frac{f(x_0+th,y_0)-f(x_0,y_0)}{t} - \delta_1 f(x_0,y_0)(h).$$

To estimate the first term, apply the first mean value inequality in Theorem 23.1.10 to the map $v \longmapsto f(x_0+th,v) - \delta_2 f(x_0,y_0)(v-y_0)$ between the points y_0 and y_0+tk. Its directional derivative in the direction tk is $\big(\delta_2 f(x_0+th,v) - \delta_2 f(x_0,y_0)\big)(tk)$. Dividing the resulting estimate by $|t|$ gives

$$\left\| \frac{f(x_0+th,y_0+tk) - f(x_0+th,y_0) - \delta_2 f(x_0,y_0)(tk)}{t} \right\|_Z \leq \sup_{\tau\in[0,1]} \|\delta_2 f(x_0+th, y_0+\tau tk) - \delta_2 f(x_0,y_0)\|_{\mathcal{L}(Y,Z)} \|k\|_Y.$$

Let $\varepsilon > 0$. By continuity of $\delta_2 f$ at (x_0,y_0), there exists $\eta \in (0,r)$ such that $\|\delta_2 f(u,v) - \delta_2 f(x_0,y_0)\|_{\mathcal{L}(Y,Z)} < \varepsilon$ whenever $(u,v) \in W$ and $\|(u-x_0, v-y_0)\|_{X\times Y} < \eta$. If $0 < |t| < \dfrac{\eta}{\|h\|_X + \|k\|_Y}$, then $\|(th,\tau tk)\|_{X\times Y} < \eta$ for every $\tau \in [0,1]$, and consequently

$$\left\| \frac{f(x_0+th,y_0+tk) - f(x_0+th,y_0)}{t} - \delta_2 f(x_0,y_0)(k) \right\|_Z \leq \varepsilon \|k\|_Y.$$

Thus

$$\lim_{t\to 0} \left(\frac{f(x_0+th,y_0+tk) - f(x_0+th,y_0)}{t} - \delta_2 f(x_0,y_0)(k) \right) = 0.$$

On the other hand, existence of $\delta_1 f(x_0,y_0)$ implies

$$\lim_{t\to 0} \left(\frac{f(x_0+th,y_0) - f(x_0,y_0)}{t} - \delta_1 f(x_0,y_0)(h) \right) = 0.$$

Combining the two limits, we conclude that the directional derivative of f at (x_0,y_0) in the direction (h,k) exists and equals

$$\delta_1 f(x_0,y_0)(h) + \delta_2 f(x_0,y_0)(k).$$

The map $(h,k) \longmapsto \delta_1 f(x_0,y_0)(h) + \delta_2 f(x_0,y_0)(k)$ is linear. Moreover,

$$\|\delta_1 f(x_0,y_0)(h) + \delta_2 f(x_0,y_0)(k)\|_Z \leq \max\big\{\|\delta_1 f(x_0,y_0)\|_{\mathcal{L}(X,Z)}, \|\delta_2 f(x_0,y_0)\|_{\mathcal{L}(Y,Z)}\big\} \|(h,k)\|_{X\times Y},$$

, so it is continuous. Thus f is Gâteaux differentiable at (x_0,y_0) and (23.1) holds.
Now suppose $D_1 f(x_0,y_0)$ exists and define the operator $L\colon X\times Y \longrightarrow Z$ by

$$L(h,k) := D_1 f(x_0,y_0)(h) + \delta_2 f(x_0,y_0)(k).$$

The operator L is linear and continuous. We will show that it is the Fréchet approximation to f at (x_0,y_0). Let $\varepsilon > 0$. By continuity of $\delta_2 f$ at (x_0,y_0), there exists $\eta \in (0,r)$ such that $\|\delta_2 f(u,v) - \delta_2 f(x_0,y_0)\|_{\mathcal{L}(Y,Z)} < \dfrac{\varepsilon}{2}$ whenever $(u,v) \in W$ and $\|(u-x_0, v-y_0)\|_{X\times Y} < \eta$. Since $D_1 f(x_0,y_0)$ exists, there exists $\rho > 0$ such that

$$\|f(x_0+h,y_0) - f(x_0,y_0) - D_1 f(x_0,y_0)(h)\|_Z \leq \frac{\varepsilon}{2}\|h\|_X$$

whenever $0 < \|h\|_X < \rho$; for $h = 0$, both sides are zero.
Let $\delta := \min\{\eta,\rho\}$ and suppose $0 < \|h\|_X + \|k\|_Y < \delta$. Then, for every $\tau \in [0,1]$, we have $\|(h,\tau k)\|_{X\times Y} < \eta < r$, so

the segment joining (x_0+h, y_0) to (x_0+h, y_0+k) is contained in W. Adding and subtracting $f(x_0+h, y_0)$ gives

$$f(x_0+h, y_0+k) - f(x_0, y_0) - L(h,k)$$

$$= f(x_0+h, y_0+k) - f(x_0+h, y_0) - \delta_2 f(x_0, y_0)(k) \quad + f(x_0+h, y_0) - f(x_0, y_0) - D_1 f(x_0, y_0)(h).$$

Apply the first mean value inequality again, now to the map $v \longmapsto f(x_0+h, v) - \delta_2 f(x_0, y_0)(v - y_0)$ between y_0 and y_0+k. This gives

$$\|f(x_0+h, y_0+k) - f(x_0+h, y_0) - \delta_2 f(x_0, y_0)(k)\|_Z \leq \sup_{\tau \in [0,1]} \|\delta_2 f(x_0+h, y_0+\tau k) - \delta_2 f(x_0, y_0)\|_{\mathcal{L}(Y,Z)} \|k\|_Y \leq \frac{\varepsilon}{2}\|k\|_Y.$$

Consequently,

$$\|f(x_0+h, y_0+k) - f(x_0, y_0) - L(h,k)\|_Z \leq \frac{\varepsilon}{2}\big(\|h\|_X + \|k\|_Y\big).$$

It follows that

$$\lim_{(h,k)\to(0,0)} \frac{\|f(x_0+h, y_0+k) - f(x_0, y_0) - L(h,k)\|_Z}{\|h\|_X + \|k\|_Y} = 0.$$

Thus f is Fréchet differentiable at (x_0, y_0), $Df(x_0, y_0) = L$, and (23.2) holds.
Finally, Proposition 23.1.19 implies that $D_2 f(x_0, y_0)$ exists and that, for every $k \in Y$,

$$D_2 f(x_0, y_0)(k) = Df(x_0, y_0)(0, k) = \delta_2 f(x_0, y_0)(k).$$

Therefore, $D_2 f(x_0, y_0) = \delta_2 f(x_0, y_0)$. □

Corollary 23.1.21. (Characterization of functions of class C^1 by partial derivatives) Let X, Y, and Z be normed spaces, let $U \subseteq X$ and $V \subseteq Y$ be open, and let $f\colon U \times V \longrightarrow Z$. Then $f \in C^1(U \times V, Z)$ if and only if $\delta_1 f$ and $\delta_2 f$ exist on $U \times V$ and are continuous as maps with values in $\mathcal{L}(X, Z)$ and $\mathcal{L}(Y, Z)$, respectively. In this case, $D_1 f = \delta_1 f$, $D_2 f = \delta_2 f$, and, for every $(x, y) \in U \times V$ and every $(h, k) \in X \times Y$,

$$Df(x, y)(h, k) = \delta_1 f(x, y)(h) + \delta_2 f(x, y)(k).$$

Proof. First suppose $f \in C^1(U \times V, Z)$. Let $\iota_X\colon X \longrightarrow X \times Y$ and $\iota_Y\colon Y \longrightarrow X \times Y$ be the continuous linear operators defined by $\iota_X(h) := (h, 0)$ and $\iota_Y(k) := (0, k)$. By Proposition 23.1.19, for each $(x, y) \in U \times V$ we have $D_1 f(x, y) = Df(x, y) \circ \iota_X$ and $D_2 f(x, y) = Df(x, y) \circ \iota_Y$. Since $Df\colon U \times V \longrightarrow \mathcal{L}(X \times Y, Z)$ is continuous, so are $D_1 f$ and $D_2 f$. Moreover, Fréchet differentiability implies Gâteaux differentiability, and therefore $D_j f = \delta_j f$ for $j \in \{1, 2\}$.
Conversely, suppose $\delta_1 f$ and $\delta_2 f$ exist and are continuous. For fixed $y \in V$, the map $x \longmapsto f(x, y)$ has Gâteaux derivative $\delta_1 f(x, y)$, continuous in the operator norm with respect to x. Corollary 23.1.17 shows that this map is of class C^1; in particular, $D_1 f(x, y)$ exists and agrees with $\delta_1 f(x, y)$. All the assumptions of Proposition 23.1.20 are now satisfied. Applying it at each point of $U \times V$, we obtain that f is Fréchet differentiable and

$$Df(x, y)(h, k) = \delta_1 f(x, y)(h) + \delta_2 f(x, y)(k).$$

Let $(x_1, y_1), (x_2, y_2) \in U \times V$. For every $(h, k) \in X \times Y$ with $\|(h, k)\|_{X \times Y} \leq 1$, we have

$$\big\|\big(Df(x_1, y_1) - Df(x_2, y_2)\big)(h, k)\big\|_Z \leq \|\delta_1 f(x_1, y_1) - \delta_1 f(x_2, y_2)\|_{\mathcal{L}(X,Z)} \|h\|_X + \|\delta_2 f(x_1, y_1) - \delta_2 f(x_2, y_2)\|_{\mathcal{L}(Y,Z)} \|k\|_Y.$$

Taking the supremum over the unit ball of $X \times Y$ gives

$$\|Df(x_1,y_1) - Df(x_2,y_2)\|_{\mathcal{L}(X\times Y,Z)} \le \|\delta_1 f(x_1,y_1) - \delta_1 f(x_2,y_2)\|_{\mathcal{L}(X,Z)} + \|\delta_2 f(x_1,y_1) - \delta_2 f(x_2,y_2)\|_{\mathcal{L}(Y,Z)}.$$

Continuity of the partial derivatives then implies continuity of Df. □

23.2 Higher derivatives and Taylor's formula

The first derivative takes values in $\mathcal{L}(X,Y)$. Differentiating it again produces operators that depend linearly on two directions, and further iterations lead to multilinear operators. Before defining higher derivatives, it is therefore useful to specify the normed structure of these spaces and their relation to the spaces of linear operators already introduced.

Definition 23.2.1. (Continuous multilinear operators) Let $X_1,\dots,X_m$ and Y be normed spaces, with $m \in \mathbb{N}$. We denote by $\mathcal{L}^m(X_1,\dots,X_m;Y)$ the vector space of continuous m-linear maps $A\colon X_1 \times \cdots \times X_m \longrightarrow Y$. For each $A \in \mathcal{L}^m(X_1,\dots,X_m;Y)$, define

$$\|A\|_{\mathcal{L}^m} := \sup\big\{\|A(h_1,\dots,h_m)\|_Y \mid h_j \in X_j,\ \|h_j\|_{X_j} \le 1 \text{ for } 1 \le j \le m\big\}.$$

If $X_1 = \cdots = X_m = X$, write $\mathcal{L}^m(X;Y)$. In particular, $\mathcal{L}^1(X;Y) = \mathcal{L}(X,Y)$. Denote by $\mathcal{L}^m_s(X;Y)$ the subspace of symmetric operators, that is, those $A \in \mathcal{L}^m(X;Y)$ such that $A(h_{\sigma(1)},\dots,h_{\sigma(m)}) = A(h_1,\dots,h_m)$ for every permutation σ of $\{1,\dots,m\}$. By convention, $\mathcal{L}^0(X;Y) := Y$.

Proposition 23.2.2. (Characterization of multilinear continuity) A m-linear map $A\colon X_1 \times \cdots \times X_m \longrightarrow Y$ is continuous if and only if there exists a constant $C \ge 0$ such that

$$\|A(h_1,\dots,h_m)\|_Y \le C \prod_{j=1}^{m} \|h_j\|_{X_j} \tag{23.3}$$

for all $h_j \in X_j$. In this case, the smallest constant for which (23.3) holds is $\|A\|_{\mathcal{L}^m}$.

Proof. First suppose (23.3) holds. If $h_j \longrightarrow 0$ in X_j for each j, the right-hand side of (23.3) tends to zero, so A is continuous at the origin. Now let $x_j, h_j \in X_j$. By multilinearity,

$$A(x_1+h_1,\dots,x_m+h_m) - A(x_1,\dots,x_m) = \sum_{\varnothing \ne I \subseteq \{1,\dots,m\}} A(z_1^I,\dots,z_m^I),$$

, where $z_j^I = h_j$ if $j \in I$ and $z_j^I = x_j$ if $j \notin I$. Inequality (23.3), applied to each summand, shows that the difference tends to zero as $(h_1,\dots,h_m) \longrightarrow (0,\dots,0)$. Thus A is continuous at every point.
Conversely, suppose A is continuous at the origin. There exists $\rho > 0$ such that $\|A(u_1,\dots,u_m)\|_Y < 1$ whenever $\|u_j\|_{X_j} < \rho$ for every j. If none of the vectors h_j is zero, define $u_j := \dfrac{\rho h_j}{2\|h_j\|_{X_j}}$. Then $\|u_j\|_{X_j} = \dfrac{\rho}{2}$ and, by multilinearity,

$$\|A(h_1,\dots,h_m)\|_Y < \left(\frac{2}{\rho}\right)^m \prod_{j=1}^{m} \|h_j\|_{X_j}.$$

If one of the vectors is zero, both sides of (23.3) vanish. This proves existence of a constant C.
Finally, if (23.3) holds with constant C, then $\|A\|_{\mathcal{L}^m} \le C$. On the other hand, for nonzero vectors, applying the definition of the norm to $\dfrac{h_j}{\|h_j\|_{X_j}}$ gives

$$\|A(h_1,\dots,h_m)\|_Y \le \|A\|_{\mathcal{L}^m} \prod_{j=1}^{m} \|h_j\|_{X_j}.$$

Thus $\|A\|_{\mathcal{L}^m}$ is the smallest possible constant. □

Proposition 23.2.3. (Completeness and association of multilinear operators) If Y is Banach, then $\mathcal{L}^m(X_1,\dots,X_m;Y)$ is Banach. Moreover, the map

$$\mathfrak{C} : \mathcal{L}^{m+1}(X_1,\dots,X_m,X_{m+1};Y) \longrightarrow \mathcal{L}\big(X_{m+1},\mathcal{L}^m(X_1,\dots,X_m;Y)\big),$$

, defined by

$$\mathfrak{C}(A)(h_{m+1})(h_1,\dots,h_m) := A(h_1,\dots,h_m,h_{m+1}),$$

, is an isometric linear isomorphism. In particular, if Y is Banach, then $\mathcal{L}^m_s(X;Y)$ is a closed subspace and hence a Banach space.

Proof. Let $(A_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^m(X_1,\dots,X_m;Y)$. For each $(h_1,\dots,h_m) \in X_1\times\cdots\times X_m$, we have

$$\|(A_j - A_k)(h_1,\dots,h_m)\|_Y \le \|A_j - A_k\|_{\mathcal{L}^m} \prod_{i=1}^{m} \|h_i\|_{X_i}.$$

Thus $(A_j(h_1,\dots,h_m))_{j\in\mathbb{N}}$ is a Cauchy sequence in Y. Define $A(h_1,\dots,h_m) := \lim\limits_{j\to\infty} A_j(h_1,\dots,h_m)$. Passage to the limit in each variable shows that A is m-linear.
Since (A_j) is Cauchy, there exists $j_0 \in \mathbb{N}$ such that $\|A_j - A_k\|_{\mathcal{L}^m} < 1$ for all $j,k \ge j_0$. Letting $k\to\infty$ in the preceding estimate gives

$$\|(A_j - A)(h_1,\dots,h_m)\|_Y \le \prod_{i=1}^{m} \|h_i\|_{X_i}$$

for every $j \ge j_0$. Proposition 23.2.2 implies that $A_j - A$ is continuous. Moreover, given $\varepsilon > 0$, there exists $j_1 \in \mathbb{N}$ such that $\|A_j - A_k\|_{\mathcal{L}^m} < \varepsilon$ for all $j,k \ge j_1$. Letting $k\to\infty$ and taking the supremum over the unit balls gives $\|A_j - A\|_{\mathcal{L}^m} \le \varepsilon$ for every $j \ge j_1$. Thus $A_j \longrightarrow A$, and the space is complete.
It is immediate that $\mathfrak{C}$ is linear. For each A, we have

$$\|\mathfrak{C}(A)\| = \sup_{\substack{h_{m+1}\in X_{m+1}\\ \|h_{m+1}\|_{X_{m+1}}\le 1}} \ \sup_{\substack{h_i\in X_i,\ 1\le i\le m\\ \|h_i\|_{X_i}\le 1}} \|A(h_1,\dots,h_m,h_{m+1})\|_Y = \|A\|_{\mathcal{L}^{m+1}},$$

, so it is an isometry. If $T \in \mathcal{L}(X_{m+1},\mathcal{L}^m(X_1,\dots,X_m;Y))$, the map $A_T(h_1,\dots,h_{m+1}) := T(h_{m+1})(h_1,\dots,h_m)$ is continuous and $(m+1)$-linear and satisfies $\mathfrak{C}(A_T) = T$. Thus $\mathfrak{C}$ is surjective.
Finally, if a sequence of symmetric operators converges in norm to A, passage to the limit in the symmetry identity shows that A is also symmetric. Thus $\mathcal{L}^m_s(X;Y)$ is closed. □

For $A \in \mathcal{L}^m(X;Y)$, write $A[h_1,\dots,h_m] := A(h_1,\dots,h_m)$ and $A[h]^m := A(h,\dots,h)$.

Proposition 23.2.4. (Differentiability of multilinear operators) Let $X_1,\dots,X_m$ and Y be normed spaces, and let $A \in \mathcal{L}^m(X_1,\dots,X_m;Y)$. Regarded as a map on the product space, A is smooth and

$$DA(x_1,\dots,x_m)(h_1,\dots,h_m) = \sum_{j=1}^{m} A(x_1,\dots,x_{j-1},h_j,x_{j+1},\dots,x_m).$$

Proof. By multilinearity,

$$A(x_1+h_1,\dots,x_m+h_m) - A(x_1,\dots,x_m)$$

is the sum of the terms in which at least one of the variables x_j has been replaced by h_j. The terms containing exactly one increment form the linear map in the statement. The remaining terms contain at least two increments. If $\max\limits_{j\in\{1,\dots,m\}} \|h_j\|_{X_j} < 1$, Proposition 23.2.2 bounds the norm of their sum by

$$\|A\|_{\mathcal{L}^m} \sum_{\substack{I\subseteq\{1,\dots,m\}\\ \#I\geq 2}} \prod_{j\in I} \|h_j\|_{X_j} \prod_{j\in\{1,\dots,m\}\setminus I} \max\{1,\|x_j\|_{X_j}\}.$$

After division by $\sum\limits_{j=1}^{m} \|h_j\|_{X_j}$, each summand tends to zero as $(h_1,\dots,h_m) \longrightarrow 0$. This proves the formula for the first derivative. To make the successive derivatives explicit, write $E := X_1 \times \cdots \times X_m$, with the sum norm, and fix $k \in \{1,\dots,m\}$. Let $\mathcal{I}_k$ be the set of injective maps $\sigma\colon \{1,\dots,k\} \longrightarrow \{1,\dots,m\}$. For $x = (x_1,\dots,x_m) \in E$ and $h^{(a)} = (h_1^{(a)},\dots,h_m^{(a)}) \in E$, with $a \in \{1,\dots,k\}$, set

$$z_i^\sigma := \begin{cases} h_i^{(a)}, & i = \sigma(a) \text{ for some } a \in \{1,\dots,k\},\\ x_i, & i \in \{1,\dots,m\} \setminus \operatorname{im}(\sigma). \end{cases}$$

Injectivity of σ makes this definition unambiguous. Define the continuous k–linear form

$$B_k(x)[h^{(1)},\dots,h^{(k)}] := \sum_{\sigma\in\mathcal{I}_k} A(z_1^\sigma,\dots,z_m^\sigma).$$

The formula already proved says that $DA = B_1$. Now suppose $1 \leq k < m$. To compute the increment of B_k when x is replaced by $x+u$, expand each summand by multilinearity. The terms linear in u are obtained by choosing an index $i \in \{1,\dots,m\} \setminus \operatorname{im}(\sigma)$ and replacing x_i by u_i. To each pair (σ,i) there corresponds exactly one map in $\mathcal{I}_{k+1}$, agreeing with σ on $\{1,\dots,k\}$ and sending $k+1$ to i. The sum of these terms is therefore $B_{k+1}(x)[h^{(1)},\dots,h^{(k)},u]$.

The remaining terms contain at least two components of u. If $\|u\|_E \leq 1$, the multilinear bound estimates them together by

$$C_{A,x,k}\|u\|_E^2 \prod_{a=1}^{k} \|h^{(a)}\|_E,$$

, where $C_{A,x,k}$ is finite: only finitely many summands, the norm of A, and the factors $\max\{1,\|x_i\|_{X_i}\}$, with $i \in \{1,\dots,m\}$, occur. Taking the supremum over $\|h^{(a)}\|_E \leq 1$ for $a \in \{1,\dots,k\}$ and dividing by $\|u\|_E$, we obtain that B_k is Fréchet differentiable as a map with values in $\mathcal{L}^k(E;Y)$. Its derivative corresponds to B_{k+1} through

Proposition 23.2.3. The same expansion, retaining all terms with at least one increment, proves norm continuity of each B_k.
This completes the step from order k to order $k+1$. For $k = m$, all indices have been occupied by the increments $h^{(a)}$, so B_m is independent of x and its derivative is zero. Subsequent derivatives are also zero. Consequently, A is smooth. □

Definition 23.2.5. (Higher Gâteaux derivatives) Let $U \subseteq X$ be open and let $f\colon U \longrightarrow Y$. Define $\delta^0 f := f$. Suppose $\delta^{m-1} f$ is defined on a neighborhood of $x \in U$ as a map with values in $\mathcal{L}^{m-1}(X;Y)$ and is Gâteaux differentiable at x. Through the isomorphism in Proposition 23.2.3, identify

$$\delta(\delta^{m-1} f)(x) \in \mathcal{L}\big(X, \mathcal{L}^{m-1}(X;Y)\big)$$

with an element $\delta^m f(x) \in \mathcal{L}^m(X;Y)$, called the **Gâteaux derivative of order** m of f at x. With the order of association fixed by $\mathfrak{C}$, we have

$$\delta^m f(x)[h_1,\dots,h_m] = D_{h_m}\big(D_{h_{m-1}} \cdots D_{h_1} f\big)(x)$$

whenever the iterated derivatives on the right-hand side are defined on a neighborhood of x.

Definition 23.2.6. (Higher Fréchet derivatives and functions of class C^k) Let $U \subseteq X$ be open and let $f\colon U \longrightarrow Y$. Define $D^0 f := f$. Suppose $D^{m-1} f$ is defined on a neighborhood of $x \in U$ as a map with values in $\mathcal{L}^{m-1}(X;Y)$ and is Fréchet differentiable at x. Identify $D(D^{m-1} f)(x)$ with an element $D^m f(x) \in \mathcal{L}^m(X;Y)$ through Proposition 23.2.3; it is called the **Fréchet derivative of order** m of f at x.
For $k \in \mathbb{N}$, we say that f is of **class** C^k on U, and write $f \in C^k(U,Y)$, if $D^j f$ exists at every point of U and is continuous as a map $U \longrightarrow \mathcal{L}^j(X;Y)$ for every $j \in \{1,\dots,k\}$. We say that f is **smooth**, and write $f \in C^\infty(U,Y)$, if $f \in C^k(U,Y)$ for every $k \in \mathbb{N}$.

Remark 23.2.7. In the definition, continuity of all derivatives through order k can be replaced by existence of $D^k f$ and continuity of the latter. For each $j < k$, the map $D^j f$ is Fréchet differentiable and hence continuous by Proposition 23.1.3. Nevertheless, the preceding formulation makes all the available regularity explicit and is more convenient when applying higher-order chain rules.

Regularity of a composition follows from the first-order chain rule and continuity of operator composition. It is useful to formulate this fact with values in normed spaces, since the derivatives themselves take values in operator spaces.

Corollary 23.2.8. (Composition of maps of class C^k) Let X, Y, and Z be normed spaces, let $U \subseteq X$ and $V \subseteq Y$ be open, and let $k \in \mathbb{N} \cup \{\infty\}$. If $f \in C^k(U,Y)$, $f(U) \subseteq V$, and $g \in C^k(V,Z)$, then $g \circ f \in C^k(U,Z)$.

Proof. The first-order chain rule gives

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x), \qquad x \in U.$$

Operator composition

$$\mathcal{C}\colon \mathcal{L}(Y,Z) \times \mathcal{L}(X,Y) \longrightarrow \mathcal{L}(X,Z), \qquad \mathcal{C}(B,A) := B \circ A,$$

is continuous and bilinear, since $\|B\circ A\|\le\|B\|\|A\|$. If f and g are of class C^1, the maps $Dg\circ f$ and Df are continuous. The preceding identity then shows that $D(g\circ f)$ is continuous, proving the case $k=1$.

Suppose the assertion has been proved for an integer $r\ge 1$ in arbitrary normed spaces, and now let f and g be of class C^{r+1}. The map Dg is of class C^r; by the induction hypothesis, so is $Dg\circ f$. Moreover, Df is of class C^r. The map $x\longmapsto (Dg(f(x)),Df(x))$ is therefore of class C^r: in a product of two normed spaces, differentiation and continuity are checked componentwise using the sum norm. By Proposition 23.2.4, $\mathcal{C}$ is smooth. Applying the induction hypothesis again to composition with $\mathcal{C}$, we conclude that $D(g\circ f)$ is of class C^r. Thus $g\circ f$ is of class C^{r+1}. Induction proves all finite orders; the case $k=\infty$ follows by applying the conclusion to each of them. □

Theorem 23.2.9. (Characterization by Gâteaux derivatives) Let X and Y be normed spaces, let $U\subseteq X$ be open, and let $k\in\mathbb{N}$. Then $f\in C^k(U,Y)$ if and only if, for each $j\in\{1,\dots,k\}$, the Gâteaux derivative $\delta^j f(x)\in\mathcal{L}^j(X;Y)$ exists for every $x\in U$ and the map $\delta^j f\colon U\longrightarrow\mathcal{L}^j(X;Y)$ is continuous. In this case, $D^j f=\delta^j f$ for every $j\in\{1,\dots,k\}$.

Proof. We proceed by induction on k. For $k=1$, the assertion is Corollary 23.1.17. Suppose the result holds for $k-1$.

If $f\in C^k(U,Y)$, then $D^{k-1}f\colon U\longrightarrow\mathcal{L}^{k-1}(X;Y)$ is of class C^1. Corollary 23.1.17, applied to this map, shows that its Gâteaux derivative exists, is continuous, and agrees with its Fréchet derivative. Through the identification in Proposition 23.2.3, this means that $\delta^k f=D^k f$. For lower orders, the conclusion follows from the induction hypothesis.

Conversely, suppose the Gâteaux derivatives in the statement exist and are continuous. By the induction hypothesis, $f\in C^{k-1}(U,Y)$ and $D^{k-1}f=\delta^{k-1}f$. The map $D^{k-1}f$ is Gâteaux differentiable, its derivative is $\delta^k f$, and this derivative is continuous. Corollary 23.1.17, applied with target space $\mathcal{L}^{k-1}(X;Y)$, implies that $D^{k-1}f$ is of class C^1 and its Fréchet derivative agrees with $\delta^k f$. Thus $f\in C^k(U,Y)$ and $D^k f=\delta^k f$. □

Theorem 23.2.10. (Symmetry of higher derivatives) If $f\in C^m(U,Y)$, with $m\ge 2$, then $D^m f(x)\in\mathcal{L}^m_s(X;Y)$ for every $x\in U$.

Proof. We proceed by induction on m. For $m=2$, fix $x\in U$, $h,k\in X$, and $\varphi\in Y'$. Define $g(s,t):=\varphi(f(x+sh+tk))$ for (s,t) in a neighborhood of $(0,0)$. The chain rule implies that g is of class C^2 and that

$$\frac{\partial^2 g}{\partial s\,\partial t}(0,0)=\varphi(D^2f(x)[k,h]),\qquad \frac{\partial^2 g}{\partial t\,\partial s}(0,0)=\varphi(D^2f(x)[h,k]).$$

The classical equality of mixed partial derivatives and the fact that Y' separates points imply $D^2f(x)[h,k]=D^2f(x)[k,h]$.

Now suppose the assertion holds at order $m-1$, with $m\ge 3$. Since $D^{m-1}f(x)$ is symmetric, the operator $D^m f(x)$ is symmetric in its first $m-1$ arguments. For each $v\in X$, the function

$$x\longmapsto D^{m-1}f(x)[h_{\sigma(1)},\dots,h_{\sigma(m-1)}]-D^{m-1}f(x)[h_1,\dots,h_{m-1}]$$

is identically zero for every permutation σ of $\{1,\dots,m-1\}$; differentiating it in the direction v gives the stated symmetry.

It remains to show that the last two arguments can be interchanged. Fix $h_1, \dots, h_{m-2} \in X$ and consider

$$G(z) := D^{m-2} f(z)[h_1, \dots, h_{m-2}].$$

Evaluation of a multilinear operator on fixed vectors is a continuous linear map; hence G is of class C^2. The case $m = 2$, applied to G, gives

$$D^m f(x)[h_1, \dots, h_{m-2}, h_{m-1}, h_m] = D^m f(x)[h_1, \dots, h_{m-2}, h_m, h_{m-1}].$$

Permutations of the first $m-1$ arguments, together with transposition of the last two, generate all permutations of m elements. Thus $D^m f(x)$ is symmetric. □

The Leibniz rule has a particularly useful formulation for the second derivative. We will use this identity in the next chapter when comparing the kinematic and operational notions of tangent space.

Proposition 23.2.11. (Leibniz rule for the second derivative) Let X be a normed space, let $U \subseteq X$ be open, and let $f, g \in C^2(U, \mathbb{R})$. Then, for every $x \in U$ and $h, k \in X$,

$$D^2(fg)(x)[h,k] = f(x)D^2 g(x)[h,k] + g(x)D^2 f(x)[h,k] + Df(x)(h)Dg(x)(k) + Df(x)(k)Dg(x)(h).$$

Equivalently,

$$D^2(fg)(x) = f(x)D^2 g(x) + g(x)D^2 f(x) + Df(x) \otimes Dg(x) + Dg(x) \otimes Df(x).$$

Proof. The product rule for the first derivative gives $D(fg)(x) = f(x)Dg(x) + g(x)Df(x)$. Differentiate this map with values in X' in the direction k and evaluate the resulting operator on h. The product rule and linearity of evaluation give

$$D^2(fg)(x)[h,k] = Df(x)(k)Dg(x)(h) + f(x)D^2 g(x)[h,k] + Dg(x)(k)Df(x)(h) + g(x)D^2 f(x)[h,k].$$

Symmetry of second derivatives, given by Theorem 23.2.10, allows the arguments to be ordered as in the statement. □

Theorem 23.2.12. (Taylor's formula with integral remainder) Let X be a normed space, let Y be a Banach space, let $U \subseteq X$ be open, and let $m \in \mathbb{N}$ and $f \in C^m(U, Y)$. Let $x \in U$ and $h \in X$ be such that $[x, x+h] \subseteq U$. Then

$$f(x+h) = \sum_{j=0}^{m-1} \frac{1}{j!} D^j f(x)[h]^j + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} D^m f(x+th)[h]^m \, dt, \tag{23.4}$$

, where $D^0 f(x)[h]^0 := f(x)$.

Proof. Define $\varphi\colon [0,1] \longrightarrow Y$ by $\varphi(t) := f(x+th)$. Since the segment $[x, x+h]$ is contained in the open set U, this formula defines the curve on an open interval containing $[0,1]$. The chain rule gives $\varphi'(t) = Df(x+th)(h)$. More generally, we will prove that $\varphi^{(j)}(t) = D^j f(x+th)[h]^j$ for every $j \in \{0, \dots, m\}$.
The case $j = 0$ is the definition of φ, and the case $j = 1$ has just been checked. Suppose the formula holds for an integer j with $1 \leq j < m$. The evaluation $A \longmapsto A[h]^j$ is a continuous linear operator from $\mathcal{L}^j(X; Y)$ to Y, with norm at most $\|h\|_X^j$. We may therefore differentiate the inductive identity by the chain rule. The

derivative of $D^j f(x+th)$ with respect to t is $D(D^j f)(x+th)(h)$. Evaluating on the j vectors h and using the association in Proposition 23.2.3 gives $D^{j+1} f(x+th)[h]^{j+1}$. This proves the induction step. Continuity of these expressions follows from continuity of each $D^j f$.

We prove by induction that every function $\varphi \in C^m([0,1], Y)$ satisfies

$$\varphi(1) = \sum_{j=0}^{m-1} \frac{\varphi^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} \varphi^{(m)}(t)\, dt. \tag{23.5}$$

For $m = 1$, this identity is Theorem B.6.5. Suppose it holds for $m-1$. The induction hypothesis gives

$$\varphi(1) = \sum_{j=0}^{m-2} \frac{\varphi^{(j)}(0)}{j!} + \frac{1}{(m-2)!} \int_0^1 (1-t)^{m-2} \varphi^{(m-1)}(t)\, dt.$$

Apply the integration by parts formula in Proposition B.6.6 to the scalar function $a(t) := -\dfrac{(1-t)^{m-1}}{(m-1)!}$ and the vector-valued function $\varphi^{(m-1)}$. Since $a'(t) = \dfrac{(1-t)^{m-2}}{(m-2)!}$, we obtain

$$\frac{1}{(m-2)!} \int_0^1 (1-t)^{m-2} \varphi^{(m-1)}(t)\, dt = \frac{\varphi^{(m-1)}(0)}{(m-1)!} + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} \varphi^{(m)}(t)\, dt.$$

Substituting this equality into the induction hypothesis gives (23.5). Finally, replacing each $\varphi^{(j)}(t)$ by $D^j f(x+th)[h]^j$ yields (23.4). □

Corollary 23.2.13. (Estimate for the Taylor remainder) Under the assumptions of Theorem 23.2.12, suppose $r > 0$ satisfies $B_{d_X}(x, r) \subseteq U$. Then, for each $h \in X$ with $\|h\|_X < r$,

$$\left\| f(x+h) - \sum_{j=0}^{m} \frac{1}{j!} D^j f(x)[h]^j \right\|_Y \leq \frac{\|h\|_X^m}{m!} \sup_{t \in [0,1]} \|D^m f(x+th) - D^m f(x)\|_{\mathcal{L}^m}.$$

In particular, for each $\varepsilon > 0$ there exists $\delta \in (0, r)$ such that

$$\left\| f(x+h) - \sum_{j=0}^{m} \frac{1}{j!} D^j f(x)[h]^j \right\|_Y \leq \varepsilon \|h\|_X^m$$

whenever $\|h\|_X < \delta$.

Proof. In (23.4), add and subtract $D^m f(x)[h]^m$ inside the integral. Since $\displaystyle\int_0^1 (1-t)^{m-1}\, dt = \frac{1}{m}$, we obtain

$$f(x+h) - \sum_{j=0}^{m} \frac{1}{j!} D^j f(x)[h]^j = \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} \big(D^m f(x+th) - D^m f(x)\big)[h]^m\, dt.$$

The inequality for the Riemann integral with values in Y and the characterization of the multilinear norm

imply the first estimate. Now let $\varepsilon > 0$. By continuity of $D^m f$ at x, there exists $\delta \in (0, r)$ such that

$$\|D^m f(x+u) - D^m f(x)\|_{\mathcal{L}^m} < m!\varepsilon$$

whenever $\|u\|_X < \delta$. If $\|h\|_X < \delta$, then $\|th\|_X < \delta$ for every $t \in [0,1]$, and the first estimate gives the final inequality. □

23.2.1 Exercises

Exercise 23.2.14. Let $A \in \mathcal{L}^m_s(X;Y)$ and define $P\colon X \longrightarrow Y$ by $P(x) := A[x]^m$. Prove that P is smooth and that, for $0 \le j \le m$,

$$D^j P(x)[h_1,\dots,h_j] = \frac{m!}{(m-j)!} A(\underbrace{x,\dots,x}_{m-j \text{ times}}, h_1,\dots,h_j).$$

Show that $D^j P = 0$ for $j > m$.

Exercise 23.2.15. Let H be a real Hilbert space. Compute the first and second Fréchet derivatives of $u \longmapsto \frac{1}{2}\|u\|_H^2$. For $p > 1$, compute the first derivative of $u \longmapsto \frac{1}{p}\|u\|_H^p$ and determine at which points the second derivative exists when $1 < p < 2$ and when $p \ge 2$.

Exercise 23.2.16. Let $\ell \in X'$ and let $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ be of class C^m. Prove that $f := \varphi \circ \ell$ belongs to $C^m(X,\mathbb{R})$ and that

$$D^j f(x)[h_1,\dots,h_j] = \varphi^{(j)}(\ell(x)) \prod_{i=1}^{j} \ell(h_i)$$

for every $1 \le j \le m$.

Exercise 23.2.17. Let $(M,\mathbf{g})$ be a compact Riemannian manifold with smooth boundary, and let $\mathbf{E} \to M$ be a smooth vector bundle with bundle metric $\mathbf{h_E}$ and compatible connection $\nabla^{\mathbf{E}}$. Consider $J(u) := \frac{1}{2}\int_M |\nabla^{\mathbf{E}}_w u|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}$ on $W^{1,2}_0(M,\mathbf{E})$. Prove that J is smooth, compute $DJ(u)$ and $D^2J(u)$, and identify the continuous bilinear operator associated with the second derivative.

Exercise 23.2.18. Let X be a unital Banach algebra. Use Taylor's formula to study the exponential map $\exp(x) := \sum_{j=0}^{\infty} \frac{x^j}{j!}$ near the origin. Compute its first derivatives at 0 and explain the simplification that occurs when the elements under consideration commute.

23.3 Fundamental theorems of differential calculus

The following local results turn information about the derivative into information about the function itself. The contraction principle provides the mechanism for existence and uniqueness; the inverse function theorem (Theorem 23.3.3) constructs nonlinear coordinates, and the implicit function theorem (Theorem 23.3.8) allows some of the variables to be solved for locally. These tools will underlie the adapted charts and regular level sets used on Banach manifolds.

Theorem 23.3.1. (Contraction principle) Let (M,d) be a nonempty complete metric space, and let $T\colon M \longrightarrow M$ be a contraction; that is, suppose there exists $q \in [0,1)$ such that $d(Tx,Ty) \le qd(x,y)$ for all $x,y \in M$. Then T has a unique fixed point x_*. Moreover, for any $x_0 \in M$, the sequence defined by $x_{n+1} := T(x_n)$ converges to x_* and

$$d(x_n,x_*) \le \frac{q^n}{1-q} d(x_1,x_0)$$

for every $n \in \mathbb{N}$.

Proof. Fix $x_0 \in M$. The contraction inequality gives $d(x_2,x_1) \le qd(x_1,x_0)$. If $d(x_{n+1},x_n) \le q^n d(x_1,x_0)$ for some $n \in \mathbb{N}$, applying T to the two points gives $d(x_{n+2},x_{n+1}) \le qd(x_{n+1},x_n) \le q^{n+1} d(x_1,x_0)$. By induction, the estimate holds for every $n \in \mathbb{N}$. If $m > n$, the triangle inequality gives

$$d(x_m,x_n) \le \sum_{j=n}^{m-1} d(x_{j+1},x_j) \le \sum_{j=n}^{m-1} q^j d(x_1,x_0) \le \frac{q^n}{1-q} d(x_1,x_0).$$

Thus (x_n) is a Cauchy sequence. Completeness of M guarantees existence of $x_* \in M$ such that $x_n \longrightarrow x_*$. Since every contraction is continuous, $T(x_*) = \lim\limits_{n\to\infty} T(x_n) = \lim\limits_{n\to\infty} x_{n+1} = x_*$. If y is another fixed point, then $d(x_*,y) = d(Tx_*,Ty) \le qd(x_*,y)$; since $q < 1$, it follows that $d(x_*,y) = 0$. Finally, letting $m \to \infty$ in the preceding estimate gives the inequality in the statement. □

Proposition 23.3.2. (Inversion of operators) Let X and Y be Banach spaces, and let $\mathrm{Iso}(X,Y) \subseteq \mathcal{L}(X,Y)$ be the set of continuous linear isomorphisms. Then $\mathrm{Iso}(X,Y)$ is open, and the map

$$\mathrm{Inv}\colon \mathrm{Iso}(X,Y) \longrightarrow \mathrm{Iso}(Y,X), \qquad \mathrm{Inv}(A) := A^{-1},$$

is smooth. Its derivative is given by

$$D\,\mathrm{Inv}(A)(H) = -A^{-1}HA^{-1}. \tag{23.6}$$

Proof. Let $A \in \mathrm{Iso}(X,Y)$. If $X = \{0\}$, existence of A forces $Y = \{0\}$, and all assertions can be checked directly in the one-point spaces. Suppose $X \neq \{0\}$; then $\|A^{-1}\|_{\mathcal{L}(Y,X)} > 0$. If $H \in \mathcal{L}(X,Y)$ satisfies $\|A^{-1}H\|_{\mathcal{L}(X)} < 1$, the Neumann series of Theorem B.2.10 converges in $\mathcal{L}(X)$ and

$$(A+H)^{-1} = \sum_{j=0}^{\infty} (-A^{-1}H)^j A^{-1}.$$

Thus $A+H$ is invertible, and $\mathrm{Iso}(X,Y)$ is open. If $\|A^{-1}H\|_{\mathcal{L}(X)} \le \frac{1}{2}$, then

$$\|(A+H)^{-1}\|_{\mathcal{L}(Y,X)} \le \frac{\|A^{-1}\|_{\mathcal{L}(Y,X)}}{1 - \|A^{-1}H\|_{\mathcal{L}(X)}} \le 2\|A^{-1}\|_{\mathcal{L}(Y,X)}.$$

The identity

$$(A+H)^{-1} - A^{-1} = -(A+H)^{-1}HA^{-1}$$

implies

$$\|(A+H)^{-1} - A^{-1}\|_{\mathcal{L}(Y,X)} \le 2\|A^{-1}\|^2_{\mathcal{L}(Y,X)}\|H\|_{\mathcal{L}(X,Y)},$$

, so the inversion map is continuous at A.
Moreover,

$$(A+H)^{-1} - A^{-1} + A^{-1}HA^{-1} = A^{-1}HA^{-1}H(A+H)^{-1}.$$

When $0 < \|H\|_{\mathcal{L}(X,Y)} \leq \dfrac{1}{2\|A^{-1}\|_{\mathcal{L}(Y,X)}}$, the preceding estimate gives

$$\frac{\|(A+H)^{-1} - A^{-1} + A^{-1}HA^{-1}\|_{\mathcal{L}(Y,X)}}{\|H\|_{\mathcal{L}(X,Y)}} \leq 2\|A^{-1}\|^3_{\mathcal{L}(Y,X)}\|H\|_{\mathcal{L}(X,Y)}.$$

The right-hand side tends to zero as $H \to 0$, proving (23.6).
The derivative formula can be written as $D\operatorname{Inv}(A)(H) = -\operatorname{Inv}(A)H\operatorname{Inv}(A)$. The map assigning to $(B,C) \in \mathcal{L}(Y,X) \times \mathcal{L}(Y,X)$ the operator $H \longmapsto -BHC$ is continuous and bilinear with values in $\mathcal{L}(\mathcal{L}(X,Y),\mathcal{L}(Y,X))$, since its norm is at most $\|B\|\|C\|$. By Proposition 23.2.4, this map is smooth. Since Inv is continuous, its derivative formula first proves continuity of $D\operatorname{Inv}$; hence Inv is of class C^1. Now suppose it is of class C^r for some integer $r \geq 1$. Composing the preceding bilinear map with $A \longmapsto (\operatorname{Inv}(A), \operatorname{Inv}(A))$, Corollary 23.2.8 shows that $D\operatorname{Inv}$ is of class C^r. Consequently, Inv is of class C^{r+1}. This induction step proves smoothness. □

Theorem 23.3.3. (Local inverse function theorem) Let X and Y be Banach spaces, let $U \subseteq X$ be open, and let $f \in C^1(U,Y)$. If $a \in U$ and $Df(a)\colon X \longrightarrow Y$ is an isomorphism, there exist open neighborhoods $U_0 \subseteq U$ of a and $V_0 \subseteq Y$ of $f(a)$ such that $f\restriction_{U_0}\colon U_0 \longrightarrow V_0$ is a diffeomorphism of class C^1. If $g\colon V_0 \longrightarrow U_0$ is its inverse, then

$$Dg(y) = Df(g(y))^{-1} \tag{23.7}$$

for every $y \in V_0$. If $f \in C^k(U,Y)$, then $g \in C^k(V_0,X)$.

Proof. If $X = \{0\}$, invertibility of $Df(a)$ implies $Y = \{0\}$, and the result is immediate. Suppose $X \neq \{0\}$. Write $A := Df(a)^{-1} \in \mathcal{L}(Y,X)$. By continuity of Df at a, there exists $r > 0$ such that $\overline{B}_{d_X}(a,2r) \subseteq U$ and

$$\|A(Df(x) - Df(a))\|_{\mathcal{L}(X)} \leq \frac{1}{2} \tag{23.8}$$

for every $x \in B_{d_X}(a,2r)$. For each $y \in Y$, define

$$T_y\colon \overline{B}_{d_X}(a,r) \longrightarrow X, \qquad T_y(x) := x - A(f(x) - y).$$

Let $x_1, x_2 \in \overline{B}_{d_X}(a,r)$. The segment $[x_1,x_2]$ is contained in $B_{d_X}(a,2r)$. The map $H(x) := x - Af(x)$ is of class C^1 and, by (23.8),

$$\|DH(x)\|_{\mathcal{L}(X)} = \|I_X - ADf(x)\|_{\mathcal{L}(X)} = \|A(Df(a) - Df(x))\|_{\mathcal{L}(X)} \leq \frac{1}{2}.$$

Theorem 23.1.10 implies

$$\|T_y(x_1) - T_y(x_2)\|_X \leq \frac{1}{2}\|x_1 - x_2\|_X. \tag{23.9}$$

Let $b := f(a)$ and choose $\delta > 0$ such that $\|A\|_{\mathcal{L}(Y,X)}\delta < \dfrac{r}{2}$. If $y \in B_{d_Y}(b,\delta)$, then

$$\|T_y(a) - a\|_X = \|A(y-b)\|_X < \frac{r}{2}.$$

For each $x \in \overline{B}_{d_X}(a,r)$, inequality (23.9) gives

$$\|T_y(x)-a\|_X \le \|T_y(x)-T_y(a)\|_X + \|T_y(a)-a\|_X < \frac{1}{2}\|x-a\|_X + \frac{r}{2} \le r.$$

Thus T_y maps $\overline{B}_{d_X}(a,r)$ into itself and is a contraction. The contraction principle gives a unique fixed point $g(y) \in \overline{B}_{d_X}(a,r)$. The equality $T_y(g(y)) = g(y)$ is equivalent to $f(g(y)) = y$. Moreover,

$$\|g(y)-a\|_X \le \frac{1}{1-\frac{1}{2}}\|T_y(a)-a\|_X < r,$$

, so $g(y) \in B_{d_X}(a,r)$.
If $y_1, y_2 \in B_{d_Y}(b,\delta)$, then, using (23.9),

$$\begin{aligned}\|g(y_1)-g(y_2)\|_X &= \|T_{y_1}(g(y_1)) - T_{y_2}(g(y_2))\|_X \\ &\le \|T_{y_1}(g(y_1)) - T_{y_1}(g(y_2))\|_X + \|A(y_1-y_2)\|_X \\ &\le \frac{1}{2}\|g(y_1)-g(y_2)\|_X + \|A\|_{\mathcal{L}(Y,X)}\|y_1-y_2\|_Y.\end{aligned}$$

Consequently,

$$\|g(y_1)-g(y_2)\|_X \le 2\|A\|_{\mathcal{L}(Y,X)}\|y_1-y_2\|_Y. \tag{23.10}$$

In particular, g is continuous.
Define $V_0 := B_{d_Y}(b,\delta)$ and $U_0 := f^{-1}(V_0) \cap B_{d_X}(a,r)$. If $x \in U_0$ and $y = f(x)$, then x is a fixed point of T_y in $\overline{B}_{d_X}(a,r)$; by uniqueness of the fixed point, $x = g(y)$. Thus $f \restriction_{U_0}$ is bijective onto V_0, with inverse g.
For each $x \in U_0$, the identity

$$ADf(x) = I_X + A(Df(x) - Df(a))$$

and (23.8) show, through Theorem B.2.10, that $ADf(x)$ is invertible. Thus $Df(x)$ is an isomorphism.
Fix $y \in V_0$ and write $x := g(y)$. Let $k \in Y$ be such that $y + k \in V_0$ and define $h := g(y+k) - g(y)$. Then $k = f(x+h) - f(x)$ and

$$g(y+k) - g(y) - Df(x)^{-1}k = -Df(x)^{-1}\big(f(x+h) - f(x) - Df(x)h\big). \tag{23.11}$$

Let $\varepsilon > 0$. By Fréchet differentiability of f at x, there exists $\eta > 0$ such that

$$\|f(x+u) - f(x) - Df(x)u\|_Y < \frac{\varepsilon}{2\big(1+\|A\|_{\mathcal{L}(Y,X)}\big)\big(1+\|Df(x)^{-1}\|_{\mathcal{L}(Y,X)}\big)}\|u\|_X$$

whenever $0 < \|u\|_X < \eta$. By (23.10), we have $\|h\|_X \le 2\|A\|_{\mathcal{L}(Y,X)}\|k\|_Y$. Consequently, if

$$0 < \|k\|_Y < \frac{\eta}{2\|A\|_{\mathcal{L}(Y,X)}},$$

, then $\|h\|_X < \eta$ and (23.11) imply

$$\frac{\|g(y+k) - g(y) - Df(x)^{-1}k\|_X}{\|k\|_Y} < \varepsilon.$$

This proves that g is Fréchet differentiable at y and that $Dg(y) = Df(g(y))^{-1}$.
Continuity of Dg follows from continuity of g, continuity of Df, and continuity of the inversion map established in Proposition 23.3.2. Thus $g \in C^1(V_0, X)$. If $f \in C^k(U, Y)$ and $k > 1$, start with the C^1 regularity of g just obtained. Suppose g is of class C^j, where $j \in \mathbb{N}$ and $1 \le j < k$. Since Df is of class C^{k-1}, it is in particular of class C^j. Corollary 23.2.8 and smoothness of the inversion map imply that $\mathrm{Inv} \circ Df \circ g$ is of class C^j. By (23.7), this is precisely Dg; hence g is of class C^{j+1}. Induction reaches $j = k$ when k is finite. If $k = \infty$, the same argument applies at every finite order and proves that g is smooth. □

Definition 23.3.4. (Diffeomorphism between open subsets of Banach spaces) Let $U \subseteq X$ and $V \subseteq Y$ be open subsets of Banach spaces. A bijection $f : U \longrightarrow V$ is a **diffeomorphism of class** C^k if $f \in C^k(U, Y)$ and $f^{-1} \in C^k(V, X)$.

Invertibility of the derivative is a local condition and alone does not guarantee global injectivity. The uniform assumption in the following result prevents local inverses from degenerating as they are continued along paths.

Lemma 23.3.5. (Path lifting) Let X and Y be Banach spaces, and let $f \in C^1(X, Y)$. Suppose $Df(x)$ is an isomorphism for every $x \in X$ and there exists $C > 0$ such that $\|Df(x)^{-1}\|_{\mathcal{L}(Y,X)} \le C$ for every $x \in X$. Let $\gamma : [0,1] \longrightarrow Y$ be of class C^1, and let $x_0 \in X$ be such that $f(x_0) = \gamma(0)$. Then there exists a unique map $\widetilde{\gamma} : [0,1] \longrightarrow X$ of class C^1 such that $\widetilde{\gamma}(0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$. Moreover,

$$\|\widetilde{\gamma}'(t)\|_X \le C \|\gamma'(t)\|_Y \tag{23.12}$$

for every $t \in [0,1]$.

Proof. Theorem 23.3.3, applied at x_0, gives a unique lift on an interval $[0, \tau_0]$ with $\tau_0 > 0$. Let

$$\mathcal{A} := \{\tau \in [0,1] \mid \gamma \restriction_{[0,\tau]} \text{ admits a lift of class } C^1 \text{ starting at } x_0\}$$

and let $\alpha := \sup \mathcal{A}$. Let us establish uniqueness on common intervals precisely. If two continuous lifts have the same initial value, the set of times when they agree is nonempty and closed by continuity. It is also open in the common interval: at a time of agreement, both paths remain, for a small interval, in a common neighborhood on which f is injective. Since they have the same image under f, they agree there. Connectedness of the interval implies that they agree throughout it. Thus the partial lifts glue together to define a unique map $\widetilde{\gamma} : [0, \alpha) \longrightarrow X$, of class C^1 on every compact interval contained in its domain.
Differentiating $f(\widetilde{\gamma}(t)) = \gamma(t)$ and applying the chain rule gives

$$Df(\widetilde{\gamma}(t))\widetilde{\gamma}'(t) = \gamma'(t), \qquad \widetilde{\gamma}'(t) = Df(\widetilde{\gamma}(t))^{-1}\gamma'(t),$$

, which proves (23.12). If $0 \le t_1 < t_2 < \alpha$, the integral mean value formula gives

$$\|\widetilde{\gamma}(t_2) - \widetilde{\gamma}(t_1)\|_X \le C \left(\max_{t \in [0,1]} \|\gamma'(t)\|_Y \right) |t_2 - t_1|.$$

This estimate shows that $\widetilde{\gamma}(t)$ is Cauchy as $t \to \alpha^-$. By completeness of X, there exists $x_\alpha := \lim_{t \to \alpha^-} \widetilde{\gamma}(t)$, whether $\alpha < 1$ or $\alpha = 1$. Define $\widetilde{\gamma}(\alpha) := x_\alpha$. The extension is continuous and satisfies $f(x_\alpha) = \gamma(\alpha)$.

Choose a local inverse $g_\alpha\colon V_\alpha \longrightarrow U_\alpha$ of f, with $x_\alpha \in U_\alpha$ and $\gamma(\alpha) \in V_\alpha$. By the convergence just proved and continuity of γ, there exists $\varepsilon > 0$ such that $\widetilde{\gamma}(t) \in U_\alpha$ and $\gamma(t) \in V_\alpha$ for $t \in (\alpha - \varepsilon, \alpha]$. On this interval, injectivity of $f\restriction_{U_\alpha}$ gives

$$\widetilde{\gamma}(t) = g_\alpha(\gamma(t)).$$

Thus the extension is of class C^1 up to α, with a one-sided derivative at the endpoint. If $\alpha < 1$, continuity of γ allows this same formula to be used on $[\alpha, \alpha+\varepsilon']$ for some $\varepsilon' > 0$ with $\alpha + \varepsilon' \le 1$. We would obtain a lift of class C^1 beyond α, contradicting the definition of the supremum. Hence $\alpha = 1$.
In particular, the value $\widetilde{\gamma}(1)$ is already defined, and the formula involving g_α proves C^1 regularity at the final endpoint. Differentiating it gives $\widetilde{\gamma}'(1) = Df(\widetilde{\gamma}(1))^{-1}\gamma'(1)$ there as well, so (23.12) holds on all of $[0,1]$. Uniqueness on the closed interval is the uniqueness established at the beginning of the proof. □

Lemma 23.3.6. (Homotopy lifting) Under the assumptions of the preceding lemma, let $H\colon [0,1]^2 \longrightarrow Y$ be of class C^1, and let $\eta\colon [0,1] \longrightarrow X$ be a continuous path such that $f(\eta(s)) = H(0,s)$ for every $s \in [0,1]$. Then there exists a unique continuous map $\widetilde{H}\colon [0,1]^2 \longrightarrow X$ such that $f \circ \widetilde{H} = H$ and $\widetilde{H}(0,s) = \eta(s)$. For each s, the path $t \longmapsto \widetilde{H}(t,s)$ is of class C^1.

Proof. For each $s \in [0,1]$, Lemma 23.3.5 gives a unique lift of the path $t \longmapsto H(t,s)$ starting at $\eta(s)$; denote it by $t \longmapsto \widetilde{H}(t,s)$. We must prove that the resulting map is continuous in both variables.
Let

$$\mathcal{T} := \{\tau \in [0,1] \mid \widetilde{H} \text{ is continuous on } [0,\tau]\times[0,1]\}.$$

We will prove that $\mathcal{T} = [0,1]$. For each $s_0 \in [0,1]$, Theorem 23.3.3 gives open neighborhoods U_{s_0} of $\eta(s_0)$ and V_{s_0} of $H(0,s_0)$ such that $f\restriction_{U_{s_0}}\colon U_{s_0} \longrightarrow V_{s_0}$ is a homeomorphism. By continuity of η and H, there exist an open interval $I_{s_0} \subseteq [0,1]$ containing s_0 and a number $\delta_{s_0} > 0$ such that $\eta(s) \in U_{s_0}$ and $H(t,s) \in V_{s_0}$ for $(t,s) \in [0,\delta_{s_0}]\times I_{s_0}$. On this rectangle, uniqueness of lifts implies

$$\widetilde{H}(t,s) = (f\restriction_{U_{s_0}})^{-1}(H(t,s)).$$

Compactness of $[0,1]$ allows us to choose finitely many intervals I_{s_0} covering it. If δ is the minimum of the corresponding numbers δ_{s_0}, the preceding formula proves that $\widetilde{H}$ is continuous on $[0,\delta]\times[0,1]$. Thus $\mathcal{T}$ is nonempty.
Let $\tau \in \mathcal{T}$ with $\tau < 1$. The set $\widetilde{H}(\{\tau\}\times[0,1])$ is compact. For each $s_0 \in [0,1]$, apply the local theorem again at $\widetilde{H}(\tau,s_0)$. Continuity of $s \longmapsto \widetilde{H}(\tau,s)$ and H allows us to find an interval I_{s_0} and a number $\delta_{s_0} > 0$ such that, for $s \in I_{s_0}$ and $t \in [\tau, \tau+\delta_{s_0}] \cap [0,1]$, the lift is given by a common local inverse. A finite subcover of $[0,1]$ and the minimum of the numbers δ_{s_0} show that there exists $\delta > 0$ such that $\widetilde{H}$ is continuous on $[0, \min\{1, \tau+\delta\}]\times[0,1]$. Thus $\mathcal{T}$ is open to the right.
To check that continuation cannot stop, let $\tau_* := \sup\mathcal{T}$. Since H is of class C^1 on the compact square, there exists

$$M := \max_{(t,s)\in[0,1]^2} \left\| \frac{\partial H}{\partial t}(t,s) \right\|_Y < \infty.$$

For each $s \in [0,1]$ and $0 \le t_1 < t_2 < \tau_*$, estimate (23.12) and the integral mean value formula give

$$\|\widetilde{H}(t_2,s) - \widetilde{H}(t_1,s)\|_X \le CM|t_2 - t_1|.$$

The estimate is uniform in s. Consequently, as $t \to \tau_*^-$, $\widetilde{H}(t,s)$ has a uniform limit in s. For each fixed s, the individual lift is already continuous at $[0,1]$, so this limit agrees with the value $\widetilde{H}(\tau_*,s)$ defined at the beginning of the proof. Moreover, for every $t < \tau_*$, there exists an element of $\mathcal{T}$ greater than t, so $s \longmapsto \widetilde{H}(t,s)$ is continuous. The uniform limit of these functions is continuous. We thus obtain continuity of $s \longmapsto \widetilde{H}(\tau_*,s)$ and, by passage to the limit in the estimate,

$$\|\widetilde{H}(t,s) - \widetilde{H}(\tau_*,s_0)\|_X \le CM(\tau_* - t) + \|\widetilde{H}(\tau_*,s) - \widetilde{H}(\tau_*,s_0)\|_X$$

for $t \in [0,\tau_*]$ and $s, s_0 \in [0,1]$. The right-hand side tends to zero as $(t,s) \longrightarrow (\tau_*,s_0)$ with $t \le \tau_*$. This proves joint continuity up to this edge, that is, $\tau_* \in \mathcal{T}$. The local argument in the preceding paragraph would extend continuity beyond τ_* if $\tau_* < 1$, contradicting the definition of the supremum. Thus $\tau_* = 1$, and $\widetilde{H}$ is continuous on the entire square.

The identity $f \circ \widetilde{H} = H$ and the initial condition hold by construction. Uniqueness follows for each fixed value of s from the argument for continuous lifts in the proof of Lemma 23.3.5. □

Theorem 23.3.7. (Global inverse function theorem) Let X and Y be Banach spaces, and let $f \in C^1(X,Y)$. Suppose $Df(x)$ is an isomorphism for every $x \in X$ and there exists $C > 0$ such that $\|Df(x)^{-1}\|_{\mathcal{L}(Y,X)} \le C$ for every $x \in X$. Then $f\colon X \longrightarrow Y$ is a global diffeomorphism.

Proof. Fix $a \in X$ and let $b := f(a)$. Given $y \in Y$, consider the segment $\gamma(t) := (1-t)b + ty$. By Lemma 23.3.5, there exists a lift $\widetilde{\gamma}\colon [0,1] \longrightarrow X$ with $\widetilde{\gamma}(0) = a$. Then $f(\widetilde{\gamma}(1)) = \gamma(1) = y$. Since y was arbitrary, f is surjective. To prove injectivity, suppose $f(x_0) = f(x_1) = y_0$. Let $\sigma(t) := (1-t)x_0 + tx_1$ and let $\gamma := f \circ \sigma$. The map

$$H(t,s) := (1-s)\gamma(t) + sy_0$$

is a homotopy between the loop γ and the constant loop y_0 and satisfies $H(0,s) = y_0$ for every s. Apply Lemma 23.3.6 with $\eta(s) := x_0$. For each $s \in [0,1]$, the point $\widetilde{H}(1,s)$ belongs to $f^{-1}(y_0)$. This fiber is discrete because f is locally injective at each of its points. Since $s \longmapsto \widetilde{H}(1,s)$ is continuous and $[0,1]$ is connected, this map is constant. For $s = 0$, uniqueness of the lift implies $\widetilde{H}(t,0) = \sigma(t)$ and hence $\widetilde{H}(1,0) = x_1$. For $s = 1$, the lift is the constant path x_0, so $\widetilde{H}(1,1) = x_0$. We conclude that $x_0 = x_1$.

Thus f is bijective. Theorem 23.3.3 shows that its inverse is of class C^1 and satisfies $D(f^{-1})(y) = Df(f^{-1}(y))^{-1}$. □

Theorem 23.3.8. (Implicit function theorem) Let X, Y, and Z be Banach spaces, let $U \subseteq X \times Y$ be open, and let $F \in C^k(U,Z)$, with $k \ge 1$. Suppose $(a,b) \in U$, $F(a,b) = 0$, and $D_2F(a,b)\colon Y \longrightarrow Z$ is an isomorphism. Then there exist open neighborhoods $U_0 \subseteq X$ of a and $V_0 \subseteq Y$ of b, and a unique function $\varphi\colon U_0 \longrightarrow V_0$ of class C^k such that, for every $(x,y) \in U_0 \times V_0$,

$$F(x,y) = 0 \quad \Longleftrightarrow \quad y = \varphi(x). \tag{23.13}$$

After shrinking U_0 and V_0 if necessary, $D_2F(x,\varphi(x))$ is an isomorphism for every $x \in U_0$ and

$$D\varphi(x) = -D_2F(x,\varphi(x))^{-1} \circ D_1F(x,\varphi(x)). \tag{23.14}$$

Proof. Define

$$\Phi\colon U \longrightarrow X \times Z, \qquad \Phi(x,y) := \big(x, F(x,y)\big).$$

Proposition 23.1.19 implies that

$$D\Phi(a,b)(h,k) = \big(h, D_1F(a,b)h + D_2F(a,b)k\big).$$

This operator is an isomorphism from $X \times Y$ onto $X \times Z$. Its inverse is given by

$$(u,v) \longmapsto \big(u, D_2F(a,b)^{-1}\big(v - D_1F(a,b)u\big)\big).$$

Theorem 23.3.3 gives open neighborhoods $\mathcal{U} \subseteq U$ of (a,b) and $\mathcal{V} \subseteq X \times Z$ of $(a,0)$ such that $\Phi\restriction_{\mathcal{U}}\colon \mathcal{U} \longrightarrow \mathcal{V}$ is a diffeomorphism of class C^k. First choose an open neighborhood $V_0 \subseteq Y$ of b such that there exists an open neighborhood $U_1 \subseteq X$ of a with $U_1 \times V_0 \subseteq \mathcal{U}$. By continuity of Φ^{-1} at $(a,0)$, there exist open neighborhoods $U_0 \subseteq U_1$ of a and $W_0 \subseteq Z$ of 0 such that $U_0 \times W_0 \subseteq \mathcal{V}$ and

$$\Phi^{-1}(U_0 \times W_0) \subseteq U_1 \times V_0.$$

For each $x \in U_0$, define $(x, \varphi(x)) := \Phi^{-1}(x,0)$. The first component of Φ is the identity on X, so the first component of $\Phi^{-1}(x,0)$ must be x, and the preceding choice ensures that $\varphi(x) \in V_0$. Moreover,

$$\Phi(x,\varphi(x)) = (x,0) \quad\Longleftrightarrow\quad F(x,\varphi(x)) = 0.$$

If $(x,y) \in U_0 \times V_0$ and $F(x,y) = 0$, then $\Phi(x,y) = (x,0)$; injectivity of $\Phi\restriction_{\mathcal{U}}$ implies $y = \varphi(x)$. This proves (23.13). Since φ is obtained by composing the map $x \longmapsto (x,0)$, the inverse Φ^{-1}, and the projection onto Y, it belongs to $C^k(U_0, Y)$.
Continuity of D_2F and openness of $\mathrm{Iso}(Y,Z)$, proved in Proposition 23.3.2, allow the neighborhoods to be shrunk so that $D_2F(x,\varphi(x))$ is invertible for every $x \in U_0$. Differentiating the identity $F(x,\varphi(x)) = 0$ and applying the chain rule and (23.2) gives

$$D_1F(x,\varphi(x)) + D_2F(x,\varphi(x)) \circ D\varphi(x) = 0.$$

Composition on the left with $D_2F(x,\varphi(x))^{-1}$ yields (23.14). □

23.4 Lagrange multipliers

Theorem 23.3.8 describes a regular level set as a graph over the kernel of the derivative of the constraint. In these coordinates, a constrained extremum becomes an ordinary extremum on the tangent space. This observation leads to the principle of Lagrange multipliers and explains why the derivative of the functional must be a combination of the derivatives of the constraints.

Theorem 23.4.1. (Lagrange multipliers for finitely many constraints) Let X be a Banach space, let $U \subseteq X$ be open, and let $J \in C^1(U,\mathbb{R})$ and $F = (F_1,\dots,F_m) \in C^1(U,\mathbb{R}^m)$. Let $x_0 \in U$ be a local extremum of $J\restriction_{F^{-1}(F(x_0))}$

and suppose $DF(x_0)\colon X \longrightarrow \mathbb{R}^m$ is surjective. Then there exists a unique functional $\lambda \in (\mathbb{R}^m)'$ such that

$$DJ(x_0) = \lambda \circ DF(x_0). \tag{23.15}$$

Equivalently, there exist unique numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$DJ(x_0) = \sum_{j=1}^{m} \lambda_j DF_j(x_0).$$

Proof. Let $K := \ker DF(x_0)$. Since $DF(x_0)$ is surjective and $\mathbb{R}^m$ is finite-dimensional, for each vector e_j in the standard basis of $\mathbb{R}^m$ there exists $v_j \in X$ such that $DF(x_0)v_j = e_j$. Define $E := \operatorname{span}\{v_1, \dots, v_m\}$. Then E is finite-dimensional and therefore closed. Moreover, $DF(x_0) \restriction_E \colon E \longrightarrow \mathbb{R}^m$ is an isomorphism. For each $v \in X$, if

$$w := \big(DF(x_0) \restriction_E\big)^{-1}(DF(x_0)v),$$

, then $v - w \in K$. Thus $X = K \oplus E$.
Consider the open set

$$\mathcal{O} := \{(u, v) \in K \times E \mid x_0 + u + v \in U\}$$

and the map $G\colon \mathcal{O} \longrightarrow \mathbb{R}^m$ defined by $G(u, v) := F(x_0 + u + v) - F(x_0)$. Moreover, $G(0, 0) = 0$ and

$$D_2 G(0, 0) = DF(x_0) \restriction_E \colon E \longrightarrow \mathbb{R}^m.$$

Theorem 23.3.8 gives a neighborhood $W \subseteq K$ of 0 and a function $\psi\colon W \longrightarrow E$ of class C^1 such that $\psi(0) = 0$ and

$$F(x_0 + u + \psi(u)) = F(x_0)$$

for every $u \in W$. Differentiating this identity at $u = 0$ gives

$$DF(x_0)\big(u + D\psi(0)u\big) = 0$$

for every $u \in K$. Since $DF(x_0)u = 0$ and $D\psi(0)u \in E$, injectivity of $DF(x_0) \restriction_E$ implies $D\psi(0)u = 0$. Thus $D\psi(0) = 0$.
Fix $u \in K$. For every t in a neighborhood of 0, the curve $\gamma(t) := x_0 + tu + \psi(tu)$ lies in $F^{-1}(F(x_0))$. Since x_0 is a local extremum of the restriction of J, the real function $J \circ \gamma$ has a local extremum at 0. Fermat's rule and the chain rule give

$$0 = (J \circ \gamma)'(0) = DJ(x_0)\big(u + D\psi(0)u\big) = DJ(x_0)u.$$

Thus $K \subseteq \ker DJ(x_0)$.
Let $R\colon \mathbb{R}^m \longrightarrow E$ be the inverse of $DF(x_0) \restriction_E$ and define $\lambda := DJ(x_0) \circ R \in (\mathbb{R}^m)'$. For each $v \in X$, the vector $v - R(DF(x_0)v)$ belongs to K; hence

$$DJ(x_0)v = DJ(x_0)R(DF(x_0)v) = \lambda(DF(x_0)v).$$

This proves (23.15). If $\mu \in (\mathbb{R}^m)'$ also satisfies $DJ(x_0) = \mu \circ DF(x_0)$, surjectivity of $DF(x_0)$ implies $\lambda = \mu$. The formulation in terms of the numbers λ_j follows by identifying $(\mathbb{R}^m)'$ with $\mathbb{R}^m$ using the standard basis. □

Corollary 23.4.2. (Lagrange multipliers for one constraint) Let X be a Banach space, let $U \subseteq X$ be open, and let $J, F \in C^1(U, \mathbb{R})$. If x_0 is a local extremum of J restricted to $\{x \in U \mid F(x) = F(x_0)\}$ and $DF(x_0) \neq 0$, there exists a unique $\lambda \in \mathbb{R}$ such that $DJ(x_0) = \lambda DF(x_0)$.

Proof. The operator $DF(x_0)\colon X \longrightarrow \mathbb{R}$ is surjective because it is linear and nonzero. Apply the preceding theorem with $m = 1$ and identify each linear functional on $\mathbb{R}$ with multiplication by a unique real number. □

23.4.1 Exercises

Exercise 23.4.3. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(0) = 0$ and $f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$. Prove that $f'(0) = 1$ but f is not injective on any neighborhood of 0. Determine which assumption of Theorem 23.3.3 fails.

Exercise 23.4.4. Let $\Phi(r, \theta) := (r\cos\theta, r\sin\theta)$ and let $u \in C^2(\mathbb{R}^2 \setminus \{0\})$. If $v := u \circ \Phi$, use the second-order chain rule to prove that

$$(\Delta u) \circ \Phi = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}.$$

Interpret this identity as a first example of the behavior of a differential operator under a change of coordinates.

Exercise 23.4.5. Let $P(z, a_0, \ldots, a_{n-1}) := z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Suppose z_0 is a simple root of the polynomial corresponding to $(a_0^0, \ldots, a_{n-1}^0)$. Apply Theorem 23.3.8 to prove that this root depends smoothly on the coefficients in a neighborhood of the given point. Compute the derivative with respect to each coefficient.

Exercise 23.4.6. Let H be a real Hilbert space and let $S := \{u \in H \mid \|u\|_H = 1\}$. Use Theorem 23.3.8 to prove that S is a Banach submanifold of codimension one and that $T_u S = u^\perp$. If $J \in C^1(H, \mathbb{R})$ has a local extremum on S, prove that there exists $\lambda \in \mathbb{R}$ such that $DJ(u)(v) = \lambda \langle u, v \rangle_H$ for every $v \in H$.

Exercise 23.4.7. Let $(M, \mathbf{g})$ be a compact Riemannian manifold with smooth boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$ and a compatible connection $\nabla^{\mathbf{E}}$. On $W_0^{1,2}(M, \mathbf{E})$, consider the functionals $J(\mathbf{u}) := \frac{1}{2}\int_M |\nabla^{\mathbf{E}}_w \mathbf{u}|^2_{\mathbf{g},\mathbf{h_E}}\, d\lambda_{\mathbf{g}}$ and $G(\mathbf{u}) := \frac{1}{2}\int_M |\mathbf{u}|^2_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}}$. Suppose $\mathbf{u} \neq 0$ minimizes J subject to the constraint $G(\mathbf{u}) = 1$. Apply the Lagrange multiplier theorem to obtain the weak identity for the eigenvalue problem associated with the Bochner Laplacian.

Exercise 23.4.8. Let X and Y be Banach spaces of sections of vector bundles, and let $\mathcal{F}\colon X \times \mathbb{R} \longrightarrow Y$ be of class C^1. Suppose $\mathcal{F}(\mathbf{u}_0, \lambda_0) = 0$ and $D_1\mathcal{F}(\mathbf{u}_0, \lambda_0)\colon X \longrightarrow Y$ is an isomorphism. Use Theorem 23.3.8 to formulate and prove existence of a local branch $\lambda \longmapsto \mathbf{u}(\lambda)$ of solutions. Compute $\mathbf{u}'(\lambda_0)$.

24

Banach manifolds

Infinite-dimensional manifolds extend the notion of a smooth manifold beyond the finite-dimensional setting. Whereas a finite-dimensional manifold is described locally by open subsets of $\mathbb{R}^n$, in infinite dimensions various topological vector spaces can serve as local models. This chapter works exclusively with Banach spaces; Fréchet and ILH structures are introduced later only when needed.

This extension is more than formal. Global analysis and differential geometry naturally give rise to function spaces equipped with an infinite-dimensional geometric structure. For example, if $\mathbf{E} \longrightarrow M$ is a vector bundle over a Riemannian manifold, the Sobolev spaces $W^{m,p}(M, \mathbf{E})$ are Banach spaces and serve as configuration spaces for numerous variational problems. Other important examples include the spaces of smooth sections $\Gamma(\mathbf{E})$, spaces of smooth maps $C^\infty(M, N)$, diffeomorphism groups $\mathrm{Diff}(M)$, loop manifolds $C^\infty(S^1, M)$, and spaces of immersions or embeddings.

This viewpoint is particularly useful in the variational study of partial differential equations. Solutions are often characterized as critical points of an energy functional defined on an infinite-dimensional function space. It is also common to restrict such a functional to subsets with a manifold structure, such as unit spheres, Nehari manifolds, submanifolds defined by integral constraints, and level sets of differentiable functionals. The geometry of these spaces then provides the appropriate language for formulating criticality conditions, applying Lagrange multipliers, studying gradient flows, and analyzing constrained minimization problems.

Certain spaces of geometric objects can likewise be understood as infinite-dimensional manifolds. They include the space of Riemannian metrics on a fixed manifold, the space of connections on a principal bundle, various spaces of curves or surfaces, and some quotients by symmetry groups. This language is central to Morse theory, gauge theory, symplectic geometry, the calculus of variations, and the study of geometric equations.

The chapter develops the foundations of Banach manifolds that will be used in nonlinear analysis and variational methods. It includes their tangent spaces, submanifolds defined by differentiable constraints, local forms of immersions and submersions, the constant rank theorem, and the preimage theorem. It also examines the assumptions under which bump functions and smooth partitions of unity exist. This last question reveals an essential difference from finite dimensions: paracompactness of the manifold alone does not suffice if the model Banach space lacks the necessary smooth functions. The resulting framework covers many situations modeled on Sobolev spaces and prepares the later study of variational problems and mapping spaces.

From finite-dimensional manifolds to configuration spaces

The idea of modeling a manifold on a function space made differential geometry applicable to problems in which the entire unknown is a function. Eells and Palais developed global tools for these spaces and showed that many finite-dimensional constructions survive under precise functional assumptions. The essential difference is that, in infinite dimensions, existence of complements, bump functions, or suitable metrics is no longer automatic. This chapter therefore keeps the assumptions needed to apply each theorem explicit [85, 111].

24.1 Banach manifolds

We begin by studying manifolds modeled on Banach spaces. They retain the differential calculus and local theorems needed to treat function spaces as geometric objects. In particular, Sobolev spaces of sections, constraints defined by differentiable functionals, and many level sets arising in variational methods have a Banach manifold structure under suitable assumptions.

Unless expressly stated otherwise, the model spaces in this chapter are real Banach spaces, and differentiability is understood over the real field.

Definition 24.1.1. Let (M,τ) be a Hausdorff topological space, and let $k \in \mathbb{N}\cup\{\infty\}$, with $k \geq 1$. A **Banach chart** on M is a pair (U,φ), where $U \subseteq M$ is nonempty and open and $\varphi\colon U \longrightarrow V$ is a homeomorphism from U onto an open subset V of a Banach space X.
Two charts $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β), modeled on Banach spaces X_α and X_β, respectively, are C^k**–compatible** if $U_\alpha \cap U_\beta = \varnothing$ or if the coordinate change

$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism of class C^k between open subsets of Banach spaces. Differentiability is understood in the Fréchet sense, as in Definition 23.2.6, and diffeomorphism has the meaning given in Definition 23.3.4. When $k=\infty$, both the coordinate change and its inverse are required to be of class C^r for every $r \in \mathbb{N}$.
A **Banach atlas of class** C^k on M is a family of charts $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ covering M whose charts are mutually C^k–compatible. A **Banach manifold without boundary of class** C^k is a Hausdorff topological space equipped with a maximal Banach atlas of class C^k. A Banach manifold of class C^∞ is called a **smooth Banach manifold**.

In this chapter, and whenever Definition 24.1.1 is invoked, the term *Banach manifold* means a manifold without boundary unless expressly stated otherwise. We adopt the same convention for finite-dimensional manifolds occurring as domains or codomains of the maps under study. This assumption is incorporated into the local models: chart images are open in Banach spaces.

Banach manifolds with boundary also exist in infinite dimensions. Their charts allow relatively open subsets of half-spaces such as $X\times[0,\infty)$, with coordinate changes differentiable up to the boundary. The product $X\times[0,\infty)$ itself is an example, with boundary $X\times\{0\}$; see [85, pp. 766–767]. The local forms and tangent construction below use the preceding convention.

The boundary of a configuration space differs from the boundary of the manifold on which its functions are defined. If a Riemannian manifold M has boundary, the space $W_0^{1,p}(M)$, with $1<p<\infty$, is still a Banach space

and, as a manifold modeled on itself, has no boundary. The Dirichlet condition here defines a vector space of admissible functions. Constraints given by regular values of functionals will be described through embedded submanifolds of these spaces.

In the preceding definition, we allow, a priori, charts modeled on different Banach spaces. However, on each connected component of the manifold, the model space is determined up to Banach space isomorphism.

Lemma 24.1.2. Let M be a Banach manifold of class C^k, with $k \geq 1$. If M is connected, all Banach spaces modeling its charts are linearly and topologically isomorphic.

Proof. If $M = \varnothing$, there are no charts to compare. Suppose $M \neq \varnothing$. First observe that if two charts in the differentiable structure have nonempty intersection, their model spaces are isomorphic as Banach spaces. Let (U, φ) and (V, ψ) be two charts such that $U \cap V \neq \varnothing$, modeled on the Banach spaces X and Y, respectively. Fix $x \in U \cap V$ and write $\xi := \varphi(x)$. Since the coordinate change $\psi \circ \varphi^{-1}$ is a diffeomorphism of class at least C^1, the operator

$$d(\psi \circ \varphi^{-1})_\xi : X \longrightarrow Y$$

is linear and continuous. Differentiating the identities between the inverse coordinate changes gives

$$d(\varphi \circ \psi^{-1})_{\psi(x)} \circ d(\psi \circ \varphi^{-1})_\xi = \mathrm{id}_X, \qquad d(\psi \circ \varphi^{-1})_\xi \circ d(\varphi \circ \psi^{-1})_{\psi(x)} = \mathrm{id}_Y .$$

Thus $d(\psi \circ \varphi^{-1})_\xi$ is a continuous linear isomorphism with continuous inverse.
Now take a fixed chart (U_0, φ_0), modeled on a Banach space X_0, and define

$$A := \{x \in M \mid \text{there exists a chart around } x \text{ modeled on a space isomorphic to } X_0\}.$$

The set A is nonempty, since $U_0 \subseteq A$, and open: if $x \in A$ and (U, φ) is a chart around x modeled on a space isomorphic to X_0, the same chart works around every point of U, so $U \subseteq A$.
The set $M \setminus A$ is also open. Let $y \in M \setminus A$ and let (V, ψ) be a chart around y modeled on a Banach space Y. If there existed $z \in V \cap A$, there would be a chart around z modeled on a space isomorphic to X_0. This chart would intersect (V, ψ) in an open set containing z, and the first part of the proof would show that Y is isomorphic to X_0. The chart (V, ψ) would then show that $y \in A$, a contradiction. Thus $V \subseteq M \setminus A$.
Hence A is open and closed. Since M is connected and $A \neq \varnothing$, we have $A = M$. Finally, if (V, ψ) is any chart, choose $y \in V$. Since $y \in A$, there exists another chart around y whose model space is isomorphic to X_0; comparing the two charts in a neighborhood of y, the first part of the proof shows that the model space of (V, ψ) is also isomorphic to X_0. This completes the proof. □

This result justifies speaking, when M is connected, of the model space of a Banach manifold up to Banach space isomorphism. For this reason, a Banach manifold is often defined from the outset as a manifold modeled on a fixed Banach space X.

Before introducing the tangent space, we specify the regularity of maps between Banach manifolds.

Definition 24.1.3. Let M and N be Banach manifolds of classes C^k and C^ℓ, respectively, and let $r \in \mathbb{N} \cup \{\infty\}$ be such that $1 \leq r \leq k$ and $1 \leq r \leq \ell$. A map $F : M \longrightarrow N$ is of **class** C^r **at** $p \in M$ if there exist a chart (U, φ) of M around p, a chart (V, ψ) of N around $F(p)$, and an open neighborhood $U_0 \subseteq U$ of p such that $F(U_0) \subseteq V$

and

$$\psi \circ F \circ \varphi^{-1} \colon \varphi(U_0) \longrightarrow \psi(V)$$

is of class C^r. The map F is of **class** C^r if it is of this class at every point of M. When $r = \infty$, we say that F is **smooth**.

This regularity does not depend on the chosen charts. Let $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ be other charts around p and $F(p)$, respectively. The representation $\psi \circ F \circ \varphi^{-1}$ is continuous and $V \cap \widetilde{V}$ is a neighborhood of $F(p)$; we may therefore restrict the domain to a neighborhood of p whose image under F is contained in $V \cap \widetilde{V}$. Restricting also to $U \cap \widetilde{U}$ gives

$$\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1} = (\widetilde{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \widetilde{\varphi}^{-1}).$$

The maps on either side of the coordinate representation of F are coordinate changes of class at least C^r. The chain rule shows that one coordinate representation of F is of class C^r if and only if every other one is.

Remark 24.1.4. (The structure of lower regularity) Let $\mathcal{A}^k$ be the maximal atlas of a manifold of class C^k, and fix $r \in \mathbb{N} \cup \{\infty\}$ with $1 \le r \le k$. The coordinate changes of $\mathcal{A}^k$ are also of class C^r. This atlas therefore determines a structure of class C^r on the same topological space; denote its maximal atlas by $\mathcal{A}^r$. Its charts are those that are C^r–compatible with the charts of $\mathcal{A}^k$. Any two such charts are compatible with each other: near each point of an overlap, insert a chart of $\mathcal{A}^k$ and write their coordinate change as a composition of two diffeomorphisms of class C^r.

When $r < k$, the atlas $\mathcal{A}^r$ may contain more charts than $\mathcal{A}^k$. A chart in $\mathcal{A}^r$ will be called a *chart of class C^r on M*. The class C^r of a map can be checked in these charts, since the argument for coordinate independence uses only coordinate changes of class C^r. This distinction allows us to construct coordinates adapted to a map with the regularity that the map possesses. In the results on local forms of maps of class C^r, the constructed charts will belong to the underlying structures of class C^r.

Since $r \ge 1$, all these coordinate changes are of class C^1. By the chain rule, two curves have the same velocity in a chart of $\mathcal{A}^k$ if and only if they do in a chart of $\mathcal{A}^r$. Thus passage to the structure of lower regularity preserves the tangent space to be defined next.

To study locally Lipschitz vector fields and their flows, we will also use regularity intermediate between C^1 and C^2.

Definition 24.1.5. (Maps and manifolds of class $C^{1,1}$) Let X, Y be Banach spaces and let $U \subseteq X$ be open. A map $f \colon U \longrightarrow Y$ is of **class** $C^{1,1}$ if it is of class C^1 and, for each $x_0 \in U$, there exist $r, L > 0$ such that $B_X(x_0, r) \subseteq U$ and

$$\|Df(x) - Df(y)\|_{\mathcal{L}(X,Y)} \le L\|x - y\|_X, \qquad x, y \in B_X(x_0, r).$$

The condition on the derivative is local: a single constant L valid on all of U is not required. A diffeomorphism is of class $C^{1,1}$ if both it and its inverse have this regularity.

Two Banach charts are $C^{1,1}$**–compatible** if their intersection is empty or the coordinate change between them is a diffeomorphism of class $C^{1,1}$. An **atlas of class** $C^{1,1}$ is a covering by mutually $C^{1,1}$–compatible charts. A **Banach manifold of class** $C^{1,1}$ is a Hausdorff space equipped with a maximal atlas of that class. Since its coordinate changes are C^1, this atlas determines a Banach manifold structure of class C^1 on the same space.

Proposition 24.1.6. (Stability of $C^{1,1}$ regularity) Every map of class C^2 between open subsets of Banach spaces is of class $C^{1,1}$. The composition of two maps of class $C^{1,1}$, whenever defined, is of class $C^{1,1}$. Moreover, if a diffeomorphism of class C^1 is of class $C^{1,1}$, so is its inverse.

Proof. If f is of class C^2, continuity of $D^2 f$ allows its norm to be bounded by a constant L on a sufficiently small ball around each point of its domain. Theorem 23.1.10, applied to the map Df on this convex ball, gives $\|Df(x) - Df(y)\| \le L\|x - y\|$. This proves the first assertion.

Now let $f\colon U \subseteq X \longrightarrow Y$ and $g\colon V \subseteq Y \longrightarrow Z$ be of class $C^{1,1}$, with $f(U) \subseteq V$. Fix $x_0 \in U$. We can choose a convex ball $B \subseteq U$ around x_0 whose image lies in a convex ball $C \subseteq V$, and constants $M_f, M_g, L_f, L_g > 0$, such that $\|Df\| \le M_f$ on B, $\|Dg\| \le M_g$ on C, and the derivatives are Lipschitz there with constants L_f, L_g, respectively. The mean value inequality gives $\|f(x) - f(y)\| \le M_f\|x - y\|$ for $x, y \in B$. By the chain rule,

$$D(g \circ f)(x) - D(g \circ f)(y) = [Dg(f(x)) - Dg(f(y))]Df(x) + Dg(f(y))[Df(x) - Df(y)].$$

Consequently,

$$\|D(g \circ f)(x) - D(g \circ f)(y)\| \le (L_g M_f^2 + M_g L_f)\|x - y\|, \qquad x, y \in B.$$

The composition is therefore of class $C^{1,1}$.

Finally, let $f\colon U \longrightarrow V$ be a diffeomorphism of class C^1 and suppose f is of class $C^{1,1}$. Write $h = f^{-1}$. Around $y_0 \in V$, choose a convex ball C such that $h(C)$ is contained in a ball on which Df is L–Lipschitz. Since Dh is continuous, we may shrink C so that $\|Dh(y)\| \le M$ there. Then $\|h(y) - h(z)\| \le M\|y - z\|$ for $y, z \in C$. The identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$, applied to $A = Df(h(y))$ and $B = Df(h(z))$, gives

$$Dh(y) - Dh(z) = Dh(y)[Df(h(z)) - Df(h(y))]Dh(z).$$

Consequently,

$$\|Dh(y) - Dh(z)\| \le LM^3\|y - z\|, \qquad y, z \in C.$$

This proves that h is of class $C^{1,1}$. □

Stability under composition allows a $C^{1,1}$ atlas to be completed to a unique maximal atlas by adding all charts compatible with the given ones. To compare two added charts around a common point, insert a chart of the original atlas and apply the preceding proposition to the two coordinate changes. Likewise, a map between $C^{1,1}$ manifolds is of class $C^{1,1}$ when its coordinate representations are; the same composition of chart changes shows that it suffices to check this in one chart around each point.

Every manifold of class at least C^2 thus has an underlying $C^{1,1}$ structure obtained by completing its atlas in this category. When working at $C^{1,1}$ regularity, we use charts of this chosen atlas, rather than all charts of the underlying C^1 atlas: a merely C^1 coordinate change need not have locally Lipschitz differential. The admissible curves and the tangent space construction below use only the first-order chain rule, so they also apply to this category through its underlying C^1 structure.

If $U \subseteq M$ is open, we equip it with the Banach manifold structure induced by that of M. Denote by $C^r(U, N)$ the set of maps of class C^r from U to N and write $C^r(U) := C^r(U, \mathbb{R})$. In particular, if $I \subseteq \mathbb{R}$ is an open interval, a curve $\gamma\colon I \longrightarrow M$ is of class C^r if for each $t_0 \in I$ there exist a chart (U, φ) around $\gamma(t_0)$, modeled on X, and an open interval $J \subseteq I$ containing t_0 such that $\gamma(J) \subseteq U$ and $\varphi \circ \gamma\restriction_J\colon J \longrightarrow X$ is of class C^r. By chart independence, it suffices to verify this condition in one chart around each point of the curve.

We now introduce the notion of tangent space. In finite dimensions there are several equivalent constructions, including one based on curves and another formulated through derivations on germs of smooth functions. On Banach manifolds, these two constructions need not agree under the canonical identification, so it is useful to distinguish them before fixing our convention.

Definition 24.1.7. Let M be a Banach manifold of class C^k, with $k \geq 1$, and let $p \in M$. An **admissible curve at** p is a map $\gamma\colon (-\varepsilon,\varepsilon) \longrightarrow M$ of class C^1 such that $\gamma(0) = p$, where $\varepsilon > 0$ may depend on γ.
Two admissible curves γ_1 and γ_2 are **equivalent at** p, and we write $\gamma_1 \sim_p \gamma_2$, if there exists a chart (U,φ) around p such that, after restricting both curves to a sufficiently small common interval around 0,

$$\left.\frac{d}{dt}\right|_{t=0} (\varphi \circ \gamma_1)(t) = \left.\frac{d}{dt}\right|_{t=0} (\varphi \circ \gamma_2)(t).$$

Proposition 24.1.8. Let M be a Banach manifold of class C^k, with $k \geq 1$, and let $p \in M$. The condition in Definition 24.1.7 is independent of the chosen chart and defines an equivalence relation on the set of admissible curves at p.

Proof. Let (U,φ) and (V,ψ) be two charts around p, and let γ_1 and γ_2 be two admissible curves at p. Since $\gamma_1(0) = \gamma_2(0) = p$ and the curves are continuous, there exists $\delta > 0$ such that $\gamma_j((-\delta,\delta)) \subseteq U \cap V$ for $j \in \{1,2\}$. Near 0, we have

$$\psi \circ \gamma_j = (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma_j),$$

, and the chain rule gives

$$(\psi \circ \gamma_j)'(0) = d(\psi \circ \varphi^{-1})_{\varphi(p)}\big((\varphi \circ \gamma_j)'(0)\big).$$

The operator $A_{\psi\varphi}(p) := d(\psi \circ \varphi^{-1})_{\varphi(p)}$ is a continuous linear isomorphism, with inverse $d(\varphi \circ \psi^{-1})_{\psi(p)}$. Consequently,

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0) \quad \Longleftrightarrow \quad (\psi \circ \gamma_1)'(0) = (\psi \circ \gamma_2)'(0).$$

Thus two curves have the same velocity in one chart around p if and only if they do in every other chart. In particular, in the preceding definition, "there exists a chart" can equivalently be replaced by "for every chart."
Reflexivity and symmetry are immediate. To check transitivity, suppose $\gamma_1 \sim_p \gamma_2$ and $\gamma_2 \sim_p \gamma_3$, and choose any chart (U,φ) around p. By chart independence,

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0) = (\varphi \circ \gamma_3)'(0),$$

, so $\gamma_1 \sim_p \gamma_3$. □

Definition 24.1.9. Let M be a Banach manifold of class C^k, with $k \geq 1$, and let $p \in M$. The set of equivalence classes of admissible curves at p is called the **kinematic tangent space** of M at p and is denoted by $T_p^{\mathrm{kin}}M$. The class of an admissible curve γ is denoted by $[\gamma]$.

Proposition 24.1.10. Let M be a Banach manifold of class C^k, with $k \geq 1$, let $p \in M$, and let (U, φ) be a chart around p modeled on a Banach space X. The map

$$\Theta_{\varphi,p} \colon T_p^{\mathrm{kin}} M \longrightarrow X, \qquad \Theta_{\varphi,p}([\gamma]) := \left.\frac{d}{dt}\right|_{t=0} (\varphi \circ \gamma)(t),$$

is a bijection. The vector operations transported from X through $\Theta_{\varphi,p}$ do not depend on the chosen chart. Each chart also determines a complete norm on $T_p^{\mathrm{kin}} M$, and norms determined by different charts satisfy the explicit inequalities in the proof.

Proof. For every admissible curve γ at p, continuity of γ allows its domain to be restricted so that its image lies in U. Chart independence, proved in Proposition 24.1.8, shows that $\Theta_{\varphi,p}$ does not depend on the representative of $[\gamma]$, so it is well defined. If $\Theta_{\varphi,p}([\gamma_1]) = \Theta_{\varphi,p}([\gamma_2])$, then $\gamma_1 \sim_p \gamma_2$ by the definition of the relation itself; thus $\Theta_{\varphi,p}$ is injective.

To prove surjectivity, let $v \in X$ and write $a := \varphi(p)$. Since $\varphi(U)$ is open, there exists $r > 0$ such that $B_X(a, r) \subseteq \varphi(U)$. If $\varepsilon := \frac{r}{1+\|v\|_X}$, then $a + tv \in \varphi(U)$ for $|t| < \varepsilon$. The curve $\gamma_v(t) := \varphi^{-1}(a + tv)$ is admissible at p and satisfies $\Theta_{\varphi,p}([\gamma_v]) = v$.

For $\xi, \eta \in T_p^{\mathrm{kin}} M$ and $\lambda \in \mathbb{R}$, define

$$\xi + \eta := \Theta_{\varphi,p}^{-1}\big(\Theta_{\varphi,p}(\xi) + \Theta_{\varphi,p}(\eta)\big), \qquad \lambda\xi := \Theta_{\varphi,p}^{-1}\big(\lambda\Theta_{\varphi,p}(\xi)\big).$$

Let (V, ψ) be another chart around p modeled on a Banach space Y. The chain rule gives

$$\Theta_{\psi,p} = A_{\psi\varphi}(p) \circ \Theta_{\varphi,p}, \qquad A_{\psi\varphi}(p) := d(\psi \circ \varphi^{-1})_{\varphi(p)}.$$

Since $A_{\psi\varphi}(p)$ is linear, the operations obtained through $\Theta_{\psi,p}$ agree with those obtained through $\Theta_{\varphi,p}$. In particular, $T_p^{\mathrm{kin}} M$ has a well-defined vector space structure; its zero vector is the class of the constant curve $t \longmapsto p$.

Finally, set $\|\xi\|_{\varphi,p} := \|\Theta_{\varphi,p}(\xi)\|_X$. This norm is complete because $\Theta_{\varphi,p}$ is a linear isometry onto X. Moreover,

$$\|\xi\|_{\psi,p} \leq \|A_{\psi\varphi}(p)\|_{\mathcal{L}(X,Y)} \|\xi\|_{\varphi,p}, \qquad \|\xi\|_{\varphi,p} \leq \|A_{\psi\varphi}(p)^{-1}\|_{\mathcal{L}(Y,X)} \|\xi\|_{\psi,p}.$$

Thus the preceding two inequalities show that the norms associated with different charts determine the same topology. The resulting linear structure and Banach topology are independent of the chart, although in general there is no distinguished norm. □

There is also an algebraic construction of the tangent space. To formulate it, we need M to be smooth.

Definition 24.1.11. Let M be a smooth Banach manifold and let $p \in M$. A **germ at p of a smooth real function** is an equivalence class of pairs (U, f), where $U \subseteq M$ is an open neighborhood of p and $f \colon U \longrightarrow \mathbb{R}$ is smooth. Two pairs (U, f) and (V, g) are equivalent if there exists an open neighborhood $W \subseteq U \cap V$ of p such that $f \restriction_W = g \restriction_W$. The set of these germs is denoted by $C_p^\infty(M)$.

The preceding relation is an equivalence relation: reflexivity and symmetry are immediate, and transitivity follows by intersecting two neighborhoods on which the corresponding representatives agree. Addition, multipli-

cation, and scalar multiplication in $C^\infty_p(M)$ are defined by choosing representatives, restricting them to a common neighborhood of p, and performing the usual operations there. These definitions do not depend on the representatives, since equivalent representatives agree on a sufficiently small neighborhood of p. Thus $C^\infty_p(M)$ is a commutative unital real algebra. Evaluation at p is well defined and is an algebra homomorphism. As usual, we use the same symbol f for a germ and any of its representatives when no confusion can arise, and simply write $f(p)$.

Definition 24.1.12. Let M be a smooth Banach manifold and let $p \in M$. An **operational tangent vector at** p is an algebraic linear functional

$$D\colon C^\infty_p(M) \longrightarrow \mathbb{R}$$

satisfying the Leibniz rule

$$D(fg) = D(f)g(p) + f(p)D(g)$$

for any $f, g \in C^\infty_p(M)$. The space of all operational tangent vectors at p is denoted by $T^{\mathrm{op}}_p M$.

No continuity condition is imposed on D, since no topology has been chosen on the germ algebra $C^\infty_p(M)$. With addition and scalar multiplication defined pointwise, $T^{\mathrm{op}}_p M$ is a vector space. Every derivation $D \in T^{\mathrm{op}}_p M$ annihilates constants: $D(1) = D(1 \cdot 1) = 2D(1)$ gives $D(1) = 0$, and linearity gives $D(c) = 0$ for every $c \in \mathbb{R}$.

Proposition 24.1.13. Let M be a smooth Banach manifold and let $p \in M$. Every kinematic tangent vector determines an operational tangent vector. More precisely, there exists a canonical injective linear map

$$\mathcal{J}_p\colon T^{\mathrm{kin}}_p M \longrightarrow T^{\mathrm{op}}_p M, \qquad \mathcal{J}_p([\gamma]) = D_{[\gamma]},$$

, where

$$D_{[\gamma]} f := \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)).$$

Proof. The expression is well defined with respect to the germ. If two representatives f and g agree on a neighborhood W of p, continuity of γ allows its domain to be restricted so that $\gamma(t) \in W$ near 0; then $f(\gamma(t)) = g(\gamma(t))$ near 0, and the two derivatives agree.

It is also independent of the curve representing $[\gamma]$. Fix a chart (U, φ) around p modeled on a Banach space X and write $a := \varphi(p)$. After restricting the representative of f and the chart to a common neighborhood of p, the chain rule gives

$$D_{[\gamma]} f = D(f \circ \varphi^{-1})(a)\big[\Theta_{\varphi,p}([\gamma])\big]. \tag{24.1}$$

The right-hand side depends only on the germ of f and the class $[\gamma]$.

Linearity of $D_{[\gamma]}$ in f follows from linearity of the derivative, and the usual product rule shows that

$$D_{[\gamma]}(fg) = D_{[\gamma]}(f)g(p) + f(p)D_{[\gamma]}(g).$$

Thus $D_{[\gamma]} \in T^{\mathrm{op}}_p M$.

Formula (24.1), linearity of $D(f \circ \varphi^{-1})(a)$, and the definition of the vector operations in $T^{\mathrm{kin}}_p M$ imply that $\mathcal{J}_p$ is linear. To prove injectivity, let $\xi \in T^{\mathrm{kin}}_p M$ be such that $\mathcal{J}_p(\xi) = 0$ and set $v := \Theta_{\varphi,p}(\xi)$. For each $\ell \in X'$,

consider the smooth function $f_\ell\colon U \longrightarrow \mathbb{R}$ defined by $f_\ell(x) := \ell(\varphi(x) - a)$. Then

$$0 = \mathcal{J}_p(\xi)(f_\ell) = \ell(v).$$

Since the continuous dual X' separates points of X, it follows that $v = 0$. Injectivity of $\Theta_{\varphi,p}$ implies $\xi = 0$, so $\mathcal{J}_p$ is injective. □

In finite dimensions, the identification in Definition 1.4.5, completed in Exercise 1.4.13, shows that $\mathcal{J}_p$ is an isomorphism. In infinite dimensions it may fail to be surjective, since the Leibniz rule alone does not force an algebraic derivation to depend only on the first derivative.

Proposition 24.1.14. (The operational tangent space may contain nonkinematic vectors) Let $E := \ell^2(\mathbb{N};\mathbb{R})$ be regarded as a smooth Banach manifold, and let $0 \in E$. Then the canonical map

$$\mathcal{J}_0\colon T_0^{\mathrm{kin}}E \longrightarrow T_0^{\mathrm{op}}E$$

is not surjective. Consequently, under the canonical identification given by $\mathcal{J}_0$, the kinematic tangent space is a proper subspace of the operational tangent space.

Proof. Let $\mathcal{L}^2_s(E;\mathbb{R})$ be the space of continuous symmetric bilinear forms on E, equipped with the norm in Definition 23.2.1, and denote by $\mathcal{F}_s(E)$ the subspace of symmetric forms of finite rank. Proposition B.3.23 identifies each $B \in \mathcal{L}^2_s(E;\mathbb{R})$ isometrically with a continuous self-adjoint linear operator $A_B\colon E \longrightarrow E$ through $B(x,y) = \langle A_B x, y\rangle_E$ and shows that $\overline{\mathcal{F}_s(E)}$ corresponds to the space of compact self-adjoint operators.
Consider the form $B_0(x,y) := \langle x, y\rangle_E$. The operator associated with B_0 is the identity on E, which is not compact because the standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$ lies in the unit ball and satisfies $\|e_n - e_m\|_E = \sqrt{2}$ for $n \neq m$. Thus

$$B_0 \notin \overline{\mathcal{F}_s(E)}.$$

Form the quotient space

$$\mathcal{Q} := \mathcal{L}^2_s(E;\mathbb{R})/\overline{\mathcal{F}_s(E)}$$

and let $\pi\colon \mathcal{L}^2_s(E;\mathbb{R}) \longrightarrow \mathcal{Q}$ be the canonical projection. By Proposition B.2.20, $\mathcal{Q}$ is a normed space and

$$\|\pi(B)\|_{\mathcal{Q}} = \operatorname{dist}_{\mathcal{L}^2_s(E;\mathbb{R})}\big(B, \overline{\mathcal{F}_s(E)}\big).$$

In particular, $b_0 := \pi(B_0)$ is nonzero and $\|b_0\|_{\mathcal{Q}} > 0$.
On the one-dimensional subspace $\operatorname{span}\{b_0\}$, define

$$\lambda_0\colon \operatorname{span}\{b_0\} \longrightarrow \mathbb{R}, \qquad \lambda_0(tb_0) := t.$$

This function is well defined because $b_0 \neq 0$: if $tb_0 = sb_0$, then $(t-s)b_0 = 0$ and therefore $t = s$. Moreover, λ_0 is linear. By absolute homogeneity of the norm, $\|tb_0\|_{\mathcal{Q}} = |t|\,\|b_0\|_{\mathcal{Q}}$, and consequently

$$|\lambda_0(tb_0)| = |t| = \frac{\|tb_0\|_{\mathcal{Q}}}{\|b_0\|_{\mathcal{Q}}}.$$

In particular, for every $t \neq 0$,

$$\frac{|\lambda_0(tb_0)|}{\|tb_0\|_{\mathcal{Q}}} = \frac{1}{\|b_0\|_{\mathcal{Q}}}.$$

Since every nonzero element of $\operatorname{span}\{b_0\}$ has the form tb_0 with $t \neq 0$, we conclude that

$$\|\lambda_0\| = \sup_{\substack{x \in \operatorname{span}\{b_0\} \\ x \neq 0}} \frac{|\lambda_0(x)|}{\|x\|_{\mathcal{Q}}} = \frac{1}{\|b_0\|_{\mathcal{Q}}}.$$

Thus λ_0 is continuous. Corollary B.2.29 gives a continuous linear extension $\widetilde{\lambda}\colon \mathcal{Q} \longrightarrow \mathbb{R}$ of λ_0. Define

$$\Lambda := \widetilde{\lambda} \circ \pi\colon \mathcal{L}^2_s(E;\mathbb{R}) \longrightarrow \mathbb{R}.$$

Then Λ is linear and continuous, annihilates $\overline{\mathcal{F}_s(E)}$, and satisfies $\Lambda(B_0) = 1$.
For a germ $f \in C^\infty_0(E)$, define

$$\mathscr{D}(f) := \Lambda(D^2 f(0)).$$

This definition is independent of the representative, since two representatives of the same germ agree on a neighborhood of 0 and have the same derivatives there. Theorem 23.2.10 ensures that $D^2 f(0) \in \mathcal{L}^2_s(E;\mathbb{R})$. Linearity of the second derivative and of Λ shows that $\mathscr{D}$ is linear.
Let $f, g \in C^\infty_0(E)$ and choose representatives defined on a common open neighborhood of 0. By Proposition 23.2.11,

$$D^2(fg)(0) = f(0)D^2 g(0) + g(0)D^2 f(0) + Df(0) \otimes Dg(0) + Dg(0) \otimes Df(0).$$

Corollary B.3.24 shows that

$$C_{f,g} := Df(0) \otimes Dg(0) + Dg(0) \otimes Df(0)$$

is a symmetric bilinear form of rank at most two. Thus $C_{f,g} \in \mathcal{F}_s(E) \subseteq \overline{\mathcal{F}_s(E)}$ and $\Lambda(C_{f,g}) = 0$. Applying Λ to the preceding formula gives

$$\mathscr{D}(fg) = f(0)\mathscr{D}(g) + g(0)\mathscr{D}(f).$$

Hence $\mathscr{D} \in T_0^{\mathrm{op}} E$.
It remains to prove that $\mathscr{D}$ is not in the image of $\mathcal{J}_0$. Consider $q\colon E \longrightarrow \mathbb{R}$, $q(x) := \|x\|_E^2$. For $x, h \in E$, we have $q(x+h) - q(x) = 2\langle x, h\rangle_E + \|h\|_E^2$, whence

$$Dq(x)[h] = 2\langle x, h\rangle_E, \qquad D^2 q(x)[h,k] = 2\langle h, k\rangle_E.$$

The second derivative is constant and all higher derivatives vanish; in particular, q is smooth, $Dq(0) = 0$, and $D^2 q(0) = 2B_0$. Consequently,

$$\mathscr{D}(q) = 2\Lambda(B_0) = 2.$$

In contrast, if $[\gamma] \in T_0^{\mathrm{kin}} E$ and $v := \gamma'(0)$, the chain rule gives

$$\mathcal{J}_0([\gamma])(q) = \frac{d}{dt}\bigg|_{t=0} q(\gamma(t)) = Dq(0)[v] = 0.$$

Thus $\mathscr{D}$ does not agree with $\mathcal{J}_0([\gamma])$ for any admissible curve γ, and $\mathcal{J}_0$ is not surjective. □

Since $\mathcal{J}_0$ is injective, it canonically identifies $T_0^{\mathrm{kin}}E$ with the subspace $\mathcal{J}_0(T_0^{\mathrm{kin}}E)$ of $T_0^{\mathrm{op}}E$; the proposition shows that this inclusion is proper. This means precisely that there are operational derivations that do not arise from velocities of curves and hence that the two tangent space constructions do not agree canonically in infinite dimensions. Failure of surjectivity of $\mathcal{J}_0$ does not by itself exclude the possible existence of an abstract noncanonical linear isomorphism between the two vector spaces: to rule out that possibility as well, one would need to compare, for example, their algebraic dimensions. Such a stronger assertion is unnecessary here; the geometrically relevant difference is that the natural identification provided by curves does not exhaust the space of derivations.

From now on, unless expressly stated otherwise, "tangent space" means the kinematic tangent space.

Definition 24.1.15. Let M be a Banach manifold of class C^k, with $k \geq 1$, and let $p \in M$. The **tangent space** of M at p is

$$T_pM := T_p^{\mathrm{kin}}M.$$

Its elements are equivalence classes of admissible curves. If (U,φ) is a chart around p modeled on a Banach space X, the associated bicontinuous linear isomorphism is

$$\Theta_{\varphi,p}\colon T_pM \longrightarrow X, \qquad [\gamma] \longmapsto \frac{d}{dt}\bigg|_{t=0} (\varphi\circ\gamma)(t).$$

Definition 24.1.16. Let $F\colon M \longrightarrow N$ be a map of class C^1 between Banach manifolds and let $p \in M$. The **differential of F at p** is the map

$$dF_p\colon T_pM \longrightarrow T_{F(p)}N, \qquad dF_p[\gamma] := [F\circ\gamma].$$

The preceding definition makes sense because $F\circ\gamma$ is a curve of class C^1 and satisfies $(F\circ\gamma)(0) = F(p)$. Moreover, it is independent of the representative of $[\gamma]$. Let (U,φ) be a chart of M around p and (V,ψ) a chart of N around $F(p)$. After restricting the domains, for every admissible curve γ at p we have

$$(\psi\circ F\circ\gamma)'(0) = d(\psi\circ F\circ\varphi^{-1})_{\varphi(p)}\big((\varphi\circ\gamma)'(0)\big).$$

If γ_1 and γ_2 represent the same vector in T_pM, their velocities in the chart φ agree, and the preceding formula shows that $F\circ\gamma_1$ and $F\circ\gamma_2$ have the same velocity in the chart ψ. Thus $[F\circ\gamma_1] = [F\circ\gamma_2]$.

The same formula gives the coordinate representation

$$\Theta_{\psi,F(p)}\circ dF_p\circ\Theta_{\varphi,p}^{-1} = d(\psi\circ F\circ\varphi^{-1})_{\varphi(p)}.$$

In particular, dF_p is a continuous linear operator. The chain rule for maps between open subsets of Banach spaces also implies that, if $G\colon N \longrightarrow P$ is of class C^1, then

$$d(G\circ F)_p = dG_{F(p)}\circ dF_p, \qquad d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM}.$$

24.2 Complementability and linear decompositions

The local forms of submanifolds and maps of regular rank depend on a linear property that goes unnoticed in finite dimensions. A closed subspace of a Banach space need not admit a closed complement. Before studying

immersions and submersions, it is therefore useful to make precise the relation between topological complements, continuous projections, and one-sided inverses.

Definition 24.2.1. Let E be a Banach space and let $Y \subseteq E$ be a closed subspace. We say that Y is **complemented** in E if there exists a closed subspace $Z \subseteq E$ such that

$$E = Y \oplus Z.$$

This means that every $x \in E$ can be written uniquely as $x = y + z$, with $y \in Y$ and $z \in Z$. The subspace Z is called a **topological complement** of Y.

The word "topological" refers to the fact that both summands are Banach spaces with the induced norms and the projections determined by the decomposition are continuous. This last property is not an additional assumption, as the following characterization shows.

Proposition 24.2.2. (Characterization by continuous projections) Let E be a Banach space and $Y \subseteq E$ a closed subspace. The following assertions are equivalent.

(a) The subspace Y is complemented in E.

(b) There exists an operator $P \in \mathcal{L}(E)$ such that $P^2 = P$ and $\mathrm{Ran}(P) = Y$.

If (b) holds, then

$$E = Y \oplus \ker(P).$$

Proof. Suppose $E = Y \oplus Z$, with Y and Z closed. Equip $Y \times Z$ with the norm $\|(y,z)\| := \|y\| + \|z\|$. The operator

$$S \colon Y \times Z \longrightarrow E, \qquad S(y,z) := y + z,$$

is linear, continuous, and bijective. Since $Y \times Z$ and E are Banach spaces, Theorem B.2.17 ensures that S^{-1} is continuous. If pr_Y and pr_Z denote the coordinate projections of $Y \times Z$, the projections determined by the direct sum are

$$P_Y = \mathrm{pr}_Y \circ S^{-1}, \qquad P_Z = \mathrm{pr}_Z \circ S^{-1}.$$

Thus both are continuous. In particular, $P_Y^2 = P_Y$ and $\mathrm{Ran}(P_Y) = Y$.
Conversely, let $P \in \mathcal{L}(E)$ be a projection with range Y. The subspace $\ker(P)$ is closed. For each $x \in E$, we have

$$x = Px + (x - Px),$$

, where $Px \in Y$ and $P(x - Px) = 0$. Thus $E = Y + \ker(P)$. If $x \in Y \cap \ker(P)$, then $x = Py$ for some $y \in E$ and

$$x = Py = P^2 y = P(Py) = Px = 0.$$

The sum is direct and $E = Y \oplus \ker(P)$. □

Definition 24.2.3. (One-sided inverses) Let E and F be Banach spaces and let $T \in \mathcal{L}(E,F)$. An operator $L \in \mathcal{L}(F,E)$ is a **left inverse** of T if $L \circ T = I_E$, and an operator $R \in \mathcal{L}(F,E)$ is a **right inverse** of T if $T \circ R = I_F$. Both are called **one-sided inverses** of T.

Proposition 24.2.4. (One-sided inverses and complementability) Let E and F be Banach spaces and let $T \in \mathcal{L}(E,F)$.

(a) There exists $L \in \mathcal{L}(F,E)$ such that $L \circ T = I_E$ if and only if T is injective and $\mathrm{Ran}(T)$ is a complemented subspace of F.

(b) There exists $R \in \mathcal{L}(F,E)$ such that $T \circ R = I_F$ if and only if T is surjective and $\ker(T)$ is a complemented subspace of E.

In the first case, $T\colon E \longrightarrow \mathrm{Ran}(T)$ is a Banach space isomorphism. In the second, if $E = \ker(T) \oplus E_1$, then $T\restriction_{E_1}\colon E_1 \longrightarrow F$ is an isomorphism.

Proof. First suppose $L \circ T = I_E$. Then T is injective. The operator $P := T \circ L \in \mathcal{L}(F)$ satisfies

$$P^2 = T(LT)L = TL = P.$$

Moreover, $P(F) \subseteq \mathrm{Ran}(T)$ and $P(Tx) = T(LT)x = Tx$ for every $x \in E$. Consequently, $\mathrm{Ran}(T) = \mathrm{Ran}(P) = \ker(I_F - P)$. This subspace is closed, and Proposition 24.2.2 shows that this range is complemented.
Now suppose T is injective and $F = \mathrm{Ran}(T) \oplus Z$. Since the range is closed, $T\colon E \longrightarrow \mathrm{Ran}(T)$ is a continuous bijective linear operator between Banach spaces. Theorem B.2.17 implies that its inverse $T^{-1}\colon \mathrm{Ran}(T) \longrightarrow E$ is continuous. If $P_{\mathrm{Ran}(T)}\colon F \longrightarrow \mathrm{Ran}(T)$ is the continuous projection associated with the decomposition, then

$$L := T^{-1} \circ P_{\mathrm{Ran}(T)}$$

belongs to $\mathcal{L}(F,E)$ and satisfies $L \circ T = I_E$. This proves (a).
Suppose $T \circ R = I_F$. In particular, T is surjective. The operator $Q := R \circ T \in \mathcal{L}(E)$ is a projection, since

$$Q^2 = R(TR)T = RT = Q.$$

If $x \in \ker(Q)$, then $Tx = T(Qx) = 0$. The reverse inclusion is immediate, so $\ker(Q) = \ker(T)$. Likewise, $\mathrm{Ran}(Q) = \mathrm{Ran}(R)$. Since Q is a continuous projection, its range and kernel are closed, and Proposition 24.2.2 gives the topological decomposition

$$E = \ker(T) \oplus \mathrm{Ran}(R).$$

Conversely, suppose T is surjective and $E = \ker(T) \oplus E_1$. The restriction $T_1 := T\restriction_{E_1}\colon E_1 \longrightarrow F$ is injective. It is also surjective: given $y \in F$, choose $x \in E$ with $Tx = y$ and write $x = k + x_1$, with $k \in \ker(T)$ and $x_1 \in E_1$; then $T_1 x_1 = y$. Theorem B.2.17 shows that $T_1^{-1}\colon F \longrightarrow E_1$ is continuous. Composing this inverse with the inclusion $E_1 \hookrightarrow E$ gives a continuous right inverse of T. □

One-sided inverses persist under sufficiently small perturbations. This property explains why the local forms encountered below persist in a neighborhood of the point under consideration.

Corollary 24.2.5. (Stability of one-sided inverses) Let E and F be Banach spaces and let $T, S \in \mathcal{L}(E,F)$.

(a) If $L \in \mathcal{L}(F,E)$ satisfies $LT = I_E$ and $\|L(S-T)\|_{\mathcal{L}(E)} < 1$, then S admits the continuous left inverse

$$(LS)^{-1}L.$$

(b) If $R \in \mathcal{L}(F,E)$ satisfies $TR = I_F$ and $\|(S-T)R\|_{\mathcal{L}(F)} < 1$, then S admits the continuous right inverse

$$R(SR)^{-1}.$$

Proof. In the first case, $LS = I_E + L(S-T)$ is invertible by Theorem B.2.10; hence $(LS)^{-1}LS = I_E$. In the second, $SR = I_F + (S-T)R$ is invertible and $SR(SR)^{-1} = I_F$. □

Proposition 24.2.6. (Complements in the finite-dimensional and Hilbert space cases) Let E be a Banach space.

(a) Every finite-dimensional subspace of E is complemented.

(b) Every closed subspace of finite codimension in E is complemented.

(c) If E is a Hilbert space, every closed subspace of E is complemented.

Proof. Let $Y \subseteq E$ be finite-dimensional and choose a basis $y_1,\dots,y_m$ of Y. The coordinate functionals $\ell_j\colon Y \longrightarrow \mathbb{R}$ are continuous. By Corollary B.2.29, each ℓ_j has a continuous extension $\widetilde{\ell}_j \in E'$. The operator

$$P(x) := \sum_{j=1}^{m} \widetilde{\ell}_j(x) y_j$$

is a continuous projection onto Y. Proposition 24.2.2 proves (a).
Now let $Y \subseteq E$ be closed and of codimension $m < \infty$. Choose $e_1,\dots,e_m \in E$ whose classes form an algebraic basis of E/Y and set $Z := \operatorname{span}\{e_1,\dots,e_m\}$. Then Z is closed, $Y \cap Z = \{0\}$, and every element of E belongs to $Y + Z$. Thus $E = Y \oplus Z$.
Finally, if E is a Hilbert space and Y is closed, Corollary B.3.8 gives $E = Y \oplus Y^{\perp}$. □

24.3 Banach submanifolds

We now turn to the notion of submanifold. In finite dimensions, a subset is a submanifold if it can locally be straightened by a change of coordinates. This principle remains valid in Banach spaces, but an additional topological condition arises: the subspace describing the tangent directions must be complemented. In infinite dimensions, a closed subspace need not have a closed complement, so this property must be part of the definition.

Definition 24.3.1. Let M be a Banach manifold of class C^k and let $1 \le r \le k$. A subset $N \subseteq M$ is an **embedded submanifold of class** C^r if, for each $p \in N$, there exist a chart (U,φ) of class C^r on M around p, in the sense of Remark 24.1.4, with $\varphi(p) = 0$, and a topological decomposition $X = Y \oplus Z$ of the chart's model space into closed subspaces such that, upon identifying X with $Y \times Z$ through $(y,z) \mapsto y + z$,

$$\varphi(U \cap N) = \varphi(U) \cap (Y \times \{0\}).$$

A chart of class C^r with these properties is called an **adapted chart** for N at M. The subspace Y is the local model space of N in this chart.

The condition in Definition 24.3.1 does more than describe the local position of N inside M: it canonically

equips N with a Banach manifold structure.

Proposition 24.3.2. (Induced manifold structure on a submanifold) Let N be an embedded submanifold of class C^r of a Banach manifold M. Then N, with the subspace topology, is a Banach manifold of class C^r. If (U,φ) is an adapted chart, $X = Y \oplus Z$ is the corresponding decomposition, and $\pi_Y : X \longrightarrow Y$ is the canonical projection, then

$$\varphi_N := \pi_Y \circ \varphi \restriction_{U\cap N}$$

is a chart on N. The canonical inclusion $\iota\colon N \hookrightarrow M$ is of class C^r and, for each $p \in N$, the differential $d\iota_p$ identifies T_pN with a closed complemented subspace of T_pM.

Proof. Let (U,φ) be an adapted chart around $p \in N$, with $\varphi(p) = 0$ and $X = Y \oplus Z$. Define

$$V_Y := \{y \in Y \mid (y,0) \in \varphi(U)\}.$$

The linear inclusion $\iota_Y : Y \longrightarrow X$, given by $\iota_Y(y) = (y,0)$, is continuous, and $V_Y = \iota_Y^{-1}(\varphi(U))$. Since $\varphi(U)$ is open in X, the set V_Y is open in Y.
By the adapted chart condition, every point of $\varphi(U \cap N)$ has the form $(y,0)$ with $y \in V_Y$. Consequently, $\varphi_N : U \cap N \longrightarrow V_Y$ is bijective. Its inverse is given by

$$\varphi_N^{-1}(y) = \varphi^{-1}(y,0), \qquad y \in V_Y.$$

Continuity of φ_N follows from continuity of $\varphi \restriction_{U\cap N}$ and Proposition 24.2.2; continuity of φ_N^{-1} follows from continuity of φ^{-1} and ι_Y. Thus φ_N is a homeomorphism for the subspace topology on N.
Consider two adapted charts (U_1,φ_1) and (U_2,φ_2) with decompositions $X_1 = Y_1 \oplus Z_1$ and $X_2 = Y_2 \oplus Z_2$. On the corresponding overlap, the coordinate change between the induced charts on N can be written as

$$\varphi_{N,2} \circ \varphi_{N,1}^{-1} = \pi_{Y_2} \circ \varphi_2 \circ \varphi_1^{-1} \circ \iota_{Y_1}.$$

The inclusions ι_{Y_1} and projections π_{Y_2} are continuous and linear, while $\varphi_2 \circ \varphi_1^{-1}$ is of class C^r. The chain rule shows that the coordinate change is of class C^r. Interchanging the indices gives the same regularity for its inverse. Thus the charts φ_N form a Banach atlas of class C^r on N. Since N is a subspace of a Hausdorff space, it is also Hausdorff, establishing its Banach manifold structure.
In the coordinates of an adapted chart, the canonical inclusion $\iota\colon N \hookrightarrow M$ is represented by the continuous linear operator

$$\iota_Y : Y \longrightarrow Y \oplus Z, \qquad \iota_Y(y) = (y,0).$$

Thus ι is of class C^r. Moreover, $d\iota_p$ is represented by ι_Y, whose image is $Y \times \{0\}$. This subspace is closed and complemented by $\{0\} \times Z$, so T_pN is identified with a closed complemented subspace of T_pM. □

Adapted charts also allow us to recognize when a map with image in N is differentiable as a map into the submanifold itself.

Lemma 24.3.3. (Maps into a submanifold) Let $N \subseteq M$ be an embedded submanifold of class C^r, let P be a Banach manifold of class at least C^s, and suppose $1 \le s \le r$. If $F : P \longrightarrow M$ has image contained in N, then F is of class C^s if and only if its corestriction $\widehat{F} : P \longrightarrow N$ is of class C^s.

Proof. If $\widehat{F}$ is of class C^s, so is $F = \iota \circ \widehat{F}$, since the inclusion $\iota\colon N \longrightarrow M$ is of class C^r. Conversely, suppose F is of class C^s. Since N has the subspace topology, the corestriction $\widehat{F}$ is continuous. Given $p \in P$, choose an adapted chart (U, φ) of M around $F(p)$, with decomposition $X = Y \oplus Z$. On the open set $F^{-1}(U)$, the representation of $\widehat{F}$ in the induced chart φ_N is

$$\varphi_N \circ \widehat{F} = \pi_Y \circ \varphi \circ F.$$

After introducing a chart of P, the right-hand side is a composition of maps of class C^s, since φ is of class C^r and π_Y is continuous and linear. Consequently, $\widehat{F}$ is of class C^s near p. Since p was arbitrary, it is of this class on all of P. □

Remark 24.3.4. By Theorem 23.3.8, the description through adapted charts is equivalent to locally representing the subset, after a change of coordinates, as the graph of a map of class C^r of the form $h\colon A \longrightarrow Z$, where $A \subseteq Y$ is open. The adapted chart description directly displays the tangent space and its complement in the ambient space.

When the complement Z can be chosen of finite dimension m, we say that N has codimension m. In particular, a submanifold of codimension one is locally modeled on the kernel of a nonzero continuous linear functional. This situation arises naturally in regular level sets and variational constraints.

24.4 Fundamental examples of Banach submanifolds

The adapted charts in Definition 24.3.1 can be constructed explicitly in several basic examples. These examples also show how complementability determines the tangent geometry.

Proposition 24.4.1. (Complemented subspaces and their translates) Let E be a Banach space, let $Y \subseteq E$ be a closed complemented subspace, and let $a \in E$. Then $a + Y$ is a smooth embedded submanifold of E. If $\iota\colon a + Y \longrightarrow E$ is the inclusion and $\Theta_{\mathrm{id}_E, x}\colon T_x E \longrightarrow E$ is the bicontinuous linear isomorphism in Proposition 24.1.10, then

$$\Theta_{\mathrm{id}_E, x}\big(d\iota_x(T_x(a + Y))\big) = Y$$

for every $x \in a + Y$.

Proof. Choose a closed complement Z such that $E = Y \oplus Z$. For each $x \in a + Y$, the translation

$$\varphi_x\colon E \longrightarrow E, \qquad \varphi_x(u) := u - x,$$

is a smooth diffeomorphism, $\varphi_x(x) = 0$, and

$$\varphi_x(a + Y) = Y \times \{0\},$$

, where E is identified with $Y \times Z$. Thus φ_x is an adapted chart. In these coordinates, the inclusion of $a + Y$ into E is represented by $y \mapsto (y, 0)$, whose differential has image Y. □

Proposition 24.4.2. (Tangent space to a graph) Let E and F be Banach spaces, let $U \subseteq E$ be open, and let $h \in C^r(U,F)$, with $r \geq 1$. Then

$$\operatorname{graf}(h) := \{(x, h(x)) \in E \times F \mid x \in U\}$$

is an embedded submanifold of class C^r of the open set $U \times F$. Moreover, if $\iota\colon \operatorname{graf}(h) \longrightarrow U \times F$ is the inclusion, then

$$\Theta_{\mathrm{id}_{U\times F},(x,h(x))}\big(d\iota_{(x,h(x))}(T_{(x,h(x))}\operatorname{graf}(h))\big) = \{(v, Dh(x)v) \mid v \in E\}.$$

Proof. The graph description of submanifolds in Remark 24.3.4 can be seen directly. Fix $x_0 \in U$ and consider the diffeomorphism

$$\Phi_{x_0}\colon U \times F \longrightarrow (U - x_0) \times F, \qquad \Phi_{x_0}(x,y) := (x - x_0, y - h(x)).$$

The map Φ_{x_0} is of class C^r, its inverse is given by

$$\Phi_{x_0}^{-1}(\xi, z) = (\xi + x_0, z + h(\xi + x_0)),$$

, and it satisfies

$$\Phi_{x_0}(x_0, h(x_0)) = (0,0), \qquad \Phi_{x_0}(\operatorname{graf}(h)) = (U - x_0) \times \{0\}.$$

Thus Φ_{x_0} is a chart of class C^r compatible with the underlying C^r structure of $U \times F$ and adapted at $(x_0, h(x_0))$. The differential of the parametrization $x \mapsto (x, h(x))$ at x_0 is given by $v \mapsto (v, Dh(x_0)v)$, yielding the tangent space formula, since x_0 was arbitrary. This space is complemented by $\{0\} \times F$, since, for each $(u,w) \in E \times F$,

$$(u,w) = (u, Dh(x_0)u) + (0, w - Dh(x_0)u).$$

□

Example 24.4.3. (Hyperplanes and spheres) Let E be a real Banach space.

(a) If $\ell \in E'$ is nonzero and $c \in \mathbb{R}$, then $\ell^{-1}(c)$ is a smooth embedded submanifold of codimension one. Indeed, if $e \in E$ satisfies $\ell(e) = 1$, then $E = \ker(\ell) \oplus \mathbb{R}e$ and

$$\ell^{-1}(c) = ce + \ker(\ell).$$

Proposition 24.4.1 shows that this affine hyperplane is an embedded submanifold. If $\iota_c\colon \ell^{-1}(c) \longrightarrow E$ is the inclusion, for every $x \in \ell^{-1}(c)$ we have

$$\Theta_{\mathrm{id}_E,x}\big(d(\iota_c)_x(T_x\ell^{-1}(c))\big) = \ker(\ell).$$

(b) Let $r \geq 1$ and suppose the norm on E is of class C^r on $E \setminus \{0\}$. Set

$$S_E := \{x \in E \mid \|x\|_E = 1\}.$$

For each $x \in S_E$, homogeneity of the norm implies

$$D\|\cdot\|_E(x)(x) = \frac{d}{dt}\bigg|_{t=0} \|(1+t)x\|_E = \|x\|_E.$$

Thus $D\|\cdot\|_E(x) \neq 0$ on the unit sphere. To obtain an adapted chart, fix $x \in S_E$ and set $Y := \ker(D\|\cdot\|_E(x))$. We have $E = Y \oplus \mathbb{R}x$. The map

$$F_x \colon \mathcal{O}_x \longrightarrow \mathbb{R}, \qquad F_x(y,t) := \|x + y + tx\|_E - 1,$$

, where $\mathcal{O}_x$ is a sufficiently small open neighborhood of $(0,0)$ in $Y \times \mathbb{R}$ that $x + y + tx \neq 0$, is of class C^r and satisfies $F_x(0,0) = 0$ and $D_2F_x(0,0)(s) = s$. Theorem 23.3.8 locally represents the set $F_x^{-1}(0)$ as the graph of a function of class C^r on Y. If h denotes this function, the derivative formula for the implicit function gives

$$Dh(0) = -D_2F_x(0,0)^{-1} \circ D_1F_x(0,0) = 0,$$

, since $D_1F_x(0,0)$ is the restriction of $D\|\cdot\|_E(x)$ to Y. In particular, the graph has tangent space $Y \times \{0\}$ at the origin. The coordinate change that subtracts this function in the second variable straightens the graph onto $Y \times \{0\}$. Thus S_E is an embedded submanifold of class C^r. Differentiating the identity defining the sphere gives, if $\iota_S \colon S_E \longrightarrow E$ denotes the inclusion,

$$\Theta_{\mathrm{id}_E,x}\big(d(\iota_S)_x(T_xS_E)\big) = \ker\big(D\|\cdot\|_E(x)\big).$$

If E is a real Hilbert space, we may take $q(x) := \|x\|_E^2$; then $Dq(x)v = 2\langle x, v\rangle_E$ and $\Theta_{\mathrm{id}_E,x}\big(d(\iota_S)_x(T_xS_E)\big) = x^\perp$.

Example 24.4.4. (The group of invertible operators) Let E be a Banach space. The set

$$\mathrm{GL}(E) := \{A \in \mathcal{L}(E) \mid A \text{ is invertible}\}$$

is open in the Banach space $\mathcal{L}(E)$ by Proposition 23.3.2. Thus $\mathrm{GL}(E)$ is a smooth Banach manifold, and

$$\Theta_{\mathrm{id}_{\mathrm{GL}(E)},A} \colon T_A\,\mathrm{GL}(E) \longrightarrow \mathcal{L}(E)$$

is a bicontinuous linear isomorphism for every $A \in \mathrm{GL}(E)$. Operator composition is continuous and bilinear and, by Proposition 23.2.4, smooth. The map $A \mapsto A^{-1}$ is also smooth and satisfies

$$D(A \mapsto A^{-1})(A)H = -A^{-1}HA^{-1}.$$

24.5 Immersions, submersions, and embeddings

In finite dimensions, injectivity or surjectivity of the differential alone provides the local inclusion and projection forms. On Banach manifolds, one must also retain the topological information contained in a one-sided inverse. Proposition 24.2.4 allows this condition to be expressed through complemented kernels and images. The resulting local forms are those commonly used in infinite-dimensional differential geometry; more general formulations may be found in [67, 2, 91].

Definition 24.5.1. Let M and N be Banach manifolds of class C^k, let $f\colon M \longrightarrow N$ be a map of class C^r, with $1 \le r \le k$, and let $p \in M$.

(a) We say that f is a **weak immersion at** p if $df_p\colon T_pM \longrightarrow T_{f(p)}N$ is injective.

(b) We say that f is a **split immersion at** p if df_p admits a continuous linear left inverse.

(c) We say that f is a **weak submersion at** p if df_p is surjective.

(d) We say that f is a **split submersion at** p if df_p admits a continuous linear right inverse.

The map f is a weak immersion, a split immersion, a weak submersion, or a split submersion if it has the corresponding property at every point of M. From now on, within the category of Banach manifolds, **immersion** and **submersion** mean the split versions unless expressly stated otherwise.

The condition is intrinsic: differentials of coordinate changes are continuous linear isomorphisms, and existence of a one-sided inverse is preserved under composition on either side with isomorphisms. Proposition 24.2.4 gives the characterizations

$$\begin{aligned} f \text{ is a split immersion at } p &\iff df_p \text{ is injective and } \operatorname{Ran}(df_p) \text{ is complemented},\\ f \text{ is a split submersion at } p &\iff df_p \text{ is surjective and } \ker(df_p) \text{ is complemented}. \end{aligned}$$

In the first equivalence, the image is automatically closed. Thus mere injectivity, injectivity with closed image, and injectivity with closed complemented image are distinct conditions.

Theorem 24.5.2. (Local form of split immersions) Let $f\colon M \longrightarrow N$ be a map of class C^r between Banach manifolds, with $r \ge 1$, and let $p \in M$. The following assertions are equivalent.

(a) The map f is a split immersion at p.

(b) There exist charts of class C^r, (U,φ) of M around p and (V,ψ) of N around $f(p)$, Banach spaces E and Z, and an identification of the model space of V with $E \oplus Z$ such that $\varphi(p) = 0$, $\psi(f(p)) = (0,0)$, and

$$(\psi \circ f \circ \varphi^{-1})(x) = (x, 0)$$

for every $x \in \varphi(U)$.

In particular, if f is a split immersion at p, after shrinking U the map $f\restriction_U$ is injective and is a split immersion at every point of U.

Proof. Suppose (a). Take initial charts around p and $f(p)$. Composing them with translations reduces the problem to a map $F\colon \Omega \subseteq E \longrightarrow \Theta \subseteq H$ of class C^r, where Ω and Θ are open sets containing the origin, such that $F(0) = 0$ and $A := DF(0)$ admits a continuous left inverse. By Proposition 24.2.4, the subspace $R := \operatorname{Ran}(A)$ is closed, and there exists a closed subspace $Z \subseteq H$ such that $H = R \oplus Z$. Denote the continuous projections by $P_R\colon H \longrightarrow R$ and $P_Z\colon H \longrightarrow Z$.
The operator $A\colon E \longrightarrow R$ is an isomorphism. Consider

$$G\colon \Omega \longrightarrow R, \qquad G(x) := P_R F(x).$$

We have $DG(0) = A$. Theorem 23.3.3 gives open sets $\Omega_0 \subseteq \Omega$ and $W \subseteq R$ containing the origin such that $G\restriction_{\Omega_0} : \Omega_0 \longrightarrow W$ is a diffeomorphism of class C^r. In the new coordinate $y = G(x)$, write

$$F\big(G^{-1}(y)\big) = y + h(y), \qquad h(y) := P_Z F\big(G^{-1}(y)\big) \in Z.$$

Since $h(0) = 0$ and Θ is open, there exist open neighborhoods $W_0 \subseteq W$ and $Z_0 \subseteq Z$ of the origin such that

$$W_0 \times Z_0 \subseteq \Theta, \qquad h(W_0) \subseteq Z_0.$$

Shrink Ω_0 to $G^{-1}(W_0)$. The triangular map

$$\Xi\colon W_0 \times Z \longrightarrow W_0 \times Z, \qquad \Xi(y,z) := (y, z - h(y)),$$

is a diffeomorphism of class C^r, with inverse $\Xi^{-1}(y,z) = (y, z + h(y))$. After restricting it to an open neighborhood $W_0 \times Z_0$ of the origin, its image $\Xi(W_0 \times Z_0)$ is open and $\Xi\restriction_{W_0\times Z_0}$ is a diffeomorphism onto that image. Inclusion $h(W_0) \subseteq Z_0$ ensures that the points in the image of F under consideration lie in the domain of this new chart, giving

$$\Xi\Big(F\big(G^{-1}(y)\big)\Big) = (y, 0).$$

Finally, the isomorphism $A\colon E \longrightarrow R$ allows us to postcompose the coordinate G with A^{-1} and the first component of the codomain coordinate with the same isomorphism. In this way, both first factors are identified with the model space E, and the preceding expression becomes $x \mapsto (x, 0)$. Transporting these coordinate changes to M and N gives (b).

Conversely, in the coordinates of (b), the differential of f at p is represented by the inclusion

$$E \longrightarrow E \oplus Z, \qquad v \longmapsto (v, 0),$$

, which admits projection onto E as a left inverse. The chart differentials are isomorphisms, so df_p also admits a continuous left inverse.

The local expression $(x, 0)$ shows both injectivity of $f\restriction_U$ and persistence of the split condition on U. □

Theorem 24.5.3. (Local form of split submersions) Let $f : M \longrightarrow N$ be a map of class C^r between Banach manifolds, with $r \geq 1$, and let $p \in M$. The following assertions are equivalent.

(a) The map f is a split submersion at p.

(b) There exist charts of class C^r, (U, φ) of M around p and (V, ψ) of N around $f(p)$, and a Banach space K, such that the model space of U is identified with $K \oplus F$, where F is the model space of V, and

$$(\psi \circ f \circ \varphi^{-1})(k, y) = y$$

for every $(k, y) \in \varphi(U)$.

In particular, every split submersion is locally open, admits local sections around every point of its image, and remains a split submersion in a neighborhood of the point under consideration.

Proof. Suppose (a) and work in centered charts, so the map is represented by $F\colon \Omega \subseteq E \longrightarrow H$, with $F(0) = 0$. Let $T := DF(0)$. By Proposition 24.2.4, there exists a closed subspace $E_1 \subseteq E$ such that

$$E = K \oplus E_1, \qquad K := \ker(T),$$

and $T_1 := T\restriction_{E_1} : E_1 \longrightarrow H$ is an isomorphism. Let $P_K : E \longrightarrow K$ be the continuous projection and define

$$\Gamma\colon \Omega \longrightarrow K \times H, \qquad \Gamma(x) := (P_K x, F(x)).$$

If $x = k + e_1$, then

$$D\Gamma(0)(k + e_1) = (k, T_1 e_1).$$

This operator is an isomorphism from E onto $K \times H$, with inverse $(k, y) \mapsto k + T_1^{-1} y$. Theorem 23.3.3 shows that Γ is a diffeomorphism of class C^r between neighborhoods of the origin. Since the second component of Γ is F, in the coordinate $(k, y) = \Gamma(x)$ we have

$$F\big(\Gamma^{-1}(k, y)\big) = y.$$

This proves (b).
If (b) holds, the differential is represented by the projection $K \oplus F \longrightarrow F$, which admits the right inverse $y \mapsto (0, y)$. This proves the equivalence.
In these coordinates, the image of any sufficiently small open product is open. Moreover, the map $y \mapsto (0, y)$, transported through the charts, provides a local section. Persistence of the property follows from the same normal form. □

Definition 24.5.4. Let M and N be Banach manifolds. A map $f\colon M \longrightarrow N$ of class C^r, with $r \geq 1$, is called a **split embedding** if it is a split immersion, is injective, and the map

$$f\colon M \longrightarrow f(M)$$

is a homeomorphism when $f(M)$ is equipped with the subspace topology from N.

Proposition 24.5.5. (Embeddings and embedded submanifolds) Let $f\colon M \longrightarrow N$ be a map of class C^r between Banach manifolds, with $r \geq 1$.

(a) If f is a split embedding, then $f(M)$ is an embedded submanifold of class C^r of N, and $f\colon M \longrightarrow f(M)$ is a diffeomorphism.

(b) If $S \subseteq N$ is an embedded submanifold of class C^r, its inclusion $S \hookrightarrow N$ is a split embedding.

(c) If $f(M)$ is an embedded submanifold and $f\colon M \longrightarrow f(M)$ is a diffeomorphism, then f is a split embedding.

Proof. Suppose f is a split embedding and fix $p \in M$. By Theorem 24.5.2, there exist neighborhoods $U \subseteq M$ of p and $V \subseteq N$ of $f(p)$, with charts $\varphi\colon U \longrightarrow \varphi(U) \subseteq E$ and $\psi\colon V \longrightarrow \psi(V) \subseteq E \oplus Z$, in which

$$(\psi \circ f \circ \varphi^{-1})(x) = (x, 0).$$

Since $f\colon M \longrightarrow f(M)$ is a homeomorphism, $f(U)$ is open in $f(M)$. Thus there exists an open set $O \subseteq N$ such that $f(U) = O \cap f(M)$. The set $\psi(V \cap O)$ is an open subset of the space $E \oplus Z$ containing the origin. Choose open neighborhoods $W_E \subseteq E$ and $W_Z \subseteq Z$ of the origin so that

$$W_E \subseteq \varphi(U), \qquad W_E \times W_Z \subseteq \psi(V \cap O).$$

Define

$$U_0 := \varphi^{-1}(W_E), \qquad V_0 := \psi^{-1}(W_E \times W_Z).$$

The local form of f shows that $f(U_0) \subseteq V_0 \cap f(M)$. Conversely, if $q \in V_0 \cap f(M)$, then $q \in O \cap f(M) = f(U)$. Writing $q = f(x)$ with $x \in U$ gives $\psi(q) = (\varphi(x), 0)$. Since $\psi(q) \in W_E \times W_Z$, it follows that $\varphi(x) \in W_E$ and therefore $x \in U_0$. We have proved the equality $f(U_0) = V_0 \cap f(M)$ and consequently

$$\psi(V_0 \cap f(M)) = W_E \times \{0\} = \psi(V_0) \cap (E \times \{0\}).$$

Thus $\psi \restriction_{V_0}$ is an adapted chart for $f(M)$. The same local expression shows that the corestriction of f and its inverse are of class C^r.

The second assertion follows from Proposition 24.3.2: the inclusion is of class C^r, is a homeomorphism onto its image, and its differential is the inclusion of a closed complemented subspace. The third assertion follows by composing the diffeomorphism $M \longrightarrow f(M)$ with the split inclusion $f(M) \hookrightarrow N$. □

Remark 24.5.6. (Phenomena specific to infinite dimensions) The conditions in Definition 24.5.1 cannot be reduced to injectivity or surjectivity of the differential.

(a) The linear inclusion $I\colon \ell^1 \longrightarrow \ell^2$ is continuous and injective, but its image is not closed. If the first N terms of $x^{(N)}$ are equal to $\frac{1}{N}$ and the remaining terms are zero, then

$$\|x^{(N)}\|_{\ell^1} = 1, \qquad \|Ix^{(N)}\|_{\ell^2} = \frac{1}{\sqrt{N}} \longrightarrow 0.$$

If the image were closed, Theorem B.2.17 applied to $I\colon \ell^1 \longrightarrow \operatorname{Ran}(I)$ would give a constant $C > 0$ such that $\|x\|_{\ell^1} \le C\|Ix\|_{\ell^2}$, contradicting this sequence. Thus the image is not closed. In particular, I admits no continuous left inverse and, regarded as a map between Banach manifolds, is a weak immersion that is not a split immersion; indeed, by linearity, $dI_x = I$ at every $x \in \ell^1$.

(b) The subspace $c_0 := \{x = (x_n) \in \ell^\infty \mid x_n \longrightarrow 0\}$ is closed in ℓ^∞, since a uniform limit of sequences tending to zero also tends to zero. However, c_0 is not complemented; this last assertion is the classical theorem of Phillips. Consequently, the inclusion $c_0 \hookrightarrow \ell^\infty$ is a linear map that is a homeomorphism onto its image and has injective differential with closed image, but is not a split immersion. Its differential at every point is the inclusion itself.

Noncomplementability of c_0 in ℓ^∞ is Phillips's Theorem 7.5 [117, p. 539]. Its proof uses the representation of the dual of ℓ^∞ by finitely additive measures and a weak convergence argument that is not needed in the rest of the book. Here it is used only to exhibit a closed subspace whose inclusion is not split; no later result depends on that dual representation.

(c) The canonical projection

$$\pi\colon \ell^\infty \longrightarrow \ell^\infty/c_0$$

is linear, continuous, and surjective. Since c_0 is closed, the quotient ℓ^∞/c_0 is a Banach space by Proposition B.2.20. At each point, the differential of the projection is π and its kernel is c_0. If it admitted a continuous linear right inverse, Proposition 24.2.4 would imply that $\ker(\pi) = c_0$ is complemented in ℓ^∞, which is false. Thus π is a weak submersion that is not a split submersion.

24.6 Constant rank and preimage theorems

For an operator between infinite-dimensional spaces, rank is not adequately described by an integer. The local form corresponding to the rank theorem requires simultaneous control of the kernel, the image, and their complements. The following assumption expresses that, near the basepoint, the images of the differentials retain a common complement in the codomain.

Theorem 24.6.1. (Constant rank with complemented decompositions) Let E and F be Banach spaces, let $\Omega \subseteq E$ be open, and let $f \in C^r(\Omega, F)$, with $r \geq 1$. Fix $a \in \Omega$ and set

$$A := Df(a), \qquad K := \ker(A), \qquad R := \operatorname{Ran}(A).$$

Suppose there exist closed subspaces $X \subseteq E$ and $C \subseteq F$ such that

$$E = K \oplus X, \qquad F = R \oplus C,$$

and these direct sums are topological. Suppose, moreover, that there exists an open neighborhood $\Omega_1 \subseteq \Omega$ of a for which

$$F = \operatorname{Ran}(Df(x)) \oplus C \tag{24.2}$$

is a topological direct sum for every $x \in \Omega_1$. Then there exist open sets $W_R \subseteq R$ and $W_K \subseteq K$ containing the origin, an open set $U_0 \subseteq \Omega_1$ containing a, an open set $V_0 \subseteq F$ containing $f(a)$, and diffeomorphisms of class C^r

$$\alpha \colon U_0 \longrightarrow W_R \times W_K, \qquad \beta \colon V_0 \longrightarrow \beta(V_0) \subseteq R \times C,$$

such that $\alpha(a) = (0,0)$, $\beta(f(a)) = (0,0)$, and

$$(\beta \circ f \circ \alpha^{-1})(y,k) = (y,0)$$

for every $(y,k) \in W_R \times W_K$. The open set V_0 can be chosen inside any previously specified open neighborhood of $f(a)$.

Proof. Denote by $P_K \colon E \longrightarrow K$, $P_X \colon E \longrightarrow X$, $P_R \colon F \longrightarrow R$, and $P_C \colon F \longrightarrow C$ the continuous projections associated with the decompositions in the statement. Since $K = \ker(A)$ and $R = \operatorname{Ran}(A)$, the restriction

$$A_X := A \restriction_X \colon X \longrightarrow R$$

is linear and bijective. Theorem B.2.17 implies that $A_X^{-1} \colon R \longrightarrow X$ is continuous.
Define

$$G \colon \Omega_1 \longrightarrow R \times K, \qquad G(x) := \big(P_R(f(x) - f(a)), P_K(x-a)\big).$$

If $v = k + \xi$, with $k \in K$ and $\xi \in X$, then

$$DG(a)v = (A_X\xi, k).$$

This operator is an isomorphism from E onto $R \times K$; its inverse is $(y,k) \mapsto k + A_X^{-1}y$. By Theorem 23.3.3, after shrinking Ω_1 there exist an open set $U_1 \subseteq \Omega_1$ containing a and an open set $W \subseteq R \times K$ containing $(0,0)$ such that $G\restriction_{U_1} : U_1 \longrightarrow W$ is a diffeomorphism of class C^r. We may choose open balls $W_R \subseteq R$ and $W_K \subseteq K$, centered at the origin, such that $W_R \times W_K \subseteq W$, and replace U_1 by the preimage of this product.
In the coordinates $(y,k) = G(x)$, set

$$\widetilde{f}(y,k) := f\left(G^{-1}(y,k)\right) - f(a).$$

The definition of G implies $P_R\widetilde{f}(y,k) = y$. Thus there exists a map $h \in C^r(W_R \times W_K, C)$ such that

$$\widetilde{f}(y,k) = (y, h(y,k)).$$

Fix $(y,k) \in W_R \times W_K$ and $\eta \in K$. We have

$$D\widetilde{f}(y,k)(0,\eta) = (0, D_2h(y,k)\eta).$$

This vector belongs to C. On the other hand, the chain rule and invertibility of $DG^{-1}(y,k)$ show that

$$\operatorname{Ran}(D\widetilde{f}(y,k)) = \operatorname{Ran}\left(Df\left(G^{-1}(y,k)\right)\right).$$

The direct sum in (24.2) implies that the intersection of this range with C is trivial. It follows that $D_2h(y,k)\eta = 0$ for every $\eta \in K$, and consequently

$$D_2h(y,k) = 0.$$

The ball W_K can be chosen convex. For $y \in W_R$ and $k \in W_K$, the integral mean value formula in Theorem 23.1.11 gives

$$h(y,k) - h(y,0) = \int_0^1 D_2h(y,tk)k\,dt = 0.$$

Thus $h(y,k) = h(y,0)$. Define the triangular change of coordinates

$$\Xi\colon W_R \times C \longrightarrow W_R \times C, \qquad \Xi(y,z) := (y, z - h(y,0)).$$

Its inverse is given by $\Xi^{-1}(y,z) = (y, z + h(y,0))$, so Ξ is a diffeomorphism of class C^r. Moreover,

$$\Xi(\widetilde{f}(y,k)) = (y,0).$$

Since $h(0,0) = 0$, choose an open ball $W_C \subseteq C$ centered at the origin and shrink W_R, if necessary, so that

$$h(y,0) \in W_C$$

for every $y \in W_R$. Given an open neighborhood of $f(a)$ in F in advance, the balls W_R and W_C can also be chosen so that their image under the sum isomorphism $R \times C \longrightarrow F$, translated by $f(a)$, is contained in that neighborhood.

Take

$$U_0 := G^{-1}(W_R \times W_K)$$

and define $\alpha := G \restriction_{U_0}$. Likewise, let

$$V_0 := f(a) + \{y + z \mid y \in W_R,\ z \in W_C\}.$$

This set is open in F because the sum map $R \times C \longrightarrow F$ is a topological isomorphism. The map

$$\beta\colon V_0 \longrightarrow \Xi(W_R \times W_C), \qquad \beta(v) := \Xi\big(P_R(v - f(a)), P_C(v - f(a))\big),$$

is a diffeomorphism of class C^r, satisfies $\beta(f(a)) = (0,0)$, and, for $(y,k) \in W_R \times W_K$, satisfies

$$(\beta \circ f \circ \alpha^{-1})(y,k) = (y,0).$$

These are the required diffeomorphisms. □

Theorem 24.6.2. (Constant rank on Banach manifolds) Let M and N be Banach manifolds and let $f : M \longrightarrow N$ be a map of class C^r, with $r \geq 1$. Fix $p \in M$. Suppose that, in charts centered at p and $f(p)$, a coordinate representation

$$F\colon \Omega \subseteq E \longrightarrow H, \qquad F(0) = 0,$$

has the following property. If

$$A := DF(0), \qquad K := \ker(A), \qquad R := \operatorname{Ran}(A),$$

, there exist closed subspaces $X \subseteq E$ and $C \subseteq H$ such that

$$E = K \oplus X, \qquad H = R \oplus C,$$

as topological direct sums, and C remains a topological complement of $\operatorname{Ran}(DF(x))$ for every x in a neighborhood of the origin. Then there exist charts of class C^r in which f has the form

$$(y,k) \longmapsto (y,0).$$

In particular, locally f factors as the projection $R \times K \longrightarrow R$ followed by the inclusion $R \hookrightarrow R \times C$. Its local image is an embedded submanifold, and its local fibers are embedded submanifolds modeled on K.

Proof. Apply Theorem 24.6.1 to the coordinate representation of f, choosing the open set V_0 inside the image of the initial codomain chart. Composing the resulting diffeomorphisms with the initial charts gives charts of class C^r, as in Remark 24.1.4. The assertions about the image, fibers, and factorization follow directly from the expression $(y,k) \mapsto (y,0)$. □

The common complement assumption in the preceding theorem replaces the numerical constancy of rank used in finite dimensions. A particular fixed complement need not be preserved under a nonlinear change of coordinates. The conclusion, however, is intrinsic: existence of charts in which the map has the form $(y,k) \mapsto (y,0)$ does not depend on the initial charts used to construct them.

Proposition 24.6.3. (Preimage of a submanifold under a submersion) Let M and N be Banach manifolds, let $f : M \longrightarrow N$ be a split submersion of class C^r, with $r \geq 1$, and let $S \subseteq N$ be an embedded submanifold of class C^r. Then $f^{-1}(S)$ is an embedded submanifold of class C^r of M. The restriction

$$f \restriction_{f^{-1}(S)} \colon f^{-1}(S) \longrightarrow S$$

is a split submersion and, for each $p \in f^{-1}(S)$, if $\iota \colon f^{-1}(S) \longrightarrow M$ and $\jmath \colon S \longrightarrow N$ are the inclusions, then

$$d\iota_p\big(T_p(f^{-1}(S))\big) = (df_p)^{-1}\big(d\jmath_{f(p)}(T_{f(p)}S)\big). \tag{24.3}$$

Proof. If $f^{-1}(S) = \varnothing$, all assertions hold vacuously. Thus suppose $f^{-1}(S) \neq \varnothing$. Fix $p \in f^{-1}(S)$ and write $q := f(p)$. Choose a chart (V, ψ) of N adapted to S at q. After translating the origin and identifying the model space with a sum $Y \oplus Z$, we have

$$\psi(V \cap S) = \psi(V) \cap (Y \times \{0\}).$$

Take an initial chart of M around p and represent f in these coordinates, retaining the adapted chart ψ in the codomain. The differential of the representation admits a continuous right inverse. The construction in the proof of Theorem 24.5.3, which leaves the codomain coordinate unchanged, gives a domain chart in which

$$f(k, y, z) = (y, z),$$

, where the domain model space is identified with $K \oplus Y \oplus Z$. In these coordinates,

$$f^{-1}(S) = \{(k, y, 0) \mid (k, y, 0) \text{ belongs to the chart domain}\}.$$

This set is the intersection of the domain with the complemented subspace $K \oplus Y \oplus \{0\}$. Thus the chart is adapted to $f^{-1}(S)$.
The restriction of f is represented by the projection

$$K \oplus Y \longrightarrow Y, \qquad (k, y) \longmapsto y,$$

, which admits the right inverse $y \mapsto (0, y)$. Finally, the differential of f in these coordinates is $(k, y, z) \mapsto (y, z)$, and the preimage of $Y \times \{0\}$ is $K \oplus Y \oplus \{0\}$. This proves (24.3). □

Definition 24.6.4. Let M and N be Banach manifolds and let $f : M \longrightarrow N$ be a map of class C^r, with $r \geq 1$. A point $q \in N$ is called a **split regular value** of f if, for each $p \in f^{-1}(q)$, the differential

$$df_p \colon T_pM \longrightarrow T_qN$$

admits a continuous linear right inverse. Equivalently, df_p is surjective and $\ker(df_p)$ is complemented in T_pM for every $p \in f^{-1}(q)$. If $f^{-1}(q) = \varnothing$, the condition is deemed satisfied.

Theorem 24.6.5. (Preimage theorem) Let M and N be Banach manifolds, let $f\colon M \longrightarrow N$ be a map of class C^r, with $r \geq 1$, and let $q \in N$ be a split regular value of f. Then $f^{-1}(q)$ is an embedded submanifold of class C^r of M and, if $\iota_q\colon f^{-1}(q) \longrightarrow M$ is the inclusion,

$$d(\iota_q)_p\big(T_p(f^{-1}(q))\big) = \ker(df_p)$$

for every $p \in f^{-1}(q)$. If N is finite-dimensional of dimension m, then $f^{-1}(q)$ has codimension m.

Proof. If the fiber is empty, the local condition in Definition 24.3.1 holds vacuously; we adopt the usual convention that the empty submanifold has the indicated codimension. Suppose the fiber is nonempty and let $p \in f^{-1}(q)$. By split regularity and Theorem 24.5.3, there exist charts centered at p and q in which f has the form

$$(k, y) \longmapsto y.$$

Consequently, the fiber of q corresponds in these coordinates to $K \times \{0\}$, where K represents $\ker(df_p)$. This subspace is closed and complemented by $\{0\}\times F$, where F is the model space of the codomain chart. The isomorphism $d\psi_q\colon T_qN \longrightarrow F$ transports the tangent directions of the codomain to this second factor. Thus the resulting charts are adapted charts for $f^{-1}(q)$. Its tangent space at p is K, which under the chart differentials corresponds precisely to $\ker(df_p)$.
If N has dimension m, the normal complement $\{0\} \times F$ has dimension m, since $d\psi_q$ is an isomorphism, and the fiber has codimension m. □

Corollary 24.6.6. (Regular values with finite-dimensional codomain) Let M be a Banach manifold, let N be a finite-dimensional manifold, let $f\colon M \longrightarrow N$ be a map of class C^r, with $r \geq 1$, and let $q \in N$. Set $m := \dim(N)$. If $df_p\colon T_pM \longrightarrow T_qN$ is surjective for every $p \in f^{-1}(q)$, then q is a split regular value. Consequently, $f^{-1}(q)$ is an embedded submanifold of codimension m and, if $\iota_q\colon f^{-1}(q) \longrightarrow M$ is the inclusion, for every $p \in f^{-1}(q)$,

$$d(\iota_q)_p\big(T_p(f^{-1}(q))\big) = \ker(df_p).$$

Proof. If $f^{-1}(q) = \varnothing$, the regularity assumption is vacuous and the conclusion is immediate under the usual convention that the empty set is an embedded submanifold of the indicated codimension; the assertion about tangent spaces is also vacuous. Suppose $f^{-1}(q) \neq \varnothing$. Fix $p \in f^{-1}(q)$ and choose a basis $e_1, \dots, e_m$ of T_qN. By surjectivity of df_p, there exist $v_1, \dots, v_m \in T_pM$ such that $df_p(v_j) = e_j$. The operator

$$R_p\colon T_qN \longrightarrow T_pM, \qquad R_p\left(\sum_{j=1}^{m} a_j e_j\right) := \sum_{j=1}^{m} a_j v_j$$

is linear and continuous because its domain is finite-dimensional, and satisfies $df_p \circ R_p = I_{T_qN}$. Thus q is a split regular value, and Theorem 24.6.5 gives the conclusion. □

Example 24.6.7. (Fibers of linear operators) Let E and F be Banach spaces and let $T \in \mathcal{L}(E, F)$ be surjective. If $\ker(T)$ is complemented, every $y \in F$ is a split regular value of T. For any $x_0 \in T^{-1}(y)$, we have

$$T^{-1}(y) = x_0 + \ker(T).$$

If $\iota_y\colon T^{-1}(y) \longrightarrow E$ is the inclusion, then

$$\Theta_{\mathrm{id}_E,x}\big(d(\iota_y)_x(T_x(T^{-1}(y)))\big) = \ker(T) \quad \text{for every } x \in T^{-1}(y).$$

Indeed, for each $x \in T^{-1}(y)$, the translation

$$\theta_x\colon \ker(T) \longrightarrow T^{-1}(y), \qquad \theta_x(k) := x + k,$$

is an affine diffeomorphism, and its differential is the inclusion $\ker(T) \hookrightarrow E$. Thus the tangent space of the fiber, regarded as a Banach manifold modeled on $\ker(T)$, is identified with $\ker(T)$. If the kernel is complemented, this inclusion admits a continuous left inverse, and the fiber is an embedded submanifold by Proposition 24.4.1. If the kernel is not complemented, the fiber is still a closed affine subspace and a Banach manifold in its own right. If the fiber were an embedded submanifold of E, the differential of its inclusion would have image exactly $\ker(T)$. Indeed, every differentiable curve γ in the fiber satisfies $T \circ \gamma = y$ and therefore $T\gamma'(0) = 0$; conversely, if $k \in \ker(T)$, the curve $t \mapsto x + tk$ remains in the fiber and has velocity k at $t = 0$. Under the assumed embedded submanifold structure, this curve is also of class C^1 as a curve with values in the fiber, by Lemma 24.3.3. Proposition 24.3.2 would then force this tangent image to be complemented in E, contradicting the assumption. Consequently, the fiber is not an embedded submanifold in the sense of Definition 24.3.1.

24.7 Regular level sets

Level sets of real functionals are the most frequent specialization of Theorem 24.6.5. In this case, a nonvanishing differential automatically provides a right inverse and a one-dimensional complement to the kernel. We will develop this situation directly to obtain explicit adapted charts and fix the formulation to be used for variational constraints.

Lemma 24.7.1. Let X be a real Banach space and let $f, g \in X'$ be nonzero functionals. Then $\ker(f)$ and $\ker(g)$ are isomorphic as Banach spaces.

Proof. First prove that there exists $a \in X$ such that $g(a) = 1$ and $f(a) \neq 0$. Since $g \neq 0$, there exists $b \in X$ such that $g(b) \neq 0$. Replacing b by $\frac{b}{g(b)}$, we may assume $g(b) = 1$. If $f(b) \neq 0$, take $a = b$. Thus suppose $f(b) = 0$. Since $f \neq 0$, there exists $x \in X$ such that $f(x) \neq 0$. Define $y := x - g(x)b$. Then $g(y) = g(x) - g(x)g(b) = 0$ and $f(y) = f(x) - g(x)f(b) = f(x) \neq 0$. Thus, taking $a := b + y$, we have $g(a) = g(b) + g(y) = 1$ and $f(a) = f(b) + f(y) = f(y) \neq 0$.

Now define $T\colon \ker(f) \longrightarrow \ker(g)$ by $T(h) := h - g(h)a$. First check that T is well defined. If $h \in \ker(f)$, then $g(T(h)) = g(h) - g(h)g(a) = g(h) - g(h) = 0$. Thus $T(h) \in \ker(g)$.

The operator T is linear and continuous, since it is the restriction to $\ker(f)$ of the continuous linear operator $x \mapsto x - g(x)a$ defined on X. We now prove that T is bijective. To this end, define $S\colon \ker(g) \longrightarrow \ker(f)$ by $S(k) := k - \dfrac{f(k)}{f(a)}a$. This formula is well defined because $f(a) \neq 0$. Moreover, if $k \in \ker(g)$, then $f(S(k)) = f(k) - \dfrac{f(k)}{f(a)}f(a) = 0$, so $S(k) \in \ker(f)$. Thus S is well defined. It is also linear and continuous.

Let us check that S and T are inverses. Let $h \in \ker(f)$. Since $T(h) = h - g(h)a$, we have $f(T(h)) = f(h) - g(h)f(a) = -g(h)f(a)$, because $f(h) = 0$. Then

$$S(T(h)) = T(h) - \frac{f(T(h))}{f(a)}a = h - g(h)a - \frac{-g(h)f(a)}{f(a)}a = h.$$

On the other hand, let $k \in \ker(g)$. Since $S(k) = k - \dfrac{f(k)}{f(a)}a$, we have $g(S(k)) = g(k) - \dfrac{f(k)}{f(a)}g(a) = -\dfrac{f(k)}{f(a)}$, because $g(k) = 0$ and $g(a) = 1$. Thus

$$T(S(k)) = S(k) - g(S(k))a = k - \frac{f(k)}{f(a)}a + \frac{f(k)}{f(a)}a = k.$$

Thus $S = T^{-1}$. Consequently, $T\colon \ker(f) \longrightarrow \ker(g)$ is a continuous linear bijection with continuous linear inverse.
Finally, since f and g are continuous, $\ker(f)$ and $\ker(g)$ are closed subspaces of X. Thus both are Banach spaces with the norm induced from X. We conclude that $\ker(f)$ and $\ker(g)$ are isomorphic as Banach spaces. □

Theorem 24.7.2. (Regular level sets in open subsets of Banach spaces) Let X be a real Banach space, let $\Omega \subseteq X$ be nonempty and open, and let $J\colon \Omega \longrightarrow \mathbb{R}$ be a function of class C^k, with $k \geq 1$. Let $c \in \mathbb{R}$ and suppose the level set $V := \{u \in \Omega \mid J(u) = c\}$ is regular, that is, $DJ(u) \neq 0$ for every $u \in V$. For $u \in \Omega$, the identity chart determines the operator

$$\mathcal{I}_u := \Theta_{\mathrm{id}_\Omega, u}\colon T_u\Omega \longrightarrow X.$$

Proposition 24.1.10 proves that $\mathcal{I}_u$ is linear, bijective, and bicontinuous. Then V is an embedded Banach submanifold of class C^k and codimension 1 of Ω. If $\iota\colon V \longrightarrow \Omega$ is the inclusion, then

$$\mathcal{I}_u\big(d\iota_u(T_uV)\big) = \ker(DJ(u))$$

for every $u \in V$.

Proof. If $V = \varnothing$, the assertion is vacuous. Suppose $V \neq \varnothing$ and fix $u \in V$. Since $DJ(u) \neq 0$, there exists $e \in X$ such that $DJ(u)(e) \neq 0$. Let $Y_u := \ker(DJ(u))$. Then Y_u is closed and

$$X = Y_u \oplus \mathbb{R}e.$$

For each $x \in X$, we have

$$x = \left(x - \frac{DJ(u)(x)}{DJ(u)(e)}e\right) + \frac{DJ(u)(x)}{DJ(u)(e)}e,$$

, where the first summand belongs to Y_u and the intersection $Y_u \cap \mathbb{R}e$ is trivial.
Consider

$$\Phi(y,t) := J(u + y + te) - c$$

on the open subset of $Y_u \times \mathbb{R}$ consisting of pairs (y,t) such that $u + y + te \in \Omega$. Moreover, $\Phi(0,0) = 0$ and $D_2\Phi(0,0)(1) = DJ(u)(e) \neq 0$. By Theorem 23.3.8, there exist open sets $U_0 \subseteq Y_u$ and $I_0 \subseteq \mathbb{R}$ containing 0 and a function $h\colon U_0 \longrightarrow I_0$ of class C^k, with $h(0) = 0$, such that

$$\Phi(y,t) = 0 \quad \Longleftrightarrow \quad t = h(y)$$

for every $(y,t) \in U_0 \times I_0$.
Let $W := \{u + y + te \mid y \in U_0,\ t \in I_0\}$. The linear operator $L_u \colon Y_u \times \mathbb{R} \longrightarrow X$, $L_u(y,t) := y + te$, is a topological isomorphism by Proposition 24.2.2; hence W is open in X. Define

$$\varphi \colon W \longrightarrow Y_u \times \mathbb{R}, \qquad \varphi(u + y + te) := (y, t - h(y)).$$

Its image is the open set

$$\varphi(W) = \{(y,s) \in U_0 \times \mathbb{R} \mid s + h(y) \in I_0\},$$

, and its inverse is given by $\varphi^{-1}(y,s) = u + y + (s + h(y))e$. Consequently, φ is a diffeomorphism of class C^k, $\varphi(u) = (0,0)$, and

$$\varphi(W \cap V) = \varphi(W) \cap (Y_u \times \{0\}).$$

Thus V is an embedded submanifold of class C^k and codimension 1.
To identify the tangent space in the natural coordinates of the open set $\Omega \subseteq X$, differentiate the identity $\Phi(y, h(y)) = 0$ at $y = 0$. For each $\eta \in Y_u$, this gives

$$0 = D\Phi(0,0)\big(\eta, Dh(0)(\eta)\big) = DJ(u)\big(\eta + Dh(0)(\eta)e\big) = Dh(0)(\eta)DJ(u)(e).$$

Since $DJ(u)(e) \neq 0$, it follows that $Dh(0) = 0$. The differential of the inverse adapted chart therefore sends $(\eta, 0)$ to η. Proposition 24.3.2 thus gives

$$\mathcal{I}_u\big(d\iota_u(T_uV)\big) = Y_u = \ker(DJ(u)).$$

Finally, if $u_0 \in V$ is fixed, Lemma 24.7.1 shows that $\ker(DJ(u))$ is isomorphic to $\ker(DJ(u_0))$ for every $u \in V$. Thus, even if V is disconnected, all its local model spaces are isomorphic. □

Lemma 24.7.3. (Differential of a restriction) Let X be a Banach space, let $\Omega \subseteq X$ be open, let $V \subseteq \Omega$ be a Banach submanifold, and let $\varphi \in C^1(\Omega, \mathbb{R})$. Then $\varphi \restriction_V \in C^1(V, \mathbb{R})$ and, for each $u \in V$,

$$d(\varphi \restriction_V)_u(v) = D\varphi(u)\big(\mathcal{I}_u(d\iota_u(v))\big)$$

for every $v \in T_uV$. In particular,

$$d(\varphi \restriction_V)_u = D\varphi(u) \circ \mathcal{I}_u \circ d\iota_u.$$

Proof. Let $\iota \colon V \longrightarrow \Omega$ be the canonical inclusion. Then $\varphi \restriction_V = \varphi \circ \iota$. Proposition 24.3.2 shows that ι is of class C^1, and the chain rule gives

$$d(\varphi \restriction_V)_u = d\varphi_u \circ d\iota_u.$$

Under the bicontinuous isomorphism $\mathcal{I}_u \colon T_u\Omega \longrightarrow X$ defined in Theorem 24.7.2, the differential $d\iota_u$ is represented by the inclusion of $\mathcal{I}_u(d\iota_u(T_uV))$ into X. Moreover, in the identity chart of Ω and the usual coordinate on $\mathbb{R}$, the differential $d\varphi_u$ is represented by the Fréchet derivative $D\varphi(u)$; that is, $d\varphi_u = D\varphi(u) \circ \mathcal{I}_u$, identifying $T_{\varphi(u)}\mathbb{R}$ with $\mathbb{R}$. Thus

$$d(\varphi \restriction_V)_u(v) = D\varphi(u)\big(\mathcal{I}_u(d\iota_u(v))\big)$$

for every $v \in T_u V$, as required. □

24.8 Bump functions and smooth partitions of unity

Constructing global objects from local data requires additional care on Banach manifolds. In finite dimensions, second countability and local compactness give paracompactness, and Euclidean bump functions yield smooth partitions of unity. In an infinite-dimensional Banach space, there are no relatively compact neighborhoods, and existence of bump functions with the prescribed regularity depends on the geometry of the model space. It is therefore useful to separate the topological and differential parts of the argument.

Let E be a topological space. Recall that the support of a continuous function $b\colon E \longrightarrow \mathbb{R}$ is the closed set

$$\operatorname{supp}(b) := \overline{\{x \in E \mid b(x) \neq 0\}}.$$

The closure is taken in E.

Definition 24.8.1. Let E be a real Banach space and let $k \in \mathbb{N} \cup \{\infty\}$. A **bump function of class** C^k on E, also called a **bump of class** C^k, is a function $b \in C^k(E, \mathbb{R})$ that is not identically zero and has bounded support.

In finite dimensions, a closed bounded support is compact, so the preceding definition agrees with the one used in earlier chapters. In infinite dimensions, compact support cannot be required, as the following result shows.

Proposition 24.8.2. Let E be an infinite-dimensional normed space. Every continuous function $b\colon E \longrightarrow \mathbb{R}$ with compact support is identically zero.

Proof. We first use the following elementary consequence of Riesz's lemma. If $Y \subsetneq E$ is a closed subspace and $0 < a < 1$, there exists $x \in E$ such that $\|x\|_E = 1$ and

$$\operatorname{dist}(x, Y) > a.$$

Indeed, take $z \notin Y$ and write $d := \operatorname{dist}(z, Y) > 0$. By the definition of the infimum, there exists $y_0 \in Y$ such that $\|z - y_0\|_E < \frac{d}{a}$. If $x := \frac{z - y_0}{\|z - y_0\|_E}$, then, for every $y \in Y$,

$$\|x - y\|_E = \frac{\|z - (y_0 + \|z - y_0\|_E y)\|_E}{\|z - y_0\|_E} \geq \frac{d}{\|z - y_0\|_E} > a.$$

Applying this observation inductively with $a = \frac{1}{2}$ and $Y_m := \operatorname{span}\{x_1, \dots, x_m\}$ gives a sequence $(x_m)_{m \in \mathbb{N}}$ of unit vectors such that $\|x_m - x_j\|_E > \frac{1}{2}$ whenever $m \neq j$. Here each Y_m is closed because it is finite-dimensional and proper because E is infinite-dimensional. Thus the closed unit ball of E is not compact.

Now suppose b is not identically zero and choose $x_0 \in E$ with $b(x_0) \neq 0$. Continuity gives $r > 0$ such that

$$|b(x)| > \frac{|b(x_0)|}{2} \qquad \text{if } \|x - x_0\|_E < r.$$

Consequently, $\overline{B}_E(x_0, r) \subseteq \operatorname{supp}(b)$. If this support were compact, so would be the closed ball $\overline{B}_E(x_0, r)$. Translation by $-x_0$ and dilation by the factor $\frac{1}{r}$ would then imply compactness of the closed unit ball,

contradicting the preceding paragraph. □

The usual normalization of a bump function remains valid with the notion of bounded support.

Lemma 24.8.3. If E admits a bump function of class C^k, it admits a function $\beta \in C^k(E,\mathbb{R})$ such that

$$0 \le \beta \le 1, \qquad \beta = 1 \text{ on a neighborhood of } 0, \qquad \operatorname{supp}(\beta) \text{ is bounded.}$$

Proof. Let b be a bump function and choose $z \in E$ such that $c := b(z) \neq 0$. There exist disjoint open intervals containing 0 and c, respectively. By Proposition 1.3.6, applied in $\mathbb{R}$, we can choose $\vartheta \in C^\infty(\mathbb{R})$ such that $0 \le \vartheta \le 1$, $\vartheta = 0$ on a neighborhood of 0, and $\vartheta = 1$ on a neighborhood of c. Define

$$\beta(x) := \vartheta(b(x+z)).$$

The function β is of class C^k by the chain rule, equals one on a neighborhood of 0, and takes values in $[0,1]$. Moreover,

$$\{x \in E \mid \beta(x) \neq 0\} \subseteq \operatorname{supp}(b) - z,$$

because $\vartheta(0) = 0$. Since the set on the right is closed and bounded, $\operatorname{supp}(\beta)$ is also bounded. □

Differentiable norms are a direct source of bump functions.

Proposition 24.8.4. Let ν be a norm on a Banach space E and suppose there exist constants $c_\nu, C_\nu > 0$ such that

$$c_\nu \|x\|_E \le \nu(x) \le C_\nu \|x\|_E, \qquad x \in E.$$

If $\nu : E \setminus \{0\} \longrightarrow \mathbb{R}$ is of class C^k, then E admits a bump function of class C^k.

Proof. Choose $\chi \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ for $t \le 1$, and $\chi(t) = 0$ for $t \ge 2$. The function

$$b(x) := \chi(\nu(x))$$

is of class C^k away from the origin and constant on the neighborhood $\{x \in E \mid \nu(x) < 1\}$ of 0; hence it is of class C^k on all of E. Moreover, $b(0) = 1$ and $\operatorname{supp}(b) \subseteq \{x \in E \mid \nu(x) \le 2\}$; the inequality $c_\nu \|x\|_E \le \nu(x)$ shows that this support is contained in $\left\{x \in E \;\middle|\; \|x\|_E \le \frac{2}{c_\nu}\right\}$. □

Existence of a bump function is necessary for constructing smooth partitions of unity, but the converse implication will not be used without additional assumptions. The precise condition needed in the geometric argument is formulated next.

Definition 24.8.5. Let E be a Banach space and let $k \in \mathbb{N} \cup \{\infty\}$. We say that E **admits partitions of unity of class** C^k if every open cover $\mathscr{U} = (U_\alpha)_{\alpha \in A}$ of E admits a partition of unity $(\psi_\alpha)_{\alpha \in A}$ subordinate to $\mathscr{U}$, in the sense of Definition 1.3.4, such that each ψ_α is of class C^k.

This property includes, in particular, the ability to separate a closed set from an open set by a smooth function.

Lemma 24.8.6. (Cutoff functions in the model space) Suppose E admits partitions of unity of class C^k. If $A \subseteq E$ is closed, $B \subseteq E$ is open, and $A \subseteq B$, there exists $\theta \in C^k(E, \mathbb{R})$ such that

$$0 \le \theta \le 1, \qquad \theta = 1 \text{ on } A, \qquad \operatorname{supp}(\theta) \subseteq B.$$

Proof. Consider the open cover $(B, E \setminus A)$ of E and a subordinate partition of unity (θ, ζ) of class C^k. Subordination gives $\operatorname{supp}(\theta) \subseteq B$ and $\operatorname{supp}(\zeta) \subseteq E \setminus A$. In particular, $\zeta = 0$ on A; since $\theta + \zeta = 1$, it follows that $\theta = 1$ on A. □

Taking $A = \overline{B}_E(0,1)$ and $B = B_E(0,2)$ in the preceding lemma gives a bump function of class C^k. Thus admitting C^k partitions is, a priori, a stronger condition than having a single bump function.

24.8.1 The global construction on a Banach manifold

The topological part of the construction is collected in a shrinking lemma. The paracompactness in its statement is paracompactness of the underlying topological space; it is not part of Definition 24.1.1 and must therefore be imposed explicitly.

Lemma 24.8.7. (Locally finite shrinking) Let X be a paracompact Hausdorff space. Then:

(a) if $F \subseteq U \subseteq X$, where F is closed and U is open, there exists an open set V such that

$$F \subseteq V \subseteq \overline{V} \subseteq U;$$

(b) if $(W_i)_{i \in I}$ is a locally finite open cover of X, there exists an open cover $(V_i)_{i \in I}$ such that

$$\overline{V_i} \subseteq W_i \qquad \text{for every } i \in I.$$

Proof. First prove that X is normal. Fix a point $x \in X$ and a closed set F not containing it. For each $y \in F$, the Hausdorff property gives disjoint open sets A_y and B_y with $x \in A_y$ and $y \in B_y$. The cover

$$\{X \setminus F\} \cup \{B_y \mid y \in F\}$$

admits a locally finite open refinement $(R_j)_{j \in J}$. Let $J_F := \{j \in J \mid R_j \cap F \neq \varnothing\}$. If $j \in J_F$, the set R_j cannot be contained in $X \setminus F$; since this is a refinement, it is contained in some B_y. Consequently, $x \notin \overline{R_j}$. The family $(\overline{R_j})_{j \in J}$ is also locally finite, and hence

$$\overline{\bigcup_{j \in J_F} R_j} \subseteq \bigcup_{j \in J_F} \overline{R_j}.$$

The open set $W := \bigcup_{j \in J_F} R_j$ contains F, while $x \notin \overline{W}$. Thus $X \setminus \overline{W}$ is a neighborhood of x whose closure is disjoint from F. This proves that X is regular.

Now let F_0 and F_1 be disjoint closed sets. For each $x \in F_0$, regularity gives an open set U_x such that $x \in U_x$ and $\overline{U_x} \cap F_1 = \varnothing$. Take a locally finite refinement of the cover

$$\{X \setminus F_0\} \cup \{U_x \mid x \in F_0\}$$

and let G be the union of those elements of the refinement that meet F_0. These elements are contained in some U_x. Local finiteness implies that the closure of their union is contained in the union of their closures; consequently,

$$F_0 \subseteq G, \qquad \overline{G} \cap F_1 = \varnothing.$$

The open sets G and $X \setminus \overline{G}$ separate F_0 and F_1. Thus X is normal. Applying this separation to the closed sets F and $X \setminus U$ gives (a).

To prove (b), well-order I and construct the open sets V_i by transfinite recursion. Suppose the V_j with $j < i$ have been chosen so that the family

$$\{V_j \mid j < i\} \cup \{W_j \mid j \geq i\}$$

covers X. Define

$$K_i := X \setminus \left(\bigcup_{j<i} V_j \cup \bigcup_{j>i} W_j \right).$$

The set K_i is closed, and the inductive property shows that $K_i \subseteq W_i$. By (a), there exists an open set V_i such that $K_i \subseteq V_i$ and $\overline{V_i} \subseteq W_i$. This choice preserves the covering property at the next successor stage.

It also preserves this property at every limit ordinal. Indeed, if a point x belonged neither to any of the already chosen V_j nor to any of the remaining W_j, local finiteness would allow us to take the largest index, among those strictly preceding the limit ordinal, of the sets W_j containing x. The covering property immediately after that index would force x to belong to one of the already chosen V_j, a contradiction. At the end of the recursion, $(V_i)_{i\in I}$ covers X and satisfies $\overline{V_i} \subseteq W_i$. Since $V_i \subseteq W_i$, the family $(V_i)_{i\in I}$ is locally finite. □

In metric spaces, paracompactness follows directly from the distance, without assumptions of separability or local compactness.

Theorem 24.8.8. (Paracompactness of metric spaces) Every open cover of a metric space admits a locally finite open refinement. In particular, every metrizable space and every subspace of it are paracompact.

Proof. Let (X,d) be a metric space and let $(U_\alpha)_{\alpha\in A}$ be an open cover. If $X = \varnothing$, the assertion is immediate. Suppose $X \neq \varnothing$ and choose a well-order on A. For each α, define

$$f_\alpha(x) := \min\{1, d(x, X \setminus U_\alpha)\}, \qquad x \in X,$$

and take $f_\alpha \equiv 1$ when $U_\alpha = X$. Distance to a nonempty set is 1–Lipschitz: from $d(x,z) \leq d(x,y) + d(y,z)$, taking infima over z in the set gives one inequality, and the other follows by interchanging x and y. Since truncation at 1 preserves this property, each f_α is 1–Lipschitz. Moreover, $f_\alpha(x) > 0$ if and only if $x \in U_\alpha$, since U_α is open.

For $n \in \mathbb{N}$, set $r_n := 2^{-n}$ and consider the closed sets

$$F_{\alpha,n} := \{x \in X \mid f_\alpha(x) \geq r_n,\ f_\beta(x) \leq r_n/2 \text{ for every } \beta < \alpha\}.$$

They are closed because they are defined as intersections of closed sets. If $\alpha < \beta$, $x \in F_{\alpha,n}$, and $y \in F_{\beta,n}$, then

$$d(x,y) \geq |f_\alpha(x) - f_\alpha(y)| \geq r_n/2.$$

Moreover, these sets cover X as α and n vary. Indeed, given $x \in X$, let α be the first index for which $x \in U_\alpha$. Then $f_\alpha(x) > 0$ and $f_\beta(x) = 0$ for every $\beta < \alpha$. Choosing n with $r_n \le f_\alpha(x)$ suffices to obtain $x \in F_{\alpha,n}$.
If $F_{\alpha,n} \neq \varnothing$, define the open set

$$V_{\alpha,n} := \{x \in X \mid d(x, F_{\alpha,n}) < r_n/8\};$$

; if $F_{\alpha,n} = \varnothing$, take $V_{\alpha,n} = \varnothing$. For $x \in V_{\alpha,n}$, there exists $u \in F_{\alpha,n}$ with $d(x,u) < r_n/8$, and hence $f_\alpha(x) \ge f_\alpha(u) - d(x,u) > 7r_n/8 > 0$. Thus $V_{\alpha,n} \subseteq U_\alpha$. If $\alpha \neq \beta$, $x \in V_{\alpha,n}$, and $y \in V_{\beta,n}$, choose points in the respective closed sets at distance less than $r_n/8$. Separation of the closed sets and the triangle inequality give $d(x,y) > r_n/4$. Consequently, each ball $B_d(z, r_n/8)$ meets at most one of the sets $V_{\alpha,n}$. For each fixed n, the family $(V_{\alpha,n})_{\alpha\in A}$ is therefore locally finite. Together, these families cover X as n varies.
To obtain local finiteness also with respect to n, set

$$O_n := \bigcup_{1\le k\le n} \bigcup_{\alpha\in A} V_{\alpha,k}, \qquad h_n(x) := \min\{1, d(x, X \setminus O_n)\},$$

, with $h_n \equiv 1$ if $O_n = X$, and define

$$P_0 := \varnothing, \qquad P_n := \{x \in X \mid h_n(x) \ge 2^{-n}\} \quad (n \in \mathbb{N}).$$

The sets O_n are open and increasing. The functions h_n are continuous and satisfy $h_n \le h_{n+1}$; hence the P_n are closed, $P_n \subseteq O_n$, and $P_n \subseteq P_{n+1}$. Their interiors cover X. Indeed, if $x \in O_m$, then $h_m(x) > 0$ and, for sufficiently large n, $n \ge m$ and $h_n(x) \ge h_m(x) > 2^{-n}$. By continuity, this strict inequality persists on a neighborhood of x contained in P_n.
Finally, define

$$W_{\alpha,n} := V_{\alpha,n} \setminus P_{n-1}.$$

Each $W_{\alpha,n}$ is open and contained in U_α. To check that they cover X, given $x \in X$, take the least n such that $x \in V_{\alpha,n}$ for some α. If $n = 1$, we have $x \notin P_0$. If $n > 1$, minimality gives $x \notin O_{n-1}$ and, since $P_{n-1} \subseteq O_{n-1}$, also $x \notin P_{n-1}$. In either case, $x \in W_{\alpha,n}$.
Now fix $x \in X$ and choose N with $x \in \operatorname{Int}(P_N)$. This neighborhood meets no $W_{\alpha,n}$ with $n > N$, because $P_N \subseteq P_{n-1}$. For each of the finitely many levels $1 \le n \le N$, local finiteness of $(V_{\alpha,n})_{\alpha\in A}$ gives a neighborhood of x meeting only finitely many of its elements. The intersection of these neighborhoods with $\operatorname{Int}(P_N)$ meets only finitely many of the $W_{\alpha,n}$. Thus the nonempty members of this family form the required locally finite open refinement. The assertion for metrizable spaces follows by choosing a compatible metric, and the assertion for their subspaces follows by applying what we have proved to the restricted metric. □

Theorem 24.8.9. (Smooth partitions on Banach manifolds) Let M be a Banach manifold of class C^k modeled on a Banach space E, where $k \in \mathbb{N} \cup \{\infty\}$. Suppose the topological space M is paracompact and E admits partitions of unity of class C^k. Then every open cover $\mathscr{U} = (U_\alpha)_{\alpha\in A}$ of M admits a partition of unity of class C^k subordinate to $\mathscr{U}$.

Proof. Lemma 24.8.7 shows, in particular, that M is regular. For each $x \in M$, choose $\alpha(x) \in A$ such that $x \in U_{\alpha(x)}$ and a chart (C_x, φ_x) around x. Applying regularity to the open set $C_x \cap U_{\alpha(x)}$ gives an open set Q_x such that

$$x \in Q_x, \qquad \overline{Q_x} \subseteq C_x \cap U_{\alpha(x)}.$$

Since $\varphi_x(Q_x)$ is an open neighborhood of $\varphi_x(x)$, there exists $r_x > 0$ such that

$$\overline{B}_E(\varphi_x(x), r_x) \subseteq \varphi_x(Q_x).$$

Set

$$P_x := \varphi_x^{-1}\big(B_E(\varphi_x(x), r_x)\big).$$

Since $P_x \subseteq Q_x$ and $\overline{Q_x} \subseteq C_x$, the closure of P_x in M remains inside the chart domain. Thus

$$\overline{P_x} = \varphi_x^{-1}\big(\overline{B}_E(\varphi_x(x), r_x)\big) \subseteq Q_x \subseteq U_{\alpha(x)}. \tag{24.4}$$

Paracompactness gives a locally finite open refinement $(W_i)_{i\in I}$ of the cover $(P_x)_{x\in M}$. For each i, fix a point $x(i)$ such that $W_i \subseteq P_{x(i)}$ and abbreviate

$$C_i := C_{x(i)}, \qquad \varphi_i := \varphi_{x(i)}, \qquad \alpha(i) := \alpha(x(i)).$$

By Lemma 24.8.7, there exists an open cover $(V_i)_{i\in I}$ with

$$\overline{V_i} \subseteq W_i \qquad \text{for every } i \in I. \tag{24.5}$$

Within the model space, consider

$$A_i := \varphi_i(\overline{V_i}), \qquad B_i := \varphi_i(W_i).$$

The set B_i is open and $A_i \subseteq B_i$. Moreover, A_i is closed in E. To verify this last point, first observe that $\overline{V_i}$ is closed in M and contained in C_i, so A_i is closed in the open set $\varphi_i(C_i)$. On the other hand, A_i is contained in the closed ball $\overline{B}_E(\varphi_i(x(i)), r_{x(i)})$, which is a closed subset of E contained in $\varphi_i(C_i)$ by (24.4). If $A_i = F_i \cap \varphi_i(C_i)$ with F_i closed in E, then

$$A_i = F_i \cap \overline{B}_E(\varphi_i(x(i)), r_{x(i)}),$$

, proving the assertion.
Lemma 24.8.6 gives a function $\theta_i \in C^k(E, \mathbb{R})$ such that

$$0 \le \theta_i \le 1, \qquad \theta_i = 1 \text{ on } A_i, \qquad \operatorname{supp}(\theta_i) \subseteq B_i. \tag{24.6}$$

Define $\eta_i : M \longrightarrow \mathbb{R}$ by

$$\eta_i(p) := \begin{cases} \theta_i(\varphi_i(p)), & p \in C_i, \\ 0, & p \notin C_i. \end{cases}$$

This extension by zero is of class C^k. Indeed, the third property in (24.6) implies that the local function vanishes outside a closed set contained in W_i, and $\overline{W_i} \subseteq \overline{P_{x(i)}} \subseteq C_i$ by (24.4); hence it is identically zero on a neighborhood of every point of $M \setminus C_i$. Moreover,

$$\operatorname{supp}(\eta_i) \subseteq W_i, \qquad \eta_i = 1 \text{ on } \overline{V_i}. \tag{24.7}$$

The family $(\operatorname{supp}(\eta_i))_{i\in I}$ is locally finite because it is subordinate to $(W_i)_{i\in I}$. Consequently, the sum

$$h := \sum_{i\in I} \eta_i$$

locally consists of finitely many functions of class C^k and therefore belongs to $C^k(M)$. Given $p \in M$, there exists i with $p \in V_i$; then (24.7) gives $\eta_i(p) = 1$. Thus $h(p) \geq 1$ for every $p \in M$. The functions

$$\rho_i := \frac{\eta_i}{h}$$

are of class C^k, are nonnegative, form a locally finite family, satisfy $\operatorname{supp}(\rho_i) \subseteq W_i$, and obey $\sum_{i\in I} \rho_i = 1$.

Finally, for each $\alpha \in A$, define

$$\psi_\alpha := \sum_{\{i\in I \mid \alpha(i)=\alpha\}} \rho_i.$$

The sum is locally finite, so ψ_α is of class C^k. Moreover,

$$\sum_{\alpha\in A} \psi_\alpha = \sum_{i\in I} \rho_i = 1.$$

To check subordination, note that any subfamily of a locally finite family of closed sets has closed union. Hence

$$\operatorname{supp}(\psi_\alpha) \subseteq \bigcup_{i\in I,\ \alpha(i)=\alpha} \operatorname{supp}(\rho_i) \subseteq \bigcup_{i\in I,\ \alpha(i)=\alpha} W_i \subseteq U_\alpha.$$

The family of these supports is also locally finite, since it is obtained by grouping the locally finite family $(\operatorname{supp}(\rho_i))_{i\in I}$ according to the elements of the original cover. Thus $(\psi_\alpha)_{\alpha\in A}$ is the required partition. □

Definition 24.8.10. A Banach manifold of class C^k is called **C^k–paracompact** if its topological space is paracompact and every open cover admits a subordinate partition of unity of class C^k.

Theorem 24.8.9 states that a paracompact manifold modeled on a space admitting partitions of class C^k is C^k–paracompact. This formulation distinguishes the two assumptions: paracompactness produces locally finite families, while the property of the model space provides differentiable cutoff functions.

24.8.2 Criteria and examples of model spaces

General criteria for passing from a single bump function to partitions of unity belong to the theory of smoothness and renorming of Banach spaces. We introduce the functional assumption appearing in one of the most useful criteria.

Definition 24.8.11. A Banach space E is an **Asplund space** if every closed separable subspace $Y \subseteq E$ has separable dual.

Theorem 24.8.12. (Fry's criterion) Let E be a Banach space and let $k \in \mathbb{N}\cup\{\infty\}$. If E admits a bump function of class C^k and E' is an Asplund space, then E admits partitions of unity of class C^k.

This result is Fry's Theorem 3.1 [63, p. 146]. Its proof uses projectional resolutions in dual spaces and constructs a topological embedding into a space $c_0(\Gamma)$ whose scalar components are of class C^k. Existence of this embedding provides the partitions of unity. The criterion thus combines the regularity of the bump function with the Asplund assumption on the dual. Theorem 24.8.9 explains how to transfer its conclusion to a paracompact manifold. Johanis [86] gives a characterization that brings together and extends several existence criteria.

In that work, sufficiency of a single bump function of class C^k on an arbitrary Banach space is formulated as an open problem. In addition to the bump function, his characterization uses a family of smooth functions that allows each point to be approximated by subspaces determined by its nonzero components. This additional information makes partitions possible in the cases covered by the criterion.

Recall why the Asplund assumption in the preceding criterion holds in the reflexive examples. If E is reflexive, so is E' by Proposition B.11.40. Let $Y \subseteq E'$ be a closed separable subspace. Since closed subspaces of a reflexive space are reflexive, Y is reflexive: indeed, the unit ball of Y is the intersection of the unit ball of E' with the closed convex subspace Y, which is weakly closed by Theorem B.11.16; thus this ball is weakly compact, and Theorem B.11.38 gives reflexivity of Y. Consequently, Y' is also reflexive, and its closed unit ball $B_{Y'}$ is weakly compact.

On the other hand, separability of Y and Theorem B.11.31 show that $B_{Y'}$ is metrizable in the weak-$*$ topology. Since Y is reflexive, the canonical embedding $J_Y\colon Y \longrightarrow Y''$, defined by $J_Y(y)(y') := y'(y)$, is surjective. Consequently, the weak topology $\sigma(Y', Y'')$ and the weak-$*$ topology $\sigma(Y', J_Y(Y))$ agree on Y'; through J_Y, the latter is precisely $\sigma(Y', Y)$. Thus $B_{Y'}$ is a compact metric space. Choose a countable weakly dense subset $D \subseteq B_{Y'}$. Its real convex hull $\operatorname{co}(D)$ has weak closure $B_{Y'}$, since it contains D and is contained in the convex ball $B_{Y'}$. The rational convex hull $\operatorname{co}_{\mathbb{Q}}(D)$ is countable and norm dense in $\operatorname{co}(D)$: simply approximate the coefficients of each finite convex combination by nonnegative rational numbers summing to one. Finally, the norm closure of $\operatorname{co}_{\mathbb{Q}}(D)$ is convex and, by Theorem B.11.16, also weakly closed. Since it contains $\operatorname{co}(D)$, it contains its weak closure, which is $B_{Y'}$; the reverse inclusion is immediate. Consequently, $B_{Y'}$ is norm separable, and so is Y'. This proves that every reflexive space is Asplund and, in particular, the dual of a reflexive space is Asplund.

Example 24.8.13. (Hilbert spaces) Let H be a real Hilbert space. The quadratic function

$$Q\colon H \longrightarrow \mathbb{R}, \qquad Q(x) := \langle x, x\rangle_H,$$

is smooth, and its only nonzero derivatives are

$$DQ(x)h = 2\langle x, h\rangle_H, \qquad D^2Q(x)(h_1, h_2) = 2\langle h_1, h_2\rangle_H.$$

If $\chi \in C^\infty(\mathbb{R})$ satisfies $0 \le \chi \le 1$, $\chi = 1$ on $(-\infty, 1]$, and $\chi = 0$ on $[4, \infty)$, then $\chi \circ Q$ is a smooth bump function. Since H is reflexive, H' is Asplund. Theorem 24.8.12 implies that H admits smooth partitions of unity. Consequently, every smooth paracompact manifold modeled on a Hilbert space admits smooth partitions of unity, without any separability assumption on the model space.

The degree of differentiability available in L^p spaces can be seen directly in the functional defining their norm.

Lemma 24.8.14. Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 < p < \infty$, over real scalars. Define

$$P_p\colon L^p(X) \longrightarrow \mathbb{R}, \qquad P_p(u) := \int_X |u|^p \, d\mu.$$

If p is an even integer, P_p is smooth. If p is not an even integer, P_p is of class C^m for every $m \in \mathbb{N}$ with $m < p$. For $1 \le j < p$, its derivatives are

$$D^j P_p(u)(h_1, \dots, h_j) = \int_X a_{p,j}(u) h_1 \cdots h_j \, d\mu, \tag{24.8}$$

, where

$$a_{p,j}(t) := p(p-1)\cdots(p-j+1)|t|^{p-j}(\operatorname{sgn} t)^j, \qquad a_{p,j}(0) := 0.$$

Proof. First suppose $1 \le j < p$. Hölder's inequality in Proposition B.7.9, with exponents $\frac{p}{p-j}, p, \dots, p$, gives

$$\left| \int_X a_{p,j}(u) h_1 \cdots h_j \, d\mu \right| \le C_{p,j} \|u\|_{L^p(X)}^{p-j} \prod_{\ell=1}^{j} \|h_\ell\|_{L^p(X)}. \tag{24.9}$$

Here and in the following estimates, $C_{p,j} > 0$ depends only on p and j; it does not depend on u, the increments, or the measure space. Thus the right-hand side of (24.8) defines a continuous j–linear form.
The Nemytskii operator

$$L^p(X) \longrightarrow L^{\frac{p}{p-j}}(X), \qquad u \longmapsto a_{p,j}(u), \tag{24.10}$$

is continuous. Indeed, if $r := p - j$, for $0 < r \le 1$ we have the pointwise inequality

$$\left| |s|^r (\operatorname{sgn} s)^j - |t|^r (\operatorname{sgn} t)^j \right| \le C_r |s - t|^r,$$

, while for $r > 1$,

$$\left| |s|^r (\operatorname{sgn} s)^j - |t|^r (\operatorname{sgn} t)^j \right| \le C_r (|s|^{r-1} + |t|^{r-1}) |s - t|.$$

In both inequalities, $C_r > 0$ can be chosen to depend only on r; in particular, it is independent of s and t. Raising to the power $\frac{p}{r}$, integrating, and applying Hölder in the second case gives, respectively,

$$\|a_{p,j}(u) - a_{p,j}(v)\|_{L^{\frac{p}{r}}(X)} \le C_{p,j} \|u - v\|_{L^p(X)}^r$$

and

$$\|a_{p,j}(u) - a_{p,j}(v)\|_{L^{\frac{p}{r}}(X)} \le C_{p,j} (\|u\|_{L^p(X)}^{r-1} + \|v\|_{L^p(X)}^{r-1}) \|u - v\|_{L^p(X)}.$$

This proves continuity of (24.10) and, together with (24.9), norm continuity of the function assigning to u the multilinear form in (24.8).
It remains to verify that these forms are indeed the Fréchet derivatives. Set $a_{p,0}(t) := |t|^p$. If $j + 1 < p$, the fundamental theorem of calculus on $\mathbb{R}$ gives, at almost every point of X,

$$a_{p,j}(u + h) - a_{p,j}(u) - a_{p,j+1}(u)h = \int_0^1 \big(a_{p,j+1}(u + th) - a_{p,j+1}(u) \big) h \, dt.$$

For $j = 0$, interpret the empty product as 1. For each $t \in [0,1]$, Hölder's inequality with exponents $\frac{p}{p-j-1}, p, \dots, p$ controls the $L^1(X)$ norm of the integrand after multiplication by $h_1 \cdots h_j$. Moreover, continuity of the Nemytskii operator makes this integrand depend continuously on t in $L^1(X)$. We may therefore integrate in t in this Banach space and apply the continuous linear functional $v \longmapsto \int_X v \, d\mu$; this operation justifies interchanging the integrals. The preceding scalar identity agrees with this identity in $L^1(X)$: the

integrand vanishes outside the union of the sets $\{|u| \geq 1/n\}$ and $\{|h| \geq 1/n\}$, with $n \in \mathbb{N}$, which have finite measure, so Fubini's theorem applies on this support of σ–finite measure. We thus obtain

$$\left| \int_X \bigl(a_{p,j}(u+h) - a_{p,j}(u) - a_{p,j+1}(u)h\bigr) h_1 \cdots h_j \, d\mu \right|$$
$$\leq \sup_{t\in[0,1]} \|a_{p,j+1}(u+th) - a_{p,j+1}(u)\|_{L^{\frac{p}{p-j-1}}(X)} \|h\|_{L^p(X)} \prod_{\ell=1}^{j} \|h_\ell\|_{L^p(X)}.$$

Continuity of the Nemytskii operator corresponding to $j+1$ shows that the supremum tends to zero as $h \to 0$ in $L^p(X)$: in the two preceding estimates, simply replace v by $u+th$, and the bounds are uniform for $0 \leq t \leq 1$. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$, depending on u, p, j and ε, such that, if $\|h\|_{L^p(X)} < \delta$, the norm of the remainder as a j–linear form is at most $\varepsilon \|h\|_{L^p(X)}$. For $j = 0$, this proves the first derivative formula. Suppose that, for an integer $j \geq 1$ with $j+1 < p$, the form in (24.8) already agrees with $D^j P_p$. The remainder estimate, taken over all $h_\ell \in L^p(X)$ with $\|h_\ell\|_{L^p(X)} \leq 1$ and $\ell \in \{1, \ldots, j\}$, proves differentiability of $D^j P_p$ in the norm of the space of j–linear forms. Through the association in Proposition 23.2.3, its derivative is exactly the form indicated for order $j+1$. Continuity of this form was already established in (24.10). This proves the induction step and hence that P_p is of class C^m for every $m \in \mathbb{N}$ with $m < p$.
If p is an even integer, $|u|^p = u^p$ and

$$(u_1, \ldots, u_p) \longmapsto \int_X u_1 \cdots u_p \, d\mu$$

is a continuous p–linear form by Hölder's inequality. Thus P_p is a continuous homogeneous polynomial of degree p. Proposition 23.2.4, applied to this form, and Corollary 23.2.8, applied to the diagonal map $u \longmapsto (u, \ldots, u)$, show that P_p is smooth. □

Corollary 24.8.15. (The models ℓ^p and L^p) Let $1 < p < \infty$.

(a) If p is an even integer, the real spaces ℓ^p and $L^p(X)$, for every σ–finite measure space $(X, \mathcal{A}, \mu)$, admit smooth partitions of unity.

(b) If p is not an even integer, these spaces admit partitions of unity of class C^m for each $m \in \mathbb{N}$ with $m < p$.

Consequently, Theorem 24.8.9 applies, with the same degree of differentiability, to every paracompact Banach manifold modeled on one of these spaces.

Proof. Let $\chi \in C^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$, equal to one on $(-\infty, 1]$ and zero on $[2, \infty)$. By Lemma 24.8.14, the function $u \mapsto \chi(P_p(u))$ is a smooth bump function if p is even and of class C^m if $m < p$ in the other cases. Theorem B.7.7 shows that the $L^p(X)$ spaces under consideration are reflexive; the case of ℓ^p follows by taking counting measure on $\mathbb{N}$. As we saw before Example 24.8.13, the dual of a reflexive space is Asplund. The conclusion follows from Theorem 24.8.12. □

The differentiability range in the corollary is not a defect of the chosen bump function. The following result shows that, for ℓ^p, the restriction is intrinsic.

Theorem 24.8.16. (Bonic–Frampton) Let $1 < p < \infty$ and suppose p is not an even integer. If $m \in \mathbb{N}$ and $m \geq p$, there exists no nonzero bump function on ℓ^p of class C^m. In particular, ℓ^p admits neither partitions of unity of class C^m for these values of m nor smooth partitions of unity.

The statement is Theorem 1 and its Corollary 1 in Bonic and Frampton [24, Theorem 1 and Corollary 1, pp. 393–394]. Their argument uses the following boundary value principle: if p is not an even integer, $D \subseteq \ell^p$ is open, connected, and bounded, and a real function is continuous on $\overline{D}$ and m times differentiable on D, with $m \geq p$, then its image on $\overline{D}$ is contained in the closure of its image on ∂D. Applied to a bounded connected component of the set where a bump function is nonzero, this principle forces the function to be identically zero. Indeed, the connected components of an open subset of ℓ^p are open, since each point has a convex ball contained in that open set. If D is a component of $\{u \mid b(u) \neq 0\}$, then $b = 0$ on ∂D: otherwise, a connected ball on which b is nonzero would meet D and lie in the same component, contradicting the fact that its center is a boundary point. Thus the closure of $b(\partial D)$ is $\{0\}$, and the cited principle gives $b(D) = \{0\}$, a contradiction. The result shows that the differentiability order in the preceding corollary is optimal. The assertion about partitions follows from Lemma 24.8.6, since a partition with this regularity would produce a bump function of the same order.

There are nonreflexive examples whose construction requires more refined techniques. Haydon proved the following result.

Definition 24.8.17. (The spaces $C(K)$ and $C_0(L)$) If K is compact and Hausdorff, $C(K)$ is the real Banach space of continuous functions $u\colon K \longrightarrow \mathbb{R}$, with the norm $\|u\|_\infty := \sup_{x \in K} |u(x)|$. If L is locally compact and Hausdorff, define

$$C_0(L) := \{u\colon L \longrightarrow \mathbb{R} | u \text{ is continuous and, for every } \varepsilon > 0,\ \{x \in L \mid |u(x)| \geq \varepsilon\} \text{ is compact}\},$$

with the supremum norm. This norm is finite: for $\varepsilon = 1$, we have $|u| < 1$ outside a compact set and, if this set is nonempty, $|u|$ attains a maximum on it. Moreover, $C_0(L)$ is complete. Indeed, a Cauchy sequence for $\|\cdot\|_\infty$ converges uniformly to a continuous bounded function u. Given $\varepsilon > 0$, if $\|u - u_n\|_\infty < \frac{\varepsilon}{2}$, then

$$\{x \in L \mid |u(x)| \geq \varepsilon\} \subseteq \{x \in L \mid |u_n(x)| \geq \frac{\varepsilon}{2}\};$$

; the set on the left is closed and the one on the right is compact, so the former is compact. Its elements are precisely the continuous functions vanishing at infinity.

Theorem 24.8.18. (Haydon) If K is a countable compact Hausdorff space, then $C(K)$ admits a smooth norm ν away from the origin and constants $c_K, C_K > 0$, which may depend on K, such that $c_K\|u\|_\infty \leq \nu(u) \leq C_K\|u\|_\infty$ for every $u \in C(K)$; moreover, it admits smooth partitions of unity. The same holds for $C_0([0,\Omega))$, where Ω is an ordinal and $[0,\Omega)$ has the order topology, with constants that may depend on Ω.

The smooth norm is constructed in Haydon's Theorem 1 and Proposition 2, pp. 457–463; the partition criterion is his Theorem 2, and the two cases in the statement follow from Corollaries 4–5, pp. 464–468 [77]. Haydon's construction uses Talagrand operators, projectional resolutions, and transfinite induction on ordinal spaces. These examples show that the model space assumption in Theorem 24.8.9 can also hold without reflexivity.

24.8.3 Geometric consequences

Partitions of unity allow constructions with values in vector spaces to be globalized. We record a form that will be used frequently.

Proposition 24.8.19. (Gluing functions with linear codomain) Let M be a C^k–paracompact Banach manifold, let $(U_\alpha)_{\alpha\in A}$ be an open cover, and let F be a Banach space. Suppose $f_\alpha\colon U_\alpha \longrightarrow F$ is of class C^k for each α. If $(\psi_\alpha)_{\alpha\in A}$ is a partition of unity of class C^k subordinate to the cover, then

$$f\colon M \longrightarrow F, \qquad f(p) := \sum_{\alpha\in A} \psi_\alpha(p) f_\alpha(p),$$

, where each product is extended by zero outside U_α, is well defined and of class C^k.

Proof. Inclusion $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ ensures that $\psi_\alpha f_\alpha$ extends by zero to a function of class C^k on all of M. Each point has a neighborhood meeting only finitely many supports $\operatorname{supp}(\psi_\alpha)$. The sum defining f is therefore locally finite and locally agrees with a finite sum of functions of class C^k. □

Proposition 24.8.20. (Fine smooth approximation) Let M be a C^k–paracompact Banach manifold, let F be a Banach space, and let $f\colon M \longrightarrow F$ be continuous. For every continuous function $\varepsilon\colon M \longrightarrow (0,\infty)$, there exists $g \in C^k(M,F)$ such that

$$\|g(p) - f(p)\|_F < \varepsilon(p) \qquad \text{for every } p \in M.$$

Proof. For each $x \in M$, continuity of f and ε allows us to choose an open neighborhood U_x of x such that, for every $p \in U_x$,

$$\|f(p) - f(x)\|_F < \frac{\varepsilon(x)}{2}, \qquad \varepsilon(p) > \frac{\varepsilon(x)}{2}.$$

Let $(\psi_x)_{x\in M}$ be a partition of unity of class C^k subordinate to $(U_x)_{x\in M}$. Define

$$g(p) := \sum_{x\in M} \psi_x(p) f(x).$$

The sum is locally finite, and each summand is the product of a function of class C^k and a fixed vector in F; thus g is of class C^k. If $\psi_x(p) \neq 0$, then $p \in \operatorname{supp}(\psi_x) \subseteq U_x$, and the preceding inequalities give

$$\|f(x) - f(p)\|_F < \frac{\varepsilon(x)}{2} < \varepsilon(p).$$

Using $\psi_x(p) \geq 0$ and $\sum_{x\in M} \psi_x(p) = 1$, we conclude that

$$\|g(p) - f(p)\|_F \leq \sum_{x\in M} \psi_x(p) \|f(x) - f(p)\|_F < \varepsilon(p).$$

□

In particular, partitions of class C^k imply fine approximation of continuous functions with linear codomain. Together with Proposition 24.8.4 and Theorem 24.8.12, this makes precise the implications used here: if a norm

that is C^k away from the origin satisfies $c\|x\| \le \nu(x) \le C\|x\|$ with $c, C > 0$, it produces a bump function; under the Asplund assumption on the dual, this function produces C^k partitions; and partitions produce smooth approximations. The converse implications are not needed; they belong to substantially more delicate renorming and approximation problems.

The same argument preserves linear affine relations whenever the local objects take values in a common Banach space. Indeed, if $L\colon F \longrightarrow G$ is continuous and linear, $b \in G$, and the local functions $f_\alpha\colon U_\alpha \longrightarrow F$ satisfy $L \circ f_\alpha = b$, then the glued function $f := \sum_{\alpha\in A} \psi_\alpha f_\alpha$ satisfies

$$L \circ f = \sum_{\alpha\in A} \psi_\alpha L(f_\alpha) = \left(\sum_{\alpha\in A} \psi_\alpha\right) b = b.$$

Thus partitions of unity allow local constructions with linear codomain to be globalized without destroying the affine constraints they satisfy.

There is no analogous operation for maps with values in an arbitrary Banach manifold, since a convex combination of points of a manifold is not defined. Globalizing such maps therefore requires additional geometric constructions. This distinction appears in the global study of submersions and mapping spaces; a modern application of smooth partitions in this setting is the Stacey–Roberts lemma for Banach manifolds, presented in [89].

25

Vector bundles, connections, and flows on Banach manifolds

Vector bundles over Banach manifolds are described by transition maps with values in groups of invertible operators. The topology used on these groups is induced by the operator norm; this choice makes Fréchet calculus applicable and avoids confusing the joint continuity of evaluation with the regularity, in the operator norm, of a family of linear transformations.

The framework of bundles with Banach fibers and the need to distinguish topological tensor products already appear in the classical formulation of infinite-dimensional global analysis [85, §§4(C)–4(G)]. Here we work in the Banach category and use Fréchet differentiability.

Differential geometry with infinite-dimensional fibers

Banach bundles allow each fiber to be a complete infinite-dimensional space. This generalization arises naturally in the study of families of sections, operators, or initial data that vary over a base. The operator norm determines the regularity of transition functions, connections, and parallel transport. We will study these constructions together with the topological conditions they require, some of which hold automatically in finite dimensions.

25.1 Banach vector bundles

Let X and F be real Banach spaces, and let M be a Banach manifold of class C^r, with $r \in \mathbb{N} \cup \{\infty\}$, modeled on X. The space F will be the model fiber. We use the convention of Definition 24.1.1: charts have open images in the model spaces. A continuous bundle is understood to satisfy the same conditions as a C^r bundle, with C^r regularity replaced by continuity in the operator norm. This variant will be used for the tangent bundle of a C^1 manifold.

Proposition 25.1.1. (General linear group of a Banach space) The set

$$\mathrm{GL}(F) := \{A \in \mathcal{L}(F) \mid A \text{ is invertible}\}$$

is an open subset of the Banach algebra $\mathcal{L}(F)$. With this structure, $\mathrm{GL}(F)$ is a Banach Lie group: composition and inversion

$$(A,B)\longmapsto AB, \qquad A\longmapsto A^{-1}$$

are smooth. Moreover,

$$D(A\mapsto A^{-1})(A)H = -A^{-1}HA^{-1}.$$

Proof. Composition is continuous and bilinear, and hence smooth. The openness of the set of invertible operators, the smoothness of inversion, and the formula for its differential are precisely the assertions of Proposition 23.3.2, applied with $X = Y = F$. □

Before taking duals of fibers, it is useful to distinguish two notions that coincide in finite dimensions but not in a general Banach space. For a vector space V, we denote its algebraic dual, consisting of all linear functionals, by V^*. For a Banach space F, we denote its continuous dual, consisting of the continuous linear functionals, by

$$F' := \mathcal{L}(F;\mathbb{R})$$

The latter is a subspace of the algebraic dual F^* and carries the norm

$$\|\lambda\|_{F'} := \sup_{v\in F,\ \|v\|_F\le 1} |\lambda(v)|.$$

Proposition 25.1.2. (Continuous dual and transpose operators) The space F' is a Banach space. If $A \in \mathcal{L}(F,G)$, its transpose

$$A'\colon G' \longrightarrow F', \qquad A'\mu := \mu\circ A,$$

is linear and continuous, and satisfies

$$\|A'\|_{\mathcal{L}(G',F')} = \|A\|_{\mathcal{L}(F,G)}.$$

If $A\colon F\longrightarrow G$ is a topological linear isomorphism, then so is $A'\colon G'\longrightarrow F'$, and $(A')^{-1} = (A^{-1})'$. Moreover, the map

$$\mathcal{L}(F,G)\longrightarrow\mathcal{L}(G',F'), \qquad A\longmapsto A',$$

is an isometric linear embedding.
If F is infinite-dimensional, the inclusion $F'\subseteq F^*$ is strict.

Proof. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in F'. For each $v\in F$, the sequence $(\lambda_n(v))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, since

$$|\lambda_n(v)-\lambda_m(v)| \le \|\lambda_n-\lambda_m\|_{F'}\|v\|_F.$$

Define $\lambda(v) := \lim_{n\to\infty}\lambda_n(v)$. Passing to the limit shows that λ is linear. Since every Cauchy sequence is bounded, there exists $C>0$ such that $\|\lambda_n\|_{F'}\le C$ for every $n\in\mathbb{N}$, and then $|\lambda(v)|\le C\|v\|_F$. Therefore, $\lambda\in F'$. Given $\varepsilon>0$, for sufficiently large n,m we have $\|\lambda_n-\lambda_m\|_{F'}<\varepsilon$; letting m tend to infinity in the corresponding pointwise inequality gives $\|\lambda_n-\lambda\|_{F'}\le\varepsilon$. Thus, F' is complete.

The inequality $\|A'\| \le \|A\|$ follows from

$$|(A'\mu)(v)| \le \|\mu\|_{G'}\|A\|\|v\|_F.$$

Conversely, the Hahn–Banach theorem applied to $Av \in G$ yields, for each $v \in F$ with $Av \neq 0$, a functional $\mu_v \in G'$ such that $\|\mu_v\|_{G'} = 1$ and $\mu_v(Av) = \|Av\|_G$. Hence,

$$\|Av\|_G \le \|A'\|\|v\|_F,$$

an inequality that also holds when $Av = 0$. Taking the supremum over the unit ball gives $\|A\| \le \|A'\|$. The assertions about the inverse follow from $(BA)' = A'B'$.

Finally, suppose that F is infinite-dimensional. There exists a linearly independent sequence $(v_n)_{n\in\mathbb{N}}$ of unit vectors. On its linear span, consisting of finite linear combinations of these vectors, we define $\lambda_0(v_n) := n$ for each $n \in \mathbb{N}$ and extend it algebraically to F by extending $(v_n)_{n\in\mathbb{N}}$ to a Hamel basis. The extension $\lambda \in F^*$ is not continuous, because it is not bounded on the unit sphere. Consequently, $\lambda \notin F'$. □

The algebraic dual does not in general carry a canonical Banach topology compatible with evaluation. For this reason, all the dual bundles that appear below are constructed using the continuous dual.

Definition 25.1.3. (Banach vector bundle) Let M be a Banach manifold of class C^r. A **Banach vector bundle of class C^r with model fiber** F over M consists of a Hausdorff topological space $\mathcal{E}$, a continuous surjective map

$$\pi\colon \mathcal{E} \longrightarrow M$$

and a Banach vector space structure on each fiber $\mathcal{E}_p := \pi^{-1}(p)$, together with an open cover $(U_\alpha)_{\alpha\in A}$ of M and homeomorphisms

$$\Phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$$

satisfying the following conditions:

(a) $\mathrm{pr}_1 \circ \Phi_\alpha = \pi$;

(b) for each $p \in U_\alpha$, the restriction

$$\Phi_{\alpha,p} := \mathrm{pr}_2 \circ \big(\Phi_\alpha \restriction_{\mathcal{E}_p}\big)\colon \mathcal{E}_p \longrightarrow F$$

is a linear and topological isomorphism;

(c) on every nonempty intersection there are maps of class C^r, with respect to the operator norm topology,

$$g_{\beta\alpha}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(F)$$

such that

$$\Phi_\beta \circ \Phi_\alpha^{-1}(p, v) = \big(p, g_{\beta\alpha}(p)v\big).$$

The family (U_α, Φ_α) is called a **family of local trivializations**. Two families determine the same C^r bundle if their union satisfies condition (c).

The regularity in Definition 25.1.3 is required in the operator norm. Since evaluation

$$\mathcal{L}(F)\times F \longrightarrow F, \qquad (A,v)\longmapsto Av,$$

is continuous and bilinear, each change of trivialization is of class C^r in the base and fiber coordinates. Combined with an atlas of M, the trivializations equip $\boldsymbol{\mathcal{E}}$ with a Banach manifold atlas modeled on $X\times F$. The projection π is then of class C^r.

Definition 25.1.3 reproduces, with Banach fibers and regularity in the operator norm, the finite-dimensional construction in Definition 1.7.2. In finite rank, continuity of a family of matrices and continuity in the operator norm are the same condition; for a Banach fiber, the latter is the condition that allows Fréchet calculus to be applied directly.

Proposition 25.1.4. (Transition maps and reconstruction) Let M be a Banach manifold of class C^r. The transition maps of a Banach vector bundle satisfy

$$g_{\alpha\alpha}(p)=\mathrm{id}_F, \qquad g_{\alpha\beta}(p)=g_{\beta\alpha}(p)^{-1}, \qquad g_{\gamma\beta}(p)g_{\beta\alpha}(p)=g_{\gamma\alpha}(p)$$

for $\alpha,\beta,\gamma\in A$ and p in the intersection of the corresponding domains.
Conversely, an open cover $(U_\alpha)_{\alpha\in A}$ of M and a family of C^r maps

$$g_{\beta\alpha}\colon U_\alpha\cap U_\beta\longrightarrow \mathrm{GL}(F)$$

satisfying these identities determine, up to a bundle isomorphism of class C^r, a Banach vector bundle with model fiber F.

Proof. The identities follow by composing the changes of trivialization. For the converse, consider the disjoint union

$$\boldsymbol{\mathcal{E}}_0:=\coprod_{\alpha\in A}(U_\alpha\times F)$$

and the relation

$$(\alpha,p,v)\sim(\beta,p,w) \quad\Longleftrightarrow\quad w=g_{\beta\alpha}(p)v.$$

The cocycle identities show that this is an equivalence relation. Let $q\colon \boldsymbol{\mathcal{E}}_0\longrightarrow \boldsymbol{\mathcal{E}}:=\boldsymbol{\mathcal{E}}_0/\!\sim$ be the projection, and equip $\boldsymbol{\mathcal{E}}$ with the quotient topology. For each α, define

$$j_\alpha\colon U_\alpha\times F\longrightarrow\boldsymbol{\mathcal{E}}, \qquad j_\alpha(p,v):=q(\alpha,p,v).$$

The map j_α is injective. Moreover, if $O\subseteq U_\alpha\times F$ is open, then

$$q^{-1}\big(j_\alpha(O)\big)=\coprod_{\beta\in A}\Big\{(\beta,p,g_{\beta\alpha}(p)v)\;\Big|\;(p,v)\in O,\ p\in U_\alpha\cap U_\beta\Big\}.$$

For each β, the set on the right is the image of the open set $O\cap((U_\alpha\cap U_\beta)\times F)$ under the homeomorphism

$$(p,v)\longmapsto(p,g_{\beta\alpha}(p)v).$$

Consequently, $q^{-1}(j_\alpha(O))$ is open, and the definition of the quotient topology implies that $j_\alpha(O)$ is open.

Therefore, j_α is a homeomorphism onto the open set $j_\alpha(U_\alpha \times F)$.
The formula

$$\pi\big(j_\alpha(p,v)\big) := p$$

is well defined, is surjective, and is continuous on each of the open sets $j_\alpha(U_\alpha \times F)$. In fact, $j_\alpha(U_\alpha \times F) = \pi^{-1}(U_\alpha)$: if $q(\beta,p,w)$ projects to $p \in U_\alpha$, it is also represented by $(\alpha,p,g_{\alpha\beta}(p)w)$. We show that $\boldsymbol{\mathcal{E}}$ is Hausdorff. If $e_1,e_2 \in \boldsymbol{\mathcal{E}}$ project to distinct points, choose disjoint open subsets of M containing their projections; their preimages under π separate e_1 and e_2. If $\pi(e_1) = \pi(e_2) = p$, choose α with $p \in U_\alpha$ and write $e_i = j_\alpha(p,v_i)$. The inequality $e_1 \neq e_2$ implies $v_1 \neq v_2$. Two disjoint balls in F about v_1 and v_2, multiplied by an open neighborhood of p contained in U_α and transported by j_α, separate the two points.
On each fiber, the operations are defined by transporting those of F via j_α; the cocycle guarantees that they do not depend on α. The trivializations

$$\Phi_\alpha : j_\alpha(U_\alpha \times F) \longrightarrow U_\alpha \times F, \qquad \Phi_\alpha := j_\alpha^{-1},$$

have as their coordinate changes precisely the maps $(p,v) \mapsto (p, g_{\beta\alpha}(p)v)$. Combining them with charts of M gives the Banach manifold atlas on the total space.
Finally, if $\widetilde{\boldsymbol{\mathcal{E}}} \longrightarrow M$ is another bundle equipped with trivializations $\widetilde{\Phi}_\alpha$ having the same transition maps, the formula

$$j_\alpha(p,v) \longmapsto \widetilde{\Phi}_\alpha^{-1}(p,v)$$

is independent of α. It defines a bundle isomorphism of class C^r, since it is the identity in each trivialization. This proves uniqueness up to isomorphism. □

Two cocycles $(g_{\beta\alpha})$ and $(\widetilde{g}_{\beta\alpha})$ over the same cover are called **equivalent** if there are maps $h_\alpha \in C^r(U_\alpha, \mathrm{GL}(F))$ such that

$$\widetilde{g}_{\beta\alpha}(p) = h_\beta(p)\, g_{\beta\alpha}(p)\, h_\alpha(p)^{-1}. \tag{25.1}$$

In that case, the formula

$$[\alpha,p,v]_g \longmapsto [\alpha,p,h_\alpha(p)v]_{\widetilde{g}}$$

is well defined by (25.1) and provides an isomorphism of the reconstructed bundles. Conversely, the local representations of a bundle isomorphism over M provide a family (h_α) satisfying that identity, after passing to a common cover. Thus, equivalence of cocycles describes exactly the change of trivializations of the same bundle.

Definition 25.1.5. (Bundle homomorphism) Let M and N be Banach manifolds. Let $\pi_{\mathcal{E}} : \boldsymbol{\mathcal{E}} \longrightarrow M$ and $\pi_{\mathcal{H}} : \boldsymbol{\mathcal{H}} \longrightarrow N$ be Banach vector bundles of class C^r, with model fibers F and G, respectively. A **bundle homomorphism of class C^r over** $f \in C^r(M,N)$ is a map $\mathscr{A} : \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{H}}$ such that

$$\pi_{\mathcal{H}} \circ \mathscr{A} = f \circ \pi_{\mathcal{E}},$$

whose restrictions

$$\mathscr{A}_p : \boldsymbol{\mathcal{E}}_p \longrightarrow \boldsymbol{\mathcal{H}}_{f(p)}$$

are linear and continuous and whose local representations are C^r families, in the operator norm,

$$A_{\beta\alpha} : U_\alpha \cap f^{-1}(V_\beta) \longrightarrow \mathcal{L}(F,G).$$

A homomorphism over id_M is called a homomorphism over M. It is a **bundle isomorphism** if every fiber operator is invertible and the inverse homomorphism is of class C^r.

Proposition 25.1.6. (Local data of a homomorphism) Let M and N be Banach manifolds, and let $\boldsymbol{\mathcal{E}} \to M$ and $\boldsymbol{\mathcal{H}} \to N$ be Banach vector bundles of class C^r. The local representations of a bundle homomorphism satisfy

$$A_{\delta\gamma}(p) = h_{\delta\beta}(f(p))A_{\beta\alpha}(p)g_{\alpha\gamma}(p), \tag{25.2}$$

whenever all the trivializations involved are defined. Here g and h are the transition maps of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$. Conversely, a family of C^r maps with values in $\mathcal{L}(F,G)$ satisfying (25.2) determines a unique bundle homomorphism of class C^r over f. Composition of homomorphisms corresponds to pointwise composition of their local operators.

Proof. The first identity follows by writing a local representation with respect to two different pairs of trivializations. For the converse, let $e \in \boldsymbol{\mathcal{E}}_p$, let Φ_α be a trivialization about p, and choose a trivialization Ψ_β of $\boldsymbol{\mathcal{H}}$ about $f(p)$. If $\Phi_\alpha(e) = (p, v)$, set

$$\mathscr{A}(e) := \Psi_\beta^{-1}\big(f(p), A_{\beta\alpha}(p)v\big).$$

Identity (25.2) shows that the result does not depend on α or β. In each pair of trivializations, $\mathscr{A}$ is represented by $(p,v) \mapsto (f(p), A_{\beta\alpha}(p)v)$, which is of class C^r because evaluation $\mathcal{L}(F,G)\times F \longrightarrow G$ is continuous and bilinear. Uniqueness and the assertion about compositions are immediate in these representations. □

Definition 25.1.7. (Complemented subbundle) Let M be a Banach manifold. Let $\boldsymbol{\mathcal{E}} \longrightarrow M$ be a Banach vector bundle with model fiber F. A **complemented vector subbundle** is a subset $\boldsymbol{\mathcal{S}} \subseteq \boldsymbol{\mathcal{E}}$ such that each $\boldsymbol{\mathcal{S}}_p := \boldsymbol{\mathcal{S}} \cap \boldsymbol{\mathcal{E}}_p$ is a vector subspace and, for each $p \in M$, there exist a trivialization

$$\Phi : \boldsymbol{\mathcal{E}} \restriction_U \longrightarrow U \times F$$

and a closed complemented subspace $F_0 \subseteq F$ for which

$$\Phi(\boldsymbol{\mathcal{S}} \restriction_U) = U \times F_0.$$

Proposition 25.1.8. Let M be a Banach manifold, and let $\boldsymbol{\mathcal{E}} \to M$ be a Banach vector bundle. Every complemented subbundle $\boldsymbol{\mathcal{S}} \subseteq \boldsymbol{\mathcal{E}}$ is a Banach vector bundle and, over each component on which its model fiber is fixed, its inclusion in $\boldsymbol{\mathcal{E}}$ is a split embedded submanifold. In the trivializations of Definition 25.1.7, the local normal bundle is modeled on any closed complement of F_0.

Proof. If two adapted trivializations intersect, their transition map sends F_0 onto the corresponding subfiber model. The restriction is a continuous linear isomorphism. To check regularity in norm, let F_0 and $\widetilde{F}_0$ be the subfiber models, and let $\widetilde{P}\colon F \to \widetilde{F}_0$ be a continuous projection. The restricted operator can be written as $\widetilde{P} \circ g_{\beta\alpha}(p) \restriction_{F_0}$. Restriction and composition with $\widetilde{P}$ are continuous linear operations between the operator spaces; hence this family is C^r. Therefore, the restrictions of the trivializations define the bundle

structure of $\mathcal{S}$. If $F = F_0 \oplus F_1$, the inclusion is locally

$$U \times F_0 \longrightarrow U \times (F_0 \oplus F_1), \qquad (p, v) \longmapsto (p, v, 0).$$

This is the local form of a split embedded submanifold. □

Proposition 25.1.9. (Quotient by a complemented subbundle) Let M be a Banach manifold, and let $\mathcal{E} \to M$ be a Banach vector bundle. If $\mathcal{S} \subseteq \mathcal{E}$ is a complemented subbundle, the union

$$\mathcal{E}/\mathcal{S} := \coprod_{p \in M} \mathcal{E}_p/\mathcal{S}_p$$

restricted to each connected component $C \subseteq M$ has a natural Banach vector bundle structure over C, with a fixed quotient model Q_C. If the spaces Q_C are all isomorphic to a single Banach space Q, the full union is a bundle over M with model fiber Q. On each component, the sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{S} \longrightarrow 0$$

is exact fiberwise and splits locally by homomorphisms of class C^r. If $F = F_0 \oplus F_1$ in an adapted trivialization, the projection $F_1 \longrightarrow F/F_0$ is a topological linear isomorphism. This identification depends on the complement, whereas the quotient bundle is canonical.

Proof. A transition map between adapted trivializations sends the subfiber model onto the corresponding model and therefore induces a continuous linear isomorphism between the quotient spaces. These induced operators depend on the base with C^r regularity. Indeed, if $q_\alpha \colon F \to F/F_{0,\alpha}$ is the quotient projection and $j_\alpha \colon F/F_{0,\alpha} \to F$ a continuous linear section obtained from a complement, the induced operator is $q_\beta g_{\beta\alpha}(p) j_\alpha$. This expression is C^r in the operator norm. Its value does not depend on the chosen section, because two representatives of a class differ by an element of $F_{0,\alpha}$, and the transition map sends that subspace into $F_{0,\beta}$. The cocycle identities follow by applying the operators to equivalence classes. On each connected component C, the Banach isomorphism type of these quotient spaces is locally constant. Indeed, trivialization domains that can be joined by a chain of nonempty intersections have isomorphic quotient models, and the unions corresponding to the different classes of chains would form a partition of C into open sets; connectedness forces there to be only one class. Moreover, C is open because a Banach manifold is locally connected. Fixing a model Q_C and topological linear identifications of the local models with Q_C, the induced operators become a cocycle with values in $\mathrm{GL}(Q_C)$. Proposition 25.1.4 then constructs the quotient bundle. If $F = F_0 \oplus F_1$, the canonical projection restricted to F_1 is bijective and continuous; its inverse is continuous by the continuity of the projections associated with the topological direct sum. This gives the local splitting. □

Definition 25.1.10. (Section) Let M be a Banach manifold, and let $\mathcal{E} \to M$ be a Banach vector bundle. A **section** of $\mathcal{E} \longrightarrow M$ over an open subset $U \subseteq M$ is a map $\mathbf{s} \colon U \longrightarrow \mathcal{E}$ such that $\pi \circ \mathbf{s} = \mathrm{id}_U$. It is of class C^r if, in every local trivialization, there is a map $s_\alpha \in C^r(U \cap U_\alpha, F)$ for which

$$\Phi_\alpha(\mathbf{s}(p)) = (p, s_\alpha(p)).$$

The vector space of global C^r sections is denoted by $\Gamma^r(\mathcal{E})$.

On an intersection we have

$$s_\beta(p) = g_{\beta\alpha}(p)s_\alpha(p),$$

so that the regularity of a section can be checked in any family of trivializations.

Proposition 25.1.11. (Local data and gluing of sections) Let M be a Banach manifold, and let $\mathcal{E} \to M$ be a Banach vector bundle of class C^r. A family of maps $s_\alpha \in C^r(U_\alpha, F)$ determines a unique section $\mathbf{s} \in \Gamma^r(\mathcal{E})$ if and only if

$$s_\beta(p) = g_{\beta\alpha}(p)s_\alpha(p)$$

on each overlap. More generally, sections of class C^r defined over an open cover and agreeing pairwise on overlaps glue to a unique section of class C^r. Moreover, $\Gamma^r(\mathcal{E})$ is a module over $C^r(M)$ under $(f\mathbf{s})(p) := f(p)\mathbf{s}(p)$, and every $e \in \mathcal{E}_p$ is the value at p of a local section of class C^r.

Proof. The compatibility condition allows us to define

$$\mathbf{s}(p) := \Phi_\alpha^{-1}(p, s_\alpha(p))$$

independently of the index α. In each trivialization this section is represented by s_α, so it is of class C^r; the same formula proves uniqueness. The argument for an arbitrary cover is identical. The module operations are checked pointwise and preserve regularity in the local representations. Finally, if $\Phi\colon \mathcal{E}\restriction_U \longrightarrow U \times F$ is a trivialization about p and $\Phi(e) = (p, v)$, the section $q \mapsto \Phi^{-1}(q, v)$ over U takes the value e at p. □

Proposition 25.1.12. (Pullback bundle) Let M and N be Banach manifolds. Let $f\colon N \longrightarrow M$ be a map of class C^r, and let $\mathcal{E} \longrightarrow M$ be a Banach vector bundle of class C^r. The set

$$f^*\mathcal{E} := \{(q, e) \in N \times \mathcal{E} \mid f(q) = \pi(e)\}, \qquad \pi_f(q, e) := q,$$

has a natural Banach vector bundle structure over N, called the **induced bundle** or **pullback**. Its transition maps are

$$g_{\beta\alpha} \circ f$$

over $f^{-1}(U_\alpha \cap U_\beta)$. This structure does not depend on the chosen family of trivializations.

Proof. Over $f^{-1}(U_\alpha)$ define

$$\widetilde{\Phi}_\alpha(q, e) := \big(q, \mathrm{pr}_2\, \Phi_\alpha(e)\big).$$

These bijections and their inverses provide the required local trivializations. On an intersection, the fiber coordinate change is $g_{\beta\alpha}(f(q))$, which is of class C^r. Compatibility with another family of trivializations follows from (25.1): if the change is given by h_α, the induced cocycles are related by $h_\alpha \circ f$. □

Proposition 25.1.13. (Direct sum) Let M be a Banach manifold. Let $\mathcal{E} \longrightarrow M$ and $\mathcal{H} \longrightarrow M$ be Banach

vector bundles of class C^r, with model fibers F and G. The fiberwise union

$$\mathcal{E} \oplus \mathcal{H} := \coprod_{p \in M} (\mathcal{E}_p \oplus \mathcal{H}_p)$$

is a Banach vector bundle with model fiber $F \oplus G$. In simultaneous trivializations, its transition maps are

$$g^{\mathcal{E}}_{\beta\alpha}(p) \oplus g^{\mathcal{H}}_{\delta\gamma}(p).$$

The canonical inclusions and projections are bundle homomorphisms.

Proof. On $F \oplus G$ one may use, for example, the norm $\|(v,w)\| = \|v\|_F + \|w\|_G$. The indicated diagonal operators are invertible and depend on the base in a C^r manner. The cocycle identities hold componentwise. □

If $\mathcal{E}$ and $\mathcal{H}$ are bundles over the same base, we write

$$\mathcal{E} \times_M \mathcal{H} := \{(e,h) \in \mathcal{E} \times \mathcal{H} \mid \pi_{\mathcal{E}}(e) = \pi_{\mathcal{H}}(h)\}$$

for their fiber product. This notation describes the total space underlying the direct sum when the linear operations are taken fiberwise.

Proposition 25.1.14. (Dual bundle) Let M and N be Banach manifolds. Let $\mathcal{E} \longrightarrow M$ be a Banach vector bundle of class C^r, with model fiber F and transition maps $g_{\beta\alpha}$. The union

$$\mathcal{E}' := \coprod_{p \in M} \mathcal{E}'_p$$

has a natural Banach vector bundle structure of class C^r, with model fiber F'. In the dual trivialization of Φ_α,

$$\Phi'_\alpha(\lambda_p) := \big(p, \lambda_p \circ \Phi^{-1}_{\alpha,p}\big),$$

its transition maps are

$$g^{\vee}_{\beta\alpha}(p) := \big(g_{\beta\alpha}(p)^{-1}\big)'. \tag{25.3}$$

Fiberwise evaluation

$$\mathrm{ev} \colon \mathcal{E}' \times_M \mathcal{E} \longrightarrow \mathbb{R}, \qquad (\lambda_p, e_p) \longmapsto \lambda_p(e_p),$$

is of class C^r.
If $\mathscr{A} \colon \mathcal{E} \longrightarrow \mathcal{H}$ is a homomorphism over $f \colon M \longrightarrow N$, there is a natural transpose homomorphism

$$\mathscr{A}' \colon f^*\mathcal{H}' \longrightarrow \mathcal{E}', \qquad \mathscr{A}'_p(\mu_{f(p)}) := \mu_{f(p)} \circ \mathscr{A}_p.$$

In particular, there is a canonical isomorphism

$$\mathfrak{d}_f \colon (f^*\mathcal{H})' \longrightarrow f^*(\mathcal{H}').$$

On the fiber over p, let $\iota_p\colon \mathcal{H}_{f(p)} \longrightarrow (f^*\mathcal{H})_p$ be the isomorphism $\iota_p(h) := (p,h)$. Then

$$(\mathfrak{d}_f)_p(\lambda) := \lambda \circ \iota_p.$$

Its inverse is $\mu \mapsto \mu \circ \iota_p^{-1}$. Both operators are linear and continuous, and in the induced trivializations they are represented by the identity on G'. Therefore, $\mathfrak{d}_f$ and its inverse are bundle homomorphisms of class C^r.

Proof. Proposition 25.1.2 shows that the operators in (25.3) belong to $\mathrm{GL}(F')$ and depend on p with C^r regularity. Moreover,

$$g^\vee_{\gamma\beta} g^\vee_{\beta\alpha} = \big(g^{-1}_{\gamma\beta}\big)'\big(g^{-1}_{\beta\alpha}\big)',$$

and, since $(BA)' = A'B'$, the last expression is

$$\big(g^{-1}_{\beta\alpha} g^{-1}_{\gamma\beta}\big)' = \big((g_{\gamma\beta} g_{\beta\alpha})^{-1}\big)' = g^\vee_{\gamma\alpha},$$

so they satisfy the cocycle identity in the correct order. Proposition 25.1.4 reconstructs the bundle. In the trivializations of $\mathcal{E}'$ and $\mathcal{E}$, evaluation is the continuous bilinear map $F' \times F \longrightarrow \mathbb{R}$; the dual transition law ensures that these local expressions agree on overlaps.
The local representation of $\mathscr{A}'$ is $A_{\beta\alpha}(p)'$. Transposition is continuous and linear in the operator norm, and hence the family is of class C^r; its compatibility follows by transposing (25.2). Finally, the transition maps of $(f^*\mathcal{H})'$ and $f^*(\mathcal{H}')$ are, in both cases,

$$q \longmapsto \big(h_{\delta\beta}(f(q))^{-1}\big)',$$

so the fiberwise identity defines the canonical isomorphism. □

Proposition 25.1.15. (Operator bundles) Let M be a Banach manifold. Let $\mathcal{E} \longrightarrow M$ and $\mathcal{H} \longrightarrow M$ be Banach vector bundles of class C^r, with model fibers F and G. The union

$$\mathcal{L}(\mathcal{E},\mathcal{H}) := \coprod_{p\in M} \mathcal{L}(\mathcal{E}_p, \mathcal{H}_p)$$

is a Banach vector bundle with model fiber $\mathcal{L}(F,G)$. In two pairs of trivializations, the coordinate change is

$$A \longmapsto h_{\delta\beta}(p) A g_{\alpha\gamma}(p). \tag{25.4}$$

Its sections of class C^r are identified with the homomorphisms $\mathcal{E} \longrightarrow \mathcal{H}$ of class C^r over id_M.
In particular,

$$\mathrm{End}(\mathcal{E}) := \mathcal{L}(\mathcal{E},\mathcal{E})$$

has fiber $\mathcal{L}(\mathcal{E}_p)$ and transition maps $A \mapsto g_{\beta\alpha} A g^{-1}_{\beta\alpha}$. Evaluation

$$\mathcal{L}(\mathcal{E},\mathcal{H}) \times_M \mathcal{E} \longrightarrow \mathcal{H}, \qquad (A_p, e_p) \longmapsto A_p e_p,$$

is a bundle map of class C^r and is bilinear in the fiber variables.

Proof. The space $\mathcal{L}(F,G)$ is Banach. Composition of operators is continuous and bilinear, so the maps (25.4) are continuous linear isomorphisms that depend on the point with C^r regularity. The cocycle identities follow from those of g and h, and reconstruction follows from Proposition 25.1.4. The identification of sections with homomorphisms is exactly the compatibility condition (25.2). Evaluation is represented locally by the continuous bilinear map $\mathcal{L}(F,G)\times F \longrightarrow G$. □

25.2 Tangent and cotangent bundles

Let M be a Banach manifold of class C^k, modeled on X, and suppose first that $k \geq 1$. Following Definition 24.1.15, set

$$TM := \coprod_{p\in M} T_pM, \qquad \pi_{TM}(\xi_p) := p.$$

Each chart (U,φ) induces a trivialization

$$\tau_\varphi : TM\restriction_U \longrightarrow U\times X, \qquad \tau_\varphi(\xi_p) := \big(p, \Theta_{\varphi,p}(\xi_p)\big).$$

If (V,ψ) is another chart and $p \in U\cap V$, the change in the fiber is

$$A_{\psi\varphi}(p) := D(\psi\circ\varphi^{-1})(\varphi(p)) \in \mathrm{GL}(X). \tag{25.5}$$

The chain rule gives the cocycle identities.

Proposition 25.2.1. (Regularity of the tangent bundle) If M is a Banach manifold of class C^k, with $2 \leq k < \infty$, the preceding trivializations make TM a Banach vector bundle of class C^{k-1}. If M is smooth, TM is a smooth Banach vector bundle. For a merely C^1 manifold, the same construction produces a continuous vector bundle.

Proof. If the coordinate change $\psi\circ\varphi^{-1}$ is of class C^k, the map assigning to each point its derivative, with values in $\mathcal{L}(X)$ equipped with the operator norm, is of class C^{k-1}. We consider the base with its underlying C^{k-1} structure, described in Remark 24.1.4. The expression (25.5), the openness of $\mathrm{GL}(X)$, and reconstruction from cocycles prove the assertion. When $k=1$, this map is continuous; when $k=\infty$, it is smooth. □

The loss of one derivative in the preceding proposition is part of the structure: for $k<\infty$, one should not in general assign C^k regularity to TM on the basis of an atlas of class C^k.

Definition 25.2.2. (Cotangent bundle) Let M be a Banach manifold of class at least C^1. The **cotangent space** at p is the continuous dual of the kinematic tangent space,

$$T'_pM := (T_pM)'.$$

The **cotangent bundle** is the union

$$T'M := \coprod_{p\in M} T'_pM, \qquad \pi_{T'M}(\alpha_p) := p.$$

In the chart (U,φ) it is trivialized by

$$\tau_{\varphi}^{*}(\alpha_p) = \bigl(p, \alpha_p \circ \Theta_{\varphi,p}^{-1}\bigr) \in U \times X'.$$

Thus, $T'M$ is the continuous dual bundle of TM in the sense of Proposition 25.1.14. We use neither the algebraic dual of T_pM nor the dual of the operational tangent space. This choice ensures that each fiber is a Banach space and that evaluation is continuous. The prime distinguishes this bundle from the finite-dimensional cotangent bundle T^*M and is consistent with the notation X' for the continuous dual of a Banach space.

On an intersection, if $A_{\psi\varphi}$ is the operator in (25.5), the cotangent transition map is

$$\alpha \longmapsto \bigl(A_{\psi\varphi}(p)^{-1}\bigr)'\alpha. \tag{25.6}$$

The map $A \mapsto A'$ is continuous and linear in the operator norm, and inversion in $\mathrm{GL}(X)$ is smooth. Therefore, $T'M$ is a vector bundle modeled on X', of class C^{k-1} when $2 \le k < \infty$, continuous when $k = 1$, and smooth when M is smooth.

Proposition 25.2.3. (Tangent–cotangent evaluation) Let M be a Banach manifold of class at least C^1. The bundle map

$$\mathrm{ev}\colon T'M \times_M TM \longrightarrow \mathbb{R}, \qquad \mathrm{ev}(\alpha_p, \xi_p) := \alpha_p(\xi_p),$$

is natural under coordinate changes. It has the same regularity as the tangent and cotangent bundles; in particular, it is smooth if M is smooth.

Proof. In a trivialization, the map is represented by continuous bilinear evaluation

$$X' \times X \longrightarrow \mathbb{R}, \qquad (\ell, v) \longmapsto \ell(v).$$

Formula (25.6) was chosen precisely to make this expression invariant. □

Definition 25.2.4. (Covector fields and the differential) Let M be a smooth Banach manifold. A **covector field** or a 1-**form** on an open subset $U \subseteq M$ is a smooth section $\boldsymbol{\alpha}\colon U \longrightarrow T'M$. If $f \in C^{\infty}(U)$, its **differential** is the covector field determined by

$$(df)_p(v) := \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)), \qquad v = [\gamma] \in T_pM.$$

The definition does not depend on the representative γ because, in a chart (U,φ), we have

$$(df)_{\varphi}(x) = D(f \circ \varphi^{-1})(x). \tag{25.7}$$

In the same chart, a covector field is represented by

$$\alpha_{\varphi}\colon \varphi(U) \longrightarrow X', \qquad \alpha_{\varphi}(\varphi(p)) := \boldsymbol{\alpha}_p \circ \Theta_{\varphi,p}^{-1},$$

and, if (V,ψ) is another chart, satisfies

$$\alpha_\psi(\psi(p)) = \big(A_{\psi\varphi}(p)^{-1}\big)'\,\alpha_\varphi(\varphi(p)). \tag{25.8}$$

Formula (25.7) shows that df is smooth. More precisely, if M and f are of class C^k, with $1 \le k < \infty$, then $T'M$ and the section df are of class C^{k-1}; the loss comes from differentiating the changes of charts and f.

Proposition 25.2.5. (Elementary properties of the differential) Let M be a smooth Banach manifold. For $f, g \in C^\infty(M)$ we have

$$d(fg) = f\,dg + g\,df.$$

Moreover, every covector is locally the differential of a function at the point: if $p \in M$ and $\lambda_p \in T_p'M$, there exist a neighborhood U of p and $f \in C^\infty(U)$ such that $(df)_p = \lambda_p$.

Proof. The Leibniz rule follows by applying the product rule to (25.7). For the second assertion, let (U,φ) be a chart about p and set

$$\ell := \lambda_p \circ \Theta_{\varphi,p}^{-1} \in X'.$$

The function $f(q) := \ell(\varphi(q) - \varphi(p))$ is smooth on U, and its local representation has constant derivative ℓ. Consequently, $(df)_p = \lambda_p$. □

Definition 25.2.6. (Pullback of covector fields) Let $f\colon M \longrightarrow N$ be a smooth map between Banach manifolds. For a 1-form $\boldsymbol{\alpha}$ on N, define

$$(f^*\boldsymbol{\alpha})_p := \boldsymbol{\alpha}_{f(p)} \circ df_p \in T_p'M.$$

The field $f^*\boldsymbol{\alpha}$ is called the **pullback** of $\boldsymbol{\alpha}$ by f.

Proposition 25.2.7. (Regularity and naturality of pullback) Let M, N, and P be smooth Banach manifolds, and let $f\colon M \to N$ be a smooth map. The pullback of a smooth 1-form is a smooth 1-form. If $g\colon N \longrightarrow P$ is smooth, then

$$(g\circ f)^* = f^* \circ g^*, \qquad \mathrm{id}_M^* = \mathrm{id}, \qquad d(h\circ f) = f^*(dh)$$

for every $h \in C^\infty(N)$.

Proof. Let φ and ψ be charts in M and N, respectively, and write

$$\widehat{f} := \psi \circ f \circ \varphi^{-1}, \qquad \widehat{\alpha} := \alpha_\psi.$$

The local representation of the pullback is

$$(f^*\boldsymbol{\alpha})_\varphi(x) = D\widehat{f}(x)'\,\widehat{\alpha}(\widehat{f}(x)).$$

The map $x \mapsto D\widehat{f}(x)$ is smooth with respect to the operator norm, transposition is continuous and linear, and evaluation is continuous and bilinear. Therefore, the expression is smooth. The three identities follow from the chain rule and the pointwise definition. □

To treat forms of higher degree without choosing a tensor completion, if $m \in \mathbb{N}$ and X_0, Y_0 are Banach spaces,

denote by

$$\mathcal{L}^m_{\mathrm{alt}}(X_0;Y_0)$$

the subspace of continuous m-linear maps $A\colon X_0^m \longrightarrow Y_0$ that are alternating, equipped with the operator norm. It is a closed subspace of $\mathcal{L}^m(X_0;Y_0)$ and therefore is a Banach space. Coordinate changes act by

$$\omega \longmapsto \omega \circ (A^{-1},\dots,A^{-1}).$$

Proposition 25.1.4 thus provides a Banach bundle, denoted by $\bigwedge^m(T'M)$, whose fiber at p is $\mathcal{L}^m_{\mathrm{alt}}(T_pM;\mathbb{R})$.

Definition 25.2.8. (Differential forms) Let M and N be smooth Banach manifolds. A **smooth differential m-form** on M is a smooth section of $\bigwedge^m(T'M)$, for $m\in\mathbb{N}$. The space of these sections is denoted by $\Omega^m(M)$; for $m=0$ we set $\Omega^0(M):=C^\infty(M)$. If $f\colon M\longrightarrow N$ is smooth and $\omega\in\Omega^m(N)$, its pullback is defined by

$$(f^*\omega)_p(v_1,\dots,v_m) := \omega_{f(p)}(df_pv_1,\dots,df_pv_m). \tag{25.9}$$

The same local proof as in Proposition 25.2.7, using continuous multilinear composition, shows that (25.9) defines a smooth m-form and that pullback is functorial. This formulation does not identify continuous forms with a completed tensor product; it thus avoids imposing a tensor norm that is not canonical.

Definitions 25.2.2 and 25.2.4, together with Proposition 25.2.5, correspond to Definitions 1.7.46 and 1.7.47 and Corollary 1.7.50. The pullback formula is the same as in Definition 1.10.19 and preserves the identities of Proposition 1.10.20. The functional-analytic difference is that here each covector belongs to the continuous dual and all regularity is checked in the operator norm; no finite-dimensional identification with a tensor product is used.

Remark 25.2.9. (Bidual and tensor products) There is a canonical isometric linear injection

$$J_X\colon X\longrightarrow X'',\qquad J_X(v)(\ell):=\ell(v),$$

but it need not be surjective. Hence, if the model space is not reflexive, there is no canonical fiberwise identification between TM and $(T'M)'$.

Nor is there a unique tensor completion implicit in the Banach category. For example, $X'\otimes F$ maps to the space of finite-rank operators from X to F, whereas its projective and injective completions may differ and do not in general coincide with $\mathcal{L}(X,F)$. Consequently, tensor bundles must be defined by specifying a complete tensor norm. When only continuous multilinear maps are required, it is canonical to work directly with fibers such as

$$\mathcal{L}^m(T_pM;\mathbb{R}) \quad\text{or}\quad \mathcal{L}(T_pM,T_pM),$$

without imposing a tensor identification that may fail.

25.3 Vector fields and integral curves

Throughout the rest of this chapter, M will be a smooth Banach manifold modeled on a Banach space X.

Definition 25.3.1. (Vector field) Let M be a smooth Banach manifold. A **vector field** on M is a section $\mathbf{X}\colon M \longrightarrow TM$. In a chart (U,φ) it is represented by

$$X_\varphi\colon \varphi(U) \longrightarrow X, \qquad X_\varphi(\varphi(p)) := \Theta_{\varphi,p}(\mathbf{X}(p)).$$

On the overlap of two charts we have

$$X_\psi(\psi(p)) = D(\psi\circ\varphi^{-1})(\varphi(p))X_\varphi(\varphi(p)). \tag{25.10}$$

The field is continuous, locally Lipschitz, or of class C^r if its coordinate representations have the corresponding property.

We denote by $\mathfrak{X}(U) := \Gamma^\infty(TU)$ the space of smooth vector fields on an open subset $U \subseteq M$. If $f \in C^\infty(U)$ and $\mathbf{X} \in \mathfrak{X}(U)$, the function $\mathbf{X}(f) \in C^\infty(U)$ is defined by

$$\mathbf{X}(f)(p) := (df)_p(\mathbf{X}(p)), \qquad p \in U.$$

If $\mathbf{X},\mathbf{Y} \in \mathfrak{X}(U)$, in a chart we define

$$[\mathbf{X},\mathbf{Y}]_\varphi(x) := DY_\varphi(x)X_\varphi(x) - DX_\varphi(x)Y_\varphi(x). \tag{25.11}$$

Proposition 25.3.2. (Bracket of vector fields) Let M be a smooth Banach manifold, and let $U \subseteq M$ be open. Expression (25.11) is independent of the chart and defines a smooth vector field $[\mathbf{X},\mathbf{Y}]$. Moreover,

$$[\mathbf{X},\mathbf{Y}] = -[\mathbf{Y},\mathbf{X}], \qquad [\mathbf{X},f\mathbf{Y}] = \mathbf{X}(f)\mathbf{Y} + f[\mathbf{X},\mathbf{Y}]$$

for every $f \in C^\infty(U)$, and the bracket satisfies the Jacobi identity.

Proof. Let χ be a coordinate change, and write $\widetilde{X}(\chi(x)) = D\chi(x)X(x)$ and $\widetilde{Y}(\chi(x)) = D\chi(x)Y(x)$. Differentiating and subtracting gives

$$\begin{aligned} &D\widetilde{Y}(\chi(x))\widetilde{X}(\chi(x)) - D\widetilde{X}(\chi(x))\widetilde{Y}(\chi(x)) \\ &\qquad = D\chi(x)\big(DY(x)X(x) - DX(x)Y(x)\big), \end{aligned}$$

because the terms containing $D^2\chi(x)(X(x),Y(x))$ cancel by the symmetry of the second derivative. This is exactly the transformation law for a vector field. The first two identities are checked in coordinates using the product rule. For Jacobi, first consider the action on a function $f \in C^\infty(U)$. If $\widehat{f} = f\circ\varphi^{-1}$, the product rule gives

$$\mathbf{X}(\mathbf{Y}(f)) = D^2\widehat{f}(X_\varphi, Y_\varphi) + D\widehat{f}\big(DY_\varphi[X_\varphi]\big)$$

in the chart coordinates. The symmetry of $D^2\widehat{f}$ implies $[\mathbf{X},\mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f))$. Applying this identity twice, the three terms of Jacobi act as

$$\begin{aligned} [\mathbf{X},[\mathbf{Y},\mathbf{Z}]](f) &= \mathbf{XYZ}(f) - \mathbf{XZY}(f) - \mathbf{YZX}(f) + \mathbf{ZYX}(f), \\ [\mathbf{Y},[\mathbf{Z},\mathbf{X}]](f) &= \mathbf{YZX}(f) - \mathbf{YXZ}(f) - \mathbf{ZXY}(f) + \mathbf{XZY}(f), \end{aligned}$$

$$[\mathbf{Z},[\mathbf{X},\mathbf{Y}]](f)=\mathbf{ZXY}(f)-\mathbf{ZYX}(f)-\mathbf{XYZ}(f)+\mathbf{YXZ}(f).$$

Each composition appears once with each sign, so the sum is zero. At each point p, every continuous covector is the differential of a local function by Proposition 25.2.5. Since the continuous dual separates points by Hahn–Banach, a vector annihilated by all these differentials is zero. The sum of the three fields is therefore the zero field, which is the Jacobi identity. □

The local Lipschitz condition is independent of the chart. Indeed, let χ be a coordinate change, and let B be a convex ball on which X_φ is L_X-Lipschitz. After shrinking B, the smoothness of χ provides constants $C_0, C_1, C_2 > 0$ such that, for $x, y \in B$,

$$\|D\chi(x)\| \le C_0, \qquad \|D\chi(x)-D\chi(y)\| \le C_1\|x-y\|, \qquad \|x-y\| \le C_2\|\chi(x)-\chi(y)\|.$$

If also $\|X_\varphi(x)\| \le C_X$ on B, the transformation law gives

$$\|X_\psi(\chi(x))-X_\psi(\chi(y))\| \le C_2(C_0L_X + C_1C_X)\|\chi(x)-\chi(y)\|.$$

This transports a Lipschitz constant between charts. In particular, every field of class C^1 is locally Lipschitz, since its derivative is bounded on a sufficiently small ball.

Definition 25.3.3. (Integral curve) Let M be a smooth Banach manifold, and let $\mathbf{X}$ be a vector field on M. If $I \subseteq \mathbb{R}$ is an open interval and $\gamma\colon I \longrightarrow M$ is a curve of class C^1, its velocity at t is

$$\dot\gamma(t) := d\gamma_t(1) \in T_{\gamma(t)}M.$$

The curve γ is an **integral curve** of $\mathbf{X}$ if

$$\dot\gamma(t) = \mathbf{X}(\gamma(t))$$

for every $t \in I$. It is **maximal** if it is not the restriction of an integral curve defined on a strictly larger interval.

In a chart, the integral curve equation becomes the autonomous ODE

$$x'(t) = X_\varphi(x(t)).$$

The transformation law (25.10) and the chain rule ensure that this equation is independent of the coordinates.

Theorem 25.3.4. (Existence and uniqueness of integral curves) Let M be a smooth Banach manifold. Let $\mathbf{X}$ be a continuous, locally Lipschitz field on M. For each $p \in M$ there exists a unique maximal integral curve

$$\gamma_p\colon I_p \longrightarrow M, \qquad 0 \in I_p, \qquad \gamma_p(0) = p.$$

The domain I_p is an open interval. Every integral curve taking the value p at time zero is a restriction of γ_p.

Proof. Theorem B.13.1, applied to a coordinate representation of $\mathbf{X}$, gives a local integral curve. Proposition B.13.2 shows that two integral curves with a common value agree near that time and, by the open-

and-closed argument on the intersection of their intervals, agree on every common component. The local solutions can therefore be glued. The same union argument as in Theorem B.13.3 produces a maximal open interval and a unique integral curve on it. □

25.4 Flows on Banach manifolds

Definition 25.4.1. (Local flow) Let M be a smooth Banach manifold. A **local flow** on M is a continuous map

$$\Phi\colon \mathcal{D} \longrightarrow M,$$

where $\mathcal{D} \subseteq \mathbb{R} \times M$ is open and contains $\{0\} \times M$. For each $p \in M$, we require that

$$\mathcal{D}_p := \{t \in \mathbb{R} \mid (t,p) \in \mathcal{D}\}$$

be an interval containing 0. The map satisfies

$$\Phi(0,p) = p$$

and, if $s \in \mathcal{D}_p$, the exact equivalence of domains

$$t \in \mathcal{D}_{\Phi(s,p)} \quad \Longleftrightarrow \quad t+s \in \mathcal{D}_p,$$

together with the law

$$\Phi(t+s,p) = \Phi(t,\Phi(s,p))$$

whenever those conditions hold. If $\mathcal{D} = \mathbb{R} \times M$, it is called a **global flow**. A flow is of class C^r, with $r \in \mathbb{N}_0 \cup \{\infty\}$, if the map Φ is of that class. It is said to have **generating field $\mathbf{X}$** if each curve $t \mapsto \Phi(t,p)$ is of class C^1 and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \Phi(t,p) = \mathbf{X}(p)$$

for every $p \in M$. A local flow with generating field $\mathbf{X}$ is **maximal** if every other local flow with generator $\mathbf{X}$ is a restriction of it. In particular, it admits no proper extension with the same generator.

For a field $\mathbf{X}$ as in Theorem 25.3.4, define

$$\mathcal{D}_{\mathbf{X}} := \{(t,p) \in \mathbb{R} \times M \mid t \in I_p\}, \qquad \Phi_{\mathbf{X}}(t,p) := \gamma_p(t).$$

Theorem 25.4.2. (Maximal flow and smooth dependence) Let M be a smooth Banach manifold. Let $\mathbf{X}$ be a continuous, locally Lipschitz field on M. Then $\mathcal{D}_{\mathbf{X}}$ is open, $\Phi_{\mathbf{X}}$ is continuous, and it is the unique maximal local flow whose orbit curves are integral curves of $\mathbf{X}$.
If $\mathbf{X}$ is of class C^r, with $r \in \mathbb{N} \cup \{\infty\}$, then $\Phi_{\mathbf{X}}$ is of class C^r. For any $p \in M$ and $s,t \in \mathbb{R}$ such that $(s,p) \in \mathcal{D}_{\mathbf{X}}$, we have the equivalence

$$(t,\Phi_{\mathbf{X}}(s,p)) \in \mathcal{D}_{\mathbf{X}} \quad \Longleftrightarrow \quad (t+s,p) \in \mathcal{D}_{\mathbf{X}},$$

and, when these conditions hold,

$$\Phi_{\mathbf{X}}(t, \Phi_{\mathbf{X}}(s,p)) = \Phi_{\mathbf{X}}(t+s,p). \tag{25.12}$$

Proof. In a chart and for sufficiently small times, the assertions about openness, continuity, and regularity are those of Theorem B.13.5, taking the initial point as parameter. To reach an arbitrary point $(t_0,p_0) \in \mathcal{D}_{\mathbf{X}}$, cover the compact image of the curve γ_{p_0} between 0 and t_0 by finitely many charts and subdivide the interval so that each segment lies in one of them. The finite composition of the local solution maps shows that $\mathcal{D}_{\mathbf{X}}$ is open about (t_0,p_0) and that $\Phi_{\mathbf{X}}$ has the asserted regularity there. More explicitly, if $t_0 > 0$, choose $0 = t_0' < t_1' < \cdots < t_m' = t_0$ so that each segment belongs to a local solution domain; here $m \in \mathbb{N}$. Start from an initial value p near p_0. Suppose that, after j segments, with $0 \le j < m$, the point reached depends on p with the asserted regularity and remains close to $\gamma_{p_0}(t_j')$. Continuity allows us to shrink the neighborhood of p_0 so that this point belongs to the domain of the next local map. Composition with that map preserves regularity and gives the point of segment $j+1$. This is the induction step, whose initial case is the identity $p \mapsto p$. In the last segment we also allow the final time to vary in a neighborhood of t_0. This gives a solution defined on an open product about (t_0,p_0), with the required regularity. For $t_0 < 0$ we apply this construction to the field $-\mathbf{X}$; at $t_0 = 0$ the local result suffices.

Now fix s with $(s,p) \in \mathcal{D}_{\mathbf{X}}$. The curves

$$t \longmapsto \Phi_{\mathbf{X}}(t, \Phi_{\mathbf{X}}(s,p)) \quad \text{and} \quad t \longmapsto \Phi_{\mathbf{X}}(t+s,p)$$

are integral curves of $\mathbf{X}$ and take the same value at $t = 0$. Uniqueness shows that they agree on their common domain. Maximality of both curves identifies their intervals of definition and proves both the equivalence of domains and (25.12). Finally, any flow with generating field $\mathbf{X}$ consists of integral curves: differentiating $\Phi(t+h,p) = \Phi(h,\Phi(t,p))$ at $h = 0$ gives $\partial_t \Phi(t,p) = \mathbf{X}(\Phi(t,p))$. Uniqueness of integral curves then shows that this flow is a restriction of $\Phi_{\mathbf{X}}$. □

The preceding notions extend Definitions 1.7.40, 1.8.1, and 1.8.13. The proof of Theorem 1.8.15 retains its uniqueness and gluing argument here; the new step is that local existence and dependence on the initial value come from ODEs in Banach spaces, Theorems B.13.1 and B.13.5, without using local compactness.

For each $t \in \mathbb{R}$, let

$$M_t := \{p \in M \mid (t,p) \in \mathcal{D}_{\mathbf{X}}\}.$$

The openness of $\mathcal{D}_{\mathbf{X}}$ implies that M_t is open. The flow law shows that

$$\Phi_{\mathbf{X}}^t := \Phi_{\mathbf{X}}(t,\cdot)\colon M_t \longrightarrow M_{-t}$$

is a homeomorphism with inverse $\Phi_{\mathbf{X}}^{-t}$. If $\mathbf{X}$ is of class C^r, both maps are of class C^r, and hence $\Phi_{\mathbf{X}}^t$ is a C^r diffeomorphism between open sets.

Remark 25.4.3. (Fields and flows with $C^{1,1}$ regularity) Let M be a Banach manifold with the $C^{1,1}$ atlas of Definition 24.1.5. A vector field is a section $\mathbf{X}\colon M \longrightarrow TM$, and we say it is locally Lipschitz if its representations in the charts of that atlas are locally Lipschitz. This property does not depend on the chosen chart. Indeed, the law (25.10) uses only first derivatives. If $\chi = \psi \circ \varphi^{-1}$, we can shrink the domain about the point under consideration so that $D\chi$ is bounded and Lipschitz, $\mathbf{X}_\varphi$ is bounded and Lipschitz, and χ^{-1} is Lipschitz on a ball in its domain. The last property follows from the continuity of $D\chi^{-1}$ and

the mean value inequality. The estimate in Section 25.3 applies with these same bounds and shows that $\mathbf{X}_\psi = (D\chi \cdot \mathbf{X}_\varphi) \circ \chi^{-1}$ is locally Lipschitz.

The definition of an integral curve and the conclusions concerning existence, uniqueness, and a continuous maximal flow in Theorems 25.3.4 and 25.4.2 remain valid at this regularity. To see this, in each chart we apply Theorem B.13.1 to the coordinate field. The chain rule for C^1 changes transforms a coordinate solution into an integral curve independent of the chart. Local uniqueness allows these solutions to be glued and the union of their intervals to be taken, as in Theorem 25.3.4. By Theorem B.13.5, the local solution maps are continuous in time and in the initial value. Composition over a finite partition of a compact segment of the orbit, constructed in the proof of Theorem 25.4.2, proves openness of the flow domain and joint continuity. Uniqueness gives the group law and the time homeomorphisms together with their inverses. No differentiability of the flow with respect to the initial value is asserted here for a field that is only locally Lipschitz.

Proposition 25.4.4. (Generating field) Let M be a smooth Banach manifold. If $\Phi\colon \mathcal{D} \longrightarrow M$ is a local flow of class C^1, then

$$\mathbf{X}_\Phi(p) := \left.\frac{\partial}{\partial t}\right|_{t=0} \Phi(t,p) \in T_pM$$

defines a continuous vector field. Each curve $t \mapsto \Phi(t,p)$ is an integral curve of $\mathbf{X}_\Phi$. If Φ is of class C^r, with $r \geq 1$, then $\mathbf{X}_\Phi$ is of class C^{r-1}; if Φ is smooth, so is $\mathbf{X}_\Phi$.

Proof. The identity $\Phi(0,p) = p$ shows that $\mathbf{X}_\Phi(p) \in T_pM$. Differentiating the flow law

$$\Phi(t+h,p) = \Phi(h,\Phi(t,p))$$

with respect to h at $h = 0$ gives

$$\frac{\partial}{\partial t}\Phi(t,p) = \mathbf{X}_\Phi(\Phi(t,p)).$$

The regularity assertion follows by taking a partial derivative with respect to time. □

Theorem 25.4.5. (Global flow of a bounded field on a Banach space) Let X be a Banach space, regarded as a smooth Banach manifold, and let $\mathbf{V}\colon X \longrightarrow X$ be a continuous, locally Lipschitz field. If there exists $C \geq 0$ such that

$$\|\mathbf{V}(x)\|_X \leq C$$

for every $x \in X$, then its maximal flow is global:

$$\Phi_{\mathbf{V}}\colon \mathbb{R} \times X \longrightarrow X.$$

Moreover,

$$\|\Phi_{\mathbf{V}}(t,x) - x\|_X \leq C|t|$$

for every $(t,x) \in \mathbb{R} \times X$. If $\mathbf{V}$ is of class C^r, with $r \in \mathbb{N} \cup \{\infty\}$, then $\Phi_{\mathbf{V}}$ is of class C^r, and each $\Phi_{\mathbf{V}}^t\colon X \longrightarrow X$ is a diffeomorphism of class C^r, with inverse $\Phi_{\mathbf{V}}^{-t}$.

Proof. Corollary B.13.6, applied to $F(t,x) := \mathbf{V}(x)$, shows that every maximal integral curve is defined for all $t \in \mathbb{R}$ and provides the indicated estimate. Regularity of the flow follows from Theorem B.13.5. The law (25.12) gives

$$\Phi_{\mathbf{V}}^{-t} \circ \Phi_{\mathbf{V}}^{t} = \mathrm{id}_X = \Phi_{\mathbf{V}}^{t} \circ \Phi_{\mathbf{V}}^{-t}.$$

□

The boundedness condition in the preceding theorem uses the distinguished norm of the space X. On an abstract Banach manifold it is not an invariant condition until additional structure is fixed, for example a continuous norm on each fiber of TM with suitable uniform control. The basic intrinsic criterion for continuation remains the analysis, in a final chart, of convergence of the curve and membership of its limit in the domain, as in Theorem B.13.4.

25.5 Linear connections on Banach bundles

Let $\pi\colon \mathcal{E} \longrightarrow M$ be a smooth Banach vector bundle with model fiber F, over a smooth Banach manifold modeled on X. In infinite dimensions it is not enough to require algebraic identities of an operator on sections: one must also require its local coefficients to depend smoothly on the point in the operator norm topology.

Definition 25.5.1. (Linear connection) Let M be a smooth Banach manifold, and let $\mathcal{E} \to M$ be a smooth Banach vector bundle. A **linear connection** on $\mathcal{E}$ is a rule that, for each open set $U \subseteq M$, each $\mathbf{X} \in \mathfrak{X}(U)$, and each $\mathbf{s} \in \Gamma^{\infty}(\mathcal{E}\restriction_U)$, assigns a section

$$\nabla_{\mathbf{X}}\mathbf{s} \in \Gamma^{\infty}(\mathcal{E}\restriction_U),$$

compatibly with restriction to smaller open sets, and satisfies

$$\begin{aligned} \nabla_{f\mathbf{X}+g\mathbf{Y}}\mathbf{s} &= f\nabla_{\mathbf{X}}\mathbf{s} + g\nabla_{\mathbf{Y}}\mathbf{s}, \\ \nabla_{\mathbf{X}}(a\mathbf{s}+b\mathbf{t}) &= a\nabla_{\mathbf{X}}\mathbf{s} + b\nabla_{\mathbf{X}}\mathbf{t}, \\ \nabla_{\mathbf{X}}(f\,\mathbf{s}) &= \mathbf{X}(f)\mathbf{s} + f\nabla_{\mathbf{X}}\mathbf{s} \end{aligned}$$

for $f, g \in C^{\infty}(U)$ and $a, b \in \mathbb{R}$. The following local regularity is also required. If (U,φ) is a chart of M and $\Phi\colon \mathcal{E}\restriction_U \longrightarrow U \times F$ is a trivialization, there is a map smooth in the operator norm,

$$\Gamma_{\varphi,\Phi}\colon \varphi(U) \longrightarrow \mathcal{L}\big(X, \mathcal{L}(F)\big)$$

such that

$$\big(\nabla_{\mathbf{X}}\mathbf{s}\big)_{\varphi,\Phi}(x) = Ds_{\varphi,\Phi}(x)[X_{\varphi}(x)] + \Gamma_{\varphi,\Phi}(x)(X_{\varphi}(x))s_{\varphi,\Phi}(x). \tag{25.13}$$

The map $\Gamma_{\varphi,\Phi}$ is called the **local coefficient of the connection** in the chosen chart and trivialization.

The first identity expresses linearity over $C^{\infty}(U)$ in the direction, whereas the third is the Leibniz rule in the section. The additional local condition ensures that the operation is a continuous differential structure and not merely an algebraic operator.

The algebraic part agrees with Definition 2.4.2. The difference is analytic: in a trivialization with infinite-dimensional fiber, the coefficient must be a smooth map with values in $\mathcal{L}(X, \mathcal{L}(F))$ equipped with the operator

norm. This hypothesis replaces the automatic control of matrix families in finite dimensions and ensures smoothness of the evaluation appearing in (25.13).

Proposition 25.5.2. (Transformation law for a connection) Let M be a smooth Banach manifold, and let $\mathcal{E} \to M$ be a smooth Banach vector bundle with connection ∇. Let (U, φ, Φ) and (V, ψ, Ψ) be two choices of chart and trivialization. On the overlap, write

$$\chi := \psi \circ \varphi^{-1}, \qquad g(x) := g_{\Psi\Phi}(\varphi^{-1}(x)).$$

If Γ and $\widetilde{\Gamma}$ are the corresponding local coefficients, then

$$\widetilde{\Gamma}(\chi(x))(D\chi(x)v) = g(x)\Gamma(x)(v)g(x)^{-1} - Dg(x)[v]g(x)^{-1} \tag{25.14}$$

for every $v \in X$. Conversely, a family of smooth maps with values in $\mathcal{L}(X, \mathcal{L}(F))$ satisfying this law determines a unique linear connection.

Proof. Let $x = \varphi(p)$, $w = s_{\varphi,\Phi}(x)$, and $v = X_\varphi(x)$. In the second coordinates we have

$$s_{\psi,\Psi}(\chi(x)) = g(x)w, \qquad X_\psi(\chi(x)) = D\chi(x)v.$$

The chain rule and product rule give

$$D(s_{\psi,\Psi} \circ \chi)(x)[v] = Dg(x)[v]w + g(x)Ds_{\varphi,\Phi}(x)[v].$$

Since $\nabla_{\mathbf{X}}\mathbf{s}$ is a section of $\mathcal{E}$, its second representation must be $g(x)$ times the first. Substituting (25.13) and canceling the term involving $Ds_{\varphi,\Phi}$ gives

$$\widetilde{\Gamma}(\chi(x))(D\chi(x)v)g(x)w = g(x)\Gamma(x)(v)w - Dg(x)[v]w.$$

Since $w \in F$ is arbitrary, this identity is equivalent to (25.14).
Conversely, define $\nabla_{\mathbf{X}}\mathbf{s}$ by (25.13) in each trivialization. The preceding calculation, read in reverse, shows that the local expressions transform as a section and therefore glue together. The identities in Definition 25.5.1 are checked directly, and the local formula proves uniqueness. □

Theorem 25.5.3. (Existence of connections) Let M be a smooth Banach manifold. Suppose that M admits smooth partitions of unity subordinate to every open cover. Then every smooth Banach vector bundle over M admits a smooth linear connection.

Proof. Simultaneously refining an atlas of M and a family of trivializations, we obtain a cover $(U_\alpha)_{\alpha \in A}$ such that each U_α is the domain of a chart φ_α and a trivialization Φ_α. Choose a smooth partition of unity $(\rho_\alpha)_{\alpha \in A}$ subordinate to it. On $\mathcal{E}\restriction_{U_\alpha}$ there is a local flat connection ∇^α whose coefficient in $(\varphi_\alpha, \Phi_\alpha)$ is zero.
Define the rule on the entire sheaf of sections. Let $W \subseteq M$ be open, let $\mathbf{X} \in \mathfrak{X}(W)$, and let $\mathbf{s} \in \Gamma^\infty(\mathcal{E}\restriction_W)$. On $W \cap U_\alpha$ set

$$T_\alpha(\mathbf{X}, \mathbf{s}) := \rho_\alpha\restriction_W \, \nabla^\alpha_{\mathbf{X}\restriction_{W\cap U_\alpha}}(\mathbf{s}\restriction_{W\cap U_\alpha}).$$

Since $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$, this section extends by zero to a smooth section of $\mathcal{E}\restriction_W$: every point of $W \setminus U_\alpha$ has a

neighborhood on which ρ_α vanishes identically. The family of these extensions is locally finite. Therefore, the sum

$$\nabla_{\mathbf{X}}\mathbf{s} := \sum_{\alpha\in A} T_\alpha(\mathbf{X},\mathbf{s})$$

is well defined and smooth on W. The same formula on an open set $W_0 \subseteq W$ is the restriction of the preceding one, so the rule is compatible with restrictions.

Each ∇^α is linear over functions in the direction and linear over $\mathbb{R}$ in the section; multiplication by ρ_α and summation preserve these two properties. For $f \in C^\infty(W)$, on each $W \cap U_\alpha$ we have

$$\begin{aligned}\nabla_{\mathbf{X}}(f\mathbf{s}) &= \sum_{\alpha\in A} \rho_\alpha\big(\mathbf{X}(f)\mathbf{s} + f\nabla^\alpha_{\mathbf{X}}\mathbf{s}\big)\\ &= \mathbf{X}(f)\mathbf{s} + f\nabla_{\mathbf{X}}\mathbf{s},\end{aligned}$$

since $\sum_{\alpha\in A}\rho_\alpha = 1$. Finally, in any chart and trivialization, each local flat connection transforms by (25.14) into a smooth coefficient with values in $\mathcal{L}(X,\mathcal{L}(F))$. The coefficient of ∇ is the locally finite sum of those coefficients multiplied by ρ_α; hence it is smooth in the operator norm. Thus all the conditions of Definition 25.5.1 hold. □

The hypothesis of the preceding theorem is substantive in the Banach category. Paracompactness alone does not produce smooth partitions of unity; the hypotheses on the model space studied in Section 24.8 are required.

Proposition 25.5.4. (Induced connections) Let M and N be smooth Banach manifolds, and let $\mathcal{E} \to M$ be a smooth Banach vector bundle. Let ∇ be a connection on $\mathcal{E}$. Then:

(i) the dual bundle $\mathcal{E}'$ has the dual connection ∇' determined by the identity

$$(\nabla'_{\mathbf{X}}\lambda)(\mathbf{s}) = \mathbf{X}\big(\lambda(\mathbf{s})\big) - \lambda(\nabla_{\mathbf{X}}\mathbf{s}); \tag{25.15}$$

here $U \subseteq M$ is open, $\mathbf{X} \in \mathfrak{X}(U)$, $\lambda \in \Gamma^\infty(\mathcal{E}'\restriction_U)$, and $\mathbf{s} \in \Gamma^\infty(\mathcal{E}\restriction_U)$;

(ii) if $\mathcal{H}$ has a connection $\nabla^{\mathcal{H}}$, the bundle $\mathcal{E}\oplus\mathcal{H}$ has the direct sum connection

$$\nabla^{\oplus}_{\mathbf{X}}(\mathbf{s},\mathbf{t}) = (\nabla_{\mathbf{X}}\mathbf{s}, \nabla^{\mathcal{H}}_{\mathbf{X}}\mathbf{t});$$

(iii) if $f\colon N \longrightarrow M$ is smooth, the bundle $f^*\mathcal{E}$ has a connection $f^*\nabla$. Let (V,ψ) be a chart of N and (U,φ) a chart of M such that $f(V) \subseteq U$, and let $\Phi\colon \mathcal{E}\restriction_U \longrightarrow U\times F$ be a trivialization. Let $\widehat{f} := \varphi\circ f\circ\psi^{-1}$, and denote by Γ the coefficient of ∇ in (φ,Φ). In the induced trivialization, if V and σ also denote their local representations, the connection has the expression

$$\big((f^*\nabla)_V\sigma\big)(y) = D\sigma(y)[V(y)] + \Gamma(\widehat{f}(y))\big(D\widehat{f}(y)[V(y)]\big)\sigma(y). \tag{25.16}$$

Proof. For (25.15) to define a covector, we must first check its dependence on the test section. If $h \in C^\infty(U)$, then

$$\mathbf{X}\big(\lambda(h\mathbf{s})\big) - \lambda(\nabla_{\mathbf{X}}(h\mathbf{s})) = \mathbf{X}\big(h\lambda(\mathbf{s})\big) - \lambda\big(\mathbf{X}(h)\mathbf{s} + h\nabla_{\mathbf{X}}\mathbf{s}\big)$$

$$= h\big(\mathbf{X}(\lambda(\mathbf{s})) - \lambda(\nabla_{\mathbf{X}}\mathbf{s})\big).$$

The right-hand side of (25.15) is therefore $C^\infty(U)$-linear in $\mathbf{s}$. To check also that its value at a point depends only on the fiber, write ℓ and s for the representations of λ and $\mathbf{s}$, and $u = X_\varphi(x)$. The derivative of evaluation, which is continuous and bilinear, gives

$$D\big(\ell(\cdot)(s(\cdot))\big)(x)[u] = D\ell(x)[u](s(x)) + \ell(x)(Ds(x)[u]).$$

Subtracting $\ell(x)(Ds(x)[u] + \Gamma(x)(u)s(x))$ cancels the terms involving Ds. The result depends only on $s(x)$ and is given by the continuous functional

$$D\ell(x)[X_\varphi(x)] - \big(\Gamma(x)(X_\varphi(x))\big)'\ell(x).$$

In particular, it is continuous in the fiber variable and depends smoothly on the point. The coefficient of the dual connection is

$$\Gamma'(x)(v) = -\big(\Gamma(x)(v)\big)',$$

which is smooth in the operator norm because transposition $\mathcal{L}(F) \longrightarrow \mathcal{L}(F')$ is continuous and linear. Differentiating the identity $\langle (g^{-1})'\ell, gw\rangle = \langle \ell, w\rangle$ and using (25.14) gives exactly the transformation law of Γ' for the dual cocycle $(g^{-1})'$. Linearity and the Leibniz rule in λ follow directly from (25.15).

The direct sum connection satisfies the identities componentwise; in simultaneous trivializations its coefficient is the diagonal operator $\Gamma^{\mathcal{E}} \oplus \Gamma^{\mathcal{H}}$.

For the pullback, suppose that N is modeled on a Banach space Y. In the coordinates and induced trivialization of the statement, define

$$\Gamma^f(y)(w) := \Gamma(\widehat{f}(y))\big(D\widehat{f}(y)[w]\big), \qquad w \in Y.$$

Composition and evaluation of operators are smooth, so $\Gamma^f : \psi(V) \longrightarrow \mathcal{L}(Y, \mathcal{L}(F))$ is smooth in the operator norm. If the domain chart, the codomain chart, and the trivialization are changed, substitute (25.14) into this formula and use

$$D(g \circ \widehat{f})(y)[w] = Dg(\widehat{f}(y))[D\widehat{f}(y)[w]].$$

The result is (25.14) for the induced transition map $g \circ \widehat{f}$; hence the local expressions glue together. The resulting local formula is (25.16). Its linearity over functions in the vector direction and the Leibniz rule in σ follow, respectively, from the linearity of $D\widehat{f}(y)$ and the product rule for $D(h\sigma)$. □

The dual connection and the induced connection extend the constructions of Proposition 2.4.27. In finite dimensions the coefficients are matrices; here the same proof additionally requires that transposition, composition, and the coefficients depend continuously in the operator norm.

25.6 Differentiation along curves and parallel transport

Let $\gamma : I \longrightarrow M$ be a curve of class C^1. A field of class C^1 along γ is a section $\mathbf{V} \in \Gamma^1(\gamma^*\boldsymbol{\mathcal{E}})$. If $\gamma(J)$ is contained in the domain of a chart and a trivialization, write $x(t)$ for the representation of γ and $v(t)$ for that of $\mathbf{V}$. The local formula (25.13) shows that $(\nabla_{\mathbf{X}}\mathbf{s})(p)$ depends on $\mathbf{X}$ only through $\mathbf{X}(p)$. Hence, if $u \in T_pM$, we write $\nabla_u\mathbf{s}$ for the

common value of $(\nabla_{\mathbf{X}}\mathbf{s})(p)$ among all local fields $\mathbf{X}$ such that $\mathbf{X}(p) = u$.

Definition 25.6.1. (Covariant derivative along a curve) Let M be a smooth Banach manifold, let $\boldsymbol{\mathcal{E}} \to M$ be a smooth Banach vector bundle with connection, and let $\gamma\colon I \to M$ be a curve of class C^1. The **covariant derivative of $\mathbf{V}$ along** γ is the field $\frac{D^\nabla \mathbf{V}}{dt}$ whose representation in the trivialization Φ is defined locally as follows. For p in the domain of Φ, denote by

$$\Phi_p := \mathrm{pr}_2 \circ \big(\Phi \restriction_{\boldsymbol{\mathcal{E}}_p}\big)\colon \boldsymbol{\mathcal{E}}_p \longrightarrow F$$

the fiber coordinate. Then

$$\Phi_{\gamma(t)}\left(\left(\frac{D^\nabla \mathbf{V}}{dt}\right)(t)\right) = v'(t) + \Gamma(x(t))(x'(t))v(t). \tag{25.17}$$

For γ and $\mathbf{V}$ of class C^1, the covariant derivative is a continuous field along γ. The field $\mathbf{V}$ is **parallel** if $\frac{D^\nabla \mathbf{V}}{dt} = 0$.

Proposition 25.6.2. (Invariance of differentiation along curves) Let M be a smooth Banach manifold, let $\boldsymbol{\mathcal{E}} \to M$ be a smooth Banach vector bundle with connection, and let $\gamma\colon I \to M$ be a curve of class C^1. Expression (25.17) does not depend on the chart or the trivialization. It is linear in $\mathbf{V}$ and satisfies

$$\frac{D^\nabla(a\mathbf{V})}{dt} = a'\mathbf{V} + a\frac{D^\nabla \mathbf{V}}{dt}$$

for every function $a \in C^1(I)$. If $\mathbf{V}(t) = \mathbf{s}(\gamma(t))$ for a smooth local section $\mathbf{s}$, then

$$\frac{D^\nabla \mathbf{V}}{dt}(t) = \nabla_{\dot\gamma(t)}\mathbf{s}.$$

Proof. Under a change of trivialization we have $\widetilde{v}(t) = g(x(t))v(t)$. Therefore,

$$\widetilde{v}'(t) = Dg(x(t))[x'(t)]v(t) + g(x(t))v'(t).$$

Adding the connection term and using (25.14) cancels the two terms containing Dg, and the result is $g(x(t))$ times the right-hand side of (25.17). Invariance under changes of base coordinates is included in the same law. The remaining assertions follow from the product rule and (25.13). □

Theorem 25.6.3. (Existence and uniqueness of parallel transport) Let M be a smooth Banach manifold, let $\boldsymbol{\mathcal{E}} \to M$ be a smooth Banach vector bundle with connection, and let $\gamma\colon I \to M$ be a curve of class C^1. Let $t_0 \in I$ and $e_0 \in \boldsymbol{\mathcal{E}}_{\gamma(t_0)}$. There exists a unique parallel field $\mathbf{V}$ along γ such that $\mathbf{V}(t_0) = e_0$. For $t_0, t_1 \in I$, the map

$$P^\nabla_{\gamma, t_0 \to t_1}\colon \boldsymbol{\mathcal{E}}_{\gamma(t_0)} \longrightarrow \boldsymbol{\mathcal{E}}_{\gamma(t_1)}, \qquad e_0 \longmapsto \mathbf{V}(t_1),$$

is a topological linear isomorphism. Moreover,

$$P^\nabla_{\gamma,t_1\to t_2} \circ P^\nabla_{\gamma,t_0\to t_1} = P^\nabla_{\gamma,t_0\to t_2},$$
$$\big(P^\nabla_{\gamma,t_0\to t_1}\big)^{-1} = P^\nabla_{\gamma,t_1\to t_0}.$$

for any $t_0, t_1, t_2 \in I$.

Proof. In a trivialization over a subinterval $J \subseteq I$, the parallelism condition is equivalent to the linear ODE in F

$$v'(t) = -A(t)v(t), \qquad A(t) := \Gamma(x(t))(x'(t)) \in \mathcal{L}(F). \tag{25.18}$$

The coefficient A is continuous in the operator norm. The existence and uniqueness theorem for ODEs in Banach spaces, Theorem B.13.1, gives a unique local solution. On every compact subinterval contained in J, the function $\|A(t)\|$ is bounded, and the integral equation together with Grönwall's lemma 1.8.19 gives

$$\|v(t)\|_F \le \|v(t_0)\|_F \exp\left(\int_{t_0}^{t} \|A(\tau)\|\, d\tau\right)$$

when $t \ge t_0$, with the analogous estimate in the reverse direction. Therefore, the solution remains bounded on every compact subinterval. If a finite endpoint c of the maximal interval belonged to the interior of J, then, on a compact set containing t_0 and c, A and v would also be bounded. The equation would imply that $v' = -Av$ is bounded; hence v would be uniformly Lipschitz and would have a limit in F as t tends to c, because F is complete. Applying the local theorem again with that limit as initial value would extend the solution beyond c, a contradiction. Thus, the solution exists throughout J. Covering the curve segment between two times by finitely many trivializations and using uniqueness on the overlaps gives a solution on all of I.

The equation is linear in the initial value, and the preceding estimate, combined over finitely many trivializations, proves continuity of the transport operator. Uniqueness shows that transport from t_0 to t_1 followed by transport from t_1 to t_2 produces the same parallel section as direct transport. Taking $t_2 = t_0$ gives the formula for the inverse. □

The notion of a section along a curve and the differentiation formula are those of Definition 2.7.3 and Theorem 2.7.5. Likewise, the composition and inversion laws correspond to Theorem 2.7.12 and Proposition 2.7.14. The new feature here is that the linear equation takes values in a Banach fiber; global existence on the interval follows from the estimate in Grönwall's lemma 1.8.19 and the completeness of the fiber, rather than from finite dimensionality.

Definition 25.6.4. (Curvature of a connection) Let M be a smooth Banach manifold, and let $\mathcal{E} \to M$ be a smooth Banach vector bundle with connection ∇. The **curvature** of ∇ is the operator

$$\mathbf{R}^{\nabla}(\mathbf{X}, \mathbf{Y})\mathbf{s} := \nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{s} - \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}} \mathbf{s} - \nabla_{[\mathbf{X},\mathbf{Y}]} \mathbf{s}.$$

The connection is **flat** if $\mathbf{R}^{\nabla} = 0$.

Proposition 25.6.5. (Tensoriality and local expression of curvature) Let M be a smooth Banach manifold, and let $\mathcal{E} \to M$ be a smooth Banach vector bundle with connection ∇. Curvature is $C^\infty(M)$-linear in $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{s}$, and alternating in $\mathbf{X}, \mathbf{Y}$. The local expression obtained below also shows that its value at p depends only on $\mathbf{X}(p)$, $\mathbf{Y}(p)$, and $\mathbf{s}(p)$. Thus, for each $p \in M$ it determines a continuous trilinear map

$$\mathbf{R}^{\nabla}_p \colon T_pM \times T_pM \times \mathcal{E}_p \longrightarrow \mathcal{E}_p.$$

Equivalently,

$$\mathbf{R}^{\nabla}_p \in \mathcal{L}^2_{\mathrm{alt}}\big(T_pM; \mathcal{L}(\boldsymbol{\mathcal{E}}_p)\big).$$

In a chart and a trivialization, if x represents p, then

$$\mathbf{R}^{\nabla}_x(u,v)w = \Omega(x)(u,v)w, \tag{25.19}$$

where

$$\begin{aligned}\Omega(x)(u,v) = {} & D\Gamma(x)[u](v) - D\Gamma(x)[v](u) \\ & + \Gamma(x)(u)\Gamma(x)(v) - \Gamma(x)(v)\Gamma(x)(u).\end{aligned}$$

Under a change of chart and trivialization as in Proposition 25.5.2, we have

$$\widetilde{\Omega}(\chi(x))(D\chi(x)u, D\chi(x)v) = g(x)\Omega(x)(u,v)g(x)^{-1}. \tag{25.20}$$

Proof. Antisymmetry is immediate. For $f \in C^\infty(M)$, we use

$$[f\mathbf{X}, \mathbf{Y}] = f[\mathbf{X}, \mathbf{Y}] - \mathbf{Y}(f)\mathbf{X}$$

and the Leibniz rules for the connection to obtain $\mathbf{R}^{\nabla}(f\mathbf{X}, \mathbf{Y})\mathbf{s} = f\mathbf{R}^{\nabla}(\mathbf{X}, \mathbf{Y})\mathbf{s}$. Linearity in the second direction follows from antisymmetry. Likewise, on expanding $\mathbf{R}^{\nabla}(\mathbf{X}, \mathbf{Y})(f\mathbf{s})$, the terms involving first derivatives of f cancel, and the remaining second-order term is

$$\big(\mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) - [\mathbf{X}, \mathbf{Y}](f)\big)\mathbf{s} = 0.$$

These identities prove linearity over functions. Pointwise dependence and continuity are checked directly using the coefficients of the connection. Denote by U, V and s the local representations of the two fields and the section; in the following equalities all maps and their derivatives are evaluated at x. Applying (25.13) twice gives

$$\begin{aligned}(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{s})_{\varphi,\Phi} = {} & D^2s[U,V] + Ds[DV[U]] + D\Gamma[U](V)s + \Gamma(DV[U])s \\ & + \Gamma(V)Ds[U] + \Gamma(U)Ds[V] + \Gamma(U)\Gamma(V)s.\end{aligned}$$

The expression for $\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{s}$ is obtained by interchanging U and V. Subtracting them cancels the terms involving D^2s by symmetry, and the two terms in which a derivative of s is multiplied by Γ appear on both sides with the same sign. The terms containing DU or DV sum to

$$Ds[DV[U] - DU[V]] + \Gamma(DV[U] - DU[V])s = (\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{s})_{\varphi,\Phi}.$$

Subtracting this last expression leaves (25.19), now obtained for arbitrary fields and sections. They may therefore be replaced by constant fields with the same values without changing the result. For $u, v \in X$ and $w \in F$, the formula gives the bound

$$\|\Omega(x)(u,v)w\|_F \le \big(2\|D\Gamma(x)\| + 2\|\Gamma(x)\|^2\big)\|u\|_X\|v\|_X\|w\|_F.$$

The norms of $\Gamma(x)$ and $D\Gamma(x)$ are those of $\mathcal{L}(X, \mathcal{L}(F))$ and $\mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(F)))$, respectively. This proves trilinear continuity; smoothness in x follows from the smoothness of Γ and continuous bilinear composition of operators. Finally, since $\mathbf{R}^{\nabla}(\mathbf{X}, \mathbf{Y})\mathbf{s}$ is a section and is tensorial in its three arguments, its local representations satisfy (25.20); equivalently, the identity follows by substituting (25.14) and canceling the terms containing derivatives of g and χ. □

The definition and proof of tensoriality correspond to Definition 2.10.1 and Proposition 2.10.2. The algebraic cancellations are exactly the same. What must additionally be checked in the Banach setting is that $D\Gamma(x)[u](v)$ and the operator products in (25.19) depend smoothly on x in the operator norm; this follows from the smoothness of Γ and the continuous bilinearity of evaluation and composition in $\mathcal{L}(F)$.

26

Hilbert geometry and Finsler structures

The existence of a Banach tangent space does not by itself provide a notion of length. It is useful to distinguish three structures that are often confused: a strong Riemannian metric on a Hilbert manifold, a weak Riemannian metric, and a Banach–Finsler structure. The first allows classical Riemannian geometry to be transported locally through the Riesz isomorphism; the second may lack even gradients, a Levi–Civita connection, or a distance that separates points; the third retains the local metric control needed for completeness and deformation flows. At the end we distinguish this theory from classical Finsler geometry, whose vertical convexity leads to the geodesic spray and the Chern connection. Manifolds are understood according to the convention in Definition 24.1.1. In Riemannian geometry and classical Finsler geometry we work with smooth differentiable structures; the lower regularity needed for variational applications will be specified in the Banach–Finsler section.

What remains of Riemannian geometry in infinite dimensions

In finite dimensions, every Riemannian metric identifies vectors and covectors and produces a distance locally equivalent to the coordinate distance. On a Hilbert manifold, that identification may fail if the metric is weak. This difference, studied in modern infinite-dimensional geometry, explains why some gradients do not exist, why the distance can degenerate, and why the Hopf–Rinow theorem loses its usual equivalences. Strong and Banach–Finsler structures recover part of the necessary control, but each conclusion must be proved under its own hypotheses.

26.1 Hilbert manifolds and strong or weak metrics

Definition 26.1.1. A **Hilbert manifold** is a Banach manifold whose model space is a real Hilbert space H. Let M be a Hilbert manifold modeled on H. A **smooth weak Riemannian metric** on M is a family $\mathbf{g} = (\mathbf{g}_x)_{x\in M}$ of continuous positive definite inner products on T_xM such that, in every trivialization of TM, the map

$$x \longmapsto \mathbf{g}_x \in \mathcal{L}^2_s(H;\mathbb{R})$$

is smooth. Its musical homomorphism is

$$\flat_{\mathbf{g}} \colon TM \longrightarrow T'M, \qquad \flat_{\mathbf{g}}(v) = \mathbf{g}(v, \cdot).$$

Here $T'M$ is the bundle of continuous duals from Definition 25.2.2; in coordinates its fibers are modeled on H'. The metric is **strong** if each norm $|v|_{\mathbf{g},x} = \mathbf{g}_x(v,v)^{\frac{1}{2}}$ induces the given topology of T_xM; it is **strictly weak** if it is not strong. Thus, "weak" here means "not necessarily strong." Positivity implies that $\flat_{\mathbf{g},x}$ is injective, but not that it is surjective.

Fix an inner product $\langle \cdot, \cdot \rangle_H$ on the model space of a Hilbert manifold. In a chart there is a unique positive, injective, self-adjoint operator $G(x) \in \mathcal{L}(H)$ such that

$$\mathbf{g}_x(u,v) = \langle G(x)u, v \rangle_H. \tag{26.1}$$

This is the representation in Proposition B.3.23, applied at each point. The map $x \mapsto G(x)$ is smooth. After identifying H' with H by Riesz, $\flat_{\mathbf{g},x}$ is represented by $G(x)$. In particular, its image is dense. Indeed, since $G(x)$ is self-adjoint, $(\operatorname{Ran} G(x))^{\perp} = \ker(G(x)^*) = \ker G(x) = \{0\}$ and hence

$$\overline{\operatorname{Ran} G(x)} = \big(\ker G(x)\big)^{\perp} = H.$$

The range may nevertheless be proper and nonclosed.

Proposition 26.1.2. For a weak Riemannian metric $\mathbf{g}$ on a Hilbert manifold, the following are equivalent:

(i) $\mathbf{g}$ is strong;

(ii) $\flat_{\mathbf{g},x} \colon T_xM \longrightarrow T_x'M$ is a topological isomorphism for every $x \in M$;

(iii) $\flat_{\mathbf{g}} \colon TM \longrightarrow T'M$ is a smooth vector bundle isomorphism;

(iv) about each x_0 there exist a chart and constants $0 < c \le C$ such that

$$c\|v\|_H \le |v|_{\mathbf{g},x} \le C\|v\|_H \quad \text{for every } x \text{ in a neighborhood of } x_0 \text{ and } v \in H.$$

(v) about every point there is a Hilbert chart in which the operator $G(x)$ of (26.1) belongs to $\mathrm{GL}(H)$ for every x in its domain; this property then holds in every chart.

Moreover, if a nonempty Banach manifold admits a strong metric, its model space is isomorphic to a Hilbert space.

Proof. The norm of (26.1) induces the topology of H if and only if there exist $c, C > 0$ such that

$$c^2\|v\|_H^2 \le \langle G(x)v, v \rangle_H \le C^2\|v\|_H^2.$$

The lower inequality is equivalent to invertibility of $G(x)$. Indeed, if it holds, $G(x)^{\frac{1}{2}}$ is bounded below and has closed dense image, so it is surjective. The converse follows from continuity of $G(x)^{-\frac{1}{2}}$. This proves the pointwise equivalences between (i), (ii), and (v).

Since $\mathrm{GL}(H)$ is open and inversion is smooth, continuity of G and G^{-1} gives, after shrinking the chart, uniform bounds for $\|G(x)\|$ and $\|G(x)^{-1}\|$. Specifically, if these bounds are A and B, respectively, we may take $C = A^{\frac{1}{2}}$ and $c = B^{-\frac{1}{2}}$ in (iv). The same observation shows that the fiberwise inverse $\sharp_{\mathbf{g}} := \flat_{\mathbf{g}}^{-1}$ is smooth. Thus, (iii) is equivalent to the preceding conditions.

For the last assertion, denote the model space by X, fix $x_0 \in M$, and transport $\mathbf{g}_{x_0}$ to X. Since the given norm and the inner product norm $|\cdot|_{\mathbf{g},x_0}$ induce the same topology, there exist $c_0, C_0 > 0$ such that

$$c_0\|v\|_X \le |v|_{\mathbf{g},x_0} \le C_0\|v\|_X, \qquad v \in X.$$

If (v_n) is Cauchy in $|\cdot|_{\mathbf{g},x_0}$, the left inequality makes it Cauchy in $\|\cdot\|_X$; it then converges to some $v \in X$, and the right inequality turns this convergence into convergence in $|\cdot|_{\mathbf{g},x_0}$. Consequently, X, equipped with this inner product, is a Hilbert space, and the identity between the two norms is a topological linear isomorphism. □

Example 26.1.3. (Hierarchy of Sobolev metrics) Let $\mathbf{E} \longrightarrow K$ be a real vector bundle of positive finite rank, equipped with a bundle metric and a compatible connection, over a compact Riemannian manifold without boundary K, and let $A = I + (\nabla^{\mathbf{E}})^*\nabla^{\mathbf{E}}$. Suppose that K is nonempty. For $s > 0$ and $0 \le r \le s$, on the Hilbert space $H^s(K,\mathbf{E})$ consider the constant metric

$$\mathbf{g}^{(r)}_{\mathbf{u}}(\mathbf{h},\mathbf{k}) := \left\langle A^{\frac{r}{2}}\mathbf{h}, A^{\frac{r}{2}}\mathbf{k}\right\rangle_{L^2(K,\mathbf{E})}. \tag{26.2}$$

Here we use the spectral norm $\|\mathbf{h}\|_{H^t_A(K,\mathbf{E})} := \|A^{\frac{t}{2}}\mathbf{h}\|_{L^2(K,\mathbf{E})}$. For each $t \ge 0$ there exist constants $c_t, C_t > 0$, depending only on K, the metrics, the connection, and t, such that

$$c_t\|\mathbf{h}\|_{H^t(K,\mathbf{E})} \le \|\mathbf{h}\|_{H^t_A(K,\mathbf{E})} \le C_t\|\mathbf{h}\|_{H^t(K,\mathbf{E})}, \qquad \mathbf{h} \in H^t(K,\mathbf{E}).$$

For even integer orders, these are the estimates of Corollary 11.5.1; the remaining orders follow through the spectral scale and interpolation. Specifically, for $0 < t < 2m$, with $m \in \mathbb{N}$ and parameter $\theta = t/(2m)$, interpolate between $L^2(K,\mathbf{E})$ and the domain of A^m. Proposition 12.4.11 identifies the interpolation space with the domain of $A^{t/2}$, whereas Proposition 13.3.11 identifies it with $H^t(K,\mathbf{E})$. Compactness of K allows the bounded geometry hypotheses of that result to be applied. This is also the case $p = 2$ of Theorem 14.5.6. Continuity of the embedding $H^s(K,\mathbf{E}) \hookrightarrow H^r(K,\mathbf{E})$ shows that $\mathbf{g}^{(r)}$ is a smooth weak metric. For $r = s$ it is strong, since its norm is one of the norms defining $H^s(K,\mathbf{E})$.

Suppose that K has positive dimension. If $A\mathbf{e}_j = \lambda_j\mathbf{e}_j$, with $(\mathbf{e}_j)_{j\in\mathbb{N}}$ orthonormal in $L^2(K,\mathbf{E})$ and $\lambda_j \to \infty$ as $j \to \infty$, then

$$\mathbf{h}_j := \lambda_j^{-\frac{s}{2}}\mathbf{e}_j, \qquad \|\mathbf{h}_j\|_{H^s_A(K,\mathbf{E})} = 1, \qquad |\mathbf{h}_j|_{\mathbf{g}^{(r)}} = \lambda_j^{\frac{r-s}{2}}.$$

Consequently, $\mathbf{g}^{(r)}$ is strictly weak for $r < s$. With respect to the inner product of $H^s(K,\mathbf{E})$, its metric operator is A^{r-s}, which is injective and has dense image, but is not surjective. In particular, there are continuous covectors on $H^s(K,\mathbf{E})$ that have no gradient with respect to $\mathbf{g}^{(r)}$.

Example 26.1.4. (The L^2 metric) The case $r = 0$ of (26.2) is

$$\mathbf{g}^{(0)}_{\mathbf{u}}(\mathbf{h},\mathbf{k}) = \int_K \langle \mathbf{h},\mathbf{k}\rangle_{\mathbf{E}}\, d\lambda_{\mathbf{g}}.$$

Its musical map is the inclusion

$$H^s(K,\mathbf{E}) \longrightarrow H^{-s}(K,\mathbf{E}), \qquad \mathbf{h} \longmapsto \big[\mathbf{k} \longmapsto (\mathbf{h},\mathbf{k})_{L^2(K,\mathbf{E})}\big].$$

If $\dim K > 0$, the image is proper, as checked in the preceding example for $r = 0 < s$. Thus, using an L^2 integral on a configuration space of regularity $H^s(K,\mathbf{E})$ does not automatically make the metric strong. In contrast, the $\mathbf{g}^{(s)}$ metric does identify $H^s(\mathbf{E})$ with its dual through Riesz.

The obstruction from the musical map is independent of the notion of smoothness chosen for function spaces. In particular, passing to convenient calculus on Fréchet spaces, developed in [88], does not make a musical map surjective if it was not already. Convenient theory and strong Hilbert theory solve different problems. A modern comparison of categories of infinite-dimensional calculus and weak Riemannian geometry can be found in [129, Chap. 4 and Appendices C and E]. In this book, however, all maps between open subsets of Banach spaces are interpreted within the Fréchet calculus established in Chapter 23; convenient calculus is not adopted as a premise.

26.2 Connections, geodesics, and variation in the strong case

Let $\mathbf{g}$ be strong, and write $Dg(x)[u](v,w)$ for the derivative of its local representative g. For $x \in U$ and $u,v \in H$, define the covector

$$\mathscr{K}_x(u,v)(w) := \frac{1}{2}\big(Dg(x)[u](v,w) + Dg(x)[v](u,w) - Dg(x)[w](u,v)\big). \tag{26.3}$$

With the multilinear operator norm we have

$$\|\mathscr{K}_x(u,v)\|_{H'} \le \frac{3}{2}\|Dg(x)\|_{\mathcal{L}(H;\mathcal{L}^2_s(H;\mathbb{R}))}\|u\|_H\|v\|_H. \tag{26.4}$$

Indeed, each of the three summands in (26.3) is bounded by the product appearing on the right before the factor $\frac{1}{2}$. Continuous regrouping of multilinear operators then shows that $x \mapsto \mathscr{K}_x \in \mathcal{L}^2_s(H;H')$ is smooth. In particular, the map $(x,u,v) \mapsto \mathscr{K}_x(u,v)$ is smooth, bilinear, and symmetric in (u,v). Since $\flat_{\mathbf{g},x}$ is surjective, there exists a unique vector $\Gamma_x(u,v) = \sharp_{\mathbf{g},x}\mathscr{K}_x(u,v)$ such that

$$2g_x\big(\Gamma_x(u,v),w\big) = Dg(x)[u](v,w) + Dg(x)[v](u,w) - Dg(x)[w](u,v). \tag{26.5}$$

Definition 2.4.8 and Theorem 2.4.10 are the finite-dimensional model for the following construction. Its local expression and coordinate change law appear in Corollary 2.4.11 and Lemma 2.4.12. The Koszul identity and the calculation of chart changes are the same; the new step is that its right-hand side is initially an element of H', and only the strength of the metric allows $\sharp_{\mathbf{g}}$ to be applied to it in the operator norm.

Theorem 26.2.1. (Strong Levi–Civita theorem) Every smooth strong Riemannian metric on a Hilbert manifold has a unique torsion-free connection ∇ compatible with $\mathbf{g}$. If X, Y are the local representatives of the fields $\mathbf{X},\mathbf{Y}$, then

$$(\nabla_{\mathbf{X}}\mathbf{Y})_\varphi(x) = DY(x)X(x) + \Gamma_x(X(x),Y(x)),$$

where Γ is determined by (26.5). Equivalently, for smooth fields $\mathbf{X},\mathbf{Y},\mathbf{Z}$ it satisfies the Koszul formula

$$2\mathbf{g}(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) = \mathbf{X}\mathbf{g}(\mathbf{Y},\mathbf{Z}) + \mathbf{Y}\mathbf{g}(\mathbf{X},\mathbf{Z}) - \mathbf{Z}\mathbf{g}(\mathbf{X},\mathbf{Y}) + \mathbf{g}([\mathbf{X},\mathbf{Y}],\mathbf{Z}) - \mathbf{g}([\mathbf{Y},\mathbf{Z}],\mathbf{X}) - \mathbf{g}([\mathbf{X},\mathbf{Z}],\mathbf{Y}).$$

If τ is a coordinate change between open subsets of Hilbert spaces, the corresponding symbols satisfy the full law

$$\widetilde{\Gamma}_{\tau(x)}\big(D\tau(x)u, D\tau(x)v\big) = D\tau(x)\Gamma_x(u,v) - D^2\tau(x)(u,v). \tag{26.6}$$

Equivalently, if $\sigma = \tau^{-1}$ and $y = \tau(x)$,

$$\widetilde{\Gamma}_y(a,b) = D\tau(\sigma(y))\Gamma_{\sigma(y)}\big(D\sigma(y)a, D\sigma(y)b\big) + D\tau(\sigma(y))D^2\sigma(y)(a,b).$$

Proof. In a chart, the musical map depends smoothly on the point and takes values in the open set of linear isomorphisms $H \longrightarrow H'$. Since inversion is smooth on that open set, $x \mapsto \sharp_{\mathbf{g},x} \in \mathcal{L}(H';H)$ is smooth. The preceding construction can be written as

$$\Gamma_x = \sharp_{\mathbf{g},x} \circ \mathscr{K}_x.$$

Continuous composition of operators, together with (26.4), shows that $x \mapsto \Gamma_x \in \mathcal{L}^2_s(H;H)$ is smooth. The local formula explicitly satisfies, for $f, h \in C^\infty(U)$ and $a, b \in \mathbb{R}$,

$$\begin{aligned}
\nabla_{f\mathbf{X}+h\mathbf{Z}}\mathbf{Y} &= f\nabla_{\mathbf{X}}\mathbf{Y} + h\nabla_{\mathbf{Z}}\mathbf{Y},\\
\nabla_{\mathbf{X}}(a\mathbf{Y}+b\mathbf{Z}) &= a\nabla_{\mathbf{X}}\mathbf{Y} + b\nabla_{\mathbf{X}}\mathbf{Z},\\
\nabla_{\mathbf{X}}(f\mathbf{Y}) &= \mathbf{X}(f)\mathbf{Y} + f\nabla_{\mathbf{X}}\mathbf{Y},
\end{aligned}$$

since DY is linear in the direction, Γ_x is bilinear, and, if $\widehat{f} = f \circ \varphi^{-1}$, the product rule gives $D(\widehat{f}Y)[X] = D\widehat{f}[X]Y + \widehat{f}DY[X]$. These are therefore exactly the axioms of Definition 25.5.1, once invariance of the local expressions has been checked.

Now let the representatives of the fields in the second coordinates be $\widetilde{X}(y) = D\tau(\sigma(y))X(\sigma(y))$ and $\widetilde{Y}(y) = D\tau(\sigma(y))Y(\sigma(y))$. The chain rule gives

$$\begin{aligned}
D\widetilde{Y}(\tau(x))\widetilde{X}(\tau(x)) = {}& D^2\tau(x)(X(x),Y(x))\\
&+ D\tau(x)DY(x)X(x).
\end{aligned}$$

Therefore, the equality $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = D\tau(\nabla_X Y)$ between local expressions is equivalent precisely to (26.6). To verify it from the metric, fix x and set $y = \tau(x)$, $A = D\tau(x)$, and $B = D^2\tau(x)$. The metric transformation is $g_x(v,w) = \widetilde{g}_y(Av, Aw)$. Differentiating in the direction u,

$$\begin{aligned}
Dg(x)[u](v,w) = {}& D\widetilde{g}(y)[Au](Av,Aw)\\
&+ \widetilde{g}_y(B(u,v),Aw) + \widetilde{g}_y(Av,B(u,w)).
\end{aligned}$$

In the combination defining $2\mathscr{K}_x(u,v)(w)$, the terms involving $B(u,w)$ cancel those involving $B(w,u)$, and those involving $B(v,w)$ cancel those involving $B(w,v)$. The two remaining terms are equal by symmetry of B. Hence,

$$\mathscr{K}_x(u,v)(w) = \widetilde{\mathscr{K}_y}(Au,Av)(Aw) + \widetilde{g}_y(B(u,v),Aw).$$

Substituting $\mathscr{K}_x(u,v)(w) = g_x(\Gamma_x(u,v),w)$ and the analogous equality for $\widetilde{\mathscr{K}}$ gives

$$\widetilde{g}_y\big(A\Gamma_x(u,v) - \widetilde{\Gamma}_y(Au,Av) - B(u,v), Aw\big) = 0.$$

Since A is surjective and $\widetilde{g}_y$ is nondegenerate, the first argument must be zero. This is (26.6); thus the local expressions glue together and define a connection.
Symmetry of Γ_x shows, in coordinates, that

$$(\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X})_\varphi = DY[X] - DX[Y] = [\mathbf{X},\mathbf{Y}]_\varphi,$$

so the torsion vanishes. On the other hand, the product rule gives

$$(\mathbf{X}\mathbf{g}(\mathbf{Y},\mathbf{Z}))\circ\varphi^{-1} = Dg[X](Y,Z) + g(DY[X],Z) + g(Y,DZ[X]).$$

Adding the identities (26.5) for (X,Y,Z) and (X,Z,Y), the terms $Dg[Y](X,Z)$ and $Dg[Z](X,Y)$ appear once with each sign; by symmetry of g they cancel, leaving

$$Dg[X](Y,Z) = g(\Gamma(X,Y),Z) + g(Y,\Gamma(X,Z)).$$

The last two formulas prove metric compatibility. Conversely, compatibility and the absence of torsion imply the Koszul formula through the same algebraic calculation as in Theorem 2.4.10. If ∇ and $\widetilde{\nabla}$ had the same properties, the Koszul formula would give $\mathbf{g}((\nabla_{\mathbf{X}} - \widetilde{\nabla}_{\mathbf{X}})\mathbf{Y},\mathbf{Z}) = 0$ for every field $\mathbf{Z}$; nondegeneracy of $\mathbf{g}$ establishes uniqueness. □

Proposition 26.2.2. (Musical criterion in the weak case) Let $\mathbf{g}$ be a weak Riemannian metric on a Hilbert manifold. A Levi–Civita connection for $\mathbf{g}$ exists if and only if, in an atlas, the covectors in (26.3) belong to $\operatorname{Im}\flat_{\mathbf{g},x}$ and their unique preimages

$$\Gamma_x(u,v) := \flat_{\mathbf{g},x}^{-1}\mathscr{K}_x(u,v)$$

depend smoothly on (x,u,v). Equivalently, smoothness of $x \mapsto \Gamma_x$ with values in $\mathcal{L}^2_s(H;H)$ equipped with its operator norm is required. When the connection exists, it is unique.

Proof. Necessity follows by applying the Koszul formula to constant fields in a chart. We check the equivalence of the two formulations of smoothness. A family smooth in norm produces smooth joint evaluation through continuous multilinear evaluation. Conversely, if $H_0(x,u,v) := \Gamma_x(u,v)$ is smooth and bilinear in (u,v), then

$$\Gamma_x(u,v) = D_v D_u H_0(x,0,0)[u,v].$$

The Fréchet derivative on the right depends smoothly on x in the bilinear operator norm. Hence Γ has exactly the regularity required in Definition 25.5.1. The hypothesis therefore allows coefficients of a connection to be defined. Under a change of coordinates, both sides of (26.6) agree, after application of $\flat_{\mathbf{g}}$, by the chain rule. Since $\flat_{\mathbf{g}}$ is injective, this law holds and the local expressions define a connection. The proof of Theorem 26.2.1 establishes compatibility, absence of torsion, and uniqueness from the Koszul formula and injectivity of the musical map. □

The criterion explains exactly the difficulty in the weak case. The constant metric $\mathbf{g}^{(r)}$ of Example 26.1.3 has $D\mathbf{g}^{(r)} = 0$ and therefore admits the flat connection $\Gamma = 0$, although it is strictly weak when $r < s$. For a varying weak metric, however, $\mathscr{K}_x(u,v)$ may lie outside the image of the musical map. There is then no vector that can play the role of $\Gamma_x(u,v)$.

Let $\gamma\colon [a,b] \longrightarrow M$ be a C^1 curve, and let $\mathbf{V}$ be a C^1 field along γ. Apply Definition 25.6.1 to the bundle TM

and write $\nabla_t \mathbf{V} := \frac{D^{\nabla}\mathbf{V}}{dt}$. In a chart,

$$\nabla_t \mathbf{V} := \dot{V}(t) + \Gamma_{\gamma(t)}(\dot{\gamma}(t), V(t)),$$

Independence of the chart and the Leibniz rule are given by Proposition 25.6.2. Metric compatibility adds, for every field $\mathbf{W}$ along γ, the second of the rules

$$\nabla_t(f\mathbf{V}) = f'\mathbf{V} + f\nabla_t\mathbf{V}, \qquad \frac{d}{dt}\mathbf{g}(\mathbf{V},\mathbf{W}) = \mathbf{g}(\nabla_t\mathbf{V},\mathbf{W}) + \mathbf{g}(\mathbf{V},\nabla_t\mathbf{W}).$$

Theorem 25.6.3, applied to TM, gives existence and uniqueness of fields satisfying $\nabla_t\mathbf{V} = 0$, as well as the topological linear isomorphisms

$$P^{\gamma}_{s,t} := P^{\nabla}_{\gamma,s\to t}\colon T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M, \qquad P^{\gamma}_{r,t}P^{\gamma}_{s,r} = P^{\gamma}_{s,t}, \qquad (P^{\gamma}_{s,t})^{-1} = P^{\gamma}_{t,s}.$$

To identify the additional property supplied by Levi–Civita, let $\mathbf{U}$ and $\mathbf{V}$ be the parallel fields with initial values u and v at time s. Metric compatibility gives

$$\frac{d}{dt}\mathbf{g}_{\gamma(t)}(\mathbf{U}(t),\mathbf{V}(t)) = \mathbf{g}(\nabla_t\mathbf{U},\mathbf{V}) + \mathbf{g}(\mathbf{U},\nabla_t\mathbf{V}) = 0.$$

Consequently, for $s,t \in [a,b]$ and $u,v \in T_{\gamma(s)}M$, $\mathbf{g}_{\gamma(t)}(P^{\gamma}_{s,t}u, P^{\gamma}_{s,t}v) = \mathbf{g}_{\gamma(s)}(u,v)$ and parallel transport is an isometry. This is the same deduction that, in finite dimensions, completes Theorems 2.7.5 and 2.7.12 and Proposition 2.7.14; existence and invertibility in the present case instead come from the linear ODE on the Banach space proved in the preceding chapter.

If $\mathbf{T}_{\gamma} := \dot{\gamma}$ denotes the tangent field, a curve C^2 is an **affine geodesic** if $\nabla_t\mathbf{T}_{\gamma} = 0$. In coordinates, this condition is

$$\ddot{\gamma} + \Gamma_{\gamma}(\dot{\gamma},\dot{\gamma}) = 0. \tag{26.7}$$

Every affine geodesic has constant speed, since $\frac{d}{dt}|\mathbf{T}_{\gamma}|^2_{\mathbf{g}} = 2\mathbf{g}(\nabla_t\mathbf{T}_{\gamma},\mathbf{T}_{\gamma}) = 0$. This definition and its local expression are the analogues of Definition 2.8.1; the subsequent analytic novelty is solving the equation in a Banach space, not the calculation that produces it.

Definition 26.2.3. (Spray) Let M be a smooth Banach manifold. Let $h_t\colon TM \longrightarrow TM$ be the fiberwise homothety $h_t(v) = tv$. A **spray** is a smooth vector field $\mathbf{S}$ on TM satisfying

$$T\pi \circ \mathbf{S} = \mathrm{id}_{TM}, \qquad \mathbf{S}(h_t(v)) = Th_t\big(t\mathbf{S}(v)\big)$$

for every $t \in \mathbb{R}$ and $v \in TM$. In a chart it is written as

$$\mathbf{S}(x,v) = \big((x,v),(v,Q_x(v))\big), \qquad Q_x(tv) = t^2Q_x(v).$$

Smoothness at the zero section implies that $Q_x(v)$ is the diagonal of a continuous symmetric bilinear form. Indeed, for fixed v, $Q_x(tv) = t^2Q_x(v)$ implies, by differentiating twice at $t = 0$,

$$Q_x(v) = \frac{1}{2}D^2_vQ_x(0)(v,v).$$

The second derivative is symmetric, bilinear, and continuous; moreover, it depends smoothly on x in the operator norm by Fréchet calculus. A field that satisfies only the first condition is called a second-order

field.

The Levi–Civita spray is given by

$$\mathbf{S}_{\mathbf{g}}(x,v) = \big((x,v),(v,-\Gamma_x(v,v))\big). \tag{26.8}$$

Law (26.6) is exactly what ensures that these expressions glue together. A curve γ satisfies (26.7) if and only if $t \mapsto \dot\gamma(t)$ is an integral curve of $\mathbf{S}_{\mathbf{g}}$.

Proposition 26.2.4. Let $(M,\mathbf{g})$ be a Hilbert manifold with a strong Riemannian metric. For each $v \in T_xM$, there exists a unique maximal geodesic γ_v with $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. Its domain is an open interval containing 0, and its dependence on (t,v) is smooth on the maximal domain. The set

$$\mathcal{D} := \{v \in TM \mid \gamma_v \text{ is defined on } [0,1]\}$$

is open, contains the zero section, and the map

$$\exp\colon \mathcal{D} \longrightarrow M, \qquad \exp(v) := \gamma_v(1)$$

is smooth. Moreover, $d(\exp_x)_0 = \mathrm{id}_{T_xM}$; therefore, $\exp_x$ is a diffeomorphism between neighborhoods of the origin and x.

Proof. In coordinates, (26.7) is equivalent to the first-order system $(\dot x, \dot v) = (v, -\Gamma_x(v,v))$. Theorem 25.4.2 applied to the spray gives existence, uniqueness, openness of the maximal domain, and smooth dependence on the initial values. Homogeneity of the spray and uniqueness give the scaling identity

$$\gamma_v(t) = \gamma_{tv}(1) = \exp_x(tv) \tag{26.9}$$

whenever the two sides are defined. In particular,

$$d(\exp_x)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_x(tv) = \dot\gamma_v(0) = v.$$

The scaling equality is the same geometric step as in Lemma 2.8.5. Theorem 23.3.3 concludes that $\exp_x$ is a diffeomorphism between neighborhoods of 0 and x. In finite dimensions, the geodesic field, maximal existence, and exponential map appear in Proposition 2.8.2, Theorem 2.8.3, and Definition 2.8.6, with their properties in Proposition 2.8.7. The differential argument is identical; here we use the flow and inverse function theorem in Banach spaces. □

For an absolutely continuous curve whose velocity has integrable norm, we define its length; if $|\dot\gamma|_{\mathbf{g}}^2$ is also integrable, we define its energy by

$$L_{\mathbf{g}}(\gamma) = \int_a^b |\dot\gamma|_{\mathbf{g}}\, dt, \qquad E_{\mathbf{g}}(\gamma) = \frac{1}{2}\int_a^b |\dot\gamma|_{\mathbf{g}}^2\, dt.$$

The Cauchy–Schwarz inequality gives

$$L_{\mathbf{g}}(\gamma)^2 \le 2(b-a)E_{\mathbf{g}}(\gamma), \tag{26.10}$$

with equality if and only if the speed is constant almost everywhere.

Proposition 26.2.5. (First variation) Let $(M, \mathbf{g})$ be a Hilbert manifold with a strong Riemannian metric and Levi–Civita connection. Let $\alpha\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ be a smooth variation, $\mathbf{T} = \partial_t\alpha$, $\mathbf{V} = \partial_s\alpha$, and $\gamma = \alpha_0$. On the following right-hand sides, $\mathbf{T}$ and $\mathbf{V}$ are evaluated at $s = 0$. Then

$$\left.\frac{d}{ds}\right|_{s=0} E_{\mathbf{g}}(\alpha_s) = \big[\mathbf{g}(\mathbf{V},\mathbf{T})\big]_a^b - \int_a^b \mathbf{g}\big(\mathbf{V},\nabla_t\mathbf{T}\big)\,dt.$$

The same formulas hold for variations of class C^2. If $\mathbf{T}$ does not vanish along γ, we also have

$$\left.\frac{d}{ds}\right|_{s=0} L_{\mathbf{g}}(\alpha_s) = \left[\frac{\mathbf{g}(\mathbf{V},\mathbf{T})}{|\mathbf{T}|_{\mathbf{g}}}\right]_a^b - \int_a^b \mathbf{g}\left(\mathbf{V},\nabla_t\frac{\mathbf{T}}{|\mathbf{T}|_{\mathbf{g}}}\right)dt. \tag{26.11}$$

Consequently, a curve C^2 is critical for energy with fixed endpoints if and only if it is an affine geodesic. A regular curve is critical for length with fixed endpoints if and only if its arc length parametrization is an affine geodesic.

Proof. Fix $0 < \delta < \varepsilon$ and restrict the variation to the compact rectangle $[-\delta,\delta]\times[a,b]$. The function $(s,t) \mapsto \mathbf{g}_{\alpha(s,t)}(\mathbf{T}(s,t),\mathbf{T}(s,t))$ is smooth and, for a C^2 variation, is at least C^1. Differentiation under the integral sign is therefore justified by uniform continuity of its partial derivative. Metric compatibility gives

$$\left.\frac{d}{ds}\right|_{s=0} E_{\mathbf{g}}(\alpha_s) = \int_a^b \mathbf{g}(\nabla_s\mathbf{T},\mathbf{T})\,dt.$$

Absence of torsion and the equality $[\partial_s,\partial_t] = 0$ on the parameter domain imply $\nabla_s\mathbf{T} = \nabla_t\mathbf{V}$. Using metric compatibility a second time,

$$\int_a^b \mathbf{g}(\nabla_t\mathbf{V},\mathbf{T})\,dt = \int_a^b \left(\frac{d}{dt}\mathbf{g}(\mathbf{V},\mathbf{T}) - \mathbf{g}(\mathbf{V},\nabla_t\mathbf{T})\right)dt = \big[\mathbf{g}(\mathbf{V},\mathbf{T})\big]_a^b - \int_a^b \mathbf{g}(\mathbf{V},\nabla_t\mathbf{T})\,dt,$$

which is the energy formula, including its boundary terms.

If $\mathbf{T}$ does not vanish along γ, compactness of $[a,b]$ and continuity allow us to shrink ε so that $|\mathbf{T}|_{\mathbf{g}}$ remains bounded away from zero. We can again differentiate under the integral sign, and

$$\partial_s|\mathbf{T}|_{\mathbf{g}} = \frac{\mathbf{g}(\nabla_s\mathbf{T},\mathbf{T})}{|\mathbf{T}|_{\mathbf{g}}} = \mathbf{g}\left(\nabla_t\mathbf{V},\frac{\mathbf{T}}{|\mathbf{T}|_{\mathbf{g}}}\right).$$

The identity

$$\frac{d}{dt}\mathbf{g}\left(\mathbf{V},\frac{\mathbf{T}}{|\mathbf{T}|_{\mathbf{g}}}\right) = \mathbf{g}\left(\nabla_t\mathbf{V},\frac{\mathbf{T}}{|\mathbf{T}|_{\mathbf{g}}}\right) + \mathbf{g}\left(\mathbf{V},\nabla_t\frac{\mathbf{T}}{|\mathbf{T}|_{\mathbf{g}}}\right)$$

and integration over $[a,b]$ prove (26.11).

To justify the variational consequence without assuming that every field along γ comes from a global variation, fix a compact subinterval $J \subset (a,b)$ whose image is contained in a chart $\varphi\colon U \longrightarrow H$. If $\mathbf{W}$ is a smooth field along γ, with support contained in the interior of J, for small $|s|$ we define on J

$$\alpha(s,t) := \varphi^{-1}\big(\varphi(\gamma(t)) + s\,d\varphi_{\gamma(t)}\mathbf{W}(t)\big)$$

and set $\alpha(s,t) = \gamma(t)$ outside J. Compactness of the support allows the same interval of values of s to be

chosen, and vanishing of $\mathbf{W}$ in a neighborhood of the endpoints of J makes the gluing smooth. If the central curve is only C^2, take C^2 fields along it, and the same gluing produces a C^2 variation; the first variation formula remains applicable. This variation has variational field $\mathbf{W}$ and fixed endpoints.
If the first variation of energy vanishes for all these variations, fix $z \in H$ and take

$$\mathbf{W}(t) = \chi(t)(d\varphi_{\gamma(t)})^{-1} z, \qquad \chi \in C_c^\infty(\mathring{J}).$$

The formula already proved and the fundamental lemma of the calculus of variations imply

$$\mathbf{g}_{\gamma(t)}\big((d\varphi_{\gamma(t)})^{-1} z, \nabla_t \mathbf{T}(t)\big) = 0, \qquad t \in \mathring{J}.$$

Since z is arbitrary and $d\varphi_{\gamma(t)}$ is an isomorphism, nondegeneracy of $\mathbf{g}$ gives $\nabla_t \mathbf{T} = 0$. For length, the same argument applied to $\nabla_t\big(\frac{\mathbf{T}}{|\mathbf{T}|_{\mathbf{g}}}\big)$ gives $\nabla_t\big(\frac{\mathbf{T}}{|\mathbf{T}|_{\mathbf{g}}}\big) = 0$. Upon parametrizing by arc length, this last field is the tangent vector, and the identity becomes the affine geodesic equation. The converses follow directly from the variation formulas.
Metric compatibility and differentiation along curves are those of Definition 2.4.8 and Theorem 2.7.5; the differentiation and integration argument is the same in finite dimensions. The construction of variations uses a Hilbert chart and cutoff functions on the parameter interval; it therefore remains valid even when the manifold is not locally compact. □

For p, q in the same connected component, let

$$d_{\mathbf{g}}(p,q) := \inf\{L_{\mathbf{g}}(\gamma) \mid \gamma \text{ is a piecewise } C^1 \text{ curve from } p \text{ to } q\}.$$

Proposition 26.2.6. On each connected component of a strong Riemannian Hilbert manifold, $d_{\mathbf{g}}$ is a finite distance inducing the manifold topology.

Proof. Fix x_0 and a chart $\varphi\colon U \longrightarrow H$ with $\varphi(x_0) = 0$. By Proposition 26.1.2, after shrinking U there exist $c, C > 0$ such that

$$c\|w\|_H \le |d\varphi_x^{-1} w|_{\mathbf{g},x} \le C\|w\|_H$$

for every $x \in U$ and $w \in H$. Choose $r > 0$ so that $B_H(0,3r) \subseteq \varphi(U)$ and set $V := \varphi^{-1}(B_H(0, \frac{r}{2}))$. If $x, y \in V$, the segment joining their coordinates remains in $B_H(0, \frac{r}{2})$, and its lift has length at most $C\|\varphi(y) - \varphi(x)\|_H$.
Now let γ be a piecewise C^1 curve from x to y. As long as γ remains in $\varphi^{-1}(B_H(0,2r))$, the chain rule gives

$$c\|\varphi(\gamma(t_1)) - \varphi(\gamma(t_0))\|_H \le L_{\mathbf{g}}\big(\gamma \restriction_{[t_0,t_1]}\big). \tag{26.12}$$

We justify that a curve leaving this ball reaches its coordinate sphere before leaving the chart. Let t_1 be the endpoint of the initial interval during which the curve remains in the ball. For $s < t < t_1$, (26.12) bounds the difference of its coordinates by c^{-1} times the length of the segment $[s,t]$. That length tends to zero as $s, t \nearrow t_1$, because the speed is integrable. The coordinates therefore have a limit in the complete space H, of norm at most $2r$. This limit belongs to $\varphi(U)$, since $\overline{B_H(0,2r)} \subset B_H(0,3r) \subset \varphi(U)$. Continuity of φ^{-1} and γ, together with the Hausdorff property of M, identifies $\gamma(t_1)$ with the preimage of the limit. If an exit occurs, its norm must be $2r$: a smaller norm would allow the interval to be extended inside the ball.
If the curve does not leave this ball, taking its endpoints gives $L_{\mathbf{g}}(\gamma) \ge c\|\varphi(y) - \varphi(x)\|_H$. If it does, continuity

of $\|\varphi\circ\gamma\|_H$ gives a first time t_1 at which its value is $2r$. Since $\|\varphi(x)\|_H < \frac{r}{2}$, inequality (26.12) implies

$$L_{\mathbf{g}}(\gamma) \geq c\big(2r - \|\varphi(x)\|_H\big) > \frac{3cr}{2}.$$

On the other hand, $\|\varphi(y)-\varphi(x)\|_H < r$, so this last bound is greater than $c\|\varphi(y)-\varphi(x)\|_H$. Taking the infimum over all curves gives

$$c\|\varphi(y)-\varphi(x)\|_H \leq d_{\mathbf{g}}(x,y) \leq C\|\varphi(y)-\varphi(x)\|_H$$

for $x, y \in V$. To control also the points outside V, apply the same exit argument to the ball of radius $r/2$, with initial point x_0. Every curve from x_0 to a point of $M \setminus V$ has length at least $cr/2$. Consequently, $B_{d_{\mathbf{g}}}(x_0, cr/2) \subseteq V$. The local upper inequality proves that convergence in coordinates implies convergence in distance; this inclusion and the lower inequality prove the converse. They also separate x_0 from every other point. Symmetry follows by reversing curves and the triangle inequality by concatenating them. Classes of points joined by piecewise C^1 curves are open; connectedness of the component shows that there is only one and proves finiteness.

The corresponding finite-dimensional definition and metric conclusion are found in Definition 2.9.2 and Theorem 2.9.4. The coordinate segment gives the upper bound; the first exit argument gives the lower bound through local comparison of norms, without requiring compact balls. □

For a weak metric, in general one obtains only a pseudodistance: positivity of the length of every curve does not prevent the existence of curves between two distinct points with arbitrarily small lengths.

The local part of Riemannian geometry survives in the strong case; the global part does not do so automatically.

Proposition 26.2.7. (The surviving implication of Hopf–Rinow) Every metrically complete strong Riemannian manifold is geodesically complete.

Proof. Let $\gamma\colon [0,b) \longrightarrow M$ be a maximal geodesic with $b < \infty$. Its speed is constant, so $d_{\mathbf{g}}(\gamma(s),\gamma(t)) \leq |t-s|\|\dot\gamma(0)|_{\mathbf{g}}$. By completeness, $\gamma(t)$ converges to a point q as $t \nearrow b$. Choose a chart $\varphi\colon U \longrightarrow H$ about q. After shrinking it, there exist $c, C, K > 0$ such that

$$c\|w\|_H \leq |d\varphi_x^{-1} w|_{\mathbf{g},x} \leq C\|w\|_H, \qquad \|\Gamma_x\|_{\mathcal{L}^2_s(H;H)} \leq K$$

for $x \in U$ and $w \in H$. For t sufficiently close to b, the curve remains in this chart. If its constant speed is a, then $\|(\varphi\circ\gamma)'(t)\|_H \leq \frac{a}{c}$ and the geodesic equation gives

$$\|(\varphi\circ\gamma)''(t)\|_H \leq K(\frac{a}{c})^2.$$

Thus, $(\varphi\circ\gamma)'(t)$ is Cauchy as $t \nearrow b$ and converges to some $v \in H$. In the coordinate trivialization, the state converges to $(\varphi(q), v)$; therefore, $(\gamma(t), \dot\gamma(t))$ converges to $(q, (d\varphi_q)^{-1}v) \in TM$. Local existence for the spray extends the geodesic beyond b, a contradiction.

In finite dimensions, this is one of the implications in Theorem 2.9.9. Here we reuse only constant speed and local existence for the ODE; we do not use compactness of closed balls, which fails in an infinite-dimensional Hilbert space. □

Example 26.2.8. (Completeness without a minimizing geodesic) Let $(e_n)_{n\in\mathbb{N}}$ be the standard basis of ℓ^2, let $a_1 = 1$ and $a_n = 1 + 2^{-n}$ for $n \geq 2$, and consider the ellipsoid

$$\mathcal{E} := \left\{ x = (x_n) \in \ell^2 \;\middle|\; \sum_{n=1}^{\infty} \frac{x_n^2}{a_n^2} = 1 \right\}$$

with the metric induced by ℓ^2. It is a Hilbert submanifold: the function $f(x) = \sum_{n=1}^{\infty} x_n^2/a_n^2$ is a continuous quadratic form, hence smooth, and $Df(x)[x] = 2$ on the level $f = 1$. Its differential there is surjective onto $\mathbb{R}$ and has complemented kernel. The regular value theorem in Chapter 24 provides the submanifold structure. The induced metric is strong, because it uses the norm of ℓ^2 on each closed tangent subspace. The operator $A(x_n) = (a_n x_n)$ is a bounded linear isomorphism, maps the unit sphere onto $\mathcal{E}$, and satisfies

$$\|v\|_{\ell^2} \leq \|Av\|_{\ell^2} \leq (1 + 2^{-2})\|v\|_{\ell^2}.$$

If $d_{\mathbb{S}}$ is the intrinsic distance on the unit sphere and $d_{\mathcal{E}}$ that on $\mathcal{E}$, comparison of lengths gives, for x, y on the sphere,

$$d_{\mathbb{S}}(x,y) \leq d_{\mathcal{E}}(Ax, Ay) \leq \frac{5}{4} d_{\mathbb{S}}(x,y).$$

The Hilbert sphere is complete: the ambient distance satisfies $\|x - y\|_{\ell^2} \leq d_{\mathbb{S}}(x,y)$ and, if $\|x - y\|_{\ell^2} < 1$, the great-circle arc contained in $\operatorname{span}\{x,y\}$ has length $2\arcsin\left(\frac{\|x-y\|_{\ell^2}}{2}\right) \leq 2\|x-y\|_{\ell^2}$. Hence an intrinsic Cauchy sequence converges first in ℓ^2 to a point of the sphere and then in the intrinsic distance. The preceding inequalities prove that $\mathcal{E}$ is metrically complete, and Proposition 26.2.7 proves its geodesic completeness. Every piecewise C^1 curve $c\colon [0,1] \to \mathbb{S}$ joining e_1 to $-e_1$ has length at least π. To see this, set $a(t) = \langle c(t), e_1 \rangle$. Since $\langle \dot{c}(t), c(t) \rangle = 0$,

$$|a'(t)| = |\langle \dot{c}(t), e_1 - a(t)c(t) \rangle| \leq \|\dot{c}(t)\|_{\ell^2} \sqrt{1 - a(t)^2}.$$

For $0 < \delta < 1$, the function $\theta_\delta(t) = \arccos((1-\delta)a(t))$ satisfies $|\theta_\delta'(t)| \leq \|\dot{c}(t)\|_{\ell^2}$ on each segment. Integrating and letting $\delta \to 0^+$, we obtain $\pi \leq L(c)$. Moreover,

$$\|A\dot{c}(t)\|_{\ell^2}^2 - \|\dot{c}(t)\|_{\ell^2}^2 = \sum_{n=2}^{\infty} (a_n^2 - 1)|\langle \dot{c}(t), e_n \rangle|^2 \geq 0.$$

If $L(A \circ c) = L(c)$, the nonnegative difference of the speeds vanishes almost everywhere. Since $a_n > 1$ for each $n \geq 2$, each component $\langle \dot{c}(t), e_n \rangle$ with that index vanishes almost everywhere; the fundamental theorem of calculus gives $\langle c(t), e_n \rangle = 0$ for every $t \in [0,1]$. The curve would remain in $\mathbb{S} \cap \mathbb{R}e_1 = \{e_1, -e_1\}$, where there is no continuous curve between those endpoints. Hence, $L(A \circ c) > L(c)$. In contrast, the semicircles

$$c_n(t) = \cos(\pi t) e_1 + \sin(\pi t) e_n$$

are considered for $n \geq 2$ and $t \in [0,1]$, and satisfy

$$L(A \circ c_n) = \pi \int_0^1 \sqrt{\sin^2(\pi t) + a_n^2 \cos^2(\pi t)}\, dt \longrightarrow \pi.$$

The integrand converges uniformly, since $a_n \to 1$; this also follows by dominated convergence with bound

5/4. Thus $d_{\mathcal{E}}(e_1,-e_1)=\pi$, but no curve realizes that distance. This is Grossman's ellipsoid [70].

Remark 26.2.9. (Limitations of Hopf–Rinow) Proposition 26.2.7 does not restore the finite-dimensional equivalences. Metric completeness does not imply that every pair of points can be joined by a minimizing geodesic, as Example 26.2.8 shows. Nor does metric completeness force $\exp_p$ to be surjective: this is Atkin's counterexample in [12]. The converse implication from global geodesic completeness to metric completeness also fails, but this belongs to a later work of Atkin.[a] These assertions distinguish three properties: continuation of every geodesic, convergence of Cauchy sequences, and existence of minimizing curves. In infinite dimensions one lacks the local compactness of closed balls that relates them in the classical theorem.

There are important partial results. Ekeland proved that, for fixed p on a strong, connected, metrically complete manifold, the set of points that can be joined to p by a unique minimizing geodesic contains a dense G_δ [58, Thm. B]. Under global hypotheses of nonpositive curvature, conclusions of Cartan–Hadamard type can also be recovered. None of these assertions justifies using the finite-dimensional Hopf–Rinow theorem without checking its additional hypotheses.

[a] C. J. Atkin, *Geodesic and metric completeness in infinite dimensions*, Hokkaido Mathematical Journal **26** (1997), 1–61, doi:10.14492/hokmj/1351257804.

26.3 Banach–Finsler structures

Definition 26.3.1. Let M be a Banach manifold of class at least C^1, modeled on X. A **Banach–Finsler structure** is a family of norms $F_x\colon T_xM\longrightarrow[0,\infty)$ satisfying:

(i) F_x is a complete norm inducing the given topology of T_xM for every x;

(ii) $(x,v)\mapsto F_x(v)$ is continuous on TM;

(iii) for each x_0 there is a chart $\varphi\colon U\longrightarrow X$ such that, if $\|w\|_x:=F_x((d\varphi_x)^{-1}w)$, for every $\varepsilon>0$ we can shrink U so that

$$(1+\varepsilon)^{-1}\|w\|_{x_0}\le\|w\|_x\le(1+\varepsilon)\|w\|_{x_0},\qquad x\in U,\ w\in X. \tag{26.13}$$

The structure F is called **reversible** if $F_x(-v)=F_x(v)$ for every $x\in M$ and every $v\in T_xM$. This property concerns the structure F, not the Banach manifold without that structure. Here "norm" has its usual meaning, so absolute homogeneity implies this equality: all Banach–Finsler structures in this definition are reversible. Differentiability of F at the zero section is not required. Classical nonreversible Finsler metrics will be introduced later through Minkowski functionals and are not instances of this definition.

Condition (26.13), sometimes called **admissibility**, is the Palais condition. It is stronger than requiring that, for each x, there exist constants $c_x,C_x>0$ such that $c_x\|w\|_X\le F_x((d\varphi_x)^{-1}w)\le C_x\|w\|_X$ for every $w\in X$: in infinite dimensions, pointwise continuity in w does not by itself provide uniformity on the unit ball. Taking $\|\cdot\|_X=\|\cdot\|_{x_0}$ and $C=1+\varepsilon$ gives

$$C^{-1}\|w\|_X\le\|w\|_x\le C\|w\|_X,\qquad x\in U,\quad w\in X. \tag{26.14}$$

The definition does not depend on the chosen admissible chart. Indeed, if ψ is another chart and $\tau=\psi\circ\varphi^{-1}$,

then

$$\|z\|_{x,\psi} = \left\| D\tau(\varphi(x))^{-1} z \right\|_{x,\varphi}.$$

Set

$$A_x := D\tau(\varphi(x))^{-1} D\tau(\varphi(x_0)).$$

Continuity of inversion in the general linear group implies $\|A_x - I\|_{\mathcal{L}(X)} \to 0$, where the operator norm is calculated with $\|\cdot\|_{x_0,\varphi}$. Given $\eta > 0$, choose $\varepsilon, \delta > 0$ so that

$$(1+\varepsilon)(1+\delta) \le 1+\eta, \qquad (1+\varepsilon)^{-1}(1-\delta) \ge (1+\eta)^{-1}.$$

After shrinking the domain, admissibility in the chart φ holds with ε and $\|A_x - I\| \le \delta$. For $w = D\tau(\varphi(x_0))^{-1} z$ we then obtain

$$\|z\|_{x,\psi} = \|A_x w\|_{x,\varphi} \le (1+\varepsilon)(1+\delta)\|w\|_{x_0,\varphi},$$
$$\|z\|_{x,\psi} \ge (1+\varepsilon)^{-1}(1-\delta)\|w\|_{x_0,\varphi}.$$

Since $\|w\|_{x_0,\varphi} = \|z\|_{x_0,\psi}$, this is (26.13) in the chart ψ, with constant $1+\eta$.

Proposition 26.3.2. If $\mathbf{g}$ is a strong Riemannian metric on a Hilbert manifold, then $F_x(v) := |v|_{\mathbf{g},x}$ is a Banach–Finsler structure. Its length and distance agree with $L_{\mathbf{g}}$ and $d_{\mathbf{g}}$.

Proof. Only admissibility needs to be checked. In a chart, fix x_0 and conjugate $G(x)$ by $G(x_0)^{-\frac{1}{2}}$. The positive operator

$$B(x) := G(x_0)^{-\frac{1}{2}} G(x) G(x_0)^{-\frac{1}{2}}$$

converges to I in norm as $x \to x_0$. For $w \in H$, set $z = G(x_0)^{\frac{1}{2}} w$. Then $\|w\|_{x_0}^2 = \|z\|_H^2$ and $\|w\|_x^2 = \langle B(x) z, z\rangle_H$. If $\|B(x) - I\| < \delta < 1$, it follows that

$$(1-\delta)\|w\|_{x_0}^2 \le \|w\|_x^2 \le (1+\delta)\|w\|_{x_0}^2.$$

Given $\varepsilon > 0$, a choice of $\delta > 0$ satisfying $1+\delta \le (1+\varepsilon)^2$ and $1-\delta \ge (1+\varepsilon)^{-2}$ gives (26.13). Equality of lengths and distances follows from the definitions. □

Definition 26.3.3. Let (M, F) be a Banach–Finsler manifold. The **dual Finsler norm** of $\xi \in T_x'M$ is

$$F_x^*(\xi) := \sup\{|\xi(v)| \mid v \in T_xM,\ F_x(v) \le 1\}.$$

For $I \in C^1(M, \mathbb{R})$ we write $\|dI(x)\|_{F,*} := F_x^*(dI(x))$.

Proposition 26.3.4. Let (M, F) be a Banach–Finsler manifold modeled on X. In a chart satisfying (26.13), for $\lambda \in X'$ and x in a smaller domain where (26.14) holds,

$$C^{-1}\|\lambda\|_{X'} \le F_x^*(\lambda \circ d\varphi_x) \le C\|\lambda\|_{X'},$$

where the norm of X' is dual to $\|\cdot\|_X = \|\cdot\|_{x_0}$. In particular, $x \mapsto \|dI(x)\|_{F,*}$ is continuous for $I \in C^1(M)$.

Proof. From (26.14) we obtain the inclusions

$$C^{-1}B_X \subseteq B_x \subseteq CB_X, \qquad B_x := \{w \in X \mid \|w\|_x \le 1\}.$$

Here $B_X := \{w \in X \mid \|w\|_X \le 1\}$ is the closed unit ball. Taking suprema of $|\lambda|$ over these sets gives the dual estimate.

For continuity, fix x_0 and $\eta > 0$. Admissibility allows the chart to be shrunk so that the dual norms at x and x_0 differ by a factor between $(1+\eta)^{-1}$ and $1+\eta$. If $\lambda_x := dI(x) \circ (d\varphi_x)^{-1}$, then $\lambda_x \to \lambda_{x_0}$ in X' and

$$\begin{aligned}\big|\|\lambda_x\|_{x,*} - \|\lambda_{x_0}\|_{x_0,*}\big| &\le \|\lambda_x - \lambda_{x_0}\|_{x,*} \\ &\quad + \big|\|\lambda_{x_0}\|_{x,*} - \|\lambda_{x_0}\|_{x_0,*}\big|.\end{aligned}$$

The precise dual comparison is

$$(1+\eta)^{-1}\|\mu\|_{x_0,*} \le \|\mu\|_{x,*} \le (1+\eta)\|\mu\|_{x_0,*}, \qquad \mu \in X'.$$

Therefore, the first term on the preceding right-hand side does not exceed $(1+\eta)\|\lambda_x - \lambda_{x_0}\|_{x_0,*}$ and the second does not exceed $\eta\|\lambda_{x_0}\|_{x_0,*}$. First choose η and then x; both terms can be made arbitrarily small. This proves continuity. □

For a piecewise C^1 curve, set

$$L_F(\gamma) = \int_a^b F_{\gamma(t)}(\dot\gamma(t))\,dt, \qquad d_F(x,y) = \inf\{L_F(\gamma) \mid \gamma \text{ is a piecewise } C^1 \text{ curve from } x \text{ to } y\}.$$

Length is invariant under increasing piecewise C^1 reparametrizations. Symmetry of the norms implies $L_F(\gamma^{-1}) = L_F(\gamma)$.

Theorem 26.3.5. On each connected component of a Banach–Finsler manifold, d_F is a finite distance inducing the manifold topology. More precisely, for each x_0 there exist a chart $\varphi\colon U \longrightarrow X$, a neighborhood $V \subseteq U$ of x_0, and $C \ge 1$ such that, for $x, y \in V$,

$$C^{-1}\|\varphi(y) - \varphi(x)\|_X \le d_F(x,y) \le C\|\varphi(y) - \varphi(x)\|_X.$$

Proof. Translating coordinates, suppose that $\varphi(x_0) = 0$. First shrink U so that (26.14) holds there, and then choose $r > 0$ so that $B_X(0,3r) \subseteq \varphi(U)$. Let

$$V := \varphi^{-1}(B_X(0, \frac{r}{2})).$$

If $x, y \in V$, the segment $z(t) = (1-t)\varphi(x) + t\varphi(y)$ remains in $B_X(0, \frac{r}{2})$, and its lift $\sigma = \varphi^{-1} \circ z$ satisfies

$$L_F(\sigma) \le C\int_0^1 \|z'(t)\|_X\,dt = C\|\varphi(y) - \varphi(x)\|_X.$$

This is the upper bound.

Now let γ be any curve from x to y. While its image remains in $\varphi^{-1}(B_X(0,2r))$, the chain rule and (26.14)

give

$$\|\varphi(\gamma(t_1))-\varphi(\gamma(t_0))\|_X \le C L_F(\gamma\restriction_{[t_0,t_1]}). \tag{26.15}$$

If γ does not leave that ball, apply the inequality to its endpoints and obtain $L_F(\gamma)\ge C^{-1}\|\varphi(y)-\varphi(x)\|_X$. If it does, there is a first time t_1 at which the norm of its coordinate equals $2r$. Indeed, estimate (26.15) makes the coordinates a Cauchy family as the endpoint of the initial segment is approached. Completeness of X provides a limit in $\overline{B_X(0,2r)}$, which is contained in $\varphi(U)$. Continuity of the curve and the Hausdorff property of M identify that limit with the coordinate of $\gamma(t_1)$. If its norm were less than $2r$, the curve would remain in the ball a little longer. This is the exit argument of Proposition 26.2.6, based on finite length and completeness of the model. Since $\|\varphi(x)\|_X<\frac{r}{2}$, the same inequality gives

$$L_F(\gamma)\ge C^{-1}\frac{3r}{2}\ge C^{-1}\|\varphi(y)-\varphi(x)\|_X,$$

because the distance between two points of $B_X(0,\frac{r}{2})$ is less than r. Taking the infimum over γ gives the lower bound, even against curves that leave the chart.

Nonnegativity, symmetry, and the triangle inequality follow, respectively, from the definition, reversing curves, and concatenating them. To include points outside V, the same argument in the ball of radius $r/2$ proves that every curve from x_0 to $M\setminus V$ has length at least $r/(2C)$. Thus, $B_{d_F}(x_0,r/(2C))\subseteq V$. This inclusion and the lower bound inside V separate x_0 from all other points and show that a sequence converging to x_0 in distance eventually enters V and converges in coordinates. The upper bound proves the converse. Finally, the relation of accessibility by piecewise C^1 curves partitions M into open sets: from an accessible point one can reach every point of a sufficiently small convex chart. By connectedness, each connected component is a single accessibility class, so d_F is finite on it. The two local bounds show that the balls of d_F and the coordinate neighborhoods generate the same topology. □

Definition 26.3.6. A Banach–Finsler manifold is **metrically complete** if each component, equipped with the restriction of d_F, is a complete metric space.

Definition 26.3.7. ($C^{1,1}$ regularity and regularity of the norm) Let (M,F) be a Banach–Finsler manifold. We say it is of class $C^{1,1}$ when M is equipped with a $C^{1,1}$ atlas in the sense of Definition 24.1.5; regularity of coordinate representations will be checked in this atlas. This condition does not impose Lipschitz regularity on the family of norms F_x. A Banach–Finsler structure is **locally Lipschitz in the base** if, for each x_0, there exist a smaller admissible chart and $L>0$ such that

$$\big|\|w\|_x-\|w\|_y\big|\le L\|\varphi(x)-\varphi(y)\|_X\,\|w\|_{x_0},\qquad x,y\in U,\quad w\in X. \tag{26.16}$$

These two conditions are independent: the first concerns the atlas and the second the family of norms. The second is expressed in charts of the chosen $C^{1,1}$ atlas, since transformation of a tangent norm contains the differential of the chart change.

The term "Finsler manifold of class $C^{1,1}$" sometimes groups together different hypotheses. The Palais–Struwe arguments use the following ingredients:

(a) M is a paracompact Banach manifold with a $C^{1,1}$ atlas;

(b) F is a continuous admissible structure in the sense of Definition 26.3.1;

(c) the covers involved admit partitions of unity locally Lipschitz in the charts of the $C^{1,1}$ atlas.

In this construction, the partitions in the last item do not constitute an additional hypothesis on the model space. On each component, Theorem 26.3.5 and Theorem 24.8.8 allow Lemma 24.8.7 to be applied. Distances to the complements of the shrunk open sets, divided by their positive, locally finite sum, produce these partitions. The construction is developed in Theorem 31.4.4. When differentiable partitions are required, one instead uses Theorem 24.8.9, with the regularity and model space hypotheses stated there.

Continuity and admissibility of F suffice for the topology of d_F and for transporting strict inequalities from one fiber to nearby fibers. The $C^{1,1}$ atlas and locally Lipschitz partitions are the ingredients that allow local fields to be glued and an ODE with uniqueness to be obtained. The additional condition (26.16) is invoked only when the norm itself needs to vary with Lipschitz control. None of these hypotheses asserts that $v \mapsto F_x(v)^2$ has a vertical Hessian or that a Chern connection exists. This is the setting of the deformation theory of Palais and Struwe [138, §II.3.7].

Example 26.3.8. (Complete submanifolds of a Banach space) Let N be a closed split submanifold of a Banach space X. The norm induced by X on each T_xN defines a complete Banach–Finsler structure. In a local parametrization $\theta\colon V \subseteq X_0 \longrightarrow N$, continuity of $D\theta$ in the operator norm and the fact that $D\theta(y_0)$ is an isomorphism onto its image allow us to use $\|w\|_0 := \|D\theta(y_0)w\|_X$. This norm is complete because $D\theta(y_0)$ is a topological isomorphism onto the closed subspace $T_{\theta(y_0)}N$. Given $\varepsilon > 0$, shrink V so that

$$\|D\theta(y) - D\theta(y_0)\|_{0,X} < \frac{\varepsilon}{1+\varepsilon}.$$

Here $\|\cdot\|_{0,X}$ is the operator norm from the space $(X_0, \|\cdot\|_0)$ to $(X, \|\cdot\|_X)$. The triangle inequality and its reverse form then give

$$(1+\varepsilon)^{-1}\|w\|_0 \le \|D\theta(y)w\|_X \le (1+\varepsilon)\|w\|_0,$$

which is exactly admissibility of the induced norms. To prove completeness, first observe that, for every curve γ in N,

$$\|\gamma(b) - \gamma(a)\|_X \le L_F(\gamma),$$

so an intrinsic Cauchy sequence is Cauchy in X. Its limit belongs to N because N is closed. In an adapted submanifold chart, the local parametrization and its differential are bounded after shrinking the chart. The coordinate segment then gives $d_F(x,p) \le C_0\|\psi(x) - \psi(p)\|_{X_0}$ for x close to the limit p, where $C_0 > 0$ depends on the smaller chart. Therefore, convergence in X, and with it convergence in the chart, implies intrinsic convergence. Global closedness and local control of lengths play different roles in this argument.

Proposition 26.3.9. (Continuation criterion) Let (M, F) be a complete Banach–Finsler manifold, and let $\mathbf{X}$ be a field defined on all of M, continuous and locally Lipschitz in coordinates. Suppose the atlas of M is at least $C^{1,1}$. If there exists $A \ge 0$ such that $F_x(\mathbf{X}(x)) \le A$ for every x, every maximal integral curve of $\mathbf{X}$ is defined on $\mathbb{R}$. The global flow is continuous, and its time maps are homeomorphisms with inverse Φ_{-t}. If the atlas is C^{r+1} and $\mathbf{X}$ is C^r, with $r \in \mathbb{N}$, the flow and its time maps are C^r; if the atlas and field are smooth, so is the flow.

Proof. If the maximal right endpoint were $b < \infty$, then, for $s < t < b$, $d_F(\gamma(s), \gamma(t)) \le L_F(\gamma\restriction_{[s,t]}) \le A(t-s)$. The curve is Cauchy and, by completeness, converges to a point p. Since d_F induces the manifold topology, the tail of γ enters a chart about p. Remark 25.4.3, applied with initial value p at time b, produces a local

solution. To check gluing at b, write the integral equation in the final chart and let the upper endpoint tend to b. The field is continuous and the curve has a limit, so the integral equation also holds with the limiting initial value; its derivative at b is the field value at p. Local uniqueness identifies this continuation with the solution starting at p. This gives an integral curve beyond b, contradicting maximality. The left endpoint is treated in the same way, or by applying the argument to $-\mathbf{X}$.

The chart argument of Theorem 25.4.2 remains applicable: changes in a $C^{1,1}$ atlas preserve the local Lipschitz condition on fields, by the estimate in Section 25.3. The continuous dependence theorem in the appendix gives continuity of the flow. If the atlas is C^{r+1}, its tangent changes are C^r, and the same construction gives the asserted C^r dependence. Uniqueness gives $\Phi_{t+s} = \Phi_t \circ \Phi_s$ and $\Phi_t^{-1} = \Phi_{-t}$; C^r dependence on the initial values gives the last assertion. □

Remark 26.3.10. (Gradient versus pseudogradient) For a strong metric, $\operatorname{grad} I = \sharp_{\mathbf{g}}(dI)$ and $dI(\operatorname{grad} I) = |\operatorname{grad} I|_{\mathbf{g}}^2$. In a Banach–Finsler structure there is generally no canonical choice of a vector that realizes the dual norm, and in a nonreflexive Banach space the supremum may not be attained. At a point where $dI \neq 0$, deformation only requires choosing locally, by the definition of the supremum, directions v with $F(v) \leq 1$ and $dI(v) > \theta \|dI\|_{F,*}$, $0 < \theta < 1$. One may also leave a strict margin in the norm. Fixing x_0, choose $\theta < \theta_1 < 1$ and $v_0 \in T_{x_0}M$ with $F_{x_0}(v_0) \leq 1$ and $dI_{x_0}(v_0) > \theta_1 \|dI(x_0)\|_{F,*}$. Multiply v_0 by $a \in (\theta/\theta_1, 1)$. Then $F_{x_0}(av_0) < 1$ and $dI_{x_0}(av_0) > \theta \|dI(x_0)\|_{F,*}$. Extend av_0 as a field with constant components in a chart. Continuity of F, dI, and the dual norm preserves both strict inequalities in a neighborhood of x_0. In a $C^{1,1}$ atlas, this field is locally Lipschitz in the other charts as well, without having to divide it by a norm that is only continuous. To glue the directions without losing this inequality, one needs a nonnegative locally finite partition of unity of the required regularity. To obtain a locally Lipschitz pseudogradient, one uses the partitions constructed from distances in Theorem 31.4.4; their existence rests on Theorem 24.8.8 and Lemma 24.8.7. The model space need not admit differentiable bump functions. Theorem 24.8.9 concerns the different problem of obtaining differentiable partitions. None of these constructions enters the definition of F or the proof that d_F induces the topology. This is the functional-analytic reason for the pseudogradient vector fields used in variational deformations.

26.4 Classical Finsler geometry in finite dimensions

26.4.1 Minkowski functionals and vertical tensors

Definition 26.4.1. (Minkowski functional) Let V be a finite-dimensional real vector space. A **Minkowski functional** is a continuous function $F\colon V \longrightarrow [0,\infty)$, smooth on $V \setminus \{0\}$, positive away from 0, and positively homogeneous, $F(\lambda v) = \lambda F(v)$ for $\lambda > 0$, such that

$$\mathbf{g}_v(u,w) := \frac{1}{2} D^2(F^2)_v(u,w)$$

is an inner product for every $v \neq 0$. It is **reversible** if $F(-v) = F(v)$.

Definition 26.4.2. On a finite-dimensional smooth manifold M, a **classical Finsler metric** is a function

$F\colon TM \longrightarrow [0,\infty)$, continuous at the zero section and smooth on the slit bundle

$$0_M := \{0_x \in T_xM \mid x \in M\}, \qquad \widetilde{TM} := TM \setminus 0_M,$$

such that each restriction $F_x := F\restriction_{T_xM}$ is a Minkowski functional. In coordinates $(x^1,\dots,x^n,v^1,\dots,v^n)$, where $n = \dim M$, the indices i,j,k range over $\{1,\dots,n\}$. The **fundamental tensor** and **Cartan tensor** are

$$g_{ij}(x,v) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^i \partial v^j}(x,v), \qquad C_{ijk}(x,v) = \frac{1}{4}\frac{\partial^3 F^2}{\partial v^i \partial v^j \partial v^k}(x,v).$$

Intrinsically, $\mathbf{g}_v = \frac{1}{2}D_v^2(F_x^2)$ and $\mathbf{C}_v = \frac{1}{4}D_v^3(F_x^2)$, where the derivatives D_v and $d(F_x)_v$ appearing in this section are always vertical; that is, they are computed in the fixed fiber T_xM.

Proposition 26.4.3. (Euler identities and the Legendre transformation) Let (M,F) be a finite-dimensional classical Finsler manifold. For $v \neq 0$ and $u,w \in T_xM$ we have

$$d(F_x)_v(u) = \frac{\mathbf{g}_v(v,u)}{F(v)}, \qquad \mathbf{g}_v(v,v) = F(v)^2, \qquad \mathbf{C}_v(v,u,w) = 0.$$

The Legendre transformation

$$\mathcal{L}_x\colon T_xM \setminus \{0\} \longrightarrow T_x^*M \setminus \{0\}, \qquad \mathcal{L}_x(v) := d\left(\frac{1}{2}F_x^2\right)_v = \mathbf{g}_v(v,\cdot),$$

is a positively homogeneous diffeomorphism.

Proof. The function F_x^2 is homogeneous of degree two. Euler's theorem gives $d(F_x^2)_v(v) = 2F_x(v)^2$. Differentiating this identity in the direction u gives

$$D^2(F_x^2)_v(v,u) = d(F_x^2)_v(u),$$

which proves the first two formulas. Since $\mathbf{g}_{\lambda v} = \mathbf{g}_v$, its radial derivative is zero, yielding $\mathbf{C}_v(v,u,w) = 0$. The vertical derivative of $\mathcal{L}_x$ at v is the isomorphism $u \mapsto \mathbf{g}_v(u,\cdot)$, so $\mathcal{L}_x$ is a local diffeomorphism. To justify injectivity also on segments passing through the origin, fix a Euclidean norm $|\cdot|$ on T_xM and set $L = \frac{1}{2}F_x^2$. Homogeneity and compactness of the Euclidean sphere give $L(v) \le C|v|^2$ and $\|dL_v\| \le C'|v|$ for constants $C, C' > 0$. Hence L is C^1 on all of T_xM, with $dL_0 = 0$. If $u \neq v$, the function $h(t) := L(u + t(v-u))$ has a continuous derivative on $[0,1]$ and

$$h''(t) = \mathbf{g}_{u+t(v-u)}(v-u, v-u) > 0$$

except, at most, at the unique parameter where the segment passes through zero. On each side of that parameter, h' is strictly increasing, and continuity prevents a jump when crossing it. Therefore, $h'(1) > h'(0)$, that is, $\big(dL_v - dL_u\big)(v-u) > 0$. This proves strict convexity of L and injectivity of its differential. Given $\xi \in T_x^*M \setminus \{0\}$, the function ξ attains a positive maximum on the compact sphere $F_x(v) = 1$. This constraint is regular, since Euler's identity gives $d(F_x)_v(v) = F_x(v) = 1$. Theorem 23.4.2 provides $\xi = \lambda\, d(F_x)_v = \lambda\mathcal{L}_x(v)$ with $\lambda > 0$; the last equality uses $F_x(v) = 1$, and positivity follows by evaluation at v. By homogeneity, $\xi = \mathcal{L}_x(\lambda v)$; thus, $\mathcal{L}_x$ is surjective. A bijection that is a local diffeomorphism has a smooth inverse. □

Strong convexity includes the triangle inequality. Indeed, the preceding Euler identities and the quotient rule give, for $v \neq 0$,

$$D^2F_v(w,w) = \frac{\mathbf{g}_v(w,w)F(v)^2 - \mathbf{g}_v(v,w)^2}{F(v)^3} \geq 0$$

by Cauchy–Schwarz for $\mathbf{g}_v$. The restriction of F to any line is convex away from the possible parameters where it passes through the origin; homogeneity describes the only missing point: on a line tw through the origin, the left slope is $-F(-w)$ and the right slope is $F(w)$. The former is smaller than the latter, so convexity is preserved across the origin as well. Therefore, F is convex on all of V, and convexity and homogeneity imply $F(u+w) \leq F(u)+F(w)$. Thus, a reversible Minkowski functional is a norm, whereas a nonreversible one is a directed norm.

The Cartan tensor measures directional dependence of the fundamental tensor. If $\mathbf{g}_v$ does not depend on v, then $\mathbf{C}_v = 0$. Conversely, if $\dim M \geq 2$, each slit fiber $T_xM \setminus \{0\}$ is connected and $\mathbf{C} = 0$ implies that $\mathbf{g}_v$ is independent of v; therefore, F_x^2 is a quadratic form. In dimension one, the same conclusion additionally requires reversibility: without it, the two rays of a slit fiber may carry different quadratic forms. If F is reversible, it satisfies Definition 26.3.1. In finite dimensions, fix a chart φ and the norm $\|w\|_0 := F_{x_0}((d\varphi_{x_0})^{-1}w)$. On the compact sphere $\{w \in \mathbb{R}^n \mid \|w\|_0 = 1\}$, uniform continuity of $F_x((d\varphi_x)^{-1}w)$ implies that, given $\varepsilon > 0$, after shrinking the chart,

$$(1+\varepsilon)^{-1}\|w\|_0 \leq F_x((d\varphi_x)^{-1}w) \leq (1+\varepsilon)\|w\|_0$$

for every w and every x in the smaller chart. This is admissibility; the compactness used in the argument is exactly what is missing from the unit ball of an infinite-dimensional Banach model.

Example 26.4.4. (Minkowski spaces) A finite-dimensional vector space equipped with a constant Minkowski functional is a Finsler manifold. Its geodesics are straight lines, and the nonlinear and Chern connections introduced below have zero coefficients in linear coordinates. The Cartan tensor may be nonzero: vanishing of the connection expresses that the geometry is locally Minkowski, not that the functional is Euclidean.

26.4.2 Spray, nonlinear connection, and Chern connection

Write $n := \dim M \geq 1$. All Latin and Greek coordinate indices in this section range over $\{1,\dots,n\}$ unless another set is specified. The matrix $(g^{ij})_{i,j=1}^n$ is the inverse of $(g_{ij})_{i,j=1}^n$. Free indices remain fixed in each equality; summed indices will be indicated in the corresponding sum.

For $L = \frac{1}{2}F^2$, the extremals satisfy

$$\frac{d}{dt}\frac{\partial L}{\partial v^i}(\gamma,\dot\gamma) - \frac{\partial L}{\partial x^i}(\gamma,\dot\gamma) = 0. \tag{26.17}$$

Since $\frac{\partial^2 L}{\partial v^i \partial v^j} = g_{ij}$, the preceding equation becomes

$$\ddot x^i + 2G^i(x,\dot x) = 0, \tag{26.18}$$

where

$$G^i(x,v) = \frac{1}{4}\sum_{\ell=1}^n g^{i\ell}(x,v)\left(\sum_{k=1}^n \frac{\partial^2 F^2}{\partial x^k \partial v^\ell}(x,v)v^k - \frac{\partial F^2}{\partial x^\ell}(x,v)\right). \tag{26.19}$$

The coefficients G^i are smooth on $\widetilde{TM}$ and homogeneous of degree two. Under a coordinate change $y = \tau(x)$ they

satisfy

$$\widetilde{G}^{\alpha}(y, D\tau(x)v) = \sum_{i=1}^{n} \frac{\partial \tau^{\alpha}}{\partial x^{i}}(x) G^{i}(x,v) - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^{2} \tau^{\alpha}}{\partial x^{i} \partial x^{j}}(x) v^{i} v^{j}, \tag{26.20}$$

as can be checked by transforming (26.18). Hence they define the **geodesic spray on the slit tangent bundle**

$$\mathbf{S}_{F}(x,v) = \sum_{i=1}^{n} v^{i} \partial_{i} - 2 \sum_{i=1}^{n} G^{i}(x,v) \partial_{v^{i}}$$

on $\widetilde{TM}$. It is a second-order field and satisfies the homogeneity identity of Definition 26.2.3 for $\lambda > 0$:

$$\mathbf{S}_{F}(h_{\lambda}(v)) = T h_{\lambda}\big(\lambda \mathbf{S}_{F}(v)\big), \qquad v \in \widetilde{TM}, \quad \lambda > 0.$$

The restriction to positive scalars is essential when F is not reversible. In general, $\mathbf{S}_{F}$ does not extend smoothly to the zero section; hence it is not a smooth spray on all of TM in the sense of that definition, and constant curves are included separately.

Definition 26.4.5. (Canonical nonlinear connection) Let (M, F) be a finite-dimensional classical Finsler manifold. The coefficients

$$N^{i}_{j}(x,v) := \frac{\partial G^{i}}{\partial v^{j}}(x,v)$$

define the **canonical nonlinear connection**. Denote the restricted tangent projection by $\widetilde{\pi}\colon \widetilde{TM} \to M$. The vertical distribution is

$$\mathcal{V}_{(x,v)} := \ker T_{(x,v)} \widetilde{\pi} = \operatorname{span}\{\partial_{v^{i}} \mid 1 \le i \le n\},$$

and the horizontal distribution is spanned by the fields

$$\boldsymbol{\delta}_{x^{j}} := \partial_{j} - \sum_{i=1}^{n} N^{i}_{j} \partial_{v^{i}},$$

and the adapted vertical coforms are written as

$$\boldsymbol{\delta v}^{i} := \mathbf{d} v^{i} + \sum_{j=1}^{n} N^{i}_{j} \mathbf{d} x^{j}.$$

In particular, $\boldsymbol{\delta v}^{i}(\boldsymbol{\delta}_{x^{j}}) = 0$ and $\boldsymbol{\delta v}^{i}(\partial_{v^{j}}) = \delta^{i}_{j}$. If $\mathcal{H}_{(x,v)}$ denotes the subspace spanned by $\boldsymbol{\delta}_{x^{j}}$, then $T_{(x,v)}\widetilde{TM} = \mathcal{H}_{(x,v)} \oplus \mathcal{V}_{(x,v)}$.

Law (26.20), differentiated with respect to v, gives the explicit transformation

$$\sum_{\beta=1}^{n} \widetilde{N}^{\alpha}_{\beta}(y, D\tau(x)v) \frac{\partial y^{\beta}}{\partial x^{j}} = \sum_{i=1}^{n} \frac{\partial y^{\alpha}}{\partial x^{i}} N^{i}_{j}(x,v) - \sum_{k=1}^{n} \frac{\partial^{2} y^{\alpha}}{\partial x^{j} \partial x^{k}} v^{k}. \tag{26.21}$$

This identity shows that the horizontal distribution is intrinsic, although the coefficients N^{i}_{j} are not those of a linear connection on TM. Homogeneity gives $\sum_{j=1}^{n} N^{i}_{j} v^{j} = 2G^{i}$.

The local flow of $\mathbf{S}_F$ produces a maximal geodesic γ_v for each $v \neq 0$. Homogeneity gives $\gamma_v(t) = \gamma_{tv}(1)$ for $t > 0$ whenever both sides are defined. Adding the constant geodesics gives a star-shaped domain $\mathcal{D}_p \subseteq T_pM$ and the exponential map

$$\exp_p \colon \mathcal{D}_p \longrightarrow M, \qquad \exp_p(v) := \gamma_v(1).$$

It is smooth on $\mathcal{D}_p \setminus \{0\}$, of class C^1 on 0, and

$$d(\exp_p)_0 = \mathrm{id}_{T_pM}.$$

To quantify this assertion, choose coordinates $\varphi \colon U \longrightarrow \mathbb{R}^n$ with $\varphi(p) = 0$ and use $d\varphi_p$ to identify T_pM with $\mathbb{R}^n$. Let $\widehat{\exp}_p := \varphi \circ \exp_p \circ (d\varphi_p)^{-1}$. Take $\rho > 0$ with $\overline{B(0,\rho)} \subseteq \varphi(U)$. Compactness of $\overline{B(0,\rho)} \times S^{n-1}$ provides numbers $M_0, M_x, M_v \geq 0$ bounding, respectively,

$$|G(x,u)|, \qquad \|D_xG(x,u)\|, \qquad \|D_vG(x,u)\|, \qquad |u| = 1.$$

By homogeneity, for $w \neq 0$ these bounds become

$$|G(x,w)| \leq M_0|w|^2, \quad \|D_xG(x,w)\| \leq M_x|w|^2, \quad \|D_vG(x,w)\| \leq M_v|w|.$$

Choose $0 < r \leq 1$ so that $2r < \rho$ and $8M_0r \leq \frac{1}{2}$. If $0 < |v| < r$, the solution x_v of $x_v'' = -2G(x_v, x_v')$, $x_v(0) = 0$, $x_v'(0) = v$ satisfies $|x_v'(t)| \leq 2|v|$ for $0 \leq t \leq 1$. Indeed, while this bound holds,

$$|x_v'(t) - v| \leq 2M_0 \int_0^t |x_v'(s)|^2 ds \leq 8M_0|v|^2 \leq \frac{1}{2}|v|,$$

which improves the initial bound and also gives $|x_v'(t)| \geq \frac{|v|}{2}$; moreover, $|x_v(t)| \leq 2|v| < \rho$. For fixed v, the state (x_v, x_v') thus remains in a compact set contained in the domain of the spray on the slit tangent bundle. Theorem B.13.4 rules out a maximal endpoint before 1, and the argument reaches all of $[0,1]$. Integrating once more gives

$$|\widehat{\exp}_p(v) - v| \leq C_0|v|^2, \qquad C_0 := 8M_0. \tag{26.22}$$

For the derivative with respect to the initial value, set $J_v(t) := D_v x_v(t)$. The variational equation is

$$J_v'' = -2D_xG(x_v, x_v')J_v - 2D_vG(x_v, x_v')J_v', \qquad J_v(0) = 0, \quad J_v'(0) = I. \tag{26.23}$$

The preceding bounds give $\|D_xG(x_v, x_v')\| \leq 4M_x|v|^2$ and $\|D_vG(x_v, x_v')\| \leq 2M_v|v|$. Applying Grönwall's lemma 1.8.19 to the first-order system for (J_v, J_v') and comparing it with (tI, I) gives

$$\|D\widehat{\exp}_p(v) - I\| \leq C_1|v|, \qquad C_1 := (8M_xr + 4M_v)\exp(1 + 8M_xr^2 + 4M_vr). \tag{26.24}$$

In detail, set $E_0 := \exp(1 + 8M_xr^2 + 4M_vr)$. Grönwall's lemma 1.8.19 applied to the sum norm of the system gives $\|J_v(t)\| + \|J_v'(t)\| \leq E_0$ for $t \in [0,1]$. Integrating (26.23), we obtain

$$\|J_v'(t) - I\| \leq \int_0^t \left(8M_x|v|^2\|J_v(s)\| + 4M_v|v|\|J_v'(s)\|\right) ds \leq (8M_xr + 4M_v)E_0|v|t.$$

Integrating again and using $J_v(0) = 0$ yields

$$\|J_v(t) - tI\| \le \frac{1}{2}(8M_x r + 4M_v)E_0|v|t^2.$$

At $t = 1$ this bound implies (26.24) with the indicated value of C_1. Estimates (26.22) and (26.24) extend the differential continuously to the origin with value I and prove that $d(\exp_p)_0 = \mathrm{id}_{T_pM}$. In general, $\exp_p$ is not C^2 at 0, reflecting that F^2 and the spray on the slit tangent bundle need not be smooth at the zero section. The constants C_0, C_1 are independent of the direction v and depend only on r and the bounds M_0, M_x, M_v on the chosen coordinate neighborhood and unit sphere. The inverse function theorem, applied to this C^1 map, also shows that $\exp_p$ restricts to a C^1 diffeomorphism between neighborhoods of 0 and p; away from 0 both the map and its local inverse are smooth.

On the slit tangent bundle, consider the **pullback bundle**

$$\widetilde{\pi}^*TM = \{((x,v),w) \mid v \in T_xM \setminus \{0\},\ w \in T_xM\}.$$

The fundamental tensor is a metric on this bundle: on the fiber over (x,v) we use $\mathbf{g}_v$. The tautological section is $\boldsymbol{\mathcal{Y}}(x,v) = v$.

Theorem 26.4.6. (Chern connection) Let (M,F) be a finite-dimensional classical Finsler manifold. There exists a unique linear connection ∇^{Ch} on $\widetilde{\pi}^*TM$ whose connection forms $\omega^i_j = \sum_{k=1}^{n} \Gamma^i_{jk}(x,v)\,\mathbf{d}x^k$ satisfy

$$\sum_{j=1}^{n} \mathbf{d}x^j \wedge \omega^i_j = 0, \tag{26.25}$$

$$\mathbf{d}g_{ij} - \sum_{k=1}^{n}\left(g_{kj}\omega^k_i + g_{ik}\omega^k_j\right) = 2\sum_{k=1}^{n} C_{ijk}\,\boldsymbol{\delta v}^k. \tag{26.26}$$

Its coefficients are given by

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{\ell=1}^{n} g^{i\ell}\left(\boldsymbol{\delta}_{x^j} g_{\ell k} + \boldsymbol{\delta}_{x^k} g_{j\ell} - \boldsymbol{\delta}_{x^\ell} g_{jk}\right), \qquad \boldsymbol{\delta}_{x^j} = \partial_j - \sum_{m=1}^{n} N^m_j \partial_{v^m}. \tag{26.27}$$

If $y = \tau(x)$ is a coordinate change, its full transformation law is

$$\sum_{\beta,\gamma=1}^{n} \widetilde{\Gamma}^{\alpha}_{\beta\gamma}(y, D\tau(x)v)\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j} = \sum_{k=1}^{n}\frac{\partial y^\alpha}{\partial x^k}\Gamma^k_{ij}(x,v) - \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}. \tag{26.28}$$

Moreover,

$$N^i_j = \sum_{k=1}^{n}\Gamma^i_{jk}v^k, \qquad 2G^i = \sum_{j,k=1}^{n}\Gamma^i_{jk}v^j v^k. \tag{26.29}$$

If F comes from a Riemannian metric, $\mathbf{C} = 0$ and ∇^{Ch} is the lift of the Levi–Civita connection.

Proof. The first condition is equivalent to $\Gamma^i_{jk} = \Gamma^i_{kj}$. Since $\frac{\partial g_{ij}}{\partial v^k} = 2C_{ijk}$, the vertical components of (26.26)

hold automatically. Its horizontal components are

$$\delta_{x^k} g_{ij} = \sum_{m=1}^{n} \bigl(g_{mj}\Gamma^m_{ik} + g_{im}\Gamma^m_{jk} \bigr).$$

Fix $i, j, k \in \{1, \dots, n\}$. The three required identities are

$$\begin{aligned}
\delta_{x^j} g_{ik} &= \sum_{m=1}^{n} \bigl(g_{mk}\Gamma^m_{ij} + g_{im}\Gamma^m_{kj} \bigr), \\
\delta_{x^k} g_{ij} &= \sum_{m=1}^{n} \bigl(g_{mj}\Gamma^m_{ik} + g_{im}\Gamma^m_{jk} \bigr), \\
\delta_{x^i} g_{jk} &= \sum_{m=1}^{n} \bigl(g_{mk}\Gamma^m_{ji} + g_{jm}\Gamma^m_{ki} \bigr).
\end{aligned}$$

Adding the first two and subtracting the third, symmetry in the lower indices cancels the terms involving g_{mk} and g_{mj}. What remains is $2\sum_{m=1}^{n} g_{im}\Gamma^m_{jk}$. Multiplication by the inverse matrix of g gives (26.27). This proves local uniqueness. Conversely, the formula is symmetric in j, k and, on substituting it into the horizontal identity, the two pairs of terms with opposite signs cancel, leaving $\delta_{x^k} g_{ij}$. The vertical components have already been checked; thus the local connection also exists.

To justify gluing, consider on $\widetilde{\pi}^{*}TM$ the fiber-valued form that sends a tangent vector Z to $T\widetilde{\pi}(Z)$. Its components are $\mathbf{d}x^i$. The first equation expresses that its exterior covariant derivative is zero, so it is invariant under coordinate changes. For the second, the vertical projection determined by the horizontal distribution has components δv^i. Law (26.21) shows that these components change by the Jacobian of the coordinate change. The tensor $\mathbf{C}$, obtained by three derivatives in a fiber where the change is linear, is covariant of order three. Hence the right-hand side of (26.26) and the covariant derivative of $\mathbf{g}$ are tensorial forms of the same type. Both conditions are preserved under coordinate changes. The change of basis in the fiber depends only on x, so it introduces no $\mathbf{d}v$ components into the connection forms. Local uniqueness gives the gluing. Finally, differentiating the change of components of a field along a curve produces $D^2\tau(\dot\gamma, X)$; moving it to the other side gives the negative sign in (26.28). This is the same transformation rule already proved in (26.6).

We prove the contraction with the tautological section $\mathcal{Y}(x, v) = v$. Write $L = \tfrac{1}{2}F^2$. The Euler identities give

$$L_{v^i} = \sum_{k=1}^{n} g_{ik} v^k, \qquad \frac{\partial g_{ik}}{\partial v^m} = 2C_{ikm}, \qquad \sum_{k=1}^{n} C_{ikm} v^k = 0.$$

The Euler–Lagrange equation, written using the spray, is

$$2\sum_{\ell=1}^{n} g_{i\ell} G^\ell = \sum_{k=1}^{n} L_{x^k v^i} v^k - L_{x^i}. \tag{26.30}$$

Differentiating it with respect to v^j gives

$$2\sum_{\ell=1}^{n} g_{i\ell} N^\ell_j = L_{x^j v^i} - L_{x^i v^j} + \sum_{k=1}^{n} \frac{\partial g_{ij}}{\partial x^k} v^k - 4\sum_{m=1}^{n} C_{ijm} G^m. \tag{26.31}$$

We abbreviate $j \in \{1,\dots,n\}$ by $\delta_j := \boldsymbol{\delta}_{x^j}$. On the other hand, contracting (26.27) in the index k with $\boldsymbol{\mathcal{Y}}^k = v^k$ gives

$$2\sum_{\ell,k=1}^{n} g_{i\ell}\Gamma^{\ell}_{jk}v^k = \sum_{k=1}^{n}\big(\delta_j g_{ik} + \delta_k g_{ij} - \delta_i g_{jk}\big)v^k$$
$$= L_{x^j v^i} - L_{x^i v^j} + \sum_{k=1}^{n}\frac{\partial g_{ij}}{\partial x^k}v^k - 4\sum_{m=1}^{n} C_{ijm}G^m.$$

The second equality used $\sum_{k=1}^{n} C_{ikm}v^k = 0$ and $\sum_{k=1}^{n} N^m_k v^k = 2G^m$. Comparison with (26.31) and invertibility of (g_{ij}) prove $N^i_j = \sum_{k=1}^{n}\Gamma^i_{jk}v^k$. The second equality in (26.29) follows by contracting once more and using $\sum_{j=1}^{n} N^i_j v^j = 2G^i$. In the Riemannian case, g_{ij} does not depend on v and (26.27) is the usual Christoffel formula. □

If γ is a regular curve and $\mathbf{W}$ is a nonvanishing field along γ, the Chern connection defines

$$D_t^{\mathbf{W}}\mathbf{X} = \sum_{i=1}^{n}\left(\dot{X}^i + \sum_{j,k=1}^{n}\Gamma^i_{jk}(\gamma,\mathbf{W})\dot{\gamma}^j X^k\right)\partial_i. \tag{26.32}$$

Almost compatibility is expressed by

$$\frac{d}{dt}\mathbf{g}_{\mathbf{W}}(\mathbf{X},\mathbf{Y}) = \mathbf{g}_{\mathbf{W}}(D_t^{\mathbf{W}}\mathbf{X},\mathbf{Y}) + \mathbf{g}_{\mathbf{W}}(\mathbf{X},D_t^{\mathbf{W}}\mathbf{Y}) + 2\mathbf{C}_{\mathbf{W}}(\mathbf{X},\mathbf{Y},D_t^{\mathbf{W}}\mathbf{W}). \tag{26.33}$$

In particular, if $\mathbf{T}_\gamma := \dot{\gamma}$ is the tangent field, a regular curve is a geodesic if and only if $D_t^{\mathbf{T}_\gamma}\mathbf{T}_\gamma = 0$; by (26.29), this is precisely (26.18).

26.4.3 First variation and directed distance

For a regular curve, set

$$E_F(\gamma) := \frac{1}{2}\int_a^b F(\dot{\gamma})^2\,dt, \qquad L_F(\gamma) := \int_a^b F(\dot{\gamma})\,dt.$$

Proposition 26.4.7. (Finsler first variation) Let (M,F) be a finite-dimensional classical Finsler manifold. Let $\alpha\colon (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$ be a smooth variation through regular curves, let $\mathbf{T} = \partial_t\alpha$, $\mathbf{V} = \partial_s\alpha$, and let $\gamma = \alpha_0$. On the right-hand sides, $\mathbf{T}$ and $\mathbf{V}$ are evaluated at $s = 0$. Then

$$\frac{d}{ds}\Big|_0 \frac{1}{2}\int_a^b F(\dot{\alpha}_s)^2\,dt = \left[\sum_{i=1}^{n}\frac{\partial L}{\partial v^i}V^i\right]_a^b + \int_a^b\sum_{i=1}^{n}\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i}\right)V^i\,dt.$$

Equivalently,

$$\frac{d}{ds}\Big|_0 E_F(\alpha_s) = \big[\mathbf{g}_{\mathbf{T}}(\mathbf{V},\mathbf{T})\big]_a^b - \int_a^b \mathbf{g}_{\mathbf{T}}(\mathbf{V},D_t^{\mathbf{T}}\mathbf{T})\,dt. \tag{26.34}$$

For length we have

$$\left.\frac{d}{ds}\right|_0 L_F(\alpha_s) = \left[\frac{\mathbf{g}_{\mathbf{T}}(\mathbf{V},\mathbf{T})}{F(\mathbf{T})}\right]_a^b - \int_a^b \mathbf{g}_{\mathbf{T}}\left(\mathbf{V}, D_t^{\mathbf{T}}\frac{\mathbf{T}}{F(\mathbf{T})}\right) dt. \tag{26.35}$$

Thus, the energy extremals with fixed endpoints are geodesics with an affine parameter. The regular length extremals are curves whose arc length parametrization is a geodesic.

Proof. Fix $0 < \delta < \varepsilon$ and restrict the variation to $[-\delta,\delta]\times[a,b]$. Since it is regular and this rectangle is compact, the map $(s,t)\mapsto(\alpha(s,t),\mathbf{T}(s,t))$ has compact image in $\widetilde{TM}$; the derivatives of $L=\frac{1}{2}F^2$ are continuous and bounded on that image. Cover the central curve by finitely many charts and subdivide $[a,b]$ so that, after shrinking ε, each strip of the variation lies in one of them. Differentiation under the integral sign is therefore justified and gives on each subinterval

$$\left.\frac{d}{ds}\right|_0 E_F(\alpha_s) = \int_a^b \sum_{i=1}^n \left(\frac{\partial L}{\partial x^i}V^i + \frac{\partial L}{\partial v^i}\dot{V}^i\right) dt.$$

Integration by parts is written explicitly as

$$\int_a^b \sum_{i=1}^n \frac{\partial L}{\partial v^i}\dot{V}^i\,dt = \left[\sum_{i=1}^n \frac{\partial L}{\partial v^i}V^i\right]_a^b - \int_a^b \sum_{i=1}^n \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)V^i\,dt,$$

and proves the first formula on summing over the subintervals: the interior boundary terms cancel because $\sum_{i=1}^n L_{v^i}V^i = d_vL(\mathbf{V})$ is the intrinsic evaluation of a covector. Now $L_{v^i} = \sum_{j=1}^n g_{ij}v^j$ and (26.30) imply

$$\frac{d}{dt}L_{v^i} - L_{x^i} = \sum_{j=1}^n g_{ij}\left(\dot{x}^j + 2G^j(x,\dot{x})\right) = \sum_{j=1}^n g_{ij}(D_t^{\mathbf{T}}\mathbf{T})^j.$$

The last equality uses (26.29). Since the boundary term is $\sum_{i=1}^n L_{v^i}V^i = \mathbf{g}_{\mathbf{T}}(\mathbf{T},\mathbf{V})$, this proves (26.34).

For length, the Euler identity in Proposition 26.4.3 gives $d(F_x)_{\mathbf{T}}(\mathbf{W}) = \frac{\mathbf{g}_{\mathbf{T}}(\mathbf{T},\mathbf{W})}{F(\mathbf{T})}$. Symmetry of the lower indices of Γ^i_{jk}, together with $\partial_s\mathbf{T} = \partial_t\mathbf{V}$, produces the zero-torsion identity along the variation

$$D_s^{\mathbf{T}}\mathbf{T} = D_t^{\mathbf{T}}\mathbf{V}.$$

Applying (26.33) with reference $\mathbf{T}$ eliminates the Cartan term because one of its arguments is $\mathbf{T}$; therefore,

$$\partial_s F(\mathbf{T}) = \mathbf{g}_{\mathbf{T}}\left(D_t^{\mathbf{T}}\mathbf{V}, \frac{\mathbf{T}}{F(\mathbf{T})}\right).$$

The same almost compatibility formula, now in the direction t, gives

$$\frac{d}{dt}\mathbf{g}_{\mathbf{T}}\left(\mathbf{V}, \frac{\mathbf{T}}{F(\mathbf{T})}\right) = \mathbf{g}_{\mathbf{T}}\left(D_t^{\mathbf{T}}\mathbf{V}, \frac{\mathbf{T}}{F(\mathbf{T})}\right) + \mathbf{g}_{\mathbf{T}}\left(\mathbf{V}, D_t^{\mathbf{T}}\frac{\mathbf{T}}{F(\mathbf{T})}\right),$$

since $\mathbf{C}_{\mathbf{T}}\left(\mathbf{V}, \frac{\mathbf{T}}{F(\mathbf{T})}, D_t^{\mathbf{T}}\mathbf{T}\right) = 0$. Integrating this identity proves (26.35), with the indicated boundary terms. It remains to justify that the fundamental lemma applies to all the required local fields. Let $J\subset(a,b)$ be a

compact subinterval contained in a chart $\varphi\colon U \longrightarrow \mathbb{R}^n$ in the sense that $\gamma(J) \subseteq U$, and let $\mathbf{W}$ be a smooth field along γ supported in the interior of J. For small $|s|$,

$$\alpha(s,t) := \varphi^{-1}\Big(\varphi(\gamma(t)) + s\, d\varphi_{\gamma(t)}\mathbf{W}(t)\Big)$$

on J, extended by γ outside J, is a smooth variation. Compactness of the support allows a uniform choice of s, the gluing is smooth because $\mathbf{W}$ vanishes near the endpoints of J, and regularity of γ ensures that nearby curves remain regular after shrinking $|s|$.

For fixed endpoints the boundary terms disappear. Fixing $z \in \mathbb{R}^n$ and $\chi \in C_c^\infty(\mathring{J})$, choose $\mathbf{W}(t) = \chi(t)(d\varphi_{\gamma(t)})^{-1}z$. The energy formula and the fundamental lemma imply

$$\mathbf{g}_{\mathbf{T}}\big((d\varphi_{\gamma(t)})^{-1}z, D_t^{\mathbf{T}}\mathbf{T}\big) = 0 \qquad \text{on } \mathring{J}.$$

The arbitrariness of z and positivity of $\mathbf{g}_{\mathbf{T}}$ give $D_t^{\mathbf{T}}\mathbf{T} = 0$, which by (26.29) is (26.18). Applied to the length formula, the same argument gives $D_t^{\mathbf{T}}\left(\frac{\mathbf{T}}{F(\mathbf{T})}\right) = 0$; after arc length parametrization, this is the affine geodesic equation. The converses follow directly from the formulas.

Metric differentiation and integration by parts agree with the Levi–Civita calculation of the finite-dimensional chapter, whose connection and derivative along curves are given in Theorem 2.4.10 and Theorem 2.7.5. Here we also use the identity $\mathbf{C}_{\mathbf{T}}(\mathbf{T},\cdot,\cdot) = 0$: it eliminates the additional terms in Chern almost compatibility and allows the first variation to be expressed through covariant acceleration. □

The **forward directed distance** is

$$d_F^+(p,q) := \inf\{L_F(\gamma) \mid \gamma \text{ is a piecewise } C^1 \text{ curve from } p \text{ to } q\}. \tag{26.36}$$

It satisfies $d_F^+(p,q) > 0$ for $p \neq q$ and the triangle inequality, but is not necessarily symmetric. The reverse metric

$$\overleftarrow{F}(v) := F(-v)$$

satisfies $d_{\overleftarrow{F}}^+(p,q) = d_F^+(q,p)$. We denote this last distance by $d_F^-(p,q)$. The functions

$$d_F^{\max}(p,q) := \max\{d_F^+(p,q), d_F^-(p,q)\}, \qquad d_F^{\mathrm{sum}}(p,q) := d_F^+(p,q) + d_F^-(p,q)$$

are finite symmetric distances on each connected component. For points in different components, we adopt the value $+\infty$; the metric assertions apply componentwise.

Proposition 26.4.8. (Topology of the directed distance) Let (M,F) be a finite-dimensional classical Finsler manifold. Forward directed balls, backward directed balls, and balls of $d_F^{\max}$ and d_F^{sum} induce the topology of M.

Proof. Fix a Euclidean norm $|\cdot|$ in a chart about x_0. By compactness of the unit sphere and continuity of F, after shrinking the chart there exist $0 < m \le M$ such that

$$m|v| \le F_x(v) \le M|v|, \qquad m|v| \le F_x(-v) \le M|v|$$

for every x in the chart and every v. The first exit argument in the proof of Theorem 26.3.5, applied

separately to F and $\overleftarrow{F}$, gives a coordinate ball V such that, for $x, y \in V$,

$$m|\varphi(y) - \varphi(x)| \le d_F^+(x,y) \le M|\varphi(y) - \varphi(x)|$$

and the same inequalities with d_F^- in place of d_F^+. Finally,

$$d_F^{\max}(x,y) \le d_F^{\mathrm{sum}}(x,y) \le 2 d_F^{\max}(x,y).$$

These estimates prove all four topological assertions. □

Definition 26.4.9. (Directed and geodesic completeness) Let (M,F) be a classical Finsler manifold. A sequence $(x_n)_{n\in\mathbb{N}}$ is **forward Cauchy** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d_F^+(x_i, x_j) < \varepsilon \qquad \text{if } i, j \in \mathbb{N},\ j \ge i \ge N.$$

The manifold is **forward metrically complete** if every forward Cauchy sequence converges. Backward metric completeness is defined by replacing F by $\overleftarrow{F}$. It is **forward geodesically complete** if every maximal geodesic with nonzero initial velocity is defined for all positive times; the backward notion is defined analogously.

Convergence in the preceding definition can be expressed in the topology of M, by $d_F^+(x_n, x) \to 0$, or by $d_F^-(x, x_n) \to 0$. Indeed, the last two quantities are equal by definition, and in a smaller chart the preceding proof gives constants $m, M > 0$ such that

$$m|\varphi(x_n) - \varphi(x)| \le d_F^+(x_n, x) \le M|\varphi(x_n) - \varphi(x)|$$

when x_n and x belong to that chart. Forward completeness does not in general imply backward completeness. If F is reversible, both distances agree and ordinary metric completeness is recovered.

Example 26.4.10. (Randers metrics) If a is Riemannian and β is a 1-form with $|\beta|_a < 1$, then

$$F(v) = \sqrt{a(v,v)} + \beta(v)$$

is classical Finsler. It is reversible if and only if $\beta = 0$. The condition $|\beta|_a < 1$ is precisely pointwise positivity preventing F from vanishing in a nonzero direction. Over a noncompact base, a uniform bound requires the additional condition $\|\beta\|_{L^\infty(M,a)} < 1$. To check vertical convexity, fix a fiber and write $\alpha(v) = \sqrt{a(v,v)}$. If $v \neq 0$, then

$$\frac{1}{2} D^2(F^2)_v(w,w) = \big(d\alpha_v(w) + \beta(w)\big)^2 + \frac{F(v)}{\alpha(v)}\left(a(w,w) - \frac{a(v,w)^2}{a(v,v)}\right).$$

The second summand is nonnegative by Cauchy–Schwarz and vanishes only when w is proportional to v. In that case, if $w = tv \neq 0$, the first summand is $t^2 F(v)^2 > 0$. If they are not proportional, the second summand is positive. Thus, the Hessian is positive definite for every $v \neq 0$. Moreover, if $b(x) := |\beta_x|_a < 1$, we have $(1 - b(x))\alpha(v) \le F(v) \le (1 + b(x))\alpha(v)$; this distinguishes pointwise positivity from uniform comparison over the whole base.

Example 26.4.11. (Funk metric) On the Euclidean ball B^n, the formula

$$F_x(v) = \frac{\sqrt{(1-|x|^2)|v|^2 + \langle x, v\rangle^2} + \langle x, v\rangle}{1-|x|^2}$$

defines a nonreversible metric. It is a Randers metric: its Riemannian part and its 1-form are

$$a_x(v,v) = \frac{|v|^2}{1-|x|^2} + \frac{\langle x, v\rangle^2}{(1-|x|^2)^2}, \qquad \beta_x(v) = \frac{\langle x, v\rangle}{1-|x|^2}.$$

The inverse matrix of a_x is $(1-|x|^2)(I - xx^{\mathsf{T}})$, so $|\beta_x|^2_{a_x} = |x|^2 < 1$. The preceding example ensures strong vertical convexity. Along a radius directed toward the boundary, $F_{te_1}(e_1) = \frac{1}{1-t}$, whose integral diverges; in the opposite direction one obtains $F_{te_1}(-e_1) = \frac{1}{1+t}$. More generally, if the ray starting at x and passing through y meets the boundary at b, we have

$$d_F^+(x,y) = \log \frac{|x-b|}{|y-b|}.$$

To verify the formula, fix $x \neq y$ and that point $b \in \partial B^n$, and set $h_b(z) = 1 - \langle b, z\rangle > 0$. For $w \neq 0$, the formula for $F_z(w)$ is the positive solution of $|z + w/F_z(w)| = 1$. Taking the inner product with b gives

$$F_z(w) \geq \frac{\langle b, w\rangle}{h_b(z)} = -d(\log h_b)_z(w).$$

The inequality also holds for $w = 0$. Every curve from x to y therefore has length at least $\log(h_b(x)/h_b(y))$. On the oriented segment from x to y equality holds, because the tangent ray meets the sphere precisely at b. Since x, y, b are collinear, $h_b(x)/h_b(y) = |x-b|/|y-b|$, proving the distance formula. In particular,

$$d_F^+(0,y) = -\log(1-|y|), \qquad \overline{B}^+(0,R) = \{y \in B^n \mid |y| \leq 1 - e^{-R}\}, \qquad R \geq 0.$$

These balls are compact. A forward Cauchy sequence lies in one of them by the triangle inequality and has a subsequence converging to $y \in B^n$. To check convergence of the whole sequence, fix $\varepsilon > 0$ and an index N beyond which $d_F^+(x_i, x_j) < \varepsilon/2$ if $j \geq i \geq N$. For each $i \geq N$, take subsequence indices greater than i and pass to the limit; continuity of the distance gives $d_F^+(x_i, y) \leq \varepsilon/2$. The topology of backward balls implies $x_i \to y$. This gives forward completeness. For the reverse metric, however, the radius $t \mapsto te_1$ has length

$$\int_0^1 F_{te_1}(-e_1)\,dt = \log 2 < \infty.$$

Thus, a sequence along this radius converging to the boundary is Cauchy for the reverse distance and does not converge in B^n; the metric is not backward complete.

Theorem 26.4.12. (Finite-dimensional Finsler Hopf–Rinow theorem) Let (M, F) be a connected finite-dimensional classical Finsler manifold. The following are equivalent:

(i) (M, F) is forward metrically complete;

(ii) (M, F) is forward geodesically complete;

(iii) $\exp_p$ is defined on all of T_pM for each $p \in M$;

(iv) every forward closed ball $\overline{B}^{+}(p,r) = \{q \in M \mid d_F^{+}(p,q) \le r\}$ is compact;

(v) every forward bounded closed subset, that is, one contained in some ball $B^{+}(p,r)$, is compact.

Under these conditions, for any $p, q \in M$ there exists a directed geodesic from p to q whose length is $d_F^{+}(p,q)$. The analogous assertions hold backward, applied to $\overleftarrow{F}$. If F is reversible, the two versions agree and give the same completeness equivalences as the Riemannian Hopf–Rinow theorem; a reversible metric need not come from an inner product.

A proof can be found in [19, §6.6, Thm. 6.6.1]; it combines local existence of geodesics with compactness arguments for minimizing sequences and properties of directed balls. Finite dimensionality is essential both for compactness of indicatrices and for the local compactness used in the argument, and cannot be omitted in view of Remark 26.2.9. The result relates, in each orientation separately, metric completeness, geodesic completeness, and existence of minimizing geodesics. It does not identify forward completeness with backward completeness.

26.5 Global results and distinctions between theories

A smooth spray on all of TM determines the symmetric connection used in the following result. In coordinates we write, as in Definition 26.2.3,

$$\mathbf{S}(x,v) = \big((x,v),(v,Q_x(v))\big).$$

Smoothness at the zero section and homogeneity for all scalars imply that Q_x is quadratic. Its polarization defines

$$\Gamma_x(u,v) := -\frac{1}{2}\big(Q_x(u+v) - Q_x(u) - Q_x(v)\big), \qquad Q_x(v) = -\Gamma_x(v,v). \tag{26.37}$$

The transformation law for the spray gives (26.6) for these coefficients, without using an inner product; hence they define a symmetric linear connection on TM. Its parallel transport is that of Theorem 25.6.3. We say that the spray is **compatible** with a Banach–Finsler structure F if this transport, along each geodesic segment, preserves the norms F_x.

Theorem 26.5.1. (Neeb's Cartan–Hadamard theorem) Let M be a connected smooth Banach manifold equipped with a Banach–Finsler structure F and a smooth spray on all of TM, compatible with F in the preceding sense. Suppose it has nonpositive curvature in the following sense: for every $x \in M$ and every $v \in \mathcal{D}_x$, where $\mathcal{D}_x$ is the domain of $\exp_x$, and every $w \in T_xM$,

$$(d\exp_x)_v \colon T_xM \longrightarrow T_{\exp_x(v)}M \quad \text{is an isomorphism and} \quad F_{\exp_x(v)}\big((d\exp_x)_v w\big) \ge F_x(w).$$

Under these hypotheses, geodesic completeness is equivalent to completeness of (M, d_F). If either holds, then for every $x \in M$ the map $\exp_x \colon T_xM \longrightarrow M$ is a covering map. If M is also simply connected, $\exp_x$ is a global diffeomorphism. In this last case radial geodesics minimize distance between their endpoints.

The notions and results that follow come from Neeb [107, pp. 118–124]. Compatible tangent norms and compatible sprays are introduced in his Definitions 1.3–1.4, pp. 118–120. The implication from metric completeness to geodesic completeness is his Lemma 1.8, and the path lifting argument is his Theorem 1.9, pp. 121–123.

The covering property of the exponential map is Theorem 1.10, and equivalence of the completeness notions is Corollary 1.11, pp. 123–124; the diffeomorphism and minimization in the simply connected case appear in Corollary 1.12, p. 124. Neeb's argument constructs path lifts with length control, extends geodesics, and verifies the covering map criterion. The estimate making that argument possible is

$$F_x\big(((d\exp_x)_v)^{-1}z\big) \le F_{\exp_x(v)}(z), \qquad z \in T_{\exp_x(v)}M.$$

Thus, the length of a lift does not exceed that of its projection; in the complete domain T_xM, a lift of finite length has a limit on reaching a finite endpoint, and a local inverse allows it to be continued. This is the path continuation mechanism used in Chapter 23. The condition on $d\exp_x$ contains two independent assertions, invertibility and expansiveness, neither of which follows from continuity of F alone. In turn, a continuous Banach–Finsler structure does not by itself determine a compatible spray.

Proof. Suppose first that (M, d_F) is complete. If a maximal geodesic γ had a finite right endpoint b, compatibility of the spray would make its speed $F_{\gamma(t)}(\dot\gamma(t)) = a$ constant. Indeed, its velocity is parallel for the connection associated with the spray, and transport along the geodesic itself preserves F. Hence, $d_F(\gamma(s),\gamma(t)) \le a|t-s|$, and completeness provides a limit q as $t \nearrow b$.
In a chart about q, local comparison of norms gives $c > 0$ such that $c\|w\|_X \le F_z((d\varphi_z)^{-1}w)$. Smoothness of the connection coefficients allows the chart to be shrunk so that $\|\Gamma_z\| \le K$, with $K > 0$. If $u(t) = \varphi(\gamma(t))$, on the tail of the curve we have

$$\|u'(t)\|_X \le a/c, \qquad \|u''(t)\|_X \le K(a/c)^2.$$

Hence u' has a limit in X as $t \nearrow b$. The state (u, u') converges to a point of the spray domain; the ODE continuation criterion in the appendix extends the geodesic. This contradicts maximality. The same proof at the left endpoint gives geodesic completeness. This is the argument of Proposition 26.2.7, with constant speed and Finsler comparison in place of Riemannian compatibility.
Now suppose geodesic completeness, fix $x \in M$, and write $E := T_xM$, with the complete norm F_x, and $f := \exp_x\colon E \to M$. The differential of f is an isomorphism at every point, so the inverse function theorem makes it a local diffeomorphism. If $c\colon [0,1] \to M$ is a piecewise C^1 curve and $v_0 \in E$ satisfies $f(v_0) = c(0)$, a local inverse starts a lift v of c with $v(0) = v_0$. Two lifts with the same value agree near that time by local injectivity of f; the open-and-closed argument extends agreement to their common interval.
On every segment where the lift is defined, the chain rule and expansiveness give

$$F_x(v'(t)) = F_x\big((df_{v(t)})^{-1}c'(t)\big) \le F_{c(t)}(c'(t)).$$

If its initial domain had an endpoint $b \le 1$, then, for $s < t < b$,

$$F_x(v(t) - v(s)) \le \int_s^t F_{c(\tau)}(c'(\tau))\,d\tau.$$

The right-hand side tends to zero as $s, t \nearrow b$. Completeness of E provides a limit v_b, and continuity of f gives $f(v_b) = c(b)$. A local inverse about v_b continues the lift, or defines it at $t = 1$ if $b = 1$. Thus, every piecewise C^1 curve has a unique lift on all of $[0,1]$ with the prescribed initial value. Since connectedness of M allows $x = f(0)$ to be joined to any point by a curve of this class, f is surjective.
We verify the covering map property. Let $q \in M$ and choose a chart $\psi\colon V \to B_X(0,r)$ with $\psi(q) = 0$ whose

image is an open ball. For $y \in V$, let $c_y(t) := \psi^{-1}(t\psi(y))$. For each $e \in f^{-1}(q)$, define $s_e(y)$ to be the endpoint of the lift of c_y starting at e. Then $f \circ s_e = \mathrm{id}_V$. These maps are smooth. To check this near a point $y_0 \in V$, cover the compact lift of c_{y_0} by finitely many domains of local inverses of f. Subdivide $[0,1]$ so that each segment lifts into one of them. On the first segment use the inverse that takes q to e. If the first j segments have been constructed for y near y_0, continuity of the endpoint and openness of the next domain allow that neighborhood of y_0 to be shrunk and the next local inverse to be used. Joint continuity of $(t,y) \mapsto c_y(t)$ and compactness of each interval of the subdivision allow this choice to be made for the entire segment at once. Uniqueness makes the two expressions agree at their common time. Repeating this step finitely many times expresses the endpoint through smooth local inverses. This proves smoothness of s_e.

Each $s_e(V)$ is open. Indeed, about $s_e(y)$ take a domain O on which f is a diffeomorphism; by continuity of s_e, there is an open set $W \subseteq V$ about y with $s_e(W) \subseteq O$. There $s_e = (f\restriction_O)^{-1}$, so $s_e(W)$ is open. If $s_e(y) = s_{e'}(y)$, lifting the reverse curve of c_y from that point, uniqueness gives $e = e'$. Hence the images of two distinct sections are disjoint. Finally, given $z \in f^{-1}(V)$, lift from z the reverse curve of $c_{f(z)}$; its endpoint is some $e \in f^{-1}(q)$, and uniqueness gives $z = s_e(f(z))$. We have proved

$$f^{-1}(V) = \coprod_{e \in f^{-1}(q)} s_e(V), \qquad f\restriction_{s_e(V)} \colon s_e(V) \to V \text{ is a diffeomorphism.}$$

This is the covering map condition.

The same lifting estimate proves metric completeness. Let $(q_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in M. Choose a subsequence $(q_{j_k})_{k\in\mathbb{N}}$ such that $d_F(q_{j_k}, q_{j_{k+1}}) < 2^{-k-1}$. By the definition of distance, there exists a curve from q_{j_k} to $q_{j_{k+1}}$ of length less than 2^{-k}. Take $v_1 \in f^{-1}(q_{j_1})$ and lift these curves successively, using the endpoint of each as the initial value of the next. If v_k is the lift of q_{j_k}, then

$$F_x(v_\ell - v_k) \le \sum_{m=k}^{\ell-1} 2^{-m} \le 2^{1-k}, \qquad k,\ell \in \mathbb{N}, \quad \ell > k.$$

Thus, v_k converges in E to a point v. Continuity of f and agreement of the topology of M with that of d_F imply $q_{j_k} \to f(v)$ in distance. A Cauchy sequence with a convergent subsequence converges to the same limit, so (M, d_F) is complete.

If M is simply connected, the preceding covering has a single sheet, because E is connected. This can be seen through lifts: a path in E between two points in the same fiber projects to a contractible loop. Lifting a contraction with fixed endpoints makes the lifted endpoint vary continuously in the discrete fiber, so it remains constant. The lift of the constant loop ends where it begins; the two points of the fiber must coincide. The lift of that contraction is constructed in the evenly covered neighborhoods in the same way as for paths, subdividing the compact parameter square into finitely many rectangles. Thus, f is bijective and its local inverses form a global smooth inverse.

It remains to prove minimization. In the simply connected case, let $\Gamma = \exp_x^{-1} \circ \gamma$ be the lift of a curve from x to y; then $\Gamma(0) = 0$. Expansiveness gives

$$L_F(\gamma) \ge \int_0^1 F_x(\dot\Gamma(t))\,dt \ge F_x(\Gamma(1)).$$

The radial curve $t \mapsto \exp_x(t\Gamma(1))$ is a geodesic, and its velocity is parallel; compatibility of the spray with F shows that its length is $F_x(\Gamma(1))$. Therefore, it is minimizing. □

Remark 26.5.2. (Four levels that must be distinguished) (a) A strong Riemannian metric exists only on Hilbertizable models and provides $\sharp_{\mathbf{g}}$, Levi–Civita, and a smooth spray.

(b) A strictly weak Riemannian metric may be geometrically natural, such as L^2 on mapping spaces of positive regularity. Its musical map fails to be surjective on at least one fiber; being weak, in this chapter's convention, does not rule out being strong.

(c) A Palais–Struwe Banach–Finsler structure is a locally uniform family of norms; it requires neither a vertical Hessian nor a Chern connection, and is the appropriate structure for completeness and deformations.

(d) A classical Finsler metric requires smoothness away from zero and strong vertical convexity. The results developed here also use compactness of spheres and finite dimensionality of the vertical tensors; their Banach analogues require additional smoothness and uniform convexity hypotheses and do not follow from an arbitrary norm.

No tensor product, musical duality, or identification of a space with its dual is regarded as canonical without first fixing the structure that produces it.

27

Uniform geometry and geometric function spaces

The preceding chapters established differential calculus and the basic geometric structures on Banach manifolds. To apply them to spaces of functions and sections, an additional difficulty remains: charts, metrics, and connections must be compatible with the chosen functional regularity, and their estimates must remain controlled as the basepoint varies.

In a mapping space, a chart is constructed pointwise using a local addition or the exponential map of the target space. In spaces of metrics and connections, the structure comes from open subsets or affine spaces of tensor sections. Gauge and diffeomorphism groups also introduce nonlinear operations with different regularity: the gauge action requires one derivative of the group element, whereas diffeomorphism composition loses derivatives in one variable. These differences determine which objects form Hilbert manifolds at fixed regularity and which must be studied through a Sobolev scale or an ILH structure.

The aim of this chapter is to bring these constructions together in a uniform framework. We will identify the charts, tangent spaces, and natural actions, while specifying in each case the regularity assumptions that make the operations well defined and smooth. This precision will be essential in the next chapter: many geometric energies use an L^2 metric that is weak relative to the Sobolev topology, and their differential gradients lower regularity. Infinite-dimensional geometry must therefore retain both the geometric structure and the analytic scale in which the equations live.

Eichhorn, Amann, and control at infinity

Noncompact geometry requires distinguishing local closeness, uniform control, and global distance. Eichhorn developed uniformities for comparing geometric objects and studying their finite-distance components; Amann organized uniformly regular atlases adapted to the analysis of function spaces. Although both languages share uniform estimates, they address different questions: one describes configuration spaces, while the other allows operators and norms to be localized without the constants deteriorating as one escapes to infinity.

27.1 Uniform control of the base manifold: uniform regularity and bounded geometry

The spaces $BC^j(U,F)$ and $BUC^j(U,F)$, including the conventions for a half-box and for $j=0,\infty$, were defined in Definition 6.4.1. Normalized and shrinkable atlases, their multiplicity, and uniformly regular atlases and structures were defined in Definitions 13.2.1 and 13.2.2. For a chart $\kappa\colon U_\kappa \longrightarrow Q_\kappa$, the coordinate representation of the metric is, by definition, $g_\kappa = (\kappa^{-1})^* g$. The uniform ellipticity constants and BUC^j bounds for g_κ and g_κ^{-1} appear explicitly in (13.8), (13.9), and (13.11); none depends on the chart.

Theorem 13.2.4 and Corollary 14.4.7 prove the direction used in this chapter, without and with boundary. In the first case, the constants depend only on the dimension, a common normal radius, and bounds on curvature derivatives of the required order. In the second, one adds the collar width, the normal radius of the boundary, and bounds on the second fundamental form and its derivatives. The full comparisons and precise references for their converses are stated in Theorems 13.2.5 and 14.4.8, respectively.

Amann's localization system was constructed in Proposition 13.2.6. Without boundary, for the analysis operators $\mathcal{S}_{\mathfrak{K}}$ and synthesis operators $\mathcal{R}_{\mathfrak{K}}$ defined in Corollary 13.3.12, we have $\mathcal{R}_{\mathfrak{K}}\mathcal{S}_{\mathfrak{K}} = \mathrm{id}$ and, for each $s\in\mathbb{R}$ and $1<p<\infty$, there exist constants $c_{s,p}, C_{s,p}, D_{s,p}>0$ such that

$$c_{s,p}\|u\|_{H^{s,p}(M)} \le \|\mathcal{S}_{\mathfrak{K}}u\|_{\ell^p(I;H^{s,p}(\mathbb{R}^m))} \le C_{s,p}\|u\|_{H^{s,p}(M)}$$

, as well as $\|\mathcal{R}_{\mathfrak{K}}v\|_{H^{s,p}(M)} \le D_{s,p}\|v\|_{\ell^p(I;H^{s,p}(\mathbb{R}^m))}$. The three constants depend only on m,s,p, the multiplicity, and the uniform bounds on coordinate changes and cutoff functions through the required orders; they do not depend on the chart or on u or v. The same corollary records Triebel–Lizorkin comparisons and, when the indices allow, Besov comparisons with explicit constants.

With boundary, Corollary 14.4.12 replaces the preceding sequence space by $\mathbb{H}^{s,p}_{\partial}(\mathfrak{K})$, whose components are modeled on $\mathbb{R}^m$ for interior charts and on $\mathbb{R}^m_+$ for boundary charts. The operators $\mathcal{S}_{\partial,\mathfrak{K}}$ and $\mathcal{R}_{\partial,\mathfrak{K}}$ satisfy $\mathcal{R}_{\partial,\mathfrak{K}}\mathcal{S}_{\partial,\mathfrak{K}} = \mathrm{id}$ and estimates (14.38)– (14.39), with positive constants $c^{\partial}_{s,p}, C^{\partial}_{s,p}, D^{\partial}_{s,p}$. These depend only on m,s,p, overlap bounds, and uniform bounds on coordinate changes and cutoff functions through the required order; they do not depend on the chart or the localized element. This transfers Euclidean embeddings, multipliers, and local operators to the charts with uniform constants. This is the role of Amann's uniform regularity in this book: to provide a coordinate formulation of bounded geometry adapted to function spaces, rather than an alternative geometry or a uniformity on a configuration space.

27.2 Uniform spaces and Eichhorn uniformities

27.2.1 Uniform structures

If X is a set and $U, V \subseteq X\times X$, write

$$U^{-1} := \{(y,x) \mid (x,y)\in U\},$$
$$U\circ V := \{(x,z)\in X\times X \mid \text{there exists } y\in X \text{ with } (x,y)\in V,\ (y,z)\in U\}.$$

The diagonal of $X\times X$ is denoted by $\Delta_X := \{(x,x)\mid x\in X\}$.

Definition 27.2.1. (Uniform structure) A *uniform structure* on X is a nonempty family $\varnothing \neq \mathscr{U} \subseteq \mathcal{P}(X \times X)$ satisfying:

(i) if $U \in \mathscr{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathscr{U}$;

(ii) if $U, V \in \mathscr{U}$, then $U \cap V \in \mathscr{U}$;

(iii) $\Delta_X \subseteq U$ for every $U \in \mathscr{U}$;

(iv) if $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$;

(v) for each $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $V \circ V \subseteq U$.

The elements of $\mathscr{U}$ are called *uniform neighborhoods*; they are also known by the French name *entourages*. The pair $(X, \mathscr{U})$ is a *uniform space*. It is separated if $\bigcap_{U \in \mathscr{U}} U = \Delta_X$.

Definition 27.2.2. (Base and cofinality) A family $\mathscr{B} \subseteq \mathscr{U}$ is a *base* for the uniformity $\mathscr{U}$ if, for each $U \in \mathscr{U}$, there exists $B \in \mathscr{B}$ such that $B \subseteq U$. Two bases $\mathscr{B}_1$ and $\mathscr{B}_2$ are *cofinal* if each element of either contains an element of the other.
A nonempty family $\mathscr{B} \subseteq \mathcal{P}(X \times X)$ generates a uniformity if all its elements contain Δ_X and the following conditions hold: for $B_1, B_2 \in \mathscr{B}$, there exists $B_3 \in \mathscr{B}$ with $B_3 \subseteq B_1 \cap B_2$; for $B \in \mathscr{B}$, there exist $B', B'' \in \mathscr{B}$ such that $(B')^{-1} \subseteq B$ and $B'' \circ B'' \subseteq B$. The generated uniformity consists of the subsets of $X \times X$ containing some element of $\mathscr{B}$.

Definition 27.2.3. (Metrizable uniformity) A uniformity is *metrizable* if a finite-valued metric generates it. A base is *countable* if it has a countable cofinal subfamily.

For $x \in X$ and $U \in \mathscr{U}$, set $U[x] := \{y \in X \mid (x, y) \in U\}$. The sets $U[x]$ form a neighborhood base for a topology on X. The uniform structure contains more information than this topology: it allows points to be compared without first choosing a basepoint.

Definition 27.2.4. (Uniform continuity) Let $(X, \mathscr{U}_X)$ and $(Y, \mathscr{U}_Y)$ be uniform spaces. A map $F\colon X \longrightarrow Y$ is *uniformly continuous* if, for each $V \in \mathscr{U}_Y$, there exists $U \in \mathscr{U}_X$ such that

$$(F \times F)(U) \subseteq V.$$

Two uniform structures are equivalent if the identity is uniformly continuous in both directions.

Definition 27.2.5. (Filters and uniform completeness) A *filter* on X is a nonempty family $\mathscr{F}$ of subsets of X that does not contain $\varnothing$, is closed under finite intersections, and contains every superset of each of its elements. A filter $\mathscr{F}$ on X is *Cauchy* if, for every $U \in \mathscr{U}$, there exists $A \in \mathscr{F}$ such that $A \times A \subseteq U$. It converges to $x \in X$ if $U[x] \in \mathscr{F}$ for every $U \in \mathscr{U}$. The uniform space is *complete* if every Cauchy filter converges.

Lemma 27.2.6. (Filters and sequences for a countable base) Suppose $\mathscr{U}$ has a countable base. A sequence (x_n) is called *Cauchy* if, for each $U \in \mathscr{U}$, there exists n_0 such that $(x_m, x_n) \in U$ whenever $m, n \geq n_0$. Then the space is complete in the sense of Definition 27.2.5 if and only if every Cauchy sequence converges.

Proof. If every Cauchy filter converges, apply the assumption to the filter of tails of a Cauchy sequence; convergence of this filter is equivalent to convergence of the sequence.
Conversely, let $\mathscr{F}$ be a Cauchy filter and let $(\mathscr{B}_k)_{k\in\mathbb{N}}$ be a countable cofinal base. For each k, there exists $A_k \in \mathscr{F}$ such that $A_k \times A_k \subseteq \mathscr{B}_k$. The sets $C_k := A_1 \cap \cdots \cap A_k$ belong to $\mathscr{F}$ and are nonempty; choose $x_k \in C_k$. Given $U \in \mathscr{U}$, there exists j with $\mathscr{B}_j \subseteq U$. If $m, n \geq j$, then $x_m, x_n \in C_j \subseteq A_j$ and therefore $(x_m, x_n) \in U$. The sequence (x_k) is Cauchy and converges, by assumption, to some $x \in X$.
Fix $U \in \mathscr{U}$. There exists a symmetric entourage V such that $V \circ V \subseteq U$: intersect an entourage whose square is contained in U with its inverse. Choose j with $\mathscr{B}_j \subseteq V$ and then $n \geq j$ with $(x, x_n) \in V$. For $y \in A_j$, we have $(x_n, y) \in \mathscr{B}_j \subseteq V$, so $(x, y) \in U$. Consequently, $A_j \subseteq U[x]$ and $U[x] \in \mathscr{F}$. This holds for each U, so $\mathscr{F}$ converges to x. □

Proposition 27.2.7. (Metric uniformity) Let (X, d) be a metric space. The sets

$$U_\varepsilon := \{(x, y) \in X \times X \mid d(x, y) < \varepsilon\}, \qquad \varepsilon > 0,$$

form a base for a separated uniform structure. Its topology is the metric topology, its Cauchy sequences are the usual ones, and the uniform space is complete if and only if (X, d) is complete.

Proof. The diagonal is contained in each U_ε and $U_\varepsilon^{-1} = U_\varepsilon$. The triangle inequality gives $U_{\frac{\varepsilon}{2}} \circ U_{\frac{\varepsilon}{2}} \subseteq U_\varepsilon$. Moreover, $U_\varepsilon[x] = B_d(x, \varepsilon)$, so the induced topology is the metric topology. Finally, $\{x_j\}$ is Cauchy for the uniformity if and only if, given $\varepsilon > 0$, there exists j_0 such that $(x_i, x_j) \in U_\varepsilon$ for $i, j \geq j_0$, which is precisely the usual metric condition. Lemma 27.2.6, applied to the base $\left(U_{\frac{1}{k}}\right)_{k\geq 1}$, identifies completeness by filters with metric completeness. □

Proposition 27.2.8. (Completion in the metrizable case) Every separated metrizable uniform space $(X, \mathscr{U})$ admits a metric completion: there exist a separated, metrizable, complete uniform space $\widehat{X}$ and a uniform embedding $\iota\colon X \longrightarrow \widehat{X}$ with dense image. Here *uniform embedding* means that ι is injective and both ι and its inverse on $\iota(X)$ are uniformly continuous. The completion is unique among separated metrizable completions: if $\jmath\colon X \longrightarrow Y$ is another, there exists a unique uniform isomorphism $\Phi\colon \widehat{X} \longrightarrow Y$ such that $\Phi \circ \iota = \jmath$.

Proof. Choose a metric d generating $\mathscr{U}$. Consider Cauchy sequences in X and identify two of them, (x_j) and (y_j), when $d(x_j, y_j) \to 0$. If $[x_j]$ denotes the resulting class, the formula

$$\widehat{d}([x_j], [y_j]) := \lim_{j\to\infty} d(x_j, y_j)$$

is well defined, since

$$\left|d(x_j, y_j) - d(x_k, y_k)\right| \leq d(x_j, x_k) + d(y_j, y_k).$$

This inequality shows that the scalar sequence is Cauchy and its limit is unchanged when representatives are replaced. Positivity, symmetry, and the triangle inequality pass to the limit, and $\widehat{d}([x_j], [y_j]) = 0$ is equivalent to the relation defining the classes. Thus $\widehat{d}$ is a metric. The map

$$\iota\colon X \longrightarrow \widehat{X}, \qquad x \longmapsto [(x, x, \ldots)],$$

is isometric and hence a uniform embedding. If (x_j) represents $\xi \in \widehat{X}$, then

$$\widehat{d}(\iota(x_j), \xi) = \lim_{k\to\infty} d(x_j, x_k) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty,$$

, so $\iota(X)$ is dense.
Let (ξ_k) be a Cauchy sequence in $\widehat{X}$. Choose a subsequence (η_j) such that $\widehat{d}(\eta_j, \eta_{j+1}) < 2^{-j-2}$ and, by density, points $z_j \in X$ such that $\widehat{d}(\iota(z_j), \eta_j) < 2^{-j-2}$. The triangle inequality gives $d(z_j, z_{j+1}) < 2^{-j}$; consequently, for $q > p$,

$$d(z_p, z_q) < \sum_{j=p}^{q-1} 2^{-j},$$

, and (z_j) is Cauchy. Let $\zeta := [z_j]$. Then

$$\widehat{d}(\eta_j, \zeta) \le 2^{-j-2} + \sum_{r=j}^{\infty} 2^{-r} \longrightarrow 0.$$

The subsequence (η_j) converges to ζ and, since (ξ_k) is Cauchy, we also have $\xi_k \to \zeta$. Thus $\widehat{X}$ is complete.
For uniqueness, we use the following construction, which also explains extension to a completion. If $F\colon X \longrightarrow Y$ is uniformly continuous and (Y, e) is a complete metric space, define, for $\xi = [x_j]$,

$$\widehat{F}(\xi) := \lim_{j\to\infty} F(x_j).$$

Uniform continuity makes $(F(x_j))$ a Cauchy sequence. If $d(x_j, y_j) \to 0$, given $\varepsilon > 0$, choose $\delta > 0$ such that $d(x, y) < \delta$ implies $e(F(x), F(y)) < \frac{\varepsilon}{2}$; for large j, we obtain $e(F(x_j), F(y_j)) < \frac{\varepsilon}{2}$. Thus the limit is independent of the representative. The same criterion shows uniform continuity of $\widehat{F}$: if $\widehat{d}(\xi, \eta) < \frac{\delta}{2}$, choose representatives (x_j) and (y_j) and, for sufficiently large j, $d(x_j, y_j) < \delta$; passage to the limit gives $e(\widehat{F}(\xi), \widehat{F}(\eta)) \le \frac{\varepsilon}{2} < \varepsilon$. The extension is unique because Y is separated and $\iota(X)$ is dense.
Now let $\jmath\colon X \longrightarrow Y$ be another separated metrizable completion. The uniform identity between $\iota(X)$ and $\jmath(X)$ extends to uniformly continuous maps $\Phi\colon \widehat{X} \longrightarrow Y$ and $\Psi\colon Y \longrightarrow \widehat{X}$. Their compositions fix the corresponding dense subsets; the preceding uniqueness gives $\Psi \circ \Phi = \mathrm{id}_{\widehat{X}}$ and $\Phi \circ \Psi = \mathrm{id}_Y$. Moreover, $\Phi \circ \iota = \jmath$, and the same density shows that no other isomorphism has this property. □

Definition 27.2.9. (Extended metric) An *extended metric* on a set X is a map

$$d\colon X \times X \longrightarrow [0, +\infty]$$

satisfying the metric axioms, with the arithmetic convention $a + (+\infty) = +\infty$.

Definition 27.2.10. (Finite-distance component) If d is an extended metric and $x \in X$, the *finite-distance component* of x is

$$X_x := \{y \in X \mid d(x, y) < +\infty\}.$$

The restriction $d \restriction_{X_x}$ is an ordinary metric. The relation defined by the condition $d(x, y) < +\infty$ is an equivalence relation, and its classes are the finite-distance components.

Proposition 27.2.11. (Decomposition into finite-distance components) Finite-distance components are open and closed in the topology generated by balls of finite radius. The completion of (X,d) is the disjoint union of the metric completions of these components.

Proof. If $y \in X_x$, every ball of finite radius centered at y is contained in X_x by the triangle inequality; hence X_x is open. Its complement is the union of the other components and is also open. A Cauchy sequence eventually lies in a single component, since for some finite radius all sufficiently late pairs have distance less than that radius. Consequently, completing each component and taking the disjoint union produces, and characterizes, the completion of the extended metric. □

Proposition 27.2.12. (Metrization by chains) Let $(V_n)_{n\in\mathbb{N}_0}$ be a decreasing cofinal base of symmetric entourages of a separated uniform space X, and suppose

$$V_{n+1} \circ V_{n+1} \circ V_{n+1} \subseteq V_n, \qquad n \geq 0.$$

Define the cost of an edge by

$$\omega(x,y) := \inf\big(\{2^{-n} \mid n \in \mathbb{N}_0,\ (x,y) \in V_n\} \cup \{+\infty\}\big)$$

and, for chains $(x_i)_{i=0}^{N}$ with all vertices in X, define the chain metric by

$$d_{\mathrm{cad}}(x,y) := \inf\left\{ \sum_{i=0}^{N-1} \omega(x_i, x_{i+1}) \,\middle|\, N \in \mathbb{N},\ x = x_0,\ x_N = y \right\}. \tag{27.1}$$

If there is no chain of finite cost, the infimum is $+\infty$. Then, for $n \geq 1$,

$$V_n \subseteq \{(x,y) \mid d_{\mathrm{cad}}(x,y) \leq 2^{-n}\} \quad \text{and} \quad \{(x,y) \mid d_{\mathrm{cad}}(x,y) < 2^{-n}\} \subseteq V_{n-1}.$$

In particular, d_{cad} is an extended metric generating the uniformity on each finite-distance component, and $\min\{1, d_{\mathrm{cad}}\}$ is a finite-valued metric generating the uniformity on the entire space.

Proof. The first inclusion follows by using a chain with one edge. For the second, we prove simultaneously for every $n \in \mathbb{N}$ the following implication: a chain of N edges with cost less than 2^{-n} has its endpoints in V_{n-1}. If $N = 1$, the definition of the cost and the dyadic nature of its values give $(x_0, x_1) \in V_{n+1} \subseteq V_{n-1}$. An edge of zero cost joins equal points, since the pair belongs to all V_j and the uniformity is separated.
Now suppose the implication holds for every chain with fewer than N edges. If the total cost c is zero, all vertices agree. If $c > 0$, choose the first edge for which the accumulated cost reaches or exceeds $c/2$. The preceding subchain has cost less than $c/2$, and the following one has cost at most $c/2$. Since $c < 2^{-n}$, both costs are less than $2^{-(n+1)}$. Each subchain has fewer than N edges, so the induction hypothesis, applied at index $n+1$, places its endpoints in V_n. If a subchain is empty, its endpoints agree and belong to the diagonal of V_n. The central edge has cost less than 2^{-n}, so it belongs to $V_{n+1} \subseteq V_n$. Inclusion $V_n \circ V_n \circ V_n \subseteq V_{n-1}$ completes the induction step.
Symmetry and the triangle inequality for d_{cad} follow by reversing and concatenating chains, respectively. If its value is zero, the second inclusion places the pair in every V_n; separation implies that the points agree.

The two inclusions show that balls and entourages are cofinal. Finally, truncation at one preserves all balls of radius less than one and assigns distance one to pairs in distinct components. □

In particular, every separated uniformity with a countable base is metrizable: intersect each base element with its inverse and refine recursively to obtain the triple-composition condition in the proposition.

27.2.2 Eichhorn uniformities

The preceding uniform regularity is a property of the base manifold and an atlas. The uniformities introduced now, by contrast, live on a set of sections, maps, metrics, or diffeomorphisms. Their role is to compare two objects globally, even when the base is noncompact. The general uniform framework and its use on complete noncompact manifolds without boundary are developed in [51]; the specific analytic constructions are formulated below.

The linear model

Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary, and let $\mathbf{E}\longrightarrow M$ be a smooth real vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla:=\nabla^{\mathbf{E}}$. For $1\le p<\infty$ and $r\in\mathbb{N}_0$, set

$$\|\mathbf{u}\|_{p,r;\nabla}:=\left(\sum_{j=0}^{r}\int_M |\nabla^j\mathbf{u}|^p_{\mathbf{g},\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}\right)^{\frac{1}{p}}.$$

On the space $L^0(M,\mathbf{E})$ of classes of measurable sections modulo almost everywhere equality, define the extended metric

$$d_{p,r;\nabla}(\mathbf{u},\mathbf{v}):=\begin{cases}\|\mathbf{u}-\mathbf{v}\|_{p,r;\nabla}, & \mathbf{u}-\mathbf{v}\in W^{r,p}(M,\mathbf{E}),\\ +\infty, & \mathbf{u}-\mathbf{v}\notin W^{r,p}(M,\mathbf{E}).\end{cases}\tag{27.2}$$

Proposition 27.2.13. (Linear components) Let $(M,\mathbf{g})$ be a complete Riemannian manifold without boundary, and let $\mathbf{E}\to M$ be a smooth real vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a compatible connection $\nabla:=\nabla^{\mathbf{E}}$. Fix $1\le p<\infty$ and $r\in\mathbb{N}_0$. Then the sets

$$V_\varepsilon:=\{(\mathbf{u},\mathbf{v})\mid d_{p,r;\nabla}(\mathbf{u},\mathbf{v})<\varepsilon\},\qquad \varepsilon>0,$$

form a base for a separated metrizable uniformity. The finite-distance component of $\mathbf{u}_0$ is the affine space $\mathbf{u}_0+W^{r,p}(M,\mathbf{E})$, which is complete when $W^{r,p}(M,\mathbf{E})$ is. If ∇' is another connection, if the spaces of sections with integrable weak derivatives of order at most r are the same for both connections, and if there exist $c_\nabla,C_\nabla>0$ such that

$$c_\nabla\|\mathbf{u}\|_{p,r;\nabla}\le\|\mathbf{u}\|_{p,r;\nabla'}\le C_\nabla\|\mathbf{u}\|_{p,r;\nabla}\qquad(\mathbf{u}\in W^{r,p}(M,\mathbf{E})),$$

, then both connections induce the same uniformity.

Proof. Symmetry and the triangle inequality for $d_{p,r;\nabla}$ are inherited from the norm on each finite-distance component. Hence $V_{\frac{\varepsilon}{2}}\circ V_{\frac{\varepsilon}{2}}\subseteq V_\varepsilon$; moreover, $V_\varepsilon^{-1}=V_\varepsilon$, $\Delta_{L^0(M,\mathbf{E})}\subseteq V_\varepsilon$, and $V_{\min\{\varepsilon,\delta\}}\subseteq V_\varepsilon\cap V_\delta$. These are all the conditions in Definition 27.2.2. The function $\min\{1,d_{p,r;\nabla}\}$ is a finite-valued metric generating the same entourages of radius less than one, proving metrizability. On the other hand, $d_{p,r;\nabla}(\mathbf{u}_0,\mathbf{u})<\infty$ is equivalent to $\mathbf{u}-\mathbf{u}_0\in W^{r,p}(M,\mathbf{E})$, identifying the component. The translation $\mathbf{u}\mapsto\mathbf{u}-\mathbf{u}_0$ is an isometry from this component onto $W^{r,p}(M,\mathbf{E})$, which gives completeness. Finally, two norms satisfying the explicit

inequalities in the statement have the same uniform balls after changing the radius by the factors c_∇, C_∇ and hence cofinal bases of entourages. □

If $\nabla' - \nabla = \mathbf{a} \in \Omega^1(\operatorname{End}\mathbf{E})$ and $\mathbf{a}, \nabla\mathbf{a}, \dots, \nabla^{r-1}\mathbf{a}$ are uniformly bounded, the Leibniz rule expresses $(\nabla')^j\mathbf{u}$, in the weak sense of Chapter 9, as $\nabla^j\mathbf{u}$ plus a sum of products of derivatives of $\mathbf{a}$ and lower-order derivatives of $\mathbf{u}$. We obtain $\|\mathbf{u}\|_{p,r;\nabla'} \le C_\nabla \|\mathbf{u}\|_{p,r;\nabla}$, where C_∇ depends on r and the bounds on $\mathbf{a}, \nabla\mathbf{a}, \dots, \nabla^{r-1}\mathbf{a}$. If the corresponding derivatives of $-\mathbf{a} = \nabla - \nabla'$ are bounded with respect to ∇', we obtain $c_\nabla \|\mathbf{u}\|_{p,r;\nabla} \le \|\mathbf{u}\|_{p,r;\nabla'}$. The same identities, applied to locally integrable sections in the distributional sense, prove both inclusions between the Sobolev domains; comparing the norms on just one of them is insufficient. In the step from order j to order $j+1$, the Leibniz rule in Proposition 9.1.40 differentiates one of the existing factors, and the term $\mathbf{a}*$ adds a new factor of order zero. Starting from $\nabla'\mathbf{u} = \nabla\mathbf{u} + \mathbf{a} * \mathbf{u}$, this describes all terms for $j \in \{0, \dots, r\}$. This observation explains why, in bounded geometry, admissible reference connections produce the same linear uniformity. For connections on a fixed bundle, the difference of two connections is already a section of $T^*M \otimes \operatorname{End}\mathbf{E}$, and (27.2) applies directly to that difference.

Maps and exponential comparison

Definition 27.2.14. (Conditions (I) and (B_k)) Let $(M, \mathbf{g})$ be a complete Riemannian manifold without boundary. We say that $(M, \mathbf{g})$ satisfies (I) if there exists $i_0 > 0$ such that $\operatorname{inj}(M, \mathbf{g}) \ge i_0$, and that it satisfies (B_k) if there exist constants $K_0, \dots, K_k > 0$ such that

$$\|\nabla^j \operatorname{Rm}_{\mathbf{g}}\|_\infty \le K_j, \qquad 0 \le j \le k.$$

In this section, *bounded geometry of order k* means that (I) and (B_k) hold. Constants in estimates will refer to $i_0, K_0, \dots, K_k$ and the other data specified in each statement.

In what follows, $(M, \mathbf{g})$ and $(N, \mathbf{h}_N)$ denote complete manifolds without boundary satisfying the conditions in Definition 27.2.14 to the required order.

If $\mathbf{E} \longrightarrow M$ is a smooth real vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_\mathbf{E}$ and a connection $\nabla := \nabla^\mathbf{E}$, and $j \in \mathbb{N}_0$, set

$$C^j_b(M, \mathbf{E}) := \left\{ \mathbf{u} \in C^j(M, \mathbf{E}) \,\middle|\, \|\mathbf{u}\|_{C^j_b(M,\mathbf{E})} := \sum_{\ell=0}^{j} \sup_{x \in M} |\nabla^\ell \mathbf{u}(x)|_{\mathbf{g}, \mathbf{h}_\mathbf{E}} < +\infty \right\}.$$

For $j = 0$, only continuity and boundedness of $\mathbf{u}$ are required. On the pullback bundle f^*TN, use the connection induced by f and the Levi–Civita connection of N.

For $q \in \mathbb{N}$, denote by $C^{\infty,q}_b(M, N)$ the set of smooth maps $f : M \longrightarrow N$ such that $\|\nabla^j df\|_\infty < +\infty$ for $0 \le j \le q-1$. When uniform constants are needed, fix $\mathbf{B} = (B_0, \dots, B_{q-1}) \in (0, +\infty)^q$ and write

$$C^{\infty,q}_{b,\mathbf{B}}(M, N) := \left\{ f \in C^{\infty,q}_b(M, N) \,\middle|\, \|\nabla^j df\|_\infty \le B_j,\ 0 \le j \le q-1 \right\}.$$

The constants will be common to all maps in a class with $\mathbf{B}$ fixed. Allowing each map its own bound does not give a common constant; this distinction matters when checking the axioms of a uniformity.

If $\mathbf{X} \in \Gamma(f^*TN)$ and $\|\mathbf{X}\|_{C^0(M, f^*TN; f^*\mathbf{h}_N)} < i_0(N)$, set $(\operatorname{Exp}_f \mathbf{X})(x) := \exp_{f(x)} \mathbf{X}(x)$ and measure its derivatives with

the induced connection on f^*TN by

$$\|\mathbf{X}\|_{b,r;f} := \sum_{j=0}^{r} \|\nabla^j \mathbf{X}\|_\infty.$$

For $0 < \varepsilon < i_0(N)/2$, define

$$\mathcal{V}_\varepsilon^{b,r} := \left\{ (f,h) \middle| h = \operatorname{Exp}_f \mathbf{X} \text{ for some } \mathbf{X} \in \Gamma(f^*TN),\ \|\mathbf{X}\|_{b,r;f} < \varepsilon \right\}. \tag{27.3}$$

Now suppose $1 < p < \infty$ and $r \in \mathbb{N}$ with $r > \dim(M)/p + 1$, and fix $0 < \rho < i_0(N)/2$. In the Sobolev regime, we use

$$\mathcal{V}_{\varepsilon,\rho}^{p,r} := \left\{ (f,h) \middle| \begin{array}{l} h = \operatorname{Exp}_f \mathbf{X} \text{ for some } \mathbf{X} \in W^{r,p}(M, f^*TN), \\ \|\mathbf{X}\|_{W^{r,p}(M,f^*TN)} < \varepsilon, \quad \|\mathbf{X}\|_{C^0(M,f^*TN;f^*\mathbf{h}_N)} < \rho \end{array} \right\}. \tag{27.4}$$

The pointwise condition keeps the comparison inside a common normal ball. For $q \geq r$, on $C^{\infty,q}_{b,\mathbf{B}}(M,N)$, localization results and Sobolev embedding give a constant $C_S = C_S(M,N,p,r,\rho,\mathbf{B}) > 0$ such that

$$\|\mathbf{X}\|_{C^0(M,f^*TN;f^*\mathbf{h}_N)} \leq C_S \|\mathbf{X}\|_{W^{r,p}(M,f^*TN)}.$$

Thus, in a class with fixed data, the condition $\|\mathbf{X}\|_{C^0(M,f^*TN;f^*\mathbf{h}_N)} < \rho$ follows from $\varepsilon < \rho/C_S$.

The central property is that inversion and composition of exponential comparisons preserve control. If $h = \operatorname{Exp}_f \mathbf{X}$, denote by $P_{\mathbf{X}} \colon f^*TN \longrightarrow h^*TN$ parallel transport along $t \mapsto \exp_{f(x)}(t\mathbf{X}(x))$.

Lemma 27.2.15. (Uniform comparison estimates) Let $(M,\mathbf{g})$ and $(N,\mathbf{h}_N)$ be complete Riemannian manifolds without boundary satisfying (I) and (B_k) to the required order. Fix a radius $\rho > 0$ small enough that balls of radius 4ρ lie in uniform normal domains and the vertical differentials of the exponential map and their inverses have common bounds. It may be chosen in terms of $i_0(N)$ and the curvature bound, with $4\rho < i_0(N)$. Also fix a profile $\mathbf{B}$ bounding the derivatives of the maps under consideration through the necessary order. For each $0 \leq j \leq r$, there exist continuous polynomials P_j, Q_j with nonnegative coefficients and no constant term, depending only on the dimensions, j, ρ, the uniform geometric data of M and N through order r, and $\mathbf{B}$, and satisfying:

(i) if $h = \operatorname{Exp}_f \mathbf{X}$, then

$$\|\nabla^j (P_{\mathbf{X}}\mathbf{X})\|_{C^0(M,T^{(0,j)}(TM)\otimes h^*TN;\mathbf{g},h^*\mathbf{h}_N)} \leq P_j\left(\|\mathbf{X}\|_{C^0(M,f^*TN;f^*\mathbf{h}_N)}, \dots, \|\nabla^j \mathbf{X}\|_{C^0(M,T^{(0,j)}(TM)\otimes f^*TN;\mathbf{g},f^*\mathbf{h}_N)} \right);$$

; since $f = \operatorname{Exp}_h(-P_{\mathbf{X}}\mathbf{X})$, this bound controls inversion;

(ii) if $h = \operatorname{Exp}_f \mathbf{X}$ and $\ell = \operatorname{Exp}_h \mathbf{Y}$ remain in a common normal ball, there exists a unique $\mathbf{Z} \in \Gamma(f^*TN)$ with $\ell = \operatorname{Exp}_f \mathbf{Z}$, and $\|\nabla^j \mathbf{Z}\|_{C^0(M,T^{(0,j)}(TM)\otimes f^*TN;\mathbf{g},f^*\mathbf{h}_N)} \leq Q_j(\|\mathbf{X}\|_{b,j;f}, \|\mathbf{Y}\|_{b,j;h})$.

We may take $P_0(t) = t$ and $Q_0(a,b) = a + b$. Moreover, given $\eta > 0$, there exists $\delta > 0$, depending only on the preceding data, such that $Q_j(a,b) < \eta$ whenever $a + b < \delta$. If $r > \dim(M)/p + 1$, there exist polynomials $\widehat{P}_r, \widehat{Q}_r$ with the same dependencies, together with the Sobolev module and embedding constants, for which

$$\|P_{\mathbf{X}}\mathbf{X}\|_{W^{r,p}(M,h^*TN)} \leq \widehat{P}_r(\|\mathbf{X}\|_{W^{r,p}(M,f^*TN)}), \qquad \|\mathbf{Z}\|_{W^{r,p}(M,f^*TN)} \leq \widehat{Q}_r\left(\|\mathbf{X}\|_{W^{r,p}(M,f^*TN)}, \|\mathbf{Y}\|_{W^{r,p}(M,h^*TN)}\right),$$

if the corresponding norms in $C^0(M, f^*TN; f^*\mathbf{h}_N)$ and $C^0(M, h^*TN; h^*\mathbf{h}_N)$ are less than ρ.

Proof. At order zero, parallel transport is isometric, so $|P_{\mathbf{X}}\mathbf{X}|_{h^*\mathbf{h}_N} = |\mathbf{X}|_{f^*\mathbf{h}_N}$. If two successive geodesic segments determine $\mathbf{Z}$, the triangle inequality gives $|\mathbf{Z}|_{f^*\mathbf{h}_N} \le |\mathbf{X}|_{f^*\mathbf{h}_N} + |\mathbf{Y}|_{h^*\mathbf{h}_N}$ after transporting the vectors to the same tangent space along the segments.

At first order, let $x(s)$ be a curve with $x(0) = x$ and $x'(0) = \mathbf{V}$, and consider $c(s,t) = \exp_{f(x(s))}(t\mathbf{X}(x(s)))$. The field $\mathbf{J}(t) = \partial_s c(0,t)$ satisfies

$$\frac{D^2\mathbf{J}}{dt^2} + \mathrm{Rm}_{\mathbf{h}_N}(\mathbf{J},\dot\gamma)\dot\gamma = 0, \qquad \mathbf{J}(0) = df(\mathbf{V}), \qquad \frac{D\mathbf{J}}{dt}(0) = \nabla_{\mathbf{V}}\mathbf{X}. \tag{27.5}$$

In a parallel frame, the integral form of this equation and Grönwall's lemma 1.8.19 imply $\sup_{t\in[0,1]} |\mathbf{J}(t)|_{\mathbf{h}_N} \le e^{K_0(N)\rho^2}(|df(\mathbf{V})|_{\mathbf{h}_N} + |\nabla_{\mathbf{V}}\mathbf{X}|_{\mathbf{h}_N})$. Integrating the Jacobi equation once gives

$$\sup_{t\in[0,1]} \left|\frac{D\mathbf{J}}{dt}(t)\right|_{\mathbf{h}_N} \le |\nabla_{\mathbf{V}}\mathbf{X}|_{\mathbf{h}_N} + K_0(N)|\mathbf{X}|^2_{f^*\mathbf{h}_N} e^{K_0(N)\rho^2}\left(|df(\mathbf{V})|_{\mathbf{h}_N} + |\nabla_{\mathbf{V}}\mathbf{X}|_{\mathbf{h}_N}\right). \tag{27.6}$$

Since the connection is torsion-free, $\nabla_{\mathbf{V}}(P_{\mathbf{X}}\mathbf{X}) = D_t\mathbf{J}(1)$, and the preceding bound proves the case $j = 1$.

To compose two exponential comparisons, write $\widetilde{\mathbf{Y}} := P_{\mathbf{X}}^{-1}\mathbf{Y} \in \Gamma(f^*TN)$. On the common normal domain, consider the smooth map

$$\Psi(p,v,y) := \exp_p^{-1}\left(\exp_{\exp_p v}(P_v y)\right), \qquad p \in N, \quad v,y \in T_pN,$$

, where $P_v\colon T_pN \longrightarrow T_{\exp_p v}N$ is parallel transport along $t \mapsto \exp_p(tv)$. The section directly comparing $\ell = \mathrm{Exp}_h\mathbf{Y}$ with f is then $\mathbf{Z}(x) = \Psi(f(x),\mathbf{X}(x),\widetilde{\mathbf{Y}}(x))$. Differentiating this identity produces the preceding Jacobi fields and the linear equation for parallel transport. In particular, there exists a constant A_1 depending only on $\dim N$, ρ, and $K_0(N)$ such that

$$|\nabla\mathbf{Z}|_{\mathbf{g},f^*\mathbf{h}_N} \le A_1\left[(1 + |\mathbf{X}|_{f^*\mathbf{h}_N} + |\widetilde{\mathbf{Y}}|_{f^*\mathbf{h}_N})(|\nabla\mathbf{X}|_{\mathbf{g},f^*\mathbf{h}_N} + |\nabla\widetilde{\mathbf{Y}}|_{\mathbf{g},f^*\mathbf{h}_N}) + (|\mathbf{X}|_{f^*\mathbf{h}_N} + |\widetilde{\mathbf{Y}}|_{f^*\mathbf{h}_N})|df|_{\mathbf{g},f^*\mathbf{h}_N}\right].$$

Derivatives of $\widetilde{\mathbf{Y}} = P_{\mathbf{X}}^{-1}\mathbf{Y}$ are estimated using the same parallel transport equation; their bounds depend polynomially on the derivatives of $\mathbf{X}$ and $\mathbf{Y}$ and introduce no constant term.

To make the induction step precise, suppose bounds have been obtained through order $j-1$, with $2 \le j \le r$. Differentiate equation (27.5) a total of $j-1$ times. The recurrence (13.4)– (13.5) separates the order-j unknown from the source: the operator acting on this unknown remains the Jacobi operator, and each source term contains only variation fields of lower orders and curvature derivatives of the orders indicated there. Proposition 9.1.40 allows the contractions to be differentiated without changing their nature. Differentiating a curvature factor raises its order by one; differentiating a field raises the order of that field by one. These are exactly the two operations in the recurrence, and the weight identity prevents the source from containing a second unknown of order j. Its factors are therefore controlled by the induction hypothesis and the fixed data.

We must also preserve vanishing when $\mathbf{X} = 0$. To do so, subtract from the differentiated equation the one corresponding to the constant variation $c(x,t) = f(x)$. The terms containing only $df,\dots,\nabla^{j-1}df$ cancel. Every remaining term contains at least one factor among $\mathbf{X},\nabla\mathbf{X},\dots,\nabla^j\mathbf{X}$. Integrating the linear equation and applying Grönwall gives the order-j bound for the final vector $P_{\mathbf{X}}\mathbf{X}$, with a polynomial having no constant term. This completes the step $j-1 \longrightarrow j$, starting from the zero and first orders already computed.

For composition, proceed with the map Ψ and the parallel transport equation. Here subtract the identity

$\Psi(p,0,0)=0$. Differentiating it with respect to p again eliminates all terms containing only derivatives of the center. The chain rule of order j leaves products of derivatives of $\mathbf{X}$ and $\mathbf{Y}$ whose total order is at most j; their coefficients are bounded by the same Jacobi recurrence and the bound for the inverse differential of the exponential map. The sum of their bounds defines Q_j and retains the absence of a constant term.
To pass to Sobolev norms, use the uniform trivializations and the norm equivalence of Proposition 13.3.14. Each term has at least one integrable factor. Products of derivatives of the sections are estimated by Hölder and Sobolev embeddings, assigning to each factor the order it retains after differentiation; the sum of the differentiation orders does not exceed r. The order-zero factors are controlled by embedding into C^1_b, and the coefficient derivatives by the geometric data and $\mathbf{B}$. The product estimate in the regime $r > m/p+1$ thus gives a polynomial in the $W^{r,p}$ norms of the sections, without a term that would require integrating a constant over M. Finite multiplicity of the cover allows the p powers of the local bounds to be summed with a common constant. This gives $\widehat{P}_r$ and $\widehat{Q}_r$.

□

Proposition 27.2.16. (Uniformity with fixed control data) Let $(M,\mathbf{g})$ and $(N,\mathbf{h}_N)$ be complete Riemannian manifolds without boundary satisfying (I) and (B_k) to the required order. For a fixed profile $\mathbf{B}$, the families $\mathcal{V}^{b,r}_\varepsilon$ restricted to $C^{\infty,r}_{b,\mathbf{B}}(M,N)$ form a base for a separated uniformity. The same holds in the Sobolev regime for $\mathcal{V}^{p,r}_{\varepsilon,\rho}$ after choosing $\varepsilon<\rho/C_S(\mathbf{B})$.

Proof. The diagonal belongs to every entourage, and monotonicity in ε controls intersections. Part (i) of the lemma allows us, for each $\varepsilon>0$, to choose $\delta>0$ such that $(\mathcal{V}^{b,r}_\delta)^{-1}\subseteq\mathcal{V}^{b,r}_\varepsilon$; part (ii) allows δ to be decreased to obtain $\mathcal{V}^{b,r}_\delta\circ\mathcal{V}^{b,r}_\delta\subseteq\mathcal{V}^{b,r}_\varepsilon$. The constants are uniform because $\mathbf{B}$ is fixed. Separation follows from pointwise control. The same argument, using the Sobolev estimates and the condition $C_S(\mathbf{B})\delta<\rho$, proves the second assertion. □

Theorem 27.2.17. (Metrization in a class with common bounds) Let $(M,\mathbf{g})$ and $(N,\mathbf{h}_N)$ be complete, without boundary, and of bounded geometry of order k. For $q\in\{1,\dots,k\}$, fix a profile $\mathbf{B}\in(0,+\infty)^q$. The sets (27.3), with $r=q$, restricted to $C^{\infty,q}_{b,\mathbf{B}}(M,N)$, generate a separated uniformity with a countable base. If $1<p<\infty$ and $k\geq r>m/p+1$, the same holds for (27.4) on $C^{\infty,r}_{b,\mathbf{B}}(M,N)$. Both uniformities are metrizable and are unchanged when the common normal radius is decreased.

Proof. Proposition 27.2.16 verifies the uniformity axioms. Positive rational radii suffice to give a countable base. Separation and metrization follow from Proposition 27.2.12. If the normal radius is decreased, Sobolev embedding with a common constant shows that entourages of sufficiently small radius in the original base belong to the new base. The reverse inclusion is immediate; the bases are cofinal. □

The need to fix the bounds can be seen directly. Take $M=S^1\times\mathbb{R}$ and $N=S^2\times\mathbb{R}$ with their product metrics, and, for $n\in\mathbb{N}$, define

$$f_n(\theta,t)=(\cos(n\theta),\sin(n\theta),0,t),\qquad \theta\in\mathbb{R}/(2\pi\mathbb{Z}),\quad t\in\mathbb{R}.$$

Let $\chi\in C^\infty_c(\mathbb{R})$ equal one on $[-1,1]$, with $0\leq\chi\leq1$, and fix $0<\alpha<\rho$. The field $\mathbf{X}_n=\alpha\chi(t)(0,0,1,0)$ along f_n has C^r_b and $W^{r,p}$ norms equal to a constant times α, independently of n: the vertical vector of the sphere is parallel along the equator, and only the factors χ are differentiated. Writing $a(t)=\alpha\chi(t)$,

$$h_n=\operatorname{Exp}_{f_n}\mathbf{X}_n=(\cos a\cos(n\theta),\cos a\sin(n\theta),\sin a,t).$$

The vector giving the inverse comparison is

$$\mathbf{Y}_n = \log_{h_n} f_n = a(\sin a\cos(n\theta), \sin a\sin(n\theta), -\cos a, 0).$$

On $S^1 \times [-1,1]$, we have $|\nabla_{\partial_\theta}\mathbf{Y}_n| = n\alpha\sin\alpha$. Consequently, the direct comparison can be arbitrarily small with a radius independent of n, whereas the inverse is not bounded with that radius. Each f_n has bounded derivatives, but the family has no common profile. Constant-radius exponential entourages on the union of all classes do not by themselves satisfy the inversion axiom. The local construction of mapping manifolds used below retains the bounds of the center of each chart; see also the chart construction in [49, §5].

In the following construction, write r for the order of either norm. Fix a class with profile $\mathbf{B}$ and one of the two uniformities in the theorem. Abbreviate its exponential entourages by $\mathcal{V}_\varepsilon$. The inversion and composition estimates allow us to choose recursively a decreasing rational sequence $\varepsilon_n \to 0^+$ such that the symmetric entourages

$$\mathcal{W}_n := \mathcal{V}_{\varepsilon_n} \cap \mathcal{V}_{\varepsilon_n}^{-1}$$

are cofinal and satisfy $\mathcal{W}_{n+1} \circ \mathcal{W}_{n+1} \circ \mathcal{W}_{n+1} \subseteq \mathcal{W}_n$. Define

$$\omega(f,h) := \inf\big(\{2^{-n} \mid n \in \mathbb{N}_0,\ (f,h) \in \mathcal{W}_n\} \cup \{+\infty\}\big)$$

and, for a chain $f = f_0, f_1, \dots, f_N = h$ whose vertices f_i, with $i \in \{0,\dots,N\}$, belong to the fixed class, set

$$d_{\mathrm{cad}}(f,h) := \inf\left\{ \sum_{i=0}^{N-1} \omega(f_i, f_{i+1}) \,\middle|\, N \in \mathbb{N},\ f = f_0,\ f_N = h \right\}, \tag{27.7}$$

, with value $+\infty$ when there is no chain of finite cost. Proposition 27.2.12 gives

$$\mathcal{W}_n \subseteq \{(f,h) \mid d_{\mathrm{cad}}(f,h) \le 2^{-n}\} \quad \text{and} \quad \{(f,h) \mid d_{\mathrm{cad}}(f,h) < 2^{-n}\} \subseteq \mathcal{W}_{n-1}, \qquad n \ge 1.$$

Thus (27.7) generates the uniformity on each finite-distance component, and its truncation at one metrizes it on the entire space. The construction does not presuppose that an entire component fits inside a single exponential chart.

Definition 27.2.18. (Slice, equivariant tube, and orbit types) Let G be a Banach Lie group acting smoothly on a Banach manifold $\mathcal{X}$, and let $x \in \mathcal{X}$. Suppose the stabilizer G_x is a closed split Banach Lie subgroup, so that the associated quotient below has its manifold structure. A *slice* at x is a submanifold $S \subseteq \mathcal{X}$ containing x, invariant under the stabilizer G_x, and such that the map

$$G \times_{G_x} S \longrightarrow \mathcal{X}, \qquad [g,s] \longmapsto g \cdot s,$$

is a diffeomorphism onto a G–invariant neighborhood of the orbit $G\cdot x$. Here $G \times_{G_x} S$ is the quotient of $G \times S$ by $(g,s) \sim (gh^{-1}, h\cdot s)$, with $h \in G_x$. The image of this diffeomorphism is called an *equivariant tube* of the orbit.

We will also use the topological version of this notion: if G is a topological group acting continuously, the same associated map is required to be a homeomorphism onto an invariant open set, giving a *topological slice*. Differentiable regularity of S or the tube is added as a separate conclusion. This distinction allows

slices to be formulated for Sobolev diffeomorphisms, whose operations lose derivatives and which do not form a Banach Lie group at a fixed level.

A *quasistratification by orbit types* is a partition $\mathcal{X} = \coprod_{(H)} \mathcal{X}_{(H)}$, where $\mathcal{X}_{(H)} := \{y \in \mathcal{X} \mid G_y \text{ is conjugate to } H\}$, satisfying the following conditions: each $\mathcal{X}_{(H)}$ is a locally closed submanifold; slices provide local models for the partition; and the frontier condition holds:

$$\mathcal{X}_{(K)} \cap \overline{\mathcal{X}_{(H)}} \neq \varnothing \quad \Longrightarrow \quad H \text{ is conjugate to a subgroup of } K.$$

The prefix *quasi* indicates that local finiteness of the family of types is not required. When an external result adds regularity or second countability to the strata and quotient, these properties will be stated as part of its assumptions and conclusions.

Lemma 27.2.19. (Products, derivatives, and approximation in the Hilbert scale) Let M be a smooth compact manifold of dimension m, with or without boundary, and let $\mathbf{E}, \mathbf{F}, \mathbf{G} \to M$ be smooth vector bundles of finite rank, equipped with smooth bundle metrics and connections. The following properties hold.

(a) For $a > m/2$, $0 \le r \le a$, and a smooth homomorphism $B : \mathbf{E} \otimes \mathbf{F} \to \mathbf{G}$, there exists $C > 0$ such that

$$\|B(\mathbf{v} \otimes \mathbf{z})\|_{H^r(M,\mathbf{G})} \le C \|\mathbf{v}\|_{H^a(M,\mathbf{E})} \|\mathbf{z}\|_{H^r(M,\mathbf{F})}. \tag{27.8}$$

(b) For $r \ge 0$, the covariant derivative is a continuous linear operator

$$\nabla : H^r(M,\mathbf{E}) \longrightarrow H^{r-1}(M, T^*M \otimes \mathbf{E}).$$

In particular, if $s > m/2 + 1$, contraction of this derivative satisfies

$$\|\nabla_{\mathbf{v}} \mathbf{z}\|_{H^{s-1}(M,TM)} \le C \|\mathbf{v}\|_{H^s(M,TM)} \|\mathbf{z}\|_{H^s(M,TM)}. \tag{27.9}$$

(c) For every $r \ge 0$, $\Gamma(\mathbf{E})$ is dense in $H^r(M,\mathbf{E})$. If $r_0, r_1 \ge 0$ and $0 < \theta < 1$, then

$$[H^{r_0}(M,\mathbf{E}), H^{r_1}(M,\mathbf{E})]_\theta = H^{(1-\theta)r_0 + \theta r_1}(M,\mathbf{E})$$

with equivalent norms.

Proof. Fix a finite cover by interior and boundary charts, smooth frames, and a subordinate partition of unity $(\rho_i)_{i=1}^N$. Choose cutoff functions χ_i supported in the same charts, with $\chi_i = 1$ on a neighborhood of $\operatorname{supp} \rho_i$. Local norm equivalence for bundles and the constructions in Theorems 6.5.2 and 14.5.9 allow $\rho_i \mathbf{v}$ to be represented continuously by functions in $H^r(\mathbb{R}^m, \mathbb{K}^{\operatorname{rank} \mathbf{E}})$: in boundary charts use the half-space extension, and in interior charts extend by zero after multiplication by the cutoff function. To return to M, restrict to the chart and multiply by χ_i. The sum of these operations recovers $\mathbf{v}$.

First prove the scalar product estimate on $\mathbb{R}^m$. For $a > m/2$, Cauchy–Schwarz gives

$$\|\widehat{v}\|_{L^1(\mathbb{R}^m)} \le \left(\int_{\mathbb{R}^m} \langle \xi \rangle^{-2a} \, d\xi \right)^{1/2} \|v\|_{H^a(\mathbb{R}^m)}.$$

The integral is finite. The inequality $\langle \xi \rangle^a \le C_a(\langle \xi - \zeta \rangle^a + \langle \zeta \rangle^a)$, the Fourier formula for products, and Young's

convolution inequality imply, initially for Schwartz functions,

$$\|vz\|_{H^a(\mathbb{R}^m)} \le C_a\big(\|v\|_{H^a(\mathbb{R}^m)}\|\widehat{z}\|_{L^1(\mathbb{R}^m)} + \|\widehat{v}\|_{L^1(\mathbb{R}^m)}\|z\|_{H^a(\mathbb{R}^m)}\big) \le C\|v\|_{H^a(\mathbb{R}^m)}\|z\|_{H^a(\mathbb{R}^m)}.$$

Density of Schwartz functions in Proposition 6.1.6 extends this product: if $v_j \to v$ and $z_j \to z$ in $H^a(\mathbb{R}^m)$, applying the inequality to $(v_j - v_k)z_j + v_k(z_j - z_k)$ proves that $(v_j z_j)$ is Cauchy. Embedding into $C^0(\mathbb{R}^m)$ identifies its limit with vz. On the other hand, $\|vz\|_{L^2(\mathbb{R}^m)} \le \|v\|_{L^\infty(\mathbb{R}^m)}\|z\|_{L^2(\mathbb{R}^m)}$. Interpolate these two bounds for the linear operator $z \mapsto vz$, holding v fixed; Theorem 6.3.11, with $p = q = 2$, gives the bound in $H^r(\mathbb{R}^m)$ for $0 \le r \le a$. Apply it componentwise and to the localized smooth coefficients of B. The finitely many localization operations above prove (a).

In a local frame, $\nabla_j \mathbf{v}$ has components $\partial_j v^b + \sum_{c=1}^{\operatorname{rank}\mathbf{E}} \Gamma^b_{jc} v^c$. Euclidean differentiation from the same proposition and continuity of smooth multipliers prove (b). For the final estimate, take $a = r = s - 1 > m/2$ in (a), use $H^s(M, TM) \hookrightarrow H^{s-1}(M, TM)$, and contract $\mathbf{v} \otimes \nabla \mathbf{z}$.

For (c), represent each $\rho_i \mathbf{v}$ by an extension V_i as at the beginning of the proof. Take $V_{i,j} \in \mathcal{S}(\mathbb{R}^m, \mathbb{K}^{\operatorname{rank}\mathbf{E}})$ with $V_{i,j} \to V_i$ in the indicated space H^r. Restricting, reconstructing in the frames, and multiplying by χ_i gives smooth sections $\mathbf{v}_j$ on all of M, including the boundary charts, and

$$\|\mathbf{v}_j - \mathbf{v}\|_{H^r(M,\mathbf{E})} \le C \sum_{i=1}^{N} \|V_{i,j} - V_i\|_{H^r(\mathbb{R}^m, \mathbb{K}^{\operatorname{rank}\mathbf{E}})} \longrightarrow 0.$$

Finally, the localization and reconstruction operators can be chosen simultaneously for r_0 and r_1: the Euclidean extension operator is universal. They are therefore a coretraction and a retraction between the section spaces and finite products of Euclidean spaces. Interpolating both identities through Theorem 6.3.11 proves the equality in (c). □

27.3 Sobolev mapping manifolds

Let $(M^m, \mathbf{g})$ and $(N^n, \mathbf{h}_N)$ be smooth finite-dimensional Riemannian manifolds, with N without boundary; M may have boundary. Let $1 < p < \infty$, $r \in \mathbb{N}$. The basic condition

$$rp > m \tag{27.10}$$

ensures that $W^{r,p}$ classes have continuous representatives. This continuity is indispensable: without it, one cannot require two maps to take values in a common normal neighborhood pointwise.

Definition 27.3.1. (Sobolev maps) Let M be a smooth compact manifold with or without boundary and let N be a smooth manifold without boundary. A continuous map $f : M \longrightarrow N$ is of class $W^{r,p}$ if, for finite collections of charts in the base and target adapted to the image of f, its coordinate representations belong to $W^{r,p}$. The resulting space is denoted by $W^{r,p}(M, N)$.

If f is only Sobolev, f^*TN is initially a continuous bundle. The notation $W^{r,p}(M, f^*TN)$ is defined using trivializations induced by charts of N: their transition matrices are smooth superpositions of f and belong locally to $W^{r,p}$. Since $rp > m$, multiplication by these matrices and their inverses is continuous in $W^{r,p}$. A finite partition of

unity thus gives equivalent norms on the section space. No smoothness is attributed to f^*TN before its construction. If the center f is smooth, this definition agrees with the covariant norm in the chapter on Sobolev spaces on bundles, by Lemma 9.1.28.

Fix $f \in W^{r,p}(M,N)$ with M compact. The map

$$\mathcal{E}\colon TN \supset \mathcal{O} \longrightarrow N \times N, \qquad v \longmapsto \big(\pi(v), \exp_{\pi(v)} v\big),$$

is a local diffeomorphism along the zero section. Since $f(M)$ is compact, there exist a neighborhood $\mathcal{O}_f$ of the zero section over $f(M)$ and a number $\rho_f > 0$ such that $|v|_{\mathbf{h}_N} < \rho_f$ implies $v \in \mathcal{O}_f$ and $\mathcal{E}$ is injective on this set. By Sobolev embedding,

$$\mathcal{V}_f := \{\mathbf{X} \in W^{r,p}(M, f^*TN) \mid \|\mathbf{X}\|_{C^0(M, f^*TN; f^*\mathbf{h}_N)} < \rho_f\}$$

is open. Define

$$\operatorname{Exp}_f(\mathbf{X})(x) := \exp^N_{f(x)} \mathbf{X}(x). \tag{27.11}$$

Lemma 27.3.2. (Superposition in the algebra regime) Let M be a smooth compact manifold with or without boundary, let $U \subseteq \mathbb{R}^a$ be open, and let $F\colon U \longrightarrow \mathbb{R}^b$ be smooth. If $rp > m$, the superposition map $u \mapsto F \circ u$ is smooth on the open set $\{u \in W^{r,p}(M,\mathbb{R}^a) \mid u(M) \subseteq U\}$. On each neighborhood whose values lie in a compact set $K \Subset U$, the estimates depend only on the geometry of M, r, p, a bound on the $W^{r,p}$ norm of the arguments, and bounds on the derivatives of F on a neighborhood of K. Its differential is

$$D(F_*)(u)v = (DF \circ u)v.$$

Proof. Since $rp > m$, Sobolev embedding gives each class a continuous representative; below we identify the class with this representative. Set

$$\mathcal{U} := \{u \in W^{r,p}(M,\mathbb{R}^a) \mid u(M) \subseteq U\}.$$

First check that $\mathcal{U}$ is open. If $u \in \mathcal{U}$, compactness of M and continuity of u imply that $u(M)$ is a compact set contained in U. If $U \neq \mathbb{R}^a$, set $d_u := \operatorname{dist}(u(M), \mathbb{R}^a \setminus U) > 0$; if $U = \mathbb{R}^a$, simply fix $d_u := 1$. Let $C_{\mathrm{emb}} > 0$ be a constant for the embedding $W^{r,p}(M,\mathbb{R}^a) \hookrightarrow C^0(M,\mathbb{R}^a)$. If $\|v\|_{W^{r,p}(M,\mathbb{R}^a)} < \frac{d_u}{2C_{\mathrm{emb}}}$, then $\|v\|_{C^0(M,\mathbb{R}^a)} < \frac{d_u}{2}$ and $(u+v)(M) \subseteq U$. Thus $\mathcal{U}$ is open.

Now fix a bounded subset $\mathcal{B} \subseteq \mathcal{U}$ whose images lie in a common compact set $K \Subset U$. Choose another compact set $K_1 \Subset U$ whose interior contains K. Decreasing $\delta > 0$, the preceding embedding ensures that, if $u \in \mathcal{B}$ and $\|v\|_{W^{r,p}(M,\mathbb{R}^a)} < \delta$, then $(u + tv)(M) \subseteq K_1$ for every $t \in [0,1]$.

For a smooth function $G\colon U \longrightarrow \mathbb{R}^c$, the Faà di Bruno formula in Theorem 1.1.19, applied in charts, expresses each derivative of order at most r of $G \circ u$ as a finite sum of terms of the form

$$(D^\ell G \circ u)\big[D^{\alpha_1}u, \dots, D^{\alpha_\ell}u\big], \qquad 1 \le \ell \le r, \quad |\alpha_1| + \cdots + |\alpha_\ell| \le r, \quad \alpha_i \in \mathbb{N}_0^m, \quad |\alpha_i| \ge 1 \quad (1 \le i \le \ell).$$

To estimate each product, work on a bounded coordinate domain $\Omega \subseteq \mathbb{R}^m$ with fixed cutoff functions, and write u for its local representation. If $d_i = |\alpha_i|$, choose $q_i = rp/d_i$. The Sobolev embedding of order $r - d_i$ gives

$$\|D^{\alpha_i} u\|_{L^{q_i}(\Omega,\mathbb{R}^a)} \le C \|u\|_{W^{r,p}(\Omega,\mathbb{R}^a)}.$$

Indeed, $1/p-(r-d_i)/m \le d_i/(rp)$ because $rp > m$; if $d_i = r$, simply use $q_i = p$. Since $\sum_{i=1}^{\ell} 1/q_i = \sum_{i=1}^{\ell} d_i/(rp) \le 1/p$, Hölder's inequality and the finite measure of the region control the product in L^p. The factor $D^\ell G \circ u$ is controlled by its supremum on K_1. This argument, also applied to Leibniz's rule, proves the algebra bound

$$\|vw\|_{W^{r,p}(M)} \le C\|v\|_{W^{r,p}(M)}\|w\|_{W^{r,p}(M)}.$$

Summing over the multi-indices and finitely many charts gives a constant $C = C(M, r, p, \mathcal{B}, K_1, G) > 0$ such that

$$\|G \circ u\|_{W^{r,p}(M,\mathbb{R}^c)} \le C \qquad (u \in \mathcal{B}), \tag{27.12}$$

and, if $u_0, u_1 \in \mathcal{B}$ and the segment joining their values remains in K_1,

$$\|G \circ u_1 - G \circ u_0\|_{W^{r,p}(M,\mathbb{R}^c)} \le C\|u_1 - u_0\|_{W^{r,p}(M,\mathbb{R}^a)}. \tag{27.13}$$

Indeed, for smooth functions the first bound follows term by term from the preceding formula. For the second, write

$$G \circ u_1 - G \circ u_0 = \int_0^1 (DG \circ (u_0 + t(u_1 - u_0)))(u_1 - u_0)\, dt$$

and apply the same product estimates. To extend the bounds, let $u \in \mathcal{B}$ and choose, by Meyers–Serrin 9.4.8 if there is boundary and 9.1.48 if there is none, a sequence $u_j \in C^\infty(M, \mathbb{R}^a)$ with $u_j \to u$ in $W^{r,p}(M, \mathbb{R}^a)$. Embedding into $C^0(M, \mathbb{R}^a)$ gives uniform convergence. After discarding finitely many terms, the values of u_j lie in K_1 and their Sobolev norms have a common bound. The segments joining $u_j(x)$ and $u_k(x)$ remain in a compact neighborhood of K contained in U when the approximation is sufficiently close. By the inequality already proved for smooth arguments,

$$\|G \circ u_j - G \circ u_k\|_{W^{r,p}(M,\mathbb{R}^c)} \le C\|u_j - u_k\|_{W^{r,p}(M,\mathbb{R}^a)} \longrightarrow 0.$$

The limit exists by completeness and agrees with $G \circ u$ by uniform convergence. Taking norms and limits gives (27.12). For two arguments u_0, u_1, choose both approximations in the same compact set and apply the smooth bound to their differences; their limits give (27.13). The constants depend on the stated Sobolev bounds and the derivatives of G on the chosen compact neighborhood, independently of j and k.
For $q \in \mathbb{N}_0$, define

$$A_q(u)[v_1, \dots, v_q] := (D^q F \circ u)[v_1, \dots, v_q],$$

, with the convention $A_0(u) = F \circ u$. The product estimate and (27.12), applied to $G = D^q F$, show that $A_q(u)$ is a continuous q–linear operator from $W^{r,p}(M, \mathbb{R}^a)^q$ to $W^{r,p}(M, \mathbb{R}^b)$ and that $u \mapsto A_q(u)$ is continuous in the operator norm. Moreover, for any $w_1, \dots, w_q \in W^{r,p}(M, \mathbb{R}^a)$, the fundamental theorem of calculus gives, pointwise,

$$\big(A_q(u+v) - A_q(u) - A_{q+1}(u)[v]\big)[w_1, \dots, w_q] = \int_0^1 (1-t)(D^{q+2} F \circ (u + tv))[v, v, w_1, \dots, w_q]\, dt.$$

Taking the operator norm and applying the preceding bounds again, there exist $\delta, C_q > 0$ such that

$$\|A_q(u+v) - A_q(u) - A_{q+1}(u)[v]\|_{\mathcal{L}^q(W^{r,p}(M,\mathbb{R}^a); W^{r,p}(M,\mathbb{R}^b))} \le C_q \|v\|^2_{W^{r,p}(M,\mathbb{R}^a)}$$

for $u \in \mathcal{B}$ and $\|v\|_{W^{r,p}(M,\mathbb{R}^a)} < \delta$. For $q = 0$, division of the remainder by $\|v\|_{W^{r,p}(M,\mathbb{R}^a)}$ proves that $DF_* = A_1$. If $D^q F_* = A_q$ has been identified, the same operator norm estimate proves that A_q is differentiable with differential A_{q+1}; the continuity already established gives $F_* \in C^{q+1}$. This step, for each $q \in \mathbb{N}_0$, identifies $A_q = D^q F_*$ at all orders. In particular,

$$D(F_*)(u)v = (DF \circ u)v,$$

, and F_* is smooth on $\mathcal{U}$. □

Corollary 27.3.3. (Superposition at real Hilbert orders) The preceding lemma holds with $W^{r,p}$ replaced by H^s for every real index $s > m/2$. Consequently, the compact constructions in this chapter using H^s also admit noninteger indices.

Proof. We explain the estimate that handles these indices. Localize in a chart using cutoff functions and use the extension operator from the trace chapter when the chart meets the boundary. It suffices to estimate superposition on functions in $H^s(\mathbb{R}^m)$, multiplying the result by a fixed cutoff function χ. Around a given argument, all values lie in a compact subset of U. Extend F from a neighborhood of this compact set to a smooth map on all of $\mathbb{R}^a$ with bounded derivatives; the extension does not change the superposition under study.

Take the low-frequency operators S_j and the blocks $\Delta_j = S_{j+1} - S_j$, for $j \in \mathbb{N}_0$, of an inhomogeneous dyadic decomposition. Convergence $S_j u \to u$ in C^0 and H^s gives the telescoping identity

$$F(u) = F(S_0 u) + \sum_{j=0}^{\infty} a_j \Delta_j u, \qquad a_j(x) = \int_0^1 DF\big(S_j u(x) + t\Delta_j u(x)\big)\, dt.$$

For each $N \in \mathbb{N}_0$, Bernstein's inequality and the chain rule give

$$\|D^\alpha a_j\|_\infty \le C_N 2^{j|\alpha|}, \qquad j \in \mathbb{N}_0, \quad \alpha \in \mathbb{N}_0^m, \quad |\alpha| \le N.$$

Here C_N depends on a bound for $\|u\|_\infty$ and the derivatives of F through order $N+1$. Applying Leibniz's rule to the product and Bernstein's inequality to the block $\Delta_j u$ gives $\|D^\alpha(\chi a_j \Delta_j u)\|_{L^2(\mathbb{R}^m,\mathbb{R}^b)} \le C_N 2^{j|\alpha|}\|\Delta_j u\|_{L^2(\mathbb{R}^m,\mathbb{R}^a)}$. If $k > j$, the vanishing moments of the kernel of Δ_k allow subtraction of the Taylor polynomial of order $N-1$ of the product. The integral Taylor remainder, together with the preceding bound, gives

$$\|\Delta_k(\chi a_j \Delta_j u)\|_{L^2(\mathbb{R}^m,\mathbb{R}^b)} \le C_N \min\{1, 2^{-N(k-j)}\}\|\Delta_j u\|_{L^2(\mathbb{R}^m,\mathbb{R}^a)}, \qquad j,k \in \mathbb{N}_0.$$

For $k \le j$, we have simply used the L^1 bound on the kernel of Δ_k and the uniform bound on a_j. Choose an integer $N > s$. After multiplication by 2^{ks}, the two regions produce the factors $2^{-(N-s)(k-j)}$ and $2^{-s(j-k)}$, respectively. Both sequences are summable. The convolution inequality in $\ell^2(\mathbb{N}_0)$ and the characterization $H^s = F^s_{2,2}$ from the chapter on intermediate regularity imply

$$\|\chi F(u)\|_{H^s(\mathbb{R}^m,\mathbb{R}^b)} \le C\big(1 + \|u\|_{H^s(\mathbb{R}^m,\mathbb{R}^a)}\big).$$

The low-frequency term is estimated by differentiating $\chi F(S_0 u)$ through order N; compact support of χ makes its L^2 norm finite.

Apply this bound also to $DF, D^2F, \ldots$. Since H^s is an algebra, the integral identity for $F(u+v) - F(u)$ gives

local Lipschitz continuity, and the second-order integral remainder gives differentiability with differential $(DF \circ u)v$. For each $q \in \mathbb{N}_0$, applying the same remainder to $D^q F$ identifies the differential of A_q with A_{q+1}, exactly as in the preceding proof. The estimates are local in the H^s norm and the cover of M is finite; combining them gives the asserted smoothness. □

Theorem 27.3.4. (Exponential charts) Let M be a smooth compact manifold with or without boundary and N a smooth finite-dimensional Riemannian manifold without boundary, and suppose $rp > m$. For sufficiently small ρ_f, the map (27.11) is a diffeomorphism from $\mathcal{V}_f$ onto an open subset of $W^{r,p}(M,N)$. These charts define a Banach manifold independent of the auxiliary metric on N. The operator

$$J_f := D(\operatorname{Exp}_f^{-1})_f \colon T_f W^{r,p}(M,N) \longrightarrow W^{r,p}(M, f^*TN)$$

is a continuous linear isomorphism with inverse $D\operatorname{Exp}_f(0)$. In particular,

$$T_f W^{r,p}(M,N) \cong W^{r,p}(M, f^*TN) \tag{27.14}$$

canonically.

The construction goes back to [85, Sections 6–8] and [111, Chapters II–III].

Proof. Let $\mathcal{W}_f := \mathcal{E}(\mathcal{O}_f)$. The inverse of Exp_f on the set of h for which $(f(x), h(x)) \in \mathcal{W}_f$ for every x is

$$\log_f(h)(x) := \mathcal{E}^{-1}(f(x), h(x)) = \exp_{f(x)}^{-1} h(x).$$

The condition on (f,h) is open in the C^0 topology because M is compact and the graph of (f,h) is compact inside $\mathcal{W}_f$; by $W^{r,p} \hookrightarrow C^0$, it is also open in the Sobolev topology.
To prove smoothness, choose a finite cover of M over which f^*TN is trivialized, and charts of N containing the relevant compact sets. In these coordinates, $\mathbf{X} \mapsto \operatorname{Exp}_f \mathbf{X}$ and $h \mapsto \log_f h$ are superposition operators given by smooth maps whose derivatives are bounded on a compact set. Lemma 27.3.2 proves that they are smooth; a finite partition of unity combines the local expressions.
If the charts are centered at f and g, the coordinate change has the intrinsic expression

$$\Theta_{gf}(\mathbf{X})(x) := \exp_{g(x)}^{-1}\big(\exp_{f(x)} \mathbf{X}(x)\big). \tag{27.15}$$

The preceding argument, applied to the smooth map $(y,z,v) \mapsto \exp_z^{-1}(\exp_y v)$ on its domain of definition, proves that (27.15) is smooth. This directly verifies compatibility of the atlas. If the auxiliary metric is changed, the same formula with the two exponential maps shows that the identity between the two atlases is smooth in both directions. Finally, $D\operatorname{Exp}_f(0)\mathbf{X} = \mathbf{X}$ pointwise. Since Exp_f and its inverse are smooth, their differentials are continuous; differentiating the two inverse identities gives $J_f D\operatorname{Exp}_f(0) = \mathrm{id}$ and $D\operatorname{Exp}_f(0) J_f = \mathrm{id}$. This identification is independent of the chart and gives (14). □

Proposition 27.3.5. (Evaluation and composition) Let M be a smooth compact manifold with or without boundary and let N and P be smooth manifolds without boundary. Under the remaining assumptions of the preceding theorem:

(i) for each $x \in M$, evaluation $\operatorname{ev}_x(f) = f(x)$ is smooth;

(ii) joint evaluation $\mathrm{ev}\colon W^{r,p}(M,N)\times M\longrightarrow N$ is continuous;

(iii) if $r>\frac{m}{p}+j$, joint evaluation is of class C^j;

(iv) if $H\colon N\longrightarrow P$ is smooth, the map $f\mapsto H\circ f$ is smooth on open sets where the required derivatives of H are uniformly bounded;

(v) the map $(f,\eta)\mapsto f\circ\eta$ loses derivatives: in the Hilbert case, for $s>\frac{m}{2}+1$,

$$H^{s+j}(M,N)\times \mathrm{Diff}^s(M)\longrightarrow H^s(M,N),\qquad (f,\eta)\longmapsto f\circ\eta,$$

is of class C^j, whereas at the same level $H^s\times\mathrm{Diff}^s\longrightarrow H^s$ only continuity is asserted in general.

The composition estimates used in the last item are those of [46, Section 2].

Proof. In exponential coordinates, evaluation has the form $(\mathbf{X},x)\mapsto \exp_{f(x)}\mathbf{X}(x)$. The continuous embedding $W^{r,p}\hookrightarrow C^0$ makes point evaluation continuous, and if $r>\frac{m}{p}+j$, embedding $W^{r,p}\hookrightarrow C^j$ allows this formula to be differentiated j times with respect to x and $\mathbf{X}$; each differential is a finite sum of evaluations of derivatives of $\mathbf{X}$ multiplied by smooth coefficients of the exponential map. This proves the first three items. The fourth is Lemma 27.3.2. For the last, the first formal derivative with respect to η contains

$$D_\eta(f\circ\eta)\mathbf{Y}=(df\circ\eta)\mathbf{Y},$$

and uses one derivative of f. Iteration uses j derivatives. The cited composition estimates prove continuity and the C^j assertions with these exact domains. □

On a noncompact base, charts require common estimates in the spatial variable, but their constants may depend on the center. It is useful to construct the atlas first and keep this issue separate from completion of a class with fixed profile.

Fix $k\ge r>m/p+1$ and a normal radius ρ as in Lemma 27.2.15. For each smooth center $f\in C^{\infty,r}_b(M,N)$, set

$$\mathcal{V}_f=\{\mathbf{X}\in W^{r,p}(M,f^*TN)\mid \|\mathbf{X}\|_\infty<\rho\},\qquad \mathcal{U}_f=\mathrm{Exp}_f(\mathcal{V}_f),\qquad \Omega^{p,r}(M,N)=\bigcup_{f\in C^{\infty,r}_b(M,N)}\mathcal{U}_f.$$

The elements of this union are actual maps; if two exponential images agree pointwise, they represent the same element. Give the union the topology of these charts. Compatibility and openness of overlaps are checked in the following proof.

Theorem 27.3.6. (Sobolev charts on a complete noncompact base) Let $(M^m,\mathbf{g})$ and $(N^n,\mathbf{h})$ be smooth, complete, without boundary, and of bounded geometry of order k. Let $1<p<\infty$ and $k\ge r>m/p+1$, with $k,r\in\mathbb{N}$. The preceding charts equip $\Omega^{p,r}(M,N)$ with a Banach manifold structure of class C^{k+1-r}, and a Hilbert manifold structure if $p=2$. In the case of bounded geometry of all orders, the atlas is smooth. If $\Phi_f(\mathbf{Y})=\mathrm{Exp}_f\mathbf{Y}$ is a chart centered at f, then

$$J_f:=D(\Phi_f^{-1})_f:\ T_f\Omega^{p,r}(M,N)\longrightarrow W^{r,p}(M,f^*TN)$$

is linear and bicontinuous, with inverse $D\Phi_f(0)$.

Proof. The Sobolev norm controls the uniform norm, so $\mathcal{V}_f$ is open. The exponential map is injective on this set by the choice of ρ. The localization estimates in the bounded geometry chapter allow us to work in balls and trivializations of a common size; their constants now depend on bounds on the derivatives of the center f.

In these representations, a superposition has the form $\mathbf{X} \mapsto F(x, \mathbf{X}(x))$. The required mixed derivatives of F are uniformly bounded on the normal domain. Through spatial order r and order $k+1-r$ of differentiation with respect to the section, these bounds follow from the Jacobi and transport equations with the recurrence (13.4); this is also the local estimate used in [49, Theorem 5.2]. Differentiating r times in x, the chain rule uses only derivatives of the center through order r and mixed derivatives of the exponential map through order $r + (k+1-r) = k+1$.

On a base of infinite volume, a constant cannot be estimated by its L^p norm. The superposition argument is therefore applied to differences. If $F(x,0) = 0$, the identity

$$F(x, \mathbf{X}(x)) = \int_0^1 D_v F(x, t\mathbf{X}(x))\mathbf{X}(x)\, dt$$

ensures that each term of its spatial derivative contains a derivative of $\mathbf{X}$, even when all remaining derivatives fall on x. Sobolev products and uniform coefficient bounds give a $W^{r,p}$ bound on each ball. The sum of the p powers of these bounds converges by finite multiplicity of the cover. For a nonzero center $\mathbf{X}_0$, use the analogous identity with $F(x,\mathbf{X}) - F(x,\mathbf{X}_0)$ and the factor $\mathbf{X} - \mathbf{X}_0$. The integral Taylor remainder in the variable v proves differentiability; repeating it for $D_v^a F$, with $a \in \{0, \dots, k-r\}$, gives the successive orders. At the last order, continuity follows from uniform continuity of the coefficients on the normal domain and the same product estimates. This yields C^{k+1-r} without integrating any constant term.

Consider an overlap and write $h = \operatorname{Exp}_f \mathbf{X}_0 = \operatorname{Exp}_g \mathbf{Y}_0$, with smooth centers f, g. Then $d(f(x), g(x)) < 2\rho$ for every x. The transition function is

$$\Psi(x, v) = \log_{g(x)}\bigl(\exp_{f(x)} v\bigr).$$

A sufficiently small neighborhood of $\mathbf{X}_0$ in $W^{r,p}$ keeps the arguments in the normal domain. To justify integrability of the transition, write

$$\Psi(x, \mathbf{X}(x)) - \Psi(x, \mathbf{X}_0(x)) = \int_0^1 D_v \Psi\bigl(x, \mathbf{X}_0(x) + t(\mathbf{X}(x) - \mathbf{X}_0(x))\bigr)(\mathbf{X}(x) - \mathbf{X}_0(x))\, dt.$$

The second term on the left is precisely $\mathbf{Y}_0 \in W^{r,p}(M, g^*TN)$. The preceding argument shows that the difference belongs to this space and depends on $\mathbf{X}$ with the stated regularity. Shrinking the neighborhood makes its uniform norm less than $\rho - \|\mathbf{Y}_0\|_\infty$. This positive margin proves openness of the overlap. Interchanging f and g gives the inverse transition.

Point evaluations are continuous in each chart. Two distinct maps differ at some point of M, and two disjoint neighborhoods of their values in N separate them; the constructed space is Hausdorff. This verifies the requirements for a Banach manifold in Definition 24.1.1.

If h has Sobolev regularity, sections of h^*TN are defined in Sobolev trivializations, as in the compact case, or transported from a smooth center of a chart containing h. The transport matrices and changes of trivialization are multipliers in $W^{r,p}$ by the preceding estimates. The resulting norms are equivalent. This allows charts to be recentered at h without assuming h^*TN is a smooth bundle. Finally, $D\Phi_h(0)\mathbf{X} = \mathbf{X}$ pointwise; differentiating the inverse identities gives the two continuous operators in the statement. □

Smooth centers are locally dense: approximate a section $\mathbf{X} \in \mathcal{V}_f$ by smooth compactly supported sections in $W^{r,p}(M, f^*TN)$. Uniform embedding keeps the approximations inside $\mathcal{V}_f$, and their images under Exp_f converge to the initial map. Chart radii and transition constants may vary with the center. This local assertion does not assign a constant-radius uniformity to all maps with bounded derivatives.

27.4 Spaces of metrics, connections, and gauge theory

27.4.1 Riemannian metrics

Let M^m be a smooth compact manifold with or without boundary and let $s > \frac{m}{2}$. Define

$$\mathrm{Met}^s(M) := \{\mathbf{g} \in H^s(M, S^2T^*M) \mid \mathbf{g}_x \text{ is positive definite for every } x \in M\}.$$

Proposition 27.4.1. (Openness of positivity) Let M^m be a smooth compact manifold with or without boundary and let $s > \frac{m}{2}$. Then $\mathrm{Met}^s(M)$ is an open subset of the Hilbert space $H^s(M, S^2T^*M)$. In particular, it is a Hilbert manifold and $T_{\mathbf{g}}\,\mathrm{Met}^s(M) = H^s(M, S^2T^*M)$.

Proof. By compactness, the smallest eigenvalue of $\mathbf{g}$ relative to an auxiliary metric $\mathbf{g}_0$ has a minimum $c_{\mathbf{g}} > 0$. The embedding

$$H^s(M, S^2T^*M) \hookrightarrow C^0(M, S^2T^*M; \mathbf{g}_0)$$

gives $C > 0$ such that

$$\|\mathbf{h}\|_{C^0(M,S^2T^*M;\mathbf{g}_0)} \le C\|\mathbf{h}\|_{H^s(M,S^2T^*M)}.$$

If $\|\mathbf{h}\|_{H^s(M,S^2T^*M)} < \frac{c_{\mathbf{g}}}{2C}$, then $(\mathbf{g}+\mathbf{h})(v,v) \ge (\frac{c_{\mathbf{g}}}{2})|v|^2_{\mathbf{g}_0}$ for every $v \in T_xM$. Thus the corresponding ball lies in $\mathrm{Met}^s(M)$. Since this set is open, its tangent space is the model space. □

On a noncompact manifold, the smallest eigenvalue may tend to zero at infinity; positivity is not open in an absolute Sobolev topology. For example, on $\mathbb{R}$ the metric $\mathbf{g} = e^{-2x^2}(\mathbf{d}x)^2$ is positive, but there exist cutoff functions χ_j translated toward j, with uniformly bounded derivatives, such that $\mathbf{h}_j = -2e^{-2x^2}\chi_j dx^2$ tends to zero in every H^s while $\mathbf{g}+\mathbf{h}_j$ is negative where $\chi_j = 1$. This difficulty disappears when working within a uniform equivalence class.

Two metrics $\mathbf{g}$ and $\mathbf{g}'$ are *quasi-isometric* if there exists $C \ge 1$ such that

$$C^{-1}\mathbf{g} \le \mathbf{g}' \le C\mathbf{g} \tag{27.16}$$

in the sense of Definition 2.1.5; that is, a common constant C satisfies

$$C^{-1}\mathbf{g}_x(v,v) \le \mathbf{g}'_x(v,v) \le C\mathbf{g}_x(v,v) \qquad (x \in M,\ v \in T_xM).$$

Equivalently, the tangent norms satisfy

$$C^{-1/2}|v|_{\mathbf{g}} \le |v|_{\mathbf{g}'} \le C^{1/2}|v|_{\mathbf{g}} \qquad (x \in M,\ v \in T_xM).$$

This condition controls the values of the metrics on all fibers with common constants; control of the connections is added in the following functional. If $\nabla^{\mathbf{g}}$ and $\nabla^{\mathbf{g}'}$ are their Levi–Civita connections, define the comparison

functional

$$\delta_{b,q;\mathbf{g}}(\mathbf{g},\mathbf{g}') := \|\mathbf{g}-\mathbf{g}'\|_{\infty;\mathbf{g}} + \sum_{j=0}^{q-1} \|(\nabla^{\mathbf{g}})^j(\nabla^{\mathbf{g}'}-\nabla^{\mathbf{g}})\|_{\infty;\mathbf{g}}. \tag{27.17}$$

The infinity norms use the metrics on the indicated bundles; when confusion could arise, the metric is added after a semicolon. The difference of connections is a tensor of type $(1,2)$, so all terms are intrinsically defined.

For $k \in \mathbb{N}_0$, denote by $\mathrm{Met}(I,B_k)(M)$ the set of smooth Riemannian metrics $\mathbf{g}$ satisfying conditions (I) and (B_k) in Definition 27.2.14; that is, there exist constants $i_{\mathbf{g}}, K_0(\mathbf{g}), \dots, K_k(\mathbf{g}) > 0$ such that

$$\mathrm{inj}(M,\mathbf{g}) \geq i_{\mathbf{g}}, \qquad \|(\nabla^{\mathbf{g}})^j \mathbf{R}_{\mathbf{g}}\|_{\infty;\mathbf{g}} \leq K_j(\mathbf{g}) \quad (0 \leq j \leq k).$$

The first condition implies that $(M,\mathbf{g})$ is complete. In the following Sobolev construction, both metrics of each pair belong to this set.

If $\mathbf{T}$ is a tensor field of fixed type and $j \in \mathbb{N}_0$, we use $\|\mathbf{T}\|_{b,j;\mathbf{g}} := \sum_{a=0}^{j} \|(\nabla^{\mathbf{g}})^a \mathbf{T}\|_{\infty;\mathbf{g}}$. For $1 < p < \infty$, we also set

$$\|\mathbf{T}\|_{p,j;\mathbf{g}} := \left(\sum_{a=0}^{j} \int_M |(\nabla^{\mathbf{g}})^a \mathbf{T}|_{\mathbf{g}}^p \, d\lambda_{\mathbf{g}} \right)^{\frac{1}{p}}.$$

Both notations are used only when the right-hand side is finite.

Theorem 27.4.2. (Uniform comparison of metrics) Let M^m be smooth and without boundary. For $q \in \mathbb{N}$, the sets

$$\mathcal{U}_{\varepsilon}^{b,q} := \{(\mathbf{g},\mathbf{g}') \mid \mathbf{g} \text{ and } \mathbf{g}' \text{ are quasi-isometric and } \delta_{b,q;\mathbf{g}}(\mathbf{g},\mathbf{g}') < \varepsilon\}$$

generate a separated metrizable uniformity on the smooth complete metrics. In the Sobolev regime, fix a reference metric $\mathbf{g}_0 \in \mathrm{Met}(I,B_k)(M)$, $1 < p < \infty$, and $k \geq r > m/p + 1$. The intrinsic comparison is written as

$$\delta_{p,r;\mathbf{g}}(\mathbf{g},\mathbf{g}')^p := \int_M \left(|\mathbf{g}-\mathbf{g}'|_{\mathbf{g}}^p + \sum_{j=0}^{r-1} |(\nabla^{\mathbf{g}})^j(\nabla^{\mathbf{g}'}-\nabla^{\mathbf{g}})|_{\mathbf{g}}^p \right) d\lambda_{\mathbf{g}}. \tag{27.18}$$

On the affine space $\mathbf{g}_0 + W^{r,p}(M, S^2T^*M; \mathbf{g}_0)$, we use the distance

$$d_{\mathbf{g}_0}(\mathbf{g},\mathbf{g}') := \|\mathbf{g}'-\mathbf{g}\|_{p,r;\mathbf{g}_0}.$$

On every set of smooth metrics satisfying, for fixed numbers $\lambda > 0$ and $R > 0$,

$$\mathbf{g} \geq \lambda \mathbf{g}_0, \qquad \|\mathbf{g}-\mathbf{g}_0\|_{p,r;\mathbf{g}_0} \leq R,$$

, there exist $c, C > 0$ common to all pairs such that

$$c\, d_{\mathbf{g}_0}(\mathbf{g},\mathbf{g}') \leq \delta_{p,r;\mathbf{g}}(\mathbf{g},\mathbf{g}') \leq C\, d_{\mathbf{g}_0}(\mathbf{g},\mathbf{g}').$$

The constants depend on λ, R, p, r and the data of $\mathbf{g}_0$.

Proof. If $\|\mathbf{g}' - \mathbf{g}\|_{\infty;\mathbf{g}} < 1/2$, for every $x \in M$ and $v \in T_xM$, Cauchy–Schwarz in the tensor inner product gives

$$|(\mathbf{g}' - \mathbf{g})_x(v,v)| \le |\mathbf{g}' - \mathbf{g}|_{\mathbf{g}}(x)|v|^2_{\mathbf{g}} \le \frac{1}{2}\mathbf{g}_x(v,v).$$

Adding $\mathbf{g}_x(v,v)$ gives

$$\frac{1}{2}\mathbf{g}_x(v,v) \le \mathbf{g}'_x(v,v) \le \frac{3}{2}\mathbf{g}_x(v,v).$$

By Proposition 7.1.9, the pointwise tensor norms are comparable with constants depending only on the tensor type. The change-of-connection formula (27.21), proved below, expresses derivatives in the new connection as finite sums of products of derivatives of the connection difference. Applied to $\mathbf{g}' - \mathbf{g}$ and $\nabla^{\mathbf{g}'} - \nabla^{\mathbf{g}}$, it gives polynomials with no constant term controlling inversion. For three metrics, add their connection differences and apply the same formula when changing connection in the second summand. This gives corresponding composition polynomials independent of the metrics. This verifies the inversion and composition axioms; the other axioms follow from the diagonal and order-zero control. Rational radii and metrization by chains complete the argument, also formulated in [50, Proposition 2.1].

For Sobolev norms, the embedding gives $\|\mathbf{g} - \mathbf{g}_0\|_{\infty;\mathbf{g}_0} \le C_S R$. The same evaluation on (v,v) gives $\mathbf{g}_x(v,v) \le (1 + C_S R)(\mathbf{g}_0)_x(v,v)$ for every x and v. Together with the lower bound in the statement and Proposition 7.1.9, this implies

$$(1 + C_S R)^{-1}\mathbf{g}_0^{-1} \le \mathbf{g}^{-1} \le \lambda^{-1}\mathbf{g}_0^{-1}, \qquad \lambda^{m/2} \le \frac{d\lambda_{\mathbf{g}}}{d\lambda_{\mathbf{g}_0}} \le (1 + C_S R)^{m/2}.$$

Comparison of the inverse metrics is evaluated on covectors, and the ratio of measures is the positive function described in that proposition. These bounds are common to all metrics in the set under consideration. The Koszul formula (27.19) expresses the connection difference in terms of one metric's inverse and the first derivative of the other. Differentiating the identity $\mathbf{g}^{-1}\mathbf{g} = I$ gives $\nabla\mathbf{g}^{-1} = -\mathbf{g}^{-1}(\nabla\mathbf{g})\mathbf{g}^{-1}$; at each subsequent order, Leibniz's rule differentiates one of the existing factors. Thus, through order $r-1$, derivatives of the inverse and the connection difference are controlled by λ, R and the product estimates. In particular, $\nabla^{\mathbf{g}} - \nabla^{\mathbf{g}_0}$ is bounded in $W^{r-1,p}$ in terms of these data.

Set $\mathbf{h} = \mathbf{g}' - \mathbf{g}$ and $\mathbf{D} = \nabla^{\mathbf{g}'} - \nabla^{\mathbf{g}}$. Koszul's formula and the product estimates give $\|\mathbf{D}\|_{p,r-1;\mathbf{g}_0} \le C\|\mathbf{h}\|_{p,r;\mathbf{g}_0}$. For the reverse inequality, use the metric compatibility identity

$$(\nabla^{\mathbf{g}}_{\mathbf{X}}\mathbf{h})(\mathbf{Y},\mathbf{Z}) = \mathbf{g}'(\mathbf{D}(\mathbf{X},\mathbf{Y}),\mathbf{Z}) + \mathbf{g}'(\mathbf{Y},\mathbf{D}(\mathbf{X},\mathbf{Z})).$$

Starting at order zero for $\mathbf{h}$, differentiate this identity through order $r-1$. In the step from order j to $j+1$, terms in which $\mathbf{g}' = \mathbf{g} + \mathbf{h}$ is differentiated contain derivatives of $\mathbf{h}$ of order at most j, already controlled; the remaining terms contain a derivative of $\mathbf{D}$ of order at most j. The module estimate for $W^{r-1,p}$, valid because $r - 1 > m/p$, controls all these products. The change-of-connection formula allows each norm to be expressed using $\mathbf{g}_0$, and the volume ratios are bounded. We obtain $\|\mathbf{h}\|_{p,r;\mathbf{g}_0} \le C(\|\mathbf{h}\|_{p,0;\mathbf{g}} + \|\mathbf{D}\|_{p,r-1;\mathbf{g}})$, which is the remaining inequality. □

The fixed reference is essential in the Sobolev assertion. On S^1, with $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, take $\mathbf{g}_a = a^2 d\theta^2$ and $\mathbf{h}_a = a^6 d\theta^2$, for $0 < a < 1$. Their connections agree, and both metrics are complete and of bounded geometry, though their data are not common as a varies. For $p = 2$ and any integer order $r \ge 1$,

$$\delta_{2,r;\mathbf{g}_a}(\mathbf{g}_a,\mathbf{h}_a) = \sqrt{2\pi}\,a^{1/2}(1 - a^4), \qquad \delta_{2,r;\mathbf{h}_a}(\mathbf{h}_a,\mathbf{g}_a) = \sqrt{2\pi}\,a^{-5/2}(1 - a^4).$$

The first side tends to zero and the second to infinity. Thus the condition $\delta_{p,r;\mathbf{g}} < \varepsilon$ on all metrics without common data does not satisfy the inversion axiom. The distance $d_{\mathbf{g}_0}$ is symmetric and satisfies the triangle inequality; the preceding comparison allows the intrinsic expressions to be used in each neighborhood with fixed bounds.

Proposition 27.4.3. (Comparison of tensor norms in a component) Let M be a smooth manifold without boundary, let $\mathbf{g}$ and $\mathbf{g}'$ be smooth quasi-isometric metrics, and let $\mathbf{T}$ be a tensor field of fixed type.

(i) If $\delta_{b,q;\mathbf{g}}(\mathbf{g},\mathbf{g}') < \infty$, then, for each $0 \le j \le q$, there exist constants $c_{b,j}, C_{b,j} > 0$ such that

$$c_{b,j}\|\mathbf{T}\|_{b,j;\mathbf{g}} \le \|\mathbf{T}\|_{b,j;\mathbf{g}'} \le C_{b,j}\|\mathbf{T}\|_{b,j;\mathbf{g}}.$$

(ii) If $\mathbf{g},\mathbf{g}'$ belong to the affine space based at $\mathbf{g}_0$ in the preceding theorem and have a positive lower bound relative to $\mathbf{g}_0$, then, for each $0 \le j \le r$, there exist constants $c_{p,j}, C_{p,j} > 0$ such that

$$c_{p,j}\|\mathbf{T}\|_{p,j;\mathbf{g}} \le \|\mathbf{T}\|_{p,j;\mathbf{g}'} \le C_{p,j}\|\mathbf{T}\|_{p,j;\mathbf{g}}.$$

The constants depend on the tensor type, a quasi-isometry constant between $\mathbf{g}$ and $\mathbf{g}'$, and the norms of $(\nabla^{\mathbf{g}})^a(\nabla^{\mathbf{g}'} - \nabla^{\mathbf{g}})$ through the required order, but not on $\mathbf{T}$.

Proof. Set $\mathbf{D} := \nabla^{\mathbf{g}'} - \nabla^{\mathbf{g}}$. The Koszul identity gives

$$2\mathbf{g}'\big(\mathbf{D}(\mathbf{X},\mathbf{Y}),\mathbf{Z}\big) = (\nabla^{\mathbf{g}}_{\mathbf{X}}\mathbf{g}')(\mathbf{Y},\mathbf{Z}) + (\nabla^{\mathbf{g}}_{\mathbf{Y}}\mathbf{g}')(\mathbf{X},\mathbf{Z}) - (\nabla^{\mathbf{g}}_{\mathbf{Z}}\mathbf{g}')(\mathbf{X},\mathbf{Y}). \tag{27.19}$$

If $C^{-1}\mathbf{g} \le \mathbf{g}' \le C\mathbf{g}$, Proposition 7.1.9, applied with constants C^{-1} and C, gives, for a tensor of type (a,b),

$$C^{-(a+b)/2}|\mathbf{T}|_{\mathbf{g}} \le |\mathbf{T}|_{\mathbf{g}'} \le C^{(a+b)/2}|\mathbf{T}|_{\mathbf{g}}, \tag{27.20}$$

, and the ratio of volume elements satisfies

$$C^{-m/2} \le \rho(x) := \frac{\sqrt{\det(\mathbf{g}'_x)}}{\sqrt{\det(\mathbf{g}_x)}} \le C^{m/2}.$$

The determinants are computed in a common chart, and the ratio is independent of it. For $1 \le p < \infty$, we have

$$\|\mathbf{T}\|^p_{L^p(M,\mathbf{g}')} = \int_M |\mathbf{T}|^p_{\mathbf{g}'}\rho\, d\lambda_{\mathbf{g}}.$$

Raising (27.20) to the power p, multiplying by the bounds on ρ, and integrating gives

$$C^{-(a+b)/2-m/(2p)}\|\mathbf{T}\|_{L^p(M,\mathbf{g})} \le \|\mathbf{T}\|_{L^p(M,\mathbf{g}')} \le C^{(a+b)/2+m/(2p)}\|\mathbf{T}\|_{L^p(M,\mathbf{g})}.$$

For the uniform norm, simply take suprema in the pointwise comparison. These estimates compare the same tensor $\mathbf{T}$; the next step compares the derivatives obtained using the two connections.
The local expansion of the covariant derivative in Lemma 9.1.28 and the Leibniz rule in Proposition 9.1.40

allow us to write, for $j \in \mathbb{N}$,

$$(\nabla^{\mathbf{g}'})^j \mathbf{T} = (\nabla^{\mathbf{g}})^j \mathbf{T} + \sum_{\ell=1}^{j} \sum_{i_0+\cdots+i_\ell=j-\ell} C_{j,\boldsymbol{i}} (\nabla^{\mathbf{g}})^{i_1} \mathbf{D} * \cdots * (\nabla^{\mathbf{g}})^{i_\ell} \mathbf{D} * (\nabla^{\mathbf{g}})^{i_0} \mathbf{T}. \tag{27.21}$$

In the inner sum, $\boldsymbol{i} = (i_0, \dots, i_\ell)$ ranges over $\mathbb{N}_0^{\ell+1}$ subject to the indicated restriction. The coefficients are universal; each term with $*$ represents the finite sum of contractions and permutations specified by the tensor type. The endomorphisms appearing here are not assumed to commute. For $j = 1$, we obtain $\nabla^{\mathbf{g}'}\mathbf{T} = \nabla^{\mathbf{g}}\mathbf{T} + \mathbf{D} * \mathbf{T}$; on differentiating again, the derivative may act on any existing factor or introduce a new factor $\mathbf{D}$. In the first case, one of the i_a, with $a \in \{0, \dots, \ell\}$, increases by one, and the index sum changes from $j-\ell$ to $(j+1)-\ell$. In the second, ℓ increases by one and a zero index is added; the sum remains equal to $(j+1)-(\ell+1)$. These are all the terms of order $j+1$, and their orders and types are those in the formula. This completes the induction step.

In the b, q uniformity, supremum bounds on the derivatives of $\mathbf{D}$ and (27.20) control (27.21) term by term. In the Sobolev uniformity, $r > m/p + 1$ implies that $W^{r-1,p}$ is an algebra and its sections act as multipliers on lower orders. The product and module theorems, applied to each contraction, give the upper bound in part (ii). Interchanging $\mathbf{g}$ and $\mathbf{g}'$ gives the reverse inequalities in both cases. □

For fixed $\mathbf{g}_0 \in \mathrm{Met}(I, B_k)(M)$, denote by $\mathrm{Met}^{p,r}_{\mathbf{g}_0}(M)$ the set

$$\left\{ \mathbf{g}_0 + \mathbf{h} \;\middle|\; \mathbf{h} \in W^{r,p}(M, S^2T^*M; \mathbf{g}_0), \quad \mathbf{g}_0 + \mathbf{h} \geq c\mathbf{g}_0 \text{ for some } c > 0 \right\}.$$

This is the positive affine component considered when writing $\mathrm{Met}^{p,r}(I, B_k)(M)$ with a fixed reference. The complete affine space is $\mathbf{g}_0 + W^{r,p}$; uniform positivity selects an open subset of it, which need not be complete for the affine distance. In the uniform version, let $B^q_{\mathbf{g}_0}$ be the completion of the smooth sections bounded through order q in $\|\cdot\|_{b,q;\mathbf{g}_0}$. Define $\mathrm{Met}^{b,q}_{\mathbf{g}_0}(M)$ by requiring $\mathbf{h} \in B^q_{\mathbf{g}_0}$ and the same uniform positive inequality.

Theorem 27.4.4. (Affine manifolds of metrics) Let M^m be smooth, complete, and without boundary with respect to $\mathbf{g}_0 \in \mathrm{Met}(I, B_k)(M)$. If $1 < p < \infty$ and $k \geq r > m/p + 1$, the space $\mathrm{Met}^{p,r}_{\mathbf{g}_0}(M)$ is an open convex subset of $\mathbf{g}_0 + W^{r,p}(M, S^2T^*M; \mathbf{g}_0)$. It is a smooth Banach manifold, and a Hilbert manifold if $p = 2$. For the affine chart $\Phi_{\mathbf{g}}(\mathbf{g}') = \mathbf{g}' - \mathbf{g}$,

$$J_{\mathbf{g}} := D\Phi_{\mathbf{g}}(\mathbf{g}) : T_{\mathbf{g}} \mathrm{Met}^{p,r}_{\mathbf{g}_0}(M) \longrightarrow H^{r,p}(M, S^2T^*M; \mathbf{g}_0)$$

is the linear bicontinuous identity. Here we use equivalence of the $W^{r,p}$ and $H^{r,p}$ norms for integer orders from the bounded geometry chapter. The uniform construction is an open convex subset modeled on $B^q_{\mathbf{g}_0}$.

Proof. If $\mathbf{g} \geq c\mathbf{g}_0$, the uniform embedding gives $\|\mathbf{k}\|_{\infty;\mathbf{g}_0} \leq C_S \|\mathbf{k}\|_{p,r;\mathbf{g}_0}$. The condition $\|\mathbf{k}\|_{p,r;\mathbf{g}_0} < c/(2C_S)$ implies $\mathbf{g} + \mathbf{k} \geq (c/2)\mathbf{g}_0$, proving openness. A convex combination of two metrics with lower bounds $c_1, c_2 > 0$ has lower bound $(1-t)c_1 + tc_2 > 0$ for $t \in [0,1]$; thus the set is convex. Changes between affine charts are translations with identity differential. The uniform assertion is proved in the same way, using the fact that the $B^q_{\mathbf{g}_0}$ norm controls the infinity norm.

In the Sobolev component, pointwise positivity already implies a uniform lower bound. To see this, $\mathbf{h} \in W^{r,p}$ has bounded first derivative and tends uniformly to zero at infinity. If the latter assertion failed, there would exist $\varepsilon > 0$ and points x_j escaping every compact set, with $|\mathbf{h}(x_j)| \geq \varepsilon$. The bound on $\nabla\mathbf{h}$ and parallel

transport would give $|\mathbf{h}| \geq \varepsilon/2$ on balls of a common positive radius around these points. A disjoint family of balls can be extracted; their volumes have a common lower bound by bounded geometry. Then the sum of the integrals of $|\mathbf{h}|^p$ would be infinite, a contradiction. Thus, outside a compact set, $\mathbf{g}_0 + \mathbf{h} \geq \frac{1}{2}\mathbf{g}_0$. Inside it, continuity and positivity give a positive minimum for the smallest eigenvalue.
Smooth compactly supported sections approximate $\mathbf{h}$ in $W^{r,p}$. Uniform convergence ensures that the approximations retain a positive lower bound. The resulting smooth metrics agree with $\mathbf{g}_0$ outside a compact set, are complete, and satisfy (I) and (B_k), with constants that may depend on the approximation. This relates the affine model to approximation by smooth metrics. It does not assert that the limit retains curvature bounds of order k: that would require control of additional metric derivatives. □

Affine charts are those used in studying spaces of metrics on open manifolds; see [50]. Proposition 27.4.3 explains why, within each neighborhood with common bounds, one may use intrinsic norms for another metric in the component. The fixed reference metric makes explicit the topology used for completion and openness of positivity.

On a compact base, or in a Hilbert Sobolev component of a noncompact base, the Ebin metric is defined by

$$G_{\mathbf{g}}(\mathbf{h}, \mathbf{k}) := \int_M \operatorname{tr}_{\mathbf{g}}(\mathbf{h} \circ \mathbf{k})\, d\lambda_{\mathbf{g}}. \tag{27.22}$$

In the noncompact case, tangent vectors belong to L^2 and the metrics are uniformly equivalent to the reference; Hölder's inequality thus ensures that the integral is finite. For a $W^{r,p}$ component with $p \neq 2$, this integrability must additionally be required. In an H^s topology with $s > 0$, this metric controls only the L^2 norm and is therefore weak. Existence of a connection or an exponential map cannot be deduced from the Koszul formula alone.

27.4.2 Connections and gauge transformations

Let $\mathbf{P} \longrightarrow M$ be a principal bundle with compact structure group G and Lie algebra $\mathfrak{g}$ equipped with a Ad–invariant inner product. Denote by h_{ad} the induced metric on the adjoint bundle and by $\operatorname{ad}\mathbf{P} := \mathbf{P} \times_{\mathrm{Ad}} \mathfrak{g}$ the adjoint bundle.

Proposition 27.4.5. (Affine structure of the space of connections) Let M be a smooth manifold with or without boundary and let $\mathbf{P} \longrightarrow M$ be a smooth principal bundle. For a fixed smooth connection A_0, every smooth connection can be written uniquely as $A_0 + \mathbf{a}$, with $\mathbf{a} \in \Omega^1(M, \operatorname{ad}\mathbf{P})$. Consequently, for every Sobolev index s for which the completion has been fixed,

$$\mathcal{A}^s(\mathbf{P}) := A_0 + H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P})$$

is an affine space and

$$T_A\mathcal{A}^s(\mathbf{P}) = H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}).$$

Proof. The smooth assertion is Theorem 2.6.23: the difference of two principal connection forms is horizontal and Ad–equivariant and therefore descends to a unique form with values in $\operatorname{ad}\mathbf{P}$. Completing the model vector space in H^s gives the second assertion. No identification of a principal connection with an arbitrary connection on a vector bundle is made here. □

Proposition 27.4.6. (Right gauge action and curvature) Let M be a smooth manifold with or without boundary and let $\mathbf{P} \longrightarrow M$ be a smooth principal bundle. The convention fixed in (2.49) defines a right action:

$$A \triangleleft \mathbf{u} := \mathrm{Ad}_{\mathbf{u}^{-1}} A + \mathbf{u}^{-1} d\mathbf{u}, \qquad (A \triangleleft \mathbf{u}) \triangleleft \mathbf{v} = A \triangleleft (\mathbf{u}\mathbf{v}). \tag{27.23}$$

In a matrix representation, the first formula is written as $A \triangleleft \mathbf{u} = \mathbf{u}^{-1} A \mathbf{u} + \mathbf{u}^{-1} d\mathbf{u}$. Moreover,

$$\mathbf{F}_{A+\mathbf{a}} = \mathbf{F}_A + d_A \mathbf{a} + \frac{1}{2}[\mathbf{a} \wedge \mathbf{a}], \tag{27.24}$$

$$\mathbf{F}_{A \triangleleft \mathbf{u}} = \mathrm{Ad}_{\mathbf{u}^{-1}} \mathbf{F}_A. \tag{27.25}$$

For the Sobolev extension on a compact base, fix $s > m/2$; on a noncompact base, impose the assumptions for the gauge component constructed below. If

$$\Phi \colon \mathcal{A}^s(\mathbf{P}) \times \mathcal{G}^{s+1}(\mathbf{P}) \longrightarrow \mathcal{A}^s(\mathbf{P}), \qquad \Phi(A, \mathbf{u}) = A \triangleleft \mathbf{u},$$

and we trivialize $T_{\mathbf{u}} \mathcal{G}^{s+1}(\mathbf{P})$ by left translation, writing a tangent vector as $\mathbf{u}\xi$, then

$$D\Phi_{(A,\mathbf{u})}(\mathbf{a}, \mathbf{u}\xi) = \mathrm{Ad}_{\mathbf{u}^{-1}} \mathbf{a} + d_{A \triangleleft \mathbf{u}} \xi. \tag{27.26}$$

Proof. The first two identities are Propositions 2.6.35 and 2.6.21, written in a local section. In particular, the order of $\mathbf{u}\mathbf{v}$ in (27.23) is not an implicit convention: it comes from the right action already established. To compute the differential, take a curve $A(t)$ with $A(0) = A$, $\dot{A}(0) = \mathbf{a}$, and the exact curve $\mathbf{u}(t) = \mathbf{u} \exp_G(t\xi)$. By the action law,

$$A(t) \triangleleft \mathbf{u}(t) = \big(A(t) \triangleleft \mathbf{u}\big) \triangleleft \exp_G(t\xi).$$

The variation of the first factor is $\mathrm{Ad}_{\mathbf{u}^{-1}} \mathbf{a}$, while the derivative of $B \triangleleft \exp_G(t\xi)$ at $t = 0$ is $d_B \xi$. Taking $B = A \triangleleft \mathbf{u}$ gives (27.26). □

Theorem 27.4.7. (Sobolev gauge group and compact Coulomb slice) Let $(M^m, \mathbf{g})$ be a smooth compact connected Riemannian manifold without boundary; let G be compact, let $\mathbf{P} \to M$ be a right principal bundle, and let A be a smooth connection. If $s > \frac{m}{2}$, then

$$\mathcal{G}^{s+1}(\mathbf{P}) := H^{s+1}(M, \mathrm{Ad}(\mathbf{P}))$$

is a Hilbert Lie group modeled on $H^{s+1}(M, \mathrm{ad}\,\mathbf{P})$, and its right action on $\mathcal{A}^s(\mathbf{P})$ is smooth. The infinitesimal action is

$$\xi \longmapsto d_A \xi. \tag{27.27}$$

Set

$$K_A := \ker\big(d_A : H^{s+1}(M, \mathrm{ad}\,\mathbf{P}) \longrightarrow H^s(M, T^*M \otimes \mathrm{ad}\,\mathbf{P})\big)$$

and, using the L^2 inner product induced by $\mathbf{g}$ and the Ad–invariant metric on $\mathrm{ad}\,\mathbf{P}$, define

$$X_A^{s+1} := H^{s+1}(M, \mathrm{ad}\,\mathbf{P}) \cap K_A^{\perp_{L^2}}, \qquad Y_A^{s-1} := \Big\{ \mathbf{f} \in H^{s-1}(M, \mathrm{ad}\,\mathbf{P}) \,\Big|\, \langle \mathbf{f}, \zeta \rangle_{\mathcal{D}', \mathcal{D}} = 0 \text{ for every } \zeta \in K_A \Big\}.$$

When $s \geq 1$, the last space is precisely $H^{s-1}(M, \mathrm{ad}\,\mathbf{P}) \cap K_A^{\perp_{L^2}}$; the annihilator formulation remains meaningful

if $s-1<0$, since K_A consists of smooth sections. Then

$$\Delta_A^{(0)} := d_A^* d_A : X_A^{s+1} \xrightarrow{\cong} Y_A^{s-1} \tag{27.28}$$

is a Hilbert space isomorphism and

$$H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}) = d_A X_A^{s+1} \oplus \ker\bigl(d_A^* : H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}) \longrightarrow H^{s-1}(M, \operatorname{ad}\mathbf{P})\bigr). \tag{27.29}$$

The stabilizer

$$\operatorname{Stab}(A) := \{\mathbf{u} \in \mathcal{G}^{s+1}(\mathbf{P}) \mid A \triangleleft \mathbf{u} = A\}$$

is a compact finite-dimensional Lie group with Lie algebra K_A. More precisely, if $p \in \mathbf{P}$ and $\operatorname{Hol}_p(A) \subseteq G$ is the holonomy group based at p, evaluation at p induces a Lie group isomorphism

$$\operatorname{Stab}(A) \xrightarrow{\cong} Z_G\bigl(\operatorname{Hol}_p(A)\bigr). \tag{27.30}$$

Finally, if

$$\mathcal{S}_A^s := A + \ker\bigl(d_A^* : H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}) \to H^{s-1}(M, \operatorname{ad}\mathbf{P})\bigr),$$

, there exist an open neighborhood $\mathcal{U}_A$ of A in $\mathcal{A}^s(\mathbf{P})$ and an open neighborhood $\mathcal{W}_A$ of $[A, e]$ in $\mathcal{S}_A^s \times_{\operatorname{Stab}(A)} \mathcal{G}^{s+1}(\mathbf{P})$ for which the restriction of the map

$$\mathcal{W}_A \longrightarrow \mathcal{U}_A, \qquad [B, \mathbf{u}] \longmapsto B \triangleleft \mathbf{u}, \tag{27.31}$$

is a diffeomorphism. Here the equivalence relation is induced by

$$(B, \mathbf{u}) \cdot \mathbf{h} = (B \triangleleft \mathbf{h}, \mathbf{h}^{-1}\mathbf{u}), \qquad \mathbf{h} \in \operatorname{Stab}(A).$$

Moreover, one can choose an open $\operatorname{Stab}(A)$-invariant neighborhood $\mathcal{S}_0$ of A in $\mathcal{S}_A^s$ such that

$$\mathcal{S}_0 \times_{\operatorname{Stab}(A)} \mathcal{G}^{s+1}(\mathbf{P}) \longrightarrow \mathcal{A}^s(\mathbf{P}), \qquad [B, \mathbf{u}] \longmapsto B \triangleleft \mathbf{u},$$

is a diffeomorphism onto an invariant open set containing the orbit of A. Every connection sufficiently close to A has a representative in $\mathcal{S}_0$, unique modulo $\operatorname{Stab}(A)$. Uniqueness refers to representatives in this Coulomb neighborhood, rather than the entire affine hyperplane $\mathcal{S}_A^s$.

Proof. Since $s > \frac{m}{2}$, we have $s+1 > \frac{m}{2}+1$ and, in particular,

$$H^{s+1}(M, \operatorname{ad}\mathbf{P}) \hookrightarrow C^0(M, \operatorname{ad}\mathbf{P}; h_{\mathrm{ad}}).$$

The Lie exponential is Ad–equivariant and therefore induces a bundle map

$$\operatorname{Exp}\colon \operatorname{ad}\mathbf{P} \longrightarrow \operatorname{Ad}(\mathbf{P}), \qquad [p, \xi] \longmapsto [p, \exp_G \xi].$$

With respect to the Ad–invariant inner product, choose an Ad–invariant ball $\mathcal{O} \subset \mathfrak{g}$ on which $\exp_G$ is a

diffeomorphism onto its image. For each section $\mathbf{u}$ of the group bundle, the formula

$$\Theta_{\mathbf{u}}(\xi) := \mathbf{u}\,\mathrm{Exp}(\xi), \qquad \|\xi\|_{C^0(M,\mathrm{ad}\,\mathbf{P};h_{\mathrm{ad}})} < \mathrm{rad}(\mathcal{O}),$$

has the pointwise inverse $v \mapsto \log_G(\mathbf{u}^{-1}v)$. In a finite cover trivializing both bundles, the two formulas are superposition operators. Lemma 27.3.2 proves that they are smooth in $H^{s+1}(M,\mathrm{ad}\,\mathbf{P})$; equivalently, the differential at zero is the identity, and Theorem 23.3.3 provides the same local inverse. These charts, translated by each $\mathbf{u}$, construct the Hilbert Lie group structure and the identification $T_{\mathbf{u}}\mathcal{G}^{s+1}(\mathbf{P}) = \mathbf{u}H^{s+1}(M,\mathrm{ad}\,\mathbf{P})$.
In these charts, multiplication and inversion are superpositions of the smooth finite-dimensional maps of G and are therefore smooth. Likewise,

$$H^{s+1}(M,\mathrm{End}(\mathrm{ad}\,\mathbf{P}))\cdot H^s(M,T^*M\otimes\mathrm{ad}\,\mathbf{P}) \longrightarrow H^s(M,T^*M\otimes\mathrm{ad}\,\mathbf{P}), \qquad \mathbf{u}\longmapsto \mathbf{u}^{-1}d_A\mathbf{u} : \mathcal{G}^{s+1}(\mathbf{P}) \longrightarrow H^s(M,T^*M\otimes\mathrm{ad}\,\mathbf{P})$$

are smooth maps in charts. Here $\mathbf{u}^{-1}d_A\mathbf{u} := A\triangleleft\mathbf{u} - A$ is a global form with values in $\mathrm{ad}\,\mathbf{P}$. The term $\mathbf{u}^{-1}d\mathbf{u}$ alone is used in a local trivialization: it combines with the connection potential to give that global form. For real indices, apply Corollary 27.3.3. This constructs the Lie group and proves smoothness of the action. Formula (27.26), with $\mathbf{u} = e$, gives (27.27).
The operator $\Delta_A^{(0)} = d_A^*d_A$ is elliptic, formally self-adjoint, and nonnegative. Its self-adjoint realization and domain follow from Corollary 11.4.5; on the closed manifold, Proposition 15.7.2 proves that its resolvent is compact. By the spectral theorem, $K_A = \ker\Delta_A^{(0)}$ is finite-dimensional, consists of smooth sections, and there exists $\lambda_A > 0$ such that

$$\langle\Delta_A^{(0)}\xi,\xi\rangle_{L^2(M,\mathrm{ad}\,\mathbf{P})} \geq \lambda_A\|\xi\|^2_{L^2(M,\mathrm{ad}\,\mathbf{P})}, \qquad \xi\in H^2(M,\mathrm{ad}\,\mathbf{P})\cap K_A^{\perp_{L^2}}.$$

The elliptic estimate and elliptic regularity, applied at each Sobolev level, prove that (27.28) is bijective with continuous inverse.
For $\mathbf{a}\in H^s(M,T^*M\otimes\mathrm{ad}\,\mathbf{P})$, define

$$\xi := (\Delta_A^{(0)}|_{X_A^{s+1}})^{-1}d_A^*\mathbf{a}, \qquad \mathbf{b} := \mathbf{a} - d_A\xi.$$

Since, for $\zeta\in K_A$, $\langle d_A^*\mathbf{a},\zeta\rangle = \langle\mathbf{a},d_A\zeta\rangle = 0$, we have $d_A^*\mathbf{a}\in Y_A^{s-1}$; the formula is well defined and $d_A^*\mathbf{b} = 0$. If $d_A\xi$ also belongs to $\ker d_A^*$, then $\Delta_A^{(0)}\xi = 0$; since $\xi\perp K_A$, it follows that $\xi = 0$. This proves the direct sum in (27.29). The preceding formula also shows that its projections are continuous.
We prove that every element of the Sobolev stabilizer is smooth. If $\mathbf{u}\in\mathcal{G}^{s+1}(\mathbf{P})$ satisfies $A\triangleleft\mathbf{u} = A$, the local formula (27.23) gives, in the weak sense,

$$\mathbf{u}^{-1}d\mathbf{u} = A - \mathrm{Ad}_{\mathbf{u}^{-1}}A.$$

Since $s+1 > \frac{m}{2}+1$, the section $\mathbf{u}$ is continuous. In a trivialization of the group bundle, denote its local representation by $\widehat{u}$. In group coordinates, the last equality is a system $\partial_j\widehat{u} = F_j(x,\widehat{u})$ with F_j smooth. To gain one derivative, suppose $\widehat{u}\in H^{s+1+j}_{\mathrm{loc}}$, with $j\in\mathbb{N}_0$. Smooth superposition, applied to the map $(x,z)\mapsto F_i(x,z)$ in a relatively compact chart, gives $F_i(x,\widehat{u})\in H^{s+1+j}_{\mathrm{loc}}$ for each $i\in\{1,\dots,m\}$. The equation then provides all first derivatives of $\widehat{u}$ in this space; the local Sobolev characterization by derivatives implies $\widehat{u}\in H^{s+2+j}_{\mathrm{loc}}$. Cutoff functions in slightly smaller charts justify each step. Starting from $j = 0$, this implication increases regularity by one at each stage. A finite cover and Sobolev embeddings give $\mathbf{u}\in C^\infty$. The equality is

then classical, and Theorem 2.6.36 shows that $\mathbf{u}$ is a parallel section and evaluation induces the group isomorphism (27.30). Conversely, a smooth parallel section belongs to H^{s+1} because M is compact, so no element is lost.

Point evaluation is smooth in H^{s+1} by embedding into C^0. In the opposite direction, the extension of $c \in Z_G(\mathrm{Hol}_p(A))$ is written in a trivialization as $h(x)^{-1} c\, h(x)$, where $h(x)$ depends smoothly on the locally chosen parallel transport. Thus the inverse is also smooth, and (27.30) is a Lie group isomorphism. Proposition 1.9.16 shows that the centralizer is a closed Lie subgroup of the compact group G and is therefore compact and finite-dimensional. Finally, (2.52) identifies its Lie algebra with K_A.

Now apply the implicit function theorem. For small $\mathbf{a}$ and ξ, with $\xi \in X_A^{s+1}$, set

$$\mathscr{F}(\mathbf{a}, \xi) := d_A^*\big((A + \mathbf{a}) \triangleleft \exp_G(-\xi) - A\big) \in Y_A^{s-1}.$$

The image belongs to Y_A^{s-1} because d_A^* takes values orthogonal to K_A. The differential formula for the action gives

$$D_\xi \mathscr{F}(0,0)\eta = -d_A^* d_A \eta = -\Delta_A^{(0)} \eta, \tag{27.32}$$

with no undetermined sign. By (27.28), Theorem 23.3.8 produces a unique smooth function $\mathbf{a} \mapsto \xi(\mathbf{a}) \in X_A^{s+1}$ such that

$$d_A^*\big((A + \mathbf{a}) \triangleleft \exp_G(-\xi(\mathbf{a})) - A\big) = 0.$$

Equivalently, the map

$$\Psi(\mathbf{b}, \xi) := (A + \mathbf{b}) \triangleleft \exp_G(\xi) - A, \qquad d_A^* \mathbf{b} = 0, \qquad \xi \in X_A^{s+1},$$

has differential

$$D\Psi_{(0,0)}(\mathbf{b}, \xi) = \mathbf{b} + d_A \xi,$$

, which is an isomorphism by (27.29). Theorem 23.3.3 gives the unique local parametrization by $(\mathbf{b}, \xi)$.

It remains to incorporate the stabilizer without confusing it with the directions K_A. Abbreviate

$$H := \mathrm{Stab}(A), \qquad \mathfrak{h} := K_A, \qquad X := X_A^{s+1}.$$

The affine Coulomb space $\mathcal{S}_A^s$ is H–invariant: if $\mathbf{h}$ is parallel, then

$$(A + \mathbf{b}) \triangleleft \mathbf{h} = A + \mathrm{Ad}_{\mathbf{h}^{-1}} \mathbf{b}, \qquad d_A^* \mathrm{Ad}_{\mathbf{h}^{-1}} \mathbf{b} = \mathrm{Ad}_{\mathbf{h}^{-1}} d_A^* \mathbf{b}.$$

The same identity shows that $\mathrm{Ad}_{\mathbf{h}}$ commutes with d_A. Since $\mathbf{h}$ is parallel and the inner product on $\mathfrak{g}$ is Ad–invariant, $\mathrm{Ad}_{\mathbf{h}}$ commutes with the covariant derivatives of the connection A. Choose the equivalent Sobolev norm constructed with this connection; for real orders, we may use powers of $I + \nabla_A^* \nabla_A$. The parallel endomorphism commutes with this operator and its powers. It is therefore an isometry of $H^{s+1}(M, \mathrm{ad}\,\mathbf{P})$. In particular, it preserves both K_A and its orthogonal complement X. Thus

$$H^{s+1}(M, \mathrm{ad}\,\mathbf{P}) = \mathfrak{h} \oplus X$$

is an Ad_H–invariant decomposition.

We now construct the required tube. By left translation, the subbundle complementary to TH in the re-

striction of the gauge group tangent bundle to H is

$$N_{\mathbf{h}}H := (dL_{\mathbf{h}})_e X, \qquad NH \cong H \times X, \qquad (\mathbf{h}, \xi) \longmapsto (dL_{\mathbf{h}})_e \xi.$$

Invariance of X makes this identification compatible also with right translations by elements of H. Consider

$$\Theta \colon H \times X \longrightarrow \mathcal{G}^{s+1}(\mathbf{P}), \qquad \Theta(\mathbf{h}, \xi) := \mathbf{h}\exp(\xi).$$

This map is smooth by the Lie group structure already constructed. In the left trivialization at $(\mathbf{h}, 0)$, its differential is

$$(dL_{\mathbf{h}^{-1}})_{\mathbf{h}} D\Theta_{(\mathbf{h},0)}\big((dL_{\mathbf{h}})_e \eta, \xi\big) = \eta + \xi, \qquad (\eta, \xi) \in \mathfrak{h} \oplus X.$$

It is an isomorphism. The inverse function theorem gives, around each $(\mathbf{h}, 0)$, an open product $W_{\mathbf{h}} \times B_X(0, r_{\mathbf{h}})$ on which Θ is a diffeomorphism onto its image, with $\mathbf{h} \in W_{\mathbf{h}} \subseteq H$ and $r_{\mathbf{h}} > 0$. The open sets $W_{\mathbf{h}}$ cover H. Choose a subcover $W_{\mathbf{h}_1}, \dots, W_{\mathbf{h}_N}$ and set

$$\varepsilon_0 := \min_{1 \le i \le N} r_{\mathbf{h}_i} > 0.$$

If $\mathbf{h} \in H$ and $\|\xi\|_{H^{s+1}(M, \mathrm{ad}\,\mathbf{P})} < \varepsilon_0$, some $W_{\mathbf{h}_i}$ contains $\mathbf{h}$ and $(\mathbf{h}, \xi) \in W_{\mathbf{h}_i} \times B_X(0, r_{\mathbf{h}_i})$. Thus Θ is a local diffeomorphism on all of $H \times B_X(0, \varepsilon_0)$. Here and below, B_X denotes a ball for the norm $H^{s+1}(M, \mathrm{ad}\,\mathbf{P})$ restricted to X.

Injectivity can be made uniform. If there were no $0 < \varepsilon \le \varepsilon_0$ for which Θ was injective, there would be sequences

$$\mathbf{h}_j, \mathbf{k}_j \in H, \qquad \xi_j, \eta_j \in X, \qquad \|\xi_j\|_{H^{s+1}(M, \mathrm{ad}\,\mathbf{P})}, \|\eta_j\|_{H^{s+1}(M, \mathrm{ad}\,\mathbf{P})} \longrightarrow 0,$$

with $(\mathbf{h}_j, \xi_j) \neq (\mathbf{k}_j, \eta_j)$ and $\Theta(\mathbf{h}_j, \xi_j) = \Theta(\mathbf{k}_j, \eta_j)$. After passing to subsequences, compactness of H gives $\mathbf{h}_j \to \mathbf{h}$ and $\mathbf{k}_j \to \mathbf{k}$. Passing to the limit gives $\mathbf{h} = \mathbf{k}$. For large j, the two pairs then belong to a common domain of a local inverse of Θ around $(\mathbf{h}, 0)$, forcing $(\mathbf{h}_j, \xi_j) = (\mathbf{k}_j, \eta_j)$, a contradiction. Consequently, for some $\varepsilon > 0$,

$$\Theta \colon H \times B_X(0, \varepsilon) \xrightarrow{\cong} \mathcal{N}_\varepsilon := \Theta\big(H \times B_X(0, \varepsilon)\big) \tag{27.33}$$

is a diffeomorphism. Its image is open because Θ is a local diffeomorphism and contains H. Thus (27.33) is indeed a tubular neighborhood of H, obtained from the complementary normal bundle NH, rather than merely a chart at the identity. The terminology agrees with Definition 2.12.2: as in Fermi coordinates, a neighborhood of the zero section of a normal bundle is identified with a neighborhood of the submanifold; here the diffeomorphism is Θ rather than a Riemannian exponential map on the Hilbert manifold.

The ball $B_X(0, \varepsilon)$ is H–invariant for the preceding norm. The tube is equivariant under the right action

$$(\mathbf{h}, \xi) \cdot \mathbf{k} = (\mathbf{h}\mathbf{k}, \mathrm{Ad}_{\mathbf{k}^{-1}} \xi), \qquad \Theta((\mathbf{h}, \xi) \cdot \mathbf{k}) = \Theta(\mathbf{h}, \xi)\mathbf{k},$$

and also under left translations. This is precisely the compatibility required by the associated quotient.

To conclude, choose an open H–invariant neighborhood

$$\mathcal{S}_0 \subset \mathcal{S}_A^s, \qquad A \in \mathcal{S}_0,$$

and shrink $\mathcal{S}_0$ and ε simultaneously so that the local diffeomorphism Ψ obtained above is injective on

$(\mathcal{S}_0 - A) \times B_X(0,\varepsilon)$. Let q be the projection onto the associated quotient and set

$$\mathcal{W}_A := q(\mathcal{S}_0 \times \mathcal{N}_\varepsilon).$$

We may take $\mathcal{S}_0$ to be a ball for an H–invariant norm obtained by averaging any equivalent norm over the compact group. The action of H on $\mathcal{S}_A^s \times \mathcal{G}^{s+1}(\mathbf{P})$ is free because of its second factor and proper because H is compact; in particular, the projection q is open and $\mathcal{W}_A$ is an open neighborhood of $[A,e]$. If $\mathbf{u} = \mathbf{h}\operatorname{Exp}(\xi)$ belongs to $\mathcal{N}_\varepsilon$, then, in the associated quotient,

$$[B, \mathbf{h}\operatorname{Exp}(\xi)] = [B \triangleleft \mathbf{h}, \operatorname{Exp}(\xi)],$$

, with $B \triangleleft \mathbf{h} \in \mathcal{S}_0$ after shrinking $\mathcal{S}_0$ invariantly. This representative is unique. Indeed, if

$$[B_1, \operatorname{Exp}(\xi_1)] = [B_2, \operatorname{Exp}(\xi_2)],$$

, there exists $\mathbf{k} \in H$ such that $B_2 = B_1 \triangleleft \mathbf{k}$ and $\operatorname{Exp}(\xi_2) = \mathbf{k}^{-1}\operatorname{Exp}(\xi_1)$. Hence

$$\Theta(e, \xi_1) = \Theta(\mathbf{k}, \xi_2),$$

, and injectivity of the tube implies $\mathbf{k} = e$, $\xi_1 = \xi_2$, and $B_1 = B_2$. Thus

$$\mathcal{S}_0 \times B_X(0,\varepsilon) \longrightarrow \mathcal{W}_A, \qquad (B,\xi) \longmapsto [B, \operatorname{Exp}(\xi)]$$

is a smooth bijective chart with smooth inverse induced by (27.33). In this chart, the map (27.31) agrees with

$$(B,\xi) \longmapsto B \triangleleft \operatorname{Exp}(\xi),$$

, that is, the parametrization Ψ translated by A. Its image $\mathcal{U}_A$ is open, and the restriction is a diffeomorphism. This proves the local parametrization. To obtain a slice for the full group, we must also control gauge transformations that have not been chosen in advance in $\mathcal{N}_\varepsilon$.

Fix a faithful unitary representation of G with associated vector bundle $\mathbf{E}$. Gauge transformations are represented as unitary endomorphisms of $\mathbf{E}$, and $d_A\mathbf{u}$ here denotes their covariant derivative before multiplication by $\mathbf{u}^{-1}$. The elliptic estimate for the fixed connection A gives

$$\|\mathbf{u}\|_{H^{s+1}(M,\operatorname{End}\mathbf{E})} \le C_A\big(\|d_A\mathbf{u}\|_{H^s(M,T^*M\otimes\operatorname{End}\mathbf{E})} + \|\mathbf{u}\|_{L^2(M,\operatorname{End}\mathbf{E})}\big).$$

This estimate applies to arbitrary endomorphisms, not just group elements. If $(A + \mathbf{b}_1) \triangleleft \mathbf{u} = A + \mathbf{b}_2$, the action formula is equivalent to

$$d_A\mathbf{u} = \mathbf{u}\mathbf{b}_2 - \mathbf{b}_1\mathbf{u}.$$

Sobolev product estimates imply

$$\|d_A\mathbf{u}\|_{H^s(M,T^*M\otimes\operatorname{End}\mathbf{E})} \le C\big(\|\mathbf{b}_1\|_{H^s(M,T^*M\otimes\operatorname{End}\mathbf{E})} + \|\mathbf{b}_2\|_{H^s(M,T^*M\otimes\operatorname{End}\mathbf{E})}\big)\|\mathbf{u}\|_{H^{s+1}(M,\operatorname{End}\mathbf{E})}.$$

Moreover, $\|\mathbf{u}\|_{L^2(M,\operatorname{End}\mathbf{E})}$ has a common bound because $\mathbf{u}$ is unitary and M has finite volume. If the norms of $\mathbf{b}_1, \mathbf{b}_2$ are sufficiently small, the first estimate absorbs the term containing $\|\mathbf{u}\|_{H^{s+1}(M,\operatorname{End}\mathbf{E})}$ and gives a uniform bound on this norm.

Suppose there exist $\mathbf{b}_{1,j}, \mathbf{b}_{2,j} \to 0$ in H^s and transformations $\mathbf{u}_j$ relating the two connections but remaining outside $\mathcal{N}_\varepsilon$. The preceding bound and Rellich's theorem allow extraction of a subsequence converging in H^s and, by embedding, in C^0, to a group-valued section $\mathbf{u}$. The differential identity shows that $d_A \mathbf{u}_j \to 0$ in H^s, so $d_A \mathbf{u} = 0$ in distributions. The stabilizer regularity argument proves that $\mathbf{u} \in H = \mathrm{Stab}(A)$. Applying the elliptic estimate to $\mathbf{u}_j - \mathbf{u}$ gives

$$\|\mathbf{u}_j - \mathbf{u}\|_{H^{s+1}(M,\mathrm{End}\,\mathbf{E})} \le C_A \big(\|d_A \mathbf{u}_j\|_{H^s(M,T^*M\otimes \mathrm{End}\,\mathbf{E})} + \|\mathbf{u}_j - \mathbf{u}\|_{L^2(M,\mathrm{End}\,\mathbf{E})} \big) \longrightarrow 0.$$

This contradicts the fact that $\mathcal{N}_\varepsilon$ is an open neighborhood of H. Consequently, after shrinking $\mathcal{S}_0$, every transformation relating two points of $\mathcal{S}_0$ belongs to $\mathcal{N}_\varepsilon$.

Write this transformation as $\mathbf{u} = \mathbf{h}\,\mathrm{Exp}\,\xi$, with $\mathbf{h} \in H$ and $\xi \in X$. If $B_1 \triangleleft \mathbf{u} = B_2$, with $B_1, B_2 \in \mathcal{S}_0$, invariance under H keeps $B_1 \triangleleft \mathbf{h}$ in $\mathcal{S}_0$. Injectivity of the parametrization Ψ compares the pairs $(B_1 \triangleleft \mathbf{h}, \xi)$ and $(B_2, 0)$ and gives $\xi = 0$, $B_2 = B_1 \triangleleft \mathbf{h}$. This proves uniqueness with respect to any group element.

Finally, the differential of the parametrization remains invertible at (B, e) for every B in a sufficiently small neighborhood of A, by continuity and openness of the set of isomorphisms. Translation by the action transfers this property to $(B, \mathbf{u})$ for any $\mathbf{u}$. The map from the associated quotient is thus a local diffeomorphism at all its points, and the preceding argument proves its injectivity. Its image is open and invariant and contains the orbit of A; being an open injective map with smooth local inverses, it is a diffeomorphism onto that image. This completes the slice construction. □

The gain of one derivative in the gauge element is visible both in $\mathbf{u}^{-1} d\mathbf{u}$ and in the elliptic isomorphism used in the Coulomb argument.

On a compact base, $d_A^* d_A$ has compact resolvent: its spectrum is discrete, each eigenspace is finite-dimensional, and there is no essential spectrum. The elliptic estimate on $(\ker d_A)^\perp$ replaces, in this case, the additional gap assumption arising on a noncompact base.

On a complete noncompact base without boundary, we say that a smooth connection B satisfies the connection condition (B_k) if there exist constants $D_0, \dots, D_k > 0$ such that

$$\|(\nabla^B)^i \mathbf{F}_B\|_\infty \le D_i, \qquad 0 \le i \le k. \tag{27.34}$$

For fixed A_0, define its affine Sobolev component by

$$\mathcal{A}^r_{A_0}(\mathbf{P}) := A_0 + H^r(M, T^*M \otimes \mathrm{ad}\,\mathbf{P}).$$

On a noncompact base, a gauge transformation is likewise not required to be integrable as a function with values in G. Fix a faithful unitary representation of G and use the connection induced by A_0. Consider sections $\mathbf{u}$ of $\mathrm{Ad}(\mathbf{P})$ of class H^{r+1}_{loc} for which

$$\mathbf{u}^{-1} d_{A_0} \mathbf{u} := A_0 \triangleleft \mathbf{u} - A_0 \in H^r(M, T^*M \otimes \mathrm{ad}\,\mathbf{P}).$$

This set is $\mathcal{G}^{r+1}(A_0)$. Its topology is described by the charts $\xi \mapsto \mathbf{u}\,\mathrm{Exp}\,\xi$, with $\xi \in H^{r+1}(M, \mathrm{ad}\,\mathbf{P})$ of small uniform norm. The norms are always based on A_0 and $\mathbf{g}$.

The definition is compatible with multiplication. If $\mathbf{b}_{\mathbf{u}} = \mathbf{u}^{-1} d_{A_0} \mathbf{u}$, the action law gives

$$\mathbf{b}_{\mathbf{uv}} = \mathrm{Ad}_{\mathbf{v}^{-1}} \mathbf{b}_{\mathbf{u}} + \mathbf{b}_{\mathbf{v}}, \qquad \mathbf{b}_{\mathbf{u}^{-1}} = -\mathrm{Ad}_{\mathbf{u}}\, \mathbf{b}_{\mathbf{u}}.$$

The coefficients $\mathbf{u}$ are bounded because G is compact. Estimates for their derivatives follow from $d_{A_0}\mathbf{u} = \mathbf{u}\mathbf{b}_{\mathbf{u}}$: in the step from order j to $j+1$, Leibniz's rule differentiates one of the factors $\mathbf{b}_{\mathbf{u}}$ or adds a new factor by differentiating $\mathbf{u}$. For $j \in \{0,\dots,r\}$, Sobolev products, with $r > m/2$, control all terms. In particular, $\mathrm{Ad}_{\mathbf{u}}$ and its inverse are multipliers in H^r. The preceding two identities preserve the membership condition, and the charts construct the Hilbert Lie group described in [55, §3]. Its action on the affine component is written as

$$(A_0 + \mathbf{a}) \triangleleft \mathbf{u} = A_0 + \mathbf{b}_{\mathbf{u}} + \mathrm{Ad}_{\mathbf{u}^{-1}}\,\mathbf{a}.$$

Let $\mathcal{G}_e^{r+1}(A_0)$ be the identity component of this group. A central constant section $c \in Z(G)$ has $d_{A_0}c = 0$, so it belongs to the full group even if it does not belong to that component. If $Z(G)$ is countable, set

$$\mathcal{G}_\bullet^{r+1}(A_0) := \bigcup_{c \in Z(G)} c\,\mathcal{G}_e^{r+1}(A_0).$$

If the center is uncountable, set $\mathcal{G}_\bullet^{r+1}(A_0) = \mathcal{G}_e^{r+1}(A_0)$. The selected components form a subgroup. Since the Hilbert model is separable, the identity component is second countable; a countable union of its translates preserves this property. This is the restriction used in the slice theorem.

Definition 27.4.8. (Essential spectrum and spectral gap) Let L be a densely defined nonnegative self-adjoint operator on a Hilbert space. Its *essential spectrum* $\sigma_{\mathrm{e}}(L)$ is the set of $\lambda \in \mathbb{R}$ for which $L - \lambda I \colon \mathcal{D}(L) \longrightarrow H$ is not Fredholm, where $\mathcal{D}(L)$ carries the graph norm. We say there is a *spectral gap above the kernel* if there exists $\gamma > 0$ such that

$$\langle Lu, u\rangle \geq \gamma \|u\|^2, \qquad u \in \mathcal{D}(L) \cap (\ker L)^{\perp}.$$

If $\inf \sigma_{\mathrm{e}}(L|_{(\ker L)^{\perp}}) > 0$, this gap also follows, with the convention $\inf \varnothing = +\infty$. Indeed, below the essential spectrum there can be only isolated eigenvalues of finite multiplicity. A sequence of positive eigenvalues tending to zero would place zero in the essential spectrum. Nor can zero be an eigenvalue of the restriction to $(\ker L)^{\perp}$. Its spectrum is therefore bounded away from zero. The spectral theorem gives the inequality, and the inverse on the complement of the kernel is bounded; it follows that the range of L is closed.

Theorem 27.4.9. (Eichhorn–Heber: gauge slice on a complete noncompact manifold) Let $(M^m, \mathbf{g})$ be a smooth complete noncompact Riemannian manifold without boundary and of bounded geometry: it has positive injectivity radius and curvature bounds (B_k). Let $\mathbf{P} \longrightarrow M$ be a principal bundle with compact group, let A_0 be a connection satisfying (27.34) and

$$\mathrm{YM}(A_0) = \frac{1}{2}\int_M |\mathbf{F}_{A_0}|^2_{\mathbf{g},h_{\mathrm{ad}}}\, d\lambda_{\mathbf{g}} < \infty,$$

, and suppose

$$k - 1 \geq r > \frac{m}{2} + 2.$$

For $A \in \mathcal{A}^r_{A_0}(\mathbf{P})$, assume the spectral gap

$$\inf \sigma_{\mathrm{e}}\left(\Delta_A^{(0)} \restriction_{(\ker \Delta_A^{(0)})^{\perp}}\right) > 0.$$

Then $d_A : H^{r+1}(M, \mathrm{ad}\,\mathbf{P}) \to H^r(M, T^*M \otimes \mathrm{ad}\,\mathbf{P})$ has closed range in H^r and admits the kernel of $d_A^* : H^r \to$

H^{r-1} as a complement. The sum is orthogonal for the L^2 inner product, and the orbit of A under $\mathcal{G}_\bullet^{r+1}(A_0)$ has an equivariant tube and a slice. If, in addition, the same gap holds for the reference connection,

$$\inf \sigma_{\mathrm{e}} \left(\Delta_{A_0}^{(0)} \restriction_{(\ker \Delta_{A_0}^{(0)})^\perp} \right) > 0,$$

, the quotient of the component by the subgroup $\mathcal{G}_\bullet^{r+1}(A_0)$ corresponding to A_0 has a regular quasistratification determined by conjugacy classes of stabilizers.

The Hodge decomposition used here is Proposition 4.12, p. 282, and the slice is Theorem 4.17, pp. 283–284, of [55], both under the gap corresponding to A. The quasistratification is their Theorem 5.13, pp. 285–286, and requires the gap at A_0. These conditions make the range of d_A a closed subspace and allow orbit directions to be separated from Coulomb directions. The full proof requires elliptic theory with Sobolev coefficients on a noncompact manifold and does not reduce to the local implicit function theorem.

There is a parallel result for the action of diffeomorphisms on metrics. For a metric $\mathbf{g}$, denote by $\Delta_{\mathbf{g}}^{(0)} = d_{\mathbf{g}}^* d$ the nonnegative self-adjoint realization of the Beltrami Laplacian on $L^2(M, d\lambda_{\mathbf{g}})$.

Theorem 27.4.10. (Eichhorn: slice for metrics of negative curvature) Let $(M^m, \mathbf{g}_0)$ be a smooth complete noncompact Riemannian manifold without boundary, with positive injectivity radius and bounded curvature and curvature derivatives, and let $r > \frac{m}{2} + 2$. Suppose $K_{\mathbf{g}_0} \le -c < 0$. Let $\mathbf{g}$ be a metric in the Sobolev component of order r of $\mathbf{g}_0$, with strictly negative sectional curvature, positive injectivity radius, and uniformly bounded $\mathrm{Rm}_{\mathbf{g}}, \nabla \mathrm{Rm}_{\mathbf{g}}$. Require, in addition, the positive equality of gaps

$$\inf \sigma_{\mathrm{e}}(\Delta_{\mathbf{g}_0}^{(0)}) = \inf \sigma_{\mathrm{e}}(\Delta_{\mathbf{g}}^{(0)}) > 0.$$

Then the identity component of the diffeomorphism group of order $r+1$ acts properly on this admissible set, and the orbit of $\mathbf{g}$ admits a topological slice. The action is regarded as continuous at the indicated Sobolev levels; differentiating it requires the level changes associated with composition and pullback.

This result is Theorem 5.8 of [52, p. 115]. The conclusion uses negative curvature, the uniform conditions (I) and (B_1), membership in the component, and equality of spectral gaps together. Its proof constructs the equivariant tube and establishes properness of the action using uniform elliptic theory.

27.5 Diffeomorphism groups, Hilbert scales, and ILH

27.5.1 The Sobolev group at fixed regularity

Let M^m be a smooth compact manifold with or without boundary and let $s > \frac{m}{2} + 1$. If $\partial M = \varnothing$, the set

$$\mathrm{Diff}^s(M) := \{ \eta \in H^s(M, M) \mid \eta \text{ is bijective and } \eta^{-1} \in H^s(M, M) \}$$

is open in $H^s(M, M)$ and is therefore a Hilbert manifold. If $\partial M \neq \varnothing$, introduce the double DM, which is a finite-dimensional manifold without boundary. In this case, $\mathrm{Diff}^s(M)$ consists of the C^1 diffeomorphisms of M whose coordinate representations and those of their inverses are H^s; in boundary charts, use the restriction spaces on the half-space. Work near one such diffeomorphism $\eta \in \mathrm{Diff}^s(M)$ within $H^s(M, DM)$. Choose on DM a product metric in a collar of ∂M; thus ∂M is totally geodesic. In the exponential chart centered at η, the condition of mapping

boundary to boundary is equivalent to

$$\operatorname{Tr}(\mathbf{X}) \in H^{s-\frac{1}{2}}(\partial M, (\eta|_{\partial M})^* T\partial M).$$

Indeed, a sufficiently short geodesic with both endpoints on ∂M is the geodesic for the induced metric on that hypersurface; its initial vector is tangent. The reverse implication follows because the hypersurface is totally geodesic.

The operator taking the normal component of the trace,

$$H^s(M, \eta^* TDM) \longrightarrow H^{s-\frac{1}{2}}(\partial M, (\eta|_{\partial M})^* N\partial M),$$

, is continuous and surjective. The trace extension operator in the collar, followed by inclusion of the normal bundle, provides a continuous right inverse. Its kernel is therefore a closed complemented subspace. Sobolev trivializations of the pullback bundle are justified using $H^{s-1/2}(\partial M)$, which is an algebra in this range.

It remains to check that maps in this chart take values in M and are diffeomorphisms. In collar coordinates with inward coordinate $t \geq 0$, the normal component of η vanishes at $t = 0$, and its normal derivative is strictly positive there. Compactness of the boundary gives a positive lower bound on a smaller collar. A sufficiently small perturbation in C^1 preserves this bound and, by integration from $t = 0$, maps the interior collar into the interior. Outside the collar, the image under η lies at positive distance from the boundary, a condition preserved by C^0 closeness.

The differential remains invertible, and the boundary restriction is a local diffeomorphism. For global injectivity, cover M by finitely many charts in which small perturbations of η are injective, using the inverse function estimate. There then exists $\delta > 0$ such that two points at distance less than δ belong to one of these charts. Pairs of points at distance at least δ form a compact set, and their images under η are separated by a positive distance. A small perturbation in C^0 preserves this separation. Thus no identification of points can arise. The image is open, including in boundary charts, and closed by compactness; it meets every component met by η, so it is all of M. The inverse is H^s by Sobolev inversion estimates.

The embedding $H^s \hookrightarrow C^1$ makes this argument applicable in a Sobolev neighborhood of η. We obtain a chart of $\operatorname{Diff}^s(M)$ on the kernel of the normal trace. Its tangent space at η consists of sections of $\eta^* TM$ whose trace is tangent to the boundary. This construction is Lemma 6.6 of [46]; it uses a Hilbert manifold without boundary modeled on that kernel.

Localization near diffeomorphisms is part of the argument. The set of all maps of pairs $M \to M$ may have additional constraints imposed by positivity of the normal coordinate. For example, at the constant zero map from $[0,1]$ to itself, every differentiable variation with a parameter of both signs has zero velocity at every point, since the value zero is a minimum. The preceding linear model is not obtained there.

Theorem 27.5.1. (Ebin–Marsden) Let M be a smooth compact manifold with or without boundary. If $s > \frac{m}{2} + 1$, the preceding space is a topological group. For $\eta \in \operatorname{Diff}^s(M)$, right translation $R_\eta(\varphi) = \varphi \circ \eta$ is smooth. Left translation and inversion, however, are not generally smooth at the same level. More precisely, for $j \in \mathbb{N}_0$,

$$L_\eta \colon \operatorname{Diff}^s(M) \longrightarrow \operatorname{Diff}^s(M) \quad \text{is } C^j \text{ if } \eta \in \operatorname{Diff}^{s+j}(M),$$

, and inversion is C^j as a map $\operatorname{Diff}^{s+j}(M) \longrightarrow \operatorname{Diff}^s(M)$. Thus $\operatorname{Diff}^s(M)$ is not a Hilbert Lie group in the usual sense.

Continuity and the regularity assertions follow from the composition lemmas α and ω in Section 2 of [46]; restriction to the boundary-preserving subspaces is established in its Section 6. The loss is already visible on differentiation:

$$DL_\eta(\varphi)\mathbf{X} = (d\eta \circ \varphi)\mathbf{X},$$

, which contains one derivative of η. The derivative of $\eta \mapsto \eta^{-1}$ follows from $\eta \circ \eta^{-1} = \mathrm{id}$ and likewise contains $d\eta^{-1}$.

Lemma 27.5.2. (Composition at lower levels and differentiation of curves) Let M be a smooth compact manifold of dimension m, with or without boundary, and let $s > m/2 + 1$. For $0 \le r \le s$, composition defines a continuous map

$$H^r(M,\mathbb{R}^d) \times \mathrm{Diff}^s(M) \longrightarrow H^r(M,\mathbb{R}^d), \qquad (F,\eta) \longmapsto F \circ \eta.$$

For each η_0, there exist a neighborhood $\mathcal{U}$ and a constant $C_{r,\mathcal{U}}$ with

$$\|F \circ \eta\|_{H^r(M,\mathbb{R}^d)} \le C_{r,\mathcal{U}} \|F\|_{H^r(M,\mathbb{R}^d)}, \qquad \eta \in \mathcal{U}. \tag{27.35}$$

If $F \in C^1(I, H^s(M,\mathbb{R}^d))$ and $\eta \in C^1(I, \mathrm{Diff}^s(M))$, then $z = F \circ \eta^{-1}$ belongs to $C(I, H^s(M,\mathbb{R}^d)) \cap C^1(I, H^{s-1}(M,\mathbb{R}^d))$ and

$$\partial_t z = (\partial_t F) \circ \eta^{-1} - dz(\mathbf{v}), \qquad \mathbf{v} = \dot\eta \circ \eta^{-1}. \tag{27.36}$$

In local frames, the same conclusions hold for sections along η; using covariant derivatives, the formula is $\partial_t \mathbf{z} = (\nabla_t \mathbf{F}) \circ \eta^{-1} - \nabla_{\mathbf{v}} \mathbf{z}$.

Proof. For $r = s$, use continuous composition from Proposition 27.3.5. Its continuity at $(0, \eta_0)$ and linearity in F give a uniform bound on the norm of the operator $R_\eta : F \mapsto F \circ \eta$ when η lies in a neighborhood of η_0: simply apply continuity on a ball of radius δ and then scale F. For $r = 0$, change of variables gives

$$\|F \circ \eta\|^2_{L^2(M,\mathbb{R}^d)} = \int_M |F(y)|^2 J_{\eta^{-1}}(y)\, d\lambda_{\mathbf{g}}(y) \le \|J_{\eta^{-1}}\|_{L^\infty(M)} \|F\|^2_{L^2(M,\mathbb{R}^d)}.$$

The Jacobian is uniformly bounded near η_0 because inversion is continuous and $H^s(M,TM) \hookrightarrow C^1(M,TM)$. Interpolate these two operators using Lemma 27.2.19 to obtain (27.35) for every $r \in [0,s]$.

To check joint continuity, let $F_j \to F$ in $H^r(M,\mathbb{R}^d)$ and $\eta_j \to \eta$ in $\mathrm{Diff}^s(M)$. Choose $G \in C^\infty(M,\mathbb{R}^d)$ with $\|F - G\|_{H^r(M,\mathbb{R}^d)} < \varepsilon$. For large j, all composition operators under consideration have a common bound, and

$$\begin{aligned}\|F_j \circ \eta_j - F \circ \eta\|_{H^r(M,\mathbb{R}^d)} \le C\|F_j - F\|_{H^r(M,\mathbb{R}^d)} + 2C\varepsilon \\ + \|G \circ \eta_j - G \circ \eta\|_{H^r(M,\mathbb{R}^d)}.\end{aligned}$$

The last term tends to zero even in $H^s(M,\mathbb{R}^d)$, by smooth superposition with G. First let $j \to \infty$ and then $\varepsilon \to 0$.

For curves, continuity already proved gives $z \in C(I, H^s(M,\mathbb{R}^d))$. Evaluation and embedding into C^1 allow pointwise differentiation of the identity $F(t,x) = z(t, \eta(t,x))$, giving (27.36) pointwise. Its right-hand side is a continuous curve in $H^{s-1}(M,\mathbb{R}^d)$: the first term is continuous by composition, and the second by the product and differentiation estimate in Lemma 27.2.19. Denote this side by $b(t)$. The integral $z(t_*) + \int_{t_*}^t b(\tau)\, d\tau$ exists in $H^{s-1}(M,\mathbb{R}^d)$. Point evaluation is continuous in this space because $s - 1 > m/2$; evaluating gives

the same integral identity satisfied by $z(t,x)$. The two functions agree, and by the fundamental theorem of calculus for the Bochner integral, z is of class C^1 in $H^{s-1}(M,\mathbb{R}^d)$ with derivative b. For sections, perform the calculation in a finite cover of frames. The connection coefficients add the smooth contractions of the covariant derivative and satisfy the same product estimates; the formulas agree on overlaps by the transformation rule for a connection. □

Theorem 27.5.3. (Sobolev diffeomorphisms of a complete noncompact manifold) Let $(M^m,\mathbf{g})$ be a smooth oriented complete noncompact Riemannian manifold without boundary and of bounded geometry of order k. For $1<p<\infty$ and $k>r>\frac{m}{p}+1$, let

$$\mathcal{D}^{p,r}(M):=\left\{\eta\in\Omega^{p,r}(M,M)\;\middle|\;\begin{array}{l}\eta \text{ is a } C^1 \text{ orientation-preserving diffeomorphism,}\\ \inf\limits_{x\in M}\lambda_{\min}(d\eta_x)>0\end{array}\right\},$$

, where $\lambda_{\min}(d\eta_x):=\inf\limits_{\substack{v\in T_xM\\ |v|_{\mathbf{g},x}=1}}|d\eta_x v|_{\mathbf{g},\eta(x)}$. Then $\mathcal{D}^{p,r}(M)$ is open in $\Omega^{p,r}(M,M)$, and each component is a Banach manifold of class C^{k+1-r}, a Hilbert manifold for $p=2$. The identity component $\mathcal{D}_0^{p,r}(M)$ is a metrizable topological group under composition and inversion.

Openness can be seen in the constructed charts. If η belongs to the preceding set and h is sufficiently close in a chart, C^1 closeness preserves a positive lower bound for dh. The uniform bound on $d\eta^{-1}$ also implies that $F=\eta^{-1}\circ h$ has small uniform displacement from the identity. A map with bounded displacement on a complete finite-dimensional manifold is proper: the preimage of a compact set lies in a closed bounded metric neighborhood of that compact set, which is compact by Hopf–Rinow. Since F is a local diffeomorphism, it is a proper covering. If the displacement is less than the injectivity radius, the homotopy $F_t(x)=\exp_x(t\log_x F(x))$ joins it to the identity. On each component, F_* therefore induces an isomorphism of fundamental groups, and the covering has one sheet. This gives injectivity and surjectivity of F and hence of h. The covering and continuation arguments are those in the chapter on infinite-dimensional calculus.

Sobolev composition estimates apply in uniform charts: the lower bound on $d\eta$ controls the change-of-variables Jacobian, and bounds on derivatives of the center control the coefficients. In a fixed neighborhood, products and inverses of elements close to the identity again belong to the Sobolev charts at the identity and depend continuously on them. For continuity, first approximate the outer factor by a smooth map, apply the uniform estimate to the error, and then use continuity for that fixed map. In a connected component, every element is a product of finitely many elements of any symmetric neighborhood of the identity: the union of these products is an open subgroup and is also closed in the component. Local estimates transfer to every finite product. This gives the topological group $\mathcal{D}_0^{p,r}(M)$ with a countable neighborhood base at the identity. Pairs (a,b) with $a^{-1}b$ in one of these neighborhoods form a base for its left uniformity. Continuity of multiplication allows symmetric neighborhoods to be chosen whose cubes lie in the preceding neighborhood; Proposition 27.2.12 gives a compatible metric. These are the properties of Theorems 6.4 and 6.5 of [49, §6] used here.

The topology retains global control in the Sobolev norms of the charts. At fixed regularity, one obtains a topological group; the differentiable properties recovered by moving to higher levels motivate the ILH tower.

27.5.2 Hilbert scales and ILH towers

The preceding loss of derivatives naturally leads to working with a scale of spaces instead of fixing a single regularity.

Definition 27.5.4. (Hilbert scale) Let $s_0 \in \mathbb{Z}$. A *Hilbert scale* is a family $(H^s)_{s\in\mathbb{Z},\, s\geq s_0}$ with continuous dense inclusions $H^{s+1} \hookrightarrow H^s$. Its smooth core is $H^\infty := \bigcap_{s\in\mathbb{Z},\, s\geq s_0} H^s$, equipped with the initial topology of the inclusions $H^\infty \longrightarrow H^s$; equivalently, $H^\infty = \varprojlim_{s\in\mathbb{Z},\, s\geq s_0} H^s$.

Definition 27.5.5. (Differentiable compatibility with loss) Fix $\ell \in \mathbb{N}_0$ and $s \in \mathbb{Z}$ with $s \geq s_0$. A family of maps $F^s \colon \mathcal{U}^{s+\ell} \longrightarrow \mathcal{N}^s$ is an *extension between levels of class* ℓ of F^∞ if its restrictions are compatible with the inclusions, each F^s is of class C^ℓ, and, for $0 \leq j \leq \ell$, its differentials, expressed in compatible charts, satisfy on neighborhoods with common control data an estimate

$$\|D^j F^s(x)(v_1,\dots,v_j)\|_{H^s} \leq C\,\mathcal{P}(\|x\|_{H^{s+\ell}}) \prod_{i=1}^{j} \|v_i\|_{H^{s+\ell}},$$

, where $\mathcal{P}$ is a polynomial independent of the smooth approximation. The constant may depend on the charts and their control data. For diffeomorphisms, in addition to Sobolev bounds, a common bound on $\|(d\eta)^{-1}\|_\infty$ is fixed; boundedness of the $H^s(\Omega,\mathbb{R}^m)$ norms of the coordinate expressions of η on the domains Ω of the fixed charts alone does not prevent the differential from approaching a singular matrix. We say that F^∞ is *smooth in the ILH tower* if it admits these extensions for every ℓ. This definition uses Banach calculus at each level and explicitly records the derivatives that must be recovered by ascending the scale.

Definition 27.5.6. (Compatible ILH tower) An *ILH tower* modeled on (H^s) is a family of Hilbert manifolds $(\mathcal{M}^s)_{s\in\mathbb{Z},\, s\geq s_0}$ with smooth dense inclusions $\mathcal{M}^{s+1} \hookrightarrow \mathcal{M}^s$ whose charts restrict compatibly and whose core is $\mathcal{M}^\infty := \bigcap_{s\in\mathbb{Z},\, s\geq s_0} \mathcal{M}^s$. An ILH group tower also has a group operation on G^∞ and extensions of multiplication and inversion between levels. The regularities of these extensions and the loss of derivatives are specified in Definition 27.5.5; the mere equality $G^\infty = \bigcap_{s\in\mathbb{Z},\, s\geq s_0} G^s$ does not suffice.

Proposition 27.5.7. If each H^{s+1} is dense in H^s, then H^∞, with the norms $\|\cdot\|_{H^s}$, is a Fréchet space. A sequence converges in H^∞ if and only if it converges at every level H^s.

Proof. The metric

$$d(u,v) := \sum_{j=0}^{\infty} 2^{-j} \frac{\|u-v\|_{H^{s_0+j}}}{1+\|u-v\|_{H^{s_0+j}}}$$

generates the projective topology. If (u_n) is Cauchy, it converges in each H^s to an element u_s. Continuity of the inclusions shows that these limits are compatible and represent a unique $u \in \bigcap_{s\in\mathbb{Z},\, s\geq s_0} H^s$; then $u_n \to u$ in all norms. □

Proposition 27.5.8. (Smooth core of the Sobolev scale) If M is a smooth compact manifold with or without boundary and $\mathbf{E} \longrightarrow M$ is a smooth vector bundle, then $\Gamma(\mathbf{E}) = \bigcap_{s\in\mathbb{Z},\ s\geq s_0} H^s(M,\mathbf{E})$ as locally convex spaces: the projective topology of the Sobolev norms agrees with the usual topology of uniform convergence of all derivatives. If M is noncompact, we always have $\bigcap_{s\in\mathbb{Z},\ s\geq s_0} H^s(M,\mathbf{E}) \subseteq \Gamma(\mathbf{E})$, but the inclusion may be strict because a smooth section need not be integrable at infinity.

Proof. First suppose M is compact. For fixed s, choose an integer $k \geq \max\{s,0\}$. Inclusion in the scale and equivalence between the H^k norm and the covariant norm of order k give constants $A_{s,k}, B_k > 0$ such that, for $\mathbf{u} \in \Gamma(\mathbf{E})$,

$$\|\mathbf{u}\|_{H^s(M,\mathbf{E})} \leq A_{s,k}\|\mathbf{u}\|_{H^k(M,\mathbf{E})} \leq B_k \lambda_{\mathbf{g}}(M)^{1/2} \sum_{\ell=0}^{k} \|\nabla^\ell \mathbf{u}\|_{C^0(M,T^{(0,\ell)}(TM)\otimes\mathbf{E};\mathbf{g},\mathbf{h}_\mathbf{E})}.$$

Thus each inclusion $\Gamma(\mathbf{E}) \longrightarrow H^s(M,\mathbf{E})$ is continuous. In the opposite direction, given j, choose $s > j + m/2$. Sobolev embedding gives $\|\mathbf{u}\|_{C^j(M,\mathbf{E})} \leq C_{s,j}\|\mathbf{u}\|_{H^s(M,\mathbf{E})}$. Thus the scale norms and the C^j seminorms generate the same topology. On a noncompact base, even a nonzero parallel section may have infinite L^2 norm, proving the last assertion. □

Definition 27.5.9. (The term *strong ILH*) We reserve the term *strong ILH Lie group* for Omori's notion [110, Chapter I, §1.1]. It requires a Hilbert chain, a compatible chart at the identity, and conditions $(N,1)$–$(N,7)$ for extensions of multiplication, translations, their iterated differentials, inversion, and conjugation by smooth elements. A compatible ILH tower in the sense of the preceding definitions will not be regarded as *strong ILH* until these axioms have been verified.

Remark 27.5.10. (Three forms of uniform control) The notions used in this chapter act on different objects. Amann's uniform regularity is a coordinate property of the atlas and the base metric; bounded geometry is its intrinsic formulation through injectivity radius, curvature, and, when there is boundary, collar data. An Eichhorn uniformity is imposed on a configuration space and serves to define global closeness, components, and completions. An ILH tower, finally, organizes a family of Hilbert completions and losses of regularity between them.

Theorems 13.2.5 and 14.4.8 relate the first two notions, without and with boundary. Neither determines by itself a uniformity on configurations, and a uniformity does not provide the charts and estimates between levels required by an ILH tower.

Proposition 27.5.11. (Operations on the diffeomorphism core) Let M be a smooth compact manifold with or without boundary and let $s_0 \in \mathbb{Z}$ with $s_0 > \frac{m}{2} + 1$. Then

$$\mathrm{Diff}^\infty(M) = \bigcap_{s\in\mathbb{Z},\ s\geq s_0} \mathrm{Diff}^s(M)$$

, and composition and inversion are smooth in the sense of Definition 27.5.5. More precisely, for each $\ell \geq 0$,

$$\mathrm{Diff}^{s+\ell}(M) \times \mathrm{Diff}^s(M) \longrightarrow \mathrm{Diff}^s(M), \qquad (\eta,\varphi) \longmapsto \eta \circ \varphi,$$

and

$$\mathrm{Diff}^{s+\ell}(M) \longrightarrow \mathrm{Diff}^{s}(M), \qquad \eta \longmapsto \eta^{-1},$$

are of class C^{ℓ}.

Proof. The set-theoretic equality follows from Proposition 27.5.8 applied in charts and the same assertion for the inverse. The estimates follow by differentiating composition. Differentiation with respect to the inner argument φ uses spatial derivatives of the outer factor η. In coordinates, for $j \in \{1,\dots,\ell\}$, its leading term is

$$D^{j}_{\varphi}(\eta\circ\varphi)[\mathbf{X}_1,\dots,\mathbf{X}_j] = (D^{j}\eta\circ\varphi)[\mathbf{X}_1,\dots,\mathbf{X}_j].$$

Dependence on the outer factor is linear in these coordinates; mixed derivatives are obtained by replacing η by its variation in this formula. Chart changes add smooth superpositions and products of the same order. The space H^s, with $s > m/2+1$, is an algebra, and composition estimates control each term in terms of the $H^{s+\ell}(\Omega,\mathbb{R}^m)$ norms of the coordinate expressions of η on the domains Ω of the fixed charts, and the common data of the neighborhood of φ. For inversion, differentiate $\eta\circ\eta^{-1} = \mathrm{id}$; each derivative of order ℓ is a finite sum of products of spatial differentials of the outer factor and factors $(d\eta\circ\eta^{-1})^{-1}$. The first step is $D(\eta^{-1})[\mathbf{X}] = -(d\eta\circ\eta^{-1})^{-1}(\mathbf{X}\circ\eta^{-1})$. In passing from order j to $j+1$, Leibniz's rule differentiates one of these factors; the rule $D(B^{-1})[C] = -B^{-1}CB^{-1}$ handles the inverse factors, and the chain rule handles the compositions. At most one additional spatial derivative of η is needed at each step, through order ℓ. The lower bound on $d\eta$ in a neighborhood of a diffeomorphism controls the matrix inverses. This proves the extensions and their bounds at each level. □

This definition does not assert that each G^s is a Hilbert Lie group. In the diffeomorphism example, G^s is a topological group with smooth right translations; the smooth operation reappears in the limit because any finite loss can be compensated by ascending finitely many levels.

Scales do not remove derivative loss; they record it. An argument at fixed regularity must close in a single H^s. An ILH argument may temporarily ascend to H^{s+j} and then descend, provided the estimates are uniform and compatible. This distinction will be decisive when interpreting the geometric differential operators of the next chapter as fields on scales of function spaces.

Remark 27.5.12. (Hierarchy of structures) A Hilbert manifold at a fixed regularity level, a compatible tower, and a *strong ILH* Lie group are successively stronger structures. The identity $G^{\infty} = \bigcap_{s\in\mathbb{Z},\ s\geq s_0} G^s$ alone provides neither compatible charts nor estimates for derivative loss, and these charts do not automatically imply Omori's axioms either. In what follows, constructions at fixed regularity are treated using Banach calculus, while operations with derivative loss are organized by means of Definition 27.5.5.

28

Geometric applications in infinite dimensions

The constructions of the preceding chapter take on their clearest meaning when an equation can be recognized as a geometric condition on a configuration space. In this chapter we apply this viewpoint to harmonic maps, the Yang–Mills equations, and the Euler equation for an incompressible fluid. Although the problems differ, in each case the unknown is no longer viewed as a collection of components, but as a point of an infinite-dimensional manifold.

The procedure will be the same in each application. First we identify the configuration space and its tangent space; we then introduce the natural functional or metric and compute its variation in an arbitrary direction. Integration by parts then reveals the corresponding differential equation. Symmetries are also part of the problem: invariance under reparametrizations or gauge transformations produces degenerate directions and requires us to distinguish the geometric equation from a particular choice of coordinates or gauge.

The L^2 metrics used in these applications are generally weak with respect to the chosen Sobolev topologies. Thus expressions such as "gradient of the energy" will initially be understood in the formal L^2 sense: the resulting operator usually loses derivatives and does not by itself define a vector field on the configuration space at that regularity. When an evolution equation or a smooth spray is needed, we shall specify the regularity scale or elliptic mechanism that recovers the required derivatives. The geometric interpretation of these equations therefore depends on estimates controlling the loss and recovery of derivatives.

Three problems, three geometries

The theory of harmonic maps developed by extending the Dirichlet energy to maps between manifolds; Yang–Mills theory made the curvature of a connection a central functional of gauge theory; Arnold and, in a precise analytic framework, Ebin and Marsden interpreted the Euler equations as geodesics on a diffeomorphism group. These three examples show that the geometry of the configuration space is not merely decorative: it determines the variables, the symmetries, and the regularity at which the equation is well posed [46].

28.1 Harmonic maps

The first example extends the Dirichlet functional from the calculus of variations to maps between manifolds. Let $(M^m, \mathbf{g})$ be a compact Riemannian manifold with or without boundary and $(N, \mathbf{h})$ a finite-dimensional Riemannian manifold without boundary. For a smooth map $f : M \longrightarrow N$, the differential is a section of $T^*M \otimes f^*TN$, and its *Dirichlet energy* is

$$\mathcal{E}(f) := \frac{1}{2}\int_M |df|^2_{\mathbf{g},\mathbf{h}}\, d\lambda_{\mathbf{g}}. \tag{28.1}$$

The Levi–Civita connection of $\mathbf{h}$ induces a connection, still denoted by ∇, on f^*TN and on $T^*M \otimes f^*TN$. The *tension field* of f is

$$\boldsymbol{\tau}(f) := \operatorname{tr}_{\mathbf{g}}(\nabla df) = \sum_{i=1}^{m} \big(\nabla_{\mathbf{e}_i} df(\mathbf{e}_i) - df(\nabla^M_{\mathbf{e}_i}\mathbf{e}_i)\big), \tag{28.2}$$

where $(\mathbf{e}_i)_{i=1}^m$ is any local orthonormal frame. This expression is tensorial and therefore independent of the chosen frame.

Proposition 28.1.1. (First variation of the energy) Let $(M^m, \mathbf{g})$ be a compact Riemannian manifold, with or without boundary, and $(N, \mathbf{h})$ a Riemannian manifold without boundary. Let f_t be a smooth variation of $f = f_0$, and let

$$\mathbf{V} := \partial_t f_t|_{t=0} \in \Gamma(f^*TN).$$

Then

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}(f_t) = -\int_M \langle \boldsymbol{\tau}(f), \mathbf{V}\rangle_{\mathbf{h}}\, d\lambda_{\mathbf{g}} + \int_{\partial M} \langle df(\boldsymbol{\nu}), \mathbf{V}\rangle_{\mathbf{h}}\, d\sigma_{\mathbf{g}}, \tag{28.3}$$

where $\boldsymbol{\nu}$ is the outward unit normal when $\partial M \neq \varnothing$.

Proof. Consider the map

$$F : (-\varepsilon, \varepsilon) \times M \longrightarrow N, \qquad F(t, x) := f_t(x).$$

Since the Levi–Civita connection of N is torsion-free and the vector fields ∂_t and $\mathbf{e}_i$ commute,

$$\nabla_{\partial_t} df_t(\mathbf{e}_i) = \nabla_{\mathbf{e}_i} \partial_t f_t.$$

Evaluating at $t = 0$ gives

$$\frac{d}{dt}\Big|_{t=0} \frac{1}{2}|df_t|^2_{\mathbf{g},\mathbf{h}} = \sum_{i=1}^{m} \langle \nabla_{\mathbf{e}_i}\mathbf{V}, df(\mathbf{e}_i)\rangle_{\mathbf{h}}.$$

Define the 1-form

$$\boldsymbol{\alpha}(\mathbf{X}) := \langle \mathbf{V}, df(\mathbf{X})\rangle_{\mathbf{h}}.$$

The Leibniz rule and the definition of the tension field give

$$\operatorname{div}_{\mathbf{g}} \boldsymbol{\alpha}^\sharp = \sum_{i=1}^{m} \langle \nabla_{\mathbf{e}_i}\mathbf{V}, df(\mathbf{e}_i)\rangle_{\mathbf{h}} + \langle \mathbf{V}, \boldsymbol{\tau}(f)\rangle_{\mathbf{h}}.$$

Consequently,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}(f_t) = \int_M \operatorname{div}_{\mathbf{g}} \boldsymbol{\alpha}^\sharp\, d\lambda_{\mathbf{g}} - \int_M \langle \mathbf{V}, \boldsymbol{\tau}(f)\rangle_{\mathbf{h}}\, d\lambda_{\mathbf{g}}.$$

Stokes' theorem transforms the first term into

$$\int_{\partial M} \boldsymbol{\alpha}(\boldsymbol{\nu})\, d\sigma_{\mathbf{g}} = \int_{\partial M} \langle \mathbf{V}, df(\boldsymbol{\nu})\rangle_{\mathbf{h}}\, d\sigma_{\mathbf{g}}.$$

□

If $\partial M = \varnothing$, or if the trace of f is fixed and only variations with $\operatorname{Tr}\mathbf{V} = 0$ are allowed, the boundary term vanishes. In this variational problem, the formal L^2 representative of the differential is

$$\operatorname{grad}_{L^2} \mathcal{E}(f) = -\boldsymbol{\tau}(f). \tag{28.4}$$

Consequently, for these variations the critical points are exactly the maps satisfying $\boldsymbol{\tau}(f) = 0$, called *harmonic maps*. Variations compactly supported in the interior establish the necessity of the equation, and the first variation establishes its sufficiency. If the trace is allowed to vary freely, one must also impose $df(\boldsymbol{\nu}) = 0$ on the boundary: once the tension field vanishes, extending any smooth vector field from the boundary makes the trace of $\mathbf{V}$ arbitrary in the remaining term. Thus, for free variations, the boundary term cannot be discarded or generally represented by an interior L^2 gradient. This variational formulation goes back to [85]; the preceding calculation also fixes the sign convention we shall use.

Proposition 28.1.2. (Linearization of the tension field and second variation) Let $(M^m, \mathbf{g})$ be a compact Riemannian manifold without boundary, $(N, \mathbf{h})$ a Riemannian manifold without boundary, and $f : M \longrightarrow N$ a smooth map. Define

$$\nabla^*\nabla := -\operatorname{tr}_{\mathbf{g}}(\nabla^2)$$

on sections of f^*TN. The derivative of the tension field is understood covariantly: transport $\boldsymbol{\tau}(f_t)$ to f^*TN along the variation and differentiate at $t = 0$. We denote this derivative by $D\boldsymbol{\tau}(f)[\mathbf{V}]$. For every variation with variational vector field $\mathbf{V}$,

$$D\boldsymbol{\tau}(f)[\mathbf{V}] = -\nabla^*\nabla\mathbf{V} + \sum_{i=1}^{m} \operatorname{Rm}_{\mathbf{h}}\big(\mathbf{V}, df(\mathbf{e}_i)\big)\, df(\mathbf{e}_i). \tag{28.5}$$

If f is harmonic, the linearization of $-\boldsymbol{\tau}$ is the Jacobi operator

$$J_f\mathbf{V} = \nabla^*\nabla\mathbf{V} - \sum_{i=1}^{m} \operatorname{Rm}_{\mathbf{h}}\big(\mathbf{V}, df(\mathbf{e}_i)\big)\, df(\mathbf{e}_i). \tag{28.6}$$

The Hessian of the energy is given by

$$D^2\mathcal{E}(f)[\mathbf{V},\mathbf{W}] = \int_M \left(\langle \nabla\mathbf{V}, \nabla\mathbf{W}\rangle_{\mathbf{g}, f^*\mathbf{h}} - \sum_{i=1}^{m} \big\langle \operatorname{Rm}_{\mathbf{h}}\big(\mathbf{V}, df(\mathbf{e}_i)\big)\, df(\mathbf{e}_i), \mathbf{W}\big\rangle_{f^*\mathbf{h}} \right) d\lambda_{\mathbf{g}}. \tag{28.7}$$

In particular,

$$D^2\mathcal{E}(f)[\mathbf{V},\mathbf{V}] = \int_M \left(|\nabla\mathbf{V}|^2_{\mathbf{g}, f^*\mathbf{h}} - \sum_{i=1}^{m} \big\langle \operatorname{Rm}_{\mathbf{h}}\big(\mathbf{V}, df(\mathbf{e}_i)\big)\, df(\mathbf{e}_i), \mathbf{V}\big\rangle_{f^*\mathbf{h}} \right) d\lambda_{\mathbf{g}}. \tag{28.8}$$

Proof. Let $F(t,x) = f_t(x)$, and extend the frame $(\mathbf{e}_i)$ locally without dependence on t. The absence of torsion implies

$$\nabla_{\partial_t} df_t(\mathbf{e}_i) = \nabla_{\mathbf{e}_i} \mathbf{V}.$$

Commuting the covariant derivatives once more gives

$$(\nabla_{\partial_t}\nabla_{\mathbf{e}_i} - \nabla_{\mathbf{e}_i}\nabla_{\partial_t}) df_t(\mathbf{e}_i) = \mathrm{Rm}_{\mathbf{h}}\big(dF(\partial_t), dF(\mathbf{e}_i)\big) df_t(\mathbf{e}_i),$$

so that, at $t = 0$,

$$\nabla_{\partial_t}\nabla_{\mathbf{e}_i} df_t(\mathbf{e}_i) = \nabla_{\mathbf{e}_i}\nabla_{\mathbf{e}_i}\mathbf{V} + \mathrm{Rm}_{\mathbf{h}}\big(\mathbf{V}, df(\mathbf{e}_i)\big) df(\mathbf{e}_i).$$

At a given point, choose the frame so that $\nabla^M_{\mathbf{e}_i}\mathbf{e}_i = 0$. Taking the trace gives (28.5); since both sides are tensorial, the identity holds at every point and for any frame.

Now suppose that $\boldsymbol{\tau}(f) = 0$. Differentiating the first variation in a second direction produces a term from the variation of the second vector field; this term is multiplied by $\boldsymbol{\tau}(f)$ and vanishes. Thus

$$D^2\mathcal{E}(f)[\mathbf{V},\mathbf{W}] = \int_M \langle J_f \mathbf{V}, \mathbf{W}\rangle_{f^*\mathbf{h}}\, d\lambda_{\mathbf{g}}.$$

Integrating $\nabla^*\nabla$ by parts gives (28.7). The symmetries of the curvature tensor show that this form is symmetric. Finally, the principal symbol of J_f is

$$\sigma_2(J_f)(x,\xi) = -|\xi|^2_{\mathbf{g}}\,\mathrm{id}_{T_{f(x)}N}, \qquad x \in M, \quad \xi \in T^*_xM \setminus \{0\},$$

with the symbol convention from the chapter on differential operators, which replaces ∂_j by ξ_j without introducing the factor i. The sign comes from the principal term $-g^{ij}\partial_i\partial_j$. Thus J_f is elliptic; integration by parts and the curvature symmetries separately establish its formal self-adjointness. □

We turn to the formulation in Sobolev spaces. Fix an integer

$$s > \frac{m}{2} + 1.$$

Exponential charts make $H^s(M,N)$ into a Hilbert manifold. More precisely, Theorem 27.3.4 constructs the isomorphism

$$\mathcal{I}_f : H^s(M, f^*TN) \longrightarrow T_f H^s(M,N), \qquad \mathcal{I}_f(\mathbf{V}) := \left.\frac{d}{dt}\right|_{t=0} \mathrm{Exp}_f(t\mathbf{V}).$$

In the chart Exp_f, this map and its inverse are represented by the identity on the model space. In particular,

$$T_f H^s(M,N) \cong H^s(M, f^*TN).$$

The tension field contains two derivatives of f, so its natural codomain has two fewer orders of regularity. To describe it, consider

$$\boldsymbol{\mathcal{T}}^{s-2} := \coprod_{\widetilde{f} \in H^s(M,N)} \Big(\{\widetilde{f}\} \times H^{s-2}(M, \widetilde{f}^*TN)\Big), \qquad \pi_{\boldsymbol{\mathcal{T}}^{s-2}}(\widetilde{f}, \mathbf{V}) := \widetilde{f}.$$

Let $\mathcal{U}_f$ be the domain of an exponential chart centered at f. If $\widetilde{f} = \operatorname{Exp}_f(u) \in \mathcal{U}_f$, parallel transport along

$$t \longmapsto \exp^N_{f(x)}(tu(x))$$

defines a bundle isometry

$$P_{f,\widetilde{f}} \colon \widetilde{f}^* TN \longrightarrow f^* TN.$$

The corresponding trivialization is

$$\Theta_f \colon \pi^{-1}_{\boldsymbol{\mathcal{T}}^{s-2}}(\mathcal{U}_f) \longrightarrow \mathcal{U}_f \times H^{s-2}(M, f^*TN), \qquad \Theta_f(\widetilde{f}, \mathbf{V}) := \big(\widetilde{f}, P_{f,\widetilde{f}}\mathbf{V}\big).$$

These trivializations equip $\boldsymbol{\mathcal{T}}^{s-2}$ with the structure of a smooth Banach bundle. The analytic point is continuity of multiplication between the coefficients of the transition maps and the sections. After multiplying by fixed cutoff functions for a trivialization and extending to $\mathbb{R}^m$, write

$$a \colon \mathbb{R}^m \to \operatorname{End}(\mathbb{R}^n), \qquad W \colon \mathbb{R}^m \to \mathbb{R}^n.$$

In these representations, we have

$$\|aW\|_{H^{s-2}(\mathbb{R}^m,\mathbb{R}^n)} \le C_s \|a\|_{H^s(\mathbb{R}^m,\operatorname{End}(\mathbb{R}^n))} \|W\|_{H^{s-2}(\mathbb{R}^m,\mathbb{R}^n)}, \tag{28.9}$$

where a is matrix-valued and W locally represents a section. The constant depends on M, s, and the trivializations, but not on a or W.

To verify the estimate, apply the Leibniz rule to each derivative of order at most $s-2$. If $\alpha, \beta \in \mathbb{N}_0^m$, $|\alpha| \le s-2$ and $\beta_i \le \alpha_i$ for $i \in \{1, \dots, m\}$, the regularities of the two factors satisfy

$$(s - |\beta|) + (s - 2 - |\alpha - \beta|) = 2s - 2 - |\alpha| \ge s > \frac{m}{2}.$$

Set $a_1 = s - |\beta|$ and $a_2 = s - 2 - |\alpha - \beta|$. Both are nonnegative and $a_1 + a_2 > m/2$. We may choose $p_1, p_2 \in [2, \infty]$ with $1/p_1 + 1/p_2 = 1/2$ so that the embeddings $H^{a_i}(\mathbb{R}^m) \hookrightarrow L^{p_i}(\mathbb{R}^m)$ are continuous. If one of the a_i is zero, take $p_i = 2$ and the other exponent to be infinity, which is allowed because the other order exceeds $m/2$. If both are positive, the strict inequality allows the exponents to be chosen away from any critical endpoint that would require L^∞. Hölder's inequality then estimates each product in L^2. Summing over the multi-indices and the finitely many charts yields (28.9).

On an overlap of charts centered at f_0 and f_1,

$$\Theta_{f_1} \circ \Theta_{f_0}^{-1}(\widetilde{f}, W) = \Big(\widetilde{f}, P_{f_1,\widetilde{f}} P^{-1}_{f_0,\widetilde{f}} W\Big).$$

The coefficient

$$\widetilde{f} \longmapsto P_{f_1,\widetilde{f}} P^{-1}_{f_0,\widetilde{f}}$$

depends smoothly on $\widetilde{f}$, with values in $H^s(M, \operatorname{Hom}(f_0^*TN, f_1^*TN))$, by Lemma 27.3.2. Estimate (28.9) proves smoothness of the transition map and its inverse.

To see which terms must be estimated, in coordinates on M and N the tension field has components

$$\tau^{\alpha}(f)=\sum_{i,j=1}^{m} g^{ij}\left(\partial_i\partial_j f^{\alpha}-\sum_{a=1}^{m}\Gamma_{ij}^{a}\partial_a f^{\alpha}+\sum_{\beta,\gamma=1}^{\dim N}\Gamma_{\beta\gamma}^{\alpha}(f)\partial_i f^{\beta}\partial_j f^{\gamma}\right),\qquad \alpha\in\{1,\dots,\dim N\}.$$

On each fixed coordinate domain $\Omega\subseteq\mathbb{R}^m$, the first term belongs to $H^{s-2}(\Omega)$; the coefficients on the base are smooth. The first derivatives of the components of f belong to $H^{s-1}(\Omega)$, which is an algebra, and $\Gamma_{\beta\gamma}^{\alpha}(f)$ depends smoothly on the components of f with values in $H^{s}(\Omega)$. Estimate (28.9) controls multiplication at the lower level. Each differentiation with respect to f replaces one factor by its variation or differentiates a superposition; it does not add a spatial derivative to the variation vectors. This proves smoothness with values in the fibers $H^{s-2}(M,f^{*}TN)$. For the energy, the same products are applied to $|df|^2$, followed by the continuous integral on L^1. Hence

$$\mathcal{E}\colon H^{s}(M,N)\longrightarrow\mathbb{R}$$

is smooth. Similarly, the tension field defines the smooth section

$$\boldsymbol{\tau}\colon H^{s}(M,N)\longrightarrow\boldsymbol{\mathcal{T}}^{s-2},\qquad f\longmapsto(f,\boldsymbol{\tau}(f)),\qquad \pi_{\boldsymbol{\mathcal{T}}^{s-2}}\circ\boldsymbol{\tau}=\mathrm{id}_{H^{s}(M,N)}.\tag{28.10}$$

To compute its vertical derivative at f, we use the trivialization by parallel transport centered precisely at f. This gives the continuous linear operator

$$D\boldsymbol{\tau}(f)\colon H^{s}(M,f^{*}TN)\longrightarrow H^{s-2}(M,f^{*}TN),$$

whose expression on smooth maps is (28.5). If f is harmonic, $J_f=-D\boldsymbol{\tau}(f)$ has the same domain and codomain. In another trivialization, differentiating the change of frame produces a term multiplied by $\boldsymbol{\tau}(f)$; this is why the linearization at a zero is independent of that choice. For nonharmonic maps, the intrinsic expression above is the covariant derivative, not the ordinary derivative of all representations simultaneously.

Thus $-\boldsymbol{\tau}(f)$ does not generally belong to the tangent space $T_fH^{s}(M,N)$, but to the fiber of lower regularity $H^{s-2}(M,f^{*}TN)$. An alternative is to use a strong metric on the mapping space. The representative of the differential with respect to such a metric is obtained by formally regularizing through

$$(1+\nabla^{*}\nabla)^{-s}(-\boldsymbol{\tau}(f)).$$

Its construction requires studying how the elliptic operators involved depend on f.

Finiteness of the energy requires only that df be square integrable, whereas the manifold structure and pointwise evaluation require regularity greater than $\frac{m}{2}$. This difference between the energy level and the geometric level also appears in compactness questions. In dimension two, for example, a sequence with bounded energy may concentrate energy and produce bubbles; on a noncompact base, it may also lose energy at infinity.

28.2 Yang–Mills theory and gauge reduction

The second application has an additional symmetry. A connection is not an isolated object: the gauge group transforms distinct connections describing the same geometry. This invariance makes the Hessian of the Yang–Mills functional degenerate in directions tangent to the gauge orbits and leads naturally to the Coulomb condition.

Let $(M^m,\mathbf{g})$ be a compact Riemannian manifold with or without boundary, and let $\mathbf{P}\longrightarrow M$ be a principal

bundle with compact structure group G. Denote its adjoint bundle by

$$\operatorname{ad}\mathbf{P} := \mathbf{P} \times_{\mathrm{Ad}} \mathfrak{g}$$

and fix an Ad-invariant inner product on $\mathfrak{g}$. We denote the induced metric on $\operatorname{ad}\mathbf{P}$ by $\mathbf{h}_{\mathrm{ad}}$.

The curvature of a connection $\mathbf{A}$ is an adjoint-bundle-valued 2-form:

$$\mathbf{F}_{\mathbf{A}} \in \Omega^2(M, \operatorname{ad}\mathbf{P}) := \Gamma\left(\Lambda^2(T^*M) \otimes \operatorname{ad}\mathbf{P}\right).$$

The superscript 2 indicates the degree of the form: for each $x \in M$, $\mathbf{F}_{\mathbf{A}}(x)$ takes two vectors in T_xM, depends alternatingly on them, and takes values in $(\operatorname{ad}\mathbf{P})_x$.

The space of connections is affine, modeled on $\Omega^1(M, \operatorname{ad}\mathbf{P})$. If $\mathbf{a}$ is an $\operatorname{ad}\mathbf{P}$-valued 1-form, the curvature of $\mathbf{A}+\mathbf{a}$ satisfies

$$\mathbf{F}_{\mathbf{A}+\mathbf{a}} = \mathbf{F}_{\mathbf{A}} + d_{\mathbf{A}}\mathbf{a} + \frac{1}{2}[\mathbf{a} \wedge \mathbf{a}]. \tag{28.11}$$

The Yang–Mills functional is

$$\mathrm{YM}(\mathbf{A}) := \frac{1}{2}\int_M |\mathbf{F}_{\mathbf{A}}|^2_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}}\, d\lambda_{\mathbf{g}}. \tag{28.12}$$

Proposition 28.2.1. (First variation of Yang–Mills) Let $(M, \mathbf{g})$ be a compact Riemannian manifold with or without boundary, let $\mathbf{P} \to M$ be a principal bundle with compact structure group, and let $\mathbf{A}$ be a smooth connection. For $\mathbf{a} \in \Omega^1(M, \operatorname{ad}\mathbf{P})$,

$$D\,\mathrm{YM}(\mathbf{A})[\mathbf{a}] = \int_M \langle d^*_{\mathbf{A}}\mathbf{F}_{\mathbf{A}}, \mathbf{a}\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}}\, d\lambda_{\mathbf{g}} + \int_{\partial M} \langle \iota_{\boldsymbol{\nu}}\mathbf{F}_{\mathbf{A}}, \mathbf{a}\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}}\, d\sigma_{\mathbf{g}}. \tag{28.13}$$

Here $\boldsymbol{\nu}$ is the outward unit normal and

$$d^*_{\mathbf{A}}\mathbf{F} = -\operatorname{tr}_{\mathbf{g}}(\nabla^{\mathbf{A}}\mathbf{F})$$

on $\operatorname{ad}\mathbf{P}$-valued 2-forms. Contraction is performed between the derivative index and the first index of the form: for $\mathbf{X} \in T_xM$,

$$(d^*_{\mathbf{A}}\mathbf{F})(\mathbf{X}) = -\sum_{i=1}^{m} (\nabla^{\mathbf{A}}_{\mathbf{e}_i}\mathbf{F})(\mathbf{e}_i, \mathbf{X}).$$

Proof. Substituting $t\mathbf{a}$ into (28.11), the part linear in t is $d_{\mathbf{A}}\mathbf{a}$. Consequently,

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{YM}(\mathbf{A} + t\mathbf{a}) = \int_M \langle \mathbf{F}_{\mathbf{A}}, d_{\mathbf{A}}\mathbf{a}\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}}\, d\lambda_{\mathbf{g}}.$$

The Green form from Chapter 9 gives the adjoint and the boundary term. To make the contraction explicit, define

$$\alpha(\mathbf{X}) := \sum_{j=1}^{m} \langle \mathbf{F}_{\mathbf{A}}(\mathbf{X}, \mathbf{e}_j), \mathbf{a}(\mathbf{e}_j)\rangle_{\mathbf{h}_{\mathrm{ad}}}.$$

In an orthonormal frame normal at the point of calculation, the Leibniz rule and antisymmetry of $\mathbf{F}_{\mathbf{A}}$ give $\operatorname{div}\alpha^\sharp = \langle \mathbf{F}_{\mathbf{A}}, d_{\mathbf{A}}\mathbf{a}\rangle - \langle d^*_{\mathbf{A}}\mathbf{F}_{\mathbf{A}}, \mathbf{a}\rangle$. Using the norm of 2-forms means that we sum once over $i < j$, rather than twice over ordered pairs. Upon integration, $\alpha(\boldsymbol{\nu}) = \langle \iota_{\boldsymbol{\nu}}\mathbf{F}_{\mathbf{A}}, \mathbf{a}\rangle$ yields exactly (28.13). □

If M has no boundary, or if the induced tangential connection on ∂M is fixed, the admissible variations make

the boundary term vanish. In the second case, we require $\iota^*\mathbf{a} = 0$, where $\iota : \partial M \hookrightarrow M$ is the inclusion; the normal component of $\mathbf{a}$ does not contribute because $(\iota_{\boldsymbol{\nu}}\mathbf{F}_{\mathbf{A}})(\boldsymbol{\nu}) = 0$. The formal L^2 representative of the differential and the interior Euler–Lagrange equation are then

$$\operatorname{grad}_{L^2} \mathrm{YM}(\mathbf{A}) = d_{\mathbf{A}}^{*}\mathbf{F}_{\mathbf{A}}, \qquad d_{\mathbf{A}}^{*}\mathbf{F}_{\mathbf{A}} = 0. \tag{28.14}$$

If the connection on the boundary may also vary freely, the critical point condition additionally includes $\iota_{\boldsymbol{\nu}}\mathbf{F}_{\mathbf{A}} = 0$ on ∂M. Interior variations first determine the Yang–Mills equation; arbitrariness of the tangential trace then determines this natural boundary condition.

To write the Hessian, we first specify the algebraic term containing the curvature. Define pointwise the endomorphism

$$\boldsymbol{\mathcal{R}}_{\mathbf{A}} \colon T^*M \otimes \operatorname{ad}\mathbf{P} \longrightarrow T^*M \otimes \operatorname{ad}\mathbf{P}$$

by

$$\langle \boldsymbol{\mathcal{R}}_{\mathbf{A}}\eta, \zeta\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}};x} := \langle \mathbf{F}_{\mathbf{A}}, [\eta \wedge \zeta]\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}};x}, \qquad \eta, \zeta \in T_x^*M \otimes (\operatorname{ad}\mathbf{P})_x.$$

The Riesz representation theorem in each fiber determines a unique $\boldsymbol{\mathcal{R}}_{\mathbf{A}}\eta$. Its coefficients are algebraic contractions of $\mathbf{F}_{\mathbf{A}}$ with the bracket on $\mathfrak{g}$, so $\boldsymbol{\mathcal{R}}_{\mathbf{A}}$ is a smooth endomorphism of order zero. For $1 \le i, j \le m$, $[\eta \wedge \zeta](\mathbf{e}_i, \mathbf{e}_j) = [\eta(\mathbf{e}_i), \zeta(\mathbf{e}_j)] - [\eta(\mathbf{e}_j), \zeta(\mathbf{e}_i)]$. Antisymmetry of the bracket implies $[\eta \wedge \zeta] = [\zeta \wedge \eta]$; thus the endomorphism $\boldsymbol{\mathcal{R}}_{\mathbf{A}}$ is symmetric in each fiber, as required for the Hessian.

Proposition 28.2.2. (Hessian and gauge-fixed operator) Let $(M, \mathbf{g})$ be a compact Riemannian manifold with or without boundary, let $\mathbf{P} \to M$ be a principal bundle with compact structure group, and let $\mathbf{A}$ be a smooth connection. For $\mathbf{a}, \mathbf{b} \in \Omega^1(M, \operatorname{ad}\mathbf{P})$,

$$D^2 \mathrm{YM}(\mathbf{A})[\mathbf{a}, \mathbf{b}] = \int_M \left(\langle d_{\mathbf{A}}\mathbf{a}, d_{\mathbf{A}}\mathbf{b}\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}} + \langle \mathbf{F}_{\mathbf{A}}, [\mathbf{a} \wedge \mathbf{b}]\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}} \right) d\lambda_{\mathbf{g}}. \tag{28.15}$$

If M is compact without boundary and $\mathbf{A}$ is a Yang–Mills connection, the Hessian is represented by

$$H_{\mathbf{A}}\mathbf{a} = d_{\mathbf{A}}^{*} d_{\mathbf{A}}\mathbf{a} + \boldsymbol{\mathcal{R}}_{\mathbf{A}}\mathbf{a}. \tag{28.16}$$

Moreover,

$$H_{\mathbf{A}}(d_{\mathbf{A}}\xi) = 0 \qquad \text{for every } \xi \in \Omega^0(M, \operatorname{ad}\mathbf{P}).$$

The gauge-fixed operator

$$L_{\mathbf{A}}\mathbf{a} = (d_{\mathbf{A}}^{*} d_{\mathbf{A}} + d_{\mathbf{A}} d_{\mathbf{A}}^{*})\mathbf{a} + \boldsymbol{\mathcal{R}}_{\mathbf{A}}\mathbf{a} \tag{28.17}$$

has principal symbol

$$\sigma_2(L_{\mathbf{A}})(x, \zeta) = -|\zeta|_{\mathbf{g}}^2 \, \mathrm{id}_{T_x^*M \otimes (\operatorname{ad}\mathbf{P})_x}, \qquad \zeta \in T_x^*M \setminus \{0\}. \tag{28.18}$$

Proof. Substitute $\mathbf{A} + t\mathbf{a} + s\mathbf{b}$ into (28.11). The coefficient of ts in

$$\frac{1}{2} \| \mathbf{F}_{\mathbf{A}+t\mathbf{a}+s\mathbf{b}} \|^2_{L^2(M, \Lambda^2(T^*M) \otimes \operatorname{ad}\mathbf{P})}$$

is

$$\int_M \langle d_{\mathbf{A}}\mathbf{a}, d_{\mathbf{A}}\mathbf{b}\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}} \, d\lambda_{\mathbf{g}} + \int_M \langle \mathbf{F}_{\mathbf{A}}, [\mathbf{a} \wedge \mathbf{b}]\rangle_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}} \, d\lambda_{\mathbf{g}},$$

which proves (28.15). When M is compact without boundary, integration by parts in the first term gives (28.16).

Let $\xi \in \Omega^0(M, \operatorname{ad}\mathbf{P})$. Gauge invariance implies, for every connection $\mathbf{B}$,

$$D\,\mathrm{YM}(\mathbf{B})[d_{\mathbf{B}}\xi] = 0.$$

Differentiate this identity at $\mathbf{B} = \mathbf{A}$ in the direction $\mathbf{b}$. The term arising from the variation of $d_{\mathbf{B}}\xi$ is multiplied by $D\,\mathrm{YM}(\mathbf{A})$ and vanishes because $\mathbf{A}$ is critical. Thus

$$D^2\,\mathrm{YM}(\mathbf{A})[\mathbf{b}, d_{\mathbf{A}}\xi] = 0$$

for every $\mathbf{b}$. Symmetry of the Hessian gives

$$H_{\mathbf{A}}(d_{\mathbf{A}}\xi) = 0.$$

Finally, for $\zeta \in T_x^*M \setminus \{0\}$,

$$\sigma_2(d_{\mathbf{A}}^* d_{\mathbf{A}})(x,\zeta) = -\iota_{\zeta^\sharp}(\zeta \wedge \cdot),$$

whereas

$$\sigma_2(d_{\mathbf{A}} d_{\mathbf{A}}^*)(x,\zeta) = -\zeta \wedge \iota_{\zeta^\sharp}(\cdot).$$

The identity

$$\iota_{\zeta^\sharp}(\zeta \wedge a) + \zeta \wedge \iota_{\zeta^\sharp} a = |\zeta|_{\mathbf{g}}^2 a$$

shows that the sum of the two symbols is $-|\zeta|_{\mathbf{g}}^2 \mathrm{id}$. Since $\boldsymbol{\mathcal{R}}_{\mathbf{A}}$ has order zero, we obtain (28.18). □

On the compact manifold without boundary, the Coulomb condition

$$d_{\mathbf{A}}^* \mathbf{a} = 0$$

selects the L^2-orthogonal complement of the infinitesimal directions $d_{\mathbf{A}}\xi$. On this subspace, the term $d_{\mathbf{A}} d_{\mathbf{A}}^* \mathbf{a}$ vanishes, so $L_{\mathbf{A}}$ agrees with the Hessian. On the full space of 1-forms, this term completes the symbol and yields an elliptic operator.

On a compact base without boundary, projection onto the Coulomb complement is obtained by solving an elliptic equation on adjoint-bundle-valued 0-forms.

Proposition 28.2.3. (Coulomb decomposition in the compact case) Let $(M, \mathbf{g})$ be a compact Riemannian manifold without boundary, let $\mathbf{P} \to M$ be a principal bundle with compact structure group, let $\mathbf{A}$ be a smooth connection, and let $s \geq 1$. Then

$$H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}) = \ker d_{\mathbf{A}}^* \oplus d_{\mathbf{A}}\left(H^{s+1}(M, \operatorname{ad}\mathbf{P}) \cap (\ker d_{\mathbf{A}})^\perp\right).$$

The two projections associated with this direct sum are continuous.

Proof. The operator

$$\Delta_{\mathbf{A}}^{(0)} := d_{\mathbf{A}}^* d_{\mathbf{A}}$$

is elliptic, nonnegative, and self-adjoint. Its resolvent maps $L^2(M,\operatorname{ad}\mathbf{P})$ continuously into $H^2(M,\operatorname{ad}\mathbf{P})$ by Corollary 11.4.4. The inclusion

$$H^2(M,\operatorname{ad}\mathbf{P}) \hookrightarrow L^2(M,\operatorname{ad}\mathbf{P})$$

is compact by Theorem 9.2.2. The spectral theorem for operators with compact resolvent provides a constant $c_{\mathbf{A}} > 0$ such that

$$\langle \Delta_{\mathbf{A}}^{(0)}\xi,\xi\rangle_{L^2(M,\operatorname{ad}\mathbf{P})} \geq c_{\mathbf{A}}\|\xi\|^2_{L^2(M,\operatorname{ad}\mathbf{P})}, \qquad \xi \in H^2(M,\operatorname{ad}\mathbf{P}) \cap (\ker d_{\mathbf{A}})^{\perp_{L^2}}.$$

Let us make the construction of the inverse precise. Let (ξ_j) be an orthonormal basis of eigensections, with $\Delta_{\mathbf{A}}^{(0)}\xi_j = \lambda_j\xi_j$. The kernel is $\ker d_{\mathbf{A}}$, since $\langle \Delta_{\mathbf{A}}^{(0)}\xi,\xi\rangle_{L^2(M,\operatorname{ad}\mathbf{P})} = \|d_{\mathbf{A}}\xi\|^2_{L^2(M,T^*M\otimes\operatorname{ad}\mathbf{P})}$. For $\mathbf{f} \in H^{s-1}(M,\operatorname{ad}\mathbf{P})$ orthogonal to this kernel, the series

$$\xi = \sum_{\lambda_j>0} \lambda_j^{-1}\langle \mathbf{f},\xi_j\rangle_{L^2(M,\operatorname{ad}\mathbf{P})}\xi_j$$

converges in $L^2(M,\operatorname{ad}\mathbf{P})$, because $\lambda_j \geq c_{\mathbf{A}} > 0$ in the sum. Moreover, $\displaystyle\sum_{j=1}^{\infty} \lambda_j^2|\langle \xi,\xi_j\rangle|^2 = \|\mathbf{f}\|^2_{L^2(M,\operatorname{ad}\mathbf{P})}$; the spectral characterization of the domain gives $\Delta_{\mathbf{A}}^{(0)}\xi = \mathbf{f}$ and $\|\xi\|_{L^2(M,\operatorname{ad}\mathbf{P})} \leq c_{\mathbf{A}}^{-1}\|\mathbf{f}\|_{L^2(M,\operatorname{ad}\mathbf{P})}$. A parametrix of order -2 and Corollary 15.6.6 give, for every real $s \geq 1$,

$$\|\xi\|_{H^{s+1}(M,\operatorname{ad}\mathbf{P})} \leq C_s\big(\|\mathbf{f}\|_{H^{s-1}(M,\operatorname{ad}\mathbf{P})} + \|\xi\|_{L^2(M,\operatorname{ad}\mathbf{P})}\big) \leq C_s'\|\mathbf{f}\|_{H^{s-1}(M,\operatorname{ad}\mathbf{P})}.$$

The same energy identity proves injectivity on the complement of the kernel. Thus

$$\Delta_{\mathbf{A}}^{(0)}\colon H^{s+1}(M,\operatorname{ad}\mathbf{P}) \cap (\ker d_{\mathbf{A}})^{\perp} \longrightarrow H^{s-1}(M,\operatorname{ad}\mathbf{P}) \cap (\ker d_{\mathbf{A}})^{\perp}$$

is a topological isomorphism. If

$$\mathbf{a} \in H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}),$$

then, for every $\eta \in \ker d_{\mathbf{A}}$,

$$\langle d_{\mathbf{A}}^*\mathbf{a},\eta\rangle_{L^2(M,\operatorname{ad}\mathbf{P})} = \langle \mathbf{a}, d_{\mathbf{A}}\eta\rangle_{L^2(M,T^*M\otimes\operatorname{ad}\mathbf{P})} = 0.$$

Therefore $d_{\mathbf{A}}^*\mathbf{a}$ belongs to the codomain of the preceding restriction. There exists a unique

$$\xi \in H^{s+1}(M,\operatorname{ad}\mathbf{P}) \cap (\ker d_{\mathbf{A}})^{\perp}$$

such that

$$\Delta_{\mathbf{A}}^{(0)}\xi = d_{\mathbf{A}}^*\mathbf{a}.$$

Set

$$\mathbf{b} := \mathbf{a} - d_{\mathbf{A}}\xi.$$

Then $d_{\mathbf{A}}^*\mathbf{b} = 0$, so $\mathbf{a} = \mathbf{b} + d_{\mathbf{A}}\xi$.
If $d_{\mathbf{A}}\xi \in \ker d_{\mathbf{A}}^*$, then

$$\|d_{\mathbf{A}}\xi\|^2_{L^2(M,T^*M\otimes\operatorname{ad}\mathbf{P})} = \langle \Delta_{\mathbf{A}}^{(0)}\xi,\xi\rangle_{L^2(M,\operatorname{ad}\mathbf{P})} = 0,$$

and hence $d_{\mathbf{A}}\xi = 0$. The sum is direct. Finally, the estimate for the inverse of $\Delta_{\mathbf{A}}^{(0)}$ proves continuity of the two projections. □

Now fix

$$s > \frac{m}{2} + 1$$

and a smooth connection $\mathbf{A}_0$. The Sobolev completion of the space of connections is the affine space

$$\mathcal{A}^s(\mathbf{P}) := \mathbf{A}_0 + H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}),$$

and

$$T_{\mathbf{A}}\mathcal{A}^s(\mathbf{P}) = H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}).$$

Identity (28.11) shows that the curvature map loses one derivative:

$$\mathcal{F}\colon \mathcal{A}^s(\mathbf{P}) \longrightarrow H^{s-1}\left(M, \Lambda^2(T^*M) \otimes \operatorname{ad}\mathbf{P}\right), \qquad \mathcal{F}(\mathbf{A}) := \mathbf{F}_{\mathbf{A}}.$$

The functional

$$\mathrm{YM}\colon \mathcal{A}^s(\mathbf{P}) \longrightarrow \mathbb{R}, \qquad \mathrm{YM}(\mathbf{A}) = \frac{1}{2}\int_M |\mathbf{F}_{\mathbf{A}}|^2_{\mathbf{g},\mathbf{h}_{\mathrm{ad}}}\, d\lambda_{\mathbf{g}},$$

is also smooth. Indeed, the linear part of $\mathcal{F}$ is a first-order differential operator, and the remaining part is quadratic. The Sobolev product estimates control $[\mathbf{a} \wedge \mathbf{a}]$, while the inclusion

$$H^{s-1}\left(M, \Lambda^2(T^*M) \otimes \operatorname{ad}\mathbf{P}\right) \hookrightarrow L^2\left(M, \Lambda^2(T^*M) \otimes \operatorname{ad}\mathbf{P}\right)$$

allows us to integrate the squared norm of the curvature.

The gauge group $\mathcal{G}^{s+1}(\mathbf{P}) = H^{s+1}(M, \mathrm{Ad}(\mathbf{P}))$ acts smoothly on $\mathcal{A}^s(\mathbf{P})$. For each smooth connection $\mathbf{A}$, the Hessian and Coulomb operators induce continuous linear maps

$$H_{\mathbf{A}}, L_{\mathbf{A}}\colon H^s(M, T^*M \otimes \operatorname{ad}\mathbf{P}) \longrightarrow H^{s-2}(M, T^*M \otimes \operatorname{ad}\mathbf{P}).$$

The Yang–Mills gradient loses two derivatives; more precisely,

$$\mathcal{Y}\colon \mathcal{A}^s(\mathbf{P}) \longrightarrow H^{s-2}(M, T^*M \otimes \operatorname{ad}\mathbf{P}), \qquad \mathcal{Y}(\mathbf{A}) := d_{\mathbf{A}}^*\mathbf{F}_{\mathbf{A}}, \tag{28.19}$$

is smooth.

Remark 28.2.4. (The noncompact case) On a complete, noncompact manifold without boundary and of bounded geometry, the compactness of the resolvent used in the preceding decomposition is no longer available. Theorem 27.4.9 assumes that G is compact, that the reference connection $\mathbf{A}_0$ has bounded geometry and finite Yang–Mills energy, and that

$$k - 1 \geq r > \frac{m}{2} + 2.$$

It also requires that the connection $\mathbf{A}$ belong to the Sobolev component under consideration and that

$$\inf \sigma_{\mathrm{e}} \left(\Delta_{\mathbf{A}}^{(0)} \restriction_{(\ker \Delta_{\mathbf{A}}^{(0)})^{\perp}} \right) > 0.$$

Under these hypotheses, one obtains a Hodge decomposition and a local slice for the action of the reduced subgroup $\mathcal{G}_{\bullet}^{r+1}(\mathbf{A}_0)$.

The range in question is that of $d_{\mathbf{A}} : H^{r+1}(M, \operatorname{ad} \mathbf{P}) \to H^r(M, T^*M \otimes \operatorname{ad} \mathbf{P})$; the complement is the kernel of $d_{\mathbf{A}}^* : H^r(M, T^*M \otimes \operatorname{ad} \mathbf{P}) \to H^{r-1}(M, \operatorname{ad} \mathbf{P})$. Orthogonality is measured in $L^2(M, T^*M \otimes \operatorname{ad} \mathbf{P})$, whereas the sum and its projections are continuous in $H^r(M, T^*M \otimes \operatorname{ad} \mathbf{P})$. The reduced subgroup is the one defined using the components of the full group in the preceding chapter. The spectral gap ensures that the range of $d_{\mathbf{A}}$ is closed and replaces the compact resolvent argument used on the compact base.

28.3 Incompressible fluids as geodesics

The third application uses a diffeomorphism group as its configuration space. Incompressibility requires these diffeomorphisms to preserve volume, and the kinetic energy defines a weak right-invariant metric on this group. The corresponding geodesic equation is the Euler equation.

Let $(M^m, \mathbf{g})$ be a compact, connected, oriented Riemannian manifold, with or without boundary, and let

$$s > \frac{m}{2} + 1.$$

Denote by $\operatorname{Diff}_{\mu}^s(M)$ the identity component of the group $\operatorname{Diff}^s(M)$ of diffeomorphisms preserving $d\lambda_{\mathbf{g}}$ and ∂M. When the boundary is empty, conditions referring to it are omitted. In this section, $\Delta_{\mathbf{g}} = \operatorname{div}_{\mathbf{g}} \nabla$ has the sign of the Laplacian on functions from the chapter on Riemannian geometry; the Hodge Laplacian $d\delta + \delta d$ used at the end is nonnegative. Define

$$\mathfrak{X}_{\mu}^s(M) := \left\{ \mathbf{u} \in H^s(M, TM) \,\middle|\, \operatorname{div}_{\mathbf{g}} \mathbf{u} = 0, \quad (\mathbf{u} \cdot \boldsymbol{\nu}) \restriction_{\partial M} = 0 \right\}. \tag{28.20}$$

The same notation $\mathfrak{X}_{\mu}^t(M)$ will be used for any $t > 1/2$, interpreting the divergence distributionally and the normal condition by means of the trace. In particular, it is defined for $t = s - 1$.

Lemma 28.3.1. (Global estimate for the Neumann problem) Let $(M, \mathbf{g})$ be a smooth compact connected Riemannian manifold, with or without boundary. For $a \geq 0$, let $f \in H^a(M)$ and, when $\partial M \neq \varnothing$, $h \in H^{a+1/2}(\partial M)$, with

$$\int_M f \, d\lambda_{\mathbf{g}} = \int_{\partial M} h \, d\sigma_{\mathbf{g}}.$$

There exists a unique mean-zero solution $q \in H^{a+2}(M)$ of $\Delta_{\mathbf{g}} q = f$, $\partial_{\boldsymbol{\nu}} q = h$, and

$$\|q\|_{H^{a+2}(M)} \leq C_a \left(\|f\|_{H^a(M)} + \|h\|_{H^{a+1/2}(\partial M)} \right). \tag{28.21}$$

When the boundary is empty, h and its norm are omitted.

Proof. Begin with $h = 0$ and mean-zero $f \in L^2(M)$. Corollary 9.6.4 makes the form $\int_M \langle \nabla v, \nabla \varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}$ coercive on the mean-zero subspace of $H^1(M)$. Theorem B.3.14, applied to the functional $\varphi \mapsto -\int_M f \varphi \, d\lambda_{\mathbf{g}}$, gives a

weak solution $v \in H^1(M)$ and

$$\|v\|_{H^1(M)} \le C\|f\|_{L^2(M)}. \tag{28.22}$$

Since f has mean zero, the weak identity holds for every $\varphi \in H^1(M)$, without any restriction on its mean or trace.

We now establish regularity up to the boundary. In an interior chart or a chart straightening the boundary, write $\omega = \sqrt{\det(\mathbf{g})}$ and $A^{ij} = \omega g^{ij}$. In a coordinate region Q, which is a ball or a half-ball, the weak identity is

$$\sum_{i,j=1}^{m} \int_Q A^{ij} \partial_j v \, \partial_i \varphi \, dx = -\int_Q \omega f \varphi \, dx. \tag{28.23}$$

Test functions need not vanish on the flat face of the half-ball; their supports remain away from the other faces. In a smaller chart, the matrix A is uniformly positive: $A^{ij}\xi_i\xi_j \ge \lambda|\xi|^2$ for some $\lambda > 0$.

Let χ be a cutoff supported away from those other faces, with $\chi = 1$ on a smaller region. For a coordinate direction α, tangential if there is a boundary, set

$$D_h^\alpha z(x) = \frac{z(x + he_\alpha) - z(x)}{h}, \qquad \varphi = -D_{-h}^\alpha(\chi^2 D_h^\alpha v).$$

For small $|h|$, this test function belongs to $H^1(Q)$ and is admissible. Changing variables in the differences gives $\int z D_{-h}^\alpha w = -\int (D_h^\alpha z) w$, while

$$D_h^\alpha(A\nabla v) = A(\cdot + he_\alpha)\nabla D_h^\alpha v + (D_h^\alpha A)\nabla v.$$

The principal term in (28.23) is therefore $\int_Q \chi^2 A^{ij}(x + he_\alpha)\partial_j D_h^\alpha v \, \partial_i D_h^\alpha v \, dx$. The remaining terms contain $\nabla\chi$, $D_h^\alpha A$, or f. We use

$$\|D_h^\alpha v\|_{L^2(\operatorname{supp}\chi)} \le \|\partial_\alpha v\|_{L^2(Q)}, \qquad \|D_h^\alpha A\|_{L^\infty(\operatorname{supp}\chi)} \le \|\partial_\alpha A\|_{L^\infty(Q)}.$$

For the term involving f, retain the difference on the test function:

$$\left| \int_Q \omega f D_{-h}^\alpha(\chi^2 D_h^\alpha v)\, dx \right| \le C\|f\|_{L^2(Q)} \big(\|\chi \nabla D_h^\alpha v\|_{L^2(Q,\mathbb{R}^m)} + \|v\|_{H^1(Q)} \big).$$

These inequalities follow from the integral formula for translations and Cauchy–Schwarz; for functions in $H^1(Q)$, they follow by the smooth approximation in Lemma 27.2.19(c), using a cutoff in a larger chart. Applying Young's inequality to each product and absorbing half the principal term yields

$$\|\chi \nabla D_h^\alpha v\|^2_{L^2(Q,\mathbb{R}^m)} \le C\big(\|f\|^2_{L^2(Q)} + \|v\|^2_{H^1(Q)} \big), \tag{28.24}$$

with C independent of h. When tested against compactly supported smooth functions, a weakly convergent subsequence of these differences has limit $\partial_\alpha \nabla v$. This gives in L^2 all second-order derivatives except, in a boundary chart, the double normal derivative.

The interior equation determines the remaining derivative:

$$A^{mm}\partial_m^2 v = \omega f - \sum_{\substack{1\le i,j\le m\\ (i,j)\neq(m,m)}} A^{ij}\partial_i\partial_j v - \sum_{i,j=1}^{m}(\partial_i A^{ij})\partial_j v. \tag{28.25}$$

All terms on the right belong to L^2, and $A^{mm}\ge\lambda$. This distributional equality proves that $\partial_m^2 v\in L^2$ on the smaller region. In an interior chart, all difference directions are allowed from the outset. A finite cover and the chosen cutoffs give $v\in H^2(M)$ and $\|v\|_{H^2(M)}\le C(\|f\|_{L^2(M)}+\|v\|_{H^1(M)})$.

For higher orders, we proceed by induction. Suppose that $f\in H^j(M)$ and that $v\in H^{j+1}(M)$ has already been obtained, with $j\ge 1$. If β is a tangential multi-index of length j, differentiate (28.23) tangentially in the distributional sense. The field $w=\partial^\beta v\in H^1(Q)$ satisfies

$$\int_Q A\nabla w\cdot\nabla\varphi\,dx = -\int_Q \partial^\beta(\omega f)\varphi\,dx + \int_Q \mathbf{G}_\beta\cdot\nabla\varphi\,dx,$$

where

$$\mathbf{G}_\beta = -\sum_{0<\gamma\le\beta}\binom{\beta}{\gamma}(\partial^\gamma A)\nabla\partial^{\beta-\gamma}v.$$

Here $\partial^\beta(\omega f)\in L^2(Q)$ and $\mathbf{G}_\beta\in H^1(Q,\mathbb{R}^m)$: each derivative of v in $\mathbf{G}_\beta$ has order at most j. The same difference quotient proof applies to w. The additional term becomes $\int_Q D_h^\alpha\mathbf{G}_\beta\cdot\nabla(\chi^2 D_h^\alpha w)\,dx$ and is estimated by

$$C\|\mathbf{G}_\beta\|_{H^1(Q,\mathbb{R}^m)}\big(\|\chi\nabla D_h^\alpha w\|_{L^2(Q,\mathbb{R}^m)}+\|w\|_{H^1(Q)}\big).$$

This gives all derivatives of order $j+2$ with at most one normal index. For the others, differentiate (28.25) j times. A derivative with $b\ge 2$ normal indices is expressed through derivatives of the same order with at most $b-1$ normal indices, derivatives of v of order at most $j+1$, and derivatives of f of order at most j. Induction on b determines them all. Combining the finitely many charts, induction on j gives

$$\|v\|_{H^{j+2}(M)}\le C_j\big(\|f\|_{H^j(M)}+\|v\|_{H^1(M)}\big),\qquad j\in\mathbb{N}_0.$$

The preceding tangential differentiations are also justified by successive differences; their distributional limits are the derivatives written above, and the bounds use only orders already obtained in the induction. For arbitrary $f\in L^2(M)$, let Tf be the mean-zero weak solution with datum $f-\overline{f}$, where $\overline{f}=\lambda_{\mathbf{g}}(M)^{-1}\int_M f\,d\lambda_{\mathbf{g}}$. We have proved that the same operator T is continuous from $H^j(M)$ to $H^{j+2}(M)$ for each integer $j\ge 0$. The interpolation in Lemma 27.2.19, applied between consecutive integers, gives $T:H^a(M)\to H^{a+2}(M)$ for every $a\ge 0$. Its weak identity remains the one defining it in $L^2(M)$.

We now treat $h\neq 0$. Theorem 14.4.18, with $p=2$ and a first-order jet, provides $b\in H^{a+2}(M)$ with

$$b|_{\partial M}=0,\qquad \partial_{\boldsymbol{\nu}} b=h,\qquad \|b\|_{H^{a+2}(M)}\le C_a\|h\|_{H^{a+1/2}(\partial M)}.$$

The theorem is formulated using the inward collar coordinate: to obtain the outward normal, prescribe the jet $(0,-h)$. The divergence identity for $b\in H^2(M)$ follows by approximation in $H^2(M)$ by smooth functions; the Laplacian converges in $L^2(M)$ and the normal derivative in $H^{1/2}(\partial M)$. Thus $\int_M(f-\Delta_{\mathbf{g}}b)\,d\lambda_{\mathbf{g}}=0$. The

function

$$q = b + T(f - \Delta_{\mathbf{g}} b) - \overline{b}$$

has the required mean, equation, and normal condition. The bounds for T, b, and $\Delta_{\mathbf{g}} : H^{a+2}(M) \to H^a(M)$ prove (28.21). If two solutions have the same data, their difference z satisfies $\int_M |\nabla z|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} = 0$ upon using z as a weak test function. Connectedness and the zero mean imply $z = 0$. □

Proposition 28.3.2. (Neumann problem and Hodge–Leray projection) Let $(M^m, \mathbf{g})$ be a smooth compact connected oriented Riemannian manifold, with or without boundary. For $r \geq 0$ there exists a continuous linear operator

$$Q : H^r(M, TM) \longrightarrow \left\{ q \in H^{r+1}(M) \;\middle|\; \int_M q \, d\lambda_{\mathbf{g}} = 0 \right\}$$

characterized by

$$\int_M \langle \nabla Q\mathbf{X}, \nabla \varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M \langle \mathbf{X}, \nabla \varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}, \qquad \varphi \in H^1(M). \tag{28.26}$$

Set

$$\mathfrak{X}^0_\mu(M) := \left\{ \mathbf{Y} \in L^2(M, TM) \;\middle|\; \int_M \langle \mathbf{Y}, \nabla \varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = 0 \quad \forall \varphi \in H^1(M) \right\}.$$

The operator $\mathbb{P}\mathbf{X} := \mathbf{X} - \nabla Q\mathbf{X}$ is the orthogonal projection of $L^2(M, TM)$ onto $\mathfrak{X}^0_\mu(M)$, and

$$\|Q\mathbf{X}\|_{H^{r+1}(M)} + \|\mathbb{P}\mathbf{X}\|_{H^r(M,TM)} \leq C_r \|\mathbf{X}\|_{H^r(M,TM)}. \tag{28.27}$$

If $r > 1/2$, its range in $H^r(M, TM)$ is $\mathfrak{X}^r_\mu(M)$, with the definition of divergence and trace given above. In particular, for $s > m/2 + 1$ and $\mathbf{X} \in H^{s-1}(M, TM)$, $q = Q\mathbf{X} \in H^s(M)$ solves

$$\Delta_{\mathbf{g}} q = \operatorname{div}_{\mathbf{g}} \mathbf{X}, \qquad \partial_{\boldsymbol{\nu}} q = \langle \operatorname{Tr} \mathbf{X}, \boldsymbol{\nu} \rangle_{\mathbf{g}}. \tag{28.28}$$

The first equality is distributional, and the second is an equality of traces. We have the direct sum, orthogonal in $L^2(M, TM)$,

$$H^{s-1}(M, TM) = \mathfrak{X}^{s-1}_\mu(M) \oplus \nabla\big(H^s(M)/\mathbb{R}\big).$$

Moreover, Q and $\mathbb{P}$ map smooth data to data smooth up to the boundary.

Proof. For $\mathbf{X} \in L^2(M, TM)$, Poincaré and Lax–Milgram, used as in the preceding lemma, apply to the functional $\varphi \mapsto \int_M \langle \mathbf{X}, \nabla \varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}$. Its norm does not exceed $\|\mathbf{X}\|_{L^2(M,TM)}$. This yields a unique mean-zero $Q\mathbf{X}$ with $\|Q\mathbf{X}\|_{H^1(M)} \leq C\|\mathbf{X}\|_{L^2(M,TM)}$. The weak identity directly proves that $\mathbb{P}\mathbf{X} \in \mathfrak{X}^0_\mu(M)$ and that $\mathbf{X} - \mathbb{P}\mathbf{X}$ is orthogonal to this subspace. If $\mathbf{X} \in \mathfrak{X}^0_\mu(M)$, uniqueness gives $Q\mathbf{X} = 0$; thus $\mathbb{P}^2 = \mathbb{P}$ and $\mathbb{P}$ is the asserted orthogonal projection.

If $\mathbf{X} \in H^k(M, TM)$, with integer $k \geq 1$, the data $f = \operatorname{div}_{\mathbf{g}} \mathbf{X}$ and $h = \langle \operatorname{Tr} \mathbf{X}, \boldsymbol{\nu} \rangle_{\mathbf{g}}$ belong to $H^{k-1}(M)$ and $H^{k-1/2}(\partial M)$, respectively. To justify their compatibility, take $\mathbf{X}_j \in \Gamma(TM)$ with $\mathbf{X}_j \to \mathbf{X}$ in $H^1(M, TM)$, by Lemma 27.2.19. Theorem 2.14.21 holds for each $\mathbf{X}_j$. Differentiation and the trace are continuous from $H^1(M, TM)$ to $L^2(M)$ and $H^{1/2}(\partial M, TM|_{\partial M})$, respectively. Moreover,

$$\left| \int_M \operatorname{div}_{\mathbf{g}}(\mathbf{X}_j - \mathbf{X}) \, d\lambda_{\mathbf{g}} \right| \leq \lambda_{\mathbf{g}}(M)^{1/2} \| \operatorname{div}_{\mathbf{g}}(\mathbf{X}_j - \mathbf{X}) \|_{L^2(M)},$$

and the boundary integral satisfies the corresponding bound by Cauchy–Schwarz on ∂M. Both errors tend to zero. This proves the divergence formula for every vector field in $H^1(M, TM)$ and, in particular, compatibility of the specified data. The preceding lemma with $a = k - 1$ provides $q \in H^{k+1}(M)$ with $\|q\|_{H^{k+1}(M)} \le C_k \|\mathbf{X}\|_{H^k(M,TM)}$. Integration by parts gives (28.26), so $q = Q\mathbf{X}$ by uniqueness. Interpolate this same operator between $k = 0, 1, 2, \ldots$; Lemma 27.2.19 gives the bounds in Q for every $r \ge 0$. Continuity of the gradient proves those for $\mathbb{P}$.

Let us make the normal condition precise. If $r > 1/2$ and $\mathbf{Y} \in H^r(M, TM)$ has zero distributional divergence, then

$$\int_M \langle \mathbf{Y}, \nabla\varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_{\partial M} \varphi \langle \operatorname{Tr} \mathbf{Y}, \boldsymbol{\nu} \rangle_{\mathbf{g}} \, d\sigma_{\mathbf{g}}, \qquad \varphi \in C^\infty(M). \tag{28.29}$$

For $r \ge 1$, this is Green's formula obtained by smooth approximation. For $1/2 < r < 1$, the construction of the trace on the half-space implies that restrictions to the collar hypersurfaces at distance t converge in $L^2(\partial M, TM|_{\partial M})$ to the trace as $t \downarrow 0$, identifying the fibers by means of the collar. In the Euclidean proof, this convergence follows by Fourier analysis in the tangential variables and Cauchy–Schwarz in the normal variable; the integral $\int_{\mathbb{R}} (1 + \xi_m^2)^{-r} \, d\xi_m$ is finite. Take a cutoff ζ_ε that is zero near the boundary, one outside the collar of width ε, and in the collar equals $\zeta(t/\varepsilon)$, with $\zeta(0) = 0$, $\zeta(1) = 1$. Applying the zero-divergence condition to $\varphi\zeta_\varepsilon$ gives

$$0 = \int_M \zeta_\varepsilon \langle \mathbf{Y}, \nabla\varphi \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} + \int_M \varphi \langle \mathbf{Y}, \nabla\zeta_\varepsilon \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}.$$

The first term converges to the interior integral in (28.29). Convergence of the collar restrictions and $\int_0^1 \zeta'(t)\, dt = 1$ make the second term converge to the negative of the boundary integral, since the collar coordinate increases inward. This proves the remaining identity.

If $\mathbf{Y} \in \mathfrak{X}^0_\mu(M) \cap H^r(M, TM)$, test functions supported in the interior give $\operatorname{div}_{\mathbf{g}} \mathbf{Y} = 0$. The preceding identity and the extension of every smooth boundary function give $\langle \operatorname{Tr} \mathbf{Y}, \boldsymbol{\nu} \rangle_{\mathbf{g}} = 0$. Conversely, these two conditions make the right-hand side of (28.29) vanish; density of $C^\infty(M)$ in $H^1(M)$ extends the identity to all test functions in that space. This identifies the range. Applied to $\mathbf{Y} = \mathbf{X} - \nabla Q\mathbf{X}$, the same argument also gives (28.28). Finally, if $\mathbf{X}$ is smooth, the bounds at each integer order give $Q\mathbf{X} \in H^{k+1}(M)$ for every k. The Sobolev embeddings at each order prove that $Q\mathbf{X}$ and $\mathbb{P}\mathbf{X}$ are smooth up to the boundary. □

Corollary 28.3.3. (Approximation preserving incompressibility) For $r \ge 0$, smooth divergence-free vector fields tangent to ∂M are dense in $\mathfrak{X}^0_\mu(M) \cap H^r(M, TM)$. In particular, they are dense in $\mathfrak{X}^{s-1}_\mu(M)$ for $s > m/2 + 1$ and in $\mathfrak{X}^0_\mu(M)$ with the $L^2(M, TM)$ norm.

Proof. Let $\mathbf{Z}$ be an element of the indicated space. Lemma 27.2.19 provides $\mathbf{V}_j \in \Gamma(TM)$ with $\|\mathbf{V}_j - \mathbf{Z}\|_{H^r(M,TM)} \to 0$. Using the mean-zero operator from the preceding proposition, define

$$q_j = Q\mathbf{V}_j, \qquad \mathbf{Z}_j = \mathbf{V}_j - \nabla q_j = \mathbb{P}\mathbf{V}_j.$$

Each q_j and each $\mathbf{Z}_j$ is smooth up to the boundary. The Neumann equations give $\operatorname{div}_{\mathbf{g}} \mathbf{Z}_j = 0$ and $\langle \mathbf{Z}_j, \boldsymbol{\nu} \rangle_{\mathbf{g}} = 0$. Since $\mathbb{P}\mathbf{Z} = \mathbf{Z}$, the explicit estimate is

$$\|\mathbf{Z}_j - \mathbf{Z}\|_{H^r(M,TM)} = \|\mathbb{P}(\mathbf{V}_j - \mathbf{Z})\|_{H^r(M,TM)} \le C_r \|\mathbf{V}_j - \mathbf{Z}\|_{H^r(M,TM)} \longrightarrow 0.$$

□

Theorem 28.3.4. (Ebin–Marsden: the volume-preserving group) Let $(M^m, \mathbf{g})$ be a compact connected oriented Riemannian manifold, with or without boundary. If

$$s > \frac{m}{2} + 1,$$

then $\mathrm{Diff}^s_\mu(M)$ is a closed *split* Hilbert submanifold of $\mathrm{Diff}^s(M)$. For each $\eta \in \mathrm{Diff}^s_\mu(M)$, the map

$$\mathcal{I}_\eta : \mathfrak{X}^s_\mu(M) \longrightarrow T_\eta \,\mathrm{Diff}^s_\mu(M), \qquad \mathcal{I}_\eta(\mathbf{u}) := \mathbf{u} \circ \eta,$$

is a topological linear isomorphism, with inverse

$$\mathbf{U} \longmapsto \mathbf{U} \circ \eta^{-1}.$$

In particular,

$$T_{\mathrm{id}} \,\mathrm{Diff}^s_\mu(M) = \mathfrak{X}^s_\mu(M).$$

The submanifold structure is proved in Theorems 4.2 and 8.1 of [46, pp. 114 and 123]. The identification of the tangent space follows from right translation and the Sobolev composition theorem for a fixed diffeomorphism.

The weak right-invariant kinetic metric is

$$\mathbf{G}_\eta(\mathbf{U}, \mathbf{V}) = \int_M \left\langle \mathbf{U} \circ \eta^{-1}, \mathbf{V} \circ \eta^{-1} \right\rangle_{\mathbf{g}} d\lambda_{\mathbf{g}}. \tag{28.30}$$

Since η preserves volume, a change of variables transforms this formula into

$$\mathbf{G}_\eta(\mathbf{U}, \mathbf{V}) = \int_M \mathbf{g}_{\eta(x)}(\mathbf{U}(x), \mathbf{V}(x))\, d\lambda_{\mathbf{g}}(x).$$

Smooth dependence of this metric is verified in the latter expression by superposition and products, without assuming smoothness of inversion on $\mathrm{Diff}^s(M)$. For a curve $\eta(t)$, define its Eulerian velocity by

$$\mathbf{u}(t) := \dot\eta(t) \circ \eta(t)^{-1}.$$

The kinetic action is

$$\mathcal{A}(\eta) := \frac{1}{2} \int_{t_0}^{t_1} \int_M |\mathbf{u}(t,x)|^2_{\mathbf{g}}\, d\lambda_{\mathbf{g}}\, dt. \tag{28.31}$$

Proposition 28.3.5. (Euler as a geodesic equation) Let $(M^m, \mathbf{g})$ be a smooth compact connected oriented Riemannian manifold, with or without boundary, and let $s > m/2 + 1$. A curve $\eta \in C^2([t_0, t_1], \mathrm{Diff}^s_\mu(M))$, with fixed endpoints, is critical for (28.31) if and only if its Eulerian velocity satisfies

$$\partial_t \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p, \qquad \mathrm{div}_{\mathbf{g}}\, \mathbf{u} = 0, \qquad \langle \mathrm{Tr}\, \mathbf{u}, \boldsymbol{\nu} \rangle_{\mathbf{g}} = 0 \quad \text{on } \partial M. \tag{28.32}$$

Here

$$\mathbf{u} \in C([t_0, t_1], H^s(M, TM)) \cap C^1([t_0, t_1], H^{s-1}(M, TM)), \qquad p \in C([t_0, t_1], H^s(M)),$$

and the first equation is interpreted in $H^{s-1}(M, TM)$. The pressure is determined at each time up to a spatial constant; the normalization $\int_M p(t,x)\, d\lambda_{\mathbf{g}}(x) = 0$ makes it unique. The boundary condition is omitted when

$\partial M = \varnothing$.

Proof. Write $I = [t_0, t_1]$. We use variations of class C^2 in (ε, t) with values in $\mathrm{Diff}^s_\mu(M)$ and fixed endpoints. Continuous composition from Lemma 27.5.2 and continuous inversion from Theorem 27.5.1 give $\mathbf{u} = \dot\eta \circ \eta^{-1} \in C(I, H^s(M, TM))$. The embedding $H^s(M, TM) \hookrightarrow C^1(M, TM)$ allows us to differentiate the identity $\dot\eta = \mathbf{u} \circ \eta$ pointwise. This gives

$$\partial_t \mathbf{u} = (\nabla_t \dot\eta) \circ \eta^{-1} - \nabla_{\mathbf{u}} \mathbf{u}. \tag{28.33}$$

The covariant acceleration $\nabla_t \dot\eta$ is continuous in the space of sections $H^s(M, \eta(t)^* TM)$ along η: in coordinates, its components are $\ddot\eta^a + \sum_{b,c=1}^{m} \Gamma^a_{bc}(\eta)\dot\eta^b \dot\eta^c$. Superposition and the product estimate in Lemma 27.2.19 control each term. By (27.9), the right-hand side of (28.33) is continuous in $H^{s-1}(M, TM)$. The last part of Lemma 27.5.2, or its proof using the Bochner integral of this right-hand side, gives $\mathbf{u} \in C^1(I, H^{s-1}(M, TM))$. Thus this regularity follows from the given curve, before imposing the Euler equation.

Let η_ε be an admissible variation, and set

$$\mathbf{W}(t) = \partial_\varepsilon \eta_\varepsilon(t)\big|_0, \qquad \mathbf{w} = \mathbf{W} \circ \eta^{-1}.$$

The tangent-space identification in Theorem 28.3.4 implies $\mathbf{w}(t) \in \mathfrak{X}^s_\mu(M)$ and $\mathbf{w}(t_0) = \mathbf{w}(t_1) = 0$. The same argument applied to $\mathbf{W}$ shows that

$$\mathbf{w} \in C(I, H^s(M, TM)) \cap C^1(I, H^{s-1}(M, TM)), \quad \partial_t \mathbf{w} = (\nabla_t \mathbf{W}) \circ \eta^{-1} - \nabla_{\mathbf{u}} \mathbf{w}.$$

Conversely, let $\mathbf{w}$ be smooth in $I \times M$, divergence-free, tangent to the boundary, and zero at the temporal endpoints. For each t, take the flow φ^t_ε of the spatial vector field $\mathbf{w}(t, \cdot)$. Theorem 1.8.15 and the dependence of ODEs on parameters give a common interval in ε and smooth dependence on (ε, t, x); a common choice is possible because $I \times M$ is compact. Tangency makes the boundary invariant by uniqueness of the flow, and Lie's formula for the volume form gives

$$\frac{d}{d\varepsilon} (\varphi^t_\varepsilon)^* d\lambda_{\mathbf{g}} = (\varphi^t_\varepsilon)^* ((\mathrm{div}_{\mathbf{g}} \mathbf{w}(t)) d\lambda_{\mathbf{g}}) = 0.$$

Moreover, $\mathbf{w}(t_i) = 0$ implies $\varphi^{t_i}_\varepsilon = \mathrm{id}$ for $i = 0, 1$. By smooth superposition,

$$\eta_\varepsilon(t) = \varphi^t_\varepsilon \circ \eta(t)$$

is an admissible variation of class C^2 with Eulerian vector field $\mathbf{w}$. The path in ε starting at $\eta(t)$ ensures that it remains in the identity component.

We now differentiate the velocity of a general variation. The version of the preceding calculation with parameter ε proves the existence of $\delta\mathbf{u} = \partial_\varepsilon \mathbf{u}_\varepsilon|_0$ in $C(I, H^{s-1}(M, TM))$. Differentiate $\dot\eta_\varepsilon = \mathbf{u}_\varepsilon \circ \eta_\varepsilon$ and $\mathbf{W} = \mathbf{w} \circ \eta$ covariantly. This gives

$$\nabla_\varepsilon \dot\eta_\varepsilon\big|_0 = (\delta\mathbf{u} + \nabla_{\mathbf{w}} \mathbf{u}) \circ \eta, \qquad \nabla_t \mathbf{W} = (\partial_t \mathbf{w} + \nabla_{\mathbf{u}} \mathbf{w}) \circ \eta.$$

The Levi–Civita connection is torsion-free, and the mixed derivatives of η_ε commute. The two left-hand

sides are equal, and we obtain Lin's identity

$$\delta\mathbf{u} = \partial_t \mathbf{w} + [\mathbf{u},\mathbf{w}] = \partial_t \mathbf{w} + \nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} \quad \text{in } C(I, H^{s-1}(M,TM)). \tag{28.34}$$

All contractions belong to this space by (27.9).
The functional $\mathbf{v} \mapsto \frac{1}{2}\|\mathbf{v}\|^2_{L^2(M,TM)}$ is differentiable, and $H^{s-1}(M,TM) \hookrightarrow L^2(M,TM)$. The preceding differentiation, uniform on the compact interval I, allows differentiation under the time integral:

$$\delta\mathcal{A} = \int_I \int_M \langle \mathbf{u}, \delta\mathbf{u}\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}\, dt.$$

Since $\mathbf{u}$ and $\mathbf{w}$ are curves of class C^1 in $L^2(M,TM)$, the product rule in that space gives

$$\int_I \int_M \langle \mathbf{u}, \partial_t \mathbf{w}\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}\, dt = -\int_I \int_M \langle \partial_t \mathbf{u}, \mathbf{w}\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}\, dt.$$

The endpoint term is zero because $\mathbf{w}(t_i) = 0$. For spatial integration, we directly use $\mathbf{u}(t), \mathbf{w}(t) \in C^1(M,TM)$, metric compatibility, and the divergence formula in $H^1(M,TM)$ justified in Proposition 28.3.2. The vector fields to which it is applied even belong to $H^s(M,TM)$ by the product estimate. Indeed,

$$\begin{aligned} \operatorname{div}_{\mathbf{g}}(\langle \mathbf{u},\mathbf{w}\rangle_{\mathbf{g}}\mathbf{u}) &= \langle \nabla_{\mathbf{u}}\mathbf{u}, \mathbf{w}\rangle_{\mathbf{g}} + \langle \mathbf{u}, \nabla_{\mathbf{u}}\mathbf{w}\rangle_{\mathbf{g}}, \\ \operatorname{div}_{\mathbf{g}}(\tfrac{1}{2}|\mathbf{u}|^2_{\mathbf{g}}\mathbf{w}) &= \langle \mathbf{u}, \nabla_{\mathbf{w}}\mathbf{u}\rangle_{\mathbf{g}}. \end{aligned}$$

We have used $\operatorname{div}_{\mathbf{g}}\mathbf{u} = \operatorname{div}_{\mathbf{g}}\mathbf{w} = 0$. The normal fluxes of both displayed vector fields are zero. Consequently,

$$\begin{aligned} \int_M \langle \mathbf{u}, \nabla_{\mathbf{u}}\mathbf{w}\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} &= -\int_M \langle \nabla_{\mathbf{u}}\mathbf{u}, \mathbf{w}\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}, \\ -\int_M \langle \mathbf{u}, \nabla_{\mathbf{w}}\mathbf{u}\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} &= -\tfrac{1}{2}\int_M \mathbf{w}(|\mathbf{u}|^2_{\mathbf{g}})\, d\lambda_{\mathbf{g}} = 0. \end{aligned}$$

Thus

$$\delta\mathcal{A} = -\int_I \int_M \langle \mathbf{F}(t), \mathbf{w}(t)\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}\, dt, \qquad \mathbf{F} = \partial_t \mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} \in C(I, H^{s-1}(M,TM)). \tag{28.35}$$

Suppose that η is critical. For a smooth vector field $\mathbf{Z} \in \mathfrak{X}^s_\mu(M)$ and $\chi \in C^\infty_c((t_0,t_1))$, the vector field $\mathbf{w}(t,x) = \chi(t)\mathbf{Z}(x)$ arises from the variation constructed above. By (28.35),

$$\int_{t_0}^{t_1} \chi(t)\langle \mathbf{F}(t), \mathbf{Z}\rangle_{L^2(M,TM)}\, dt = 0.$$

The scalar function obtained by pairing is continuous. The fundamental lemma of the calculus of variations makes it vanish on (t_0,t_1), and continuity makes it vanish at the endpoints as well. Fix $t \in I$. By Corollary 28.3.3, there exist smooth vector fields $\mathbf{Z}_j \in \mathfrak{X}^s_\mu(M)$ with

$$\mathbf{Z}_j \longrightarrow \mathbb{P}\mathbf{F}(t) \quad \text{in } H^{s-1}(M,TM).$$

By Cauchy–Schwarz and orthogonality of $\mathbb{P}$,

$$0 = \lim_{j\to\infty} \langle \mathbf{F}(t), \mathbf{Z}_j\rangle_{L^2(M,TM)} = \langle \mathbf{F}(t), \mathbb{P}\mathbf{F}(t)\rangle_{L^2(M,TM)} = \|\mathbb{P}\mathbf{F}(t)\|^2_{L^2(M,TM)}.$$

Thus $\mathbf{F}(t) = \nabla Q\mathbf{F}(t)$. Define $p(t) = -Q\mathbf{F}(t)$. Proposition 28.3.2 gives the equation, the zero mean, and, for $t,\tau \in I$,

$$\|p(t) - p(\tau)\|_{H^s(M)} \le C_{s-1}\|\mathbf{F}(t) - \mathbf{F}(\tau)\|_{H^{s-1}(M,TM)}.$$

This proves continuity of the pressure in time. If two pressures produce the same equation, their difference has zero gradient; by connectedness it is constant, and its zero mean makes it vanish.
Conversely, suppose that (28.32). For every admissible variation, $\mathbf{w}(t) \in \mathfrak{X}^0_\mu(M)$ and $p(t) \in H^s(M) \subseteq H^1(M)$. The definition of $\mathfrak{X}^0_\mu(M)$ implies $\int_M \langle \nabla p(t), \mathbf{w}(t)\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} = 0$. Substituting $\mathbf{F} = -\nabla p$ into (28.35) gives $\delta\mathcal{A} = 0$. □

The subspace $\mathfrak{X}^0_\mu(M)$ is closed in $L^2(M,TM)$, since it is the intersection of the kernels of the continuous functionals $\mathbf{v} \mapsto \langle \mathbf{v}, \nabla\varphi\rangle_{L^2(M,TM)}$, with $\varphi \in H^1(M)$. Since $\mathbf{u}(t)$ belongs to this subspace, so do its difference quotients. Their convergence to $\partial_t\mathbf{u}(t)$ in $H^{s-1}(M,TM)$, and hence in $L^2(M,TM)$, gives $\partial_t\mathbf{u}(t) \in \mathfrak{X}^{s-1}_\mu(M)$. Thus $\mathbb{P}\partial_t\mathbf{u} = \partial_t\mathbf{u}$ and $Q\partial_t\mathbf{u} = 0$. Applying $\mathbb{P}$ to (28.32) eliminates the pressure gradient and gives the reduced equation

$$\partial_t\mathbf{u} = -\mathbb{P}(\nabla_{\mathbf{u}}\mathbf{u}). \tag{28.36}$$

For $\mathbf{X} = \nabla_{\mathbf{u}}\mathbf{u}$, let q be the function from Proposition 28.3.2. The identity

$$\nabla_{\mathbf{u}}\mathbf{u} = \mathbb{P}(\nabla_{\mathbf{u}}\mathbf{u}) + \nabla q,$$

, substituted into (28.36), shows that $p = -q$ and that

$$\Delta_{\mathbf{g}} p = -\operatorname{div}_{\mathbf{g}}(\nabla_{\mathbf{u}}\mathbf{u}), \qquad \partial_\nu p = -\langle \nabla_{\mathbf{u}}\mathbf{u}, \nu\rangle_{\mathbf{g}}.$$

In Eulerian coordinates, for the fixed metric $\mathbf{g}$ and the boundary conditions incorporated in $\mathbb{P}_{\mathbf{g}}$, the right-hand side defines the smooth quadratic map

$$\mathscr{E}_{\mathbf{g}} \colon \mathfrak{X}^s_\mu(M) \longrightarrow \mathfrak{X}^{s-1}_\mu(M), \qquad \mathscr{E}_{\mathbf{g}}(\mathbf{u}) := -\mathbb{P}_{\mathbf{g}}(\nabla_{\mathbf{u}}\mathbf{u}).$$

Since

$$s - 1 > \frac{m}{2},$$

the Sobolev product estimate gives

$$\|\nabla_{\mathbf{u}}\mathbf{u}\|_{H^{s-1}(M,TM)} \le C_{M,\mathbf{g},s}\|\mathbf{u}\|^2_{H^s(M,TM)}.$$

Proposition 28.3.2 proves continuity of $\mathbb{P}_{\mathbf{g}}$ on $H^{s-1}(M,TM)$. Thus the Eulerian formulation loses one derivative. Its linearization along a solution is

$$\partial_t\mathbf{v} + \nabla_{\mathbf{u}}\mathbf{v} + \nabla_{\mathbf{v}}\mathbf{u} = -\nabla q, \qquad \operatorname{div}_{\mathbf{g}}\mathbf{v} = 0, \qquad (\mathbf{v}\cdot\nu)\restriction_{\partial M} = 0. \tag{28.37}$$

Proposition 28.3.6. (Regularity of the pressure) Under the hypotheses of Proposition 28.3.5, for each $\mathbf{u} \in \mathfrak{X}^s_\mu(M)$ the mean-zero function

$$p(\mathbf{u}) = -Q(\nabla_{\mathbf{u}}\mathbf{u})$$

belongs to $H^{s+1}(M)$. If $\mathrm{II}_{\mathrm{ext}}(\mathbf{V},\mathbf{W}) = \langle \nabla_{\mathbf{V}}\boldsymbol{\nu}, \mathbf{W}\rangle_{\mathbf{g}}$ for vectors tangent to the boundary, then

$$\Delta_{\mathbf{g}} p(\mathbf{u}) = -\sum_{i,j=1}^{m} (\nabla_i u^j)(\nabla_j u^i) - \mathrm{Ric}_{\mathbf{g}}(\mathbf{u},\mathbf{u}), \tag{28.38}$$

$$\partial_{\boldsymbol{\nu}} p(\mathbf{u}) = \mathrm{II}_{\mathrm{ext}}(\mathrm{Tr}\,\mathbf{u}, \mathrm{Tr}\,\mathbf{u}). \tag{28.39}$$

The interior datum belongs to $H^{s-1}(M)$ and the boundary datum to $H^{s-1/2}(\partial M)$. There exist constants C_s, C_s' such that

$$\|p(\mathbf{u})\|_{H^{s+1}(M)} + \|\nabla p(\mathbf{u})\|_{H^s(M,TM)} \le C_s \|\mathbf{u}\|^2_{H^s(M,TM)}, \tag{28.40}$$

$$\|p(\mathbf{u}) - p(\mathbf{v})\|_{H^{s+1}(M)} \le C_s'\big(\|\mathbf{u}\|_{H^s(M,TM)} + \|\mathbf{v}\|_{H^s(M,TM)}\big)\|\mathbf{u}-\mathbf{v}\|_{H^s(M,TM)}. \tag{28.41}$$

In particular, the normalized pressure of an Euler solution belongs to $C(I, H^{s+1}(M))$.

Proof. Begin with a smooth divergence-free vector field tangent to the boundary. In an orthonormal frame normal at the point of calculation, write $\nabla_i u^j = \langle \nabla_{\mathbf{e}_i}\mathbf{u}, \mathbf{e}_j\rangle_{\mathbf{g}}$. The Leibniz rule gives

$$\mathrm{div}_{\mathbf{g}}(\nabla_{\mathbf{u}}\mathbf{u}) = \sum_{i,j=1}^{m} (\nabla_i u^j)(\nabla_j u^i) + \sum_{i,j=1}^{m} u^j \nabla_i \nabla_j u^i.$$

The definition of curvature as a commutator implies

$$\sum_{i=1}^{m} \nabla_i \nabla_j u^i = \nabla_j \left(\sum_{i=1}^{m} \nabla_i u^i \right) + \sum_{k=1}^{m} \mathrm{Ric}_{kj}\, u^k.$$

Multiplying by u^j and summing, the first part vanishes because $\mathrm{div}_{\mathbf{g}}\,\mathbf{u} = 0$. We obtain

$$\mathrm{div}_{\mathbf{g}}(\nabla_{\mathbf{u}}\mathbf{u}) = \sum_{i,j=1}^{m} (\nabla_i u^j)(\nabla_j u^i) + \mathrm{Ric}_{\mathbf{g}}(\mathbf{u},\mathbf{u}).$$

On the boundary, differentiate $\langle \mathbf{u}, \boldsymbol{\nu}\rangle_{\mathbf{g}} = 0$ tangentially in the direction $\mathbf{u}$ and use metric compatibility:

$$0 = \langle \nabla_{\mathbf{u}}\mathbf{u}, \boldsymbol{\nu}\rangle_{\mathbf{g}} + \langle \mathbf{u}, \nabla_{\mathbf{u}}\boldsymbol{\nu}\rangle_{\mathbf{g}}.$$

This gives both pressure data in the smooth case.

For a vector field $\mathbf{u} \in \mathfrak{X}^s_\mu(M)$, Corollary 28.3.3 gives smooth vector fields $\mathbf{u}_j \in \mathfrak{X}^s_\mu(M)$ converging in $H^s(M,TM)$. Let $f(\mathbf{u})$ and $h(\mathbf{u})$ be the right-hand sides of (28.38) and (28.39). The contraction defining f is independent of the frame, as the smooth identity shows. Lemma 27.2.19, with $s-1 > m/2$, gives

$$\|f(\mathbf{u})\|_{H^{s-1}(M)} \le C_s \|\mathbf{u}\|^2_{H^s(M,TM)},$$

$$\|f(\mathbf{u}_j)-f(\mathbf{u})\|_{H^{s-1}(M)} \le C_s\big(\|\mathbf{u}_j\|_{H^s(M,TM)}+\|\mathbf{u}\|_{H^s(M,TM)}\big)\|\mathbf{u}_j-\mathbf{u}\|_{H^s(M,TM)}.$$

To verify the second bound, simply decompose each difference of products as $(A_j-A)B_j+A(B_j-B)$ and use the first product estimate. On the other hand, $\nabla_{\mathbf{u}_j}\mathbf{u}_j \to \nabla_{\mathbf{u}}\mathbf{u}$ in $H^{s-1}(M,TM)$, so their divergences converge in $H^{s-2}(M)$. The smooth identity and the stronger convergence of the $f(\mathbf{u}_j)$ identify, in the distributional sense, $-\operatorname{div}_{\mathbf{g}}(\nabla_{\mathbf{u}}\mathbf{u}) = f(\mathbf{u})$.
The continuous trace from Theorem 14.5.9 gives

$$\operatorname{Tr}\mathbf{u}_j \to \operatorname{Tr}\mathbf{u} \quad \text{in } H^{s-1/2}(\partial M, TM|_{\partial M}).$$

Since $s-1/2 > (m-1)/2$, the same product lemma, now on ∂M, yields

$$\|h(\mathbf{u})\|_{H^{s-1/2}(\partial M)} \le C_s\|\mathbf{u}\|^2_{H^s(M,TM)},$$
$$\|h(\mathbf{u}_j)-h(\mathbf{u})\|_{H^{s-1/2}(\partial M)} \le C_s\big(\|\mathbf{u}_j\|_{H^s(M,TM)}+\|\mathbf{u}\|_{H^s(M,TM)}\big)\|\mathbf{u}_j-\mathbf{u}\|_{H^s(M,TM)}.$$

The trace of $\nabla_{\mathbf{u}_j}\mathbf{u}_j$ converges in $H^{s-3/2}(\partial M, TM|_{\partial M})$, since $s-1>1/2$. Taking limits in the smooth boundary identity gives $-\langle \operatorname{Tr}(\nabla_{\mathbf{u}}\mathbf{u}), \boldsymbol{\nu}\rangle_{\mathbf{g}} = h(\mathbf{u})$, with the higher regularity just proved. If $m=1$, the boundary is finite and its spaces of sections are finite-dimensional; the same conclusion is immediate.
For each j, the divergence formula gives $\int_M f(\mathbf{u}_j)\,d\lambda_{\mathbf{g}} = \int_{\partial M} h(\mathbf{u}_j)\,d\sigma_{\mathbf{g}}$. The preceding convergences allow us to take limits in both integrals. Thus the data $f(\mathbf{u}), h(\mathbf{u})$ satisfy Neumann compatibility. Lemma 28.3.1, with $a=s-1$, gives a mean-zero solution in $H^{s+1}(M)$ and the bound (28.40). By weak uniqueness, it agrees with $-Q(\nabla_{\mathbf{u}}\mathbf{u})$. Applying the same Neumann estimate to differences of the data, with the product bounds already written, proves (28.41). This last inequality, applied to $\mathbf{u}(t)$ and $\mathbf{u}(\tau)$, proves continuity in time in $H^{s+1}(M)$. Finally, $\nabla_t\dot\eta = -(\nabla p)\circ\eta$ by (28.33), and the Lagrangian acceleration takes values at the $H^s(M,\eta^*TM)$ level of sections. □

In Lagrangian coordinates, this loss disappears from the geodesic vector field. The spray is a smooth map

$$\mathbf{S}\colon T\operatorname{Diff}^s_\mu(M) \longrightarrow T\big(T\operatorname{Diff}^s_\mu(M)\big), \qquad T\pi\circ\mathbf{S} = \mathrm{id}_{T\operatorname{Diff}^s_\mu(M)}.$$

Besides the preceding gain, differentiable dependence on η is needed. An estimate at a single point does not suffice for this. Precisely, the Ebin–Marsden result constructs the projection on the tangent bundles by means of the Neumann problem conjugated by Lagrangian charts, and proves that this projection depends smoothly on the chart. Differentiation is performed on operators between fixed spaces; for their inverses, one uses $D(B^{-1})[H] = -B^{-1}HB^{-1}$. The highest-order terms arising when differentiating composition combine with those from the projection. The cancellation expressed by the preceding divergence and boundary identities allows the acceleration to be estimated in $H^s(M,\eta^*TM)$. The precise result, including the boundary, is that of [46, Theorems 10.2 and 11.1, pp. 134–135]; it provides the smooth connection and the displayed spray, not merely a formal operator with a loss of derivatives.

In a tangent bundle chart, the spray has the form $\mathbf{S}(x,v) = (v, \mathcal{Q}(x,v))$, with $\mathcal{Q}$ smooth and quadratic in v. In a neighborhood of each initial datum, its differential is bounded, by continuity at that datum; the mean value formula makes it locally Lipschitz. The ODE theorem from the chapter on Banach flows therefore provides an interval $(-T,T)$ and a unique Lagrangian solution with the prescribed initial datum. Its Eulerian velocity belongs to $C((-T,T),\mathfrak{X}^s_\mu(M)) \cap C^1((-T,T),\mathfrak{X}^{s-1}_\mu(M))$. That theorem is not applied directly to the right-hand side of (28.36),

whose codomain has one fewer order of regularity. Theorem 12.1 of [46, p. 137] treats propagation of additional regularity separately.

The pressure is recovered by taking the divergence in (28.32). The resulting equation is a Neumann problem, whose boundary condition comes from the normal component of Euler's equation. This elliptic problem is local in time, but local existence does not imply completeness of the kinetic metric or global control of solutions. In dimension three, global continuation of a smooth solution is a question distinct from the geodesic interpretation.

On the complete manifold without boundary, let $\Delta_1 = d\delta + \delta d$ be the nonnegative Hodge Laplacian on $L^2(M, T^*M)$, with its self-adjoint realization. Here δ is the formal adjoint of d, and the complement of $\ker \Delta_1$ is taken in $L^2(M, T^*M)$.

Theorem 28.3.7. (Eichhorn–Schmid: Euler on a noncompact base) Let $(M^m, \mathbf{g})$ be smooth, oriented, complete, connected, noncompact, and without boundary, with bounded geometry of order k. Suppose that

$$\inf \sigma_{\mathrm{e}} \left(\Delta_1 \restriction_{(\ker \Delta_1)^\perp} \right) > 0, \qquad k > r > \frac{m}{2} + 1, \qquad k, r \in \mathbb{N}.$$

The identity component $\mathcal{D}^{2,r}_{\mu,0}(M)$ of the group of volume-preserving Sobolev diffeomorphisms is a split submanifold of class C^{k+1-r} of $\mathcal{D}^{2,r}_0(M)$. Each datum $\mathbf{u}_0 \in H^r(M, TM)$ with $\operatorname{div}_{\mathbf{g}} \mathbf{u}_0 = 0$ determines a unique local Euler solution, with

$$\mathbf{u} \in C((-T, T), H^r(M, TM)) \cap C^1((-T, T), H^{r-1}(M, TM)), \qquad \mathbf{u}(0) = \mathbf{u}_0,$$

for some $T > 0$. The pressure is chosen in $H^{r+1}(M)$; its differential therefore belongs to the closed range of $d : H^1(M) \to L^2(M, T^*M)$. This condition is part of the class of solutions in which uniqueness is asserted. No mean normalization is imposed on the base of infinite volume. If the geometry is bounded to all orders, the kinetic metric has a smooth connection and spray on this submanifold. In that case, its Lagrangian flow is the flow of a smooth ODE on the Hilbert tangent bundle.

Proof. The hypothesis $k > r$ allows us to use the diffeomorphism manifold from Theorem 27.5.3; in particular, $k + 1 - r \geq 2$. The submanifold result for the volume form is [56, Theorem 3.3], applied to the identity component. The spectral condition provides closed range for the Hodge operators and a complement to the directions changing volume. The norms and topology are those of the Sobolev charts at the identity; there is no need to extend to all maps a uniformity with radii independent of their bounds.

Let $\gamma > 0$ be a gap constant for Δ_1 on $(\ker \Delta_1)^\perp$, and let $\Delta_0 = \delta d = -\Delta_{\mathbf{g}}$ in $L^2(M)$. If $v \in C^\infty_c(M)$, the form dv is orthogonal to the kernel of Δ_1, and commutation of d with the Hodge Laplacians gives

$$\|\Delta_0 v\|^2_{L^2(M)} = \langle \Delta_1 dv, dv \rangle_{L^2(M, T^*M)} \geq \gamma \|dv\|^2_{L^2(M, T^*M)} = \gamma \langle \Delta_0 v, v \rangle_{L^2(M)}.$$

Theorem 12.1.10 states that $C^\infty_c(M)$ is a core for Δ_0. Thus, for $v \in \mathcal{D}(\Delta_0)$, choose $v_j \in C^\infty_c(M)$ with $v_j \to v$ and $\Delta_0 v_j \to \Delta_0 v$ in $L^2(M)$. The squared norms and inner product in the preceding inequality converge, so it holds for $v \in \mathcal{D}(\Delta_0)$. If E_0 is the spectral resolution of Δ_0, this inequality is

$$\int_{[0,\infty)} \lambda(\lambda - \gamma)\, d\langle E_0(\lambda) v, v \rangle_{L^2(M)} \geq 0.$$

A nonzero spectral projection on a compact interval $[a, b] \subset (0, \gamma)$ would give a nonzero vector in $\mathcal{D}(\Delta_0)$ for

which the integral is strictly negative. All such projections therefore vanish, and $\sigma(\Delta_0) \subseteq \{0\} \cup [\gamma, \infty)$. The volume of a connected noncompact manifold of bounded geometry is infinite: one can choose infinitely many disjoint balls of a common radius with volumes bounded below. A function in $L^2(M)$ in the kernel of Δ_0 has zero gradient and is constant, hence zero. Since an isolated point in the spectrum of a self-adjoint operator is an eigenvalue, zero does not belong to its spectrum either. Thus Δ_0^{-1} is bounded on $L^2(M)$.
The Hodge decomposition and uniform elliptic estimates give

$$\mathbb{P}\mathbf{X} = \mathbf{X} - \big(d\Delta_0^{-1}\delta(\mathbf{X}^\flat)\big)^\sharp.$$

This projection is continuous on $H^{r-1}(M, TM)$ and $H^r(M, TM)$. For $\mathbf{X} = \nabla_{\mathbf{u}}\mathbf{u}$, take

$$p = -\Delta_0^{-1}\delta\big((\nabla_{\mathbf{u}}\mathbf{u})^\flat\big).$$

The divergence cancellation computed above places the argument of Δ_0^{-1} in $H^{r-1}(M)$; the elliptic estimate gives $p \in H^{r+1}(M)$. The local bounds are summed over a uniform cover of finite multiplicity, and the gap controls the norm of the inverse on $L^2(M)$. This choice preserves orthogonality between the pressure and divergence-free vector fields. Allowing the gradient of an arbitrary harmonic potential, without requiring it to come from a function in $L^2(M)$, would change the class of solutions and is not the pressure condition used in this theorem.
Existence and uniqueness for geometry of finite order is the analytic result of [56, Corollary 4.6], in the strict range adopted here. Continuity in $H^r(M, TM)$ is part of that solution; the projected equation and continuity of the product $H^r(M, TM) \times H^r(M, TM) \to H^{r-1}(M, TM)$ give the continuous time derivative in $H^{r-1}(M, TM)$. This assertion does not presuppose that an atlas of finite regularity is a smooth atlas.
With geometric bounds at all orders, the charts, the pointwise connection operator, and the conjugated Hodge projection have continuous derivatives of all orders in the Sobolev charts. The construction in [56, §4] then gives a smooth connection and spray on $T\mathcal{D}^{2,r}_{\mu,0}(M)$. The ODE argument already explained applies on the tangent bundle modeled on $H^r(M, TM)$: the spray is, in particular, C^1 and locally Lipschitz. Its projection is a Lagrangian curve η of class C^2, with $\eta(0) = \mathrm{id}$ and $\dot\eta(0) = \mathbf{u}_0$. The equality $\mathbf{u} = \dot\eta \circ \eta^{-1}$ recovers the preceding solution, and uniqueness of the ODE implies Lagrangian uniqueness. The result with finite geometric bounds and the interpretation through a smooth spray are thus formulated with their respective hypotheses. □

Loss of regularity and choice of variables

The three examples show that a variational formulation does not by itself determine the appropriate function space. In the harmonic map problem, the tension field contains two derivatives of the map. In Yang–Mills theory, the curvature loses one derivative and the gradient $d^*_{\mathbf{A}}\mathbf{F}_{\mathbf{A}}$ loses two. In Euler's equation, the Eulerian expression loses one derivative, whereas Lagrangian variables produce a smooth spray in the compact case and in the noncompact version with geometric bounds of all orders specified above.

Symmetries also determine the form of the analysis. Gauge invariance requires separating the directions $d_{\mathbf{A}}\xi$; invariance under reparametrizations leads to working on a diffeomorphism group; and ellipticity controls local regularity, but does not by itself provide global compactness. On a noncompact manifold, uniform conditions and often spectral hypotheses enter the picture.

A strong metric may change the representative of a differential within the chosen tangent space, but it does not eliminate the loss of regularity of the original geometric operator. This distinction between the energy level, the geometric level, and the level at which the equation lives will be essential in the variational applications of the following chapters.

Part VII

Variational methods on Riemannian manifolds

Introduction to Part VII

Many nonlinear equations can be written as the condition that an energy remain unchanged to first order under perturbations of the unknown. In this case, solving the equation amounts to finding critical points of a functional defined on a space of functions or sections. Computing the first variation identifies the equation; the difficulty lies in proving that these critical points exist and in understanding the mechanisms that prevent the sequences used to find them from escaping.

The direct method studies minima. A minimizing sequence is controlled through coercivity, a subsequence is extracted by reflexivity, and the minimum is recovered through lower semicontinuity. Superlinear problems require a different geometry: the desired critical point may be a saddle point rather than a minimum. Palais–Smale conditions, pseudogradients, and deformation lemmas then allow minimax arguments to be constructed, including the mountain pass theorem.

The Nehari manifold introduces a third perspective. When the functional has a suitable radial geometry, each positive ray may intersect a natural constraint at a unique point. The problem is thus reduced to a more controlled geometry without artificially imposing an external condition. These chapters bring together the tools developed throughout the book: Sobolev spaces specify the domain of the functional, embeddings control the nonlinearities, integration by parts identifies the weak equation, and Banach geometry makes it possible to formulate constraints, tangent spaces, and deformation flows. Variational theory therefore emerges as a natural continuation of the analysis developed in the preceding parts.

From Euler and Lagrange to Palais and Smale

The calculus of variations began with problems about curves and surfaces in which a geometric or physical quantity had to be minimized. Euler and Lagrange developed the differential language that makes it possible to pass from an energy to an equation. With functional analysis, the unknown could be understood as a point of an infinite-dimensional space. The direct method organized existence arguments for minima through compactness and semicontinuity; Morse theory and minimax methods extended the search to critical points that are not minima; and Palais and Smale introduced a compactness condition adapted to these problems. The Nehari constraint, in turn, made a radial identity into a particularly effective tool for nonlinear equations.

29

Variational formulation of elliptic problems

Variational calculus allows partial differential equations to be studied through functionals defined on spaces of functions or sections. This perspective applies both on Euclidean domains and on Riemannian manifolds and, more generally, on smooth vector bundles. The common idea is first to formulate a notion of weak solution, obtained by multiplying the equation by test functions or sections and integrating by parts, and then to construct an energy functional whose Fréchet derivative reproduces that weak identity exactly. In this way, weak solutions arise as critical points of the functional.

The Green and integration-by-parts formulas developed in the preceding chapters provide the link between a differential operator and its weak formulation. In this part we will use them to identify the test spaces and recognize the Fréchet derivative of the associated energy functional.

For Poisson's equation, the quadratic functional

$$u \longmapsto \frac{1}{2}\int_M |du|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} - \int_M f\,u \, d\lambda_{\mathbf{g}}$$

produces the weak equation $-\Delta_{\mathbf{g}} u = f$. Replacing the quadratic term by $\frac{1}{p}\int_M |du|_{\mathbf{g}}^p \, d\lambda_{\mathbf{g}}$ gives the p-Laplacian, and the natural space changes from $W^{1,2}$ to $W^{1,p}$. Zeroth-order nonlinearities will be treated using Nemytski operators: the Carathéodory conditions and growth bounds will determine when the map $u \mapsto f(\cdot, u)$ acts continuously between the required Lebesgue spaces.

From an equation to an energy

The calculus of variations arose from the study of curves that extremize lengths, times, or actions. In modern elliptic equations, the same principle is formulated on function spaces: the first variation of an energy produces a weak identity, and critical points represent solutions. This translation is powerful because it replaces a pointwise equation by global properties of the functional, but it works only when integrability, differentiability, and boundary terms are justified in the chosen space.

29.1 Carathéodory functions and superposition operators

In variational problems, the spatial variable and the dependent variable play different roles. Often, the coefficients of the equation depend only measurably on the base point, whereas the dependence on the unknown must retain enough continuity for compositions to be measurable and integral functionals to be well defined. The Carathéodory conditions formalize precisely this situation.

Definition 29.1.1. (Carathéodory function) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let V and W be finite-dimensional real vector spaces. A function $f : M\times V \longrightarrow W$ is a **Carathéodory function** if, for each $\xi \in V$, the function $x \longmapsto f(x,\xi)$ is $\lambda_{\mathbf{g}}$-measurable and, for almost every $x \in M$, the function $\xi \longmapsto f(x,\xi)$ is continuous.

Proposition 29.1.2. (Measurability of compositions) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, let $f : M \times V \longrightarrow W$ be a Carathéodory function, and let $u : M \longrightarrow V$ be a measurable function. Then the function $x \longmapsto f(x,u(x))$ is $\lambda_{\mathbf{g}}$-measurable.

Proof. Since V is finite-dimensional, there exists a sequence $(u_j)_{j\in\mathbb{N}}$ of measurable simple functions such that $u_j(x) \to u(x)$ for almost every $x \in M$. For each $j \in \mathbb{N}$, we can write

$$u_j = \sum_{i=1}^{N_j} \xi_{i,j}\mathbf{1}_{E_{i,j}},$$

where the sets $E_{1,j},\dots,E_{N_j,j}$ are measurable and pairwise disjoint. Adding the set $M \setminus \bigcup_{i=1}^{N_j} E_{i,j}$ with value $0 \in V$ if necessary, we may assume that $M = \bigcup_{i=1}^{N_j} E_{i,j}$. Therefore,

$$f(x,u_j(x)) = \sum_{i=1}^{N_j} f(x,\xi_{i,j})\mathbf{1}_{E_{i,j}}(x).$$

Since $x \longmapsto f(x,\xi_{i,j})$ is measurable for each i, the function $x \longmapsto f(x,u_j(x))$ is measurable.
There is a measurable set $N \subseteq M$, with $\lambda_{\mathbf{g}}(N) = 0$, such that $f(x,\cdot)$ is continuous and $u_j(x) \to u(x)$ for every $x \in M \setminus N$. Consequently, $f(x,u_j(x)) \to f(x,u(x))$ for every $x \in M \setminus N$. Since $\lambda_{\mathbf{g}}$ is complete, an almost everywhere pointwise limit of measurable functions with values in a finite-dimensional space is measurable. Therefore, $x \longmapsto f(x,u(x))$ is measurable. □

The preceding formulation extends naturally to nonlinearities acting on sections of vector bundles. In this case, both the unknown and its image belong to fibers that depend on the base point. The next definition expresses the Carathéodory condition without choosing trivializations.

Definition 29.1.3. (Fiberwise Carathéodory map) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{F} \to M$ and $\mathbf{G} \to M$ be smooth real vector bundles of finite rank. A fiber-preserving map $\mathscr{G} : \mathbf{F} \longrightarrow \mathbf{G}$ is called a **fiberwise Carathéodory map** if the following conditions hold:

(a) there exists a measurable set $M_0 \subseteq M$, with $\lambda_{\mathbf{g}}(M \setminus M_0) = 0$, such that $\mathscr{G}_x \colon \mathbf{F}_x \longrightarrow \mathbf{G}_x$ is continuous for every $x \in M_0$;

(b) for every open set $U \subseteq M$ and every measurable section $\mathbf{u} \colon U \longrightarrow \mathbf{F} \restriction_U$, the section $x \longmapsto \mathscr{G}_x(\mathbf{u}(x))$ is measurable on $\mathbf{G} \restriction_U$.

The second condition in the preceding definition can be checked using trivializations. The next result shows that the intrinsic definition agrees with the usual Carathéodory condition for the local representations.

Proposition 29.1.4. (Local characterization of Carathéodory maps) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{F} \to M$ and $\mathbf{G} \to M$ be smooth real vector bundles of finite rank, and let $\mathscr{G} \colon \mathbf{F} \longrightarrow \mathbf{G}$ be a fiber-preserving map. Then $\mathscr{G}$ is a fiberwise Carathéodory map if and only if, for every open set $U \subseteq M$ over which $\mathbf{F}$ and $\mathbf{G}$ admit smooth trivializations

$$\Phi_{\mathbf{F}} \colon \mathbf{F} \restriction_U \longrightarrow U \times \mathbb{R}^r, \qquad \Phi_{\mathbf{G}} \colon \mathbf{G} \restriction_U \longrightarrow U \times \mathbb{R}^s,$$

the local representation $\mathscr{G}_U \colon U \times \mathbb{R}^r \longrightarrow \mathbb{R}^s$, determined by

$$\Phi_{\mathbf{G}} \circ \mathscr{G} \circ \Phi_{\mathbf{F}}^{-1}(x, \xi) = \big(x, \mathscr{G}_U(x, \xi)\big),$$

is a Carathéodory function.

Proof. Suppose first that $\mathscr{G}$ is a fiberwise Carathéodory map. Fix $\xi \in \mathbb{R}^r$ and consider the smooth section $\mathbf{u}_\xi \colon U \longrightarrow \mathbf{F} \restriction_U$ defined by $\mathbf{u}_\xi(x) := \Phi_{\mathbf{F}}^{-1}(x, \xi)$. By the second condition in Definition 29.1.3, the section $x \longmapsto \mathscr{G}_x(\mathbf{u}_\xi(x))$ is measurable on $\mathbf{G} \restriction_U$. Its local representation with respect to $\Phi_{\mathbf{G}}$ is the function $x \longmapsto \mathscr{G}_U(x, \xi)$, which is therefore measurable.

On the other hand, there is a measurable set $M_0 \subseteq M$, with $\lambda_{\mathbf{g}}(M \setminus M_0) = 0$, such that $\mathscr{G}_x \colon \mathbf{F}_x \longrightarrow \mathbf{G}_x$ is continuous for every $x \in M_0$. Since the restrictions of $\Phi_{\mathbf{F}}$ and $\Phi_{\mathbf{G}}$ to the fibers are linear isomorphisms, the function $\xi \longmapsto \mathscr{G}_U(x, \xi)$ is continuous for every $x \in U \cap M_0$. Therefore, $\mathscr{G}_U$ is a Carathéodory function.

Conversely, suppose that the local representations of $\mathscr{G}$ are Carathéodory functions. Since M is second countable, we can choose a countable cover $(U_k)_{k \in \mathbb{N}}$ by common trivialization domains for $\mathbf{F}$ and $\mathbf{G}$. For each $k \in \mathbb{N}$, there exists a measurable set $N_k \subseteq U_k$, with $\lambda_{\mathbf{g}}(N_k) = 0$, such that the local representation $\mathscr{G}_{U_k}(x, \cdot)$ is continuous for every $x \in U_k \setminus N_k$. If $N := \bigcup_{k \in \mathbb{N}} N_k$, then $\lambda_{\mathbf{g}}(N) = 0$ and $\mathscr{G}_x \colon \mathbf{F}_x \longrightarrow \mathbf{G}_x$ is continuous for every $x \in M \setminus N$.

Now let $U \subseteq M$ be an open set, and let $\mathbf{u} \colon U \longrightarrow \mathbf{F} \restriction_U$ be a measurable section. For each $k \in \mathbb{N}$, let $u_k \colon U \cap U_k \longrightarrow \mathbb{R}^r$ be the local representation of $\mathbf{u}$. Since $\mathscr{G}_{U_k}$ is a Carathéodory function, Proposition 29.1.2 shows that the function

$$x \longmapsto \mathscr{G}_{U_k}(x, u_k(x))$$

is measurable on $U \cap U_k$. This function is the local representation of the section $x \longmapsto \mathscr{G}_x(\mathbf{u}(x))$. Since the sets $U \cap U_k$ form a countable cover of U, it follows that $x \longmapsto \mathscr{G}_x(\mathbf{u}(x))$ is a measurable section of $\mathbf{G} \restriction_U$. Therefore, $\mathscr{G}$ is a fiberwise Carathéodory map. □

Remark 29.1.5. To verify that a fiber-preserving map is fiberwise Carathéodory, it suffices to check that its local representations are Carathéodory functions with respect to a countable trivializing cover. Proposition 29.1.4 also shows that this property does not depend on the chosen trivializations.
In particular, every continuous fiber-preserving map $\mathscr{G}\colon \mathbf{F} \longrightarrow \mathbf{G}$ is a fiberwise Carathéodory map. If $\mathbf{F} = M \times V$ and $\mathbf{G} = M \times W$ are trivial bundles, the definition reduces to the usual Carathéodory condition for a function $f : M \times V \longrightarrow W$.

Definition 29.1.6. (Nemytski operator between bundles) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{F} \to M$ and $\mathbf{G} \to M$ be smooth real vector bundles of finite rank, and let $\mathscr{G}\colon \mathbf{F} \longrightarrow \mathbf{G}$ be a fiberwise Carathéodory map. The operator defined pointwise by $\mathcal{N}_{\mathscr{G}}(\mathbf{u})(x) := \mathscr{G}_x(\mathbf{u}(x))$, whenever the resulting section belongs to the target space under consideration, is called the **Nemytski operator**, **superposition operator**, or **substitution operator** associated with $\mathscr{G}$.

The next result gives a growth condition ensuring that the superposition operator acts continuously between Lebesgue spaces of sections. The independent term belonging to $L^q(M)$ allows manifolds of infinite volume to be included.

Theorem 29.1.7. (Continuity of superposition operators between bundles) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary, let $\mathbf{F} \to M$ and $\mathbf{G} \to M$ be smooth real vector bundles of finite rank equipped with bundle metrics $\mathbf{h}_{\mathbf{F}}$ and $\mathbf{h}_{\mathbf{G}}$, and let $\mathscr{G}\colon \mathbf{F} \longrightarrow \mathbf{G}$ be a fiberwise Carathéodory map. Let $1 \le p, q < \infty$ and suppose that there exist a measurable function $a\colon M \longrightarrow [0, \infty)$, with $a \in L^q(M)$, and a constant $C \ge 0$ such that

$$|\mathscr{G}_x(\xi)|_{\mathbf{h}_{\mathbf{G}}} \le a(x) + C|\xi|_{\mathbf{h}_{\mathbf{F}}}^{\frac{p}{q}}$$

for almost every $x \in M$ and every $\xi \in \mathbf{F}_x$. Then the operator

$$\mathcal{N}_{\mathscr{G}}\colon L^p(M, \mathbf{F}) \longrightarrow L^q(M, \mathbf{G}), \qquad \mathcal{N}_{\mathscr{G}}(\mathbf{u})(x) := \mathscr{G}_x(\mathbf{u}(x)),$$

is well defined, is continuous, and maps bounded sets to bounded sets.

Proof. Let $\mathbf{u} \in L^p(M, \mathbf{F})$. Definition 29.1.3 ensures that $\mathcal{N}_{\mathscr{G}}(\mathbf{u})$ is a measurable section of $\mathbf{G}$. Moreover, by the growth hypothesis and the inequality $(\alpha + \beta)^q \le 2^{q-1}(\alpha^q + \beta^q)$, we have

$$|\mathscr{G}_x(\mathbf{u}(x))|_{\mathbf{h}_{\mathbf{G}}}^q \le 2^{q-1} a(x)^q + 2^{q-1} C^q |\mathbf{u}(x)|_{\mathbf{h}_{\mathbf{F}}}^p$$

for almost every $x \in M$. Therefore,

$$\|\mathcal{N}_{\mathscr{G}}(\mathbf{u})\|_{L^q(M,\mathbf{G})}^q \le 2^{q-1} \|a\|_{L^q(M)}^q + 2^{q-1} C^q \|\mathbf{u}\|_{L^p(M,\mathbf{F})}^p.$$

In particular, $\mathcal{N}_{\mathscr{G}}(\mathbf{u}) \in L^q(M, \mathbf{G})$. Furthermore, if two sections agree almost everywhere, their images under $\mathcal{N}_{\mathscr{G}}$ also agree almost everywhere, so the operator is well defined on the equivalence classes of $L^p(M, \mathbf{F})$.
The preceding estimate also shows that $\mathcal{N}_{\mathscr{G}}$ maps bounded sets to bounded sets. If $B \subseteq L^p(M, \mathbf{F})$ is bounded, there exists $R > 0$ such that $\|\mathbf{u}\|_{L^p(M,\mathbf{F})} \le R$ for every $\mathbf{u} \in B$, and then

$$\|\mathcal{N}_{\mathscr{G}}(\mathbf{u})\|_{L^q(M,\mathbf{G})}^q \le 2^{q-1} \|a\|_{L^q(M)}^q + 2^{q-1} C^q R^p$$

for every $\mathbf{u} \in B$.

We prove continuity. Let $(\mathbf{u}_j)_{j\in\mathbb{N}} \subseteq L^p(M,\mathbf{F})$ be such that $\mathbf{u}_j \to \mathbf{u}$ in $L^p(M,\mathbf{F})$. Consider an arbitrary subsequence $(\mathbf{u}_{j_\ell})_{\ell\in\mathbb{N}}$. Passing to a further subsequence, denoted by $(\mathbf{v}_\ell)_{\ell\in\mathbb{N}}$, we may assume that $\|\mathbf{v}_\ell - \mathbf{u}\|_{L^p(M,\mathbf{F})} \le 2^{-\ell}$ for every $\ell \in \mathbb{N}$.

Define

$$H(x) := \sum_{\ell=1}^{\infty} |\mathbf{v}_\ell(x) - \mathbf{u}(x)|_{\mathbf{h}_\mathbf{F}}.$$

Minkowski's inequality (Theorem B.7.3) implies

$$\|H\|_{L^p(M)} \le \sum_{\ell=1}^{\infty} \|\mathbf{v}_\ell - \mathbf{u}\|_{L^p(M,\mathbf{F})} \le \sum_{\ell=1}^{\infty} 2^{-\ell} < \infty.$$

Consequently, $H(x) < \infty$ for almost every $x \in M$. At each point of this set of full measure, we have $\mathbf{v}_\ell(x) \to \mathbf{u}(x)$ in $\mathbf{F}_x$ and

$$|\mathbf{v}_\ell(x)|_{\mathbf{h}_\mathbf{F}} \le |\mathbf{u}(x)|_{\mathbf{h}_\mathbf{F}} + H(x).$$

Continuity of $\mathscr{G}_x$ for almost every $x \in M$ gives

$$\mathscr{G}_x(\mathbf{v}_\ell(x)) \longrightarrow \mathscr{G}_x(\mathbf{u}(x))$$

for almost every $x \in M$.

On the other hand, the triangle inequality and the growth hypothesis imply the existence of a constant $C_0 > 0$, independent of ℓ, such that

$$|\mathscr{G}_x(\mathbf{v}_\ell(x)) - \mathscr{G}_x(\mathbf{u}(x))|^q_{\mathbf{h}_\mathbf{G}} \le C_0\left(a(x)^q + |\mathbf{v}_\ell(x)|^p_{\mathbf{h}_\mathbf{F}} + |\mathbf{u}(x)|^p_{\mathbf{h}_\mathbf{F}}\right)$$

for almost every $x \in M$. Since

$$|\mathbf{v}_\ell(x)|^p_{\mathbf{h}_\mathbf{F}} \le 2^{p-1}\left(|\mathbf{u}(x)|^p_{\mathbf{h}_\mathbf{F}} + H(x)^p\right),$$

there is a constant $C_1 > 0$, independent of ℓ, such that

$$|\mathscr{G}_x(\mathbf{v}_\ell(x)) - \mathscr{G}_x(\mathbf{u}(x))|^q_{\mathbf{h}_\mathbf{G}} \le C_1\left(a(x)^q + |\mathbf{u}(x)|^p_{\mathbf{h}_\mathbf{F}} + H(x)^p\right)$$

for almost every $x \in M$. The right-hand side belongs to $L^1(M)$. Theorem B.6.14 gives

$$\|\mathcal{N}_\mathscr{G}(\mathbf{v}_\ell) - \mathcal{N}_\mathscr{G}(\mathbf{u})\|^q_{L^q(M,\mathbf{G})} \longrightarrow 0.$$

We have shown that every subsequence of $(\mathbf{u}_j)_{j\in\mathbb{N}}$ has a subsequence whose image converges to $\mathcal{N}_\mathscr{G}(\mathbf{u})$ in $L^q(M,\mathbf{G})$. By the subsequence criterion, $\mathcal{N}_\mathscr{G}(\mathbf{u}_j) \to \mathcal{N}_\mathscr{G}(\mathbf{u})$ in $L^q(M,\mathbf{G})$. Therefore, $\mathcal{N}_\mathscr{G}$ is continuous. □

The case of trivial bundles yields the usual result for vector-valued Carathéodory functions.

Corollary 29.1.8. (Nemytski operator between Lebesgue spaces) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, let $1 \le p, q < \infty$, and let $f : M \times \mathbb{R}^N \longrightarrow \mathbb{R}^L$ be a Carathéodory function. Suppose that

there exist a measurable function $a\colon M \longrightarrow [0,\infty)$, with $a \in L^q(M)$, and a constant $C \geq 0$ such that

$$\|f(x,\xi)\| \leq a(x) + C\|\xi\|^{\frac{p}{q}}$$

for almost every $x \in M$ and every $\xi \in \mathbb{R}^N$. Then the operator

$$\mathcal{N}_f\colon L^p(M,\mathbb{R}^N) \longrightarrow L^q(M,\mathbb{R}^L), \qquad \mathcal{N}_f(u)(x) := f(x,u(x)),$$

is well defined, is continuous, and maps bounded sets to bounded sets.

Proof. Simply apply Theorem 29.1.7 to the trivial bundles $\mathbf{F} = M\times\mathbb{R}^N$ and $\mathbf{G} = M\times\mathbb{R}^L$, equipped with their Euclidean bundle metrics. □

Corollary 29.1.9. (The case of conjugate exponents) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with or without boundary, let $\mathbf{F} \to M$ and $\mathbf{G} \to M$ be smooth real vector bundles of finite rank equipped with bundle metrics $\mathbf{h_F}$ and $\mathbf{h_G}$, and let $\mathscr{G}\colon \mathbf{F} \longrightarrow \mathbf{G}$ be a continuous fiber-preserving map. Suppose that $m > 1$ and that there exists a constant $C > 0$ such that

$$|\mathscr{G}_x(\xi)|_{\mathbf{h_G}} \leq C\big(1 + |\xi|_{\mathbf{h_F}}^{m-1}\big)$$

for every $x \in M$ and every $\xi \in \mathbf{F}_x$. If $m' := \dfrac{m}{m-1}$, then the operator $\mathcal{N}_{\mathscr{G}}\colon L^m(M,\mathbf{F}) \longrightarrow L^{m'}(M,\mathbf{G})$ is well defined, is continuous, and maps bounded sets to bounded sets.

Proof. Since $\mathscr{G}$ is continuous and preserves fibers, it is a fiberwise Carathéodory map. Moreover, $\dfrac{m}{m'} = m-1$. Since M is compact, the constant function $a(x) := C$ belongs to $L^{m'}(M)$. The result follows from Theorem 29.1.7 by taking $p = m$ and $q = m'$. □

Remark 29.1.10. (The compact case) If M is compact, then $\lambda_{\mathbf{g}}(M) < \infty$ and every constant function belongs to $L^q(M)$. Consequently, in Theorem 29.1.7 the growth hypothesis can be replaced by

$$|\mathscr{G}_x(\xi)|_{\mathbf{h_G}} \leq C\big(1 + |\xi|_{\mathbf{h_F}}^{\frac{p}{q}}\big).$$

Similarly, in Corollary 29.1.8 it suffices to assume that $\|f(x,\xi)\| \leq C(1 + \|\xi\|^{\frac{p}{q}})$ for almost every $x \in M$ and every $\xi \in \mathbb{R}^N$.

Corollary 29.1.11. (Differentiability of powers of the norm) Let $(M,\mathbf{g})$ be a Riemannian manifold with or without boundary, and let $\mathbf{E} \to M$ be a smooth real vector bundle of finite rank, equipped with a bundle metric $\mathbf{h_E}$. For $1 < \ell < \infty$, the functional

$$Q_\ell(\mathbf{u}) := \frac{1}{\ell}\|\mathbf{u}\|_{L^\ell(M,\mathbf{E})}^\ell$$

is of class C^1 on $L^\ell(M,\mathbf{E})$ and

$$DQ_\ell(\mathbf{u})(\mathbf{v}) = \int_M |\mathbf{u}|_{\mathbf{h_E}}^{\ell-2}\langle \mathbf{u},\mathbf{v}\rangle_{\mathbf{h_E}}\, d\lambda_{\mathbf{g}}.$$

The integrand is interpreted as zero where $\mathbf{u} = 0$.

Proof. In each fiber, the map $\xi \mapsto |\xi|^\ell/\ell$ is of class C^1 and its differential is represented by $b_\ell(\xi) := |\xi|^{\ell-2}\xi$, with $b_\ell(0) = 0$. At zero, differentiability follows from $|\xi|^\ell/|\xi| \to 0$, since $\ell > 1$; away from zero, apply the chain rule. Theorem 29.1.7, with exponents ℓ and $\ell' := \ell/(\ell-1)$ and zero independent term, shows that b_ℓ is continuous from $L^\ell(M,\mathbf{E})$ to $L^{\ell'}(M,\mathbf{E})$.
By Hölder, the proposed formula defines a continuous linear functional of norm at most $\|\mathbf{u}\|^{\ell-1}_{L^\ell(M,\mathbf{E})}$. The fundamental theorem of calculus in each fiber and Fubini give

$$Q_\ell(\mathbf{u}+\mathbf{h}) - Q_\ell(\mathbf{u}) = \int_0^1 \int_M \langle b_\ell(\mathbf{u}+t\mathbf{h}), \mathbf{h}\rangle_{\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \, dt.$$

Indeed, the integral of the absolute value is at most $(\|\mathbf{u}\|_{L^\ell(M,\mathbf{E})} + \|\mathbf{h}\|_{L^\ell(M,\mathbf{E})})^{\ell-1}\|\mathbf{h}\|_{L^\ell(M,\mathbf{E})}$, by Hölder. Subtracting the proposed functional, we obtain

$$\begin{aligned}\left| Q_\ell(\mathbf{u}+\mathbf{h}) - Q_\ell(\mathbf{u}) - \int_M \langle b_\ell(\mathbf{u}), \mathbf{h}\rangle_{\mathbf{h}_\mathbf{E}} \, d\lambda_\mathbf{g} \right| \\ \le \|\mathbf{h}\|_{L^\ell(M,\mathbf{E})} \sup_{t\in[0,1]} \|b_\ell(\mathbf{u}+t\mathbf{h}) - b_\ell(\mathbf{u})\|_{L^{\ell'}(M,\mathbf{E})}.\end{aligned}$$

Continuity of b_ℓ at $\mathbf{u}$ makes the supremum tend to zero as $\mathbf{h} \to 0$: all the points $\mathbf{u}+t\mathbf{h}$ lie at distance at most $\|\mathbf{h}\|_{L^\ell(M,\mathbf{E})}$ from $\mathbf{u}$. This proves Fréchet differentiability. Finally, Hölder gives

$$\|DQ_\ell(\mathbf{u}) - DQ_\ell(\mathbf{w})\|_{(L^\ell)'} \le \|b_\ell(\mathbf{u}) - b_\ell(\mathbf{w})\|_{L^{\ell'}(M,\mathbf{E})},$$

so the differential is continuous. □

29.2 Classical, strong, and weak solutions

First consider the scalar problem

$$\begin{cases} -\Delta_\mathbf{g} u = f(x,u) & \text{in } M, \\ u = 0 & \text{on } \partial M, \end{cases} \tag{29.1}$$

where $(M,\mathbf{g})$ is a compact Riemannian manifold of dimension n with nonempty smooth boundary and $f : M\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function. This equation will serve as a model for introducing the basic notions. We adopt the convention

$$2^* := \begin{cases} \dfrac{2n}{n-2}, & n \ge 3, \\ +\infty, & n \in \{1,2\}. \end{cases}$$

When $|f(x,t)|$ grows like $|t|^{r-1}$, the growth is said to be sublinear if $1 < r < 2$ and superlinear subcritical if $2 < r < 2^*$. If $n \ge 3$, the case $r = 2^*$ is called critical and the case $r > 2^*$ supercritical. For $n \in \{1,2\}$, no finite critical exponent occurs in the first-order embeddings. These terms depend on the function space under consideration: 2^* is the limiting exponent for the embedding of $W^{1,2}_0(M)$.

Definition 29.2.1. (Classical solution) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let $f\colon M\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function. A classical solution of (29.1) is a function $u\in C^2(M)$ satisfying the equation at every point of M and whose restriction to ∂M is zero.

Definition 29.2.2. (Strong solution) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let $f\colon M\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function. A strong solution of (29.1) is a function $u\in W^{2,2}(M)\cap W_0^{1,2}(M)$ such that $-\Delta_{\mathbf{g}}u=f(x,u)$ almost everywhere on M.

Green's formula, obtained earlier from Theorem 2.14.21, shows that every sufficiently regular solution satisfies

$$\int_M \langle\nabla u,\nabla v\rangle_{\mathbf{g}}\,d\lambda_{\mathbf{g}}=\int_M f(x,u)v\,d\lambda_{\mathbf{g}}.$$

This leads to the notion natural in Sobolev spaces.

Definition 29.2.3. (Weak solution) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let $f\colon M\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function. A weak solution of (29.1) is a function $u\in W_0^{1,2}(M)$ such that

$$\int_M \langle\nabla u,\nabla v\rangle_{\mathbf{g}}\,d\lambda_{\mathbf{g}}=\int_M f(x,u)v\,d\lambda_{\mathbf{g}}$$

for every $v\in W_0^{1,2}(M)$, provided that the right-hand side is well defined.

Every classical solution is both strong and weak. Passing from a weak solution to a strong or classical solution requires elliptic regularity results and depends on the geometry of M, the regularity of the boundary, and the hypotheses on the nonlinearity.

The same procedure applies to sections. Let $\mathbf{E}\to M$ be a smooth real vector bundle of finite rank, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a connection $\nabla^{\mathbf{E}}$ compatible with $\mathbf{h}_{\mathbf{E}}$. An equation in $\mathbf{E}$ is tested against sections $\varphi\in W_0^{1,q}(M,\mathbf{E})$, and integration by parts is performed using the induced connection and its formal adjoint. The boundary condition is incorporated through the characterization of $W_0^{1,q}(M,\mathbf{E})$ as the kernel of the trace operator.

29.3 Energy functionals and critical points

Let X be a real Banach space, and let $I\colon X\longrightarrow\mathbb{R}$. We use the notation $DI(u)\in X'$ for the Fréchet derivative. A point $u\in X$ is critical if $DI(u)=0$.

Proposition 29.3.1. (Scalar energy functional) Let $(M,\mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty boundary. Suppose that $f\colon M\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function and that there exist $C>0$ and $r\in(1,2^*)$ such that $|f(x,t)|\le C(1+|t|^{r-1})$. Define $F(x,t):=\displaystyle\int_0^t f(x,s)\,ds$ and

$$I(u):=\frac{1}{2}\int_M|\nabla u|_{\mathbf{g}}^2\,d\lambda_{\mathbf{g}}-\int_M F(x,u)\,d\lambda_{\mathbf{g}}$$

on $X = W_0^{1,2}(M)$. Then $I \in C^1(X, \mathbb{R})$ and

$$DI(u)(v) = \int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} - \int_M f(x,u) v \, d\lambda_{\mathbf{g}}$$

for all $u, v \in X$. In particular, the critical points of I are exactly the weak solutions of (29.1).

Proof. Write $I = Q - P$, where

$$Q(u) := \frac{1}{2} \int_M |\nabla u|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}}, \qquad P(u) := \int_M F(x,u) \, d\lambda_{\mathbf{g}}.$$

The functional Q is of class C^1 and

$$DQ(u)(v) = \int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}.$$

The growth hypothesis implies

$$|F(x,t)| \le C|t| + \frac{C}{r}|t|^r.$$

Since $X \hookrightarrow L^1(M) \cap L^r(M)$ continuously, P is well defined. Likewise, Corollary 29.1.8, applied with $p = r$ and $q = r' := \dfrac{r}{r-1}$, shows that

$$u \longmapsto f(\cdot, u)$$

is continuous from $L^r(M)$ to $L^{r'}(M)$.
For $u \in X$, define $K_u \in X'$ by

$$K_u(v) := \int_M f(x,u) v \, d\lambda_{\mathbf{g}}.$$

If $C_S > 0$ is a constant for the embedding $X \hookrightarrow L^r(M)$, then

$$|K_u(v)| \le \|f(\cdot,u)\|_{L^{r'}(M)} \|v\|_{L^r(M)} \le C_S \|f(\cdot,u)\|_{L^{r'}(M)} \|v\|_X,$$

so K_u is linear and continuous.
Let $h \in X$. For almost every $x \in M$, Theorem B.6.5, applied to $s \longmapsto F(x, u(x) + sh(x))$, gives

$$F(x,u+h) - F(x,u) = \int_0^1 f(x, u+sh) h \, ds.$$

Hölder's inequality (Proposition B.7.9) and the growth bound justify applying Fubini's theorem for Bochner integrals B.6.26, in its scalar case, to $(M, \lambda_{\mathbf{g}}) \times ([0,1], ds)$. Consequently,

$$P(u+h) - P(u) - K_u(h) = \int_0^1 \int_M \big(f(x,u+sh) - f(x,u) \big) h \, d\lambda_{\mathbf{g}} \, ds$$

and

$$|P(u+h) - P(u) - K_u(h)| \le C_S \|h\|_X \int_0^1 \|f(\cdot, u+sh) - f(\cdot,u)\|_{L^{r'}(M)} \, ds.$$

Let $(h_j)_{j \in \mathbb{N}} \subseteq X$ be such that $h_j \to 0$ in X. For each $s \in [0,1]$, we have $u + sh_j \to u$ in $L^r(M)$, and continuity of

the Nemytski operator implies

$$\|f(\cdot, u+sh_j) - f(\cdot, u)\|_{L^{r'}(M)} \longrightarrow 0.$$

Moreover, there exists $R > 0$ such that $\|u+sh_j\|_{L^r(M)} \le R$ for every j and every $s \in [0,1]$. The estimate obtained in the first part of the proof of Corollary 29.1.8 provides a constant $C_R > 0$ such that

$$\|f(\cdot, u+sh_j) - f(\cdot, u)\|_{L^{r'}(M)} \le C_R$$

for every j and every $s \in [0,1]$. Theorem B.6.14, applied in the variable s, gives

$$\int_0^1 \|f(\cdot, u+sh_j) - f(\cdot, u)\|_{L^{r'}(M)}\, ds \longrightarrow 0.$$

Therefore,

$$\lim_{h\to 0} \frac{|P(u+h) - P(u) - K_u(h)|}{\|h\|_X} = 0,$$

so P is Fréchet differentiable and $DP(u) = K_u$.
Finally, if $u_j \to u$ in X, then $u_j \to u$ in $L^r(M)$ and

$$\|DP(u_j) - DP(u)\|_{X'} \le C_S \|f(\cdot, u_j) - f(\cdot, u)\|_{L^{r'}(M)} \longrightarrow 0.$$

Thus, $P \in C^1(X, \mathbb{R})$, $I = Q - P \in C^1(X, \mathbb{R})$, and the asserted formula for DI holds. Comparing it with Definition 29.2.3, we conclude that $DI(u) = 0$ if and only if u is a weak solution of (29.1). □

Proposition 29.3.2. (Second derivative of the potential functional) Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension n with nonempty boundary, and let f be the nonlinearity in Proposition 29.3.1. Suppose in addition that $f(x,\cdot)$ is of class C^1 for almost every x, that f_t is a Carathéodory function, and that there exist $C > 0$ and $r \in (2, 2^*)$ such that

$$|f_t(x,t)| \le C\left(1 + |t|^{r-2}\right)$$

for almost every $x \in M$ and every $t \in \mathbb{R}$. Then $I \in C^2(X, \mathbb{R})$ and

$$D^2 I(u)(v,w) = \int_M \langle \nabla v, \nabla w \rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}} - \int_M f_t(x,u) v w\, d\lambda_{\mathbf{g}}.$$

Proof. Let $P(u) := \displaystyle\int_M F(x,u)\, d\lambda_{\mathbf{g}}$. By the preceding proposition,

$$DP(u)(v) = \int_M f(x,u) v\, d\lambda_{\mathbf{g}}.$$

Define $s := \dfrac{r}{r-2}$. Then $\frac{1}{s} + \frac{2}{r} = 1$. Corollary 29.1.8, applied with source exponent r and target exponent s, shows that

$$u \longmapsto f_t(\cdot, u)$$

is continuous from $L^r(M)$ to $L^s(M)$, since $\frac{r}{s} = r - 2$.

For $u \in X$, define the bilinear form

$$B_u(h,v) := \int_M f_t(x,u)hv \, d\lambda_{\mathbf{g}}.$$

By Hölder's inequality (Proposition B.7.9) and the embedding $X \hookrightarrow L^r(M)$,

$$|B_u(h,v)| \le \|f_t(\cdot,u)\|_{L^s(M)}\|h\|_{L^r(M)}\|v\|_{L^r(M)} \le C_S^2\|f_t(\cdot,u)\|_{L^s(M)}\|h\|_X\|v\|_X.$$

Therefore, $B_u \in \mathcal{L}^2(X;\mathbb{R})$.
For $h, v \in X$, Theorem B.6.5, applied in the second variable, gives

$$f(x,u+h) - f(x,u) = \int_0^1 f_t(x,u+\tau h)h \, d\tau$$

for almost every $x \in M$. Thus,

$$\big(DP(u+h) - DP(u) - B_u(h,\cdot)\big)(v) = \int_0^1 \int_M \big(f_t(x,u+\tau h) - f_t(x,u)\big)hv \, d\lambda_{\mathbf{g}} \, d\tau.$$

Taking the supremum over $\|v\|_X \le 1$ and applying Hölder, we obtain

$$\|DP(u+h) - DP(u) - B_u(h,\cdot)\|_{X'} \le C_S^2\|h\|_X \int_0^1 \|f_t(\cdot,u+\tau h) - f_t(\cdot,u)\|_{L^s(M)} \, d\tau.$$

If $h_j \to 0$ in X, then $u + \tau h_j \to u$ in $L^r(M)$ for each $\tau \in [0,1]$. Continuity of the Nemytski operator associated with f_t implies that the integrand converges to zero. Moreover, the family $(u+\tau h_j)_{j,\tau}$ is bounded in $L^r(M)$, and the boundedness estimate of Corollary 29.1.8 provides a constant independent of j and τ that dominates the integrand. By Theorem B.6.14, applied in $[0,1]$,

$$\int_0^1 \|f_t(\cdot,u+\tau h_j) - f_t(\cdot,u)\|_{L^s(M)} \, d\tau \longrightarrow 0.$$

Consequently,

$$\lim_{h\to 0} \frac{\|DP(u+h) - DP(u) - B_u(h,\cdot)\|_{X'}}{\|h\|_X} = 0.$$

This proves that DP is Fréchet differentiable and that $D^2P(u) = B_u$.
If $u_j \to u$ in X, then $u_j \to u$ in $L^r(M)$ and

$$\|D^2P(u_j) - D^2P(u)\|_{\mathcal{L}^2(X;\mathbb{R})} \le C_S^2\|f_t(\cdot,u_j) - f_t(\cdot,u)\|_{L^s(M)} \longrightarrow 0.$$

Therefore, $P \in C^2(X,\mathbb{R})$. Since the second derivative of the quadratic term is the form $(v,w) \mapsto \int_M \langle \nabla v, \nabla w\rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}}$, the formula for D^2I follows from $I = Q - P$. □

In the subcritical regime, the compact embedding of $W_0^{1,2}(M)$ into $L^r(M)$ makes the potential term stable under the weak convergences that occur in the direct method and in the Nehari method.

30

The direct method in the calculus of variations

The direct method turns a minimization problem into a combination of three properties: coercivity, compactness, and lower semicontinuity. Coercivity keeps minimizing sequences in bounded sets; weak compactness allows a convergent subsequence to be extracted; and lower semicontinuity ensures that the limit does not increase the energy. In reflexive spaces, this scheme gives a general criterion for the existence of minimizers.

The distinction between weak compactness and weak convergence of sequences allows two versions of the method to be formulated. After establishing them, we will apply them to a sublinear elliptic problem. In that example, the Dirichlet condition controls the quadratic part of the energy, whereas compactness of a Sobolev embedding allows passage to the limit in the nonlinear term.

The direct method and the power of weak convergence

The direct method does not attempt to solve the Euler–Lagrange equation explicitly. It starts from a sequence whose energy approaches the infimum and seeks enough compactness to obtain a limit. Weak convergence is decisive because every bounded sequence in a reflexive Banach space admits a weakly convergent subsequence; lower semicontinuity prevents the energy from jumping upward at the limit. This combination, developed in the modern calculus of variations, turns structural properties of the space and the functional into an existence theorem.

30.1 Lower semicontinuity

A minimizing sequence often converges only weakly. For its limit to remain a minimizer, the energy must not increase on passing to the limit; this is precisely the role of lower semicontinuity.

Definition 30.1.1. Let X be a topological space. A function $\varphi\colon X \longrightarrow \mathbb{R}$ is said to be lower semicontinuous if, for every $a \in \mathbb{R}$, the set $\{x \in X \mid \varphi(x) > a\}$ is open in X.

Example 30.1.2. Every continuous function $\varphi\colon X \longrightarrow \mathbb{R}$ is lower semicontinuous, since the preimage of any open set is an open set in X.

Example 30.1.3. Consider the function $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then the function φ is lower semicontinuous. Indeed, if $a < 0$, the set $\{x \in \mathbb{R} \mid \varphi(x) > a\}$ is $\mathbb{R}$; if $0 \leq a < 1$, it is $(0, +\infty)$; and if $a \geq 1$, it is empty. All three sets are open. The discontinuity at 0 is compatible with lower semicontinuity because the value at that point agrees with the smaller of the two one-sided values.

Lemma 30.1.4. Let X be a first-countable topological space, and let $\psi\colon X \longrightarrow \mathbb{R}$ be a functional. Then ψ is lower semicontinuous if and only if

$$\psi(x) \leq \liminf_{n\to\infty} \psi(x_n),$$

for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$ in X.

Proof. Suppose first that ψ is lower semicontinuous. We wish to prove that $\psi(x) \leq \liminf_{n\to\infty} \psi(x_n)$ for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$. Let $\varepsilon > 0$. Consider the set

$$V_\varepsilon := \{y \in X \mid \psi(y) > \psi(x) - \varepsilon\} = \psi^{-1}\big((\psi(x) - \varepsilon, +\infty)\big).$$

Since ψ is lower semicontinuous, the set V_ε is open. Since $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V_\varepsilon$ for every $n \geq n_0$. Therefore,

$$\psi(x_n) > \psi(x) - \varepsilon, \quad \text{for every } n \geq n_0,$$

which implies $\liminf_{n\to\infty} \psi(x_n) \geq \psi(x) - \varepsilon$.
Since $\varepsilon > 0$ is arbitrary, it follows that $\psi(x) \leq \liminf_{n\to\infty} \psi(x_n)$.
Conversely, suppose that $\psi(x) \leq \liminf_{n\to\infty} \psi(x_n)$ for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$. We show that ψ is lower semicontinuous. Let $a \in \mathbb{R}$ and consider the set $F := \{x \in X \mid \psi(x) \leq a\}$. We claim that F is closed. Suppose, for a contradiction, that it is not. Then there exists $x_0 \in \overline{F} \setminus F$; that is, x_0 is a point of the closure of F but $\psi(x_0) > a$. Since X is first countable, x_0 has a countable base of open neighborhoods $(U_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, set $V_n := \bigcap_{j=1}^{n} U_j$. Since $x_0 \in \overline{F}$, we can choose $x_n \in F \cap V_n$. If U is a neighborhood of x_0, there exists $j \in \mathbb{N}$ such that $U_j \subseteq U$; for every $n \geq j$, we have $x_n \in V_n \subseteq U_j \subseteq U$. Therefore, $x_n \to x_0$. By hypothesis, $\psi(x_0) \leq \liminf_{n\to\infty} \psi(x_n)$.
Since $x_n \in F$, we have $\psi(x_n) \leq a$ for every $n \in \mathbb{N}$, and hence $\liminf_{n\to\infty} \psi(x_n) \leq a$. Consequently, $\psi(x_0) \leq a$, contradicting $\psi(x_0) > a$. Therefore, F is closed for every $a \in \mathbb{R}$, which implies that ψ is lower semicontinuous. □

Theorem 30.1.5. Let X be a nonempty compact topological space, and let $\varphi\colon X \longrightarrow \mathbb{R}$ be a lower semicontinuous functional. Then φ is bounded below and attains its infimum; that is, there exists $u_0 \in X$ such

that

$$\varphi(u_0) = \inf_{x \in X} \varphi(x).$$

Proof. Observe that $X = \bigcup_{n=1}^{\infty} \varphi^{-1}((-n, +\infty))$.
Since φ is lower semicontinuous, each set $\varphi^{-1}((-n, +\infty))$ is open in X.
Since X is compact, the preceding cover admits a finite subcover. If $n_0 \in \mathbb{N}$ is the largest of its indices, then $X = \bigcup_{n=1}^{n_0} \varphi^{-1}((-n, +\infty))$.
Therefore, for every $x \in X$ we have $\varphi(x) > -n_0$, which implies that φ is bounded below.
Since $X \neq \varnothing$, the set $\varphi(X)$ is nonempty and bounded below. Completeness of $\mathbb{R}$ then provides a real number $c = \inf_{x \in X} \varphi(x)$.
Suppose, for a contradiction, that $\varphi(x) > c$ for every $x \in X$. Then $X = \bigcup_{n=1}^{\infty} \varphi^{-1}\left(\left(c + \frac{1}{n}, +\infty\right)\right)$.
Again, compactness provides a finite subcover. Taking $k \in \mathbb{N}$ to be the largest index of that subcover, we obtain $X = \bigcup_{n=1}^{k} \varphi^{-1}\left(\left(c + \frac{1}{n}, +\infty\right)\right)$.
It follows that $\varphi(x) > c + \frac{1}{k}$ for every $x \in X$, which implies that $c + \frac{1}{k}$ is a lower bound for $\varphi(X)$. This contradicts the definition of the infimum, since it would imply $c \geq c + \frac{1}{k}$. Therefore, there exists $u_0 \in X$ such that $\varphi(u_0) = \inf_{x \in X} \varphi(x)$.

□

Definition 30.1.6. Let X be a normed space. A function $\varphi \colon X \longrightarrow \mathbb{R}$ is said to be weakly lower semicontinuous if it is lower semicontinuous with respect to the weak topology $\sigma(X, X')$. In particular, if a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ satisfies $x_n \rightharpoonup x$ in X, then

$$\varphi(x) \leq \liminf_{n \to \infty} \varphi(x_n).$$

The implication from lower semicontinuity to the inequality for sequences does not require first countability: the first part of the proof of Lemma 30.1.4 suffices. The converse does require additional justification when working with the weak topology.

Definition 30.1.7. Let X be a normed space. A functional $\varphi \colon X \longrightarrow \mathbb{R}$ is sequentially weakly lower semicontinuous if, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ and every $x \in X$ such that $x_n \rightharpoonup x$, we have

$$\varphi(x) \leq \liminf_{n \to \infty} \varphi(x_n).$$

The weak topology of an infinite-dimensional Banach space need not be first countable. We therefore keep the two notions distinct. In the applications, it will suffice to check the sequential condition and use a version of the direct method formulated for it.

Theorem 30.1.8. (Direct method in the calculus of variations) Let E be a reflexive Banach space, and let $\varphi \colon E \longrightarrow \mathbb{R}$ be a functional satisfying:

(i) φ is weakly lower semicontinuous;

(ii) φ is coercive, that is, $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$: for every $a \in \mathbb{R}$ there exists $R_a > 0$ such that $\varphi(u) > a$ for every $u \in E$ with $\|u\| \geq R_a$.

Then φ is bounded below and attains its infimum; that is, there exists $u_0 \in E$ such that

$$\varphi(u_0) = \inf_{u \in E} \varphi(u).$$

Proof. Let $R > 0$ be such that

$$\varphi(u) \geq \varphi(0), \quad \text{for every } u \in E \text{ with } \|u\| \geq R.$$

Since E is a reflexive Banach space, Theorem B.11.38 implies that the closed ball $\overline{B}_{d_E}(0,R) = \{u \in E \mid \|u\| \leq R\}$, where d_E is the metric induced by the norm, is weakly compact.
Moreover, the restriction of φ to $\overline{B}_{d_E}(0,R)$ is weakly lower semicontinuous. Therefore, applying Theorem 30.1.5 shows that φ is bounded below on $\overline{B}_{d_E}(0,R)$ and that there exists $u_0 \in \overline{B}_{d_E}(0,R)$ such that $\varphi(u_0) = \inf_{u \in \overline{B}_{d_E}(0,R)} \varphi(u)$.
By the choice of R, for every $u \in E$ with $\|u\| \geq R$ we have $\varphi(u) \geq \varphi(0) \geq \varphi(u_0)$, which implies that $\varphi(u_0) = \inf_{u \in E} \varphi(u)$.

□

Coercivity is equivalent to boundedness of every sublevel set $\{u \in E \mid \varphi(u) \leq a\}$, with $a \in \mathbb{R}$. Indeed, the radius R_a in the definition encloses that sublevel set in a ball. Conversely, if the sublevel set is contained in a ball of radius r_a, any $R_a > r_a$ satisfies the coercivity condition. This observation allows us to work directly with minimizing sequences.

Theorem 30.1.9. (Sequential form of the direct method) Let E be a reflexive Banach space, and let $\varphi\colon E \longrightarrow \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous functional. Then φ is bounded below and there exists $u_0 \in E$ such that

$$\varphi(u_0) = \inf_{u \in E} \varphi(u).$$

Proof. First we rule out the possibility that the infimum is $-\infty$. If it were, for each $n \in \mathbb{N}$ we could choose $u_n \in E$ with $\varphi(u_n) \leq -n$. The entire sequence would lie in the sublevel set $\{u \in E \mid \varphi(u) \leq -1\}$, which is bounded by coercivity. Corollary B.11.39 gives a subsequence $(u_{n_j})_{j \in \mathbb{N}}$ and a point $u \in E$ such that $u_{n_j} \rightharpoonup u$. Sequential semicontinuity would give

$$\varphi(u) \leq \liminf_{j \to \infty} \varphi(u_{n_j}) = -\infty,$$

which is impossible because φ takes real values. Consequently, $c := \inf_{u \in E} \varphi(u)$ is finite; moreover, $c \leq \varphi(0) < +\infty$.
By the definition of the infimum, we can now choose, for each $n \in \mathbb{N}$, a point $v_n \in E$ such that

$$c \leq \varphi(v_n) < c + \frac{1}{n}.$$

This sequence lies in the sublevel set at height $c+1$ and is therefore bounded. Applying the cited corollary

again, we obtain $u_0 \in E$ and a subsequence $(v_{n_j})_{j\in\mathbb{N}}$ such that $v_{n_j} \rightharpoonup u_0$. Then

$$c \le \varphi(u_0) \le \liminf_{j\to\infty} \varphi(v_{n_j}) = c.$$

The two inequalities force $\varphi(u_0) = c$. □

The extraction of weakly convergent subsequences rests on Eberlein–Šmulian and reflexivity, through the cited corollary in Appendix B. This is also the formulation of the direct method used in [138, chap. I, §1, Theorem 1.2].

Under the hypotheses of the last theorem, topological lower semicontinuity can also be recovered. Fix $a \in \mathbb{R}$ and set $K_a := \{u \in E \mid \varphi(u) \le a\}$. If K_a is nonempty, every sequence in K_a is bounded and admits a weakly convergent subsequence in E. Sequential semicontinuity ensures that its limit remains in K_a. Thus, K_a is sequentially weakly compact and, by Theorem B.11.36, weakly compact. Since the weak topology is Hausdorff, K_a is weakly closed. The empty sublevel set is also closed, so all sublevel sets are closed. Proposition B.11.15 then implies that φ is weakly lower semicontinuous. This argument uses both coercivity and reflexivity; it has not assumed that the two notions of semicontinuity agree in an arbitrary weak topology.

We turn to an application in which the energy is not convex. Its quadratic part will be weakly lower semicontinuous, and a compact embedding will allow us to handle the remaining part.

30.2 Solving a sublinear problem

Let $(M, \mathbf{g})$ be a nonempty compact Riemannian manifold of dimension $n \ge 1$, with smooth boundary, and suppose that every connected component of M has nonempty boundary. We will work with real functions. The condition on the components allows the norm of a function with zero trace to be controlled by its gradient.

We will use the Poincaré inequality of Theorem 9.6.8, applied to the trivial real bundle with $p = 2$. Its hypothesis on each component is precisely the one just imposed, and its range $1 \le p < \infty$ includes this exponent in every dimension $n \ge 1$. The constant depends on M and $\mathbf{g}$.

Consider the sublinear problem

$$\begin{cases} -\Delta_{\mathbf{g}} u = |u|^{q-2}u & \text{in } M, \\ u = 0 & \text{on } \partial M, \end{cases} \tag{30.1}$$

where $1 < q < 2$. We use the sign $\Delta_{\mathbf{g}} = \operatorname{div}_{\mathbf{g}} \nabla$, so that $-\Delta_{\mathbf{g}}$ is nonnegative under the Dirichlet condition. The expression $|t|^{q-2}t$ is defined to be zero when $t = 0$; with this convention it is continuous on all of $\mathbb{R}$. The condition $u = 0$ on ∂M is understood in the trace sense.

The aim is to find $u \in W_0^{1,2}(M)$ such that

$$\int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} = \int_M |u|^{q-2} u \, v \, d\lambda_{\mathbf{g}}$$

for every $v \in W_0^{1,2}(M)$. The associated energy functional is

$$I(u) = \frac{1}{2} \int_M |\nabla u|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} - \frac{1}{q} \int_M |u|^q \, d\lambda_{\mathbf{g}}.$$

Proposition 30.2.1. (Application to the sublinear problem) Suppose that every connected component of the compact manifold M has nonempty smooth boundary. If $1 < q < 2$, the functional I attains its global minimum on $W_0^{1,2}(M)$, and every minimizer is a nontrivial weak solution of problem (30.1).

Proof. The nonlinearity $f(x,t) = |t|^{q-2}t$ satisfies the hypotheses of Proposition 29.3.1 with exponent $r = q$: it is continuous and its growth is $|t|^{q-1}$, with $1 < q < 2 \le 2^*$. Hence, $I \in C^1(W_0^{1,2}(M), \mathbb{R})$ and its critical points are precisely the weak solutions. More explicitly,

$$DI(u)(v) = \int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} - \int_M |u|^{q-2} u \, v \, d\lambda_{\mathbf{g}}$$

for all $u, v \in W_0^{1,2}(M)$. The second term is finite by Hölder, since $|u|^{q-2}u \in L^{q/(q-1)}(M)$ and $v \in L^q(M)$.
Let C_P be the Poincaré constant from Theorem 9.6.8. Since M has finite volume, Hölder and Poincaré give

$$\|u\|_{L^q(M)} \le \lambda_{\mathbf{g}}(M)^{\frac{1}{q}-\frac{1}{2}} \|u\|_{L^2(M)}, \qquad \|u\|^2_{W^{1,2}(M)} \le (1 + C_P^2) \|\nabla u\|^2_{L^2(M,TM)}.$$

Consequently, we can take $C_1 := \dfrac{1}{2(1+C_P^2)}$ and $C_2 := \dfrac{1}{q} \lambda_{\mathbf{g}}(M)^{1-q/2}$ to obtain

$$I(u) \ge C_1 \|u\|^2_{W^{1,2}(M)} - C_2 \|u\|^q_{W^{1,2}(M)}.$$

If $t := \|u\|_{W^{1,2}(M)} > 0$, the right-hand side of this estimate is $t^2(C_1 - C_2 t^{q-2})$. Since $q - 2 < 0$, there exists $R > 0$ such that $C_2 t^{q-2} \le C_1/2$ for every $t \ge R$. Thus, $I(u) \ge C_1 \|u\|^2_{W^{1,2}(M)}/2$ as $\|u\|_{W^{1,2}(M)} \ge R$, proving coercivity.
To study passage to the limit, let $(u_j)_{j\in\mathbb{N}} \subseteq W_0^{1,2}(M)$ be a sequence such that $u_j \rightharpoonup u$ in that space. By Proposition B.11.20, there exists $R_0 > 0$ such that $\|u_j\|_{W^{1,2}(M)} \le R_0$ for every $j \in \mathbb{N}$. The preceding estimate implies $I(u_j) \ge -C_2 R_0^q$, whereas the definition of I gives $I(u_j) \le R_0^2/2$. Hence, $\ell := \liminf_{j\to\infty} I(u_j)$ is a real number. Choose a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ such that $I(u_{j_k}) \to \ell$. Corollary 9.4.29, applied to the trivial real bundle with $m = 1$ and $p = 2$, gives compactness of $W^{1,2}(M) \hookrightarrow L^2(M)$. Since $\lambda_{\mathbf{g}}(M) < \infty$, Hölder gives the continuous embedding

$$\|w\|_{L^q(M)} \le \lambda_{\mathbf{g}}(M)^{\frac{1}{q}-\frac{1}{2}} \|w\|_{L^2(M)} \qquad (1 < q < 2).$$

By composition, $W^{1,2}(M) \hookrightarrow L^q(M)$ is compact. Passing to a subsequence of $(u_{j_k})_{k\in\mathbb{N}}$, which we continue to denote in the same way, we obtain strong convergence in $L^q(M)$. Its limit is u: the continuous embedding into $L^q(M)$ carries weak convergence in u_{j_k} to weak convergence in $L^q(M)$, and every strong limit is also a weak limit. Uniqueness of the weak limit identifies the two limits. In particular, $\|u_{j_k}\|_{L^q(M)} \to \|u\|_{L^q(M)}$, and hence

$$\int_M |u_{j_k}|^q \, d\lambda_{\mathbf{g}} \longrightarrow \int_M |u|^q \, d\lambda_{\mathbf{g}}.$$

The operator $u \mapsto \nabla u$ is linear and continuous from $W_0^{1,2}(M)$ to $L^2(M, TM)$, so $\nabla u_{j_k} \rightharpoonup \nabla u$. Weak lower semicontinuity of the norm, Proposition B.11.20, and convergence of the nonlinear term give

$$I(u) \le \liminf_{k\to\infty} \left(\frac{1}{2} \|\nabla u_{j_k}\|^2_{L^2(M,TM)} - \frac{1}{q} \|u_{j_k}\|^q_{L^q(M)} \right) = \ell.$$

Thus, I is sequentially weakly lower semicontinuous. The space $W_0^{1,2}(M)$ is Hilbert, being a closed subspace

of $W^{1,2}(M)$, and in particular is reflexive. Theorem 30.1.9 then provides $u_0 \in W_0^{1,2}(M)$ such that $I(u_0) = \inf_{u \in W_0^{1,2}(M)} I(u)$.

It remains to prove that $u_0 \neq 0$. Take $v \in C_c^\infty(\mathrm{Int}(M)) \setminus \{0\}$. For $t > 0$,

$$I(tv) = \frac{t^2}{2} \int_M |\nabla v|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} - \frac{t^q}{q} \int_M |v|^q \, d\lambda_{\mathbf{g}}.$$

Write $A := \displaystyle\int_M |\nabla v|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}}$ and $B := \displaystyle\int_M |v|^q \, d\lambda_{\mathbf{g}}$. We have $B > 0$, since $v \neq 0$, and $A > 0$ by Poincaré. The preceding equality becomes $I(tv) = t^q(At^{2-q}/2 - B/q)$. Since $2 - q > 0$, any

$$0 < t_0 < \left(\frac{2B}{qA}\right)^{\frac{1}{2-q}}$$

satisfies $I(t_0 v) < 0 = I(0)$. Consequently, $I(u_0) \leq I(t_0 v) < 0$ and $u_0 \neq 0$. For each $v \in W_0^{1,2}(M)$, the real map $t \mapsto I(u_0 + tv)$ has a minimum at $t = 0$, so its derivative there is zero. Thus, $DI(u_0)(v) = 0$ for every $v \in W_0^{1,2}(M)$, which is the asserted weak solution identity. □

If only $\partial M \neq \varnothing$ were assumed, there could be a closed component M_0. For the function u_t equal to t on M_0 and zero on the other components, we would have

$$I(u_t) = -\frac{\lambda_{\mathbf{g}}(M_0)}{q} |t|^q \longrightarrow -\infty \qquad \text{as } |t| \to \infty.$$

In that case the energy has no global minimum on $W_0^{1,2}(M)$. Nevertheless, a nontrivial weak solution can still be obtained if there is some component with boundary: apply the proposition on that component and extend the solution by zero to the remaining ones. The weak identity on M reduces to the identity on the chosen component, since both the function and its gradient vanish on the others. This extension does not assert that the solution minimizes the energy on all of M.

30.3 Exercises

Exercise 30.3.1. Let E be a reflexive Banach space, and let $C \subset E$ be a nonempty closed convex set. Prove that if $\psi \colon E \longrightarrow \mathbb{R}$ is a weakly lower semicontinuous and coercive functional, then ψ attains its infimum on C; that is, there exists $x_0 \in C$ such that

$$\psi(x_0) = \inf_{x \in C} \psi(x).$$

Hint. Theorem B.11.16 implies that C is weakly closed. Repeat the sequential proof of the direct method with sequences contained in C; their weak limits will also belong to C. To bound the infimum above, use the value of ψ at a fixed point of C, which exists because $C \neq \varnothing$.

Exercise 30.3.2. Let E be a real Hilbert space, and let $a \colon E \times E \longrightarrow \mathbb{R}$ be a continuous bilinear form satisfying the coercivity condition

$$a(u,u) \geq \alpha \|u\|^2, \quad \text{for every } u \in E,$$

for some $\alpha > 0$. Let also $f : E \longrightarrow \mathbb{R}$ be a continuous linear functional.
Given a nonempty closed convex set $C \subset E$, prove that there exists a unique $u_0 \in C$ such that

$$\psi(u_0) = \inf_{u \in C} \psi(u),$$

where the functional $\psi \colon E \longrightarrow \mathbb{R}$ is defined by

$$\psi(u) = \frac{1}{2} a(u,u) - f(u).$$

Hint. It is not necessary to assume that a is symmetric. Define $a_s(u,v) := (a(u,v)+a(v,u))/2$ and observe that $a(u,u) = a_s(u,u)$. The quadratic part is continuous and strictly convex; by Corollary B.11.17, it is weakly lower semicontinuous. To make uniqueness precise, check that

$$\psi((1-t)u + tv) = (1-t)\psi(u) + t\psi(v) - \frac{t(1-t)}{2} a(v-u, v-u)$$

for all $u, v \in E$ and every $t \in (0,1)$. Also characterize the minimum by $a_s(u_0, v-u_0) \geq f(v-u_0)$ for every $v \in C$.

Exercise 30.3.3. Consider the problem

$$\begin{cases} -\Delta u = u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{30.2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain. Work in $W_0^{1,2}(\Omega)$, and interpret the Dirichlet condition through the definition of this space as the closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$. This formulation does not require assigning a pointwise trace to an arbitrary boundary.

(a) Define an energy functional associated with problem (30.2) such that, if this functional is of class C^1, its critical points are weak solutions of (30.2).

(b) Prove that this functional is of class C^1.

Hint. Use $F(t) = t^3/3$, whose derivative is t^2. The potential $|t|^3/3$ would have a different derivative. The embedding $W_0^{1,2}(\Omega) \hookrightarrow L^3(\Omega)$ allows the integral of the potential and its derivative to be controlled. Also check that this energy tends to $-\infty$ along tv as $t \to +\infty$, when $v \in C_c^\infty(\Omega) \setminus \{0\}$ is nonnegative. This distinguishes the variational formulation from the hypotheses needed to obtain a global minimum.

Exercise 30.3.4. Let $(M, \mathbf{g})$ be a nonempty compact Riemannian manifold of dimension $n \geq 1$, with smooth boundary, and suppose that every connected component has nonempty boundary. Let $1 < q < 2$. Suppose that the elliptic estimate

$$\|u\|_{H^2(M)} \leq C \|\Delta_{\mathbf{g}} u\|_{L^2(M)}$$

holds on $H_0^2(M)$. Here $H_0^2(M) = W_0^{2,2}(M)$, and ν is the outward unit normal. The conditions $u = 0$ and $\partial_\nu u = 0$ are interpreted in the trace sense: Theorem 9.4.23 requires vanishing of the traces up to order one,

and tangential derivatives of the zero trace of u also vanish. Consider the problem with clamped boundary conditions

$$\begin{cases} \Delta_{\mathbf{g}}^2 u = |u|^{q-2}u & \text{in } M, \\ u = 0, \quad \partial_\nu u = 0 & \text{on } \partial M. \end{cases} \tag{30.3}$$

(a) Rigorously formulate the notion of a weak solution in $H_0^2(M)$.

(b) Prove that the functional

$$I(u) := \frac{1}{2}\int_M |\Delta_{\mathbf{g}} u|^2 \, d\lambda_{\mathbf{g}} - \frac{1}{q}\int_M |u|^q \, d\lambda_{\mathbf{g}}$$

is well defined and belongs to $C^1(H_0^2(M), \mathbb{R})$, and compute $DI(u)$.

(c) Prove that I is coercive and sequentially weakly lower semicontinuous, and deduce that it attains its global minimum.

(d) Prove that the minimizer is nonzero and is a weak solution of (30.3).

Hint. Use the elliptic estimate to work with the norm $\|\Delta_{\mathbf{g}} u\|_{L^2(M)}$, the compact embedding of $H_0^2(M)$ into $L^q(M)$, and the fact that $q < 2$. To prove that the minimizer is nonzero, evaluate $I(tv)$ with $v \in C_c^\infty(\mathrm{Int}(M)) \setminus \{0\}$ and small $t > 0$.

Exercise 30.3.5. Let $(M, \mathbf{g})$ be a nonempty compact Riemannian manifold of dimension $n \geq 1$, with smooth boundary, such that every connected component has nonempty boundary. Let $-\Delta_{\mathbf{g},D}$ be the nonnegative self-adjoint Dirichlet realization of the Laplacian. Let $0 < s < 1$ and $1 < q < 2$, and define $H_{0,D}^s(M) := \mathcal{D}((-\Delta_{\mathbf{g},D})^{s/2})$. Suppose it is a Hilbert space under the norm

$$\|u\|_{H_{0,D}^s(M)} := \|(-\Delta_{\mathbf{g},D})^{\frac{s}{2}} u\|_{L^2(M)}$$

and that the inclusion $H_{0,D}^s(M) \hookrightarrow L^q(M)$ is compact. Consider

$$(-\Delta_{\mathbf{g},D})^s u = |u|^{q-2}u \quad \text{in } M. \tag{30.4}$$

Formulate the corresponding weak identity and prove, using the direct method, that there exists a nontrivial weak solution.

Hint. Study the functional

$$I(u) := \frac{1}{2}\int_M |(-\Delta_{\mathbf{g},D})^{\frac{s}{2}} u|^2 \, d\lambda_{\mathbf{g}} - \frac{1}{q}\int_M |u|^q \, d\lambda_{\mathbf{g}}.$$

Prove separately that it is well defined, that it is of class C^1, that it is coercive, and that it is sequentially weakly lower semicontinuous. The assumed compactness gives strong convergence in $L^q(M)$ along a subsequence of any weakly convergent sequence in $H_{0,D}^s(M)$. Apply Theorem 30.1.9. The space $H_{0,D}^s(M)$ is defined through the spectral Dirichlet realization: no pointwise trace condition is added for every $s \in (0,1)$, nor is it identified, without another result, with the closure of $C_c^\infty(\mathrm{Int}(M))$ in a different fractional norm. In the weak formulation, the powers of order $s/2$ act on u and on the test function; it is not initially required that u belong to $\mathcal{D}((-\Delta_{\mathbf{g},D})^s)$.

Exercise 30.3.6. Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold of dimension $n \geq 1$, with smooth boundary, such that every connected component has nonempty boundary. Let $1 < q < 2$, and let $\mathscr{M}\colon [0,+\infty) \longrightarrow (0,+\infty)$ be continuous and nondecreasing. Suppose that there exists $m_0 > 0$ such that $\mathscr{M}(t) \geq m_0$ for every $t \geq 0$, and define

$$\widehat{\mathscr{M}}(t) := \int_0^t \mathscr{M}(s)\,ds.$$

Consider the problem

$$\begin{cases} -\mathscr{M}\Big(\displaystyle\int_M |\nabla u|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}}\Big)\,\Delta_{\mathbf{g}} u = |u|^{q-2}u & \text{in } M, \\ u = 0 & \text{on } \partial M. \end{cases} \tag{30.5}$$

Prove that the functional

$$I(u) := \frac{1}{2}\widehat{\mathscr{M}}\Big(\int_M |\nabla u|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}}\Big) - \frac{1}{q}\int_M |u|^q \, d\lambda_{\mathbf{g}}, \qquad u \in W_0^{1,2}(M),$$

is of class C^1, compute its derivative, and prove that it has a nonzero global minimizer, which is a weak solution of (30.5).

Hint. Monotonicity of $\mathscr{M}$ implies that $\widehat{\mathscr{M}}$ is convex, whereas $\mathscr{M} \geq m_0$ gives $\widehat{\mathscr{M}}(t) \geq m_0 t$. Moreover, $\widehat{\mathscr{M}}$ is increasing because $\mathscr{M} > 0$. Combine this property with lower semicontinuity of $\|\nabla u\|^2_{L^2(M,TM)}$, and use the compact embedding $W_0^{1,2}(M) \hookrightarrow L^q(M)$ for the nonlinear term. Apply the chain rule to compute DI. To obtain a negative energy value, fix $v \neq 0$ and observe that, by continuity of $\mathscr{M}$ at 0, there exists $C_0 > 0$ such that $\widehat{\mathscr{M}}(t^2\|\nabla v\|^2_{L^2(M,TM)}) \leq C_0 t^2 \|\nabla v\|^2_{L^2(M,TM)}$ for every sufficiently small $t > 0$. Then compare t^2 with t^q.

Exercise 30.3.7. Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold of positive dimension, with or without smooth boundary, and let $K\colon M \times M \longrightarrow (0,+\infty)$ be a measurable symmetric kernel. Suppose that

$$\|u\|_{X_K}^2 := \int_M \int_M |u(x) - u(y)|^2 K(x,y)\, d\lambda_{\mathbf{g}}(x)\, d\lambda_{\mathbf{g}}(y)$$

defines a norm on a nonzero real Hilbert space $X_K(M)$ of measurable functions, identified when they agree almost everywhere, and that, for some $1 < q < 2$, the inclusion $X_K(M) \hookrightarrow L^q(M)$ is continuous, injective, and compact. The requirement that this be a norm excludes nonzero constants from $X_K(M)$, because their difference $u(x) - u(y)$ would be zero. We do not work with classes of functions modulo constants: the term $\displaystyle\int_M |u|^q \, d\lambda_{\mathbf{g}}$ would not be well defined on that quotient. Consider the bilinear form

$$\mathcal{B}_K(u,v) := \int_M \int_M (u(x) - u(y))(v(x) - v(y)) K(x,y)\, d\lambda_{\mathbf{g}}(x)\, d\lambda_{\mathbf{g}}(y).$$

(a) Define $\mathcal{L}_K\colon X_K(M) \longrightarrow X_K(M)'$ by $\langle \mathcal{L}_K u, v\rangle := \mathcal{B}_K(u,v)$ for all $u, v \in X_K(M)$, and formulate the weak equation $\mathcal{L}_K u = |u|^{q-2}u$ as an equality in $X_K(M)'$. On the right-hand side, a function in $L^{q/(q-1)}(M)$ acts on $v \in X_K(M)$ by integration.

(b) Prove that

$$I(u) := \frac{1}{2}\mathcal{B}_K(u,u) - \frac{1}{q}\int_M |u|^q \, d\lambda_{\mathbf{g}}$$

belongs to $C^1(X_K(M),\mathbb{R})$ and that its critical points are precisely the weak solutions.

(c) Apply the direct method to prove the existence of a nontrivial weak solution.

Hint. Cauchy–Schwarz on the measure space $K(x,y)\,d\lambda_{\mathbf{g}}(x)\,d\lambda_{\mathbf{g}}(y)$ gives $|\mathcal{B}_K(u,v)| \le \|u\|_{X_K}\|v\|_{X_K}$. The quadratic part is convex and weakly lower semicontinuous. For the nonlinear part, use the assumed compact embedding and differentiability of the functional $u \mapsto \|u\|_{L^q(M)}^q/q$. Nontriviality follows by comparing $I(tv)$ with $I(0)$ for a fixed element $v \in X_K(M)\setminus\{0\}$ and sufficiently small $t > 0$. The equation is asserted only against test functions in $X_K(M)$; if this space incorporates a constraint, an identity against all smooth test functions does not follow without further argument.

Exercise 30.3.8. Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold of dimension $n \ge 1$ with smooth boundary, and suppose that every connected component has nonempty boundary. Let $1 < p < \infty$ and $1 < q < p$, and consider

$$\begin{cases} -\Delta_{p,\mathbf{g}} u = |u|^{q-2}u & \text{in } M,\\ u = 0 & \text{on } \partial M, \end{cases} \qquad \Delta_{p,\mathbf{g}} u := \operatorname{div}_{\mathbf{g}}\big(|\nabla u|_{\mathbf{g}}^{p-2}\nabla u\big).$$

Formulate the weak problem in $W_0^{1,p}(M)$ and prove that it has a nontrivial weak solution by minimizing

$$I(u) := \frac{1}{p}\int_M |\nabla u|_{\mathbf{g}}^p \, d\lambda_{\mathbf{g}} - \frac{1}{q}\int_M |u|^q \, d\lambda_{\mathbf{g}}.$$

Hint. Corollary 29.1.11 and the chain rule give differentiability of both terms. Use Theorem 9.6.8 for the trivial real bundle. Corollary 9.4.29, with $m = 1$, gives compactness $W^{1,p}(M) \hookrightarrow L^p(M)$ for every $1 \le p < \infty$. Since $q < p$ and the volume is finite, its composition with $L^p(M) \hookrightarrow L^q(M)$ gives the required compact embedding. Convexity of the principal term provides its weak lower semicontinuity; $1 < p < \infty$ ensures reflexivity of $W_0^{1,p}(M)$. Apply Theorem 30.1.9 and compare t^p with t^q to prove nontriviality. In the expression for the p-Laplacian, define $|\xi|^{p-2}\xi = 0$ when $\xi = 0$, and all derivatives in the weak problem are understood in the sense of distributions.

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Palais–Smale, pseudogradients, and deformations

The direct method produces critical points by minimization. Minimax methods require a different mechanism: the sublevel sets of a functional must be deformed within bands containing no critical points. On a Hilbert space this deformation can be constructed, under sufficient regularity, using the negative gradient flow. On a general Banach space there is no canonical identification with the dual, so the gradient is replaced by a descent field retaining only the indispensable estimates.

We will construct these fields and prove the deformation lemma first on Banach spaces and then on complete Banach–Finsler manifolds. The Palais–Smale condition will appear exactly where it is necessary to prevent an almost critical sequence from escaping without converging. The exposition follows Struwe [138, chap. II, §3, pp. 81–87]: the linear case corresponds to Lemma II.3.2 and Theorem II.3.4, pp. 81–84; the intrinsic version to §§II.3.7–II.3.10 and Theorem II.3.11, pp. 85–87.

Palais and Smale: compactness in the search for saddle points

Finite-dimensional Morse theory uses gradient flow to deform sublevel sets without crossing critical points. Palais and Smale identified a compactness condition that preserves this strategy in infinite-dimensional spaces. A sequence with controlled energy and small derivative must have a convergent subsequence. When this holds, topological changes in the sublevel sets are tied to actual critical points, and minimax arguments acquire analytic content [113, 112].

31.1 Sublevel sets, critical points, and the Palais–Smale condition

Let X be a real Banach space, and let $I \in C^1(X, \mathbb{R})$. We denote by X' the continuous dual of X and by $\|DI(u)\|_{X'}$ the dual norm of the derivative of I at u.

Definition 31.1.1. (Critical sets and sublevel sets) The critical set of I, the critical set at level $\beta \in \mathbb{R}$, and

the open sublevel set at height $a \in \mathbb{R}$ are, respectively,

$$\operatorname{Crit}(I) := \{u \in X \mid DI(u) = 0\}, \qquad K_\beta := \{u \in X \mid I(u) = \beta,\ DI(u) = 0\},$$

and

$$I^{<a} := \{u \in X \mid I(u) < a\}.$$

The closed sublevel set is denoted by $I^{\le a} := \{u \in X \mid I(u) \le a\}$.

Definition 31.1.2. (Palais–Smale condition) A sequence $(u_j) \subset X$ is a *Palais–Smale sequence* for I if $(I(u_j))$ is bounded and

$$\|DI(u_j)\|_{X'} \longrightarrow 0.$$

The functional I satisfies the Palais–Smale condition, abbreviated *(PS)*, if every Palais–Smale sequence has a norm-convergent subsequence.
For $\beta \in \mathbb{R}$, we say that I satisfies $(PS)_\beta$ if every sequence (u_j) such that

$$I(u_j) \longrightarrow \beta, \qquad \|DI(u_j)\|_{X'} \longrightarrow 0$$

has a convergent subsequence in X.

The global condition (PS) implies $(PS)_\beta$ for every β. The levelwise version is exactly the compactness involved in a deformation around a fixed height. These formulations and their relation to the original condition of Palais and Smale are discussed in [138, chap. II, §2, pp. 77–80].

For $\delta, \rho > 0$ set

$$N_{\beta,\delta} := \{u \in X \,\big|\, |I(u) - \beta| < \delta,\ \|DI(u)\|_{X'} < \delta\},$$
$$U_{\beta,\rho} := \left\{u \in X \,\middle|\, \operatorname{dist}(u, K_\beta) < \rho\right\}.$$

If $K_\beta = \varnothing$, adopt the convention $U_{\beta,\rho} = \varnothing$.

Lemma 31.1.3. (Local consequence of Palais–Smale) Suppose that I satisfies $(PS)_\beta$. Then:

(a) K_β is compact;

(b) $(U_{\beta,\rho})_{\rho>0}$ is a neighborhood base of K_β;

(c) $(N_{\beta,\delta})_{\delta>0}$ is a neighborhood base of K_β.

In particular, if N is an open neighborhood of K_β, there exist $\delta, \rho \in (0, 1]$ such that

$$N \supseteq U_{\beta,2\rho} \supseteq U_{\beta,\rho} \supseteq N_{\beta,\delta}. \tag{31.1}$$

Proof. Every sequence in K_β is a $(PS)_\beta$ sequence. By hypothesis it has a convergent subsequence, and continuity of I and DI shows that its limit belongs to K_β. Since X is metrizable, K_β is compact.
Let N be an open neighborhood of K_β. If no $U_{\beta,\rho}$ were contained in N, there would exist $\rho_j \to 0^+$ and $u_j \in U_{\beta,\rho_j} \setminus N$. For each j we could choose $v_j \in K_\beta$ with $\|u_j - v_j\| < 2\rho_j$. Compactness of K_β would allow us to

assume that $v_j \to v \in K_\beta$; then also $u_j \to v$, contradicting openness of N.
If no $N_{\beta,\delta}$ were contained in N, we could choose $u_j \in N_{\beta,\frac{1}{j}} \setminus N$. This would be a $(PS)_\beta$ sequence. A subsequence would converge to a point of $K_\beta \subset N$, a contradiction. This proves the three assertions. Finally, choose first $\rho \in (0,1]$ so that $U_{\beta,2\rho} \subseteq N$ and then $\delta \in (0,1]$ so that $N_{\beta,\delta} \subseteq U_{\beta,\rho}$. □

Corollary 31.1.4. (Uniform separation of the derivative from zero) If the hypotheses of the preceding lemma hold and N is an open neighborhood of K_β, there exists $\delta > 0$ such that

$$|I(u) - \beta| < \delta, \quad u \notin N \quad \Longrightarrow \quad \|DI(u)\|_{X'} \geq \delta.$$

Proof. Take δ so that $N_{\beta,\delta} \subseteq N$. If the first two conditions hold and $\|DI(u)\|_{X'} < \delta$, then $u \in N_{\beta,\delta} \subseteq N$, a contradiction. □

31.2 Construction of pseudogradients on Banach spaces

Write

$$\widetilde{X} := X \setminus \mathrm{Crit}(I).$$

This set is open because DI is continuous.

Definition 31.2.1. (Pseudogradient vector field) A field $v\colon \widetilde{X} \longrightarrow X$ is a *pseudogradient* for I if it is locally Lipschitz and, for every $u \in \widetilde{X}$, satisfies

$$\|v(u)\| < 2\min\{\|DI(u)\|_{X'}, 1\}, \tag{31.2}$$

$$DI(u)[v(u)] > \min\{\|DI(u)\|_{X'}, 1\}\|DI(u)\|_{X'}. \tag{31.3}$$

The second inequality expresses that $-v$ is a descent direction. The first simultaneously provides control near critical points and a global speed bound. Replacing the gradient by a controlled descent direction is the functional-analytic mechanism used in [112].

Theorem 31.2.2. (Existence of pseudogradients) Every functional $I \in C^1(X,\mathbb{R})$ admits a pseudogradient vector field on $\widetilde{X}$.

Proof. Fix $u \in \widetilde{X}$ and write

$$q_u := \|DI(u)\|_{X'} > 0, \qquad m_u := \min\{q_u, 1\}.$$

By the definition of the dual norm, there exists $z_u \in X$, with $\|z_u\| = 1$, such that, after changing its sign if necessary,

$$DI(u)[z_u] > \frac{3}{4} q_u.$$

Define $w_u := \frac{3}{2} m_u z_u$. Then

$$\|w_u\| = \frac{3}{2} m_u < 2m_u, \qquad DI(u)[w_u] > \frac{9}{8} m_u q_u > m_u q_u.$$

Since DI is continuous and the inequalities are strict, there exists an open neighborhood $W_u \subseteq \widetilde{X}$ of u such

that, for every $y \in W_u$,

$$\|w_u\| < 2\min\{\|DI(y)\|_{X'}, 1\}, \tag{31.4}$$

$$DI(y)[w_u] > \min\{\|DI(y)\|_{X'}, 1\}\,\|DI(y)\|_{X'}. \tag{31.5}$$

The metric space $\widetilde{X}$ is paracompact by Theorem 24.8.8. Choose a locally finite open refinement $(A_i)_{i\in J}$ of the cover $(W_u)_{u\in\widetilde{X}}$ and, for each i, a point u_i such that $A_i \subseteq W_{u_i}$. Applying the locally finite shrinking in Lemma 24.8.7 gives an open cover $(B_i)_{i\in J}$ of $\widetilde{X}$ such that

$$\overline{B_i}^{\widetilde{X}} \subseteq A_i \qquad (i \in J).$$

This shrinking is essential: the distance functions used below have support contained in the closure of B_i, not necessarily in B_i.

For $y \in \widetilde{X}$ set

$$r_i(y) := \operatorname{dist}\bigl(y, \widetilde{X} \setminus B_i\bigr).$$

If $B_i = \widetilde{X}$, interpret $r_i \equiv 1$. Each r_i is Lipschitz, is positive precisely on B_i, and

$$\operatorname{supp}(r_i) \subseteq \overline{B_i}^{\widetilde{X}} \subseteq A_i.$$

The sum $r := \displaystyle\sum_{i\in J} r_i$ is locally finite and strictly positive. Therefore,

$$\varphi_i := \frac{r_i}{r}$$

defines a locally Lipschitz partition of unity on $\widetilde{X}$, with $\operatorname{supp}(\varphi_i) \subseteq A_i$.

Define

$$v(y) := \sum_{i\in J} \varphi_i(y)\, w_{u_i}.$$

The sum is locally finite, so v is locally Lipschitz. If $\varphi_i(y) > 0$, then $y \in A_i \subseteq W_{u_i}$ and the inequalities (31.4) and (31.5) hold for w_{u_i} at y. Taking the convex combination gives (31.2) and (31.3). This proves the theorem. □

31.3 The deformation lemma on a Banach space

Two formulations appear in the literature and should be distinguished. The first starts from a uniform derivative bound throughout an energy band and requires no compactness hypothesis. The second, due to Struwe, uses Palais–Smale to obtain this bound outside a neighborhood of the critical set. We will state and prove both, then make their logical relationship precise. The first form is customary in quantitative treatments of minimax; see, for example, [152].

31.3.1 A quantitative lemma without Palais–Smale

Theorem 31.3.1. (Quantitative deformation lemma) Let X be a real Banach space, let $I \in C^1(X,\mathbb{R})$, and fix $\beta \in \mathbb{R}$, $a > 0$, and $\sigma > 0$. Suppose that

$$|I(u) - \beta| \le 2a \quad \Longrightarrow \quad \|DI(u)\|_{X'} \ge \sigma. \tag{31.6}$$

Then, for each $\varepsilon \in (0, a)$, there exists a continuous map

$$H : [0,1] \times X \longrightarrow X, \qquad H_t(u) := H(t,u),$$

satisfying:

(i) $H_0 = \mathrm{id}_X$ and each H_t is a homeomorphism of X;

(ii) if $|I(u) - \beta| \ge 2a$, then $H_t(u) = u$ for every $t \in [0,1]$;

(iii) $t \mapsto I(H_t(u))$ is nonincreasing for every $u \in X$;

(iv)
$$H_1\big(I^{\le \beta+\varepsilon}\big) \subseteq I^{\le \beta-\varepsilon};$$

(v) if $\mu := \min\{\sigma, 1\}\sigma$, then
$$\|H_t(u) - u\| \le \frac{4\varepsilon}{\mu} \qquad (t \in [0,1],\ u \in X).$$

In particular, the conclusion is quantitative both in the decrease of energy and in the displacement of points.

Proof. Let v be the pseudogradient from Theorem 31.2.2. Choose a Lipschitz function $\chi : \mathbb{R} \to [0,1]$ such that

$$\chi(s) = 1 \quad \text{if } |s - \beta| \le a, \qquad \chi(s) = 0 \quad \text{if } |s - \beta| \ge 2a.$$

For example, one can interpolate linearly on each of the two intermediate intervals. Define

$$e(u) := \begin{cases} -\chi(I(u))\, v(u), & u \notin \mathrm{Crit}(I), \\ 0, & u \in \mathrm{Crit}(I). \end{cases} \tag{31.7}$$

We carefully verify that e is locally Lipschitz on all of X. This property is immediate outside $\mathrm{Crit}(I)$, since I is locally Lipschitz, χ is Lipschitz, and v is locally Lipschitz. By (31.6) there are no critical points in the closed band $|I - \beta| \le 2a$. Hence, if u is critical, then $|I(u) - \beta| > 2a$; continuity of I shows that $\chi \circ I$ vanishes in a neighborhood of u, and there $e \equiv 0$. Moreover, by (31.2),

$$\|e(u)\| \le 2 \qquad (u \in X). \tag{31.8}$$

Theorem 25.4.5 provides a global flow $\Phi : X \times \mathbb{R} \to X$ for e. Uniqueness of solutions implies that each Φ_s is a homeomorphism with inverse Φ_{-s}, and also that a point where e vanishes remains fixed. By the chain

rule,

$$\frac{d}{ds} I(\Phi_s(u)) = -\chi(I(\Phi_s(u)))DI(\Phi_s(u))[v(\Phi_s(u))] \le 0. \tag{31.9}$$

At critical points, the right-hand side of (31.9) is interpreted as zero. In particular, energy does not increase. Set

$$\mu := \min\{\sigma, 1\}\sigma > 0, \qquad T := \frac{2\varepsilon}{\mu}, \qquad H_t(u) := \Phi_{tT}(u).$$

Properties (i)–(iii) follow from the preceding argument. The speed bound (31.8) gives

$$\|H_t(u) - u\| \le \int_0^{tT} \|e(\Phi_s(u))\|\, ds \le 2tT \le \frac{4\varepsilon}{\mu},$$

which is (v).

Only (iv) remains to be proved. Let $u \in I^{\le \beta+\varepsilon}$. If $I(u) \le \beta - \varepsilon$, monotonicity suffices. Suppose, otherwise, that $I(u) > \beta - \varepsilon$ and $I(\Phi_T(u)) > \beta - \varepsilon$. Then, for every $s \in [0, T]$,

$$\beta - \varepsilon < I(\Phi_s(u)) \le I(u) \le \beta + \varepsilon.$$

Since $\varepsilon < a$, along the entire trajectory we have $\chi(I(\Phi_s(u))) = 1$ and, by (31.6), $\|DI(\Phi_s(u))\|_{X'} \ge \sigma$. The descent property of the pseudogradient then implies

$$\frac{d}{ds} I(\Phi_s(u)) \le -\mu.$$

Integrating between 0 and T gives

$$I(\Phi_T(u)) \le I(u) - \mu T \le \beta + \varepsilon - 2\varepsilon = \beta - \varepsilon,$$

a contradiction. Consequently, $H_1(u) = \Phi_T(u) \in I^{\le \beta-\varepsilon}$. □

Hypothesis (31.6) says that the entire band is regular, with the derivative uniformly bounded away from zero. It does not assert compactness of sequences with bounded energy. This is why the preceding lemma applies even when (PS) fails, provided that the slope bound can be verified by another argument.

31.3.2 Struwe's general lemma

The following is the quantitative form of Theorem II.3.4 in [138, pp. 83–84]. The proof uses continuation of the flow in both time directions and estimates how long a trajectory remains in the region of uniform descent, with the constants in the statement.

Theorem 31.3.2. (Struwe's deformation lemma) Let X be a real Banach space, and let $I \in C^1(X, \mathbb{R})$ be a functional satisfying (PS). Fix $\beta \in \mathbb{R}$, $\varepsilon_0 > 0$, and an open neighborhood N of K_β. Then there exist $\varepsilon \in (0, \varepsilon_0)$ and a continuous flow

$$\Phi\colon X \times \mathbb{R} \longrightarrow X, \qquad \Phi_t(u) := \Phi(u, t),$$

with the following properties:

(i) $\Phi_0 = \mathrm{id}_X$ and each Φ_t is a homeomorphism, with $\Phi_t^{-1} = \Phi_{-t}$;

(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for every $s, t \in \mathbb{R}$;

(iii) if $u \in \mathrm{Crit}(I)$ or $|I(u) - \beta| \geq \varepsilon_0$, then $\Phi_t(u) = u$ for every $t \in \mathbb{R}$;

(iv) $t \mapsto I(\Phi_t(u))$ is nonincreasing for every $u \in X$;

(v)
$$\Phi_1\left(I^{<\beta+\varepsilon} \setminus N\right) \subseteq I^{<\beta-\varepsilon}, \qquad \Phi_1\left(I^{<\beta+\varepsilon}\right) \subseteq I^{<\beta-\varepsilon} \cup N.$$

In particular, the restriction of $(\Phi_t)_{t\in\mathbb{R}}$ to $t \geq 0$ is the sublevel deformation semiflow.

Proof. By Lemma 31.1.3 we can choose $\delta, \rho \in (0, 1]$ such that

$$N \supseteq U_{\beta,2\rho} \supseteq U_{\beta,\rho} \supseteq N_{\beta,\delta}. \tag{31.10}$$

Let $\eta\colon X \longrightarrow [0, 1]$ be a locally Lipschitz function such that

$$\eta = 0 \quad \text{in } N_{\beta,\frac{\delta}{2}}, \qquad \eta = 1 \quad \text{in } X \setminus N_{\beta,\delta}. \tag{31.11}$$

It can be constructed without any additional regularity of the functional. Indeed,

$$C_{\beta,\frac{\delta}{2}} := \left\{ u \in X \,\middle|\, |I(u) - \beta| \leq \frac{\delta}{2}, \quad \|DI(u)\|_{X'} \leq \frac{\delta}{2} \right\}, \qquad A := C_{\beta,\frac{\delta}{2}}, \qquad B := X \setminus N_{\beta,\delta}$$

are closed by continuity of I and $u \mapsto \|DI(u)\|_{X'}$. Moreover,

$$A \cap B = \varnothing, \qquad N_{\beta,\frac{\delta}{2}} \subseteq A \subseteq N_{\beta,\delta}.$$

When A and B are nonempty, define

$$\eta(u) := \frac{d_A(u)}{q(u)}, \qquad d_A(u) := \mathrm{dist}(u, A), \quad d_B(u) := \mathrm{dist}(u, B), \quad q(u) := d_A(u) + d_B(u).$$

The functions d_A and d_B are 1-Lipschitz. Since A and B are closed and disjoint, $q(u) > 0$ for every u: equality $q(u) = 0$ would imply $u \in A \cap B$. Fixing $u_0 \in X$ and writing $q_0 := q(u_0) > 0$, if $\|u - u_0\| < \frac{q_0}{4}$, then

$$q(u) \geq q_0 - |d_A(u) - d_A(u_0)| - |d_B(u) - d_B(u_0)| > \frac{q_0}{2}.$$

Therefore, the denominator is locally bounded away from zero. For u and $\widetilde{u}$ in that ball, write $a = d_A(u)$, $\widetilde{a} = d_A(\widetilde{u})$, $q = q(u)$, and $\widetilde{q} = q(\widetilde{u})$. Then

$$\left| \frac{a}{q} - \frac{\widetilde{a}}{\widetilde{q}} \right| \leq \frac{|a - \widetilde{a}|}{q} + \frac{\widetilde{a}\,|q - \widetilde{q}|}{q\widetilde{q}}.$$

Here $\widetilde{a}$ is locally bounded, whereas d_A, d_B, and q are Lipschitz. Thus, η is locally Lipschitz. On A we have $d_A = 0$, and on B we have $d_B = 0$; disjointness and closedness show, respectively, that the other summand is positive. Consequently, $\eta = 0$ on A and $\eta = 1$ on B, proving (31.11).
If $A = \varnothing \neq B$, take $\eta \equiv 1$. If $B = \varnothing$, take $\eta \equiv 0$, including the case where both are empty. Since $N_{\beta,\frac{\delta}{2}} \subseteq A$, these conventions preserve the two required values in (31.11).

Set

$$a := \min\{\varepsilon_0, \frac{\delta}{4}\}$$

and define the Lipschitz cutoff $\chi : \mathbb{R} \longrightarrow [0,1]$ by

$$\chi(s) := \begin{cases} 1, & |s-\beta| \le \frac{a}{2}, \\ 2 - \dfrac{2|s-\beta|}{a}, & \frac{a}{2} < |s-\beta| < a, \\ 0, & |s-\beta| \ge a. \end{cases}$$

Let v be a pseudogradient provided by Theorem 31.2.2. Define the truncated field

$$e(u) := \begin{cases} -\eta(u)\chi(I(u))v(u), & u \notin \mathrm{Crit}(I), \\ 0, & u \in \mathrm{Crit}(I). \end{cases} \tag{31.12}$$

The field e is locally Lipschitz. On $\widetilde{X}$, the pseudogradient v is so by construction, η is locally Lipschitz, and $\chi \circ I$ is as well, since χ is Lipschitz and I is of class C^1 and therefore locally Lipschitz. On a ball where all three factors are bounded and have finite Lipschitz constants, the estimate

$$\|abc - a'b'c'\| \le |a-a'||b||c\| + |a'||b-b'||c\| + |a'||b'||c-c'\|$$

for $a, a', b, b' \in \mathbb{R}$ and $c, c' \in X$, applied to $a = \eta(u)$, $b = \chi(I(u))$, and $c = v(u)$, proves that their product is locally Lipschitz. If u is critical and $|I(u)-\beta| < \frac{\delta}{2}$, the cutoff η vanishes in a neighborhood of u. If $|I(u)-\beta| \ge \frac{\delta}{2}$, then $|I(u)-\beta| > a$ and $\chi \circ I$ vanishes in a neighborhood of u. Consequently, e is identically zero near every critical point; its regularity is not being deduced through a pointwise extension of v. Moreover,

$$\|e(u)\| \le 2 \qquad (u \in X). \tag{31.13}$$

Applying Theorem 25.4.5 to the autonomous problem

$$\frac{d}{dt}\Phi_t(u) = e(\Phi_t(u)), \qquad \Phi_0(u) = u, \tag{31.14}$$

gives a solution defined for all $t \in \mathbb{R}$, continuous with respect to (u,t). Uniqueness gives

$$\Phi_{t+s} = \Phi_t \circ \Phi_s, \qquad \Phi_t \circ \Phi_{-t} = \Phi_0 = \mathrm{id}_X .$$

Thus, $\Phi_t^{-1} = \Phi_{-t}$ and each Φ_t is a homeomorphism. Existence only for $t \ge 0$ would have produced a semiflow but would not suffice to conclude surjectivity of Φ_t.

If u is critical or $|I(u)-\beta| \ge \varepsilon_0$, then $e(u) = 0$; uniqueness implies that the solution starting at u is constant. By the chain rule and the pseudogradient properties,

$$\begin{aligned} \frac{d}{dt} I(\Phi_t(u)) &= DI(\Phi_t(u)) \left[\frac{d}{dt}\Phi_t(u)\right] \\ &= -\eta(\Phi_t(u))\chi(I(\Phi_t(u)))DI(\Phi_t(u))[v(\Phi_t(u))] \le 0. \end{aligned} \tag{31.15}$$

At a critical point the right-hand side is interpreted as zero. This proves the first four properties.
Now choose

$$0 < \varepsilon < \min\left\{\frac{a}{2}, \frac{\rho\delta^2}{4}\right\}. \tag{31.16}$$

Let $u \in I^{<\beta+\varepsilon}$ and suppose that $I(\Phi_1(u)) \geq \beta - \varepsilon$. Monotonicity implies, for $0 \leq t \leq 1$,

$$\beta - \varepsilon \leq I(\Phi_t(u)) < \beta + \varepsilon.$$

Hence $\chi(I(\Phi_t(u))) = 1$. Define

$$A_u := \{t \in [0,1] \mid \Phi_t(u) \notin N_{\beta,\delta}\}.$$

If $t \in A_u$, then $|I(\Phi_t(u)) - \beta| < \varepsilon < \delta$, so exclusion from $N_{\beta,\delta}$ forces $\|DI(\Phi_t(u))\|_{X'} \geq \delta$. Since $\delta \leq 1$ and $\eta = 1$ outside $N_{\beta,\delta}$, from (31.15) we obtain

$$\frac{d}{dt} I(\Phi_t(u)) \leq -\delta^2 \qquad (t \in A_u).$$

Integrating,

$$I(\Phi_1(u)) < \beta + \varepsilon - \delta^2 |A_u|, \tag{31.17}$$

where $|A_u|$ denotes Lebesgue measure.
Suppose that $u \notin N$ or $\Phi_1(u) \notin N$. Then

$$|A_u| \geq \frac{\rho}{2}. \tag{31.18}$$

Indeed, if $K_\beta = \varnothing$, δ can be chosen with $N_{\beta,\delta} = \varnothing$, and the assertion is immediate because $\rho \leq 1$. Suppose $K_\beta \neq \varnothing$. By (31.10), an endpoint outside N is outside $U_{\beta,2\rho}$, whereas $N_{\beta,\delta} \subseteq U_{\beta,\rho}$. The function $x \mapsto \operatorname{dist}(x, K_\beta)$ is 1-Lipschitz, and (31.13) gives

$$\|\Phi_t(u) - \Phi_s(u)\| \leq \int_s^t \|e(\Phi_\tau(u))\|\, d\tau \leq 2|t-s|.$$

Thus, passing from the exterior of $U_{\beta,2\rho}$ to $N_{\beta,\delta} \subseteq U_{\beta,\rho}$, or making the reverse journey, requires at least $\frac{\rho}{2}$ units of time. Before the first entry, or after the last exit, respectively, the trajectory remains outside $N_{\beta,\delta}$. If it never enters, $A_u = [0,1]$. This proves (31.18).
Estimates (31.17), (31.18), and (31.16) yield

$$I(\Phi_1(u)) < \beta + \varepsilon - \frac{\rho\delta^2}{2} < \beta - \varepsilon,$$

contradicting the preceding assumption. If $u \in I^{<\beta+\varepsilon} \setminus N$, apply the argument to the initial endpoint and obtain the first inclusion in (v). For an arbitrary $u \in I^{<\beta+\varepsilon}$, either $\Phi_1(u) \in N$ or the same argument applied to the final endpoint gives $\Phi_1(u) \in I^{<\beta-\varepsilon}$. This proves the second inclusion. □

Remark 31.3.3. (Levelwise hypotheses) Condition (PS) was used only in Lemma 31.1.3. Therefore, in the preceding theorem it suffices to assume $(PS)_\beta$. If $K_\beta = \varnothing$, there exists $\delta > 0$ such that $N_{\beta,\delta} = \varnothing$; in this case we may take $N = \varnothing$ and obtain a uniform decrease in energy in a band around β.

31.3.3 Relationship between the two deformation lemmas

The difference between the two statements lies in the source of the lower bound for $\|DI\|$. The quantitative lemma assumes it; Struwe's lemma deduces it from $(PS)_\beta$ outside a neighborhood of K_β. The next result formalizes this relationship and specifies exactly which part of Palais–Smale is used.

Proposition 31.3.4. (From compactness to a slope bounded away from zero) Let $I \in C^1(X,\mathbb{R})$ and let $\beta \in \mathbb{R}$.

(a) If I satisfies $(PS)_\beta$ and $K_\beta = \varnothing$, there exist $a, \sigma > 0$ such that

$$|I(u) - \beta| \le 2a \quad \Longrightarrow \quad \|DI(u)\|_{X'} \ge \sigma.$$

Therefore, Lemma 31.3.1 uniformly deforms the sublevel set just above β into one just below it.

(b) If I satisfies $(PS)_\beta$ and N is an open neighborhood of K_β, there exist $a, \sigma > 0$ such that

$$|I(u) - \beta| \le 2a, \qquad u \notin N \quad \Longrightarrow \quad \|DI(u)\|_{X'} \ge \sigma.$$

Struwe's lemma is the relative version of the same mechanism: an additional cutoff stops the flow near K_β and preserves uniform decrease outside N.

Proof. For (a), suppose there are no such a and σ. For each $j \in \mathbb{N}$ we could choose $u_j \in X$ with

$$|I(u_j) - \beta| \le \frac{2}{j}, \qquad \|DI(u_j)\|_{X'} < \frac{1}{j}.$$

The sequence (u_j) would be a $(PS)_\beta$ sequence. A subsequence would converge to some $u \in X$, and continuity of I and DI would give $I(u) = \beta$ and $DI(u) = 0$, contradicting $K_\beta = \varnothing$.
For (b), Corollary 31.1.4 provides $\delta > 0$ such that

$$|I(u) - \beta| < \delta, \qquad u \notin N \quad \Longrightarrow \quad \|DI(u)\|_{X'} \ge \delta.$$

It suffices to take, for example, $a = \delta/4$ and $\sigma = \delta$. The proof of Theorem 31.3.2 then shows why it does not suffice to turn off the field inside N: the nested neighborhoods $U_{\beta,\rho} \subset U_{\beta,2\rho}$ are used to control also the trajectories entering or leaving N. □

Thus, the statements are not equivalent without additional hypotheses. If $K_\beta \neq \varnothing$, the global hypothesis of the quantitative lemma is impossible, because DI vanishes on K_β. If $K_\beta = \varnothing$, however, $(PS)_\beta$ turns the quantitative lemma into the case of Struwe's lemma needed for minimax contradiction arguments.

31.4 The intrinsic Banach–Finsler formulation

Now consider a reversible Banach–Finsler manifold without boundary M with the regularity and local compatibility of norms from Section 26.3 in Chapter 26. Denote by $\|\xi\|_p$ the norm of $\xi \in T_pM$, by d the length distance, and by $\|\alpha\|_{p,*}$ the dual norm of $\alpha \in T_p'M$ defined in Definition 26.3.3. In particular,

$$\|\alpha\|_{p,*} = \sup\left\{ |\alpha(\xi)| \,\middle|\, \xi \in T_pM,\ \|\xi\|_p \le 1 \right\}.$$

Reversibility is the property of the structure F specified in Definition 26.3.1. When class $C^{1,1}$ is required, we will use the atlas of Definition 24.1.5, in accordance with Definition 26.3.7. Locally Lipschitz fields and their flows are understood as in Remark 25.4.3. Since all arguments apply componentwise, we assume below that M is connected.

Definition 31.4.1. (Intrinsic Palais–Smale condition) Let M be a reversible Banach–Finsler manifold without boundary, and let $I \in C^1(M,\mathbb{R})$. A sequence $(p_j) \subset M$ is a Palais–Smale sequence if $(I(p_j))$ is bounded and

$$\|dI_{p_j}\|_{p_j,*} \longrightarrow 0.$$

The functional satisfies (PS) if every Palais–Smale sequence has a convergent subsequence in M. It satisfies $(PS)_\beta$ if the same conclusion holds for every sequence such that

$$I(p_j) \longrightarrow \beta, \qquad \|dI_{p_j}\|_{p_j,*} \longrightarrow 0.$$

Define intrinsically

$$\begin{aligned} K_\beta &:= \{p \in M \mid I(p) = \beta,\ dI_p = 0\},\\ N_{\beta,\delta} &:= \{p \in M \mid |I(p)-\beta| < \delta,\ \|dI_p\|_{p,*} < \delta\},\\ U_{\beta,\rho} &:= \{p \in M \mid d(p,K_\beta) < \rho\}. \end{aligned}$$

We also retain the notation $I^{<a} := \{p \in M \mid I(p) < a\}$ for open sublevel sets and $I^{\le a} := \{p \in M \mid I(p) \le a\}$ for closed ones. Local compatibility of the Finsler structure implies that d induces the topology of M and that $p \mapsto \|dI_p\|_{p,*}$ is continuous. Hence the argument of Lemma 31.1.3, which uses only metrizability, continuity, and the definition of $(PS)_\beta$, gives the following result.

Lemma 31.4.2. (Neighborhoods of the critical set on a Finsler manifold) Let M be a reversible Banach–Finsler manifold without boundary, and let $I \in C^1(M,\mathbb{R})$. If I satisfies $(PS)_\beta$, then K_β is compact and the families $(U_{\beta,\rho})_{\rho>0}$ and $(N_{\beta,\delta})_{\delta>0}$ are neighborhood bases of K_β. In particular, for every open neighborhood N of K_β, there exist $\delta,\rho \in (0,1]$ such that

$$N \supseteq U_{\beta,2\rho} \supseteq U_{\beta,\rho} \supseteq N_{\beta,\delta}.$$

Proof. A sequence in K_β is a $(PS)_\beta$ sequence, and by continuity every subsequential limit remains in K_β; therefore, K_β is compact. If a neighborhood N contained no $N_{\beta,\delta}$, a choice $p_j \in N_{\beta,\frac{1}{j}} \setminus N$ would give a $(PS)_\beta$ sequence whose convergent subsequence would have a limit in $K_\beta \subset N$, a contradiction. The assertion for $U_{\beta,\rho}$ follows from compactness of K_β and the same metric argument: for points at decreasing distance from K_β, choose nearby points in K_β and extract a convergent subsequence. The final chain is obtained by choosing first ρ and then δ. □

Definition 31.4.3. (Finsler pseudogradient) Let M be a reversible Banach–Finsler manifold without boundary, and let $I \in C^1(M,\mathbb{R})$. Let

$$\widetilde{M} := \{p \in M \mid dI_p \neq 0\}.$$

A locally Lipschitz vector field $\mathbf{v}\colon \widetilde{M} \longrightarrow TM$ is a *Finsler pseudogradient* of I if $\mathbf{v}(p) \in T_pM$ and

$$\|\mathbf{v}(p)\|_p < 2\min\{\|dI_p\|_{p,*}, 1\},$$
$$dI_p[\mathbf{v}(p)] > \min\{\|dI_p\|_{p,*}, 1\}\,\|dI_p\|_{p,*}$$

for every $p \in \widetilde{M}$.

Theorem 31.4.4. (Existence of a Finsler pseudogradient) Let M be a paracompact Banach–Finsler manifold of class $C^{1,1}$. Then every $I \in C^1(M,\mathbb{R})$ admits a Finsler pseudogradient on $\widetilde{M}$.

Proof. Fix $p \in \widetilde{M}$ and write $q_p := \|dI_p\|_{p,*}$ and $m_p := \min\{q_p, 1\}$. By the definition of the dual norm, there exists $\xi_p \in T_pM$, $\|\xi_p\|_p = 1$, such that

$$dI_p[\xi_p] > \frac{3}{4}q_p.$$

Therefore,

$$\left\|\frac{3}{2}m_p\xi_p\right\|_p = \frac{3}{2}m_p < 2m_p, \qquad dI_p\left[\frac{3}{2}m_p\xi_p\right] > \frac{9}{8}m_pq_p > m_pq_p.$$

Choose a chart (U,φ) of the $C^{1,1}$ atlas about p, with $U \subseteq \widetilde{M}$, and the tangent trivialization $\tau_q := d\varphi_q\colon T_qM \longrightarrow X$. Define

$$\mathbf{w}_p(q) := \tau_q^{-1}\tau_p\left(\frac{3}{2}m_p\xi_p\right), \qquad q \in U.$$

Its components in the chart φ are constant, and $\mathbf{w}_p(p) = \frac{3}{2}m_p\xi_p$. By admissibility of the Finsler structure, Definition 26.3.1, and comparison (26.14), after shrinking U there exist constants $c_p, C_p > 0$ such that

$$c_p\|\tau_q(z)\|_X \le \|z\|_q \le C_p\|\tau_q(z)\|_X$$

for every point q in the domain and every $z \in T_qM$. The functions $q \mapsto \|\mathbf{w}_p(q)\|_q$ and $q \mapsto dI_q[\mathbf{w}_p(q)]$ are continuous by continuity of F, dI, and the local field in this trivialization. Moreover, $q \mapsto \|dI_q\|_{q,*}$ is continuous by Proposition 26.3.4, as is its minimum with 1. Since both inequalities at p are strict, we can shrink the domain again to a neighborhood W_p on which $\mathbf{w}_p$ satisfies

$$\|\mathbf{w}_p(q)\|_q < 2\min\{\|dI_q\|_{q,*}, 1\},$$
$$dI_q[\mathbf{w}_p(q)] > \min\{\|dI_q\|_{q,*}, 1\}\,\|dI_q\|_{q,*}.$$

In the chart φ, the field has constant components and is locally Lipschitz. Remark 25.4.3 shows that this property is preserved in all charts of the $C^{1,1}$ atlas.

The open set $\widetilde{M}$, with the restricted Finsler distance, is metrizable by Theorem 26.3.5 and paracompact by Theorem 24.8.8. Choose a locally finite open refinement $(A_i)_{i\in J}$ of $(W_p)_{p\in\widetilde{M}}$ and, by Lemma 24.8.7, a shrinking $(B_i)_{i\in J}$ with $\overline{B_i}^{\widetilde{M}} \subseteq A_i$. Define

$$r_i(q) := d\big(q, \widetilde{M}\setminus B_i\big), \qquad r(q) := \sum_{i\in J} r_i(q), \qquad \varphi_i(q) := \frac{r_i(q)}{r(q)}.$$

If $B_i = \widetilde{M}$, take $r_i \equiv 1$. Each r_i is Lipschitz with respect to d and positive precisely on B_i. The sum is locally

finite and positive because $(B_i)_{i\in J}$ covers $\widetilde{M}$. Near each point, r is bounded away from zero; hence the quotients φ_i are locally d-Lipschitz. Theorem 26.3.5 transfers this regularity to the charts. Moreover,

$$\sum_{i\in J}\varphi_i = 1, \qquad \operatorname{supp}(\varphi_i) \subseteq \overline{B_i}^{\widetilde{M}} \subseteq A_i.$$

For each i choose p_i with $A_i \subseteq W_{p_i}$ and set

$$\mathbf{v}(q) := \sum_{i\in J}\varphi_i(q)\mathbf{w}_{p_i}(q).$$

In this formula, each product $\varphi_i\mathbf{w}_{p_i}$ is extended by zero outside W_{p_i}. This extension is locally Lipschitz on $\widetilde{M}$: inside W_{p_i} it is a product of locally Lipschitz functions and fields, which are locally bounded in coordinates; outside W_{p_i} it is identically zero on a neighborhood, since $\operatorname{supp}(\varphi_i) \subseteq A_i \subseteq W_{p_i}$ and the support is closed in $\widetilde{M}$. Each sum is finite on a neighborhood of the point under consideration. Therefore, $\mathbf{v}$ is locally Lipschitz; since it is a convex combination of directions satisfying the two inequalities on the corresponding support, it is a Finsler pseudogradient. □

Completeness enters at a single point: it turns uniform speed control into continuation of integral curves.

Proposition 31.4.5. (Bounded fields on complete Finsler manifolds) Let (M, d) be a reversible complete Banach–Finsler manifold without boundary of class $C^{1,1}$, and let $\mathbf{e}$ be a vector field locally Lipschitz in charts. If there exists $C > 0$ such that

$$\|\mathbf{e}(p)\|_p \le C \qquad (p \in M),$$

then all its maximal integral curves are defined on $\mathbb{R}$. The flow $\Phi\colon M\times\mathbb{R} \longrightarrow M$ is continuous, satisfies

$$\Phi_{t+s} = \Phi_t \circ \Phi_s, \qquad \Phi_t^{-1} = \Phi_{-t},$$

and each Φ_t is a homeomorphism.

Proof. Remark 25.4.3 provides local existence, uniqueness, and continuous dependence in the chosen atlas. Let $\gamma\colon (a,b) \longrightarrow M$ be a maximal integral curve. For $a < s < t < b$, the definition of the length distance gives

$$d(\gamma(s),\gamma(t)) \le L(\gamma\restriction_{[s,t]}) = \int_s^t \|\mathbf{e}(\gamma(\tau))\|_{\gamma(\tau)}\,d\tau \le C|t-s|. \tag{31.19}$$

If $b < +\infty$, the estimate shows that $\gamma(t)$ is a Cauchy family as $t \to b^-$. Completeness provides $p_b \in M$ such that $\gamma(t) \to p_b$. Since the Finsler distance induces the manifold topology, we can work in a chart about p_b. Local existence for differential equations in Banach spaces produces an integral curve starting at p_b at time b. Continuity of $\mathbf{e}$ and the integral equation allow it to be concatenated with γ, and local uniqueness shows that this concatenation extends the original curve. This contradicts maximality. Thus, $b = +\infty$. Applying the same argument as $t \to a^+$ gives $a = -\infty$.

Continuous dependence on the initial value produces a continuous flow. Uniqueness of integral curves gives the group law and, taking $s = -t$, the identity $\Phi_t^{-1} = \Phi_{-t}$. Hence each Φ_t is a homeomorphism. Observe that the argument uses metric completeness at (31.19), not compactness of a closed ball. □

Theorem 31.4.6. (Banach–Finsler deformation lemma) Let M be a reversible, complete, paracompact Banach–Finsler manifold without boundary, of class $C^{1,1}$, satisfying the hypotheses of Theorem 31.4.4. Let $I \in C^1(M,\mathbb{R})$ be a functional satisfying (PS). Given $\beta \in \mathbb{R}$, $\varepsilon_0 > 0$, and an open neighborhood N of K_β, there exist $\varepsilon \in (0,\varepsilon_0)$ and a continuous flow $\Phi\colon M \times \mathbb{R} \longrightarrow M$ such that:

(i) each Φ_t is a homeomorphism, $\Phi_t^{-1} = \Phi_{-t}$, and $\Phi_{t+s} = \Phi_t \circ \Phi_s$;

(ii) $\Phi_t(p) = p$ if $dI_p = 0$ or $|I(p) - \beta| \geq \varepsilon_0$;

(iii) $I(\Phi_t(p))$ is nonincreasing in t;

(iv)
$$\Phi_1\left(I^{<\beta+\varepsilon} \setminus N\right) \subseteq I^{<\beta-\varepsilon}, \qquad \Phi_1\left(I^{<\beta+\varepsilon}\right) \subseteq I^{<\beta-\varepsilon} \cup N.$$

Here too it suffices to assume $(PS)_\beta$.

Proof. Lemma 31.4.2 provides $\delta, \rho \in (0,1]$ such that

$$N \supseteq U_{\beta,2\rho} \supseteq U_{\beta,\rho} \supseteq N_{\beta,\delta}.$$

Take
$$a := \min\{\varepsilon_0, \frac{\delta}{4}\}$$
and construct the spatial cutoff entirely from the Finsler distance. Let

$$C^M_{\beta,\frac{\delta}{2}} := \left\{ p \in M \,\middle|\, |I(p) - \beta| \leq \frac{\delta}{2}, \quad \|dI_p\|_{p,*} \leq \frac{\delta}{2} \right\}, \qquad A_M := C^M_{\beta,\frac{\delta}{2}}, \qquad B_M := M \setminus N_{\beta,\delta}.$$

Continuity of I and $p \mapsto \|dI_p\|_{p,*}$ shows that A_M and B_M are closed. Moreover,

$$A_M \cap B_M = \varnothing, \qquad N_{\beta,\frac{\delta}{2}} \subseteq A_M \subseteq N_{\beta,\delta}.$$

If both sets are nonempty, set

$$\eta(p) := \frac{d^M_A(p)}{q_M(p)}, \quad d^M_A(p) := d(p, A_M), \quad d^M_B(p) := d(p, B_M), \quad q_M(p) := d^M_A(p) + d^M_B(p). \tag{31.20}$$

The functions d^M_A and d^M_B are 1-Lipschitz with respect to d. Closedness and disjointness imply $q_M(p) > 0$ for every p. Fixing $p_0 \in M$ and writing $q_0 := q_M(p_0) > 0$, if $d(p,p_0) < \frac{q_0}{4}$, then

$$q_M(p) \geq q_0 - |d^M_A(p) - d^M_A(p_0)| - |d^M_B(p) - d^M_B(p_0)| > \frac{q_0}{2}.$$

Thus the denominator is locally bounded away from zero. For p and $\widetilde{p}$ in that ball, with $a = d^M_A(p)$, $\widetilde{a} = d^M_A(\widetilde{p})$, $q = q_M(p)$, and $\widetilde{q} = q_M(\widetilde{p})$, we have

$$\left| \frac{a}{q} - \frac{\widetilde{a}}{\widetilde{q}} \right| \leq \frac{|a - \widetilde{a}|}{q} + \frac{\widetilde{a}\,|q - \widetilde{q}|}{q\widetilde{q}}.$$

The factor $\widetilde{a}$ is locally bounded, so η is locally d-Lipschitz. Local equivalence between d and the norm of a Finsler chart gives the required local regularity in charts. Since $d^M_A = 0$ on A_M and $d^M_B = 0$ on B_M, while the

other summand is positive, we obtain $\eta = 0$ on A_M and $\eta = 1$ on B_M.
If $A_M = \varnothing \neq B_M$, take $\eta \equiv 1$. If $B_M = \varnothing$, take $\eta \equiv 0$, including the case where both are empty. In every case, η equals zero on $N_{\beta,\frac{\delta}{2}}$ and one outside $N_{\beta,\delta}$.
Let $\chi\colon \mathbb{R} \longrightarrow [0,1]$ be the Lipschitz function already defined in the Banach proof: it equals one on $[\beta - \frac{a}{2}, \beta + \frac{a}{2}]$ and zero outside $(\beta - a, \beta + a)$.
Let $\mathbf{v}$ be the pseudogradient from Theorem 31.4.4 and define

$$\mathbf{e}(p) := \begin{cases} -\eta(p)\chi(I(p))\mathbf{v}(p), & dI_p \neq 0, \\ 0, & dI_p = 0. \end{cases}$$

On the open set $\widetilde{M}$, the three factors η, $\chi \circ I$, and $\mathbf{v}$ are locally Lipschitz in charts and locally bounded; the product inequality used in the Banach proof shows that $\mathbf{e}$ is locally Lipschitz there. Critical points remain to be checked. If $dI_p = 0$ and $|I(p) - \beta| < \frac{\delta}{2}$, then p belongs to the open set $N_{\beta,\frac{\delta}{2}}$, where $\eta = 0$. If $dI_p = 0$ and $|I(p) - \beta| \geq \frac{\delta}{2}$, then $|I(p) - \beta| \geq \frac{\delta}{2} > a$; by continuity, $\chi \circ I$ vanishes on a neighborhood of p. In both cases $\mathbf{e}$ is identically zero on a neighborhood of the critical point. Therefore, the definition by cases is not a mere pointwise extension of the pseudogradient, and $\mathbf{e}$ is locally Lipschitz on all of M. Moreover,

$$\|\mathbf{e}(p)\|_p \leq 2. \tag{31.21}$$

Proposition 31.4.5 gives a global flow of homeomorphisms defined in both time directions. In particular, completeness is used explicitly in the estimate

$$d(\Phi_s(p), \Phi_t(p)) \leq \int_s^t \|\mathbf{e}(\Phi_\tau(p))\|_{\Phi_\tau(p)}\, d\tau \leq 2|t - s|. \tag{31.22}$$

The intrinsic chain rule gives

$$\begin{aligned} \frac{d}{dt} I(\Phi_t(p)) &= dI_{\Phi_t(p)}\big[\partial_t \Phi_t(p)\big] \\ &= -\eta(\Phi_t(p))\chi(I(\Phi_t(p)))dI_{\Phi_t(p)}[\mathbf{v}(\Phi_t(p))] \leq 0. \end{aligned}$$

Choose

$$0 < \varepsilon < \min\left\{\frac{a}{2}, \frac{\rho\delta^2}{4}\right\}.$$

If $p \in I^{<\beta+\varepsilon}$ and $I(\Phi_1(p)) \geq \beta - \varepsilon$, the entire trajectory between times zero and one remains in the band $|I - \beta| \leq \varepsilon < a/2$, where $\chi = 1$. For

$$A_p := \{t \in [0,1] \mid \Phi_t(p) \notin N_{\beta,\delta}\}$$

we have, using $\delta \leq 1$,

$$\frac{d}{dt} I(\Phi_t(p)) \leq -\delta^2 \quad (t \in A_p), \qquad I(\Phi_1(p)) < \beta + \varepsilon - \delta^2 |A_p|.$$

If $p \notin N$ or $\Phi_1(p) \notin N$, then $|A_p| \geq \frac{\rho}{2}$. Indeed, an endpoint outside N is outside $U_{\beta,2\rho}$, whereas $N_{\beta,\delta} \subseteq U_{\beta,\rho}$. Distance to K_β is 1-Lipschitz, and (31.22) shows that crossing the strip between the two neighborhoods requires at least $\frac{\rho}{2}$ units of time. During that interval the trajectory lies outside $N_{\beta,\delta}$. If $K_\beta = \varnothing$, choose δ

with $N_{\beta,\delta} = \varnothing$, and the conclusion is immediate. Therefore,

$$I(\Phi_1(p)) < \beta + \varepsilon - \frac{\rho\delta^2}{2} < \beta - \varepsilon.$$

Applying this contradiction first when $p \notin N$ and then when $\Phi_1(p) \notin N$ gives the two inclusions. The fixed-point properties follow from uniqueness of the flow at zeros of $\mathbf{e}$. □

This theorem is the version of Theorem II.3.11 in [138, p. 87]. The preceding proof displays its three global ingredients separately: Palais–Smale sequential compactness, the locally finite partition producing the pseudogradient, and Finsler completeness, which prevents trajectories from escaping in finite time.

31.5 Motivation and the mountain pass theorem

The direct method seeks the bottom of the energy. In many superlinear problems, however, the functional is not bounded below: along certain rays the potential part eventually dominates and the energy tends to $-\infty$. At the same time, the origin may be a strict local minimum. To travel from this local valley to a region of negative energy, every path must climb over a barrier. The mountain pass selects, among all such paths, the smallest possible maximum height. It does not minimize I over X; it minimizes the maximum of I along each path.

Ambrosetti and Rabinowitz: a critical point by minimax

The mountain pass theorem, introduced by Ambrosetti and Rabinowitz, turned this geometric picture into an existence principle in infinite dimensions [8]. Topology forces paths to cross a barrier; the deformation lemma shows that the lowest crossing height cannot be regular; and Palais–Smale prevents almost critical sequences from escaping. The three ingredients are distinct, and each plays a precise role.

31.5.1 Paths, barriers, and the minimax level

Definition 31.5.1. (Path minimax level) Let $u_0, u_1 \in X$. The class of paths with fixed endpoints is

$$\Gamma(u_0,u_1) := \{\gamma \in C([0,1],X) \mid \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$

For $I \in C^1(X,\mathbb{R})$, its minimax level is

$$c(u_0,u_1) := \inf_{\gamma\in\Gamma(u_0,u_1)} \max_{t\in[0,1]} I(\gamma(t)). \tag{31.23}$$

The class is nonempty because it contains the segment $t \mapsto (1-t)u_0 + tu_1$. For each path, the maximum of $I \circ \gamma$ exists: it is the maximum of a continuous function on the compact set $[0,1]$. Moreover, all these maxima are at least $\max\{I(u_0), I(u_1)\}$, and the segment between the endpoints provides a finite upper bound for their infimum. Thus the number in (31.23) is real.

Lemma 31.5.2. (Forced crossing of the barrier) Let $u_0, u_1 \in X$ and $r > 0$, with $\|u_1 - u_0\| > r$. Every $\gamma \in \Gamma(u_0,u_1)$ intersects the sphere

$$S_r(u_0) := \{u \in X \mid \|u - u_0\| = r\}.$$

Consequently, if

$$b := \inf_{u \in S_r(u_0)} I(u),$$

then $c(u_0, u_1) \geq b$.

Proof. The function $h(t) := \|\gamma(t) - u_0\|$ is continuous, $h(0) = 0$, and $h(1) = \|u_1 - u_0\| > r$. The intermediate value theorem gives $t_r \in (0,1)$ with $h(t_r) = r$. Hence,

$$\max_{t \in [0,1]} I(\gamma(t)) \geq I(\gamma(t_r)) \geq b.$$

Taking the infimum over γ proves the last assertion. □

31.5.2 The abstract theorem and its proof

Theorem 31.5.3. (Mountain pass theorem) Let X be a real Banach space, and let $I \in C^1(X, \mathbb{R})$. Suppose there exist $u_0, u_1 \in X$ and $r > 0$ such that

$$\|u_1 - u_0\| > r$$

and

$$\max\{I(u_0), I(u_1)\} < b := \inf_{\|u - u_0\| = r} I(u). \tag{31.24}$$

Let

$$c := c(u_0, u_1) = \inf_{\gamma \in \Gamma(u_0, u_1)} \max_{t \in [0,1]} I(\gamma(t)).$$

Then $c \geq b$ and, if I satisfies $(PS)_c$, there exists $u_c \in X$ such that

$$I(u_c) = c, \qquad DI(u_c) = 0.$$

Proof. Lemma 31.5.2 gives $c \geq b$. On the other hand, the path class contains the segment between the endpoints, on which the energy has a finite maximum. Therefore, $c \in \mathbb{R}$. Moreover,

$$d := c - \max\{I(u_0), I(u_1)\} > 0. \tag{31.25}$$

Suppose, for a contradiction, that $K_c = \varnothing$. Since I satisfies $(PS)_c$, Proposition 31.3.4 provides $a_0, \sigma > 0$ such that

$$|I(u) - c| \leq 2a_0 \quad \Longrightarrow \quad \|DI(u)\|_{X'} \geq \sigma.$$

Shrinking a_0 preserves this property. Take

$$0 < a < \min\left\{a_0, \frac{d}{2}\right\} \qquad \text{and} \qquad 0 < \varepsilon < a.$$

Lemma 31.3.1, applied with $\beta = c$, produces a deformation H. By (31.25),

$$|I(u_i) - c| = c - I(u_i) \geq d > 2a \qquad (i = 0, 1),$$

so H_1 fixes both endpoints.

The definition of the infimum ensures that there exists $\gamma_\varepsilon \in \Gamma(u_0,u_1)$ such that

$$\max_{t\in[0,1]} I(\gamma_\varepsilon(t)) < c+\varepsilon. \tag{31.26}$$

Define

$$\widetilde{\gamma}_\varepsilon(t) := H_1(\gamma_\varepsilon(t)).$$

Continuity of H_1 and the fact that the endpoints are fixed show that $\widetilde{\gamma}_\varepsilon \in \Gamma(u_0,u_1)$. Moreover, (31.26) and the conclusion of the quantitative lemma give

$$I(\widetilde{\gamma}_\varepsilon(t)) \le c-\varepsilon \qquad (0\le t\le 1).$$

Consequently,

$$c \le \max_{t\in[0,1]} I(\widetilde{\gamma}_\varepsilon(t)) \le c-\varepsilon,$$

which is impossible. Therefore, $K_c \neq \varnothing$ and any of its elements is the desired point u_c. □

The proof displays the role of each hypothesis. The topological barrier puts c above the endpoint energies; $(PS)_c$ turns the absence of critical points into a uniform slope bound; and the deformation lemma lowers an almost optimal path without moving its endpoints, contradicting the definition of c. One may also directly apply Struwe's lemma 31.3.2 with $N=\varnothing$; Proposition 31.3.4 explains why both arguments use the same mechanism in this case.

Corollary 31.5.4. (Usual form with the origin and negative energy) Let $I \in C^1(X,\mathbb{R})$ and suppose that:

(i) $I(0)=0$;

(ii) there exist $\rho,\alpha>0$ such that $I(u)\ge\alpha$ when $\|u\|=\rho$;

(iii) there exists $e\in X$ such that $\|e\|>\rho$ and $I(e)<0$.

If

$$c_e := \inf_{\gamma\in\Gamma(0,e)} \max_{t\in[0,1]} I(\gamma(t))$$

and I satisfies $(PS)_{c_e}$, then $c_e\ge\alpha$ and there exists $u\in X\setminus\{0\}$ with

$$DI(u)=0, \qquad I(u)=c_e.$$

Proof. Apply the theorem with $u_0=0$, $u_1=e$, $r=\rho$, and $b\ge\alpha$. Since $c_e\ge\alpha>0=I(0)$, the resulting critical point cannot be the origin. □

31.5.3 Model: the Beltrami p-Laplacian with a pure power

Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold of dimension n, with smooth boundary, such that every connected component has nonempty boundary. Suppose

$$1<p<n, \qquad p<r<p^* := \frac{np}{n-p},$$

and equip

$$X := W_0^{1,p}(M)$$

with the equivalent norm

$$\|u\| := \left(\int_M |\nabla u|_{\mathbf{g}}^p \, d\lambda_{\mathbf{g}} \right)^{1/p}.$$

Equivalence with the Sobolev norm follows from the Poincaré inequality, Theorem 9.6.8. The condition $p < n$ is retained here to define the finite critical exponent p^*; coercivity does not require that restriction. Consider

$$I(u) := \frac{1}{p}\|u\|^p - \frac{1}{r}\|u\|_{L^r(M)}^r. \tag{31.27}$$

Corollary 29.1.11, applied to TM with exponent p and to the trivial real bundle with exponent r, together with the continuous linear operators $u \mapsto \nabla u$ and $X \hookrightarrow L^r(M)$, allows the chain rule to be applied. Thus, the functional belongs to $C^1(X, \mathbb{R})$ and

$$DI(u)[v] = \int_M |\nabla u|_{\mathbf{g}}^{p-2} \langle \nabla u, \nabla v \rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} - \int_M |u|^{r-2} u v \, d\lambda_{\mathbf{g}}. \tag{31.28}$$

Its critical points are therefore exactly the weak solutions of

$$\begin{cases} -\operatorname{div}_{\mathbf{g}}\left(|\nabla u|_{\mathbf{g}}^{p-2} \nabla u \right) = |u|^{r-2} u & \text{in } M, \\ u = 0 & \text{on } \partial M. \end{cases} \tag{31.29}$$

Proposition 31.5.5. (Mountain pass geometry) The functional in (31.27) satisfies the geometric hypotheses of Corollary 31.5.4.

Proof. The continuous embedding $X \hookrightarrow L^r(M)$ provides $C_S > 0$ such that $\|u\|_{L^r(M)} \le C_S \|u\|$. Therefore,

$$I(u) \ge \frac{1}{p}\|u\|^p - \frac{C_S^r}{r}\|u\|^r. \tag{31.30}$$

Since $r > p$, we can choose $\rho > 0$ so that

$$\frac{C_S^r}{r}\rho^{r-p} \le \frac{1}{2p}.$$

If $\|u\| = \rho$, the preceding estimate gives

$$I(u) \ge \frac{\rho^p}{2p} =: \alpha > 0.$$

On the other hand, fix $w \in X \setminus \{0\}$. For $t > 0$,

$$I(tw) = \frac{t^p}{p}\|w\|^p - \frac{t^r}{r}\|w\|_{L^r(M)}^r.$$

The second coefficient is positive and $r > p$; dividing by t^r shows that $I(tw)/t^r \to -\|w\|_{L^r(M)}^r/r$. Consequently, $I(tw) \to -\infty$. For sufficiently large t, $e := tw$ satisfies $\|e\| > \rho$ and $I(e) < 0$. □

The strong convergence required by the Palais–Smale condition will follow from the next monotonicity property.

Lemma 31.5.6. (Property of type (S_+)) Define $\mathcal{A}_p\colon X \to X'$ by

$$\mathcal{A}_p(u)[v] := \int_M |\nabla u|_{\mathbf{g}}^{p-2}\langle \nabla u, \nabla v\rangle_{\mathbf{g}}\, d\lambda_{\mathbf{g}}.$$

If $u_j \rightharpoonup u$ in X and

$$\limsup_{j\to\infty} \mathcal{A}_p(u_j)[u_j - u] \le 0,$$

then $u_j \to u$ in X.

Proof. Let $F(v) := \|v\|^p/p$. This functional is convex, is of class C^1, and $DF(v) = \mathcal{A}_p(v)$. The supporting inequality for a differentiable convex functional, applied at u_j, gives

$$F(u) \ge F(u_j) + DF(u_j)[u - u_j].$$

Therefore,

$$\limsup_{j\to\infty} F(u_j) \le F(u).$$

Weak lower semicontinuity of the norm gives the opposite inequality for the lower limit. It follows that $\|u_j\| \to \|u\|$.

With the chosen norm, X is identified isometrically through $u \mapsto \nabla u$ with a closed subspace of $L^p(M, TM)$. The latter is uniformly convex for $1 < p < \infty$, and uniform convexity passes to subspaces. Recall why, in a uniformly convex space, weak convergence together with convergence of norms implies strong convergence. If $u = 0$, the assertion is immediate. If $u \neq 0$, normalize $y_j := u_j/\|u_j\|$ and $y := u/\|u\|$. If a subsequence satisfied $\|y_j - y\| \ge \eta > 0$, uniform convexity would give $\|(y_j + y)/2\| \le 1 - \delta$ for some $\delta > 0$. By Hahn–Banach there exists $\ell \in X'$ with $\|\ell\| = 1$ and $\ell(y) = 1$; since $y_j \rightharpoonup y$,

$$\left\|\frac{y_j + y}{2}\right\| \ge \ell\left(\frac{y_j + y}{2}\right) \longrightarrow 1,$$

a contradiction. Thus, $y_j \to y$ and, using $\|u_j\| \to \|u\|$, we obtain $u_j \to u$. □

Proposition 31.5.7. (Palais–Smale for the subcritical power) The functional in (31.27) satisfies condition (PS).

Proof. Let $(u_j) \subset X$ be a Palais–Smale sequence. There exist $C > 0$ and a sequence $d_j \to 0^+$ such that

$$|I(u_j)| \le C, \qquad \|DI(u_j)\|_{X'} \le d_j.$$

The exact identity

$$I(u_j) - \frac{1}{r} DI(u_j)[u_j] = \left(\frac{1}{p} - \frac{1}{r}\right)\|u_j\|^p \tag{31.31}$$

implies

$$\left(\frac{1}{p} - \frac{1}{r}\right)\|u_j\|^p \le C + \frac{d_j}{r}\|u_j\|.$$

The sequence is bounded. Indeed, if a subsequence had norm tending to infinity, dividing the last inequality by $\|u_j\|^p$ would leave the left-hand side positive while the right-hand side tended to zero, since $p > 1$.

Since X is reflexive, after passing to a subsequence there exists $u \in X$ such that $u_j \rightharpoonup u$. Corollary 9.4.31 gives the compact embedding because $r < p^*$; therefore,

$$u_j \longrightarrow u \quad \text{in } L^r(M).$$

From (31.28) we obtain

$$\mathcal{A}_p(u_j)[u_j - u] = DI(u_j)[u_j - u] + \int_M |u_j|^{r-2} u_j (u_j - u)\, d\lambda_{\mathbf{g}}.$$

The first term tends to zero because $(u_j - u)$ is bounded in X and $\|DI(u_j)\|_{X'} \to 0$. For the second, Hölder gives

$$\left| \int_M |u_j|^{r-2} u_j (u_j - u)\, d\lambda_{\mathbf{g}} \right| \le \|u_j\|_{L^r(M)}^{r-1} \|u_j - u\|_{L^r(M)} \longrightarrow 0.$$

Lemma 31.5.6 implies $u_j \to u$ in X. Every Palais–Smale sequence therefore has a convergent subsequence. □

Corollary 31.5.8. (Nontrivial solution of the power problem) Problem (31.29) has a nontrivial weak solution $u \in W_0^{1,p}(M)$.

Proof. Propositions 31.5.5 and 31.5.7 allow Corollary 31.5.4 to be applied. The resulting critical point has energy at least $\alpha > 0$, so it is not the origin, and (31.28) is precisely the weak formulation of (31.29). □

This example clarifies when the mountain pass contributes something that Nehari reduction does not automatically provide. The minimax argument uses only a barrier, a low-energy direction, and compactness at the selected level; it does not require every radial function $t \mapsto I(tu)$ to have a unique maximum or $\{u \neq 0 \mid DI(u)[u] = 0\}$ to be a submanifold. In the next chapter we will compare the two methods and prove that, for the pure power, their levels agree exactly.

The same abstract argument works on a complete Banach–Finsler manifold whenever the minimax class is preserved by composition of its elements with the deformation in Theorem 31.4.6. This is the starting point for more general geometric minimax principles.

32

Variational constraints and the Nehari method

Many energy functionals cannot be minimized directly on the entire ambient space: the origin may be a trivial minimum, the functional may lack coercivity, or the relevant geometry may be concentrated on a constraint set. In such situations it is useful to study critical points of the functional on a suitable submanifold.

The structure of regular level sets and the description of their tangent spaces were established in Chapter 24. Corollary 23.4.2, obtained from Theorem 23.3.8, provides the Lagrange multiplier principle in Banach spaces. Here we apply that framework to variational constraints and, in particular, to the Nehari manifold.

In the Nehari method, the constraint is obtained by setting the derivative of the functional in the radial direction u equal to zero. This condition eliminates the direction in which the energy may lose coercivity or lead to the trivial critical point, while retaining the tangential variations relevant to locating nonzero solutions.

Zeev Nehari and natural constraints

The method associated with Nehari uses the radial derivative of the energy to select a set where an unproductive direction disappears [108]. Under suitable hypotheses, this constraint is natural: a critical point of the restricted functional is also critical on the whole space. The advantage is not merely technical. The radial identity reflects the balance between terms of different homogeneities and turns the search for nontrivial solutions into a geometric problem on a submanifold of the function space.

32.1 The Nehari method

The Nehari manifold turns a necessary variational condition into a useful geometric constraint. Let $I\colon X \longrightarrow \mathbb{R}$ be an energy functional on a Banach space suitable for seeking weak solutions. Every critical point u satisfies $DI(u) = 0$ in X' and, in particular, $DI(u)(u) = 0$.

This motivates considering the set

$$\mathcal{N} := \{u \in X \setminus \{0\} \mid DI(u)(u) = 0\},$$

called the *Nehari manifold associated with I*. The idea of the method is to study first the functional I restricted to $\mathcal{N}$ and then to show that, under suitable hypotheses, critical points of $I\restriction_{\mathcal{N}}$ are in fact critical points of I on all of X.

The results of Section 24.7 provide a geometric description of this constraint. If $V = \{u \in \Omega \mid J(u) = c\}$ is a regular level set in an open set $\Omega \subseteq X$, then $T_u V = \ker(DJ(u))$ and

$$d(\varphi\restriction_V)_u = D\varphi(u)\restriction_{T_u V}.$$

For this reason, define

$$\|D\varphi(u)\|_{*,V} := \|D\varphi(u)\restriction_{T_u V}\|_{\mathcal{L}(T_u V,\mathbb{R})}.$$

If $T_u V \neq \{0\}$, this norm can also be written as

$$\|D\varphi(u)\|_{*,V} = \sup_{\substack{v\in T_u V\\ \|v\|=1}} |D\varphi(u)(v)|.$$

The norm on $T_u V$ is induced by its identification with the subspace $\ker(DJ(u)) \subseteq X$ and measures the slope of φ only in admissible directions.

Lemma 32.1.1. Let X be a real vector space, and let $f, g\colon X \longrightarrow \mathbb{R}$ be linear functionals, with $g \neq 0$. If $\ker(g) \subset \ker(f)$, then there exists a unique $\lambda \in \mathbb{R}$ such that $f = \lambda g$.

Proof. Since $g \neq 0$, there exists $x_0 \in X$ such that $g(x_0) \neq 0$. Define $\lambda := \dfrac{f(x_0)}{g(x_0)}$. We show that $f(x) = \lambda g(x)$ for every $x \in X$. Let $x \in X$ and consider $y := x - \dfrac{g(x)}{g(x_0)}x_0$. Then $g(y) = g(x) - \dfrac{g(x)}{g(x_0)}g(x_0) = 0$, so $y \in \ker(g)$. By hypothesis, $y \in \ker(f)$, and then $0 = f(y) = f(x) - \dfrac{g(x)}{g(x_0)}f(x_0)$. Therefore, $f(x) = \dfrac{f(x_0)}{g(x_0)}g(x) = \lambda g(x)$. Since this holds for every $x \in X$, it follows that $f = \lambda g$. Uniqueness is immediate: if $\lambda_1 g = \lambda_2 g$, then $(\lambda_1 - \lambda_2)g = 0$, and since $g \neq 0$, it follows that $\lambda_1 = \lambda_2$. □

Lemma 32.1.2. (Duality lemma) Let X be a real Banach space, and let $f, g \in X'$, with $g \neq 0$. Then

$$\|f\restriction_{\ker(g)}\|_{(\ker(g))'} = \min_{\lambda\in\mathbb{R}} \|f - \lambda g\|_{X'}.$$

If $\ker(g) \neq \{0\}$, the left-hand side agrees with $\sup\limits_{\substack{y\in\ker(g)\\ \|y\|=1}} |f(y)|$.

Proof. Let $\lambda \in \mathbb{R}$. Since $(f - \lambda g)\restriction_{\ker(g)} = f\restriction_{\ker(g)}$, we have

$$\|f\restriction_{\ker(g)}\|_{(\ker(g))'} \leq \|f - \lambda g\|_{X'}.$$

Taking the infimum over λ, we obtain

$$\|f\restriction_{\ker(g)}\|_{(\ker(g))'} \leq \inf_{\lambda\in\mathbb{R}} \|f - \lambda g\|_{X'}.$$

By Corollary B.2.29, there exists $\widetilde{f} \in X'$ extending $f \restriction_{\ker(g)}$ and satisfying

$$\|\widetilde{f}\|_{X'} = \|f \restriction_{\ker(g)}\|_{(\ker(g))'}.$$

Since $f - \widetilde{f}$ vanishes on $\ker(g)$, Lemma 32.1.1 provides $\lambda_0 \in \mathbb{R}$ such that $f - \widetilde{f} = \lambda_0 g$. Therefore,

$$\inf_{\lambda \in \mathbb{R}} \|f - \lambda g\|_{X'} \leq \|f - \lambda_0 g\|_{X'} = \|\widetilde{f}\|_{X'} = \|f \restriction_{\ker(g)}\|_{(\ker(g))'}.$$

The two inequalities prove equality and also show that the infimum is attained at λ_0. The last assertion is the usual characterization of the norm of a continuous linear functional on a nontrivial normed space. □

Proposition 32.1.3. (Characterization of constrained critical points) Let X be a real Banach space, let $\Omega \subset X$ be an open set, and let $J \in C^1(\Omega, \mathbb{R})$ and $c \in \mathbb{R}$. Suppose that $V := \{u \in \Omega \mid J(u) = c\}$ is a regular level set, that is, $DJ(u) \neq 0$ for every $u \in V$. If $\varphi \in C^1(\Omega, \mathbb{R})$ and $u \in V$, then

$$\|D\varphi(u)\|_{*,V} = \min_{\lambda \in \mathbb{R}} \|D\varphi(u) - \lambda DJ(u)\|_{X'}.$$

Moreover, u is a critical point of the restriction $\varphi \restriction_V$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$D\varphi(u) = \lambda DJ(u).$$

Proof. Since V is a regular level set of J in the open set Ω, Theorem 24.7.2 implies that $T_u V = \ker(DJ(u))$. Moreover, by Lemma 24.7.3,

$$d(\varphi \restriction_V)_u = D\varphi(u) \restriction_{T_u V}.$$

Therefore,

$$\|D\varphi(u)\|_{*,V} = \|D\varphi(u) \restriction_{\ker(DJ(u))}\|_{(\ker(DJ(u)))'}.$$

Applying Lemma 32.1.2 with $f = D\varphi(u)$ and $g = DJ(u)$ gives

$$\|D\varphi(u)\|_{*,V} = \min_{\lambda \in \mathbb{R}} \|D\varphi(u) - \lambda DJ(u)\|_{X'}.$$

We now prove the characterization of critical points. By definition, u is a critical point of $\varphi \restriction_V$ if and only if $d(\varphi \restriction_V)_u = 0$. By Lemma 24.7.3, this is equivalent to $D\varphi(u)(v) = 0$ for every $v \in T_u V$. Since $T_u V = \ker(DJ(u))$, this is equivalent to $\ker(DJ(u)) \subset \ker(D\varphi(u))$. Since $DJ(u) \neq 0$, Lemma 32.1.1 gives $\lambda \in \mathbb{R}$ such that $D\varphi(u) = \lambda DJ(u)$.
Conversely, if there exists $\lambda \in \mathbb{R}$ such that $D\varphi(u) = \lambda DJ(u)$, then for every $v \in T_u V = \ker(DJ(u))$ we have $D\varphi(u)(v) = \lambda DJ(u)(v) = 0$. Therefore, $d(\varphi \restriction_V)_u = 0$ and u is a critical point of $\varphi \restriction_V$. This proves the equivalence. □

In particular, for the Nehari manifold take $\Omega := X \setminus \{0\}$, which is open in X, and define $J \colon \Omega \longrightarrow \mathbb{R}$ by $J(u) := DI(u)(u)$. Then

$$\mathcal{N} = \{u \in \Omega \mid J(u) = 0\} = \{u \in X \setminus \{0\} \mid DI(u)(u) = 0\}.$$

Therefore, if $J \in C^1(\Omega, \mathbb{R})$ and $DJ(u) \neq 0$ for every $u \in \mathcal{N}$, Theorem 24.7.2 implies that $\mathcal{N}$ is an embedded

Banach submanifold of class C^1 and codimension 1. Furthermore, for each $u \in \mathcal{N}$ we have

$$T_u\mathcal{N} = \ker(DJ(u)).$$

If I is also of class C^2 on Ω, then J is of class C^1 and, for every $v \in X$,

$$DJ(u)(v) = D^2I(u)(v,u) + DI(u)(v).$$

In the next section we will apply this framework to the superlinear subcritical problem on a compact Riemannian manifold.

32.2 The Nehari manifold in a model case

Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold with smooth boundary. Consider $X = W_0^{1,2}(M)$ and

$$I(u) := \frac{1}{2}\int_M |\nabla u|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} - \frac{1}{r}\int_M |u|^r \, d\lambda_{\mathbf{g}}, \qquad 2 < r < 2^*.$$

Then

$$J(u) := DI(u)(u) = \int_M |\nabla u|_{\mathbf{g}}^2 \, d\lambda_{\mathbf{g}} - \int_M |u|^r \, d\lambda_{\mathbf{g}},$$

and the Nehari manifold is $\mathcal{N} = J^{-1}(0) \setminus \{0\}$.

Proposition 32.2.1. (The Nehari manifold as a hypersurface) Let $(M,\mathbf{g})$ be a compact Riemannian manifold with nonempty boundary, and let X, I, J, and $\mathcal{N}$ be the objects defined at the beginning of this section. The set $\mathcal{N}$ is an embedded Banach submanifold of class C^1 and codimension 1 in $X \setminus \{0\}$. Moreover, for each $u \in \mathcal{N}$,

$$T_u\mathcal{N} = \ker DJ(u).$$

Proof. By Proposition 29.3.2, $I \in C^2(X,\mathbb{R})$ and $J \in C^1(X \setminus \{0\},\mathbb{R})$. For $u, v \in X$,

$$DJ(u)(v) = 2\int_M \langle \nabla u, \nabla v\rangle_{\mathbf{g}} \, d\lambda_{\mathbf{g}} - r\int_M |u|^{r-2}uv \, d\lambda_{\mathbf{g}}.$$

If $u \in \mathcal{N}$, then

$$DJ(u)(u) = (2-r)\int_M |u|^r \, d\lambda_{\mathbf{g}} < 0.$$

Therefore, $DJ(u) \neq 0$ for every $u \in \mathcal{N}$. Theorem 24.7.2 implies that $\mathcal{N}$ is an embedded submanifold of codimension 1 and that $T_u\mathcal{N} = \ker DJ(u)$. □

This example explains why the term Nehari manifold is used when the functional is of class C^2. In quasilinear problems, however, the functional may be only of class C^1; in that situation it is not legitimate to assume that $u \mapsto DI(u)(u)$ is differentiable. The radial method of Szulkin and Weth works directly with the topological structure of the constraint.

32.3 The radial reduction of Szulkin and Weth

The Nehari method can be developed without assuming that the constraint set is a differentiable submanifold. The idea is to select on each positive ray the unique point where the functional attains its maximum and to transfer the variational problem to the unit sphere. We present the abstract framework of Szulkin and Weth [139, §3.1, Propositions 8–9 and Corollary 10] in the notation adopted in this book.

Let X be a uniformly convex real Banach space, let $I \in C^1(X,\mathbb{R})$, and suppose that $I(0) = 0$. Denote the unit sphere by

$$S_X := \{u \in X \mid \|u\|_X = 1\}$$

A function $\varphi\colon [0,+\infty) \longrightarrow [0,+\infty)$ is called a **normalization function** if it is continuous, $\varphi(0) = 0$, it is strictly increasing, and $\varphi(t) \to +\infty$ as $t \to +\infty$. We consider the following hypotheses:

(A1) There exists a normalization function φ such that the functional

$$\psi(u) := \int_0^{\|u\|_X} \varphi(t)\,dt, \qquad u \in X,$$

belongs to $C^1(X \setminus \{0\},\mathbb{R})$, the duality map $\mathcal{J} := D\psi\colon X \setminus \{0\} \longrightarrow X'$ maps bounded sets to bounded sets, and

$$\mathcal{J}(w)(w) = 1$$

for every $w \in S_X$.

(A2) For each $w \in X \setminus \{0\}$ there exists $s_w > 0$ such that, if $\alpha_w\colon (0,+\infty) \longrightarrow \mathbb{R}$ is given by $\alpha_w(s) := I(sw)$, then

$$\alpha_w'(s) > 0 \quad \text{if } 0 < s < s_w, \qquad \alpha_w'(s) < 0 \quad \text{if } s > s_w.$$

(A3) There exists $\delta > 0$ such that $s_w \geq \delta$ for every $w \in S_X$ and, for each compact set $K \subseteq S_X$, there exists $C_K > 0$ such that $s_w \leq C_K$ for every $w \in K$.

Hypothesis (A2) implies that s_w is the unique global maximum of α_w and that $s_w w$ is the unique point of the positive ray generated by w belonging to the Nehari set

$$\mathcal{N} := \{u \in X \setminus \{0\} \mid DI(u)(u) = 0\}.$$

Theorem 32.3.1. (Radial reduction principle) Suppose hypotheses (A1)–(A3) hold. Define

$$\widehat{m}\colon X \setminus \{0\} \longrightarrow \mathcal{N}, \qquad \widehat{m}(w) := s_w w,$$

and let $m := \widehat{m}\restriction_{S_X}$. Then the following assertions hold:

(a) the map $\widehat{m}$ is continuous;

(b) $m\colon S_X \longrightarrow \mathcal{N}$ is a homeomorphism and

$$m^{-1}(u) = \frac{u}{\|u\|_X}$$

for every $u \in \mathcal{N}$;

(c) the functional $\widehat{\Psi} := I \circ \widehat{m}$ belongs to $C^1(X \setminus \{0\}, \mathbb{R})$ and

$$D\widehat{\Psi}(w)(z) = \frac{\|\widehat{m}(w)\|_X}{\|w\|_X} DI(\widehat{m}(w))(z) \tag{32.1}$$

for all $w \in X \setminus \{0\}$ and $z \in X$;

(d) the sphere S_X is an embedded Banach submanifold of class C^1, the functional $\Psi := I \circ m$ belongs to $C^1(S_X, \mathbb{R})$, and

$$D\Psi(w)(z) = \|m(w)\|_X DI(m(w))(z) \tag{32.2}$$

for all $w \in S_X$ and $z \in T_w S_X$;

(e) a point $w \in S_X$ is critical for Ψ if and only if $m(w)$ is a nontrivial critical point of I;

(f) the minimax characterization

$$\inf_{u \in \mathcal{N}} I(u) = \inf_{w \in S_X} \Psi(w) = \inf_{w \in X \setminus \{0\}} \max_{s > 0} I(sw).$$

holds.

Proof. Begin with a scaling property. If $w \in X \setminus \{0\}$ and $t > 0$, then $\alpha_{tw}(s) = I(stw) = \alpha_w(st)$. Uniqueness of the maximum given by (A2) implies

$$s_{tw} = \frac{s_w}{t} \tag{32.3}$$

and hence $\widehat{m}(tw) = \widehat{m}(w)$.

The first part of (A3) shows that $\mathcal{N}$ is bounded away from the origin. If $u \in \mathcal{N}$ and $w := \frac{u}{\|u\|_X} \in S_X$, then $u = s_w w$ and $\|u\|_X = s_w \geq \delta$. Moreover, $\mathcal{N}$ is closed in X. If $u_j \in \mathcal{N}$ and $u_j \to u$ in X, separation from the origin implies $u \neq 0$, whereas continuity of $DI \colon X \longrightarrow X'$ gives

$$DI(u)(u) = \lim_{j \to \infty} DI(u_j)(u_j) = 0.$$

Consequently, $u \in \mathcal{N}$.

We prove continuity of $\widehat{m}$. Let $w_j \to w \neq 0$. By (32.3), we may replace each w_j and w by their normalizations and assume that $w_j, w \in S_X$. The set $K := \{w\} \cup \{w_j \mid j \in \mathbb{N}\}$ is compact. By (A3), the sequence (s_{w_j}) is bounded and bounded away from zero. Every subsequence therefore has a further subsequence, denoted in the same way, such that $s_{w_j} \to s > 0$. Hence,

$$\widehat{m}(w_j) = s_{w_j} w_j \longrightarrow sw.$$

Since $\mathcal{N}$ is closed, $sw \in \mathcal{N}$. Uniqueness of the intersection of the positive ray generated by w with $\mathcal{N}$ implies $s = s_w$, and therefore $\widehat{m}(w_j) \to \widehat{m}(w)$. The subsequence criterion proves continuity of $\widehat{m}$.

Continuity of m is immediate. If $u \in \mathcal{N}$ and $w := \frac{u}{\|u\|_X}$, then $w \in S_X$ and, by radial uniqueness, $s_w = \|u\|_X$; thus, $m(w) = u$. This proves that m is bijective and that its inverse is given by $m^{-1}(u) = \frac{u}{\|u\|_X}$. Since normalization is continuous on $X \setminus \{0\}$, m^{-1} is continuous and m is a homeomorphism.

We now study regularity of $\widehat{\Psi}$. Fix $w \in X \setminus \{0\}$ and $z \in X$. For sufficiently small $t \neq 0$, write $s_t := s_{w+tz}$. Since

$s_w w$ maximizes $s \mapsto I(sw)$ and $s_t(w+tz)$ maximizes $s \mapsto I(s(w+tz))$, we have

$$I(s_w(w+tz)) - I(s_w w) \le \widehat{\Psi}(w+tz) - \widehat{\Psi}(w) \le I(s_t(w+tz)) - I(s_t w).$$

Applying the mean value theorem to the real functions obtained by restricting I to the corresponding segments gives $\eta_t, \tau_t \in (0,1)$ such that

$$I(s_w(w+tz)) - I(s_w w) = t s_w DI\big(s_w(w+\eta_t tz)\big)(z)$$

and

$$I(s_t(w+tz)) - I(s_t w) = t s_t DI\big(s_t(w+\tau_t tz)\big)(z).$$

Continuity of $w \mapsto s_w$ and DI implies that both quotients, after division by t, converge to $s_w DI(s_w w)(z)$ as $t \to 0$. For $t > 0$ the result follows directly by squeezing, and for $t < 0$ the inequalities reverse on division by t and produce the same limit. Therefore, the directional derivative exists and

$$D_z \widehat{\Psi}(w) = s_w DI(s_w w)(z).$$

Since $s_w = \frac{\|\widehat{m}(w)\|_X}{\|w\|_X}$, we obtain (32.1). The right-hand side is linear and continuous in z, and depends continuously on w in the norm of X'. By Corollary 23.1.17, $\widehat{\Psi} \in C^1(X \setminus \{0\}, \mathbb{R})$.

Let $c_\varphi := \displaystyle\int_0^1 \varphi(t)\,dt$. Since φ is strictly increasing, $S_X = \psi^{-1}(c_\varphi)$. Moreover, (A1) gives $D\psi(w)(w) = \mathcal{J}(w)(w) = 1$ for every $w \in S_X$. The value c_φ is regular, and Theorem 24.7.2 implies that S_X is an embedded submanifold of class C^1 and

$$T_w S_X = \ker \mathcal{J}(w).$$

Identity (32.2) follows by restricting (32.1) to S_X, since $\|w\|_X = 1$.

If $DI(m(w)) = 0$, then (32.2) shows that $D\Psi(w) = 0$. Conversely, suppose that $D\Psi(w) = 0$. Given $v \in X$, define

$$\alpha := \mathcal{J}(w)(v), \qquad z := v - \alpha w.$$

Since $\mathcal{J}(w)(w) = 1$, we have $\mathcal{J}(w)(z) = 0$, so $z \in T_w S_X$. The Nehari identity gives $DI(m(w))(m(w)) = 0$ and, since $m(w) = \|m(w)\|_X w$, also $DI(m(w))(w) = 0$. On the other hand, (32.2) and $D\Psi(w) = 0$ imply $DI(m(w))(z) = 0$. Consequently,

$$DI(m(w))(v) = DI(m(w))(z) + \alpha DI(m(w))(w) = 0.$$

Since v was arbitrary, $DI(m(w)) = 0$.

Finally, bijectivity of m gives

$$\inf_{u \in \mathcal{N}} I(u) = \inf_{w \in S_X} \Psi(w).$$

By (A2), $I(s_w w) = \max_{s>0} I(sw)$, and this quantity depends only on the positive ray generated by w. Normalizing each nonzero vector then yields

$$\inf_{w \in S_X} \Psi(w) = \inf_{w \in X \setminus \{0\}} \max_{s>0} I(sw).$$

□

Radial reduction allows minimization on $\mathcal{N}$, which need not be a differentiable submanifold, to be replaced by a problem of class C^1 on the unit sphere.

32.4 Mountain pass and Nehari in the power model

Return to the power problem of Chapter 31. Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold of dimension n, with smooth boundary, such that every connected component has nonempty boundary, and suppose

$$1 < p < n, \qquad p < r < p^* := \frac{np}{n-p}.$$

On

$$X := W_0^{1,p}(M), \qquad \|u\| := \|\nabla u\|_{L^p(M)},$$

consider

$$I(u) := \frac{1}{p}\|u\|^p - \frac{1}{r}\|u\|_{L^r(M)}^r. \tag{32.4}$$

Its derivative was calculated in (31.28). Introduce

$$G(u) := DI(u)[u] = \|u\|^p - \|u\|_{L^r(M)}^r$$

and

$$\mathcal{N} := \{u \in X \setminus \{0\} \mid G(u) = 0\}.$$

The scaling argument allows each positive ray to be projected onto $\mathcal{N}$ and relates this projection to energy minimization.

Theorem 32.4.1. (Radial projection, minimizer, and ground state) For the functional in (32.4), the following assertions hold.

(a) For each $u \in X \setminus \{0\}$ there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$, given by

$$t_u = \left(\frac{\|u\|^p}{\|u\|_{L^r(M)}^r} \right)^{1/(r-p)}. \tag{32.5}$$

Moreover,

$$I(t_u u) = \max_{t>0} I(tu).$$

(b) The Nehari level

$$m := \inf_{u \in \mathcal{N}} I(u)$$

is positive and is attained at some $u_0 \in \mathcal{N}$.

(c) The minimizer u_0 is a nontrivial critical point of I and is a ground state:

$$I(u_0) = m = \inf\{I(v) \mid v \neq 0,\ DI(v) = 0\}.$$

(d) The radial characterization

$$m = \inf_{u \in X \setminus \{0\}} \max_{t>0} I(tu). \tag{32.6}$$

holds.

Proof. Fix $u \neq 0$ and define $\varphi_u(t) := I(tu)$ for $t > 0$. Then

$$\varphi_u(t) = \frac{t^p}{p}\|u\|^p - \frac{t^r}{r}\|u\|_{L^r(M)}^r,$$
$$\varphi_u'(t) = t^{p-1}\left(\|u\|^p - t^{r-p}\|u\|_{L^r(M)}^r\right).$$

Both coefficients are positive. Since $t \mapsto t^{r-p}$ is strictly increasing, φ_u' vanishes exactly once, precisely at the value in (32.5); it is positive before and negative afterward. This point is therefore the unique global maximum of φ_u. Moreover,

$$G(tu) = t\varphi_u'(t),$$

which proves (a).
If $u \in \mathcal{N}$, the Nehari identity gives

$$\|u\|^p = \|u\|_{L^r(M)}^r, \qquad I(u) = \left(\frac{1}{p} - \frac{1}{r}\right)\|u\|^p. \tag{32.7}$$

Let $C_S > 0$ be a constant for the embedding $X \hookrightarrow L^r(M)$. Then

$$\|u\|^p = \|u\|_{L^r(M)}^r \leq C_S^r\|u\|^r.$$

Since $u \neq 0$, it follows that

$$\|u\| \geq C_S^{-r/(r-p)} =: \delta > 0.$$

Together with (32.7), this shows that

$$m \geq \left(\frac{1}{p} - \frac{1}{r}\right)\delta^p > 0. \tag{32.8}$$

Now let $(u_j) \subset \mathcal{N}$ be a minimizing sequence. By (32.7), (u_j) is bounded in X. After passing to a subsequence,

$$u_j \rightharpoonup u \quad \text{in } X, \qquad u_j \to u \quad \text{in } L^r(M),$$

where the second convergence follows from the subcritical compact embedding. If $\kappa := 1/p - 1/r > 0$, then

$$m = \lim_{j\to\infty} I(u_j) = \kappa \lim_{j\to\infty} \|u_j\|^p = \kappa \lim_{j\to\infty} \|u_j\|_{L^r(M)}^r = \kappa\|u\|_{L^r(M)}^r. \tag{32.9}$$

Positivity of m implies $u \neq 0$. Weak lower semicontinuity gives

$$\|u\|^p \leq \liminf_{j\to\infty} \|u_j\|^p = \|u\|_{L^r(M)}^r. \tag{32.10}$$

If the inequality were strict, formula (32.5) would give $0 < t_u < 1$ and $t_u u \in \mathcal{N}$. Using (32.7) and (32.9) again, we would obtain

$$m \leq I(t_u u) = \kappa t_u^p \|u\|^p < \kappa\|u\|_{L^r(M)}^r = m,$$

a contradiction. Thus equality holds in (32.10); consequently, $u \in \mathcal{N}$ and $I(u) = m$. We have proved (b), with $u_0 := u$.

We now prove that the constraint is natural. The functional G belongs to $C^1(X, \mathbb{R})$ and, for $v \in \mathcal{N}$,

$$DG(v)[v] = p\|v\|^p - r\|v\|^r_{L^r(M)} = (p-r)\|v\|^p < 0.$$

In particular, $DG(v) \neq 0$ on $\mathcal{N}$, so $\mathcal{N}$ is a hypersurface of class C^1. Since u_0 minimizes I on this hypersurface, the Lagrange multiplier principle provides $\lambda \in \mathbb{R}$ such that

$$DI(u_0) = \lambda DG(u_0).$$

Evaluating at u_0 gives

$$0 = DI(u_0)[u_0] = \lambda(p-r)\|u_0\|^p,$$

and hence $\lambda = 0$ and $DI(u_0) = 0$. Every nontrivial critical point v of I satisfies $G(v) = DI(v)[v] = 0$, and therefore belongs to $\mathcal{N}$ with $I(v) \geq m = I(u_0)$. This proves (c).

Finally, by (a), each positive ray meets $\mathcal{N}$ exactly once and attains its maximum there. Taking all nonzero directions gives

$$\inf_{u \neq 0} \max_{t>0} I(tu) = \inf_{v \in \mathcal{N}} I(v) = m,$$

which is (d). □

32.4.1 Conceptual comparison

The two methods organize the same energy in different ways. The following table compares their hypotheses, constructions, and conclusions.

Table 32.1: The mountain pass and Nehari methods.

	Mountain pass	**Nehari method**
Main idea	Minimax over paths	Minimization on a natural constraint
Main object	The class $\Gamma(0, e)$	The manifold $\mathcal{N}$
Variational level	$c_e = \inf_{\gamma \in \Gamma(0,e)} \max_t I(\gamma(t))$	$m = \inf_{u \in \mathcal{N}} I(u)$
Geometry	Mountain pass barrier	Radial geometry
Construction	Paths from the origin to a point of negative energy	Points satisfying $DI(u)[u] = 0$
Compactness	Usually $(PS)_{c_e}$	Compactness of minimizing sequences
Typical result	A nontrivial critical point	Often a ground state
Main strength	Flexible minimax principle; does not require radial uniqueness	Precise description of least-energy solutions
Main difficulty	Verifying Palais–Smale at the minimax level	Proving naturalness, radial projection, and attainment of the minimum

The decisive difference arises when the radial function $t \mapsto I(tu)$ may have several critical points or none. The mountain pass method may still work if the global barrier and the Palais–Smale condition remain in place. Nehari reduction, on the other hand, requires hypotheses controlling the intersection of each ray with $\mathcal{N}$. When this intersection is unique, Nehari's method can be viewed as a radial minimax; the mountain pass method allows

competition among all paths, not only rays.

32.4.2 Exact equality of the levels

For each direction u, one can construct a radial path ending at a large multiple of u. If the class $\Gamma(0,e)$ has a fixed endpoint e, this multiple must be connected to e. The region of negative energy allows this connection to be made without increasing the maximum height of the path.

Theorem 32.4.2. (Equality of the mountain pass and Nehari levels) Let $e \in X$ satisfy $I(e) < 0$, and define

$$c_e := \inf_{\gamma \in \Gamma(0,e)} \max_{t \in [0,1]} I(\gamma(t)).$$

Then

$$c_e = m = \inf_{u \in \mathcal{N}} I(u) = \inf_{u \neq 0} \max_{t>0} I(tu).$$

In particular, the level does not depend on the choice of the endpoint with negative energy.

Proof. We divide the proof into the two inequalities.
First inequality: $m \le c_e$. Let C_S be a constant for the embedding $X \hookrightarrow L^r(M)$. For $v \in X$,

$$G(v) = \|v\|^p - \|v\|_{L^r(M)}^r \ge \|v\|^p - C_S^r \|v\|^r.$$

Choose $\eta > 0$ so small that $C_S^r \eta^{r-p} < 1$. Then

$$0 < \|v\| \le \eta \quad \Longrightarrow \quad G(v) > 0. \tag{32.11}$$

On the other hand, $I(e) < 0$ implies

$$\frac{1}{p}\|e\|^p < \frac{1}{r}\|e\|_{L^r(M)}^r,$$

and, since $r/p > 1$,

$$G(e) = \|e\|^p - \|e\|_{L^r(M)}^r < 0. \tag{32.12}$$

Decrease η, if necessary, so that $\eta < \|e\|$.
Let $\gamma \in \Gamma(0,e)$. The set

$$A_\gamma := \{t \in [0,1] \mid \|\gamma(t)\| = \eta\}$$

is nonempty and compact. Let $t_0 := \max A_\gamma$. For $t > t_0$, we have $\|\gamma(t)\| > \eta$: if the norm became less than or equal to η again, the intermediate value theorem would yield another point of A_γ after t_0. By (32.11), $G(\gamma(t_0)) > 0$, whereas (32.12) gives $G(\gamma(1)) < 0$. Thus there exists $t_* \in (t_0, 1)$ such that $G(\gamma(t_*)) = 0$. Since $\|\gamma(t_*)\| > \eta$, the point $\gamma(t_*)$ is nonzero and belongs to $\mathcal{N}$. Consequently,

$$\max_{t \in [0,1]} I(\gamma(t)) \ge I(\gamma(t_*)) \ge m.$$

Taking the infimum over γ gives $c_e \ge m$.

Second inequality: $c_e \le m$. Fix $u \in \mathcal{N}$ and set

$$a := \frac{u}{\|u\|}, \qquad b := \frac{e}{\|e\|}.$$

The unit sphere of X is path connected. Indeed, if $b \neq -a$, the path

$$s \longmapsto \frac{(1-s)a + sb}{\|(1-s)a + sb\|}$$

joins a to b; if $b = -a$, choose a unit vector $z \notin \operatorname{span}\{a\}$ and concatenate the normalized paths from a to z and from z to b. Denote one of these paths on the unit sphere by $h\colon [0,1] \to X$.
Such a vector z exists because $X = W_0^{1,p}(M)$ is infinite-dimensional: a sequence of nonzero smooth functions supported in disjoint balls in the interior of M is linearly independent. In particular, $\dim X \ge 2$.
The function $s \mapsto \|h(s)\|_{L^r(M)}^r$ is continuous and strictly positive. By compactness of $[0,1]$, there exists

$$q := \min_{s\in[0,1]} \|h(s)\|_{L^r(M)}^r > 0.$$

Choose $R > \max\{\|u\|, \|e\|\}$ so large that

$$\frac{R^p}{p} - \frac{qR^r}{r} < 0.$$

Then

$$I(Rh(s)) \le \frac{R^p}{p} - \frac{qR^r}{r} < 0 \qquad (0 \le s \le 1). \tag{32.13}$$

We construct $\gamma_u \in \Gamma(0, e)$ in three segments:

$$\gamma_u(t) := \begin{cases} 3tRa, & 0 \le t \le \frac{1}{3}, \\ Rh(3t-1), & \frac{1}{3} \le t \le \frac{2}{3}, \\ \big((3-3t)R + (3t-2)\|e\|\big)\, b, & \frac{2}{3} \le t \le 1. \end{cases}$$

The values agree at the joining points, because $h(0) = a$ and $h(1) = b$. On the first segment, the radial function attains its unique maximum at $\|u\|a = u$; thus the energy is at most $I(u)$. The second segment has negative energy by (32.13). Finally, for $s \ge \|e\|$,

$$I(sb) = \frac{s^p}{p} - \frac{s^r}{r}\|b\|_{L^r(M)}^r < 0,$$

since this inequality holds at $s = \|e\|$ and its equivalent form $s^{r-p} > r/(p\|b\|_{L^r(M)}^r)$ remains valid as s increases. Thus the third segment also has negative energy, and

$$\max_{t\in[0,1]} I(\gamma_u(t)) = I(u).$$

It follows that $c_e \le I(u)$ for every $u \in \mathcal{N}$. Taking the infimum gives $c_e \le m$. Equality with the radial minimax was already proved in (32.6). □

By Theorem 32.4.1, the common level $c_e = m$ is attained at a ground state. By Corollary 31.5.4 and the (PS) condition established in Proposition 31.5.7, it is also a critical level obtained by the mountain pass method. The

two constructions need not select the same function when there are several critical points, but they do select the same minimum energy in this homogeneous model.

32.5 A quasilinear equation on a vector bundle

We now apply the preceding theory to the quasilinear problem studied in [150]. The characterization of the Dirichlet space by the trace allows us to interpret the homogeneous boundary condition, while the Sobolev and Rellich–Kondrashov embeddings provide the integrability and compactness required by the energy functional. We shall establish its Fréchet differentiability, the geometry of its radial functions, and the existence of a ground state by the Nehari method of Szulkin and Weth.

Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold of dimension n, with smooth boundary, such that each connected component has nonempty boundary. Let $\mathbf{E}\to M$ be a smooth real vector bundle of positive finite rank, equipped with a bundle metric $\mathbf{h_E}$ and a connection $\nabla^{\mathbf{E}}$ compatible with $\mathbf{h_E}$. The metric $\mathbf{g}$ and the bundle metric $\mathbf{h_E}$ induce the bundle metric $\mathbf{g}\otimes\mathbf{h_E}$ on $T^*M\otimes\mathbf{E}$. Fix

$$2\le p\le q<n,\qquad q^*:=\frac{nq}{n-q},$$

and consider the problem

$$\begin{cases}(\nabla^{\mathbf{E}})^*\Big(a(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h_E}})|\nabla^{\mathbf{E}}\mathbf{u}|^{p-2}_{\mathbf{g}\otimes\mathbf{h_E}}\nabla^{\mathbf{E}}\mathbf{u}\Big)=f(\mathbf{u}) & \text{in } M,\\ \mathbf{u}=0 & \text{on } \partial M.\end{cases}\tag{32.14}$$

Define $A(t):=\displaystyle\int_0^t a(s)\,ds$. We assume that:

(a1) $a\colon[0,+\infty)\longrightarrow[0,+\infty)$ is continuous and there exist $k_1,k_2,k_3,k_4\ge 0$, with $k_1>0$ and $k_2>0$ when $p<q$, such that

$$k_1t^p+k_2t^q\le a(t^p)t^p\le k_3t^p+k_4t^q$$

for every $t>0$;

(a2) the function $t\longmapsto\dfrac{a(t^p)}{t^{q-p}}$ is nonincreasing on $(0,+\infty)$;

(a3) the function $t\longmapsto A(t^p)$ is convex on $[0,+\infty)$.

Let $\mathcal{F}\colon\mathbf{E}\longrightarrow\mathbb{R}$ be a continuous function whose restriction $\mathcal{F}_x\colon\mathbf{E}_x\longrightarrow\mathbb{R}$ to each fiber is of class C^1. We denote by $f_x\colon\mathbf{E}_x\longrightarrow\mathbf{E}_x$ its gradient with respect to $\mathbf{h_E}$, that is,

$$D\mathcal{F}_x(\xi)(\zeta)=\langle f_x(\xi),\zeta\rangle_{\mathbf{h_E}}.$$

We also assume that the bundle map $f\colon\mathbf{E}\longrightarrow\mathbf{E}$, defined by $f(\xi):=f_{\pi_{\mathbf{E}}(\xi)}(\xi)$, is continuous and that:

(f1) $\displaystyle\lim_{|\xi|_{\mathbf{h_E}}\to0}\frac{|f_x(\xi)|_{\mathbf{h_E}}}{|\xi|^{p-1}_{\mathbf{h_E}}}=0$ uniformly with respect to $x\in M$; that is, for each $\varepsilon>0$ there exists $\delta>0$, independent of x, such that

$$|f_x(\xi)|_{\mathbf{h_E}}\le\varepsilon|\xi|^{p-1}_{\mathbf{h_E}}$$

if $x\in M$ and $0<|\xi|_{\mathbf{h_E}}<\delta$;

(f2) there exist $C_f > 0$ and $r \in (q, q^*)$ such that

$$|f_x(\xi)|_{\mathbf{h_E}} \le C_f\left(1 + |\xi|_{\mathbf{h_E}}^{r-1}\right)$$

for every $x \in M$ and every $\xi \in \mathbf{E}_x$;

(f3) there exists $\theta \in (q, r]$ such that

$$0 < \theta \mathcal{F}_x(\xi) \le \langle f_x(\xi), \xi\rangle_{\mathbf{h_E}}$$

for every $x \in M$ and every $\xi \in \mathbf{E}_x \setminus \{0_x\}$;

(f4) for each $x \in M$ and each $\xi \in \mathbf{E}_x \setminus \{0_x\}$, the function

$$t \longmapsto \frac{\langle f_x(t\xi), t\xi\rangle_{\mathbf{h_E}}}{t^q}$$

is strictly increasing on $(0, +\infty)$.

The preceding hypotheses also determine the value of the potential on the zero section. Fix $x \in M$ and $\xi \in \mathbf{E}_x \setminus \{0_x\}$. By (f3), for every $t > 0$,

$$0 < \theta \mathcal{F}_x(t\xi) \le \langle f_x(t\xi), t\xi\rangle_{\mathbf{h_E}}.$$

Let $\varepsilon > 0$. By (f1), there exists $\delta > 0$ such that, if $0 < t|\xi|_{\mathbf{h_E}} < \delta$, then

$$|f_x(t\xi)|_{\mathbf{h_E}} \le \varepsilon t^{p-1}|\xi|_{\mathbf{h_E}}^{p-1}.$$

Consequently,

$$0 < \mathcal{F}_x(t\xi) \le \frac{\varepsilon}{\theta} t^p |\xi|_{\mathbf{h_E}}^p$$

for every sufficiently small $t > 0$. Thus $\mathcal{F}_x(t\xi) \to 0$ as $t \to 0^+$. Since $\mathcal{F}_x$ is continuous on the fiber, we conclude that $\mathcal{F}_x(0_x) = 0$. Theorem B.6.5, applied to $t \longmapsto \mathcal{F}_x(t\xi)$, then gives

$$\mathcal{F}_x(\xi) = \int_0^1 \langle f_x(t\xi), \xi\rangle_{\mathbf{h_E}}\, dt. \tag{32.15}$$

Global continuity of $\mathcal{F} : \mathbf{E} \longrightarrow \mathbb{R}$ is part of the hypotheses. This condition does not follow merely from continuity of each restriction $\mathcal{F}_x$ and will be used, in particular, when taking minima and maxima over the compact unit sphere and ball bundles.

Take $X := W_0^{1,q}(M, \mathbf{E})$. This space contains a nonzero section: simply multiply an element of a local frame of the bundle by a nonzero smooth function compactly supported in the interior of a trivializing chart. By the Poincaré inequality in Theorem 9.6.8, there exists a constant $C_P > 0$, depending only on M, $\mathbf{g}$, $\mathbf{E}$, $\mathbf{h_E}$, $\nabla^{\mathbf{E}}$, and q, such that, if

$$\|\mathbf{u}\|_X := \left(\int_M |\nabla_w^{\mathbf{E}}\mathbf{u}|_{\mathbf{g}\otimes\mathbf{h_E}}^q\, d\lambda_{\mathbf{g}}\right)^{\frac{1}{q}},$$

then

$$\|\mathbf{u}\|_X \le \|\mathbf{u}\|_{W^{1,q}(M,\mathbf{E})} \le (1 + C_P^q)^{\frac{1}{q}}\|\mathbf{u}\|_X$$

for every $\mathbf{u} \in X$. We use $\|\cdot\|_X$ as the norm on X. Henceforth, we write $\nabla^{\mathbf{E}}\mathbf{u}$ for the weak covariant derivative of $\mathbf{u} \in X$. Since $p \le q$ and the volume is finite, Hölder's inequality gives $W_0^{1,q}(M, \mathbf{E}) \hookrightarrow W_0^{1,p}(M, \mathbf{E})$: a sequence

of compactly supported smooth sections converging in $W^{1,q}$ also converges in $W^{1,p}$. Thus Poincaré's inequality applies to both exponents p and q.

Definition 32.5.1. (Weak solution of the quasilinear problem) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. A section $\mathbf{u} \in X$ is a weak solution of (32.14) if

$$\int_M a(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}})|\nabla^{\mathbf{E}}\mathbf{u}|^{p-2}_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\langle\nabla^{\mathbf{E}}\mathbf{u},\nabla^{\mathbf{E}}\boldsymbol{\varphi}\rangle_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}} = \int_M \langle f_x(\mathbf{u}),\boldsymbol{\varphi}\rangle_{\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}$$

for every $\boldsymbol{\varphi} \in X$.

32.5.1 Preliminary estimates

The monotonicity in (a2) relates the primitive A to the term appearing in the Nehari identity.

Lemma 32.5.2. For every $\rho \geq 0$, we have

$$A(\rho^p) \geq \frac{p}{q}a(\rho^p)\rho^p.$$

Proof. The assertion is immediate for $\rho = 0$. Let $\rho > 0$. If $0 < \tau \leq \rho$, hypothesis (a2) implies $\dfrac{a(\tau^p)}{\tau^{q-p}} \geq \dfrac{a(\rho^p)}{\rho^{q-p}}$, and hence $a(\tau^p) \geq a(\rho^p)\left(\dfrac{\tau}{\rho}\right)^{q-p}$.
Using the change of variable $s = \tau^p$, we obtain

$$A(\rho^p) = p\int_0^\rho a(\tau^p)\tau^{p-1}\,d\tau \geq p\,a(\rho^p)\rho^{p-q}\int_0^\rho \tau^{q-1}\,d\tau = \frac{p}{q}a(\rho^p)\rho^p.$$

□

Lemma 32.5.3. (Fiberwise growth estimates) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. For each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f_x(\xi)|_{\mathbf{h}_{\mathbf{E}}} \leq \varepsilon|\xi|^{p-1}_{\mathbf{h}_{\mathbf{E}}} + C_\varepsilon|\xi|^{r-1}_{\mathbf{h}_{\mathbf{E}}} \tag{32.16}$$

for every $x \in M$ and every $\xi \in \mathbf{E}_x$. Moreover,

$$|\mathcal{F}_x(\xi)| \leq \frac{\varepsilon}{p}|\xi|^p_{\mathbf{h}_{\mathbf{E}}} + \frac{C_\varepsilon}{r}|\xi|^r_{\mathbf{h}_{\mathbf{E}}}. \tag{32.17}$$

Proof. Let $\varepsilon > 0$. By (f1), there exists $\delta \in (0,1)$ such that

$$|f_x(\xi)|_{\mathbf{h}_{\mathbf{E}}} \leq \varepsilon|\xi|^{p-1}_{\mathbf{h}_{\mathbf{E}}}$$

if $x \in M$ and $0 < |\xi|_{\mathbf{h}_{\mathbf{E}}} \leq \delta$. Letting $\xi \to 0_x$ and using continuity of f_x gives $f_x(0_x) = 0$, so the same inequality holds for $\xi = 0_x$.

If $|\xi|_{\mathbf{h_E}} \geq \delta$, hypothesis (f2) gives

$$|f_x(\xi)|_{\mathbf{h_E}} \leq C_f\left(1 + |\xi|_{\mathbf{h_E}}^{r-1}\right) \leq C_f\left(1 + \delta^{1-r}\right)|\xi|_{\mathbf{h_E}}^{r-1}.$$

Taking $C_\varepsilon \geq C_f(1 + \delta^{1-r})$ yields (32.16) in both regions.
Since $\mathcal{F}_x(0_x) = 0$ and f_x is the fiberwise gradient of $\mathcal{F}_x$, Theorem B.6.5, applied to $t \longmapsto \mathcal{F}_x(t\xi)$, gives

$$\mathcal{F}_x(\xi) = \int_0^1 \langle f_x(t\xi), \xi\rangle_{\mathbf{h_E}}\, dt.$$

By (32.16),

$$|\mathcal{F}_x(\xi)| \leq \int_0^1 |f_x(t\xi)|_{\mathbf{h_E}}|\xi|_{\mathbf{h_E}}\, dt \leq \int_0^1 \left(\varepsilon t^{p-1}|\xi|_{\mathbf{h_E}}^{p-1} + C_\varepsilon t^{r-1}|\xi|_{\mathbf{h_E}}^{r-1}\right)|\xi|_{\mathbf{h_E}}\, dt = \frac{\varepsilon}{p}|\xi|_{\mathbf{h_E}}^{p} + \frac{C_\varepsilon}{r}|\xi|_{\mathbf{h_E}}^{r}.$$

□

Lemma 32.5.4. (Lower bound for the potential) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. There exist constants $D > 0$ and $D' \geq 0$ such that

$$\mathcal{F}_x(\xi) \geq D|\xi|_{\mathbf{h_E}}^{\theta} - D'$$

for every $x \in M$ and every $\xi \in \mathbf{E}_x$.

Proof. Fix $x \in M$ and $\eta \in \mathbf{E}_x \setminus \{0_x\}$. For $t > 0$, define

$$H_{x,\eta}(t) := \frac{\mathcal{F}_x(t\eta)}{t^\theta}.$$

By (f3), $\mathcal{F}_x(t\eta) > 0$. Moreover,

$$\frac{d}{dt}\log H_{x,\eta}(t) = \frac{\langle f_x(t\eta), t\eta\rangle_{\mathbf{h_E}} - \theta\mathcal{F}_x(t\eta)}{t\mathcal{F}_x(t\eta)} \geq 0.$$

Thus $H_{x,\eta}$ is nondecreasing.
The unit sphere bundle

$$S(\mathbf{E}) := \{\eta \in \mathbf{E} \mid |\eta|_{\mathbf{h_E}} = 1\}$$

is compact, because M is compact and $\mathbf{E}$ has finite rank. The function $\eta \longmapsto \mathcal{F}_{\pi_{\mathbf{E}}(\eta)}(\eta)$ is continuous and, by (f3), positive on $S(\mathbf{E})$. Consequently,

$$D := \min_{\eta \in S(\mathbf{E})} \mathcal{F}_{\pi_{\mathbf{E}}(\eta)}(\eta) > 0.$$

If $|\xi|_{\mathbf{h_E}} \geq 1$, write $\xi = \rho\eta$, with $\rho = |\xi|_{\mathbf{h_E}} \geq 1$ and $|\eta|_{\mathbf{h_E}} = 1$. Monotonicity of $H_{x,\eta}$ implies

$$\mathcal{F}_x(\xi) = \mathcal{F}_x(\rho\eta) \geq \rho^\theta \mathcal{F}_x(\eta) \geq D|\xi|_{\mathbf{h_E}}^{\theta}.$$

The closed unit ball bundle $\{\xi \in \mathbf{E} \mid |\xi|_{\mathbf{h_E}} \leq 1\}$ is compact. Thus the continuous function

$$\xi \longmapsto D|\xi|_{\mathbf{h_E}}^{\theta} - \mathcal{F}_{\pi_{\mathbf{E}}(\xi)}(\xi)$$

attains a maximum $D' \geq 0$ on this set. Hence $\mathcal{F}_x(\xi) \geq D|\xi|^\theta_{\mathbf{h_E}} - D'$ also when $|\xi|_{\mathbf{h_E}} \leq 1$. □

32.5.2 The energy functional

Define

$$I(\mathbf{u}) := \frac{1}{p}\int_M A(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h_E}})\,d\lambda_{\mathbf{g}} - \int_M \mathcal{F}_x(\mathbf{u}(x))\,d\lambda_{\mathbf{g}}, \qquad \mathbf{u}\in X. \tag{32.18}$$

Proposition 32.5.5. (Differentiability and weak solutions) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E}\to M$ be the vector bundle specified there. The functional $I\colon X \longrightarrow \mathbb{R}$ is well defined and belongs to $C^1(X,\mathbb{R})$. For all $\mathbf{u},\mathbf{v}\in X$,

$$DI(\mathbf{u})(\mathbf{v}) = \int_M a(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h_E}})|\nabla^{\mathbf{E}}\mathbf{u}|^{p-2}_{\mathbf{g}\otimes\mathbf{h_E}}\langle\nabla^{\mathbf{E}}\mathbf{u},\nabla^{\mathbf{E}}\mathbf{v}\rangle_{\mathbf{g}\otimes\mathbf{h_E}}\,d\lambda_{\mathbf{g}} - \int_M \langle f_x(\mathbf{u}(x)),\mathbf{v}(x)\rangle_{\mathbf{h_E}}\,d\lambda_{\mathbf{g}}. \tag{32.19}$$

Consequently, the critical points of I are exactly the weak solutions of (32.14).

Proof. Write $I = J_1 - J_2$, where

$$J_1(\mathbf{u}) := \frac{1}{p}\int_M A(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h_E}})\,d\lambda_{\mathbf{g}}, \qquad J_2(\mathbf{u}) := \int_M \mathcal{F}_x(\mathbf{u}(x))\,d\lambda_{\mathbf{g}}.$$

We first check that both terms are well defined. By (a1), for each $\rho>0$ we have $a(\rho^p)\leq k_3 + k_4\rho^{q-p}$. Thus

$$A(\rho^p) = p\int_0^\rho a(\tau^p)\tau^{p-1}\,d\tau \leq k_3\rho^p + \frac{pk_4}{q}\rho^q.$$

Since $\nabla^{\mathbf{E}}\mathbf{u}\in L^q(M,T^*M\otimes\mathbf{E})$, $p\leq q$, and $\lambda_{\mathbf{g}}(M)<\infty$, we obtain $A(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h_E}})\in L^1(M)$.
By Lemma 32.5.3, there exists $C>0$ such that $|\mathcal{F}_x(\xi)|\leq C\big(|\xi|^p_{\mathbf{h_E}} + |\xi|^r_{\mathbf{h_E}}\big)$.
The continuous embeddings $X\hookrightarrow L^p(M,\mathbf{E})$ and $X\hookrightarrow L^r(M,\mathbf{E})$ show that $\mathcal{F}_x(\mathbf{u}(x))\in L^1(M)$. Thus I is well defined.
We now prove that $J_1\in C^1(X,\mathbb{R})$. Let $\mathbf{V} := T^*M\otimes\mathbf{E}$, equipped with the bundle metric $\mathbf{g}\otimes\mathbf{h_E}$, and define the bundle maps

$$\Psi_x\colon \mathbf{V}_x\longrightarrow\mathbf{V}_x, \qquad \Psi_x(\eta) := a(|\eta|^p_{\mathbf{g}\otimes\mathbf{h_E}})|\eta|^{p-2}_{\mathbf{g}\otimes\mathbf{h_E}}\eta, \qquad \Psi_x(0_x) := 0_x,$$

and $\Phi_x\colon \mathbf{V}_x\longrightarrow\mathbb{R}$, $\Phi_x(\eta) := \dfrac{1}{p}A(|\eta|^p_{\mathbf{g}\otimes\mathbf{h_E}})$.
The map $\Psi\colon\mathbf{V}\longrightarrow\mathbf{V}$ is continuous and

$$D\Phi_x(\eta)(\zeta) = \langle\Psi_x(\eta),\zeta\rangle_{\mathbf{g}\otimes\mathbf{h_E}}.$$

Moreover, by (a1),

$$|\Psi_x(\eta)|_{\mathbf{g}\otimes\mathbf{h_E}} \leq k_3|\eta|^{p-1}_{\mathbf{g}\otimes\mathbf{h_E}} + k_4|\eta|^{q-1}_{\mathbf{g}\otimes\mathbf{h_E}} \leq C\big(1+|\eta|^{q-1}_{\mathbf{g}\otimes\mathbf{h_E}}\big).$$

Corollary 29.1.9, applied with $m=q$, implies that $\eta\longmapsto\Psi(\eta)$ defines a continuous operator from $L^q(M,\mathbf{V})$ to $L^{q'}(M,\mathbf{V})$, where $q' := \dfrac{q}{q-1}$.
For $\mathbf{u}\in X$, define $L_{\mathbf{u}}\in X'$ by

$$L_{\mathbf{u}}(\mathbf{v}) := \int_M \langle\Psi_x(\nabla^{\mathbf{E}}\mathbf{u}),\nabla^{\mathbf{E}}\mathbf{v}\rangle_{\mathbf{g}\otimes\mathbf{h_E}}\,d\lambda_{\mathbf{g}}.$$

Hölder's inequality (Proposition B.7.9) gives

$$|L_{\mathbf{u}}(\mathbf{v})| \le \|\Psi(\nabla^{\mathbf{E}}\mathbf{u})\|_{L^{q'}(M,\mathbf{V})}\|\nabla^{\mathbf{E}}\mathbf{v}\|_{L^{q}(M,\mathbf{V})} = \|\Psi(\nabla^{\mathbf{E}}\mathbf{u})\|_{L^{q'}(M,\mathbf{V})}\|\mathbf{v}\|_X,$$

so $L_{\mathbf{u}}$ is linear and continuous.
Let $\mathbf{h} \in X$. For almost every $x \in M$, Theorem B.6.5, applied to

$$s \longmapsto \Phi_x(\nabla^{\mathbf{E}}\mathbf{u}(x) + s\nabla^{\mathbf{E}}\mathbf{h}(x))$$

gives

$$\Phi_x(\nabla^{\mathbf{E}}\mathbf{u} + \nabla^{\mathbf{E}}\mathbf{h}) - \Phi_x(\nabla^{\mathbf{E}}\mathbf{u}) = \int_0^1 \langle \Psi_x(\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}), \nabla^{\mathbf{E}}\mathbf{h}\rangle_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\, ds.$$

Hölder's inequality (Proposition B.7.9) and the preceding bounds show that the integrand belongs to $L^1(M\times[0,1])$. We may apply Fubini's theorem for Bochner integrals B.6.26, in its scalar case, to $(M,\lambda_{\mathbf{g}})\times([0,1],ds)$ and obtain

$$J_1(\mathbf{u}+\mathbf{h}) - J_1(\mathbf{u}) - L_{\mathbf{u}}(\mathbf{h}) = \int_0^1 \int_M \langle \Psi_x(\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}) - \Psi_x(\nabla^{\mathbf{E}}\mathbf{u}), \nabla^{\mathbf{E}}\mathbf{h}\rangle_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\, d\lambda_{\mathbf{g}}\, ds.$$

Consequently,

$$|J_1(\mathbf{u}+\mathbf{h}) - J_1(\mathbf{u}) - L_{\mathbf{u}}(\mathbf{h})| \le \|\mathbf{h}\|_X \int_0^1 \|\Psi(\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}) - \Psi(\nabla^{\mathbf{E}}\mathbf{u})\|_{L^{q'}(M,\mathbf{V})}\, ds.$$

Let $(\mathbf{h}_j)_{j\in\mathbb{N}} \subseteq X$ satisfy $\mathbf{h}_j \to 0$ in X. For each $s \in [0,1]$,

$$\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}_j \longrightarrow \nabla^{\mathbf{E}}\mathbf{u} \quad \text{in } L^q(M,\mathbf{V}),$$

and continuity of the superposition operator induced by Ψ implies

$$\|\Psi(\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}_j) - \Psi(\nabla^{\mathbf{E}}\mathbf{u})\|_{L^{q'}(M,\mathbf{V})} \longrightarrow 0.$$

Since $(\mathbf{h}_j)$ is bounded in X, there exists $R > 0$ such that $\|\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}_j\|_{L^q(M,\mathbf{V})} \le R$ for every j and every $s \in [0,1]$. The growth bound for Ψ provides a constant $C_R > 0$ such that

$$\|\Psi(\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}_j)\|_{L^{q'}(M,\mathbf{V})} + \|\Psi(\nabla^{\mathbf{E}}\mathbf{u})\|_{L^{q'}(M,\mathbf{V})} \le C_R$$

for every j and every $s \in [0,1]$. Theorem B.6.14, applied in the variable s, implies

$$\int_0^1 \|\Psi(\nabla^{\mathbf{E}}\mathbf{u} + s\nabla^{\mathbf{E}}\mathbf{h}_j) - \Psi(\nabla^{\mathbf{E}}\mathbf{u})\|_{L^{q'}(M,\mathbf{V})}\, ds \longrightarrow 0.$$

Thus

$$\lim_{\mathbf{h}\to 0} \frac{|J_1(\mathbf{u}+\mathbf{h}) - J_1(\mathbf{u}) - L_{\mathbf{u}}(\mathbf{h})|}{\|\mathbf{h}\|_X} = 0.$$

Hence J_1 is Fréchet differentiable and $DJ_1(\mathbf{u}) = L_{\mathbf{u}}$.

If $\mathbf{u}_j \to \mathbf{u}$ in X, then $\nabla^{\mathbf{E}}\mathbf{u}_j \to \nabla^{\mathbf{E}}\mathbf{u}$ in $L^q(M,\mathbf{V})$ and

$$\Psi(\nabla^{\mathbf{E}}\mathbf{u}_j) \to \Psi(\nabla^{\mathbf{E}}\mathbf{u}) \quad \text{in } L^{q'}(M,\mathbf{V}).$$

Consequently,

$$\|DJ_1(\mathbf{u}_j) - DJ_1(\mathbf{u})\|_{X'} \le \|\Psi(\nabla^{\mathbf{E}}\mathbf{u}_j) - \Psi(\nabla^{\mathbf{E}}\mathbf{u})\|_{L^{q'}(M,\mathbf{V})} \longrightarrow 0.$$

This proves that $J_1 \in C^1(X,\mathbb{R})$.
We now prove that $J_2 \in C^1(X,\mathbb{R})$. By (32.16) and $p < r$, there exists $C > 0$ such that

$$|f_x(\xi)|_{\mathbf{h}_\mathbf{E}} \le C\big(1 + |\xi|_{\mathbf{h}_\mathbf{E}}^{r-1}\big).$$

Corollary 29.1.9, applied with $m = r$, implies that the operator $\mathbf{u} \longmapsto f(\mathbf{u})$ is continuous from $L^r(M,\mathbf{E})$ to $L^{r'}(M,\mathbf{E})$, where $r' := \dfrac{r}{r-1}$.
For $\mathbf{u} \in X$, define $K_\mathbf{u} \in X'$ by

$$K_\mathbf{u}(\mathbf{v}) := \int_M \langle f_x(\mathbf{u}(x)), \mathbf{v}(x)\rangle_{\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}.$$

By Hölder's inequality (Proposition B.7.9) and the embedding $X \hookrightarrow L^r(M,\mathbf{E})$, there exists a constant $C_S > 0$ such that

$$|K_\mathbf{u}(\mathbf{v})| \le \|f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})}\|\mathbf{v}\|_{L^r(M,\mathbf{E})} \le C_S\|f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})}\|\mathbf{v}\|_X.$$

For $\mathbf{h} \in X$, Theorem B.6.5, applied in each fiber, gives

$$\mathcal{F}_x(\mathbf{u}+\mathbf{h}) - \mathcal{F}_x(\mathbf{u}) = \int_0^1 \langle f_x(\mathbf{u}+s\mathbf{h}), \mathbf{h}\rangle_{\mathbf{h}_\mathbf{E}}\, ds.$$

The embeddings and the growth bound allow us to apply Fubini's theorem for Bochner integrals B.6.26, in its scalar case, to $(M,\lambda_\mathbf{g})\times([0,1],ds)$. Thus

$$J_2(\mathbf{u}+\mathbf{h}) - J_2(\mathbf{u}) - K_\mathbf{u}(\mathbf{h}) = \int_0^1 \int_M \langle f_x(\mathbf{u}+s\mathbf{h}) - f_x(\mathbf{u}), \mathbf{h}\rangle_{\mathbf{h}_\mathbf{E}}\, d\lambda_\mathbf{g}\, ds,$$

and

$$|J_2(\mathbf{u}+\mathbf{h}) - J_2(\mathbf{u}) - K_\mathbf{u}(\mathbf{h})| \le C_S\|\mathbf{h}\|_X \int_0^1 \|f(\mathbf{u}+s\mathbf{h}) - f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})}\, ds.$$

If $\mathbf{h}_j \to 0$ in X, then $\mathbf{u} + s\mathbf{h}_j \to \mathbf{u}$ in $L^r(M,\mathbf{E})$ for each $s \in [0,1]$. Continuity of the superposition operator induced by f implies

$$\|f(\mathbf{u}+s\mathbf{h}_j) - f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})} \longrightarrow 0.$$

Since $(\mathbf{h}_j)$ is bounded in X and the embedding $X \hookrightarrow L^r(M,\mathbf{E})$ is continuous, there exists $R > 0$ such that

$$\|\mathbf{u} + s\mathbf{h}_j\|_{L^r(M,\mathbf{E})} \le R$$

for every j and every $s \in [0,1]$. The boundedness estimate in Remark 29.1.10 provides $C_R > 0$ such that

$$\|f(\mathbf{u}+s\mathbf{h}_j) - f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})} \le \|f(\mathbf{u}+s\mathbf{h}_j)\|_{L^{r'}(M,\mathbf{E})} + \|f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})} \le C_R$$

for every j and every $s \in [0,1]$. By Theorem B.6.14, applied in the variable s,

$$\int_0^1 \|f(\mathbf{u}+s\mathbf{h}_j)-f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})}\,ds \longrightarrow 0.$$

It follows that J_2 is Fréchet differentiable and $DJ_2(\mathbf{u}) = K_{\mathbf{u}}$.
Finally, if $\mathbf{u}_j \to \mathbf{u}$ in X, then $\mathbf{u}_j \to \mathbf{u}$ in $L^r(M,\mathbf{E})$ and $f(\mathbf{u}_j) \to f(\mathbf{u})$ in $L^{r'}(M,\mathbf{E})$. Hence

$$\|DJ_2(\mathbf{u}_j)-DJ_2(\mathbf{u})\|_{X'} \le C_S\|f(\mathbf{u}_j)-f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})} \longrightarrow 0.$$

Therefore $J_2 \in C^1(X,\mathbb{R})$. Identity (32.19) follows from $I = J_1 - J_2$. Comparing it with Definition 32.5.1, we conclude that $DI(\mathbf{u}) = 0$ if and only if $\mathbf{u}$ is a weak solution. □

32.5.3 Radial geometry and the Nehari manifold

Define

$$\mathcal{N} := \{\mathbf{u} \in X \setminus \{0\} \mid DI(\mathbf{u})(\mathbf{u}) = 0\}.$$

Thus $\mathbf{u} \in \mathcal{N}$ if and only if $\mathbf{u} \neq 0$ and

$$\int_M a(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}})|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}} = \int_M \langle f_x(\mathbf{u}),\mathbf{u}\rangle_{\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}. \tag{32.20}$$

Proposition 32.5.6. (Projection onto the Nehari manifold) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. For each $\mathbf{u} \in X \setminus \{0\}$ there exists a unique $t_{\mathbf{u}} > 0$ such that $t_{\mathbf{u}}\mathbf{u} \in \mathcal{N}$. Moreover, $t \longmapsto I(t\mathbf{u})$ is strictly increasing on $(0, t_{\mathbf{u}})$ and strictly decreasing on $(t_{\mathbf{u}}, +\infty)$.

Proof. Fix $\mathbf{u} \in X \setminus \{0\}$ and define $\gamma_{\mathbf{u}}(t) := I(t\mathbf{u})$ for $t > 0$. By the chain rule,

$$\gamma_{\mathbf{u}}'(t) = DI(t\mathbf{u})(\mathbf{u}) = \frac{1}{t}DI(t\mathbf{u})(t\mathbf{u}).$$

We first show that $\gamma_{\mathbf{u}}(t) > 0$ for sufficiently small $t > 0$. By Lemma 32.5.2 and (a1),

$$\frac{1}{p}A(t^p|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}) \ge \frac{1}{q}a(t^p|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}})t^p|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}} \ge \frac{k_1}{q}t^p|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}} + \frac{k_2}{q}t^q|\nabla^{\mathbf{E}}\mathbf{u}|^q_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}},$$

where the norms of the derivatives are taken with respect to $\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}$. By (32.17),

$$\mathcal{F}_x(t\mathbf{u}) \le \frac{\varepsilon}{p}t^p|\mathbf{u}|^p_{\mathbf{h}_{\mathbf{E}}} + \frac{C_\varepsilon}{r}t^r|\mathbf{u}|^r_{\mathbf{h}_{\mathbf{E}}}.$$

Let $C_P > 0$ satisfy $\|\mathbf{u}\|_{L^p(M,\mathbf{E})} \le C_P\|\nabla^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}$. Take $\varepsilon := \dfrac{pk_1}{2qC_P^p}$.
Integrating the preceding inequalities gives

$$\gamma_{\mathbf{u}}(t) \ge \frac{k_1}{2q}t^p\|\nabla^{\mathbf{E}}\mathbf{u}\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + \frac{k_2}{q}t^q\|\nabla^{\mathbf{E}}\mathbf{u}\|^q_{L^q(M,T^*M\otimes\mathbf{E})} - \frac{C_\varepsilon}{r}t^r\|\mathbf{u}\|^r_{L^r(M,\mathbf{E})}.$$

Since $\mathbf{u} \neq 0$, Poincaré's inequality implies $\|\nabla^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})} > 0$. Moreover, $r > q \geq p$. Thus there exists $t_0 > 0$ such that $\gamma_{\mathbf{u}}(t) > 0$ for every $t \in (0, t_0)$.

On the other hand, the upper bound in (a1) implies $A(\rho^p) \leq k_3\rho^p + \dfrac{pk_4}{q}\rho^q$.

Lemma 32.5.4 gives

$$\gamma_{\mathbf{u}}(t) \leq \frac{k_3}{p}t^p\|\nabla^{\mathbf{E}}\mathbf{u}\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + \frac{k_4}{q}t^q\|\nabla^{\mathbf{E}}\mathbf{u}\|^q_{L^q(M,T^*M\otimes\mathbf{E})} - Dt^\theta\|\mathbf{u}\|^\theta_{L^\theta(M,\mathbf{E})} + D'\lambda_{\mathbf{g}}(M).$$

Since $\theta > q \geq p$ and $\mathbf{u} \neq 0$, we have $\gamma_{\mathbf{u}}(t) \to -\infty$ as $t \to +\infty$. We must also rule out a maximizing sequence escaping toward 0. The upper bounds in (a1) and (32.17) give, for $0 < t \leq 1$, a constant $C_{\mathbf{u}} > 0$ such that

$$|\gamma_{\mathbf{u}}(t)| \leq C_{\mathbf{u}}(t^p + t^q + t^r),$$

and hence $\gamma_{\mathbf{u}}(t) \to 0$ as $t \to 0^+$.

Choose $\tau > 0$ with $a := \gamma_{\mathbf{u}}(\tau) > 0$. By the two limits, there exist $0 < \delta < \tau < R$ such that

$$\gamma_{\mathbf{u}}(t) < \frac{a}{2} \quad \text{if } 0 < t \leq \delta \qquad \text{or if } t \geq R.$$

The continuous function $\gamma_{\mathbf{u}}$ attains a maximum on the compact set $[\delta, R]$. Since its value at τ is a, this maximum exceeds all values outside that set and is therefore global. Moreover, it is not attained at the endpoints δ or R; thus there exists $t_{\mathbf{u}} \in (\delta, R)$ with $\gamma'_{\mathbf{u}}(t_{\mathbf{u}}) = 0$, that is, $t_{\mathbf{u}}\mathbf{u} \in \mathcal{N}$.

We now prove uniqueness. Suppose that $0 < s < t$ and $s\mathbf{u}, t\mathbf{u} \in \mathcal{N}$. Dividing the two Nehari identities by s^q and t^q, respectively, gives

$$\int_M a(t^p|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}})t^{p-q}|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}} = \int_M \frac{\langle f_x(t\mathbf{u}), t\mathbf{u}\rangle_{\mathbf{h}_{\mathbf{E}}}}{t^q}\,d\lambda_{\mathbf{g}}$$

and the analogous identity with s.

If $|\nabla^{\mathbf{E}}\mathbf{u}(x)|_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}} > 0$, apply (a2) to the numbers $s|\nabla^{\mathbf{E}}\mathbf{u}(x)|_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}$ and $t|\nabla^{\mathbf{E}}\mathbf{u}(x)|_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}$. Since $s < t$, we obtain

$$a(t^p|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}})t^{p-q} \leq a(s^p|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}})s^{p-q}.$$

The inequality also holds when $|\nabla^{\mathbf{E}}\mathbf{u}(x)|_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}} = 0$. Thus the principal term corresponding to t is less than or equal to the one corresponding to s.

By contrast, (f4) implies that, for each x such that $\mathbf{u}(x) \neq 0_x$,

$$\frac{\langle f_x(t\mathbf{u}), t\mathbf{u}\rangle_{\mathbf{h}_{\mathbf{E}}}}{t^q} > \frac{\langle f_x(s\mathbf{u}), s\mathbf{u}\rangle_{\mathbf{h}_{\mathbf{E}}}}{s^q}.$$

The set $\{x \in M \mid \mathbf{u}(x) \neq 0_x\}$ has positive measure, since $\mathbf{u} \neq 0$ in $L^q(M, \mathbf{E})$. Upon integration, the potential term corresponding to t is strictly greater than the one corresponding to s. This contradicts the two Nehari identities. Therefore $t_{\mathbf{u}}$ is unique.

Finally, define $H_{\mathbf{u}}(t) := \dfrac{DI(t\mathbf{u})(t\mathbf{u})}{t^q}$. The preceding argument, applied to any $0 < s < t$, shows that $H_{\mathbf{u}}(t) < H_{\mathbf{u}}(s)$. Since $H_{\mathbf{u}}(t_{\mathbf{u}}) = 0$, we have $H_{\mathbf{u}}(t) > 0$ if $0 < t < t_{\mathbf{u}}$ and $H_{\mathbf{u}}(t) < 0$ if $t > t_{\mathbf{u}}$. Because $\gamma'_{\mathbf{u}}(t) = t^{q-1}H_{\mathbf{u}}(t)$, the function $\gamma_{\mathbf{u}}$ is strictly increasing on $(0, t_{\mathbf{u}})$ and strictly decreasing on $(t_{\mathbf{u}}, +\infty)$. □

Lemma 32.5.7. (Positivity of the energy on $\mathcal{N}$) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. For every $\mathbf{u} \in \mathcal{N}$, we have $I(\mathbf{u}) > 0$.

Proof. Since $DI(\mathbf{u})(\mathbf{u}) = 0$, Lemma 32.5.2 and (f3) give

$$\begin{aligned} I(\mathbf{u}) &= I(\mathbf{u}) - \frac{1}{q} DI(\mathbf{u})(\mathbf{u}) \\ &= \int_M \left[\frac{1}{p} A(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}) - \frac{1}{q} a(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}) |\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}} \right] d\lambda_{\mathbf{g}} \\ &\quad + \int_M \left[\frac{1}{q} \langle f_x(\mathbf{u}), \mathbf{u} \rangle_{\mathbf{h}_{\mathbf{E}}} - \mathcal{F}_x(\mathbf{u}) \right] d\lambda_{\mathbf{g}} \\ &\geq \left(\frac{\theta}{q} - 1 \right) \int_M \mathcal{F}_x(\mathbf{u})\, d\lambda_{\mathbf{g}} > 0. \end{aligned}$$

The last integral is positive because $\mathbf{u} \neq 0$ and (f3) implies $\mathcal{F}_x(\mathbf{u}(x)) > 0$ when $\mathbf{u}(x) \neq 0_x$. □

Lemma 32.5.8. (Separation from the origin) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. There exists $\rho > 0$ such that $\|\mathbf{u}\|_X \geq \rho$ for every $\mathbf{u} \in \mathcal{N}$.

Proof. Let $\mathbf{u} \in \mathcal{N}$. By (32.20), (a1), and (32.16),

$$k_1 \|\nabla^{\mathbf{E}}\mathbf{u}\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + k_2 \|\mathbf{u}\|^q_X \leq \varepsilon \|\mathbf{u}\|^p_{L^p(M,\mathbf{E})} + C_\varepsilon \|\mathbf{u}\|^r_{L^r(M,\mathbf{E})}.$$

Let $C_p, C_r > 0$ be constants such that

$$\|\mathbf{u}\|_{L^p(M,\mathbf{E})} \leq C_p \|\nabla^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})}, \qquad \|\mathbf{u}\|_{L^r(M,\mathbf{E})} \leq C_r \|\mathbf{u}\|_X.$$

Take $\varepsilon := \dfrac{k_1}{2C_p^p}$. Then

$$\frac{k_1}{2} \|\nabla^{\mathbf{E}}\mathbf{u}\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + k_2 \|\mathbf{u}\|^q_X \leq C_\varepsilon C_r^r \|\mathbf{u}\|^r_X. \tag{32.21}$$

If $p < q$, hypothesis (a1) ensures that $k_2 > 0$. Dropping the first term on the left-hand side of (32.21) gives $k_2 \|\mathbf{u}\|^q_X \leq C_\varepsilon C_r^r \|\mathbf{u}\|^r_X$. Since $\mathbf{u} \neq 0$ and $r > q$, $\|\mathbf{u}\|_X \geq \left(\dfrac{k_2}{C_\varepsilon C_r^r} \right)^{\frac{1}{r-q}}$.

If $p = q$, we have $\|\nabla^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})} = \|\mathbf{u}\|_X$. Dropping the term containing k_2 gives $\dfrac{k_1}{2}\|\mathbf{u}\|^q_X \leq C_\varepsilon C_r^r \|\mathbf{u}\|^r_X$, and hence $\|\mathbf{u}\|_X \geq \left(\dfrac{k_1}{2C_\varepsilon C_r^r} \right)^{\frac{1}{r-q}}$.

Taking ρ to be the smaller of the constants corresponding to the possible cases completes the proof. □

Lemma 32.5.9. (Boundedness of Nehari sequences) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. If $(\mathbf{u}_j)_{j\in\mathbb{N}} \subseteq \mathcal{N}$ and $(I(\mathbf{u}_j))_{j\in\mathbb{N}}$ is bounded above, then $(\mathbf{u}_j)$ is bounded in X.

Proof. For each $\mathbf{u} \in \mathcal{N}$, using (f3), Lemma 32.5.2, and (a1), we obtain

$$I(\mathbf{u}) = I(\mathbf{u}) - \frac{1}{\theta} DI(\mathbf{u})(\mathbf{u}) \geq \int_M \left[\frac{1}{p} A(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}) - \frac{1}{\theta} a(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}) |\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}} \right] d\lambda_{\mathbf{g}}$$

$$\geq \left(\frac{1}{q} - \frac{1}{\theta} \right) \int_M a(|\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}) |\nabla^{\mathbf{E}}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}} \geq \left(\frac{1}{q} - \frac{1}{\theta} \right) \left(k_1 \|\nabla^{\mathbf{E}}\mathbf{u}\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + k_2 \|\mathbf{u}\|^q_X \right).$$

If $p < q$, then $k_2 > 0$, and $I(\mathbf{u}) \geq \left(\frac{1}{q} - \frac{1}{\theta} \right) k_2 \|\mathbf{u}\|^q_X$.

If $p = q$, then $\|\nabla^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})} = \|\mathbf{u}\|_X$ and $I(\mathbf{u}) \geq \left(\frac{1}{q} - \frac{1}{\theta} \right) k_1 \|\mathbf{u}\|^q_X$.

In both cases there exists $c_0 > 0$, independent of $\mathbf{u} \in \mathcal{N}$, such that $I(\mathbf{u}) \geq c_0 \|\mathbf{u}\|^q_X$.

The upper bound for $I(\mathbf{u}_j)$ then implies boundedness of $\|\mathbf{u}_j\|_X$. □

Lemma 32.5.10. (Convergence of the nonlinear terms) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. Let $(\mathbf{u}_j)_{j\in\mathbb{N}} \subseteq L^r(M,\mathbf{E})$ satisfy $\mathbf{u}_j \to \mathbf{u}$ in $L^r(M,\mathbf{E})$. Then $f(\mathbf{u}_j) \to f(\mathbf{u})$ in $L^{r'}(M,\mathbf{E})$, $\int_M \langle f_x(\mathbf{u}_j), \mathbf{u}_j \rangle_{\mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}} \longrightarrow \int_M \langle f_x(\mathbf{u}), \mathbf{u} \rangle_{\mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}}$, and $\int_M \mathcal{F}_x(\mathbf{u}_j) \, d\lambda_{\mathbf{g}} \longrightarrow \int_M \mathcal{F}_x(\mathbf{u}) \, d\lambda_{\mathbf{g}}$.

Proof. The first convergence follows from Corollary 29.1.9, applied to the bundle map $f : \mathbf{E} \longrightarrow \mathbf{E}$ with $m = r$.

Since $\mathbf{u}_j \to \mathbf{u}$ in $L^r(M,\mathbf{E})$, the sequence $(\mathbf{u}_j)$ is bounded in $L^r(M,\mathbf{E})$. By Hölder's inequality (Proposition B.7.9),

$$\left| \int_M \langle f_x(\mathbf{u}_j), \mathbf{u}_j \rangle_{\mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}} - \int_M \langle f_x(\mathbf{u}), \mathbf{u} \rangle_{\mathbf{h}_{\mathbf{E}}} \, d\lambda_{\mathbf{g}} \right|$$
$$\leq \|f(\mathbf{u}_j) - f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})} \|\mathbf{u}_j\|_{L^r(M,\mathbf{E})} + \|f(\mathbf{u})\|_{L^{r'}(M,\mathbf{E})} \|\mathbf{u}_j - \mathbf{u}\|_{L^r(M,\mathbf{E})}.$$

and the right-hand side converges to zero.

For the potential, Theorem B.6.5, applied in each fiber, gives

$$\mathcal{F}_x(\mathbf{u}_j) - \mathcal{F}_x(\mathbf{u}) = \int_0^1 \langle f_x(\mathbf{u} + s(\mathbf{u}_j - \mathbf{u})), \mathbf{u}_j - \mathbf{u} \rangle_{\mathbf{h}_{\mathbf{E}}} \, ds.$$

By Fubini's theorem for Bochner integrals B.6.26, applied in the scalar case, and Hölder's inequality (Proposition B.7.9),

$$\left| \int_M \mathcal{F}_x(\mathbf{u}_j) \, d\lambda_{\mathbf{g}} - \int_M \mathcal{F}_x(\mathbf{u}) \, d\lambda_{\mathbf{g}} \right| \leq \|\mathbf{u}_j - \mathbf{u}\|_{L^r(M,\mathbf{E})} \int_0^1 \|f(\mathbf{u} + s(\mathbf{u}_j - \mathbf{u}))\|_{L^{r'}(M,\mathbf{E})} \, ds.$$

The family $\mathbf{u} + s(\mathbf{u}_j - \mathbf{u})$ is bounded in $L^r(M,\mathbf{E})$ uniformly with respect to j and $s \in [0,1]$. Remark 29.1.10 shows that the norms of $f(\mathbf{u} + s(\mathbf{u}_j - \mathbf{u}))$ in $L^{r'}(M,\mathbf{E})$ are uniformly bounded. Since $\|\mathbf{u}_j - \mathbf{u}\|_{L^r(M,\mathbf{E})} \to 0$, the difference of the potentials converges to zero. □

32.5.4 Radial reduction and existence of a minimizer

Proposition 32.5.6 allows us to define $m\colon S_X \longrightarrow \mathcal{N}$ by $m(\mathbf{w}) = t_{\mathbf{w}}\mathbf{w}$. To apply Theorem 32.3.1, it remains to verify the uniform bounds required there.

Proposition 32.5.11. (Hypotheses for radial reduction) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. The space X and the functional I satisfy hypotheses (A1)–(A3) for the radial reduction of Szulkin and Weth.

Proof. Consider the linear map

$$\mathcal{G}\colon X \longrightarrow L^q(M, T^*M \otimes \mathbf{E}), \qquad \mathcal{G}(\mathbf{u}) = \nabla^{\mathbf{E}}\mathbf{u}.$$

The Poincaré inequality in Theorem 9.6.8 shows that $\mathcal{G}$ is injective. Moreover, by the definition of the norm fixed in this section,

$$\|\mathcal{G}(\mathbf{u})\|_{L^q(M,T^*M\otimes\mathbf{E})} = \|\mathbf{u}\|_X.$$

The norm $\|\cdot\|_X$ is equivalent to the norm on $W_0^{1,q}(M,\mathbf{E})$; since the latter space is closed in $W^{1,q}(M,\mathbf{E})$ and complete, $(X, \|\cdot\|_X)$ is a Banach space. Consequently, the image $\mathcal{G}(X)$ is closed in $L^q(M, T^*M\otimes\mathbf{E})$. The proof of Proposition 9.1.18 is pointwise in the fibers and uses only that they are Hilbert spaces; it therefore applies without change on the manifold with boundary and shows that $L^q(M, T^*M \otimes \mathbf{E})$ is uniformly convex. This property is inherited by closed subspaces and preserved under isometries. Thus X is uniformly convex.
To verify (A1), take the normalization function $\varphi(t) := t^{q-1}$. The associated functional is

$$\psi_X(\mathbf{u}) := \int_0^{\|\mathbf{u}\|_X} t^{q-1}\,dt = \frac{1}{q}\|\mathbf{u}\|_X^q = \frac{1}{q}\int_M |\nabla^{\mathbf{E}}\mathbf{u}|^q_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}.$$

The functional ψ_X belongs to $C^1(X,\mathbb{R})$. This is the case $p = q$, $a \equiv 1$, and $A(s) = s$ of the calculation for J_1 in the proof of Proposition 32.5.5. In this specialization,

$$D\psi_X(\mathbf{u})(\mathbf{v}) = \int_M |\nabla^{\mathbf{E}}\mathbf{u}|^{q-2}_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\langle\nabla^{\mathbf{E}}\mathbf{u}, \nabla^{\mathbf{E}}\mathbf{v}\rangle_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}}.$$

Hölder's inequality (Proposition B.7.9) implies $|D\psi_X(\mathbf{u})(\mathbf{v})| \le \|\mathbf{u}\|_X^{q-1}\|\mathbf{v}\|_X$, so the map $\mathcal{J}_X := D\psi_X\colon X \longrightarrow X'$ sends bounded sets to bounded sets. If $\mathbf{w} \in S_X$, then

$$\mathcal{J}_X(\mathbf{w})(\mathbf{w}) = \int_M |\nabla^{\mathbf{E}}\mathbf{w}|^q_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\,d\lambda_{\mathbf{g}} = \|\mathbf{w}\|_X^q = 1.$$

Thus (A1) holds.
Proposition 32.5.6 proves that, for each $\mathbf{u} \in X \setminus \{0\}$, there exists a unique $t_{\mathbf{u}} > 0$ such that $t_{\mathbf{u}}\mathbf{u} \in \mathcal{N}$, and that the function $t \mapsto I(t\mathbf{u})$ is strictly increasing on $(0, t_{\mathbf{u}})$ and strictly decreasing on $(t_{\mathbf{u}}, +\infty)$. This is precisely hypothesis (A2).
We now verify (A3). If $\mathbf{w} \in S_X$, then $t_{\mathbf{w}}\mathbf{w} \in \mathcal{N}$. By Lemma 32.5.8, there exists $\rho > 0$, independent of $\mathbf{w}$, such that $t_{\mathbf{w}} = \|t_{\mathbf{w}}\mathbf{w}\|_X \ge \rho$. This provides the uniform lower bound.
Now let $K \subseteq S_X$ be nonempty and compact; for $K = \varnothing$ the upper bound is vacuous. The function $\mathbf{w} \longmapsto \|\mathbf{w}\|^{\theta}_{L^{\theta}(M,\mathbf{E})}$ is continuous on X and positive on S_X, since $\|\mathbf{w}\|_{L^{\theta}(M,\mathbf{E})} = 0$ would imply $\mathbf{w} = 0$. By compactness,

$\mu_K := \min_{\mathbf{w}\in K} \|\mathbf{w}\|^\theta_{L^\theta(M,\mathbf{E})} > 0$.
By the upper bound for A and Lemma 32.5.4, there exists $C_K > 0$ such that, for every $\mathbf{w} \in K$ and every $t > 0$,

$$I(t\mathbf{w}) \le C_K(t^p + t^q) - D\mu_K t^\theta + D'\lambda_{\mathbf{g}}(M).$$

Since $\theta > q \ge p$, the right-hand side converges to $-\infty$ as $t \to +\infty$, uniformly with respect to $\mathbf{w} \in K$. There exists $R_K > 0$ such that $I(t\mathbf{w}) < 0$ for every $\mathbf{w} \in K$ and every $t \ge R_K$. However, $I(t_{\mathbf{w}}\mathbf{w}) > 0$ by Lemma 32.5.7. Thus $t_{\mathbf{w}} < R_K$ for every $\mathbf{w} \in K$, proving (A3). □

Proposition 32.5.12. (Existence of a minimizer on $\mathcal{N}$) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. There exists $\mathbf{u}_0 \in \mathcal{N}$ such that

$$I(\mathbf{u}_0) = \inf_{\mathbf{u}\in\mathcal{N}} I(\mathbf{u}).$$

Proof. Define $c := \inf_{\mathbf{u}\in\mathcal{N}} I(\mathbf{u})$. By Lemmas 32.5.7, 32.5.8, and the estimate in Lemma 32.5.9, we have $c > 0$.
Let $(\mathbf{u}_j)_{j\in\mathbb{N}} \subseteq \mathcal{N}$ be a sequence such that $I(\mathbf{u}_j) \to c$. Lemma 32.5.9 implies that $(\mathbf{u}_j)$ is bounded in X. Since $1 < q < \infty$, the space X is reflexive; passing to a subsequence, there exists $\mathbf{u} \in X$ such that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in X.
By Corollary 9.4.31, and since $p, r < q^*$, $\mathbf{u}_j \to \mathbf{u}$ in $L^p(M,\mathbf{E}) \cap L^r(M,\mathbf{E})$.
We shall prove that $\mathbf{u} \neq 0$. Suppose otherwise. By Lemma 32.5.10, $\int_M \langle f_x(\mathbf{u}_j), \mathbf{u}_j\rangle_{\mathbf{h}_\mathbf{E}}\, d\lambda_{\mathbf{g}} \longrightarrow 0$.
Fix $C_p > 0$ such that $\|\mathbf{v}\|^p_{L^p(M,\mathbf{E})} \le C_p \|\nabla^{\mathbf{E}}\mathbf{v}\|^p_{L^p(M,T^*M\otimes\mathbf{E})}$ for every $\mathbf{v} \in X$, and choose $\varepsilon := \dfrac{k_1}{2C_p}$. By the Nehari identity, (a1), and (32.16),

$$\begin{aligned} k_1\|\nabla^{\mathbf{E}}\mathbf{u}_j\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + k_2\|\mathbf{u}_j\|^q_X &\le \int_M \langle f_x(\mathbf{u}_j), \mathbf{u}_j\rangle_{\mathbf{h}_\mathbf{E}}\, d\lambda_{\mathbf{g}} \\ &\le \varepsilon\|\mathbf{u}_j\|^p_{L^p(M,\mathbf{E})} + C_\varepsilon\|\mathbf{u}_j\|^r_{L^r(M,\mathbf{E})} \\ &\le \frac{k_1}{2}\|\nabla^{\mathbf{E}}\mathbf{u}_j\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + C_\varepsilon\|\mathbf{u}_j\|^r_{L^r(M,\mathbf{E})}. \end{aligned}$$

Thus

$$\frac{k_1}{2}\|\nabla^{\mathbf{E}}\mathbf{u}_j\|^p_{L^p(M,T^*M\otimes\mathbf{E})} + k_2\|\mathbf{u}_j\|^q_X \le C_\varepsilon\|\mathbf{u}_j\|^r_{L^r(M,\mathbf{E})}.$$

If $p < q$, we have $k_2 > 0$ and the right-hand side converges to zero, so $\|\mathbf{u}_j\|_X \to 0$. If $p = q$, the first term on the left-hand side is $\dfrac{k_1}{2}\|\mathbf{u}_j\|^q_X$, giving the same conclusion. This contradicts $\|\mathbf{u}_j\|_X \ge \rho$. Hence $\mathbf{u} \neq 0$.
By Proposition 32.5.6, there exists a unique $t_{\mathbf{u}} > 0$ such that $t_{\mathbf{u}}\mathbf{u} \in \mathcal{N}$. Since $\mathbf{u}_j \in \mathcal{N}$ is the maximum along its own positive ray, $I(t_{\mathbf{u}}\mathbf{u}_j) \le I(\mathbf{u}_j)$ for every j.
Set $\varphi(s) := A(s^p)$ for $s \ge 0$. By (a3), φ is convex; since $a \ge 0$, A and φ are also nondecreasing. The triangle inequality and these two properties give, for $0 \le \lambda \le 1$,

$$\begin{aligned} A\big(|\lambda\eta + (1-\lambda)\zeta|^p_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}}\big) &= \varphi\big(|\lambda\eta + (1-\lambda)\zeta|_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}}\big) \\ &\le \varphi\big(\lambda|\eta|_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}} + (1-\lambda)|\zeta|_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}}\big) \\ &\le \lambda A(|\eta|^p_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}}) + (1-\lambda)A(|\zeta|^p_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}}). \end{aligned}$$

Thus $\eta \mapsto p^{-1}A(|\eta|^p_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}})$ is convex and continuous. Consequently, for the fixed number $t_\mathbf{u} > 0$, the functional

$$G(\mathbf{v}) := \frac{1}{p}\int_M A(t_\mathbf{u}^p|\nabla^\mathbf{E}\mathbf{v}|^p_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}})\,d\lambda_\mathbf{g}$$

is convex and continuous on X. By Corollary B.11.17, G is weakly lower semicontinuous; therefore,

$$\frac{1}{p}\int_M A(t_\mathbf{u}^p|\nabla^\mathbf{E}\mathbf{u}|^p_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}})\,d\lambda_\mathbf{g} \le \liminf_{j\to\infty}\frac{1}{p}\int_M A(t_\mathbf{u}^p|\nabla^\mathbf{E}\mathbf{u}_j|^p_{\mathbf{g}\otimes\mathbf{h}_\mathbf{E}})\,d\lambda_\mathbf{g}.$$

Moreover, $t_\mathbf{u}\mathbf{u}_j \to t_\mathbf{u}\mathbf{u}$ in $L^r(M,\mathbf{E})$, so Lemma 32.5.10 implies

$$\int_M \mathcal{F}_x(t_\mathbf{u}\mathbf{u}_j)\,d\lambda_\mathbf{g} \longrightarrow \int_M \mathcal{F}_x(t_\mathbf{u}\mathbf{u})\,d\lambda_\mathbf{g}.$$

Consequently, $I(t_\mathbf{u}\mathbf{u}) \le \liminf_{j\to\infty} I(t_\mathbf{u}\mathbf{u}_j) \le \lim_{j\to\infty} I(\mathbf{u}_j) = c$.
Since $t_\mathbf{u}\mathbf{u} \in \mathcal{N}$, the definition of c gives $c \le I(t_\mathbf{u}\mathbf{u})$. Thus $I(t_\mathbf{u}\mathbf{u}) = c$. Taking $\mathbf{u}_0 := t_\mathbf{u}\mathbf{u}$ completes the proof. □

Theorem 32.5.13. (Existence of a ground state) Let M be the compact Riemannian manifold with nonempty boundary fixed in this section, and let $\mathbf{E} \to M$ be the vector bundle specified there. Under hypotheses (a1)–(a3) and (f1)–(f4), problem (32.14) has a ground state solution: there exists $\mathbf{u}_0 \in X \setminus \{0\}$ such that $DI(\mathbf{u}_0) = 0$ and

$$I(\mathbf{u}_0) = \inf\{I(\mathbf{u}) \mid \mathbf{u} \in X \setminus \{0\},\ DI(\mathbf{u}) = 0\}.$$

Proof. By Proposition 32.5.12, there exists $\mathbf{u}_0 \in \mathcal{N}$ minimizing I on $\mathcal{N}$. Propositions 32.5.6 and 32.5.11 allow us to apply Theorem 32.3.1. Let $\mathbf{w}_0 := m^{-1}(\mathbf{u}_0) \in S_X$. Since m is a homeomorphism and $\Psi = I \circ m$, the point $\mathbf{w}_0$ minimizes Ψ on the submanifold S_X. Fermat's theorem on manifolds gives $D\Psi(\mathbf{w}_0) = 0$; by part 5 of Theorem 32.3.1, $\mathbf{u}_0 = m(\mathbf{w}_0)$ is a nontrivial critical point of I. By Proposition 32.5.5, $\mathbf{u}_0$ is a weak solution of (32.14).
Every nontrivial critical point $\mathbf{u}$ of I satisfies $DI(\mathbf{u})(\mathbf{u}) = 0$ and therefore belongs to $\mathcal{N}$. It follows that

$$I(\mathbf{u}_0) = \inf_{\mathbf{u}\in\mathcal{N}} I(\mathbf{u}) = \inf\{I(\mathbf{u}) \mid \mathbf{u} \neq 0,\ DI(\mathbf{u}) = 0\}.$$

□

32.5.5 Examples of operators covered by the result

The following examples show that the hypotheses on a encompass several familiar geometric operators. In all of them, we may take

$$\mathcal{F}_x(\xi) := \frac{1}{r}|\xi|^r_{\mathbf{h}_\mathbf{E}}, \qquad f_x(\xi) = |\xi|^{r-2}_{\mathbf{h}_\mathbf{E}}\xi, \qquad r \in (q, q^*).$$

These conditions are verified directly, since $\dfrac{|f_x(\xi)|_{\mathbf{h}_\mathbf{E}}}{|\xi|^{p-1}_{\mathbf{h}_\mathbf{E}}} = |\xi|^{r-p}_{\mathbf{h}_\mathbf{E}} \longrightarrow 0$ uniformly with respect to x, because $r > q \ge p$; moreover, $|f_x(\xi)|_{\mathbf{h}_\mathbf{E}} = |\xi|^{r-1}_{\mathbf{h}_\mathbf{E}}$, $\langle f_x(\xi),\xi\rangle_{\mathbf{h}_\mathbf{E}} = |\xi|^r_{\mathbf{h}_\mathbf{E}} = r\mathcal{F}_x(\xi)$, and, for $\xi \neq 0_x$, $\dfrac{\langle f_x(t\xi), t\xi\rangle_{\mathbf{h}_\mathbf{E}}}{t^q} = t^{r-q}|\xi|^r_{\mathbf{h}_\mathbf{E}}$, which is strictly increasing. Thus (f1)–(f4) hold for any $\theta \in (q, r]$.

Example 32.5.14. (Bochner Laplacian of type p) Suppose that $p = q$ and $a(s) := 1$. Then $a(t^p)t^p = t^p$, so (a1) holds with $k_1 = k_3 = 1$ and $k_2 = k_4 = 0$. Since $\dfrac{a(t^p)}{t^{q-p}} = 1$, (a2) holds. Finally, $A(s) = s$ and $A(t^p) = t^p$ is convex, so (a3) also holds. The principal term in (32.14) reduces to $(\nabla^{\mathbf{E}})^*\big(|\nabla^{\mathbf{E}}\mathbf{u}|^{p-2}_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\nabla^{\mathbf{E}}\mathbf{u}\big)$.

Example 32.5.15. (Bochner operator with (p,q) growth) Let $a(s) := 1 + s^{\frac{q-p}{p}}$. Then $a(t^p)t^p = t^p + t^q$, so (a1) holds with $k_1 = k_2 = k_3 = k_4 = 1$ when $p < q$. If $p = q$, then $a(s) = 2$ for every $s \geq 0$, and (a1) holds, for example, with $k_1 = k_3 = 2$ and $k_2 = k_4 = 0$. For $p < q$, $\dfrac{a(t^p)}{t^{q-p}} = 1 + t^{p-q}$, which is strictly decreasing; for $p = q$ it is constant. Moreover,

$$A(s) = s + \frac{p}{q}s^{\frac{q}{p}}, \qquad A(t^p) = t^p + \frac{p}{q}t^q,$$

and the latter function is convex. Thus (a1)–(a3) hold, and the principal term is the sum $(\nabla^{\mathbf{E}})^*\big(|\nabla^{\mathbf{E}}\mathbf{u}|^{p-2}_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\nabla^{\mathbf{E}}\mathbf{u}\big) + (\nabla^{\mathbf{E}})^*\big(|\nabla^{\mathbf{E}}\mathbf{u}|^{q-2}_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\nabla^{\mathbf{E}}\mathbf{u}\big)$.

Example 32.5.16. (Perturbation of mean curvature type) Suppose that $q = p$ and

$$a(s) := 1 + \frac{1}{(1+s)^{\frac{p-1}{p}}}.$$

Since $1 \leq a(s) \leq 2$, we have $t^p \leq a(t^p)t^p \leq 2t^p$, so (a1) holds with $k_1 = 1$, $k_2 = 0$, $k_3 = 2$, and $k_4 = 0$. Because $q - p = 0$ and $t \mapsto a(t^p)$ is nonincreasing, (a2) holds. Direct integration gives $A(s) = s + p\big((1+s)^{\frac{1}{p}} - 1\big)$, and hence $A(t^p) = t^p + p\big((1+t^p)^{\frac{1}{p}} - 1\big)$.

The function $t \mapsto t^p$ is convex, and $t \mapsto (1+t^p)^{\frac{1}{p}}$ is the restriction to $[0,+\infty)$ of the ℓ^p norm of the vector $(1,t)$; consequently, $A(t^p)$ is convex and (a3) holds. In this case, the additional term

$$(\nabla^{\mathbf{E}})^*\left(\frac{|\nabla^{\mathbf{E}}\mathbf{u}|^{p-2}_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}}\nabla^{\mathbf{E}}\mathbf{u}}{(1+|\nabla^{\mathbf{E}}\mathbf{u}|^{p}_{\mathbf{g}\otimes\mathbf{h}_{\mathbf{E}}})^{\frac{p-1}{p}}}\right).$$

appears.

Remark 32.5.17. The condition $k_2 > 0$ is required only when $p < q$. This distinction is necessary in the coercive estimates on $\mathcal{N}$: if $p < q$, the term $k_2\|\mathbf{u}\|_X^q$ directly controls the norm of X; if $p = q$, this control is provided by the term with coefficient k_1, because $\|\nabla^{\mathbf{E}}\mathbf{u}\|_{L^p(M,T^*M\otimes\mathbf{E})} = \|\mathbf{u}\|_X$.

32.6 Exercises

Exercise 32.6.1. Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold with smooth boundary, such that each connected component has nonempty boundary. For the functional in the model case, prove that for each $u \neq 0$ there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$, and compute t_u explicitly. Also show that $u \mapsto t_u u$ is a homeomorphism between the unit sphere of $W^{1,2}_0(M)$ and $\mathcal{N}$.

Exercise 32.6.2. Let $(M,\mathbf{g})$ be a nonempty compact Riemannian manifold with smooth boundary, such that each connected component has nonempty boundary, and let $\mathbf{E}\to M$ be a smooth real vector bundle of positive finite rank, equipped with a bundle metric $\mathbf{h}_{\mathbf{E}}$ and a connection $\nabla^{\mathbf{E}}$ compatible with $\mathbf{h}_{\mathbf{E}}$. Suppose that a satisfies (a1)–(a3) and that $\mathcal{F}$ satisfies (f1)–(f4). Prove directly, without using the existence of the minimizer, that for each $\mathbf{u}\in X\setminus\{0\}$ there exist numbers $0<\alpha_{\mathbf{u}}<\beta_{\mathbf{u}}$ such that $I(t\mathbf{u})>0$ if $0<t<\alpha_{\mathbf{u}}$ and $I(t\mathbf{u})<0$ if $t>\beta_{\mathbf{u}}$. As a hint, first obtain explicit constants $C_1(\mathbf{u}),C_2(\mathbf{u})>0$ for which

$$I(t\mathbf{u})\geq C_1(\mathbf{u})t^{p}-C_2(\mathbf{u})t^{r}$$

when $t>0$, and then use the lower bound of order θ for the potential. Explain why this argument alone does not establish uniqueness of the critical point of the radial function, and identify the precise role of (a2) and (f4) in the proof of uniqueness.

Appendices

A

Topology

This appendix collects the topological notions used in constructing manifolds, bundles, and quotient spaces, together with the results from algebraic topology needed for index theory. For the latter results, we specify the conventions used in the main chapters and give precise references for the topological proofs themselves.

Before beginning, we state the maximality principle that will be used to construct separated nets in metric spaces.

Theorem A.0.1. (Zorn's lemma) Let $(P, \preccurlyeq)$ be a nonempty partially ordered set. If every chain $\mathcal{C} \subseteq P$ has an upper bound in P, then P contains a maximal element.

Remark A.0.2. Zorn's lemma is equivalent to the axiom of choice. In this text, it is used only to construct maximal families of separated balls.

A.1 Topological spaces

Definition A.1.1. (Topological space) A *topological space* is a pair (X, τ), where X is a set and τ is a family of subsets of X such that:

(a) $\varnothing, X \in \tau$,

(b) every arbitrary union of sets in τ belongs to τ,

(c) every finite intersection of sets in τ belongs to τ.

The elements of τ are called *open subsets* of X.

Definition A.1.2. (Subspace topology) Let (X, τ) be a topological space and let $A \subseteq X$. The *subspace topology* on A is

$$\tau_A = \{U \cap A \mid U \in \tau\}.$$

Then (A, τ_A) is a topological space called a *subspace* of X.

Definition A.1.3. (Quotient map and quotient topology) Let (X, τ_X) be a topological space, let Y be a set, and let $q\colon X \longrightarrow Y$ be a surjective map. The *quotient topology* on Y induced by q is defined by

$$\tau_q = \{V \subseteq Y \mid q^{-1}(V) \in \tau_X\}.$$

With this topology, $q\colon (X, \tau_X) \longrightarrow (Y, \tau_q)$ is continuous, and q is called a *quotient map*. Equivalently, a continuous surjective map between topological spaces is a quotient map if $U \subseteq Y$ is open whenever $q^{-1}(U)$ is open in X.

Definition A.1.4. (Quotient space) Let X be a topological space and let $\sim$ be an equivalence relation on X. The *quotient space* $X/\sim$ is defined as the set of equivalence classes

$$X/\sim = \{[x] \mid x \in X\},$$

equipped with the quotient topology induced by the natural projection

$$q\colon X \longrightarrow X/\sim, \qquad q(x) = [x].$$

Thus, $U \subseteq X/\sim$ is open if and only if $q^{-1}(U)$ is open in X.

Example A.1.5. (Circle) The circle $\mathbb{S}^1$ can be described as the quotient

$$\mathbb{S}^1 \cong [0,1]/\sim,$$

where $0 \sim 1$ and all other points of $[0,1]$ remain distinct.

Example A.1.6. (Cylinder) The cylinder is obtained as

$$C \cong \big([0,1] \times [0,1]\big)/\sim,$$

where we identify $(0, y) \sim (1, y)$ for every $y \in [0,1]$. In this case the vertical sides are glued together without a twist.

Example A.1.7. (Möbius strip) The Möbius strip is obtained similarly as

$$M \cong \big([0,1] \times [0,1]\big)/\sim,$$

where the equivalence relation is

$$(0, y) \sim (1, 1-y), \qquad y \in [0,1].$$

This corresponds to gluing the vertical sides of the square after giving one of them a half-turn.

Lemma A.1.8. (Characterization of quotient maps) Let $q\colon X \longrightarrow Y$ be a continuous surjective map between topological spaces. If q is closed, then q is a quotient map; that is, a subset $U \subseteq Y$ is open if and only if $q^{-1}(U)$ is open in X.

Proof. Continuity proves one implication. Conversely, if $q^{-1}(U)$ is open, then $q^{-1}(Y \setminus U)$ is closed. Since q is closed and surjective,

$$Y \setminus U = q\big(q^{-1}(Y \setminus U)\big)$$

is closed. Therefore, U is open. □

A.2 Covering spaces

Definition A.2.1. (Covering map) Let $p\colon Y \longrightarrow X$ be a continuous surjective map between topological spaces. We say that p is a *covering map* if for every $x \in X$ there is an open subset $U \subseteq X$ with $x \in U$ such that

$$p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha,$$

where each V_α is open in Y and the restriction $p\restriction_{V_\alpha}\colon V_\alpha \longrightarrow U$ is a homeomorphism.

Definition A.2.2. (Universal covering map) A covering map $p\colon \widetilde{X} \longrightarrow X$ is called a *universal covering map* if $\widetilde{X}$ is simply connected.

Proposition A.2.3. (Universal covering map of the circle) The map $\varepsilon\colon \mathbb{R} \longrightarrow \mathbb{S}^1$ defined by $\varepsilon(x) = e^{2\pi i x}$ is a universal covering map of $\mathbb{S}^1$.

Proof. Let $z_0 = e^{2\pi i a}$. If

$$U = \left\{ e^{2\pi i t} \,\middle|\, a - \frac{1}{4} < t < a + \frac{1}{4} \right\},$$

then

$$\varepsilon^{-1}(U) = \coprod_{k \in \mathbb{Z}} \left(a - \frac{1}{4} + k, a + \frac{1}{4} + k \right),$$

and each interval projects homeomorphically onto U. Thus, ε is a covering map. Moreover, $\mathbb{R}$ is contractible by the homotopy $H(x,t) = (1-t)x$ and is therefore simply connected. □

Definition A.2.4. (Evenly covered open subsets) Let $p\colon Y \longrightarrow X$ be a covering map of topological spaces. We say that a connected open subset $U \subseteq X$ is *evenly covered* by p if each connected component $\widetilde{U}$ of $p^{-1}(U)$ projects homeomorphically onto U under the restriction $p\restriction_{\widetilde{U}}\colon \widetilde{U} \longrightarrow U$.

Theorem A.2.5. (Arcs covered by the exponential) Let $U \subsetneq \mathbb{S}^1$ be an open arc. Then $\varepsilon^{-1}(U)$ is a disjoint union of open intervals, and the restriction

$$\varepsilon\restriction_{\widetilde{U}}\colon \widetilde{U} \longrightarrow U$$

is a homeomorphism for each component $\widetilde{U}$ of $\varepsilon^{-1}(U)$. Thus every proper open arc is evenly covered in the sense of Definition A.2.4.

Proof. There exist $a \in \mathbb{R}$ and $0 < \ell \leq 1$ such that

$$U = \{e^{2\pi i t} \mid a < t < a + \ell\}.$$

Consequently,

$$\varepsilon^{-1}(U) = \coprod_{k \in \mathbb{Z}} (a + k, a + \ell + k).$$

On each interval in this union, ε is continuous and injective, and its inverse is a continuous branch of the argument divided by 2π. Thus each restriction is a homeomorphism onto U. □

A.3 Complex bundles and stable equivalence

In what follows, topological vector bundles have finite rank and complex fibers. Compact base spaces are assumed to be Hausdorff. Noncompact base spaces are assumed to be locally compact, Hausdorff, and paracompact. Paracompactness is the hypothesis that allows the construction of the partitions of unity used to complement and smooth bundles; local compactness and separation ensure that the one-point compactification is Hausdorff. All manifolds in this book satisfy these conditions.

Theorem A.3.1. (Stable complement) Let $\mathbf{E} \longrightarrow X$ be a complex vector bundle over a compact Hausdorff space. There exist a bundle $\mathbf{E}' \longrightarrow X$ and an integer $N \geq 0$ such that

$$\mathbf{E} \oplus \mathbf{E}' \cong X \times \mathbb{C}^N.$$

In particular, every stable bundle class can be represented as the difference between a bundle and a trivial bundle.

The proof, including the construction of an embedding of $\mathbf{E}$ into a trivial bundle by means of a finite partition of unity, can be found in [76, Proposition 1.4, pp. 13–14]. The equivalent formulation in terms of the Grothendieck group appears in [21, Theorem 10.5 and Exercises 10.9–10.12, pp. 262–265].

Theorem A.3.2. (Homotopy invariance of pullback) Let $\mathbf{E} \longrightarrow Y$ be a complex vector bundle and let $f_0, f_1 \colon X \longrightarrow Y$ be homotopic maps. If X is paracompact, then

$$f_0^* \mathbf{E} \cong f_1^* \mathbf{E}.$$

Consequently, the induced homomorphisms $f_0^*, f_1^* \colon K^0(Y) \longrightarrow K^0(X)$ agree.

A proof that trivializes the induced bundle over pieces of $X \times [0,1]$ and glues the consecutive identifications is given in [76, Theorem 1.6, pp. 19–21].

Proposition A.3.3. (Gluing bundles) Let $X = X_0 \cup X_1$, with X_0 and X_1 closed, and let $\mathbf{E}_i \longrightarrow X_i$ be complex bundles. Each isomorphism

$$\varphi \colon \mathbf{E}_0 \restriction_{X_0 \cap X_1} \longrightarrow \mathbf{E}_1 \restriction_{X_0 \cap X_1}$$

determines a bundle $\mathbf{E}_0 \cup_\varphi \mathbf{E}_1 \longrightarrow X$, unique up to isomorphism. A homotopy of gluing isomorphisms produces isomorphic bundles. In particular, bundles over a suspension are described by transition functions

on the equator.

The quotient construction and the verification of local triviality are developed in [76, Section 1.2, pp. 22–27]. The parametrized form used in Bott periodicity can be found in [21, Exercise 10.19, pp. 269–270].

A.4 Relative K-theory and compact support

For a compact pointed space (X, x_0), define

$$\widetilde{K}^0(X) := \ker\bigl(i_{x_0}^* \colon K^0(X) \longrightarrow K^0(\{x_0\})\bigr).$$

If $A \subseteq X$ is closed, set

$$K^0(X, A) := \widetilde{K}^0(X/A).$$

For a locally compact space Y, denote its one-point compactification by $Y^+ = Y \cup \{\infty\}$ and set

$$K_c^0(Y) := \widetilde{K}^0(Y^+).$$

If SX is the reduced suspension of a pointed space X, we adopt the convention

$$\widetilde{K}^1(X) := \widetilde{K}^0(SX), \qquad K_c^1(Y) := \widetilde{K}^1(Y^+).$$

Proposition A.4.1. (Suspension and product with the line) For every locally compact Hausdorff paracompact space Y, there is a natural isomorphism

$$K_c^1(Y) \cong K_c^0(\mathbb{R} \times Y). \tag{A.1}$$

This isomorphism is compatible with pullbacks by proper maps, the external product, and extensions associated with open inclusions.

Proof. The one-point compactification of the product admits a pointed homeomorphism

$$(\mathbb{R} \times Y)^+ \cong S^1 \wedge Y^+.$$

To see this, take the one-point compactification of $\mathbb{R}$ and consider the quotient map

$$S^1 \times Y^+ \longrightarrow \frac{S^1 \times Y^+}{S^1 \times \{\infty\} \cup \{\infty\} \times Y^+} = S^1 \wedge Y^+.$$

Its restriction to $\mathbb{R} \times Y$ is a homeomorphism onto the complement of the base point. Neighborhoods of the base point correspond exactly to complements of compact subsets of $\mathbb{R} \times Y$; the universal property of the one-point compactification gives the stated homeomorphism. Consequently,

$$K_c^0(\mathbb{R} \times Y) = \widetilde{K}^0\bigl((\mathbb{R} \times Y)^+\bigr) \cong \widetilde{K}^0(S^1 \wedge Y^+) = \widetilde{K}^1(Y^+) = K_c^1(Y).$$

All these identifications arise from natural quotient maps. Naturality with respect to proper maps and compatibility with external products follow. For an open inclusion, the homeomorphism preserves the

base point and the complement of the open subset, so it also commutes with extension by zero. □

It is also useful to have a description in which the support remains explicit. If $K \subseteq Y$ is compact, let $\mathcal{K}^0_K(Y)$ denote the group of classes of complexes

$$\mathbf{E}^0 \xrightarrow{a} \mathbf{E}^1$$

of complex bundles over Y for which a_y is invertible when $y \notin K$. Two complexes are identified if, after adding complexes invertible throughout Y, they can be joined by isomorphisms and a homotopy of bundle homomorphisms that remains invertible outside K. For the spaces above, there is a natural identification

$$K^0_c(Y) \cong \varinjlim_{K \Subset Y} \mathcal{K}^0_K(Y).$$

Support is a notion belonging to this model by complexes, in which the bundle homomorphism is defined on all of Y; no support is assigned to a relative triple whose bundle homomorphism is defined only on the subspace of the pair.

Theorem A.4.2. (Exactness, excision, and external product) The following properties hold.

(i) For every compact pair (X, A) with A closed, the cyclic sequence

$$K^0(X,A) \longrightarrow K^0(X) \longrightarrow K^0(A) \longrightarrow K^1(X,A) \longrightarrow K^1(X) \longrightarrow K^1(A) \longrightarrow K^0(X,A)$$

is exact.

(ii) If $U \subseteq X$ is open and $A \subseteq X$ is closed, with $\overline{U} \subseteq \operatorname{int}(A)$, the quotient of pairs induces the corresponding excision isomorphisms. In particular, if $j\colon V \hookrightarrow Y$ is an open inclusion of locally compact spaces, there is an extension homomorphism

$$j_!\colon K^q_c(V) \longrightarrow K^q_c(Y), \qquad q \in \{0,1\},$$

that depends only on the inclusion and agrees with extension by zero for complexes exact outside a compact set. More precisely, if $K \Subset V$, restriction to V and extension to Y induce inverse isomorphisms

$$\mathcal{K}^0_K(V) \xrightarrow{\cong} \mathcal{K}^0_K(Y).$$

Taking the limit over compact subsets $K \Subset V$ gives $j_!$.

(iii) For locally compact spaces X, Y, there is a natural associative external product

$$\boxtimes\colon K^p_c(X) \otimes K^q_c(Y) \longrightarrow K^{p+q}_c(X \times Y),$$

where the degree is taken modulo two. The product commutes with proper pullbacks and extensions associated with open inclusions.

Exactness in degree zero is proved in [76, Proposition 2.9, pp. 51–52]; suspension and periodicity produce the full cyclic sequence in [76, Section 2.2, pp. 51–59]. The construction using one-point compactification and the external product can be found in [21, Section 10.4 and Theorem 10.16, pp. 266–268]. The description by

complexes with support and excision in the form just stated are proved in [21, Definition 12.6 and subsequent discussion, pp. 287–289].

Theorem A.4.3. (Relative representation by triples) Let X be a compact Hausdorff space and let $A \subseteq X$ be closed. The classes of $K^0(X, A)$, with the convention $K^0(X, \varnothing) = K^0(X)$, are represented by triples $(\mathbf{E}, \mathbf{F}, \alpha)$, where $\mathbf{E}, \mathbf{F} \longrightarrow X$ are complex bundles and

$$\alpha : \mathbf{E}\restriction_A \longrightarrow \mathbf{F}\restriction_A$$

is an isomorphism. Two triples represent the same class if, after adding triples that extend to isomorphisms over X, they are related by an isomorphism and a homotopy of the maps over A. Addition is direct sum, and

$$[(\mathbf{E}, \mathbf{F}, \alpha)] + [(\mathbf{F}, \mathbf{G}, \gamma)] = [(\mathbf{E}, \mathbf{G}, \gamma \circ \alpha)].$$

If $Y = X \setminus A$ and $X/A \cong Y^{+}$, this model identifies the group with $K^0_c(Y)$. Under this identification, the model by complexes described above allows us to choose a bundle homomorphism defined on Y that is invertible outside a compact set.

The construction using the double of X, gluing over A, and the difference element can be found in [21, Definition 12.6, pp. 287–289]; the relative model is developed in [9, Chapter I, Sections 2–3]. For compact Hausdorff spaces, extension of a trivialization from a closed subset and exactness are established in [76, Proposition 2.9 and Section 2.2, pp. 51–55]. The proof of Theorem 17.2.4 constructs the isomorphism using the double and verifies the relations, surjectivity, and injectivity by means of these properties. In particular, the statement applies to $(X \times \Lambda, A \times \Lambda)$ and its homotopies for any compact Hausdorff space Λ, without requiring Λ to be CW.

Theorem A.4.4. (Smoothing bundles and homomorphisms) Let M be a compact smooth manifold with or without boundary.

(i) Every continuous complex vector bundle over M is isomorphic, as a continuous bundle, to a smooth complex vector bundle.

(ii) If $\mathbf{E}, \mathbf{F} \longrightarrow M$ are smooth, every continuous bundle homomorphism $a : \mathbf{E} \longrightarrow \mathbf{F}$ can be uniformly approximated by smooth bundle homomorphisms.

(iii) If a is an isomorphism and the approximation a_1 satisfies

$$\sup_{x \in M} \| a_1(x) - a(x) \| < \left(\sup_{x \in M} \| a(x)^{-1} \| \right)^{-1},$$

then the segment $a_t = (1 - t)a + t a_1$ consists of isomorphisms. In particular, a and a_1 define the same relative class.

The first assertion and approximation compatible with trivializations are proved in [82, p. 101]; see also [21, Exercise B.13(b), pp. 719–720]. The final assertion follows from

$$a_t = a\big(I + t\, a^{-1}(a_1 - a)\big)$$

and the Neumann series, since the norm of the second summand is less than one uniformly in x and t.

A.5 Bott periodicity and complex orientation

Let $\mathbf{H} \longrightarrow \mathbb{CP}^1 \cong \mathbb{S}^2$ be the tautological line bundle. We identify $\mathbb{R}^2$ with $\mathbb{S}^2 \setminus \{\infty\}$ and take as the Bott class

$$\beta := [\mathbf{H}^*] - [\mathbb{C}] \in \widetilde{K}^0(\mathbb{S}^2) = K_c^0(\mathbb{R}^2).$$

With the gluing in Definition 18.1.1, the tautological bundle is $\mathbf{H} = \mathbf{E}_1$ and its dual is $\mathbf{H}^* = \mathbf{E}_{-1}$; hence β agrees with the class b used in Chapter 18. This identification is checked directly, since dualizing a bundle inverts its gluing function. The description of the tautological bundle through gluing by z is given in [21, Exercise B.11, p. 719]. In the index theory chapters, this choice is retained in all Thom isomorphisms. In particular, when a complex line is identified with $\mathbb{R}^2$ by $(u, v) \mapsto u + iv$ and $J(u, v) = (-v, u)$ is used, the Thom class satisfies $\lambda_{\mathbb{C}} = -\beta$. This convention simultaneously fixes the sign of the Bott generator, that of the Euclidean model operator, and that of the normal orientation in the embedding proof.

Theorem A.5.1. (Complex Bott periodicity) For every locally compact space X and every $q \in \{0, 1\}$, external product with β defines a natural isomorphism

$$\mathcal{B}_X \colon K_c^q(X) \longrightarrow K_c^q(X \times \mathbb{R}^2), \qquad \mathcal{B}_X(a) = a \boxtimes \beta.$$

In particular,

$$K_c^0(\mathbb{R}^{2r}) \cong \mathbb{Z}, \qquad K_c^0(\mathbb{R}^{2r+1}) = 0,$$

and $\beta^{\boxtimes r}$ generates the first group.

The proof based on families of Wiener–Hopf operators, including the normalization of the generator and the verification that the two homomorphisms are inverses, is given in [21, Theorems 10.20 and 10.22, pp. 270–272]. Another complete proof, formulated using the fundamental product, appears in [76, Theorems 2.2 and 2.11, pp. 41–55]. The original source for the formulation by complex bundles is [10, pp. 229–247].

Constructing the pushforward for a nontrivial normal bundle requires the equivariant version of this normalization. If a compact group G acts on a space Y, let $K_{G,c}^0(Y)$ denote the group defined by G-equivariant complexes that are exact outside a compact set. For a point, one obtains the complex representation ring

$$K_G^0(\{*\}) = R(G).$$

Theorem A.5.2. (Equivariant Bott normalization) Let $G = O(r)$ act on $\mathbb{R}^r$ by the standard representation, and let $i \colon \{0\} \hookrightarrow \mathbb{R}^r$. The equivariant Thom class

$$\beta_r := i_!(1) \in K_{G,c}^0(T^*\mathbb{R}^r)$$

has a G-equivariant elliptic representative on the compactification $S^r = \mathbb{R}^r \cup \{\infty\}$, elementary near ∞, whose equivariant index satisfies

$$\operatorname{Ind}_G(\beta_r) = [\mathbb{C}] \quad \text{in } R(G),$$

where G acts trivially on $\mathbb{C}$. The result is natural under formation of bundles associated with a principal bundle $O(r)$.

The construction of the class using the frame bundle, equivariant descent, and conic cutoffs is developed in [21, equations (12.8)–(12.16), pp. 293–301]. The equivariant index identity is formulated in [21, p. 295]. The precise role of [11, Proposition 4.4, p. 505] is as follows: it proves that axioms (B1), (B2′), and (B3″) imply the Bott axiom (B2). Within that proof, the normalization

$$\operatorname{Ind}_G(i_!1) = 1 \quad \text{in } R(G),$$

is established first for blocks of rank one and two, using $O(1)$ and $SO(2)$, and then in every rank by block decomposition and multiplicativity. Thus the proposition does not assume the identity needed here: it obtains that identity as the normalization step recovering the full Bott axiom.

This normalization agrees with the conventions of the book. In rank one, we write $z = u+iv$ and $J(u,v) = (-v,u)$; the associated Koszul symbol has model index $+1$. The gluing function chosen for b has the opposite orientation, so $\lambda_{\mathbb{C}} = -b$, as in (18.7). Consequently, $i_!(1) = \lambda_{\mathbb{C}}$ has equivariant index equal to the trivial representation, with no isolated change of sign. This is the equivariant normalization used for the normal bundle of the embedding.

Now let $\pi\colon \mathbf{V} \longrightarrow X$ be a complex Hermitian bundle of rank r. On $\mathbf{V}$, consider the graded bundles

$$\pi^*\Lambda^{\mathrm{ev}}\mathbf{V} := \bigoplus_{j \text{ even}} \pi^*\Lambda^j\mathbf{V}, \qquad \pi^*\Lambda^{\mathrm{odd}}\mathbf{V} := \bigoplus_{j \text{ odd}} \pi^*\Lambda^j\mathbf{V}.$$

At the point $v \in \mathbf{V}_x$, exterior multiplication by v and its adjoint define

$$c_{\mathbf{V}}(v) := \varepsilon(v) - \varepsilon(v)^*\colon \Lambda^{\mathrm{ev}}\mathbf{V}_x \longrightarrow \Lambda^{\mathrm{odd}}\mathbf{V}_x.$$

The exterior algebra relations give

$$c_{\mathbf{V}}(v)^* c_{\mathbf{V}}(v) = \|v\|^2 \operatorname{Id},$$

so $c_{\mathbf{V}}(v)$ is invertible for $v \neq 0$. The triple consisting of the two bundles and $c_{\mathbf{V}}$ determines the Thom class

$$\lambda_{\mathbf{V}} \in K^0_X(\mathbf{V}),$$

where the subscript indicates support on the zero section. If X is compact, this group is identified with $K^0_c(\mathbf{V})$. For the trivial line identified with $\mathbb{R}^2$ by $(u,v) \mapsto u + iv$ and with $J(u,v) = (-v,u)$, this convention satisfies

$$\lambda_{\mathbb{C}} = -\beta.$$

The verification using transition functions is given in (18.7).

Theorem A.5.3. (Thom isomorphism in K-theory) For every complex vector bundle $\pi\colon \mathbf{V} \longrightarrow X$ over a compact base, the homomorphism

$$\mathrm{Th}_{\mathbf{V}}\colon K^0(X) \longrightarrow K^0_c(\mathbf{V}), \qquad \mathrm{Th}_{\mathbf{V}}(a) = \pi^*a \cdot \lambda_{\mathbf{V}},$$

is an isomorphism. It is natural with respect to pullbacks, multiplicative with respect to direct sums, and compatible with the external product. If the base is merely locally compact, the class with vertical support induces the isomorphism

$$\mathrm{Th}_{\mathbf{V}}\colon K^0_c(X) \longrightarrow K^0_c(\mathbf{V}).$$

The statement, its normalization, and the comparison with Bott appear in [21, Theorem 12.8 and Remark 12.9, pp. 289–290]. A proof using Clifford modules is developed in [92, Appendix C, Theorem C.8 and Remark C.13]. Independence of the Hermitian metric also follows directly from the contractibility of the space of such metrics.

A.6 Pushforwards in K-theory

Let $f\colon X \hookrightarrow Y$ be a proper embedding of smooth manifolds, and let $\nu_f \longrightarrow X$ be its normal bundle. When ν_f has a complex structure, a tubular neighborhood identifies a neighborhood of the zero section of ν_f with an open subset $U \subseteq Y$. Define

$$f_!\colon K_c^0(X) \longrightarrow K_c^0(Y), \qquad f_! := j_! \circ \mathrm{Th}_{\nu_f},$$

where $j\colon U \hookrightarrow Y$ is the open inclusion.

Theorem A.6.1. (Properties of the pushforward) The preceding homomorphism is independent of the metric, tubular neighborhood, and auxiliary identifications. Moreover:

(i) isotopic embeddings, with compatible complex orientations of their normal bundles, induce the same map;

(ii) if $f\colon X \hookrightarrow Y$ and $g\colon Y \hookrightarrow Z$ are proper embeddings with compatible orientations, then $(g\circ f)_! = g_! \circ f_!$;

(iii) for $a \in K_c^0(X)$ and $b \in K_c^0(Y)$, the projection formula holds:

$$f_!\big(a \cdot f^*b\big) = f_!(a)\cdot b;$$

(iv) pushforwards commute with external products.

The proofs using homotopies of tubular neighborhoods, naturality of Thom, and excision are given in [92, Chapter III, §§12–13]. The construction in the form used for the index theorem appears in [21, (12.3)–(12.4), pp. 290–291].

For an embedding $f\colon M \hookrightarrow \mathbb{R}^N$, the tangent embedding $Tf\colon TM \hookrightarrow T\mathbb{R}^N$ has normal bundle isomorphic to

$$\pi^*(\nu_f \oplus \nu_f) \longrightarrow TM.$$

The rule

$$J(u,v) = (-v,u)$$

canonically makes it a complex bundle. Thus, without imposing a complex structure on M, there is a map

$$(Tf)_!\colon K_c^0(TM) \longrightarrow K_c^0(T\mathbb{R}^N) \cong K_c^0(\mathbb{R}^{2N}).$$

This is the complex orientation that enters the definition of the general topological index.

Theorem A.6.2. (Whitney embedding theorem) Every Hausdorff second-countable smooth manifold of dimension n, with or without boundary, admits a proper smooth embedding into $\mathbb{R}^{2n+1}$. If the manifold is compact, its image is automatically closed.

The proof using local embeddings, partitions of unity, and generic projections can be found in [94, Theorem 6.15]. The existence and construction of the required tubular neighborhoods were already proved in Theorem 2.12.3.

A.7 Characteristic classes and comparison of Thom isomorphisms

Let $\mathbf{E} \longrightarrow X$ be a complex bundle of rank r. The splitting principle allows us, after a pullback that is injective in cohomology, to work as though

$$\mathbf{E} = \mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_r.$$

If $x_j = c_1(\mathbf{L}_j)$, the elements $x_1, \ldots, x_r$ are called formal Chern roots. The following symmetric expressions descend to well-defined classes on X:

$$c(\mathbf{E}) = \prod_{j=1}^{r} (1 + x_j),$$

$$\operatorname{ch}(\mathbf{E}) = \sum_{j=1}^{r} e^{x_j},$$

$$\operatorname{Td}(\mathbf{E}) = \prod_{j=1}^{r} \frac{x_j}{1 - e^{-x_j}}.$$

In each fixed dimension, the series truncate, so the last two expressions belong to $H^{\mathrm{ev}}(X;\mathbb{Q})$. For a formal difference, set

$$\operatorname{ch}([\mathbf{E}] - [\mathbf{F}]) = \operatorname{ch}(\mathbf{E}) - \operatorname{ch}(\mathbf{F}).$$

Theorem A.7.1. (Chern character) The Chern character defines a natural transformation of rings

$$\operatorname{ch}\colon K^0(X) \longrightarrow H^{\mathrm{ev}}(X;\mathbb{Q}).$$

If X is a finite CW complex, it induces an isomorphism

$$K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^{\mathrm{ev}}(X;\mathbb{Q}).$$

There is also a compactly supported version

$$\operatorname{ch}_c\colon K_c^0(Y) \longrightarrow H_c^{\mathrm{ev}}(Y;\mathbb{Q})$$

compatible with external products, proper pullbacks, and extensions associated with open inclusions.

Chern classes, the splitting principle, and the Chern character are constructed in [76, Sections 2.3, 3.1 and 4.1, pp. 66–87 and 109–112]. The compactly supported version and its compatibility with Thom are developed in [21, Section 13.1, pp. 311–315].

Theorem A.7.2. (Cohomological Thom isomorphism and noncompact fundamental class) Let $\pi\colon \mathbf{V} \longrightarrow X$ be an oriented real bundle of rank r over a compact base of finite CW type. There is a class $U_{\mathbf{V}} \in H_c^r(\mathbf{V};\mathbb{Z})$ whose restriction to each oriented fiber generates $H_c^r(\mathbb{R}^r;\mathbb{Z})$, and product with this class defines an

isomorphism

$$\mathrm{Th}^{H}_{\mathbf{V}}\colon H^{q}(X;R)\longrightarrow H^{q+r}_{c}(\mathbf{V};R),\qquad \mathrm{Th}^{H}_{\mathbf{V}}(\omega)=\pi^{*}\omega\smile U_{\mathbf{V}},$$

for $R=\mathbb{Z},\mathbb{Q}$ or $\mathbb{R}$. The Thom class is natural under oriented pullbacks and multiplicative for oriented direct sums.

If Y is an oriented smooth manifold without boundary of dimension d, not necessarily compact, it has a Borel–Moore fundamental class $[Y]_{\mathrm{BM}}\in H^{\mathrm{BM}}_{d}(Y;\mathbb{Z})$. The pairing

$$H^{d}_{c}(Y;\mathbb{R})\times H^{\mathrm{BM}}_{d}(Y;\mathbb{Z})\longrightarrow\mathbb{R}$$

agrees, under the compactly supported de Rham theorem, with $[\omega]\mapsto\displaystyle\int_{Y}\omega$.

The construction of the cohomological orientation, the compactly supported Thom isomorphism, its naturality, and evaluation on the fundamental class are developed in [21, Section 13.1, pp. 311–315].

With the Thom class fixed above, comparison of the Thom isomorphisms in K-theory and cohomology introduces the Todd factor. Under the exterior multiplication convention of $\lambda_{\mathbf{V}}$, for a complex bundle of rank r one has

$$\mathrm{ch}_{c}(\lambda_{\mathbf{V}})=(-1)^{r}\,\mathrm{Th}^{H}_{\mathbf{V}}\big(\mathrm{Td}(\overline{\mathbf{V}})^{-1}\big),$$

where $\mathrm{Th}^{H}_{\mathbf{V}}$ is the Thom isomorphism in cohomology with rational coefficients.

Theorem A.7.3. (Cohomological formula for the topological index) Let M be a closed smooth manifold of dimension n, and let $u\in K^{0}_{c}(T^{*}M)$. Identify $T^{*}M$ with TM using a metric and orient TM by the almost complex structure that takes the horizontal directions as real and the vertical directions as imaginary. If $a=(\sharp_{g})_{*}u\in K^{0}_{c}(TM)$, then

$$\mathrm{ind}_{t}(u)=(-1)^{n}\int_{TM}\mathrm{ch}_{c}(a)\smile\pi^{*}\,\mathrm{Td}(TM\otimes_{\mathbb{R}}\mathbb{C}).$$

The integrand has compact support in the fiber direction; the integral means evaluation of its degree-$2n$ component on the fundamental class of TM.

The full comparison of the two Thom classes, including the conjugate bundle and the sign $(-1)^{n}$ for these conventions, is given in [21, Theorem 13.1, pp. 313–315]. This is the formulation used in the chapter devoted to the Atiyah–Singer theorem.

A.8 De Rham cohomology, spin structures, and the Euler class

The results in this section are used only to identify the indices of geometric operators topologically. Their proofs belong to algebraic and differential topology and are not reproduced here.

We first recall the obstruction that arises for spin structures. For a real vector bundle $\mathbf{V}\to X$ of rank r over a paracompact space, its *Stiefel–Whitney classes* are the classes

$$w_{j}(\mathbf{V})\in H^{j}(X;\mathbb{Z}_{2}),\qquad 0\le j\le r,$$

characterized by naturality, $w_0(\mathbf{V}) = 1$, the Whitney formula

$$w(\mathbf{V} \oplus \mathbf{W}) = w(\mathbf{V}) \smile w(\mathbf{W}), \qquad w(\mathbf{V}) := 1 + w_1(\mathbf{V}) + \cdots + w_r(\mathbf{V}),$$

and the normalization $w_1(\gamma^1) \neq 0$ for the real tautological bundle $\gamma^1 \to \mathbb{RP}^\infty$. In particular,

$$w_2(\mathbf{V}) \in H^2(X; \mathbb{Z}_2)$$

is the degree-two component of the total class. The vanishing of $w_1(\mathbf{V})$ is equivalent to orientability of $\mathbf{V}$; once an orientation has been chosen, $w_2(\mathbf{V})$ is the obstruction to lifting the principal bundle of oriented frames through the covering map $\mathrm{Spin}(r) \to \mathrm{SO}(r)$.

The universal construction of these classes, the orientation criterion, and the interpretation of w_2 as the spin obstruction can be found in [21, Definition 17.18 and Proposition 17.20, pp. 527–529]. We will use only those topological results; Clifford algebras, the spin covering map, and spinor bundles are constructed geometrically in Section 2.17.

Theorem A.8.1. (De Rham theorem and Euler characteristic) Let M be a smooth manifold with or without boundary. Integration of forms over smooth chains induces, for each q, a natural isomorphism

$$H^q_{\mathrm{dR}}(M; \mathbb{C}) \cong H^q_{\mathrm{sing}}(M; \mathbb{C}).$$

If M is a closed manifold of dimension n, these spaces are finite-dimensional and

$$\chi(M) = \sum_{q=0}^{n} (-1)^q \dim_{\mathbb{C}} H^q_{\mathrm{dR}}(M; \mathbb{C}).$$

The identification with singular cohomology and its relation to the Euler characteristic are presented in [21, Section 13.4, in particular (13.18), pp. 319–323]; the Hodge–de Rham decomposition that complements this identification appears in [21, Theorem 17.62 and Corollary 17.63, pp. 602–604].

Theorem A.8.2. (Euler class and Euler characteristic) Let $\mathbf{V} \longrightarrow X$ be an oriented real vector bundle of rank r over a compact base. If $\mathbf{s}_0 \colon X \longrightarrow \mathbf{V}$ is the zero section and $U_{\mathbf{V}}$ is the cohomological Thom class, the Euler class is defined by

$$e(\mathbf{V}) := \mathbf{s}_0^* U_{\mathbf{V}} \in H^r(X; \mathbb{Z}).$$

This class is natural under oriented pullbacks and multiplicative with respect to oriented direct sums. If M is a closed oriented smooth manifold of dimension n, then

$$\langle e(TM), [M] \rangle = \chi(M).$$

The construction using the zero section and evaluation on the fundamental class can be found in [21, Theorem 13.6(d), pp. 320–321] when the dimension is even. In odd dimensions, Poincaré duality pairs Betti numbers in complementary degrees and gives $\chi(M) = 0$. On the other hand, $-\mathrm{Id}_{TM}$ reverses the orientation of each fiber, so naturality of the Thom class implies $e(TM) = -e(TM)$. Since $H^n(M; \mathbb{Z})$ is free for a closed oriented manifold, it follows that $e(TM) = 0$. This proves the statement in every dimension. This equality is the general topological form of the Gauss–Bonnet–Chern theorem.

Let

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

be the exact sequence of coefficients, and let

$$\beta \colon H^2(X;\mathbb{Z}_2) \longrightarrow H^3(X;\mathbb{Z})$$

be its Bockstein homomorphism. For an oriented real bundle $\mathbf{V} \to X$, define the third integral Stiefel–Whitney class by

$$W_3(\mathbf{V}) := \beta\big(w_2(\mathbf{V})\big).$$

By exactness, $W_3(\mathbf{V}) = 0$ if and only if $w_2(\mathbf{V})$ is the reduction modulo two of a class in $H^2(X;\mathbb{Z})$.

Theorem A.8.3. (Spin^c orientation) An oriented real bundle $\mathbf{V} \longrightarrow X$ over a paracompact base admits a Spin^c structure if and only if $W_3(\mathbf{V}) = 0$; equivalently, if and only if $w_2(\mathbf{V})$ is the reduction modulo two of an integral class of degree two. The determinant line $\mathbf{L}$ of a chosen structure satisfies

$$c_1(\mathbf{L}) \bmod 2 = w_2(\mathbf{V}).$$

If the rank of $\mathbf{V}$ is even, Clifford multiplication between the half-spinor bundles defines a Thom class in $K^0_X(\mathbf{V})$, the group with support on the zero section; when X is compact, this is identified with $K^0_c(\mathbf{V})$. Every complex bundle $\mathbf{E}$, regarded as an oriented real bundle, has a canonical Spin^c structure induced by $U(r) \to \mathrm{Spin}^c(2r)$; its determinant line is $\det_{\mathbb{C}} \mathbf{E}$ and its Thom class agrees with the class constructed using the exterior algebra.

The existence criterion is proved in [21, Theorem 18.35, pp. 655–656] and in [92, Appendix D, Theorem D.2]. The canonical orientation of a complex bundle and comparison of the two Thom classes are developed in [92, Appendix D, Example D.6; Appendix C, Theorems C.8 and C.12].

Theorem A.8.4. (Spin obstruction and half-spinor bundles) Let $\mathbf{V} \to X$ be an oriented real bundle of rank n over a paracompact base. Its oriented frame bundle admits a principal lift with group $\mathrm{Spin}(n)$ if and only if

$$w_2(\mathbf{V}) = 0 \in H^2(X;\mathbb{Z}_2).$$

When such a lift exists, the set of equivalence classes of spin structures is a principal homogeneous space for $H^1(X;\mathbb{Z}_2)$. If $n = 2m$, each spin structure determines Hermitian half-spinor bundles $\mathbf{S}^+$ and $\mathbf{S}^-$, and Clifford multiplication by a nonzero vector defines an isomorphism

$$c(v) \colon \mathbf{S}^+_x \longrightarrow \mathbf{S}^-_x, \qquad v \in \mathbf{V}_x \setminus \{0\}.$$

The class of this isomorphism is the spinorial Thom orientation of $\mathbf{V}$ in complex K-theory, with support on the zero section. If the formal roots of $\mathbf{V} \otimes_{\mathbb{R}} \mathbb{C}$ are ordered as $x_1, -x_1, \dots, x_m, -x_m$, then

$$\mathrm{ch}(\mathbf{S}^+) - \mathrm{ch}(\mathbf{S}^-) = \prod_{j=1}^{m} \left(e^{-\frac{x_j}{2}} - e^{\frac{x_j}{2}} \right), \qquad \widehat{A}(\mathbf{V}) = \prod_{j=1}^{m} \frac{\frac{x_j}{2}}{\sinh\left(\frac{x_j}{2}\right)}.$$

Here the orientation is chosen so that $e(\mathbf{V}) = x_1 \cdots x_m$; the order of the two exponents is determined by the

chirality convention in Section 2.17.

The construction of w_2 and the existence criterion can be found in [21, Definition 17.18 and Proposition 17.20, pp. 527–529]. The bundles $\mathbf{S}^{\pm}$, Clifford multiplication, and the twisted Dirac operator are constructed in [21, Definitions 17.22–17.23, pp. 531–532]; the comparison of its symbol class with the Thom class is developed in [92, Appendix C, Theorem C.12]. The calculation of the weights of the half-spinor representations that produces $\operatorname{ch}(\mathbf{S}^+)-\operatorname{ch}(\mathbf{S}^-)$ is given in [21, pp. 519–522]; the class $\widehat{A}$ and its relation to the Dirac index appear in [21, equations (17.43)–(17.45), pp. 545–546].

Theorem A.8.5. (Chern defect of the spinorial Thom class) Let $\mathbf{V} \to X$ be a real spin bundle of rank $2m$ over a compact base, and let

$$s(\mathbf{V}) = [\pi^*\mathbf{S}^+, \pi^*\mathbf{S}^-; c(v)] \in K_c^0(\mathbf{V})$$

be its spinorial Thom class with the preceding orientations. Then

$$(\mathrm{Th}_{\mathbf{V}}^{H})^{-1} \operatorname{ch}_c\big(s(\mathbf{V})\big) = (-1)^m \widehat{A}(\mathbf{V})^{-1}. \tag{A.2}$$

The formula, including the sign determined by the choice of chirality, is Proposition 12.5 in Chapter III, §12, of [92].

Proposition A.8.6. (Topological facts for extinction by surgery) Let $M_1, \dots, M_k$ be closed connected manifolds of dimension at least three. Then

$$\pi_1(M_1 \# \cdots \# M_k) \cong \pi_1(M_1) * \cdots * \pi_1(M_k).$$

If $N = S^3/\Gamma$ is a three-dimensional spherical space form, where Γ is finite and acts freely, then $\pi_1(N) \cong \Gamma$; in particular, N is simply connected if and only if $N \cong S^3$. If $Z \to S^1$ is a bundle with fiber S^2, trivial or twisted, then $\pi_1(Z) \cong \mathbb{Z}$.
Consequently, if a closed connected manifold M is a connected sum of spherical space forms and S^2-bundles over S^1, and $\pi_1(M) = 0$, there are no bundle factors and all the groups Γ are trivial. Since S^3 is the identity for connected sum, $M \cong S^3$.

Proof. The first identity is Seifert–van Kampen applied to the neck of the connected sum. The second follows because S^3 is the universal covering space of S^3/Γ, and the third follows from the homotopy exact sequence of the bundle, since $\pi_1(S^2) = 0$. A free product is trivial only if each of its factors is trivial; this gives the conclusion. □

Every simply connected manifold is orientable. Moreover, a closed two-sided surface in an orientable 3-manifold inherits an orientation from the ambient orientation and a global normal; hence such a manifold contains no two-sided embedded $\mathbb{RP}^2$. The topological effect of surgeries and the reconstruction by connected sums that we will use are formulated in [104, Chapter 15]; the extinction used at the end is treated in Chapters 18–19 of the same reference.

Theorem A.8.7. (Gauss–Bonnet for surfaces) Let $(M, \mathbf{g})$ be a closed Riemannian surface and let $K_{\mathbf{g}}$ be its

Gaussian curvature. Then

$$\int_M K_{\mathbf{g}}\, d\lambda_{\mathbf{g}} = 2\pi\chi(M).$$

Since $R_{\mathbf{g}} = 2K_{\mathbf{g}}$, it follows that

$$\int_M R_{\mathbf{g}}\, d\lambda_{\mathbf{g}} = 4\pi\chi(M).$$

The formulation using the Euler class can be found in [21, Theorem 13.6(d), pp. 320–321]; a proof using the Gauss–Bonnet–Chern form appears in [21, Theorem 17.68, pp. 612–613]. If M is nonorientable, apply the formula to its orientable two-sheeted covering space: the integral and the Euler characteristic are both multiplied by two, so the same identity descends to M.

B

Mathematical analysis

This appendix collects the results from real analysis, metric spaces, functional analysis, measure theory, and Lebesgue spaces used throughout the book. Its purpose is not to replace a systematic treatment of these subjects, but to fix notation, recall frequently used tools, and provide precise internal references for the main chapters. More extensive treatments can be found in [4, 68, 16, 38, 45, 28, 87, 123, 120].

The order follows the natural dependencies between the topics. We begin with some facts from real analysis and metric spaces; continue with normed spaces, linear operators, and Hilbert spaces; and then study measure theory, the Lebesgue and Bochner integrals, and L^p spaces. Finally, we introduce topological vector spaces and weak topologies, which play an essential role in the compactness and semicontinuity arguments used in the calculus of variations.

B.1 Real analysis and metric spaces

We begin with some basic facts from real analysis and metric spaces. These results will be used mainly to control local arguments, completions, and convergence.

Proposition B.1.1. Let $f:(a,b)\longrightarrow\mathbb{R}$ be a function differentiable on $(a,b)\setminus\{c\}$, with $c\in(a,b)$, and continuous at (a,b), such that $\lim_{x\to c^+} f'(x) = L = \lim_{x\to c^-} f'(x)$. Then f is differentiable at c and $f'(c) = L$.

Proof. Let $x\in(a,b)\setminus\{c\}$. If $x>c$, the mean value theorem applied to f on $[c,x]$ gives a point $\xi_x\in(c,x)$ such that

$$\frac{f(x)-f(c)}{x-c} = f'(\xi_x).$$

As $x\to c^+$, necessarily $\xi_x\to c^+$, and therefore

$$\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} = L.$$

If $x < c$, the same argument applied to the interval $[x, c]$ produces $\xi_x \in (x, c)$ and gives

$$\frac{f(x) - f(c)}{x - c} = f'(\xi_x).$$

Now $\xi_x \to c^-$ as $x \to c^-$, so the left-hand limit of the difference quotient is also L. The two one-sided limits agree; consequently, f is differentiable at c and $f'(c) = L$. □

Proposition B.1.2. For every $x \in \mathbb{R}^n$, the following hold:

(a) $\|x\|_r \le \|x\|_s$ if $1 \le s \le r \le \infty$,

(b) $\|x\|_s \le n^{\frac{r-s}{sr}} \|x\|_r$ if $1 \le s \le r < \infty$,

(c) $\|x\|_\infty \le \|x\|_s \le n^{\frac{1}{s}} \|x\|_\infty$ if $1 \le s < \infty$.

In particular, all norms $\|\cdot\|_p$ with $p \in [1, \infty]$ are equivalent on $\mathbb{R}^n$.

Proof. If $x = 0$, all the inequalities are immediate. Suppose that $x \neq 0$. First let $1 \le s \le r < \infty$ and set

$$a_i := \frac{|x_i|}{\|x\|_s}, \qquad 1 \le i \le n.$$

Then $0 \le a_i \le 1$ and $\sum_{i=1}^n a_i^s = 1$. Since $r \ge s$, we have $a_i^r \le a_i^s$, and hence

$$\frac{\|x\|_r^r}{\|x\|_s^r} = \sum_{i=1}^n a_i^r \le \sum_{i=1}^n a_i^s = 1.$$

This proves $\|x\|_r \le \|x\|_s$. For $r = \infty$, the same inequality follows directly from $|x_i|^s \le \sum_{j=1}^n |x_j|^s$ for each i.

For the reverse inequality, the case $s = r$ is trivial. If $1 \le s < r < \infty$, apply Hölder's inequality with conjugate exponents r/s and $r/(r-s)$:

$$\sum_{i=1}^n |x_i|^s \le \left(\sum_{i=1}^n |x_i|^r \right)^{s/r} \left(\sum_{i=1}^n 1^{r/(r-s)} \right)^{(r-s)/r} = \|x\|_r^s n^{1-s/r}.$$

Taking sth roots gives

$$\|x\|_s \le n^{1/s - 1/r} \|x\|_r = n^{(r-s)/(sr)} \|x\|_r.$$

Finally, for $1 \le s < \infty$,

$$\|x\|_\infty^s \le \sum_{i=1}^n |x_i|^s \le n \|x\|_\infty^s,$$

and taking roots gives part (c). The inequalities in parts (a)–(c) show that the identity between any two of these normed spaces is continuous in both directions; hence all the indicated norms are equivalent. □

Proposition B.1.3. If $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \longrightarrow \mathbb{R}^m$ is of class C^1, then for every compact convex set $K \subseteq \Omega$, the restriction $f \restriction_K$ is Lipschitz.

Proof. If $K = \varnothing$, there is nothing to prove. Since Df is continuous and K is compact, there exists

$$L := \max_{z \in K} \|Df(z)\|_{\mathrm{op}} < \infty.$$

Given $x, y \in K$, convexity of K implies that the segment $\gamma(t) := x + t(y - x)$, $0 \le t \le 1$, is contained in K. The chain rule and the fundamental theorem of calculus, applied componentwise to $f \circ \gamma$, give

$$f(y) - f(x) = \int_0^1 \frac{d}{dt} f(\gamma(t))\, dt = \int_0^1 Df(\gamma(t))(y - x)\, dt.$$

Consequently,

$$\|f(y) - f(x)\| \le \int_0^1 \|Df(\gamma(t))\|_{\mathrm{op}} \|y - x\|\, dt \le L\|y - x\|.$$

Thus $f \restriction_K$ is L-Lipschitz. □

Remark B.1.4. The convexity hypothesis can be replaced by uniform control of the lengths of curves in K joining any two points of K. In particular, the result applies locally on closed balls contained in Ω, which is the case used most frequently.

Definition B.1.5. Let X be a set. A **metric** on X is a function $d \colon X \times X \longrightarrow [0, \infty)$ such that:

(a) $d(x, y) = 0$ if and only if $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A **metric space** is a pair (X, d), where X is a set and d is a metric on X.

Theorem B.1.6. (McShane extension theorem) Let (X, d) be a metric space, $A \subseteq X$ a nonempty subset, and $f \colon A \longrightarrow \mathbb{R}$ an L-Lipschitz function, with $L \ge 0$. Then there is an extension $F \colon X \longrightarrow \mathbb{R}$ of f that is also L-Lipschitz. An explicit choice is

$$F(x) := \inf_{a \in A} \big(f(a) + L\, d(x, a) \big).$$

Proof. Fix $a_0 \in A$. For any $x \in X$ and $a \in A$, the Lipschitz property and the triangle inequality give

$$f(a) + L\, d(x, a) \ge f(a_0) - L\, d(a, a_0) + L\, d(x, a) \ge f(a_0) - L\, d(x, a_0).$$

On the other hand, taking $a = a_0$ gives $F(x) \le f(a_0) + L\, d(x, a_0)$. Thus the infimum defining $F(x)$ is finite. If $x \in A$, the inequality $f(x) \le f(a) + L\, d(x, a)$ holds for every $a \in A$, so $f(x) \le F(x)$; choosing $a = x$ gives the reverse inequality. Thus $F \restriction_A = f$. Finally, for any $x, y \in X$ and $a \in A$,

$$f(a) + L\, d(x, a) \le f(a) + L\, d(y, a) + L\, d(x, y).$$

Taking the infimum over a gives $F(x) \le F(y) + L\, d(x, y)$. Interchanging x and y yields $|F(x) - F(y)| \le L\, d(x, y)$. □

Definition B.1.7. A sequence in a metric space $X = (X, d)$ is a function $x\colon \mathbb{N} \longrightarrow X$. The value of this function at k is called the k**th term** of the sequence and is denoted by x_k. A sequence is usually denoted by (x_k) or $\{x_k\}_{k\in\mathbb{N}}$, or by similar notation according to context.
We say that (x_k) converges to a point $x \in X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $d(x_k, x) < \varepsilon$.
We say that a sequence (x_k) is Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $k, l \geq N$, then $d(x_k, x_l) < \varepsilon$.

In every metric space, convergent sequences are Cauchy, but a Cauchy sequence need not converge. This leads to the following definition:

Definition B.1.8. A metric space (X, d) is a **complete metric space** if every Cauchy sequence in X converges in X.

Definition B.1.9. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $\varphi\colon X \longrightarrow Y$ is an **isometry** if $d_Y(\varphi(x), \varphi(y)) = d_X(x, y)$ for all $x, y \in X$.

Definition B.1.10. Let X be a metric space. A complete metric space $\overline{X}$ is called a ***completion*** of X if there is an isometry $i\colon X \longrightarrow \overline{X}$ such that $i(X)$ is dense in $\overline{X}$.

We will prove that every metric space admits a completion up to isometry, but first we need some lemmas.

Lemma B.1.11. Let (X, d) be a metric space, (b_n) a Cauchy sequence in X, and (a_n) a sequence in X such that $d(a_n, b_n) < \dfrac{1}{n}$ for every $n \in \mathbb{N}$. Then:

(a) (a_n) is a Cauchy sequence in X.

(b) $a_n \to x \in X$ if and only if $b_n \to x \in X$.

Proof. For (a), the triangle inequality gives

$$d(a_n, a_m) \leq d(a_n, b_n) + d(b_n, b_m) + d(b_m, a_m)$$

, and by hypothesis there exists $N_1 \in \mathbb{N}$ such that if $n, m \geq N_1$, then $d(a_n, b_n) < \dfrac{\varepsilon}{3}$ and $d(b_m, a_m) < \dfrac{\varepsilon}{3}$.
Finally, since (b_n) is Cauchy, there exists $N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$, then $d(b_n, b_m) < \dfrac{\varepsilon}{3}$. Thus, if $n, m \geq \max\{N_1, N_2\}$, then $d(a_n, a_m) < 3\dfrac{\varepsilon}{3} = \varepsilon$.
For (b), if $a_n \to x$, then $d(a_n, x) \to 0$ whenever $n \to \infty$. By the triangle inequality, $d(b_n, x) \leq d(b_n, a_n) + d(a_n, x) < \dfrac{1}{n} + d(a_n, x)$. Taking $n \to \infty$, we conclude that $d(b_n, x) \to 0$, and hence $b_n \to x$. The converse is proved analogously. □

Theorem B.1.12. Every metric space admits a completion, unique up to isometry.

Proof. Let $CS(X)$ be the set of Cauchy sequences in X. On $CS(X)$, define the relation

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Reflexivity and symmetry are immediate. For transitivity, if $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$, the triangle inequality gives $0 \le d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$; passing to the limit yields $(x_n) \sim (z_n)$. Thus $\sim$ is an equivalence relation.

Let $\overline{X} := CS(X)/\sim$. For $[(x_n)], [(y_n)] \in \overline{X}$, set

$$\overline{d}\big([(x_n)], [(y_n)]\big) := \lim_{n\to\infty} d(x_n, y_n).$$

The limit exists, since for any $m, n \in \mathbb{N}$ the triangle inequality implies

$$\big|d(x_n, y_n) - d(x_m, y_m)\big| \le d(x_n, x_m) + d(y_n, y_m),$$

and the right-hand side tends to zero because (x_n) and (y_n) are Cauchy. This estimate shows that $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$.

The definition is independent of the representatives. If $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, then

$$\big|d(x_n, y_n) - d(x'_n, y'_n)\big| \le d(x_n, x'_n) + d(y_n, y'_n),$$

so the two limits agree. The metric properties follow directly from the corresponding properties of d; in particular, the triangle inequality is obtained by passing to the limit in $d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$. Thus $(\overline{X}, \overline{d})$ is a metric space.

For each $x \in X$, let $x^* := [(x, x, \dots)]$ and define $i\colon X \longrightarrow \overline{X}$ by $i(x) := x^*$. If $x, y \in X$, then

$$\overline{d}(i(x), i(y)) = \lim_{n\to\infty} d(x, y) = d(x, y),$$

so i is an isometric embedding. Its image is dense. Given a class $[(x_n)] \in \overline{X}$, the sequence (x^*_m) satisfies

$$\overline{d}(x^*_m, [(x_n)]) = \lim_{n\to\infty} d(x_m, x_n) \longrightarrow 0$$

as $m \to \infty$, since (x_n) is Cauchy.

We now prove that $\overline{X}$ is complete. Let (a_n) be a Cauchy sequence in $\overline{X}$. By density of $i(X)$, for each n we can choose $x_n \in X$ such that $\overline{d}(i(x_n), a_n) < \frac{1}{n}$. Lemma B.1.11 shows that $(i(x_n))$ is Cauchy in $\overline{X}$. Since i is an isometry, (x_n) is Cauchy in X, and the construction of $\overline{X}$ gives $i(x_n) \to [(x_n)]$. Another application of the same lemma yields $a_n \to [(x_n)]$. This proves that $\overline{X}$ is a completion of X.

It remains to verify uniqueness. Let (Y, d_Y) be another completion of X, and let $j\colon X \longrightarrow Y$ be an isometric embedding with dense image. For each $y \in Y$, choose a sequence (x_n) in X such that $j(x_n) \to y$ and define

$$G(y) := [(x_n)] \in \overline{X}.$$

The map G is well defined: if also $j(x'_n) \to y$, then $d(x_n, x'_n) = d_Y(j(x_n), j(x'_n)) \to 0$, so both sequences determine the same class. Moreover, for $y, y' \in Y$ and sequences $j(x_n) \to y$, $j(x'_n) \to y'$, we have

$$\overline{d}(G(y), G(y')) = \lim_{n\to\infty} d(x_n, x'_n) = \lim_{n\to\infty} d_Y(j(x_n), j(x'_n)) = d_Y(y, y').$$

Therefore G is an isometry. It is surjective because every class $[(x_n)] \in \overline{X}$ is obtained from the limit in Y of the Cauchy sequence $(j(x_n))$. Consequently, G is a bijective isometry and the completion is unique up to

isometry. □

Completeness yields one of the structural principles underlying much of functional analysis. We first recall the corresponding topological notion.

Definition B.1.13. A topological space is called a **Baire space** if every countable intersection of open dense subsets is dense. Equivalently, no nonempty open subset can be written as a countable union of closed sets with empty interior.

Theorem B.1.14. (Baire category theorem) Every complete metric space is a Baire space.

Proof. Let (X,d) be a complete metric space, and let $(G_n)_{n\in\mathbb{N}}$ be open dense subsets of X. We must prove that $\bigcap_{n=1}^{\infty} G_n$ meets every nonempty open subset $U \subseteq X$. Choose $x_1 \in U \cap G_1$ and $r_1 \in (0,1)$ such that $\overline{B}_d(x_1,r_1) \subseteq U \cap G_1$. Once x_n and r_n have been chosen, density of G_{n+1} allows us to choose $x_{n+1} \in B_d(x_n, \frac{r_n}{2}) \cap G_{n+1}$ and $r_{n+1} \in (0, 2^{-n-1})$ so that $\overline{B}_d(x_{n+1},r_{n+1}) \subseteq B_d(x_n,\frac{r_n}{2}) \cap G_{n+1}$.
This construction implies that $\overline{B}_d(x_{n+1},r_{n+1}) \subseteq \overline{B}_d(x_n,r_n)$ and $d(x_{n+1},x_n) < 2^{-n}$. Therefore (x_n) is Cauchy and converges to some $x \in X$. For each n, all terms x_m with $m \geq n$ belong to $\overline{B}_d(x_n,r_n)$; since this ball is closed, $x \in \overline{B}_d(x_n,r_n) \subseteq G_n$. Moreover, $x \in \overline{B}_d(x_1,r_1) \subseteq U$. Consequently, $x \in U \cap \bigcap_{n=1}^{\infty} G_n$. □

B.2 Normed spaces

Normed spaces provide the basic setting for measuring the size of vectors and operators. In this section we fix the notation for $\mathcal{L}(X,Y)$, recall the fundamental theorems on continuous linear operators, and introduce closed and compact operators. The exposition follows, among other references, [45, Chapters 1 and 2] and [28, 87, 123].

Definition B.2.1. Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A **norm** on X is a function $\|\cdot\|_X \colon X \longrightarrow [0,\infty)$ such that, for all $x,y \in X$ and $\lambda \in \mathbb{K}$,

(a) $\|x\|_X = 0$ if and only if $x = 0$;

(b) $\|\lambda x\|_X = |\lambda| \|x\|_X$;

(c) $\|x+y\|_X \leq \|x\|_X + \|y\|_X$.

The pair $(X, \|\cdot\|_X)$ is called a **normed space**. The distance induced by the norm is $d_X(x,y) := \|x-y\|_X$.

Definition B.2.2. A normed space is called a **Banach space** if it is complete with respect to the distance induced by its norm.

The metric completion of a normed space naturally retains the vector space operations.

Theorem B.2.3. (Completion of a normed space) For every normed space $(X, \|\cdot\|_X)$, there exist a Banach space $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ and a linear isometry $\iota \colon X \longrightarrow \widehat{X}$ with dense image. This completion is unique up to linear isometries that fix the copy of X.

Proof. Let $(\widehat{X}, \widehat{d})$ be a completion of the metric space (X, d_X), and let $\iota\colon X \longrightarrow \widehat{X}$ be the isometry with dense image provided by Theorem B.1.12. Given $u, v \in \widehat{X}$, choose sequences (x_n) and (y_n) in X such that $\iota(x_n) \to u$ and $\iota(y_n) \to v$, and define

$$u + v := \lim_{n\to\infty} \iota(x_n + y_n), \qquad \lambda u := \lim_{n\to\infty} \iota(\lambda x_n).$$

The inequalities $\|(x_n + y_n) - (x_m + y_m)\|_X \leq \|x_n - x_m\|_X + \|y_n - y_m\|_X$ and $\|\lambda x_n - \lambda x_m\|_X = |\lambda|\|x_n - x_m\|_X$ show that the limits exist. The same estimates show that they do not depend on the chosen sequences. These operations make $\widehat{X}$ a vector space and make ι linear.
Define $\|u\|_{\widehat{X}} := \widehat{d}(u, 0)$. If $\iota(x_n) \to u$, then $\|u\|_{\widehat{X}} = \lim_{n\to\infty} \|x_n\|_X$. The norm properties follow by taking limits in the corresponding properties of $\|\cdot\|_X$. Since $\widehat{X}$ is complete with respect to $\widehat{d}$, it is a Banach space. Metric uniqueness of the completion and density of $\iota(X)$ imply that the isometry between two completions is necessarily linear. □

B.2.1 Continuous linear operators

Definition B.2.4. Let X and Y be vector spaces over the same field $\mathbb{K}$. A map $T\colon X \longrightarrow Y$ is called a **linear operator** if $T(x + y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$.

Definition B.2.5. Let X and Y be normed spaces. A linear operator $T\colon X \longrightarrow Y$ is called **bounded** if there exists $C \geq 0$ such that $\|T(x)\|_Y \leq C\|x\|_X$ for every $x \in X$.
Let $\mathcal{L}(X, Y)$ denote the vector space of all continuous linear operators from X to Y, and write $\mathcal{L}(X) := \mathcal{L}(X, X)$. The **operator norm** is given by

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{\substack{x\in X \\ \|x\|_X \leq 1}} \|T(x)\|_Y = \sup_{x\in X\setminus\{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

When there is no risk of confusion, we will simply write $\|T\|$. The identity operator on X is denoted by I_X, or by I when the space is clear from context.

Definition B.2.6. The **continuous dual** of a normed space X is $X' := \mathcal{L}(X, \mathbb{K})$. Its elements are called **continuous linear functionals**.

Proposition B.2.7. Let X and Y be normed spaces, and let $T\colon X \longrightarrow Y$ be linear. The following statements are equivalent:

(a) T is continuous on X;

(b) T is continuous at 0;

(c) T is bounded.

In this case, $\|T(x)\|_Y \leq \|T\|_{\mathcal{L}(X,Y)}\|x\|_X$ for every $x \in X$.

Proof. The implication $(a) \Rightarrow (b)$ is immediate. Suppose that (b). There exists $\delta > 0$ such that $\|T(x)\|_Y < 1$ whenever $\|x\|_X < \delta$. If $x \neq 0$, then $\left\| \frac{\delta x}{2\|x\|_X} \right\|_X = \frac{\delta}{2}$ and, by linearity, $\|T(x)\|_Y < \frac{2}{\delta}\|x\|_X$. The same inequality is

trivial for $x = 0$, so T is bounded. Finally, if (c) holds with a constant C, then $\|T(x) - T(y)\|_Y \le C\|x - y\|_X$ for all $x, y \in X$, and hence T is Lipschitz and continuous. □

Proposition B.2.8. Let X, Y, and Z be normed spaces. If $T, S \in \mathcal{L}(X, Y)$, $R \in \mathcal{L}(Y, Z)$, and $\lambda \in \mathbb{K}$, then

$$\|T + S\| \le \|T\| + \|S\|, \qquad \|\lambda T\| = |\lambda|\|T\|, \qquad \|R \circ T\| \le \|R\|\|T\|.$$

In particular, $\mathcal{L}(X)$ is a normed algebra under composition.

Proof. The first two identities follow directly from the definition. For the third, $\|R(Tx)\|_Z \le \|R\|\|Tx\|_Y \le \|R\|\|T\|\|x\|_X$ for every $x \in X$. □

Theorem B.2.9. Let X be a normed space and Y a Banach space. Then $\mathcal{L}(X, Y)$ is a Banach space. In particular, X' is a Banach space for every normed space X.

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{L}(X, Y)$. For each $x \in X$, the inequality $\|T_n x - T_m x\|_Y \le \|T_n - T_m\|\|x\|_X$ shows that $(T_n x)$ is Cauchy in Y. Define $T(x) := \lim_{n\to\infty} T_n x$. The map T is linear. Moreover, there exists $N \in \mathbb{N}$ such that $\|T_n - T_m\| < 1$ for $n, m \ge N$, so $\|T_n\| \le 1 + \|T_N\|$ for $n \ge N$. Taking limits gives $\|T(x)\|_Y \le (1 + \|T_N\|)\|x\|_X$, and hence $T \in \mathcal{L}(X, Y)$.
Given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\|T_n - T_m\| < \varepsilon$ for $n, m \ge N_\varepsilon$. Letting $m \to \infty$, we obtain $\|(T_n - T)x\|_Y \le \varepsilon\|x\|_X$ for every $x \in X$ and $n \ge N_\varepsilon$. Therefore, $\|T_n - T\| \le \varepsilon$. □

Theorem B.2.10. (Neumann series) Let X be a Banach space and let $T \in \mathcal{L}(X)$ satisfy $\|T\| < 1$. Then $I - T$ is invertible and

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$$

in the norm of $\mathcal{L}(X)$. Moreover, $\|(I - T)^{-1}\| \le \dfrac{1}{1 - \|T\|}$.

Proof. The series converges in $\mathcal{L}(X)$ because $\sum_{n=0}^{\infty} \|T^n\| \le \sum_{n=0}^{\infty} \|T\|^n < \infty$ and $\mathcal{L}(X)$ is complete. If $S_N := \sum_{n=0}^{N} T^n$, then $(I - T)S_N = S_N(I - T) = I - T^{N+1}$. Letting $N \to \infty$ gives $(I - T)S = S(I - T) = I$, where $S := \sum_{n=0}^{\infty} T^n$. The norm estimate follows from the geometric series. □

Corollary B.2.11. Let X be a Banach space, let $T \in \mathcal{L}(X)$ be invertible, and let $S \in \mathcal{L}(X)$ satisfy $\|T^{-1}(S - T)\| < 1$. Then S is invertible.

Proof. Since $S = T[I + T^{-1}(S - T)]$, the conclusion follows from Theorem B.2.10 applied to $-T^{-1}(S - T)$. □

Definition B.2.12. Let X be a complex Banach space and let $T \in \mathcal{L}(X)$. The **resolvent set** of T is

$$\rho(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is invertible in } \mathcal{L}(X)\},$$

and the **spectrum** of T is $\sigma(T) := \mathbb{C} \setminus \rho(T)$. For $\lambda \in \rho(T)$, the operator $R(\lambda, T) := (\lambda I - T)^{-1}$ is called the **resolvent** of T.

Proposition B.2.13. If X is a complex Banach space and $T \in \mathcal{L}(X)$, then $\{\lambda \in \mathbb{C} \mid |\lambda| > \|T\|\} \subseteq \rho(T)$ and $\rho(T)$ is open. In particular, $\sigma(T)$ is compact and is contained in $\{\lambda \in \mathbb{C} \mid |\lambda| \le \|T\|\}$.

Proof. If $|\lambda| > \|T\|$, then $\lambda I - T = \lambda\left(I - \frac{T}{\lambda}\right)$ and the Neumann series gives its inverse. If $\lambda_0 \in \rho(T)$, we write

$$\lambda I - T = (\lambda_0 I - T)\big[I + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1}\big].$$

The second factor is invertible when $|\lambda - \lambda_0|\|(\lambda_0 I - T)^{-1}\| < 1$, so $\rho(T)$ is open. The remaining assertions are immediate. □

Theorem B.2.14. If $X \neq \{0\}$ is a complex Banach space and $T \in \mathcal{L}(X)$, then $\sigma(T)$ is nonempty.

The proof uses Liouville's theorem applied to the scalar functions $\lambda \mapsto \varphi((\lambda I - T)^{-1}x)$, with $x \in X$ and $\varphi \in X'$, and can be found in [45, Section 2.1] or [123].

B.2.2 Fundamental principles of operator theory

Theorem B.2.15. (Banach–Steinhaus) Let X be a Banach space, let Y be a normed space, and let $\mathcal{F} \subseteq \mathcal{L}(X, Y)$. If $\sup_{T \in \mathcal{F}} \|T(x)\|_Y < \infty$ for every $x \in X$, then $\sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} < \infty$.

Proof. For each $n \in \mathbb{N}$, define $E_n := \{x \in X \mid \|T(x)\|_Y \le n \text{ for every } T \in \mathcal{F}\}$. Each E_n is closed and, by pointwise boundedness, $X = \bigcup_{n=1}^{\infty} E_n$. Theorem B.1.14 gives $N \in \mathbb{N}$, $x_0 \in X$, and $r > 0$ such that $B_{d_X}(x_0, r) \subseteq E_N$, where d_X is the metric induced by the norm of X.
If $\|h\|_X < r$, then $x_0 + h, x_0 \in E_N$ and $\|T(h)\|_Y \le \|T(x_0 + h)\|_Y + \|T(x_0)\|_Y \le 2N$ for every $T \in \mathcal{F}$. For $x \neq 0$, we apply this estimate to $h := \dfrac{rx}{2\|x\|_X}$ and obtain $\|T(x)\|_Y \le \dfrac{4N}{r}\|x\|_X$. Therefore, $\|T\| \le \frac{4N}{r}$ for every $T \in \mathcal{F}$. □

Theorem B.2.16. (Open mapping theorem) Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be surjective. Then T is an open map.

Proof. Let $\mathbb{B}_X := B_{d_X}(0, 1)$, where d_X is the metric induced by the norm. Since $Y = \bigcup_{n=1}^{\infty} T(n\mathbb{B}_X)$, Theorem B.1.14 implies that the closure of $T(n_0\mathbb{B}_X)$ has nonempty interior for some $n_0 \in \mathbb{N}$. Using linearity and the symmetry of balls, we find $r > 0$ such that $B_{d_Y}(0, r) \subseteq \overline{T(\mathbb{B}_X)}$.
We claim that $B_{d_Y}(0, \frac{r}{2}) \subseteq T(\mathbb{B}_X)$. Let $y \in B_{d_Y}(0, \frac{r}{2})$. Since $B_{d_Y}(0, \frac{r}{2}) \subseteq \overline{T(B_{d_X}(0, \frac{1}{2}))}$, there exists $x_1 \in B_{d_X}(0, \frac{1}{2})$ such that $\|y - Tx_1\|_Y < \frac{r}{4}$. Inductively, we choose $x_n \in B_{d_X}(0, 2^{-n})$ so that

$$\left\| y - T\left(\sum_{j=1}^{n} x_j\right)\right\|_Y < \frac{r}{2^{n+1}}.$$

The series $\sum_{n=1}^{\infty} x_n$ converges in X to an element x with $\|x\|_X < 1$, and the continuity of T implies that $T(x) = y$. Thus, $T(\mathbb{B}_X)$ contains a ball around the origin. By linearity, the image of every open ball is open, and hence so is the image of any open set. □

Corollary B.2.17. (Bounded inverse theorem) Let X and Y be Banach spaces. If $T \in \mathcal{L}(X, Y)$ is bijective, then $T^{-1} \in \mathcal{L}(Y, X)$.

Proof. Theorem B.2.16 implies that the inverse operator T^{-1} is continuous. □

Proposition B.2.18. Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be injective. Then $\operatorname{Ran}(T)$ is closed if and only if there exists $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for every $x \in X$.

Proof. If the estimate holds and $Tx_n \to y$ in Y, then (x_n) is Cauchy and converges to some $x \in X$; by continuity, $Tx = y$, so the range is closed. Conversely, if $\operatorname{Ran}(T)$ is closed, it is a Banach space and $T\colon X \longrightarrow \operatorname{Ran}(T)$ is bijective. Theorem B.2.17 implies that T^{-1} is continuous, and therefore $\|x\|_X \leq \|T^{-1}\|\|Tx\|_Y$. □

Quotient spaces allow us to factor out a closed subspace while retaining a normed structure and, when the original space is Banach, completeness. Closedness is essential: without it, the expression below defines only a seminorm.

Definition B.2.19. Let X be a vector space and let $Y \subseteq X$ be a subspace. The **quotient vector space** X/Y is the set of classes

$$X/Y := \{x + Y \mid x \in X\}, \qquad [x] := x + Y,$$

with the operations

$$[x] + [z] := [x + z], \qquad \alpha[x] := [\alpha x] \quad (\alpha \in \mathbb{K}).$$

These operations do not depend on the representatives: if $x - x' \in Y$ and $z - z' \in Y$, then $(x + z) - (x' + z') \in Y$ and $\alpha x - \alpha x' \in Y$.

Proposition B.2.20. (Quotient norm) Let X be a normed space and let $Y \subseteq X$ be a closed subspace. For $[x] = x + Y \in X/Y$, set

$$\|[x]\|_{X/Y} := \inf_{y \in Y} \|x - y\|_X = \operatorname{dist}_X(x, Y). \tag{B.1}$$

Then (B.1) defines a norm on X/Y. The canonical projection

$$\pi\colon X \longrightarrow X/Y, \qquad \pi(x) = [x],$$

is linear, continuous, and open, and satisfies $\|\pi\| \leq 1$. If X is a Banach space, then X/Y is also a Banach space.

Proof. If $x' = x + y_0$ with $y_0 \in Y$, then

$$\inf_{y \in Y} \|x' - y\|_X = \inf_{y \in Y} \|x - (y - y_0)\|_X = \inf_{z \in Y} \|x - z\|_X,$$

so the expression does not depend on the representative of the class. Homogeneity is immediate. Given $x, z \in X$ and $y_1, y_2 \in Y$, we have

$$\operatorname{dist}_X(x+z, Y) \le \|(x+z)-(y_1+y_2)\|_X \le \|x-y_1\|_X + \|z-y_2\|_X.$$

Taking the infimum over y_1 and y_2 gives the triangle inequality. Moreover, $\|[x]\|_{X/Y} = 0$ if and only if $x \in \overline{Y} = Y$, that is, if and only if $[x] = 0$. Therefore, (B.1) is a norm.

The linearity of π follows from the quotient operations, and $\|\pi(x)\|_{X/Y} \le \|x\|_X$, so π is continuous and $\|\pi\| \le 1$. If $r > 0$, then for every class $[z]$ with $\|[z]\|_{X/Y} < r$ there exists $y \in Y$ such that $\|z-y\|_X < r$ and $[z] = [z-y]$. Consequently, every class of norm less than r belongs to $\pi(B_{d_X}(0,r))$. The reverse inclusion follows from $\|\pi(x)\|_{X/Y} \le \|x\|_X$. Therefore, the image of the open ball of radius r in X is the open ball of radius r in X/Y. Translations and linearity show that π is open.

Finally, suppose that X is complete and let $([x_n])$ be a Cauchy sequence in X/Y. Choose a subsequence $([x_{n_j}])$ such that

$$\|[x_{n_{j+1}}] - [x_{n_j}]\|_{X/Y} < 2^{-j-1} \qquad (j \in \mathbb{N}).$$

By the definition of the infimum, for each j there exists $y_j \in Y$ such that

$$\|(x_{n_{j+1}} - x_{n_j}) - y_j\|_X < 2^{-j}.$$

Define $z_1 := x_{n_1}$ and, for $j \ge 2$,

$$z_j := x_{n_j} - \sum_{i=1}^{j-1} y_i.$$

Then $[z_j] = [x_{n_j}]$ and

$$\|z_{j+1} - z_j\|_X = \|(x_{n_{j+1}} - x_{n_j}) - y_j\|_X < 2^{-j}.$$

Consequently, (z_j) is Cauchy in X and converges to some $z \in X$. The continuity of π implies that $[x_{n_j}] = [z_j] \to [z]$. Since the original sequence is Cauchy, the entire sequence converges to $[z]$. This proves the completeness of X/Y. □

Differential operators are usually unbounded when considered on a single function space. To handle them, we must distinguish the domain of the operator from the ambient space.

Definition B.2.21. Let X and Y be normed spaces. A **linear operator** from X to Y is a linear map $A\colon \mathcal{D}(A) \subseteq X \longrightarrow Y$, where $\mathcal{D}(A)$ is a vector subspace called the **domain** of A; we also write $\operatorname{Dom}(A)$. The operator is **densely defined** if $\overline{\mathcal{D}(A)} = X$.

The **graph** of A is $\mathcal{G}(A) := \{(x, Ax) \mid x \in \mathcal{D}(A)\} \subseteq X \times Y$. We say that A is **closed** if $\mathcal{G}(A)$ is closed. Equivalently, if $x_n \in \mathcal{D}(A)$, $x_n \to x$ in X, and $Ax_n \to y$ in Y, then $x \in \mathcal{D}(A)$ and $Ax = y$.

Theorem B.2.22. (Closed graph theorem) Let X and Y be Banach spaces and let $A\colon X \longrightarrow Y$ be linear. Then A is continuous if and only if its graph is closed.

Proof. If A is continuous, its graph is immediately seen to be closed. Conversely, suppose that $\mathcal{G}(A)$ is closed. Then $\mathcal{G}(A)$ is a Banach space with the norm inherited from $X \times Y$. The projection $\pi_1\colon \mathcal{G}(A) \longrightarrow X$, given by $\pi_1(x, Ax) = x$, is linear, continuous, and bijective. By Theorem B.2.17, π_1^{-1} is continuous. Since $A = \pi_2 \circ \pi_1^{-1}$,

where $\pi_2 \colon X \times Y \longrightarrow Y$ is the second projection, A is continuous. □

Proposition B.2.23. Let X and Y be Banach spaces and let $A \colon \mathcal{D}(A) \subseteq X \longrightarrow Y$ be linear. The formula $\|x\|_A := \|x\|_X + \|Ax\|_Y$ defines a norm on $\mathcal{D}(A)$. The operator A is closed if and only if $(\mathcal{D}(A), \|\cdot\|_A)$ is a Banach space.

Proof. The map $J \colon \mathcal{D}(A) \longrightarrow X \times Y$, given by $J(x) = (x, Ax)$, is an isometry from $(\mathcal{D}(A), \|\cdot\|_A)$ onto $\mathcal{G}(A)$ when $X \times Y$ is equipped with the norm $\|(x,y)\| := \|x\|_X + \|y\|_Y$. Therefore, $\mathcal{D}(A)$ is complete if and only if $\mathcal{G}(A)$ is closed in the Banach space $X \times Y$. □

Definition B.2.24. A linear operator $A \colon \mathcal{D}(A) \subseteq X \longrightarrow Y$ is called **closable** if the closure of $\mathcal{G}(A)$ is the graph of an operator. This operator is denoted by $\overline{A}$ and is called the **closure** of A.

Proposition B.2.25. A linear operator $A \colon \mathcal{D}(A) \subseteq X \longrightarrow Y$ is closable if and only if, for every sequence (x_n) in $\mathcal{D}(A)$, the convergences $x_n \to 0$ in X and $Ax_n \to y$ in Y imply $y = 0$.

Proof. The closure of $\mathcal{G}(A)$ is the graph of an operator if and only if it contains no element $(0, y)$ with $y \neq 0$. Since $X \times Y$ is metrizable, a point belongs to the closure if and only if it is the limit of a sequence of elements of $\mathcal{G}(A)$, which gives precisely the stated condition. □

B.2.3 The Hahn–Banach theorem and duality

The Hahn–Banach theorem allows us to extend linear functionals while retaining the estimates that ensure their continuity. Its consequences will be used both in the dual characterization of the norm and in separation arguments.

Remark B.2.26. A function $p \colon X \longrightarrow \mathbb{R}$ defined on a real vector space is called **sublinear** if $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$ and $p(\lambda x) = \lambda p(x)$ for every $x \in X$ and $\lambda \geq 0$.

Theorem B.2.27. (Hahn–Banach theorem, analytic form) Let X be a real vector space, let $M \subseteq X$ be a subspace, and let $p \colon X \longrightarrow \mathbb{R}$ be sublinear. If $\ell \colon M \longrightarrow \mathbb{R}$ is linear and $\ell(x) \leq p(x)$ for every $x \in M$, then there exists a linear functional $\widetilde{\ell} \colon X \longrightarrow \mathbb{R}$ such that $\widetilde{\ell}\,|_M = \ell$ and $\widetilde{\ell}(x) \leq p(x)$ for every $x \in X$.

Proof. We begin with extension by one dimension. Let $N \subsetneq X$ be a subspace, let $f \colon N \to \mathbb{R}$ be linear and dominated by p, and let $x_0 \in X \setminus N$. Define

$$\alpha := \sup_{y \in N} \bigl(f(y) - p(y - x_0)\bigr), \qquad \beta := \inf_{z \in N} \bigl(p(z + x_0) - f(z)\bigr).$$

For any $y, z \in N$, the linearity of f, domination by p, and subadditivity of p imply

$$f(y) + f(z) = f(y+z) \leq p(y+z) \leq p(y - x_0) + p(z + x_0).$$

Therefore,

$$f(y) - p(y - x_0) \leq p(z + x_0) - f(z)$$

for every pair $y, z \in N$. In particular, $\alpha \le \beta$, and we may choose $a \in [\alpha, \beta]$.
Every element of $N + \mathbb{R}x_0$ has a unique representation as $y + tx_0$, with $y \in N$ and $t \in \mathbb{R}$. Define

$$f_1(y + tx_0) := f(y) + ta.$$

The map f_1 is linear and extends f. Let us verify domination. If $t > 0$, then $a \le \beta$ and, taking $z = y/t$, we obtain

$$f_1(y + tx_0) = t\big(f(y/t) + a\big) \le t\, p(y/t + x_0) = p(y + tx_0).$$

If $t < 0$, write $t = -s$ with $s > 0$. Since $a \ge \alpha$, applied now to y/s, we have

$$f_1(y - sx_0) = s\big(f(y/s) - a\big) \le s\, p(y/s - x_0) = p(y - sx_0).$$

The case $t = 0$ is precisely the hypothesis $f \le p$ on N. Thus, f_1 is dominated by p on all of $N + \mathbb{R}x_0$.
Consider the set $\mathcal{E}$ of all pairs (N, f) such that $M \subseteq N \subseteq X$, $f\colon N \to \mathbb{R}$ is linear, $f \restriction_M = \ell$, and $f \le p$ on N. Order this set by extension:

$$(N_1, f_1) \le (N_2, f_2) \quad \Longleftrightarrow \quad N_1 \subseteq N_2 \text{ and } f_2 \restriction_{N_1} = f_1.$$

The set is nonempty because it contains (M, ℓ). If $\mathcal{C} \subseteq \mathcal{E}$ is a chain, the union of its domains

$$N_{\mathcal{C}} := \bigcup_{(N,f)\in\mathcal{C}} N$$

is a subspace. On this subspace, define $f_{\mathcal{C}}(x) := f(x)$ using any element of the chain whose domain contains x. The definition is independent of this choice, since any two elements of a chain are comparable; moreover, $f_{\mathcal{C}}$ is linear, extends ℓ, and is dominated by p. Thus every chain has an upper bound in $\mathcal{E}$.
Zorn's lemma, Theorem A.0.1, gives a maximal element $(N_{\max}, f_{\max})$. If $N_{\max} \neq X$, we may choose $x_0 \in X \setminus N_{\max}$ and apply the preceding one-dimensional construction, obtaining a proper extension of $f_{\max}$ that is still dominated by p. This contradicts maximality. Hence $N_{\max} = X$, and $\widetilde{\ell} := f_{\max}$ is the desired extension. □

Remark B.2.28. On a complex vector space, the corresponding version is obtained by assuming that p is subadditive, that $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{C}$ and $x \in X$, and that $|\ell(x)| \le p(x)$ on the initial subspace. Indeed, regard X and M as real vector spaces and apply the theorem to $u := \operatorname{Re}\ell$. We obtain a real-linear extension $U\colon X \to \mathbb{R}$ such that $U \le p$. Since $p(-x) = p(x)$, we also have

$$-U(x) = U(-x) \le p(-x) = p(x),$$

so $|U(x)| \le p(x)$. Define

$$\widetilde{\ell}(x) := U(x) - iU(ix).$$

The real linearity of U shows that

$$\widetilde{\ell}(ix) = U(ix) - iU(-x) = i\widetilde{\ell}(x),$$

and therefore $\widetilde{\ell}$ is complex-linear. If $x \in M$, then

$$U(x) = \operatorname{Re}\ell(x), \qquad U(ix) = \operatorname{Re}(i\ell(x)) = -\operatorname{Im}\ell(x),$$

so $\widetilde{\ell}(x) = \ell(x)$.
Finally, given $x \in X$, choose $\lambda \in \mathbb{C}$, $|\lambda| = 1$, so that $\lambda\widetilde{\ell}(x) = |\widetilde{\ell}(x)|$. Since $\operatorname{Re}\widetilde{\ell} = U$,

$$|\widetilde{\ell}(x)| = \operatorname{Re}\widetilde{\ell}(\lambda x) = U(\lambda x) \le p(\lambda x) = p(x).$$

This proves the complex version and, in particular, the estimate used in the following corollary.

Corollary B.2.29. (Norm-preserving extension) Let X be a real or complex normed space, let $M \subseteq X$ be a subspace equipped with the induced norm, and let $\ell \in M'$. Then there exists $\widetilde{\ell} \in X'$ such that $\widetilde{\ell}\restriction_M = \ell$ and $\|\widetilde{\ell}\|_{X'} = \|\ell\|_{M'}$.

Proof. Define $p(x) := \|\ell\|_{M'}\|x\|_X$. In the real case, $\ell(x) \le |\ell(x)| \le p(x)$ for every $x \in M$, so Theorem B.2.27 gives an extension $\widetilde{\ell}$ dominated by p. In the complex case, we apply Remark B.2.28. In both cases, $|\widetilde{\ell}(x)| \le \|\ell\|_{M'}\|x\|_X$, so $\|\widetilde{\ell}\|_{X'} \le \|\ell\|_{M'}$. The reverse inequality follows from $\widetilde{\ell}\restriction_M = \ell$. □

Corollary B.2.30. (Dual characterization of the norm) Let X be a real or complex normed space. For each $x \in X$,

$$\|x\|_X = \max\left\{|\varphi(x)| \,\middle|\, \varphi \in X',\ \|\varphi\|_{X'} \le 1\right\}.$$

More precisely, if $x \neq 0$, there exists $\varphi \in X'$ such that $\|\varphi\|_{X'} = 1$ and $\varphi(x) = \|x\|_X$.

Proof. If $x \neq 0$, define $\ell\colon \operatorname{span}\{x\} \longrightarrow \mathbb{K}$ by $\ell(\lambda x) := \lambda\|x\|_X$. Then $\|\ell\| = 1$. Corollary B.2.29 gives $\varphi \in X'$ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|_X$. On the other hand, $|\psi(x)| \le \|\psi\|\|x\|_X \le \|x\|_X$ for every $\psi \in X'$ with $\|\psi\| \le 1$. The case $x = 0$ is immediate. □

B.2.4 Adjoint and compact operators on Banach spaces

For compactness of the adjoint and the Riesz–Schauder alternative, we follow [45, Proposition 2.2.8 and Theorem 2.2.9].

Definition B.2.31. Let X and Y be normed spaces and let $T \in \mathcal{L}(X, Y)$. The **Banach adjoint** of T is the operator $T'\colon Y' \longrightarrow X'$ defined by $T'(\varphi) := \varphi \circ T$.

Proposition B.2.32. Let X, Y, and Z be normed spaces, and let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$. Then $T' \in \mathcal{L}(Y', X')$, $\|T'\| = \|T\|$, and $(S \circ T)' = T' \circ S'$.

Proof. For $\varphi \in Y'$ and $x \in X$, $|T'\varphi(x)| \le \|\varphi\|\|T\|\|x\|_X$, so $\|T'\| \le \|T\|$. By the dual characterization of the norm, for each $x \in X$ with $\|x\|_X \le 1$ there exists $\varphi \in Y'$ with $\|\varphi\| = 1$ and $|\varphi(Tx)| = \|Tx\|_Y$. Therefore, $\|Tx\|_Y \le \|T'\|$, and taking the supremum gives $\|T\| \le \|T'\|$. The identity for compositions is verified by evaluating both sides at each functional. □

Definition B.2.33. Let X and Y be normed spaces. A linear operator $K\colon X \longrightarrow Y$ is called **compact** if, for every bounded sequence (x_n) in X, the sequence (Kx_n) has a subsequence converging in Y. We denote the space of compact operators from X to Y by $\mathcal{K}(X,Y)$ and write $\mathcal{K}(X) := \mathcal{K}(X,X)$.

Proposition B.2.34. Every compact linear operator is bounded. Moreover, every *continuous* linear operator of finite rank is compact.

Proof. If a compact operator K were unbounded, there would exist a sequence (x_n) with $\|x_n\|_X \leq 1$ and $\|Kx_n\|_Y \geq n$, which would prevent (Kx_n) from having a convergent subsequence. If K is continuous and has finite rank, the image of every bounded sequence is bounded in the finite-dimensional space $\operatorname{Ran}(K)$ and therefore has a convergent subsequence. □

Proposition B.2.35. Let X, Y, and Z be normed spaces.

(a) If $K \in \mathcal{K}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$, then $S \circ K \in \mathcal{K}(X,Z)$.

(b) If $T \in \mathcal{L}(Z,X)$ and $K \in \mathcal{K}(X,Y)$, then $K \circ T \in \mathcal{K}(Z,Y)$.

(c) If Y is Banach, then $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$.

Proof. The first two assertions follow directly from the definition. For the third, let $K_n \to K$ in the operator norm, with $K_n \in \mathcal{K}(X,Y)$, and let (x_j) be a bounded sequence in X. A diagonal argument gives a subsequence, still denoted by (x_j), such that $(K_n x_j)_j$ converges for each n. If $M := \sup_{j\in\mathbb{N}} \|x_j\|_X$, then

$$\|Kx_j - Kx_k\|_Y \leq 2M\|K - K_n\| + \|K_n x_j - K_n x_k\|_Y.$$

First choose n so that the first term is less than $\frac{\varepsilon}{2}$, and then j,k so that the second is less than $\frac{\varepsilon}{2}$. Thus, (Kx_j) is Cauchy and converges in Y. □

Theorem B.2.36. (Continuous and compact extension from a dense subspace) Let X be a normed space, let $D \subseteq X$ be a dense vector subspace, and let Y be a Banach space. Every continuous linear operator $T\colon D \longrightarrow Y$ has a unique continuous linear extension $\widetilde{T}\colon X \longrightarrow Y$. Moreover,

$$\|\widetilde{T}\|_{\mathcal{L}(X,Y)} = \|T\|_{\mathcal{L}(D,Y)}.$$

If T is compact, then $\widetilde{T}$ is also compact.

Proof. Let $x \in X$ and choose a sequence $(x_n) \subseteq D$ such that $x_n \to x$. The estimate $\|T(x_n) - T(x_m)\|_Y \leq \|T\|\|x_n - x_m\|_X$ shows that $(T(x_n))$ is Cauchy in Y. Since Y is complete, we may define $\widetilde{T}(x) := \lim_{n\to\infty} T(x_n)$. If $(y_n) \subseteq D$ also converges to x, then $\|T(x_n) - T(y_n)\|_Y \leq \|T\|\|x_n - y_n\|_X \to 0$, so the limit does not depend on the chosen sequence.
Linearity follows by approximating two points of X by sequences in D and passing to the limit. Similarly,

$$\|\widetilde{T}(x)\|_Y = \lim_{n\to\infty} \|T(x_n)\|_Y \leq \|T\| \lim_{n\to\infty} \|x_n\|_X = \|T\|\|x\|_X,$$

so $\widetilde{T}$ is continuous and $\|\widetilde{T}\| \le \|T\|$. The reverse inequality follows from $\widetilde{T} \restriction_D = T$. Uniqueness follows from the density of D: two continuous extensions that agree on D agree on all of X.
Now suppose that T is compact and let (x_n) be a bounded sequence in X. For each n, choose $y_n \in D$ such that $\|x_n - y_n\|_X < \frac{1}{n}$. The sequence (y_n) is also bounded, so there exists a subsequence (y_{n_k}) for which $(T(y_{n_k}))$ converges in Y. On the other hand,

$$\|\widetilde{T}(x_{n_k}) - T(y_{n_k})\|_Y \le \|T\|\|x_{n_k} - y_{n_k}\|_X < \frac{\|T\|}{n_k},$$

and the right-hand side tends to zero. Consequently, $(\widetilde{T}(x_{n_k}))$ converges in Y, proving that $\widetilde{T}$ is compact. □

Corollary B.2.37. (Extension to the completions) Let X and Y be normed spaces, and let $\widehat{X}$ and $\widehat{Y}$ be their completions. Every operator $T \in \mathcal{L}(X,Y)$ has a unique extension $\widehat{T} \in \mathcal{L}(\widehat{X},\widehat{Y})$ compatible with the canonical inclusions. We have $\|\widehat{T}\| = \|T\|$ and, if T is compact, then $\widehat{T}$ is compact.

Proof. Identify X and Y with their isometric images in $\widehat{X}$ and $\widehat{Y}$. The result follows by applying Theorem B.2.36 to the operator $X \longrightarrow \widehat{Y}$ induced by T. □

Theorem B.2.38. (Schauder's theorem) Let X and Y be Banach spaces and let $K \in \mathcal{L}(X,Y)$. Then K is compact if and only if $K' \colon Y' \longrightarrow X'$ is compact.

Proof. First suppose that K is compact and set

$$C := \overline{\{Kx \mid \|x\|_X \le 1\}} \subseteq Y.$$

Every sequence in C has a convergent subsequence. Indeed, approximate its term j by an element Kx_j, with $\|x_j\|_X \le 1$, at a distance less than $1/j$. The compactness of K gives a convergent subsequence of (Kx_j) and hence of the original sequence, whose limit belongs to C because this set is closed. In particular, C has a finite net of any radius $\delta > 0$: if no finite union of balls of that radius covered C, we could inductively choose a sequence of points whose pairwise distances are at least δ, contradicting the preceding property.
Let (φ_j) be a bounded sequence in Y' and choose $M \ge 1$ such that $\|\varphi_j\|_{Y'} \le M$ for every j. The restrictions $f_j := \varphi_j \restriction_C$ satisfy

$$|f_j(y)| \le M\|K\|, \qquad |f_j(y) - f_j(z)| \le M\|y - z\|_Y, \qquad y, z \in C.$$

Let us give the uniform compactness argument needed here. For each integer $q \ge 1$, choose a finite net of radius $1/q$ in C. The union of these nets is a countable dense subset of C. A diagonal argument, applied to the bounded scalar sequences at its points, gives a subsequence of (f_j) that converges at every point of that union. Retain the same indices for the subsequence.
Given $\varepsilon > 0$, choose from these points a finite net $\{y_1, \dots, y_l\}$ of radius less than $\varepsilon/(4M)$. For sufficiently large j, k, $|f_j(y_i) - f_k(y_i)| < \varepsilon/2$ at every point of the net. If $y \in C$ and $\|y - y_i\|_Y < \varepsilon/(4M)$, then

$$|f_j(y) - f_k(y)| \le 2M\|y - y_i\|_Y + |f_j(y_i) - f_k(y_i)| < \varepsilon.$$

Thus the subsequence is uniformly Cauchy on C. Consequently,

$$\|K'\varphi_j - K'\varphi_k\|_{X'} = \sup_{\substack{x\in X\\ \|x\|_X\le 1}} |(\varphi_j-\varphi_k)(Kx)| \le \sup_{y\in C}|f_j(y)-f_k(y)| \longrightarrow 0.$$

Theorem B.2.9 shows that X' is complete; the subsequence of $(K'\varphi_j)$ converges in X'. This proves that K' is compact.

Conversely, suppose that K' is compact. The duals Y' and X' are Banach spaces, so the implication already proved, applied to K', shows that $K''\colon X''\to Y''$ is compact. For a normed space Z, define

$$J_Z\colon Z\longrightarrow Z'', \qquad (J_Z z)(\ell) := \ell(z), \quad \ell\in Z'.$$

Corollary B.2.30 gives $\|J_Z z\|_{Z''} = \|z\|_Z$, so J_Z is a linear isometry. Moreover, evaluating at $\varphi\in Y'$ gives

$$(K''J_X x)(\varphi) = (J_X x)(K'\varphi) = \varphi(Kx) = (J_Y Kx)(\varphi).$$

Hence $K''J_X = J_Y K$. If (x_j) is bounded in X, the compactness of K'' gives a subsequence along which $(J_Y Kx_j)$ converges in Y''. Since J_Y is an isometry, the corresponding subsequence of (Kx_j) is Cauchy in Y and converges because Y is complete. Thus K is compact. □

Theorem B.2.39. (Riesz–Schauder theorem) Let X be a Banach space and let $K\in\mathcal{K}(X)$. Then $\ker(I-K)$ is finite-dimensional, $\operatorname{Ran}(I-K)$ is closed, and

$$\operatorname{Ran}(I-K)=X \quad\Longleftrightarrow\quad \ker(I-K)=\{0\}.$$

Moreover, $\dim\ker(I-K)=\dim\ker(I-K')$.

Proof. Set $T := I-K$ and $N := \ker T$. We first use an elementary separation fact. If F is a proper closed subspace of a normed space E, choose $z\in E\setminus F$ and set $d := \operatorname{dist}(z,F) > 0$. There exists $y\in F$ such that $\|z-y\| < 2d$. The vector

$$u := \frac{z-y}{\|z-y\|}$$

satisfies $\|u\|=1$ and $\operatorname{dist}(u,F) = d/\|z-y\| > 1/2$. Recall also that every finite-dimensional subspace is closed and that its coordinate functionals are continuous: for a basis $e_1,\dots,e_l$, the function $\|\sum_{i=1}^{l} a_i e_i\|$ attains a positive minimum on the Euclidean sphere $\sum_{i=1}^{l}|a_i|^2=1$. Homogeneity gives equivalence with the coordinate norm, from which both assertions follow.

The subspace N is closed because T is continuous. If it were infinite-dimensional, we could apply the separation fact successively to the subspaces spanned by the vectors already chosen. We would obtain a sequence $(u_j)\subseteq N$, with $\|u_j\|_X=1$, such that $\|u_j-u_k\|_X > 1/2$ for $j\neq k$. But $Ku_j = u_j$, contradicting the compactness of K. Thus, $\dim N<\infty$.

We now prove the estimate

$$\operatorname{dist}_X(x,N)\le C\|Tx\|_X, \qquad x\in X,$$

for some constant $C > 0$. If it were false, after normalizing and subtracting an element of N, we could choose $x_j \in X$ such that

$$\operatorname{dist}_X(x_j, N) = 1, \qquad \|x_j\|_X < 2, \qquad \|Tx_j\|_X < 1/j.$$

In detail, take a vector whose distance to N is greater than $j\|Tx\|_X$, divide by this distance, and choose a representative of the same class modulo N with norm less than two. The compactness of K allows us to pass to a subsequence along which Kx_j converges. Since $x_j = Tx_j + Kx_j$, x_j also converges, say to x. By continuity, $Tx = 0$ and $\operatorname{dist}_X(x, N) = 1$, a contradiction. By Proposition B.2.20, the estimate is a lower bound for the operator induced by T on X/N.

Now let $Tx_j \to y$ in X. Choose $n_j \in N$ so that

$$z_j := x_j - n_j, \qquad \|z_j\|_X \le 2\operatorname{dist}_X(x_j, N) + 1/j \le 2C\|Tx_j\|_X + 1/j.$$

The sequence (z_j) is bounded. Along a subsequence, Kz_j converges; since $Tz_j = Tx_j \to y$, $z_j = Tz_j + Kz_j$ also converges to some z. The continuity of T gives $Tz = y$. Hence $\operatorname{Ran} T$ is closed.

Let us prove the equivalence between injectivity and surjectivity. For $j \ge 0$, write

$$R_j := \operatorname{Ran} T^j, \qquad N_j := \ker T^j, \qquad R_0 = X, \quad N_0 = \{0\}.$$

Each N_j is closed. For $j \ge 1$, the operator

$$L_j := I - T^j = \sum_{l=1}^{j} (-1)^{l+1} \binom{j}{l} K^l$$

is compact by Proposition B.2.35. The closed range argument just proved, applied to $T^j = I - L_j$, shows that each R_j is closed. Moreover, K commutes with T and therefore preserves all the spaces R_j and N_j.

Suppose that T is injective but not surjective. The inclusions $R_{j+1} \subseteq R_j$ are then strict for every $j \ge 0$. Indeed, if $R_j = R_{j+1}$, then for each $x \in X$ there would exist $y \in X$ with $T^j x = T^{j+1} y$; the injectivity of T^j would imply $x = Ty$, a contradiction. The separation fact gives vectors

$$x_j \in R_{j-1}, \qquad \|x_j\|_X = 1, \qquad \operatorname{dist}_X(x_j, R_j) > 1/2, \qquad j \ge 1.$$

If $k > j$, we have $Tx_j \in R_j$ and $Kx_k \in R_{k-1} \subseteq R_j$. Consequently,

$$\|Kx_j - Kx_k\|_X = \|x_j - (Tx_j + Kx_k)\|_X > 1/2,$$

contradicting the compactness of K. Thus injectivity implies surjectivity.

Conversely, suppose that T is surjective and $N_1 \neq \{0\}$. The inclusions $N_j \subseteq N_{j+1}$ are strict for every $j \ge 0$. If $N_j = N_{j+1}$ and $0 \neq z \in N_1$, the surjectivity of T^j would give an y with $T^j y = z$. Then $y \in N_{j+1} = N_j$, which would force $z = 0$, a contradiction. By separation, choose

$$x_j \in N_j, \qquad \|x_j\|_X = 1, \qquad \operatorname{dist}_X(x_j, N_{j-1}) > 1/2, \qquad j \ge 1.$$

If $j > k$, then $Tx_j \in N_{j-1}$ and $Kx_k \in N_k \subseteq N_{j-1}$. The same identity gives $\|Kx_j - Kx_k\|_X > 1/2$, again contradicting compactness. This proves the stated equivalence for every compact perturbation of the identity.

It remains to compare the dimensions. By Theorem B.2.38, K' is compact on the Banach space X'. The finite-dimensional kernel argument at the beginning of this proof, applied to $I - K' = T'$, gives

$$m := \dim \ker T' < \infty.$$

Set $R := \operatorname{Ran} T$ and $Z := X/R$, with the quotient norm. The map

$$Z' \longrightarrow \ker T', \qquad \ell \longmapsto \ell \circ \pi,$$

where $\pi\colon X \to X/R$ is the projection, is a linear isomorphism. Indeed, $T'\varphi = 0$ is equivalent to φ vanishing on R; in that case, $\ell([x]) := \varphi(x)$ is well-defined and

$$|\ell([x])| = |\varphi(x - r)| \le \|\varphi\|_{X'} \|x - r\|_X, \qquad r \in R.$$

Taking the infimum gives continuity in the quotient norm. The canonical injection $Z \to Z''$ is injective by Corollary B.2.30. Since $\dim Z' = m$, it follows that Z is finite-dimensional; in finite dimensions, the dual has the same dimension. Therefore,

$$\dim(X/R) = \dim Z = \dim Z' = m.$$

Let $n := \dim N$ and choose a basis $e_1, \dots, e_n$ of N. The coordinate functionals of this basis extend to functionals $f_1, \dots, f_n \in X'$ by Corollary B.2.29. Then

$$Px := \sum_{i=1}^{n} f_i(x) e_i, \qquad X_0 := \ker P,$$

define a continuous projection onto N and a decomposition $X = N \oplus X_0$. The operator $T \restriction_{X_0}$ is a bijection from X_0 onto R: it is injective because $N \cap X_0 = \{0\}$ and surjective because $T(x - Px) = Tx$.
Choose $y_1, \dots, y_m \in X$ whose classes form a basis of X/R. With $Y_0 := \operatorname{span}\{y_1, \dots, y_m\}$, we have $X = R \oplus Y_0$. Set $r := \min\{n, m\}$ and define the continuous finite-rank operator

$$Fx := \sum_{i=1}^{r} f_i(x) y_i.$$

Empty sums are interpreted as zero. By Propositions B.2.34 and B.2.35, $K - F$ is compact. The operator $S := T + F = I - (K - F)$ satisfies the equivalence between injectivity and surjectivity already proved. For $x = x_0 + \sum_{i=1}^{n} a_i e_i$, with $x_0 \in X_0$, we have

$$Sx = Tx_0 + \sum_{i=1}^{r} a_i y_i.$$

The direct sum $X = R \oplus Y_0$ and the bijectivity of $T \restriction_{X_0}$ give

$$\ker S = \operatorname{span}\{e_{r+1}, \dots, e_n\}, \qquad \operatorname{Ran} S = R \oplus \operatorname{span}\{y_1, \dots, y_r\}.$$

If $n \le m$, then $r = n$ and S is injective; hence it is surjective, which forces $r = m$. If $m \le n$, then $r = m$ and S

is surjective; hence it is injective, which forces $r = n$. In both cases, $n = m$, that is,

$$\dim\ker(I-K) = \dim\ker(I-K').$$

This completes the proof. □

Definition B.2.40. Let X and Y be Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is called **Fredholm** if $\ker T$ is finite-dimensional, $\operatorname{Ran}(T)$ is closed, and $Y/\operatorname{Ran}(T)$ is finite-dimensional. Its **index** is

$$\operatorname{ind}(T) := \dim\ker T - \dim\big(Y/\operatorname{Ran}(T)\big).$$

In particular, Theorem B.2.39 implies that $I-K$ is Fredholm of index zero for every $K \in \mathcal{K}(X)$.

B.3 Hilbert spaces

Hilbert spaces combine completeness with a Euclidean geometry given by an inner product. This structure allows us to define orthogonal complements, projections, adjoints, and gradients, and provides the natural setting for many linear and quadratic variational problems. We mainly follow [45, Section 1.2], [28, 87, 120].

Definition B.3.1. Let V be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An **inner product** on V is a map $\langle\cdot,\cdot\rangle_V : V\times V \longrightarrow \mathbb{K}$ such that, for all $x,y,z \in V$ and $\lambda \in \mathbb{K}$,

(a) $\langle x+y,z\rangle_V = \langle x,z\rangle_V + \langle y,z\rangle_V$ and $\langle \lambda x,z\rangle_V = \lambda\langle x,z\rangle_V$;

(b) $\langle x,y\rangle_V = \overline{\langle y,x\rangle_V}$;

(c) $\langle x,x\rangle_V \geq 0$, and $\langle x,x\rangle_V = 0$ if and only if $x = 0$.

Thus, in the complex case, we adopt the convention of linearity in the first variable and antilinearity in the second.

Theorem B.3.2. (Cauchy–Schwarz inequality) For any $x,y \in V$, we have $|\langle x,y\rangle_V| \leq \langle x,x\rangle_V^{\frac{1}{2}}\langle y,y\rangle_V^{\frac{1}{2}}$.

Proof. If $y = 0$, the assertion is immediate. If $y \neq 0$, positivity of the inner product applied to $x - \dfrac{\langle x,y\rangle_V}{\langle y,y\rangle_V}y$ gives

$$0 \leq \langle x,x\rangle_V - \frac{|\langle x,y\rangle_V|^2}{\langle y,y\rangle_V},$$

from which the inequality follows. □

Proposition B.3.3. The formula $\|x\|_V := \langle x,x\rangle_V^{\frac{1}{2}}$ defines a norm on V. Moreover, the parallelogram identity holds:

$$\|x+y\|_V^2 + \|x-y\|_V^2 = 2\|x\|_V^2 + 2\|y\|_V^2.$$

Proof. Only the triangle inequality requires justification. By Cauchy–Schwarz,

$$\|x+y\|_V^2 = \|x\|_V^2 + 2\,\mathrm{Re}\langle x,y\rangle_V + \|y\|_V^2 \le (\|x\|_V + \|y\|_V)^2.$$

The parallelogram identity follows by expanding both sides. □

Definition B.3.4. A vector space with an inner product is called a **Hilbert space** if it is complete in the norm induced by that inner product.

B.3.1 Orthogonality and projections

Definition B.3.5. Let H be a Hilbert space and let $M \subseteq H$. The **orthogonal complement** of M is

$$M^{\perp} := \{x \in H \mid \langle m, x\rangle_H = 0 \text{ for every } m \in M\}.$$

Two subsets $A, B \subseteq H$ are orthogonal if $\langle a, b\rangle_H = 0$ for all $a \in A$ and $b \in B$.

Proposition B.3.6. For every $M \subseteq H$, the set $M^{\perp}$ is a closed subspace. If M is a subspace, then $M \cap M^{\perp} = \{0\}$.

Proof. For each $m \in M$, the map $x \mapsto \langle m, x\rangle_H$ is continuous, so $M^{\perp}$ is an intersection of closed kernels. If $x \in M \cap M^{\perp}$, then $\|x\|_H^2 = \langle x, x\rangle_H = 0$, and therefore $x = 0$. □

Theorem B.3.7. (Projection onto closed convex sets) Let H be a Hilbert space and let $C \subseteq H$ be nonempty, closed, and convex. For each $x \in H$, there exists a unique $P_C x \in C$ such that

$$\|x - P_C x\|_H = \inf_{z \in C} \|x - z\|_H.$$

This element is characterized by

$$\mathrm{Re}\langle x - P_C x, z - P_C x\rangle_H \le 0 \qquad \text{for every } z \in C.$$

Proof. Let $d := \inf_{z\in C} \|x-z\|_H$ and choose (z_n) in C such that $\|x - z_n\|_H^2 < d^2 + \frac{1}{n}$. The parallelogram identity and convexity of C imply

$$\|z_n - z_m\|_H^2 = 2\|x - z_n\|_H^2 + 2\|x - z_m\|_H^2 - 4\left\|x - \frac{z_n + z_m}{2}\right\|_H^2 \le \frac{2}{n} + \frac{2}{m}.$$

Therefore, (z_n) is Cauchy and converges to some $y \in C$; necessarily, $\|x - y\|_H = d$. The same identity applied to two minimizers proves uniqueness.
Let $z \in C$. For $t \in (0,1)$, the point $y + t(z-y)$ belongs to C, so

$$\|x-y\|_H^2 \le \|x - y - t(z-y)\|_H^2 = \|x-y\|_H^2 - 2t\,\mathrm{Re}\langle x-y, z-y\rangle_H + t^2\|z-y\|_H^2.$$

Dividing by t and letting $t \to 0^+$ gives the characterizing inequality. Conversely, this inequality implies $\|x-z\|_H^2 \ge \|x-y\|_H^2$ for every $z \in C$. □

Corollary B.3.8. (Orthogonal decomposition) If M is a closed subspace of a Hilbert space H, then

$$H = M \oplus M^{\perp}.$$

For each $x \in H$, the element $P_M x$ is the unique vector in M such that $x - P_M x \in M^{\perp}$. The operator $P_M \in \mathcal{L}(H)$ is called the **orthogonal projection** onto M and satisfies $P_M^2 = P_M$, $\|P_M\| = 1$ if $M \neq \{0\}$, and

$$\langle P_M x, y\rangle_H = \langle x, P_M y\rangle_H \qquad \text{for all } x, y \in H.$$

Proof. Apply Theorem B.3.7 to $C = M$. Since M is a subspace, the characterizing inequality applied to $z = P_M x \pm m$ and, in the complex case, to $z = P_M x \pm im$ implies $\langle m, x - P_M x\rangle_H = 0$ for every $m \in M$. Uniqueness of the decomposition is immediate. The linearity of P_M follows from this characterization, and $\|P_M x\|_H \leq \|x\|_H$ follows from the Pythagorean theorem. The last identity follows by decomposing x and y into their components in M and $M^{\perp}$. □

Corollary B.3.9. For every subset $M \subseteq H$, we have

$$(M^{\perp})^{\perp} = \overline{\operatorname{span}(M)}.$$

Proof. The inclusion $\overline{\operatorname{span}(M)} \subseteq (M^{\perp})^{\perp}$ is immediate. Applying Corollary B.3.8 to the closed subspace $N := \overline{\operatorname{span}(M)}$, we write $x = P_N x + (x - P_N x)$. If $x \in (M^{\perp})^{\perp}$, then $x - P_N x \in N^{\perp} = M^{\perp}$, and therefore $\|x - P_N x\|_H^2 = \langle x, x - P_N x\rangle_H = 0$. Thus, $x = P_N x \in N$. □

B.3.2 Orthonormal systems and Riesz representation

Definition B.3.10. A family $(e_j)_{j\in J}$ in a Hilbert space H is **orthonormal** if $\langle e_j, e_k\rangle_H = 0$ for $j \neq k$ and $\|e_j\|_H = 1$. It is called an **orthonormal basis** if $\overline{\operatorname{span}\{e_j \mid j \in J\}} = H$.

Theorem B.3.11. (Bessel and Parseval) If $(e_j)_{j\in J}$ is an orthonormal family in H, then for every $x \in H$ and every finite subset $F \subseteq J$,

$$\sum_{j\in F} |\langle x, e_j\rangle_H|^2 \leq \|x\|_H^2.$$

If $(e_j)_{j\in J}$ is an orthonormal basis, then

$$x = \sum_{j\in J} \langle x, e_j\rangle_H e_j, \qquad \|x\|_H^2 = \sum_{j\in J} |\langle x, e_j\rangle_H|^2,$$

where only countably many coefficients can be nonzero.

Proof. For finite $F \subseteq J$, the vector $p_F := \sum_{j\in F} \langle x, e_j\rangle_H e_j$ satisfies $x - p_F \perp p_F$. By the Pythagorean theorem, $\|x\|_H^2 = \|x - p_F\|_H^2 + \sum_{j\in F} |\langle x, e_j\rangle_H|^2$. Parseval's identity follows by enlarging F and using the density of the subspace spanned by the family. □

Theorem B.3.12. (Riesz representation theorem) Let H be a Hilbert space and let $\ell \in H'$. There exists a unique $v \in H$ such that

$$\ell(x) = \langle x, v\rangle_H \qquad \text{for every } x \in H.$$

Moreover, $\|\ell\|_{H'} = \|v\|_H$. In the real case, the map $J\colon H \longrightarrow H'$, given by $J(v)(x) = \langle x, v\rangle_H$, is a linear isometry; in the complex case, it is an antilinear isometry.

Proof. If $\ell = 0$, take $v = 0$. Suppose that $\ell \neq 0$ and let $M := \ker \ell$, which is a closed subspace. By Corollary B.3.8, $H = M \oplus M^\perp$. The space $M^\perp$ is one-dimensional: the restriction of ℓ to $M^\perp$ is injective and takes values in $\mathbb{K}$. Choose $e \in M^\perp$ with $\|e\|_H = 1$. Each $x \in H$ has a unique representation as $x = m + \alpha e$, with $m \in M$, and $\ell(x) = \alpha\ell(e)$.
In the real case, define $v := \ell(e)e$, and in the complex case, $v := \overline{\ell(e)}e$. With our convention of linearity in the first variable, $\langle x, v\rangle_H = \alpha\ell(e) = \ell(x)$. Uniqueness follows because $\langle x, v - w\rangle_H = 0$ for every $x \in H$ implies $v = w$. Cauchy–Schwarz gives $\|\ell\| \leq \|v\|_H$, while, if $v \neq 0$, evaluating at $\dfrac{v}{\|v\|_H}$ gives the reverse inequality. □

Corollary B.3.13. (Complex-linear form of Riesz representation) Let H be a complex Hilbert space with inner product linear in the first variable. Then

$$\mathscr{R}_H\colon \overline{H} \longrightarrow H', \qquad \mathscr{R}_H(\overline{v})(x) := \langle x, v\rangle_H,$$

is a complex-linear isometric isomorphism. In particular, every complex-linear identification of a complex Hilbert space with its dual obtained from its inner product must have the conjugate space $\overline{H}$ as its domain; the direct map $H \to H'$ is antilinear.

Proof. Theorem B.3.12 proves bijectivity and equality of norms. If $\lambda \in \mathbb{C}$, scalar multiplication in $\overline{H}$ satisfies $\lambda\overline{v} = \overline{\overline{\lambda}v}$; therefore,

$$\mathscr{R}_H(\lambda\overline{v})(x) = \langle x, \overline{\lambda}v\rangle_H = \lambda\langle x, v\rangle_H = \lambda\mathscr{R}_H(\overline{v})(x).$$

□

Theorem B.3.14. (Lax–Milgram theorem) Let H be a real Hilbert space and let $a\colon H \times H \longrightarrow \mathbb{R}$ be bilinear. Suppose there exist constants $C \geq 0$ and $\alpha > 0$ such that

$$|a(u, v)| \leq C\|u\|_H\|v\|_H, \qquad a(u, u) \geq \alpha\|u\|_H^2$$

for all $u, v \in H$. Then, for every $F \in H'$, there exists a unique $u \in H$ such that $a(u, v) = F(v)$ for every $v \in H$. Moreover, $\|u\|_H \leq \dfrac{\|F\|_{H'}}{\alpha}$.

Proof. By Theorem B.3.12, for each $u \in H$ there exists a unique $Tu \in H$ such that $a(u, v) = \langle v, Tu\rangle_H$ for every $v \in H$. The operator T is linear and continuous. Coercivity implies

$$\alpha\|u\|_H^2 \leq \langle u, Tu\rangle_H \leq \|u\|_H\|Tu\|_H,$$

so $\|Tu\|_H \geq \alpha\|u\|_H$. In particular, T is injective and its range is closed. If $w \perp \operatorname{Ran}(T)$, then $0 = \langle w, Tw\rangle_H =$

$a(w,w) \geq \alpha \|w\|_H^2$, so $w = 0$. Therefore, the range of T is dense and closed, hence is all of H.
Let $f \in H$ be the Riesz representative of F. There exists a unique $u \in H$ such that $Tu = f$, and then $a(u,v) = \langle v, f\rangle_H = F(v)$. Finally, $\alpha\|u\|_H \leq \|Tu\|_H = \|f\|_H = \|F\|_{H'}$. □

B.3.3 Adjoint operators on Hilbert spaces

Theorem B.3.15. Let H_1 and H_2 be Hilbert spaces and let $T \in \mathcal{L}(H_1, H_2)$. There exists a unique operator $T^* \in \mathcal{L}(H_2, H_1)$ such that

$$\langle Tx, y\rangle_{H_2} = \langle x, T^* y\rangle_{H_1} \qquad \text{for all } x \in H_1,\ y \in H_2.$$

Moreover, $\|T^*\| = \|T\|$, $(S \circ T)^* = T^* \circ S^*$, and $(T^*)^* = T$.

Proof. For each $y \in H_2$, the map $x \mapsto \langle Tx, y\rangle_{H_2}$ belongs to H_1'. Theorem B.3.12 gives a unique vector $T^* y \in H_1$ satisfying the identity. Uniqueness allows us to verify the linearity of T^*, and Cauchy–Schwarz gives $\|T^* y\|_{H_1} \leq \|T\|\|y\|_{H_2}$. Applying the same construction to T^* and using the defining identity gives $(T^*)^* = T$, and therefore $\|T\| \leq \|T^*\|$. The remaining identities are verified using the inner product. □

Definition B.3.16. Let H be a Hilbert space and let $T \in \mathcal{L}(H)$.

(a) T is **self-adjoint** if $T = T^*$.

(b) T is **positive** if $\langle Tx, x\rangle_H \geq 0$ for every $x \in H$.

(c) T is **unitary** if $T^*T = TT^* = I$.

For unbounded operators, the definition of the adjoint must include its domain.

Definition B.3.17. Let H_1 and H_2 be Hilbert spaces and let $A\colon \mathcal{D}(A) \subseteq H_1 \longrightarrow H_2$ be a densely defined linear operator. Define

$$\mathcal{D}(A^*) := \left\{ y \in H_2 \,\middle|\, \text{there exists } z \in H_1 \text{ such that } \langle Ax, y\rangle_{H_2} = \langle x, z\rangle_{H_1} \text{ for every } x \in \mathcal{D}(A) \right\}.$$

For $y \in \mathcal{D}(A^*)$, the vector z is unique, and we define $A^* y := z$.

Proposition B.3.18. The adjoint of every densely defined linear operator is closed.

Proof. Let $y_n \in \mathcal{D}(A^*)$ satisfy $y_n \to y$ in H_2 and $A^* y_n \to z$ in H_1. For each $x \in \mathcal{D}(A)$,

$$\langle Ax, y\rangle_{H_2} = \lim_{n\to\infty} \langle Ax, y_n\rangle_{H_2} = \lim_{n\to\infty} \langle x, A^* y_n\rangle_{H_1} = \langle x, z\rangle_{H_1}.$$

Therefore, $y \in \mathcal{D}(A^*)$ and $A^* y = z$. □

Theorem B.3.19. Let $A\colon \mathcal{D}(A) \subseteq H_1 \longrightarrow H_2$ be linear and densely defined. Then A is closable if and only if $\mathcal{D}(A^*)$ is dense in H_2. In this case, $\overline{A} = A^{**}$.

The proof relies on identifying H_j with H_j' by Theorem B.3.12 and on the relation between the graph of A and its orthogonal complement. It can be found in [120, 28].

Definition B.3.20. Let H be a Hilbert space and let $A\colon \mathcal{D}(A)\subseteq H \longrightarrow H$ be linear and densely defined. We say that A is **symmetric** if $\langle Ax,y\rangle_H = \langle x,Ay\rangle_H$ for all $x,y\in\mathcal{D}(A)$, which is equivalent to $A\subseteq A^*$. We say that A is **self-adjoint** if $\mathcal{D}(A)=\mathcal{D}(A^*)$ and $A=A^*$ on this domain.

Proposition B.3.21. Let H be a complex Hilbert space and let $A\colon \mathcal{D}(A)\subseteq H\longrightarrow H$ be a densely defined symmetric operator. The following assertions are equivalent:

(a) A is self-adjoint.

(b) A is closed and $\ker(A^*-iI)=\ker(A^*+iI)=\{0\}$.

(c) $\operatorname{Ran}(A-iI)=\operatorname{Ran}(A+iI)=H$.

Proof. For every $x\in\mathcal{D}(A)$, symmetry implies

$$\|(A\pm iI)x\|_H^2 = \|Ax\|_H^2+\|x\|_H^2.$$

If A is closed, this estimate shows that the ranges of $A\pm iI$ are closed. Moreover,

$$\operatorname{Ran}(A+iI)^\perp=\ker(A^*-iI),\qquad \operatorname{Ran}(A-iI)^\perp=\ker(A^*+iI).$$

If A is self-adjoint, it is closed and the preceding kernels are trivial; therefore, $(a)\Rightarrow(b)$. If (b) holds, both ranges are closed and their orthogonal complements are trivial, so both equal H. Thus, $(b)\Rightarrow(c)$.
Suppose (c) and let $y\in\mathcal{D}(A^*)$. Choose $x\in\mathcal{D}(A)$ such that $(A+iI)x=(A^*+iI)y$. Since $A\subseteq A^*$, we have $(A^*+iI)(y-x)=0$. Now $\ker(A^*+iI)=\operatorname{Ran}(A-iI)^\perp=\{0\}$, so $y=x\in\mathcal{D}(A)$. Consequently, $\mathcal{D}(A^*)\subseteq\mathcal{D}(A)$ and A is self-adjoint. This proves $(c)\Rightarrow(a)$. □

B.3.4 Compact self-adjoint operators

Proposition B.3.22. Let H_1 and H_2 be Hilbert spaces. An operator $K\in\mathcal{L}(H_1,H_2)$ is compact if and only if there exists a sequence of finite-rank operators (K_n) such that $\|K_n-K\|_{\mathcal{L}(H_1,H_2)}\to 0$.

Proof. If K is the norm limit of finite-rank operators, then it is compact by Propositions B.2.34 and B.2.35. Conversely, suppose that K is compact and set

$$C:=\overline{K(B_{d_{H_1}}(0,1))},\qquad M:=\overline{\operatorname{span}(C)}.$$

The set C is compact because $K(B_{d_{H_1}}(0,1))$ is relatively compact. For each $r\in\mathbb{N}$, choose a finite set $F_r\subseteq C$ such that the balls of radius $\dfrac{1}{r}$ centered at the points of F_r cover C. The union $D:=\bigcup_{r=1}^{\infty}F_r$ is countable and dense in C.
Enumerate $D=\{y_1,y_2,\dots\}$ and let $\mathbb{K}_0:=\mathbb{Q}$ in the real case and $\mathbb{K}_0:=\mathbb{Q}+i\mathbb{Q}$ in the complex case. The set

$$D_0:=\left\{\sum_{j=1}^{s}a_j y_{n_j}\,\middle|\, s\in\mathbb{N},\ n_1,\dots,n_s\in\mathbb{N},\ a_1,\dots,a_s\in\mathbb{K}_0\right\}$$

is countable, since it is a countable union of images of the countable sets $\mathbb{N}^s \times \mathbb{K}_0^s$.
Let us verify that D_0 is dense in M. It suffices to approximate the elements of $\operatorname{span}(C)$, since this subspace is dense in M by definition. Let $z = \sum_{j=1}^{s} \alpha_j c_j$, with $c_j \in C$, $\alpha_j \in \mathbb{K}$, and let $\varepsilon > 0$. For each j, choose $y_{n_j} \in D$ sufficiently close to c_j that

$$\sum_{j=1}^{s} |\alpha_j| \|c_j - y_{n_j}\|_{H_2} < \frac{\varepsilon}{2}.$$

Then choose $a_j \in \mathbb{K}_0$ so that

$$\sum_{j=1}^{s} |\alpha_j - a_j| \|y_{n_j}\|_{H_2} < \frac{\varepsilon}{2};$$

if any of the vectors y_{n_j} is zero, the corresponding term imposes no condition on a_j. Then $w := \sum_{j=1}^{s} a_j y_{n_j}$ belongs to D_0 and

$$\|z - w\|_{H_2} \le \sum_{j=1}^{s} |\alpha_j| \|c_j - y_{n_j}\|_{H_2} + \sum_{j=1}^{s} |\alpha_j - a_j| \|y_{n_j}\|_{H_2} < \varepsilon.$$

Consequently, D_0 is countable and dense in M, so M is separable.
For every $x \in H_1$, we have $Kx \in M$: if $x \neq 0$, simply write $Kx = 2\|x\|_{H_1} K\left(\frac{x}{2\|x\|_{H_1}}\right)$, and the vector in parentheses belongs to $B_{d_{H_1}}(0,1)$; the case $x = 0$ is immediate. If M is finite-dimensional, this shows that K already has finite rank. Thus, suppose that M is infinite-dimensional. Applying the Gram–Schmidt procedure to a dense sequence in M, and at each step discarding the vectors that belong to the subspace spanned by the preceding ones, gives a countable orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of M. Let P_n be the orthogonal projection from H_2 onto $\operatorname{span}\{e_1, \dots, e_n\}$. Since $K(H_1) \subseteq M$, the operator $K_n := P_n K$ has finite rank.
By Theorem B.3.11, $P_n y \to y$ for every $y \in M$. This convergence is uniform on C. To verify this, let $\varepsilon > 0$ and choose a finite net $\{z_1, \dots, z_s\} \subseteq C$ of radius $\frac{\varepsilon}{2}$. There exists $N \in \mathbb{N}$ such that $\|(I - P_n)z_i\|_{H_2} < \frac{\varepsilon}{2}$ for every $i \in \{1, \dots, s\}$ and $n \ge N$. Given $y \in C$, choose z_i with $\|y - z_i\|_{H_2} < \frac{\varepsilon}{2}$. Since $I - P_n$ is an orthogonal projection and therefore has norm at most one,

$$\|(I - P_n)y\|_{H_2} \le \|(I - P_n)(y - z_i)\|_{H_2} + \|(I - P_n)z_i\|_{H_2} < \varepsilon.$$

Consequently,

$$\|K - K_n\| = \sup_{\substack{x \in H_1 \\ \|x\|_{H_1} \le 1}} \|(I - P_n)Kx\|_{H_2} \le \sup_{y \in C} \|(I - P_n)y\|_{H_2} \longrightarrow 0.$$

□

Proposition B.3.23. (Bilinear forms and self-adjoint operators) Let H be a real Hilbert space. For every continuous bilinear form $B\colon H \times H \longrightarrow \mathbb{R}$, there exists a unique operator $A_B \in \mathcal{L}(H)$ such that

$$B(x, y) = \langle A_B x, y \rangle_H$$

for all $x, y \in H$. The correspondence $B \longmapsto A_B$ is a linear isometric isomorphism between $\mathcal{L}^2(H; \mathbb{R})$ and $\mathcal{L}(H)$. Moreover, B is symmetric if and only if A_B is self-adjoint.
Define the rank of B to be the rank of A_B. With this convention, symmetric bilinear forms of finite rank

correspond exactly to self-adjoint operators of finite rank. If $\mathcal{F}_s(H)$ denotes the subspace of these forms, then its closure in $\mathcal{L}^2_s(H;\mathbb{R})$ corresponds to the space of compact self-adjoint operators. In particular, if H is infinite-dimensional and $B_0(x,y) := \langle x, y\rangle_H$, then $B_0 \notin \overline{\mathcal{F}_s(H)}$.

Proof. For fixed $x \in H$, the map $y \longmapsto B(x,y)$ belongs to H'. Theorem B.3.12 gives a unique vector $A_B x \in H$ satisfying $B(x,y) = \langle A_B x, y\rangle_H$ for every $y \in H$. Uniqueness of the Riesz representatives shows that A_B is linear. Similarly,

$$\|A_B x\|_H = \sup_{\substack{y\in H\\ \|y\|_H\le 1}} |B(x,y)| \le \|B\|\,\|x\|_H,$$

so $A_B \in \mathcal{L}(H)$ and $\|A_B\| \le \|B\|$. The reverse inequality follows from $|B(x,y)| \le \|A_B\|\|x\|_H\|y\|_H$, so $\|A_B\| = \|B\|$. Linearity and surjectivity of the correspondence follow by associating the form $B_A(x,y) := \langle Ax, y\rangle_H$ to $A \in \mathcal{L}(H)$.

Symmetry of B is equivalent to the identity $\langle A_B x, y\rangle_H = \langle x, A_B y\rangle_H$ for all $x, y \in H$, which is precisely condition $A_B = A_B^*$ of Theorem B.3.15. The assertion about rank is then immediate from the definition adopted.

It remains to identify the closure of $\mathcal{F}_s(H)$. Every norm limit of finite-rank self-adjoint operators is compact by Proposition B.2.35 and self-adjoint by the continuity of the adjoint established in Theorem B.3.15. Conversely, let $K \in \mathcal{K}(H)$ be self-adjoint. Proposition B.3.22 gives finite-rank operators T_n such that $T_n \to K$ in norm. The adjoint T_n^* also has finite rank. Indeed, $\operatorname{Ran}(T_n)$ is closed because it is finite-dimensional; if P_n is the orthogonal projection onto this range, then $T_n = P_n T_n$, and therefore $T_n^* = T_n^* P_n$. Thus the range of T_n^* is contained in the image under T_n^* of the finite-dimensional space $\operatorname{Ran}(P_n)$. Consequently, the operators $S_n := \dfrac{1}{2}(T_n + T_n^*)$ are self-adjoint and have finite rank. Moreover, using $K^* = K$ and $\|T_n^* - K^*\| = \|T_n - K\|$ gives $\|S_n - K\| \le \|T_n - K\| \to 0$.

Finally, if H is infinite-dimensional, an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ can be constructed inductively. Since $\|e_n - e_m\|_H = \sqrt{2}$ for $n \neq m$, the sequence (e_n) has no convergent subsequences. The identity on H is therefore not compact. Since B_0 corresponds to the identity, we conclude that $B_0 \notin \overline{\mathcal{F}_s(H)}$. □

Corollary B.3.24. Let H be a real Hilbert space and let $\ell, m \in H'$. The symmetric bilinear form

$$B_{\ell,m} := \ell \otimes m + m \otimes \ell, \qquad B_{\ell,m}(x,y) = \ell(x)m(y) + m(x)\ell(y),$$

has rank at most two.

Proof. Let $v_\ell, v_m \in H$ be the Riesz representatives of ℓ and m. The operator associated with $B_{\ell,m}$ is $x \longmapsto \ell(x)v_m + m(x)v_\ell$, whose image is contained in $\operatorname{span}\{v_\ell, v_m\}$. □

Theorem B.3.25. (Spectral theorem for compact self-adjoint operators) Let H be a Hilbert space and let $K \in \mathcal{K}(H)$ be self-adjoint. The nonzero eigenvalues of K are real, have finite multiplicity, and can accumulate only at 0. There exists an orthonormal family $(e_j)_{j\in J}$ of eigenvectors, with real eigenvalues $(\lambda_j)_{j\in J}$, such that

$$\overline{\operatorname{Ran}(K)} = \overline{\operatorname{span}\{e_j \mid j \in J\}}, \qquad Kx = \sum_{j\in J} \lambda_j \langle x, e_j\rangle_H e_j$$

for every $x \in H$, and for each $\varepsilon > 0$ there are only finitely many indices j with $|\lambda_j| \ge \varepsilon$. In particular, if H is separable, this family can be extended by an orthonormal basis of $\ker K$ to obtain an orthonormal basis of

H consisting of eigenvectors of K.

The proof can be found in [45, Theorem 2.2.16] and [120, 28].

B.3.5 Weak and strong regularity of Hilbert-space-valued maps

Scalar regularity of all the components of a Hilbert-space-valued map determines almost all of its norm regularity. Although the conclusion is formulated in terms of differentiability, the argument relies on the uniform boundedness principle and Riesz representation; it does not require differential calculus on Banach manifolds. For this reason, we include the result in this section.

Theorem B.3.26. (Weak and strong differentiability) Let $U \subseteq \mathbb{R}^n$ be open, let H be a Hilbert space, and let $k \in \mathbb{N}$. Suppose that $F\colon U \longrightarrow H$ is of class C^k in the weak sense, that is, for each $h \in H$ the scalar function $x \mapsto \langle F(x), h\rangle_H$ belongs to $C^k(U)$. Then F is of class C^{k-1} in norm. More precisely, for each multiindex α with $|\alpha| \le k$ there exists a unique map $F_\alpha\colon U \longrightarrow H$ such that

$$\langle F_\alpha(x), h\rangle_H = D^\alpha \langle F(\cdot), h\rangle_H(x), \qquad h \in H,$$

and, if $|\alpha| \le k-1$, the map F_α is norm-continuous and equals $D^\alpha F$. In particular, every weakly smooth map with values in H is smooth in norm. The same assertion holds for maps defined on a finite-dimensional manifold with or without boundary, after working in charts.

Another formulation of this result can be found in [74, Appendix B.5].

Proof. For $h \in H$, write $\varphi_h(y) := \langle F(y), h\rangle_H$. We begin by constructing the vectors $F_\alpha(x)$. If $\alpha = 0$, set $F_0 := F$. Now fix $x \in U$ and a multiindex α of length $r := |\alpha| \ge 1$. Choose indices $i_1, \dots, i_r \in \{1, \dots, n\}$ in which each i occurs exactly α_i times and, for a function G defined near y, denote its finite difference in the direction e_i by $\Delta_{t,i}G(y) := G(y + te_i) - G(y)$.
Take a sequence $t_\nu > 0$ tending to zero and small enough that all the points appearing in the following differences belong to U. Define

$$v_\nu := \frac{1}{t_\nu^r} \Delta_{t_\nu, i_1} \cdots \Delta_{t_\nu, i_r} F(x) \in H.$$

The fundamental theorem of calculus, applied successively to the scalar function φ_h, gives

$$\langle v_\nu, h\rangle_H = \int_{[0,1]^r} D^\alpha \varphi_h \left(x + t_\nu \sum_{j=1}^{r} s_j e_{i_j} \right) ds_1 \cdots ds_r.$$

Since $D^\alpha \varphi_h$ is continuous, it follows that $\langle v_\nu, h\rangle_H \longrightarrow D^\alpha \varphi_h(x)$ for every $h \in H$. In particular, the sequence of continuous antilinear functionals $T_\nu\colon H \longrightarrow \mathbb{K}$, given by $T_\nu(h) := \langle v_\nu, h\rangle_H$, is pointwise bounded. The Banach–Steinhaus theorem, Theorem B.2.15, applied to the linear functionals $h \mapsto \overline{T_\nu(h)}$, implies that there exists $C_{x,\alpha} > 0$ such that $\|T_\nu\| \le C_{x,\alpha}$ for every ν. Consequently, the pointwise limit

$$T_{x,\alpha}(h) := D^\alpha \varphi_h(x)$$

is antilinear and continuous and satisfies $|T_{x,\alpha}(h)| \le C_{x,\alpha}\|h\|_H$. The antilinear version of the Riesz representation theorem, Theorem B.3.12, obtained by conjugating the functional, determines a unique vector

$F_\alpha(x) \in H$ such that $T_{x,\alpha}(h) = \langle F_\alpha(x), h\rangle_H$ for every $h \in H$. Since x was arbitrary, this constructs $F_\alpha \colon U \longrightarrow H$ for each $|\alpha| \le k$. Uniqueness follows because the inner product separates points. Moreover, the identity

$$\langle F_\alpha(x), h\rangle_H = D^\alpha \varphi_h(x)$$

shows that each F_α is weakly continuous.

We now prove norm continuity for $|\alpha| \le k-1$. Fix a closed convex Euclidean ball $K \subseteq U$. For each $h \in H$ and each $i \in \{1,\dots,n\}$, the function $z \mapsto \langle F_{\alpha+e_i}(z), h\rangle_H = D^{\alpha+e_i}\varphi_h(z)$ is continuous and therefore bounded on K. Applying Banach–Steinhaus to the linear functionals $h \mapsto \overline{\langle F_{\alpha+e_i}(z), h\rangle_H}$, with $z \in K$ and $i \in \{1,\dots,n\}$, gives a constant $C_{K,\alpha} > 0$ such that

$$\|F_{\alpha+e_i}(z)\|_H \le C_{K,\alpha}$$

for every $z \in K$ and every i. If $x, y \in K$, the segment $x + t(y-x)$ is contained in K. The fundamental theorem of calculus applied to $D^\alpha\varphi_h$ gives

$$\langle F_\alpha(y) - F_\alpha(x), h\rangle_H = \int_0^1 \sum_{i=1}^n (y_i - x_i)\langle F_{\alpha+e_i}(x + t(y-x)), h\rangle_H \, dt.$$

Taking the supremum over $\|h\|_H \le 1$ yields

$$\|F_\alpha(y) - F_\alpha(x)\|_H \le C_{K,\alpha} \sum_{i=1}^n |y_i - x_i| \le \sqrt{n}\, C_{K,\alpha}\|y - x\|.$$

Thus, F_α is locally Lipschitz and, in particular, norm-continuous for $|\alpha| \le k-1$.

It remains to identify these maps with the strong derivatives of F. Let $|\alpha| \le k-2$ and define $A_\alpha(x) \in \mathcal{L}(\mathbb{R}^n, H)$ by

$$A_\alpha(x)v := \sum_{i=1}^n v_i F_{\alpha+e_i}(x).$$

Fix $x \in U$ and take v small enough that the segment $x + tv$, with $0 \le t \le 1$, is contained in a closed ball $K \subseteq U$. For every $h \in H$, the fundamental theorem of calculus gives

$$\begin{aligned}\langle F_\alpha(x+v) - F_\alpha(x) - A_\alpha(x)v, h\rangle_H &\\ = \int_0^1 \sum_{i=1}^n v_i \left\langle F_{\alpha+e_i}(x+tv) - F_{\alpha+e_i}(x), h\right\rangle_H dt.&\end{aligned}$$

Taking the supremum over $\|h\|_H \le 1$ again, we obtain

$$\frac{\|F_\alpha(x+v) - F_\alpha(x) - A_\alpha(x)v\|_H}{\|v\|_{\mathbb{R}^n}} \le \sqrt{n} \max_{1\le i\le n} \sup_{0\le t\le 1} \|F_{\alpha+e_i}(x+tv) - F_{\alpha+e_i}(x)\|_H.$$

The right-hand side tends to zero as $v \to 0$, because the maps $F_{\alpha+e_i}$ are norm-continuous. This proves that F_α is Fréchet differentiable and that $DF_\alpha(x) = A_\alpha(x)$. Since the $F_{\alpha+e_i}$ are continuous, $x \mapsto A_\alpha(x)$ is also continuous in the operator norm. Induction on $|\alpha|$ shows that $F \in C^{k-1}(U, H)$ and that its coordinate derivatives are the corresponding maps F_α.

If the domain is a finite-dimensional manifold, the argument applies to F in each chart; since norm regularity is a local property invariant under smooth changes of coordinates, the conclusions fit together glob-

ally. □

B.4 Closed forms, spectral calculus, and contraction semigroups

The elliptic operators appearing in global analysis are generally unbounded. Their domain is part of the definition and cannot be omitted. In L^2, closed quadratic forms provide a particularly stable way to construct self-adjoint realizations; in L^p, strongly continuous semigroups and their generators play the corresponding role. We collect here the results used in constructing the heat semigroup and Bessel potentials. For closed forms we follow [74, Appendix B], and for semigroups, [60, Chapter II].

B.4.1 Closed forms and the Friedrichs realization

Definition B.4.1. Let H be a complex Hilbert space. A **sesquilinear form** on H is a map

$$\mathfrak{a}\colon V \times V \longrightarrow \mathbb{C},$$

where $V \subseteq H$ is a dense vector subspace, linear in the first variable and antilinear in the second. The form is called:

(a) **symmetric** if $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$;

(b) **nonnegative** if $\mathfrak{a}(u,u) \geq 0$;

(c) **bounded below** if there exists $c \in \mathbb{R}$ such that $\mathfrak{a}(u,u) \geq c\|u\|_H^2$;

(d) **closed** if V is complete with respect to the norm

$$\|u\|_{\mathfrak{a}}^2 := \mathfrak{a}(u,u) + (1-c)\|u\|_H^2,$$

where c is any lower bound strictly less than 1.

The space V is called the domain of the form and is denoted by $\mathcal{D}(\mathfrak{a})$.

Up to equivalence, the preceding norm does not depend on the chosen lower bound. For a nonnegative form we simply use

$$\|u\|_{\mathfrak{a}}^2 := \|u\|_H^2 + \mathfrak{a}(u,u).$$

Theorem B.4.2. (First representation theorem for closed forms) Let $\mathfrak{a}$ be a densely defined, symmetric, nonnegative, closed sesquilinear form on a Hilbert space H. There exists a unique nonnegative self-adjoint operator

$$A\colon \mathcal{D}(A) \subseteq H \longrightarrow H$$

such that $\mathcal{D}(A) \subseteq \mathcal{D}(\mathfrak{a})$ and

$$\mathfrak{a}(u,v) = \langle Au, v\rangle_H, \qquad u \in \mathcal{D}(A), \quad v \in \mathcal{D}(\mathfrak{a}).$$

More precisely,

$$\mathcal{D}(A) = \{u \in \mathcal{D}(\mathfrak{a}) \mid \text{there exists } f \in H \text{ such that } \mathfrak{a}(u,v) = \langle f, v\rangle_H \text{ for every } v \in \mathcal{D}(\mathfrak{a})\},$$

and then $Au = f$.

Proof. Equip $V := \mathcal{D}(\mathfrak{a})$ with the inner product

$$\langle u, v\rangle_{\mathfrak{a}} := \langle u, v\rangle_H + \mathfrak{a}(u, v).$$

Since the form is closed, V is a Hilbert space. For each $f \in H$, the antilinear functional $v \mapsto \langle f, v\rangle_H$ is continuous on V, since $\|v\|_H \le \|v\|_{\mathfrak{a}}$. The antilinear version of Theorem B.3.12 provides a unique $Rf \in V$ such that

$$\langle Rf, v\rangle_H + \mathfrak{a}(Rf, v) = \langle f, v\rangle_H, \qquad v \in V.$$

The operator $R\colon H \longrightarrow H$, obtained by composing with the inclusion $V \hookrightarrow H$, is linear and bounded. It is injective: if $Rf = 0$, then $\langle f, v\rangle_H = 0$ for every $v \in V$, and density of V gives $f = 0$. For $f, h \in H$, apply the defining identity to (f, Rh) and (h, Rf); symmetry of $\mathfrak{a}$ gives

$$\langle f, Rh\rangle_H = \langle Rf, h\rangle_H.$$

Thus R is self-adjoint. Taking $v = Rf$ gives

$$\langle f, Rf\rangle_H = \|Rf\|_H^2 + \mathfrak{a}(Rf, Rf) \ge 0,$$

so R is positive. Its range is dense: if $h \perp \mathrm{Ran}(R)$, self-adjointness implies $Rh = 0$ and injectivity gives $h = 0$. Define

$$A := R^{-1} - I, \qquad \mathcal{D}(A) := \mathrm{Ran}(R).$$

If $u = Rf$, the identity defining R becomes

$$\mathfrak{a}(u, v) = \langle f - u, v\rangle_H = \langle Au, v\rangle_H, \qquad v \in V.$$

This formula proves the characterization of the domain and nonnegativity. Moreover, A is symmetric and $A + I = R^{-1}$ is surjective. If $y \in \mathcal{D}(A^*)$, choose $x \in \mathcal{D}(A)$ with

$$(A + I)x = (A^* + I)y.$$

Since $A \subseteq A^*$, we have $(A^* + I)(y - x) = 0$. But

$$\ker(A^* + I) = \mathrm{Ran}(A + I)^{\perp} = \{0\},$$

so $y = x \in \mathcal{D}(A)$. Thus $A = A^*$, without using the spectral calculus introduced in the next subsection. Uniqueness follows from the characterization of the domain.

□

Corollary B.4.3. (Variational characterization of the resolvent) Let A be the operator associated with the form $\mathfrak{a}$ in Theorem B.4.2. If $\lambda > 0$ and $f \in H$, then $u = (\lambda I + A)^{-1} f$ is the unique element of $\mathcal{D}(\mathfrak{a})$ satisfying

$$\mathfrak{a}(u, v) + \lambda\langle u, v\rangle_H = \langle f, v\rangle_H$$

for every $v \in \mathcal{D}(\mathfrak{a})$.

Proof. The expression $\langle u, v\rangle_{\mathfrak{a},\lambda} := \mathfrak{a}(u,v) + \lambda\langle u, v\rangle_H$ defines an inner product on $\mathcal{D}(\mathfrak{a})$ whose norm is equivalent to the form norm. For $f \in H$, the functional $v \mapsto \langle f, v\rangle_H$ is continuous with respect to that norm. The Riesz representation theorem, Theorem B.3.12, provides a unique $u \in \mathcal{D}(\mathfrak{a})$ satisfying the identity in the statement. Rewriting it as $\mathfrak{a}(u,v) = \langle f - \lambda u, v\rangle_H$ and using the characterization of $\mathcal{D}(A)$ in Theorem B.4.2 gives $u \in \mathcal{D}(A)$ and $Au = f - \lambda u$. Therefore $(\lambda I + A)u = f$, and the uniqueness already proved shows that $u = (\lambda I + A)^{-1} f$. □

Definition B.4.4. (Friedrichs realization) Let S be a symmetric, densely defined operator bounded below on H, and choose $c \in \mathbb{R}$ such that $S \geq cI$. The nonnegative form

$$\mathfrak{b}_c(u,v) := \langle Su, v\rangle_H - c\langle u, v\rangle_H, \qquad u, v \in \mathcal{D}(S),$$

is closable. Let B_c be the operator associated with its closure by Theorem B.4.2. The operator

$$S_F := B_c + cI, \qquad \mathcal{D}(S_F) := \mathcal{D}(B_c),$$

is called the **Friedrichs realization** of S. It is independent of the chosen lower bound c and is a self-adjoint extension of S with the same lower bound.

Proof. Complete $\mathcal{D}(S)$ with respect to the inner product

$$(u,v)_c := \langle u, v\rangle_H + \mathfrak{b}_c(u,v)$$

and denote the resulting Hilbert space by V_c. The inclusion into H extends to a continuous operator $J_c \colon V_c \to H$. We prove that it is injective. If $w \in \ker J_c$, choose $u_j \in \mathcal{D}(S)$ such that $u_j \to w$ in V_c; then $u_j \to 0$ in H. For each $v \in \mathcal{D}(S)$, symmetry of S gives

$$(w,v)_c = \lim_{j\to\infty} (u_j, v)_c = \lim_{j\to\infty} \langle u_j, (S + (1-c)I)v\rangle_H = 0.$$

Density of $\mathcal{D}(S)$ in V_c implies $w = 0$. We may therefore identify V_c with a subspace of H. The limit of $\mathfrak{b}_c(u_j, v_j)$ for sequences converging in V_c defines a closed form on this subspace. It is precisely the closure of $\mathfrak{b}_c$, since $\mathcal{D}(S)$ is dense in the form norm.

If $u \in \mathcal{D}(S)$, the identity $\mathfrak{b}_c(u,v) = \langle (S - cI)u, v\rangle_H$, initially for $v \in \mathcal{D}(S)$, passes to the limit for every $v \in V_c$. The characterization of the domain of the associated operator shows that $B_c u = (S - cI)u$; thus S_F extends S. For two lower bounds c, d, the form norms are equivalent, since their squares differ by $(d-c)\|u\|_H^2$ and both dominate $\|u\|_H^2$. Their completions are therefore identified with the same subspace of H. The closed forms satisfy

$$\overline{\mathfrak{b}_c}(u,v) + c\langle u, v\rangle_H = \overline{\mathfrak{b}_d}(u,v) + d\langle u, v\rangle_H.$$

The variational characterization of their operators gives $B_c + cI = B_d + dI$, including their domains. This proves independence of the choice of lower bound. □

Proposition B.4.5. (Criterion for essential self-adjointness) Let S be a symmetric, densely defined operator on a complex Hilbert space. Suppose that $S \geq 0$. Then S is essentially self-adjoint if and only if

$$\ker(S^* + I) = \{0\}.$$

More generally, if $S \geq -cI$, one may replace I by λI with $\lambda > c$.

Proof. For $u \in \mathcal{D}(S)$,

$$\|(S+I)u\|_H \|u\|_H \geq \operatorname{Re}\langle (S+I)u, u\rangle_H \geq \|u\|_H^2,$$

so $\|(S+I)u\|_H \geq \|u\|_H$. The same estimate holds for the closure $\overline{S}$, so $\operatorname{Ran}(\overline{S}+I)$ is closed. On the other hand,

$$\operatorname{Ran}(S+I)^{\perp} = \ker(S^* + I).$$

If this kernel is trivial, $\operatorname{Ran}(S+I)$ is dense. The preceding estimate also shows that

$$\operatorname{Ran}(\overline{S}+I) = \overline{\operatorname{Ran}(S+I)} = H.$$

Let $y \in \mathcal{D}(S^*) = \mathcal{D}((\overline{S})^*)$. Choose $x \in \mathcal{D}(\overline{S})$ such that

$$(\overline{S}+I)x = (S^*+I)y.$$

Since $\overline{S} \subseteq S^*$, we obtain $(S^*+I)(y-x) = 0$. The hypothesis gives $y = x$, so $\mathcal{D}(S^*) \subseteq \mathcal{D}(\overline{S})$ and $\overline{S}$ is self-adjoint. Conversely, if $\overline{S}$ is self-adjoint and nonnegative, then -1 belongs to its resolvent set; hence $\ker(S^*+I) = \ker(\overline{S}+I) = \{0\}$. For $S \geq -cI$, apply the same argument to $S + cI$ and any shift $\lambda > c$. □

B.4.2 Spectral calculus for self-adjoint operators

Theorem B.4.6. (Spectral theorem) Let A be a self-adjoint operator on a complex Hilbert space H. There exists a unique spectral measure E_A on $\mathbb{R}$ such that

$$A = \int_{\mathbb{R}} \lambda \, dE_A(\lambda).$$

If $m\colon \mathbb{R} \longrightarrow \mathbb{C}$ is Borel measurable, define

$$m(A) := \int_{\mathbb{R}} m(\lambda)\, dE_A(\lambda)$$

with domain

$$\mathcal{D}(m(A)) = \left\{ u \in H \,\middle|\, \int_{\mathbb{R}} |m(\lambda)|^2 \, d\mu_u(\lambda) < \infty \right\},$$

where $\mu_u(B) := \langle E_A(B)u, u\rangle_H$. We have

$$\|m(A)u\|_H^2 = \int_{\mathbb{R}} |m(\lambda)|^2 \, d\mu_u(\lambda).$$

If m is bounded, then $m(A) \in \mathcal{L}(H)$ and

$$\|m(A)\|_{\mathcal{L}(H)} \le \|m\|_{\infty,\mathbb{R}} := \sup_{\lambda \in \mathbb{R}} |m(\lambda)|.$$

Moreover, the calculus respects sums, products, and conjugation on their natural domains.

A complete proof, in this formulation using spectral measures and Borel functional calculus, is given in [74, Theorems B.4 and B.5]. The norm formula and the rules of the functional calculus are first obtained for simple functions, then for bounded Borel functions, and finally by spectral truncation.

Corollary B.4.7. (Square-root domain associated with a closed form) Let $\mathfrak{a}$ be a form as in Theorem B.4.2, and let A be its associated self-adjoint operator. Then

$$\mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(\mathfrak{a}), \qquad \mathfrak{a}(u,v) = \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} v \rangle_H.$$

Proof. Return to $V = \mathcal{D}(\mathfrak{a})$ and the resolvent R constructed in the proof of Theorem B.4.2. The subspace $\mathcal{D}(A) = \mathrm{Ran}(R)$ is dense in V for the form norm. Indeed, if $w \in V$ is orthogonal to $\mathrm{Ran}(R)$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$, then, for every $f \in H$,

$$0 = \langle Rf, w \rangle_{\mathfrak{a}} = \langle f, w \rangle_H,$$

and hence $w = 0$.
For $u \in \mathcal{D}(A)$, the spectral calculus and the representation of the form give

$$\mathfrak{a}(u,u) = \langle Au, u \rangle_H = \|A^{\frac{1}{2}} u\|_H^2.$$

Moreover, $\mathcal{D}(A)$ is a core for $A^{\frac{1}{2}}$. If $u \in \mathcal{D}(A^{\frac{1}{2}})$ and $u_N := E_A([0,N])u$, then $u_N \in \mathcal{D}(A)$ and

$$\|u_N - u\|_H^2 + \|A^{\frac{1}{2}}(u_N - u)\|_H^2 = \int_{(N,+\infty)} (1+\lambda)\, d\mu_u(\lambda) \longrightarrow 0.$$

Thus V and $\mathcal{D}(A^{\frac{1}{2}})$ are the completions of $\mathcal{D}(A)$ with respect to the same norm. They agree as subspaces of H, and the sesquilinear identity follows by polarization. □

Corollary B.4.8. Let A be self-adjoint and nonnegative. For $s \in \mathbb{R}$, define $(I+A)^s$ and, for $s \ge 0$, A^s by the spectral calculus. If $s > 0$, then $(I+A)^{-s}$ is bounded, injective, and

$$\mathrm{Ran}\big((I+A)^{-s}\big) = \mathcal{D}\big((I+A)^{s}\big).$$

The operators $(I+A)^s$ and $(I+A)^{-s}$ are inverses of one another on these domains.

Proof. The assertions are verified on each spectral fiber using the functions $(1+\lambda)^{\pm s}$. Injectivity follows from the fact that $(1+\lambda)^{-s} > 0$ for every $\lambda \ge 0$. □

Proposition B.4.9. (Spectral semigroup) Let A be self-adjoint and nonnegative. Then $T_t := e^{-tA}$, $t \ge 0$, is a strongly continuous semigroup of self-adjoint contractions. For each $u \in H$, $T_t u \in \mathcal{D}(A^m)$ if $t > 0$ and $m \in \mathbb{N}$,

and

$$\frac{d}{dt}T_t u = -AT_t u \qquad (t > 0).$$

If $u \in \mathcal{D}(A)$, the right derivative at $t = 0$ also exists and equals $-Au$.

Proof. The algebraic properties follow from $e^{-(t+s)\lambda} = e^{-t\lambda}e^{-s\lambda}$, and the bound $|e^{-t\lambda}| \le 1$ gives contractivity. For strong continuity, observe that

$$\|T_t u - u\|_H^2 = \int_{[0,+\infty)} |e^{-t\lambda} - 1|^2 \, d\mu_u(\lambda).$$

The integrand converges pointwise to zero as $t \to 0^+$ and is dominated by the constant 4, which is integrable with respect to the finite measure μ_u; dominated convergence proves $T_t u \to u$.
If $t > 0$, then

$$\|A^m T_t u\|_H^2 = \int_{[0,+\infty)} \lambda^{2m} e^{-2t\lambda} \, d\mu_u(\lambda) \le \left(\sup_{\lambda \ge 0} \lambda^{2m} e^{-2t\lambda} \right) \|u\|_H^2,$$

so $T_t u \in \mathcal{D}(A^m)$. Now fix $t_0 > 0$ and take $|h| < \frac{t_0}{2}$. The mean value theorem gives

$$\left| \frac{e^{-(t_0+h)\lambda} - e^{-t_0\lambda}}{h} \right| \le \lambda e^{-\frac{t_0}{2}\lambda}.$$

Thus the difference between this quotient and $-\lambda e^{-t_0\lambda}$ is dominated by $2\lambda e^{-\frac{t_0}{2}\lambda}$, a bounded function of λ. Dominated convergence with respect to μ_u proves differentiability at t_0 and identity $\frac{d}{dt}T_t u = -AT_t u$.
Finally, if $u \in \mathcal{D}(A)$, then

$$\left| \frac{e^{-t\lambda} - 1}{t} \right| \le \lambda$$

for $t > 0$. The square of the dominating function is integrable with respect to μ_u, precisely because $u \in \mathcal{D}(A)$. One last application of dominated convergence gives the right derivative at $t = 0$. □

B.4.3 Strongly continuous semigroups on Banach spaces

Definition B.4.10. Let X be a Banach space. A family $(T_t)_{t \ge 0} \subseteq \mathcal{L}(X)$ is a **strongly continuous semigroup**, or a C_0-semigroup, if

$$T_0 = I, \qquad T_{t+s} = T_t T_s, \qquad \lim_{t \to 0^+} \|T_t x - x\|_X = 0$$

for every $x \in X$. It is a **contraction semigroup** if $\|T_t\| \le 1$ for every $t \ge 0$.

Definition B.4.11. The **infinitesimal generator** of a C_0-semigroup (T_t) is the operator

$$Gx := \lim_{t \to 0^+} \frac{T_t x - x}{t}$$

defined on the set $\mathcal{D}(G)$ of elements for which the limit exists in X.

Proposition B.4.12. Let G be the generator of a C_0-semigroup $(T_t)_{t\ge 0}$ on X. Then G is closed and densely defined, and $\mathcal{D}(G^m)$ is dense in X for every $m \in \mathbb{N}$. Moreover, $T_t\mathcal{D}(G) \subseteq \mathcal{D}(G)$, $GT_t x = T_t G x$ for $x \in \mathcal{D}(G)$, and

$$T_t x - x = \int_0^t T_s G x \, ds, \qquad x \in \mathcal{D}(G).$$

Proof. For $x \in X$ and $h > 0$, define $x_h := \frac{1}{h}\int_0^h T_s x \, ds$. Strong continuity shows that $x_h \to x$. A calculation using the semigroup property gives

$$\frac{T_t x_h - x_h}{t} = \frac{1}{h}\left(\frac{1}{t}\int_h^{h+t} T_s x \, ds - \frac{1}{t}\int_0^t T_s x \, ds\right),$$

and letting $t \to 0^+$ yields $x_h \in \mathcal{D}(G)$. Thus the domain is dense. To obtain density for the powers, define recursively $S_h x := \frac{1}{h}\int_0^h T_s x \, ds$ and $S_{h_1,\dots,h_m} x := S_{h_m} \cdots S_{h_1} x$. The preceding calculation shows that $S_h(X) \subseteq \mathcal{D}(G)$ and, since S_h commutes with T_t, it also commutes with G on $\mathcal{D}(G)$. Induction gives $S_{h_1,\dots,h_m} x \in \mathcal{D}(G^m)$. If we first let $h_m \to 0^+$, then $h_{m-1} \to 0^+$, and so on, strong continuity implies $S_{h_1,\dots,h_m} x \to x$. Hence $\mathcal{D}(G^m)$ is dense in X.

If $x_n \to x$ and $Gx_n \to y$, the integral identity gives

$$T_t x_n - x_n = \int_0^t T_s G x_n \, ds.$$

Passing to the limit yields $T_t x - x = \int_0^t T_s y \, ds$. Dividing by t and letting $t \to 0^+$ shows that $x \in \mathcal{D}(G)$ and $Gx = y$, so G is closed. The remaining identities follow by using $T_{t+h} = T_t T_h$ and differentiating in the strong topology. □

Definition B.4.13. Let $A\colon \mathcal{D}(A) \subseteq X \longrightarrow X$ be densely defined. The operator A is called **dissipative** if for each $x \in \mathcal{D}(A)$ there exists $x^* \in X'$ such that

$$\|x^*\|_{X'} = \|x\|_X, \qquad x^*(x) = \|x\|_X^2, \qquad \operatorname{Re} x^*(Ax) \le 0.$$

It is called **accretive** if $-A$ is dissipative. A dissipative operator is **maximally dissipative** if $\operatorname{Ran}(\lambda I - A) = X$ for some, and hence every, $\lambda > 0$.

Proposition B.4.14. (Resolvent as a Laplace transform) Let $(T_t)_{t\ge 0}$ be a contraction C_0-semigroup with generator G. For each $\lambda > 0$ and $x \in X$, the Bochner integral

$$R_\lambda x := \int_0^\infty e^{-\lambda t} T_t x \, dt$$

converges, and

$$R_\lambda = (\lambda I - G)^{-1}, \qquad \|R_\lambda\| \le \frac{1}{\lambda}.$$

Proof. Convergence and the bound follow from

$$\int_0^\infty e^{-\lambda t}\|T_t x\|_X\,dt \le \frac{1}{\lambda}\|x\|_X.$$

If $x \in \mathcal{D}(G)$, then $\dfrac{d}{dt}T_t x = T_t G x$. Integrating by parts gives

$$R_\lambda(\lambda I - G)x = -\int_0^\infty \frac{d}{dt}\left(e^{-\lambda t}T_t x\right)dt = x.$$

On the other hand, the truncated integrals belong to $\mathcal{D}(G)$, and an analogous calculation shows that $(\lambda I - G)R_\lambda x = x$. Closedness of G allows passage to the limit at the infinite endpoint. □

Theorem B.4.15. (Lumer–Phillips) Let A be a closed, densely defined linear operator on a Banach space X. The following assertions are equivalent:

(a) A is the generator of a contraction C_0-semigroup;

(b) A is maximally dissipative;

(c) A is dissipative and $\operatorname{Ran}(\lambda I - A) = X$ for some $\lambda > 0$.

In this case,

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \le \frac{1}{\lambda}, \qquad \lambda > 0.$$

Proof. First suppose that A generates a contraction semigroup. For $x \in \mathcal{D}(A)$, choose by Hahn–Banach an element $x^* \in X'$ with $\|x^*\| = \|x\|$ and $x^*(x) = \|x\|^2$. Since $\operatorname{Re} x^*(T_t x) \le \|x\|^2$, dividing the difference by $t > 0$ and passing to the limit gives $\operatorname{Re} x^*(Ax) \le 0$. Proposition B.4.14 proves surjectivity of $\lambda I - A$ for every $\lambda > 0$.
Now suppose (c). Dissipativity implies, for $x \in \mathcal{D}(A)$ and $\lambda > 0$,

$$\lambda\|x\|^2 \le \operatorname{Re} x^*((\lambda I - A)x) \le \|x\|\,\|(\lambda I - A)x\|.$$

Thus $\lambda I - A$ is injective and its inverse on its range has norm at most $1/\lambda$. The set I of positive parameters for which it is surjective is nonempty by hypothesis. It is open by the Neumann series. It is also closed in $(0,\infty)$: if $\lambda_j \in I$ and $\lambda_j \to \lambda > 0$, the resolvent identity and its bounds show that $R_{\lambda_j} := (\lambda_j I - A)^{-1}$ is Cauchy in norm, since

$$\|R_{\lambda_j} - R_{\lambda_k}\| \le \frac{|\lambda_j - \lambda_k|}{\lambda_j \lambda_k}.$$

If its limit is R, for $y \in X$ we have $R_{\lambda_j} y \to Ry$ and $AR_{\lambda_j} y = \lambda_j R_{\lambda_j} y - y \to \lambda Ry - y$. Closedness of A gives $(\lambda I - A)Ry = y$. Since $(0,\infty)$ is connected, $I = (0,\infty)$.
For each $\lambda > 0$, consider the Yosida approximants

$$A_\lambda := \lambda A(\lambda I - A)^{-1}.$$

They are bounded, since $A_\lambda = \lambda^2 R_\lambda - \lambda I$, and

$$\|e^{tA_\lambda}\| \le e^{-\lambda t}e^{t\lambda^2\|R_\lambda\|} \le 1, \qquad t \ge 0.$$

Moreover, $\lambda R_\lambda y \to y$ for every $y \in X$. Indeed, for $y \in \mathcal{D}(A)$,

$$\lambda R_\lambda y - y = R_\lambda A y, \qquad \|R_\lambda A y\| \le \lambda^{-1}\|Ay\|,$$

and density of $\mathcal{D}(A)$, together with $\|\lambda R_\lambda\| \le 1$, extends the limit to every X. Thus $A_\lambda x = \lambda R_\lambda A x \to Ax$ if $x \in \mathcal{D}(A)$.

The resolvent identity shows that A_λ and A_μ commute. Differentiating $s \mapsto e^{(t-s)A_\lambda} e^{sA_\mu} x$ and integrating gives

$$\|e^{tA_\lambda} x - e^{tA_\mu} x\| \le t\|(A_\lambda - A_\mu)x\|, \qquad x \in \mathcal{D}(A).$$

Thus the contractions converge strongly, uniformly for $0 \le t \le T$, on the dense subspace $\mathcal{D}(A)$. The uniform bound by one extends this convergence to all of X. Denote the limit by T_t. Passing to the limit in $e^{(t+s)A_\lambda} = e^{tA_\lambda} e^{sA_\lambda}$ gives the semigroup property. For $x \in \mathcal{D}(A)$, $\|T_t x - x\| \le t\|Ax\|$; density and contractivity give strong continuity at zero for every $x \in X$.

If G is its generator, the identity

$$e^{tA_\lambda} x - x = \int_0^t e^{sA_\lambda} A_\lambda x \, ds$$

passes to the limit, for $x \in \mathcal{D}(A)$, as $T_t x - x = \displaystyle\int_0^t T_s A x \, ds$. Dividing by t gives $Gx = Ax$, so $A \subseteq G$. For $y \in \mathcal{D}(G)$, choose $x \in \mathcal{D}(A)$ with $(\lambda I - A)x = (\lambda I - G)y$. Then $(\lambda I - G)(y - x) = 0$. The semigroup resolvent formula implies $y = x$, and hence $G = A$. This proves (a). The argument also proves that surjectivity for one parameter is equivalent to surjectivity for all parameters, completing the equivalence with (b) and the resolvent bound. □

Proposition B.4.16. (Exponential formula) If G generates a contraction C_0-semigroup (T_t), then

$$T_t x = \lim_{n\to\infty} \left(I - \frac{t}{n} G\right)^{-n} x$$

for every $x \in X$, uniformly for t in compact intervals of $[0, \infty)$.

Proof. For $h > 0$, the Laplace formula for the resolvent gives

$$(I - hG)^{-1} x = \int_0^\infty e^{-r} T_{hr} x \, dr.$$

Iterating this identity, Fubini and the semigroup property yield

$$(I - hG)^{-n} x = \frac{1}{(n-1)!} \int_0^\infty e^{-r} r^{n-1} T_{hr} x \, dr.$$

To verify the factor, the step from n to $n+1$ integrates over $0 < s < r$ and uses $\displaystyle\int_0^r s^{n-1} \, ds = r^n/n$. The measure $d\nu_n(r) := e^{-r} r^{n-1} dr/(n-1)!$ has mass one, mean n, and variance n, as is checked by integrating by parts the moments of orders zero, one, and two. Fix $T > 0$ and take $h = t/n$, with $0 < t \le T$. Then

$$\int_0^\infty \left|\frac{tr}{n} - t\right|^2 d\nu_n(r) = \frac{t^2}{n} \le \frac{T^2}{n}.$$

The function $s \mapsto T_s x$ is uniformly continuous on $[0, T+1]$. Given $0 < \delta < 1$, split the integral according to $|tr/n - t| < \delta$ or $|tr/n - t| \geq \delta$. If $\omega_x(\delta)$ is its modulus of continuity on that interval, contractivity and Chebyshev's inequality give

$$\sup_{0<t\leq T} \|(I - (t/n)G)^{-n} x - T_t x\| \leq \omega_x(\delta) + \frac{2\|x\|T^2}{n\delta^2}.$$

First let $n \to \infty$, then $\delta \downarrow 0$. For $t = 0$, both operators are the identity, so convergence is uniform on $[0, T]$. □

Proposition B.4.17. (Bounded perturbations of generators) Let G be the generator of a C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X, and suppose that

$$\|T_t\|_{\mathcal{L}(X)} \leq e^{\omega t}, \qquad t \geq 0,$$

for some $\omega \in \mathbb{R}$. If $B \in \mathcal{L}(X)$, then the operator

$$G + B, \qquad \mathcal{D}(G + B) = \mathcal{D}(G),$$

generates a C_0-semigroup $(S_t)_{t\geq 0}$. This semigroup is given by the Dyson–Phillips series

$$S_t = \sum_{j=0}^{\infty} S_j(t), \qquad S_0(t) := T_t, \qquad S_{j+1}(t)x := \int_0^t T_{t-\tau} B S_j(\tau) x \, d\tau, \tag{B.2}$$

which converges in operator norm uniformly for t in compact intervals. Moreover,

$$\|S_t\|_{\mathcal{L}(X)} \leq e^{(\omega + \|B\|)t}.$$

In particular, if (T_t) is a contraction semigroup and $\mu \geq \|B\|$, then $(e^{-\mu t} S_t)_{t\geq 0}$ is a contraction C_0-semigroup with generator $G + B - \mu I$.

Proof. The estimate follows by induction. For $j = 0$, it is the hypothesis on T_t. If it holds for j, then

$$\begin{aligned} \|S_{j+1}(t)\| &\leq \int_0^t e^{\omega(t-\tau)} \|B\| e^{\omega\tau} \frac{\|B\|^j \tau^j}{j!} \, d\tau \\ &= e^{\omega t} \frac{\|B\|^{j+1} t^{j+1}}{(j+1)!}. \end{aligned}$$

Thus the series in (B.2) converges absolutely in $\mathcal{L}(X)$, uniformly as t ranges over a compact interval, and

$$\|S_t\| \leq e^{\omega t} \sum_{j=0}^{\infty} \frac{(t\|B\|)^j}{j!} = e^{(\omega + \|B\|)t}.$$

Strong continuity follows from continuity of T_t, the preceding estimate, and uniform convergence of the series. The semigroup identity is first verified term by term using Fubini and the relation $T_t T_s = T_{t+s}$, after which the absolutely convergent series are summed.

If $x \in \mathcal{D}(G)$, the first iteration satisfies

$$\frac{1}{t} S_1(t)x = \frac{1}{t} \int_0^t T_{t-\tau} B T_\tau x \, d\tau \longrightarrow Bx$$

as $t \to 0^+$. For the terms with $j \geq 2$ and $0 < t \leq 1$, the bound already proved gives

$$\left\| \sum_{j=2}^{\infty} S_j(t) \right\| \leq e^{\max\{\omega,0\}} \sum_{j=2}^{\infty} \frac{(t\|B\|)^j}{j!} \leq \frac{1}{2} e^{\max\{\omega,0\}+\|B\|} \|B\|^2 t^2.$$

The last inequality follows from $e^a - 1 - a \leq \frac{1}{2} e^{\|B\|} a^2$ for $0 \leq a \leq \|B\|$. Thus, upon dividing the sum by t, its norm tends to zero as $t \to 0^+$. Together with $\frac{T_t x - x}{t} \to Gx$, this shows that the generator of (S_t) extends $G + B$ on $\mathcal{D}(G)$. Denote this generator by H. The integral formula

$$S_t x = T_t x + \int_0^t T_{t-\tau} B S_\tau x \, d\tau$$

also allows us to prove the reverse inclusion of domains. Indeed, if $x \in \mathcal{D}(H)$, then

$$\frac{T_t x - x}{t} = \frac{S_t x - x}{t} - \frac{1}{t} \int_0^t T_{t-\tau} B S_\tau x \, d\tau.$$

The first term on the right-hand side converges to Hx as $t \to 0^+$. For the integrand of the second term, write

$$T_{t-\tau} B S_\tau x - Bx = T_{t-\tau} B (S_\tau x - x) + (T_{t-\tau} - I) Bx.$$

If $0 < t \leq 1$, the norm of the first summand is bounded by $e^{\max\{\omega,0\}} \|B\| \sup_{0 \leq s \leq t} \|S_s x - x\|$, and that of the second by $\sup_{0 \leq s \leq t} \|(T_s - I)Bx\|$. Both bounds tend to zero by strong continuity at time zero. Thus

$$\left\| \frac{1}{t} \int_0^t T_{t-\tau} B S_\tau x \, d\tau - Bx \right\| \leq \sup_{0 \leq \tau \leq t} \|T_{t-\tau} B S_\tau x - Bx\| \longrightarrow 0.$$

It follows that $\lim_{t \to 0^+} (T_t x - x)/t = Hx - Bx$. The definition of G implies that $x \in \mathcal{D}(G)$ and $Gx = Hx - Bx$. Hence $\mathcal{D}(H) \subseteq \mathcal{D}(G)$; together with the preceding inclusion, this proves that $\mathcal{D}(H) = \mathcal{D}(G)$ and $H = G + B$. Finally, $e^{-\mu t} S_t$ has generator $G + B - \mu I$ and, if $\omega = 0$ and $\mu \geq \|B\|$, its norm does not exceed one. □

B.4.4 Bessel potentials associated with a semigroup

Lemma B.4.18. (Beta–gamma identity) If $a, b > 0$, then

$$\int_0^1 \theta^{a-1} (1-\theta)^{b-1} \, d\theta = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Proof. By Tonelli's theorem,

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty x^{a-1} y^{b-1} e^{-(x+y)} \, dy \, dx.$$

With the change of variables $r = x+y$, $\theta = \dfrac{x}{x+y}$, whose Jacobian is r, the right-hand side becomes

$$\left(\int_0^\infty r^{a+b-1}e^{-r}\,dr\right)\left(\int_0^1 \theta^{a-1}(1-\theta)^{b-1}\,d\theta\right).$$

The first integral is $\Gamma(a+b)$. □

Theorem B.4.19. (Bessel potentials of a contraction semigroup) Let $(T_t)_{t\geq 0}$ be a contraction C_0-semigroup on a Banach space X, and let $-A$ be its generator. For $s > 0$, define

$$J_s x := \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}T_t x\,dt.$$

Then:

(a) $J_s \in \mathcal{L}(X)$ and $\|J_s\| \leq 1$;

(b) $J_s J_r = J_{s+r}$ for $r, s > 0$;

(c) $J_2 = (I+A)^{-1}$ and $J_{2m} = (I+A)^{-m}$ for $m \in \mathbb{N}$;

(d) each J_s is injective and has dense range in X.

For $s > 0$, let $X_s := J_s(X)$ with the norm $\|J_s x\|_{X_s} := \|x\|_X$; let $X_0 := X$ and, for $s < 0$, let X_s be the completion of X with respect to $\|x\|_{X_s} := \|J_{-s}x\|_X$. Then each X_s is a Banach space, $X_{s_1} \hookrightarrow X_{s_0}$ continuously if $s_1 > s_0$, and J_r extends uniquely to an isometric isomorphism

$$J_r \colon X_s \longrightarrow X_{s+r}$$

for any $s \in \mathbb{R}$ and $r > 0$.

Proof. The estimate

$$\|J_s x\|_X \leq \frac{\|x\|_X}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}\,dt = \|x\|_X$$

proves (a). For (b), Fubini and the semigroup property give

$$J_s J_r x = \frac{1}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{r}{2}\right)}\int_0^\infty\int_0^\infty t^{\frac{s}{2}-1}\tau^{\frac{r}{2}-1}e^{-(t+\tau)}T_{t+\tau}x\,d\tau\,dt.$$

The change of variables $a = t+\tau$, $\theta = \frac{t}{a}$ and Lemma B.4.18 transform this integral into $J_{s+r}x$. Part (c) follows from Proposition B.4.14 with $G = -A$ and $\lambda = 1$.
If $J_s x = 0$, choose m with $2m > s$. Then

$$J_{2m}x = J_{2m-s}J_s x = 0.$$

Since $J_{2m} = (I+A)^{-m}$ is injective, $x = 0$. Moreover, $\mathrm{Ran}(J_s)$ contains $\mathrm{Ran}(J_{2m}) = \mathcal{D}((I+A)^m)$, which is dense in X. This proves (d).
For $s > 0$, $J_s \colon X \longrightarrow X_s$ is a surjective isometry, so X_s is complete. The negative-order spaces are complete by

construction. If $s_1 > s_0 \geq 0$ and $u = J_{s_1} x$, then

$$u = J_{s_0} J_{s_1 - s_0} x, \qquad \|u\|_{X_{s_0}} = \|J_{s_1 - s_0} x\|_X \leq \|x\|_X = \|u\|_{X_{s_1}}.$$

This identity defines J_r on the positive-order spaces. On the negative-order spaces, the estimate $\|J_r x\|_{X_{s+r}} = \|x\|_{X_s}$ is first verified on the dense subspace X and allows the operator to be extended to the completion. Density of the ranges of the J_s shows that these extensions have closed and dense range; hence they are surjective. The semigroup identities prove that they are isometries and that their inverses are the corresponding shifts in the scale. □

Lemma B.4.20. (Integral formula for the square root of a generator) Let $-A$ be the generator of a contraction C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space X, let $a > 0$, and set $B := aI + A$. Define the bounded operator

$$C := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-tB}\, dt.$$

Then $C^2 = B^{-1}$; in particular, C is injective and has dense range. Set

$$B^{-1/2} := C, \qquad \mathcal{D}(B^{1/2}) := \mathrm{Ran}(C), \qquad B^{1/2} := C^{-1} : \mathrm{Ran}(C) \longrightarrow X.$$

For every $x \in \mathcal{D}(B)$, the Bochner integral

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-\frac{3}{2}} \left(x - e^{-tB} x\right) dt \tag{B.3}$$

converges in X and equals $B^{\frac{1}{2}} x$. Since $\mathcal{D}(B)$ is dense in $\mathcal{D}(B^{\frac{1}{2}})$ for the norm $\|x\|_{1/2} := \|B^{1/2} x\|_X$, which is equivalent to the graph norm, this identity determines $B^{\frac{1}{2}}$ by closure.

Proof. If $x \in \mathcal{D}(B)$, then

$$x - e^{-tB} x = \int_0^t e^{-sB} B x\, ds.$$

Thus

$$\|x - e^{-tB} x\|_X \leq \min\{t\|Bx\|_X, 2\|x\|_X\}.$$

This bound proves convergence of (B.3) both near zero and at infinity.
The bound $\|e^{-tB}\| \leq e^{-at}$ allows Fubini's theorem to be applied to the integral defining C^2. With $r = t + s$ and $\theta = t/r$, the beta identity gives

$$\begin{aligned} C^2 x &= \frac{1}{\pi} \int_0^\infty \int_0^\infty t^{-1/2} s^{-1/2} e^{-(t+s)B} x\, ds\, dt \\ &= \frac{1}{\pi} \int_0^\infty e^{-rB} x \left(\int_0^1 \theta^{-1/2} (1-\theta)^{-1/2}\, d\theta \right) dr \\ &= \int_0^\infty e^{-rB} x\, dr = B^{-1} x. \end{aligned}$$

Hence C is injective and $\mathrm{Ran}(C) \supset \mathrm{Ran}(C^2) = \mathrm{Ran}(B^{-1}) = \mathcal{D}(B)$ is dense.

Let $f \in X$ and take $x = B^{-1} f$. Fubini's theorem for Bochner integrals and the preceding identity give

$$\begin{aligned}
&\frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-\frac{3}{2}} \left(B^{-1} f - e^{-tB} B^{-1} f \right) dt \\
&\quad = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-\frac{3}{2}} \int_0^t e^{-sB} f \, ds \, dt \\
&\quad = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-\frac{1}{2}} e^{-sB} f \, ds = B^{-\frac{1}{2}} f.
\end{aligned}$$

In the last equality we used the definition of C. Since $B^{-1} f = C^2 f = C(Cf)$, the computed value is $Cf = C^{-1} x = B^{1/2} x$. Finally, under the isometry

$$C^{-1} \colon (\mathcal{D}(B^{1/2}), \|\cdot\|_{1/2}) \longrightarrow X,$$

the subspace $\mathcal{D}(B) = \operatorname{Ran}(C^2)$ corresponds to $\operatorname{Ran}(C)$, which is dense in X. This proves the asserted density. Moreover, $\|x\|_X \le \|C\|_{\mathcal{L}(X)} \|x\|_{1/2}$, so $\|\cdot\|_{1/2}$ is equivalent to the graph norm. □

Proposition B.4.21. (Laplace calculus for a contraction semigroup) Let $(T_t)_{t \ge 0}$ be a contraction C_0-semigroup on a Banach space X. If ν is a finite real or complex Borel measure on $[0, \infty)$, then the Bochner integral

$$\mathcal{L}_\nu(T) x := \int_{[0,\infty)} T_t x \, d\nu(t)$$

defines a bounded operator on X, and

$$\|\mathcal{L}_\nu(T)\|_{\mathcal{L}(X)} \le |\nu|([0,\infty)).$$

If μ is another finite measure and $\nu * \mu$ denotes their convolution, then

$$\mathcal{L}_\nu(T) \mathcal{L}_\mu(T) = \mathcal{L}_{\nu * \mu}(T).$$

In particular, the product of two Laplace transforms of finite measures is represented by the product of the corresponding operators.

Proof. Strong continuity of $t \mapsto T_t x$ ensures Bochner measurability, and contractivity gives

$$\int_{[0,\infty)} \|T_t x\|_X \, d|\nu|(t) \le |\nu|([0,\infty)) \|x\|_X.$$

This proves existence of the integral and the bound. For the product rule, Fubini's theorem for Bochner integrals gives

$$\begin{aligned}
\mathcal{L}_\nu(T) \mathcal{L}_\mu(T) x &= \int_{[0,\infty)} \int_{[0,\infty)} T_{t+s} x \, d\mu(s) \, d\nu(t) \\
&= \int_{[0,\infty)} T_r x \, d(\nu * \mu)(r),
\end{aligned}$$

where we used $T_t T_s = T_{t+s}$. The last assertion follows by taking scalar Laplace transforms. □

Lemma B.4.22. (Independence of the positive shift) Let $-A$ be the generator of a contraction C_0-semigroup $(e^{-tA})_{t\ge 0}$ on a Banach space X. For $a>0$ and $s>0$, define

$$J_s^{(a)}x := \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty t^{\frac{s}{2}-1}e^{-at}e^{-tA}x\,dt.$$

Each $J_s^{(a)}$ is injective. Equip its range with the norm

$$\left\|J_s^{(a)}x\right\|_{\mathrm{Ran}(J_s^{(a)})} := \|x\|_X.$$

If $a,b>0$, then

$$\mathrm{Ran}(J_s^{(a)}) = \mathrm{Ran}(J_s^{(b)}),$$

and the two range norms are equivalent.

Proof. Set $\alpha = \frac{s}{2}$. The Laplace formula for negative powers gives

$$J_s^{(a)} = (aI+A)^{-\alpha}.$$

The semigroup property of these powers and the identity $J_{2m}^{(a)} = (aI+A)^{-m}$ show that $J_s^{(a)}$ is injective. Indeed, if $J_s^{(a)}x = 0$, choose $m\in\mathbb{N}$ with $2m>s$ and obtain

$$(aI+A)^{-m}x = J_{2m-s}^{(a)}J_s^{(a)}x = 0.$$

The resolvent of $-A$ is injective, so $x=0$. Thus the norm in the statement is well defined. Begin with two shifts $c,d>0$ satisfying $|d-c|<c$. For $\lambda\ge 0$, the binomial series converges absolutely, and

$$\left(\frac{d+\lambda}{c+\lambda}\right)^\alpha = \left(1+\frac{d-c}{c+\lambda}\right)^\alpha = 1+\sum_{k=1}^\infty \binom{\alpha}{k}(d-c)^k(c+\lambda)^{-k}.$$

For $k\ge 1$, we have

$$(c+\lambda)^{-k} = \int_0^\infty e^{-\lambda t}\frac{t^{k-1}e^{-ct}}{(k-1)!}\,dt.$$

Thus the preceding function is the Laplace transform of

$$\nu_{c,d} := \delta_0 + \sum_{k=1}^\infty \binom{\alpha}{k}(d-c)^k\frac{t^{k-1}e^{-ct}}{(k-1)!}\,dt.$$

This series converges in total variation because

$$\sum_{k=1}^\infty \left|\binom{\alpha}{k}\right|\left(\frac{|d-c|}{c}\right)^k < \infty.$$

Proposition B.4.21 then defines a bounded operator $Q_{c,d}$ whose scalar multiplier is $\left(\frac{d+\lambda}{c+\lambda}\right)^\alpha$. The product rule gives

$$(cI+A)^{-\alpha} = (dI+A)^{-\alpha}Q_{c,d}. \tag{B.4}$$

For arbitrary shifts $a, b > 0$, choose $N \in \mathbb{N}$ so that $\left(\frac{b}{a}\right)^{\frac{1}{N}} < 2$, and set

$$a_j := a\left(\frac{b}{a}\right)^{\frac{j}{N}}, \qquad j \in \{0, \dots, N\}.$$

Then $|a_{j+1} - a_j| < a_j$ for every j. Iterating (B.4) gives a bounded operator $Q_{a,b}$ such that

$$(aI + A)^{-\alpha} = (bI + A)^{-\alpha} Q_{a,b}.$$

If $u = J_s^{(a)} x$, this identity writes $u = J_s^{(b)} Q_{a,b} x$ and, consequently,

$$\|u\|_{\mathrm{Ran}(J_s^{(b)})} \le \|Q_{a,b}\| \, \|u\|_{\mathrm{Ran}(J_s^{(a)})}.$$

Interchanging a and b gives the reverse inclusion and estimate. □

Proposition B.4.23. (Complex-linear duality of consistent scales) Let X and Y be complex Banach spaces, and let $\mathfrak{b}\colon X \times Y \to \mathbb{C}$ be a continuous bilinear form such that

$$Y \longrightarrow X', \qquad w \longmapsto \mathfrak{b}(\cdot, w),$$

is a complex-linear isometric isomorphism. Let $(T_t)_{t\ge 0}$ and $(S_t)_{t\ge 0}$ be contraction semigroups on X and Y, respectively, and suppose that

$$\mathfrak{b}(T_t u, w) = \mathfrak{b}(u, S_t w) \tag{B.5}$$

for $t \ge 0$ and u, w in respective dense invariant subspaces. If X_s and Y_s are the scales from Theorem B.4.19, then the duality $\mathfrak{b}$ extends uniquely to a complex-linear isometric isomorphism

$$Y_{-s} \xrightarrow{\cong} (X_s)', \qquad s \in \mathbb{R}.$$

In the application to a Hermitian bundle $\mathbf{E}$, take $X = L^p(M, \mathbf{E})$, $Y = L^{p'}(M, \overline{\mathbf{E}})$, and $\mathfrak{b}(u, \overline{v}) = \displaystyle\int_M \mathbf{h}_{\mathbf{E}}(u, v)\, d\lambda_{\mathbf{g}}$; thus the space representing the complex dual of a scale on $\mathbf{E}$ is a scale on $\overline{\mathbf{E}}$, not another complex-linear copy of the scale on $\mathbf{E}$.

Proof. The identity (B.5), first on the dense subspaces and then by continuity, and the Bochner integral defining the Bessel potentials imply, for every $r > 0$,

$$\mathfrak{b}(J_r^X u, w) = \mathfrak{b}(u, J_r^Y w). \tag{B.6}$$

First suppose that $s > 0$. If $w \in Y$ and $v = J_s^X u \in X_s$, then

$$|\mathfrak{b}(v, w)| = |\mathfrak{b}(u, J_s^Y w)| \le \|v\|_{X_s} \|w\|_{Y_{-s}}.$$

By density, the map $w \mapsto \mathfrak{b}(\cdot, w)$ extends isometrically from Y_{-s} to $(X_s)'$. To see that it is surjective, let $F \in (X_s)'$. The functional $u \mapsto F(J_s^X u)$ on X is represented by a unique $z \in Y$. Since $J_s^Y\colon Y_{-s} \to Y$ is an isometric isomorphism, there exists a unique $w \in Y_{-s}$ with $J_s^Y w = z$. Identity (B.6), extended by density,

gives $F(v) = \mathfrak{b}(v, w)$ and preserves the norm.
Now let $s = -r < 0$. If $w = J_r^Y z \in Y_r$, then, for $u \in X$,

$$|\mathfrak{b}(u, w)| = |\mathfrak{b}(J_r^X u, z)| \le \|u\|_{X_{-r}} \|z\|_Y = \|u\|_{X_s} \|w\|_{Y_r}.$$

This extends w to an element of $(X_s)'$. Conversely, if $F \in (X_{-r})'$, its restriction to X is represented by some $w \in Y$. The bound on F makes the functional $J_r^X u \mapsto \mathfrak{b}(u, w)$ continuous on $\operatorname{Ran}(J_r^X)$; since that range is dense in X, it extends to a functional represented by some $z \in Y$. Then (B.6) gives $\mathfrak{b}(u, w) = \mathfrak{b}(u, J_r^Y z)$ for every $u \in X$, hence $w = J_r^Y z \in Y_r$. The argument preserves norms. The case $s = 0$ is the duality $Y \cong X'$ already mentioned. □

B.5 Measure spaces

We now turn to the basic concepts of measure theory. The aim of this section is to establish a unified language for defining Lebesgue and Bochner integrals, L^p spaces, and measures induced by geometric structures.

Definition B.5.1. Let X be a set. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a **σ-algebra on** X if $X \in \mathcal{A}$, if $X \setminus A \in \mathcal{A}$ for each $A \in \mathcal{A}$, and if $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ for every sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$. In this case, $(X, \mathcal{A})$ is called a **measurable space**, and the elements of $\mathcal{A}$ are called **$\mathcal{A}$-measurable** sets, or simply **measurable** sets when the σ-algebra is understood.

Definition B.5.2. Let $(X, \mathcal{A})$ be a measurable space. A **measure** on $(X, \mathcal{A})$ is a function $\mu\colon \mathcal{A} \longrightarrow [0, \infty]$ such that $\mu(\varnothing) = 0$ and $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ for every sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ of pairwise disjoint sets. The triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Before constructing specific measures, we recall how the σ-algebras determining measurable sets are generated.

Definition B.5.3. Let X be a set, and let $\mathcal{S} \subseteq \mathcal{P}(X)$. The **$\sigma$-algebra generated by** $\mathcal{S}$ is $\sigma(\mathcal{S}) := \bigcap \{\mathcal{A} \subseteq \mathcal{P}(X) \mid \mathcal{S} \subseteq \mathcal{A} \text{ and } \mathcal{A} \text{ is a } \sigma\text{-algebra on } X\}$. If (X, τ) is a topological space, the **Borel σ-algebra** of X is $\mathcal{B}(X) := \sigma(\tau)$, and its elements are called Borel sets.

Definition B.5.4. Let $(X, \mathcal{A}, \mu)$ be a measure space. A set $N \in \mathcal{A}$ is **μ-null** if $\mu(N) = 0$. The measure μ, or equivalently the space $(X, \mathcal{A}, \mu)$, is **complete** if every subset of a μ-null set belongs to $\mathcal{A}$; by monotonicity, that subset is then also μ-null.

Definition B.5.5. Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that μ is **finite** if $\mu(X) < \infty$, and **σ-finite** if there exists a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for every $n \in \mathbb{N}$.

Carathéodory's construction produces a complete measure from a function initially defined on all subsets of X. Complete proofs of this construction, regularity of Lebesgue measure, and the signed measure decomposition theorems stated below can be found in [68, 22, 40].

Definition B.5.6. An **outer measure** on a set X is a function $\mu^*\colon \mathcal{P}(X) \longrightarrow [0,\infty]$ such that $\mu^*(\varnothing) = 0$, $\mu^*(A) \le \mu^*(B)$ whenever $A \subseteq B$, and $\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mu^*(A_k)$ for every sequence $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{P}(X)$.

Theorem B.5.7. For a compact rectangle $R = \prod_{i=1}^{n} [a_i, b_i]$, define $\mathrm{vol}_n(R) := \prod_{i=1}^{n} (b_i - a_i)$. Given $A \subseteq \mathbb{R}^n$, let $\mathcal{C}(A)$ be the family of sequences $(R_j)_{j\in\mathbb{N}}$ of compact rectangles such that $A \subseteq \bigcup_{j=1}^{\infty} R_j$, and define $\lambda_n^*(A) := \inf_{(R_j)\in\mathcal{C}(A)} \sum_{j=1}^{\infty} \mathrm{vol}_n(R_j)$. Then λ_n^* is an outer measure on $\mathbb{R}^n$, called n-**dimensional Lebesgue outer measure**.

Not every subset of a space equipped with an outer measure is necessarily measurable. The following condition selects precisely those on which the outer measure becomes countably additive.

Definition B.5.8. Let μ^* be an outer measure on X. A subset $A \subseteq X$ is μ^*-**measurable** if $\mu^*(D) = \mu^*(D \cap A) + \mu^*(D \setminus A)$ for each $D \subseteq X$. Denote by $\mathcal{A}(\mu^*)$ the family of all μ^*-measurable subsets of X.

Theorem B.5.9. (Carathéodory) Let μ^* be an outer measure on X. Then $\mathcal{A}(\mu^*)$ is a σ-algebra on X, and the restriction $\mu^* \restriction_{\mathcal{A}(\mu^*)}$ is a complete measure. This restriction is called the **measure induced by** μ^*.

Definition B.5.10. The σ-algebra of **Lebesgue measurable** subsets of $\mathbb{R}^n$ is $\mathcal{L}(\mathbb{R}^n) := \mathcal{A}(\lambda_n^*)$, and n-**dimensional Lebesgue measure** is the complete measure $\lambda_n := \lambda_n^* \restriction_{\mathcal{L}(\mathbb{R}^n)}$. When there is no risk of confusion, we simply write dx in place of $d\lambda_n(x)$.

To relate measure theory to topology, we introduce regularity and local finiteness.

Definition B.5.11. Let X be a topological space, and let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mathcal{B}(X) \subseteq \mathcal{A}$. The measure μ is **regular** if, for each $A \in \mathcal{A}$, we have $\mu(A) = \inf\{\mu(U) \mid U \subseteq X \text{ is open and } A \subseteq U\} = \sup\{\mu(K) \mid K \subseteq A \text{ is compact}\}$. The measure μ is **locally finite** if each $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$.

Proposition B.5.12. Let X be a topological space, let $(X, \mathcal{A}, \mu)$ be a measure space, and suppose that $\mathcal{B}(X) \subseteq \mathcal{A}$. If μ is locally finite, then $\mu(K) < \infty$ for every compact set $K \subseteq X$. Conversely, if each point of X has a compact neighborhood and $\mu(K) < \infty$ for every compact set $K \subseteq X$, then μ is locally finite.

Proof. If $K \subseteq X$ is compact, for each $x \in K$ there exists an open neighborhood U_x with $\mu(U_x) < \infty$. By compactness, there exist $x_1, \dots, x_r \in K$ such that $K \subseteq \bigcup_{i=1}^{r} U_{x_i}$, so $\mu(K) \le \sum_{i=1}^{r} \mu(U_{x_i}) < \infty$. Conversely, let $x \in X$, and let K be a compact neighborhood of x. There exists an open set U such that $x \in U \subseteq K$, and then $\mu(U) \le \mu(K) < \infty$. □

Definition B.5.13. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $P(x)$ be a property defined for $x \in X$. We say that P holds μ-**almost everywhere**, or for μ-**almost every** $x \in X$, if there exists a set $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $P(x)$ holds for every $x \in X \setminus N$. When the measure is understood, we simply write **almost everywhere**.

Theorem B.5.14. Lebesgue measure λ_n is regular and locally finite.

Regularity of λ_n allows any Lebesgue measurable set to be approximated by Borel sets, changing it only on a null set. Recall that an F_σ set is a countable union of closed sets, and a G_δ set is a countable intersection of open sets.

Theorem B.5.15. Let $A \in \mathcal{L}(\mathbb{R}^n)$. There exist an F_σ set, denoted by F, and a G_δ set, denoted by G, such that $F \subseteq A \subseteq G$ and $\lambda_n(A \setminus F) = \lambda_n(G \setminus A) = 0$. In particular, $\lambda_n(F) = \lambda_n(A) = \lambda_n(G)$. If A is bounded, G may be chosen bounded.

Definition B.5.16. Let X be a topological space.

(a) We say that X is **locally compact** if each point of X has a compact neighborhood.

(b) We say that X is **σ-compact** if there exists a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of X such that $X = \bigcup_{j=1}^{\infty} K_j$.

(c) If X is locally compact and Hausdorff, a measure μ defined on a σ-algebra $\mathcal{A}$ with $\mathcal{B}(X) \subseteq \mathcal{A}$ is called a **Radon measure** if it is regular and locally finite. We say that μ is **massive** if, in addition, it is complete and $\mu(U) > 0$ for every nonempty open set $U \subseteq X$.

The following two properties of Lebesgue measurability will be used when formulating the change-of-variables theorem on subsets that need not be open.

Proposition B.5.17. Let $\Omega \subseteq \mathbb{R}^n$ be open, $A \in \mathcal{L}(\mathbb{R}^n)$ with $A \subseteq \Omega$, and $f \in C^1(\Omega, \mathbb{R}^m)$, where $m \geq n$. Then $f(A) \in \mathcal{L}(\mathbb{R}^m)$.

Proof. By Theorem B.5.15, there exists an F_σ set such that $F \subseteq A$ and $N := A \setminus F$ is λ_n-null. Writing $F = \bigcup_{j=1}^{\infty} C_j$ with each C_j closed and observing that $C_j \subseteq \Omega$, we obtain $F = \bigcup_{j,k=1}^{\infty} (C_j \cap \overline{B}_{\mathrm{euc}}(0,k))$ as a countable union of compact sets. By continuity, $f(F)$ is an F_σ set and, in particular, is Borel. On the other hand, Ω admits a countable cover by convex open balls $(U_r)_{r\in\mathbb{N}}$ with $\overline{U_r} \subseteq \Omega$, and the restriction of f to each U_r is Lipschitz, since Df is bounded on $\overline{U_r}$. A Lipschitz map from a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$, with $m \geq n$, sends λ_n-null sets to λ_m-null sets; to check this, cover the null set by cubes of arbitrarily small diameters and use the fact that the image of each cube is contained in a ball whose diameter is bounded by the Lipschitz constant times the diameter of the cube. Consequently, each $f(N \cap U_r)$ is λ_m-null, and so is $f(N) = \bigcup_{r=1}^{\infty} f(N \cap U_r)$. Finally, $f(A) = f(F) \cup f(N)$ is Lebesgue measurable. □

Proposition B.5.18. A subset $A \subseteq \mathbb{R}^n$ is Lebesgue measurable if and only if for each $x \in A$ there exists an open set $U_x \subseteq \mathbb{R}^n$ such that $x \in U_x$ and $A \cap U_x$ is Lebesgue measurable.

Proof. Necessity is immediate upon taking $U_x = \mathbb{R}^n$. Conversely, the family $(U_x)_{x\in A}$ is an open cover of A. Since $\mathbb{R}^n$ is second countable and hence Lindelöf, there exists a countable subcover $(U_{x_j})_{j\in\mathbb{N}}$. Thus $A = \bigcup_{j=1}^{\infty}(A \cap U_{x_j})$ is Lebesgue measurable. □

B.5.1 Signed measures

To represent linear functionals that need not be positive, we must allow measures taking values of both signs. We combine the finitely additive and countably additive versions in a single definition.

Definition B.5.19. Let X be a nonempty set, and let $\mathcal{A}$ be an algebra of subsets of X, that is, a nonempty family closed under complements relative to X and finite unions. A function $\lambda\colon \mathcal{A} \longrightarrow [-\infty,\infty]$ is a **finitely additive signed measure** if $\lambda(\varnothing) = 0$, it takes at most one of the values ∞ and $-\infty$, and $\lambda(E\cup F) = \lambda(E)+\lambda(F)$ whenever $E, F \in \mathcal{A}$ are disjoint. If $\mathcal{A}$ is a σ-algebra and, for every sequence $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ of pairwise disjoint sets, $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n)$ holds, then λ is called a **signed measure**.

Definition B.5.20. Two positive measures μ and ν on a measurable space $(X,\mathcal{A})$ are **mutually singular**, written $\mu \perp \nu$, if there exists $P \in \mathcal{A}$ such that $\mu(X \setminus P) = 0$ and $\nu(P) = 0$.

The Hahn–Jordan theorem shows that every signed measure decomposes canonically as the difference of two mutually singular positive measures.

Theorem B.5.21. (Hahn–Jordan decomposition) Let $(X,\mathcal{A})$ be a measurable space, and let $\lambda\colon \mathcal{A} \longrightarrow [-\infty,\infty]$ be a signed measure. There exist unique positive measures λ^+ and λ^- such that $\lambda^+ \perp \lambda^-$ and $\lambda = \lambda^+ - \lambda^-$; moreover, one of them is finite. For each $E \in \mathcal{A}$, we have $\lambda^+(E) = \sup\{\lambda(F) \mid F \in \mathcal{A},\ F \subseteq E\}$ and $\lambda^-(E) = -\inf\{\lambda(F) \mid F \in \mathcal{A},\ F \subseteq E\}$. The measures λ^+ and λ^- are called the **positive variation** and **negative variation** of λ, respectively, while $|\lambda| := \lambda^+ + \lambda^-$ is called the **total variation**. We say that λ is finite if $|\lambda|(X) < \infty$.

Definition B.5.22. Let X be a locally compact Hausdorff space. A signed measure ν on $\mathcal{B}(X)$ is called a **signed Radon measure** if its variations ν^+ and ν^- are positive Radon measures.
A **local signed Radon measure** is a family $(\nu_U)_U$, indexed by open sets $U \subseteq X$ with compact closure, such that each ν_U is a finite signed Radon measure on U and

$$\nu_V \restriction_U = \nu_U \qquad \text{if } U \subseteq V.$$

For $u \in C_c(X)$, choose one of these open sets U containing $\operatorname{supp}(u)$ and define

$$\int_X u\,d\nu := \int_U u\,d\nu_U.$$

The value does not depend on U: two choices may be compared on their union, which also has compact closure. The same compatibility and uniqueness in Hahn–Jordan allow the variations ν^+, ν^-, and $|\nu|$ to be defined on relatively compact Borel sets through the variations of ν_U. In particular, $|\nu|(K) < \infty$ for each compact set $K \subseteq X$. A global signed Radon measure determines such a family by restriction, but a local

family need not admit values on all Borel subsets of X.

In what follows, $C_c(X)$ denotes the vector space of continuous functions $u\colon X \longrightarrow \mathbb{R}$ with compact support. On a general locally compact space, the regularity of representing measures should be specified separately. When X is σ-compact, the positive measures in the following theorem are Radon measures under the regularity convention fixed above.

Theorem B.5.23. (Riesz representation theorem on C_c) Let X be a locally compact Hausdorff topological space, and let $L\colon C_c(X) \longrightarrow \mathbb{R}$ be a linear functional.

(a) If L is positive, that is, $L(u) \geq 0$ for every $u \in C_c(X)$ with $u \geq 0$, then there exists a unique positive Borel measure μ, finite on compact sets, outer regular on Borel sets, and inner regular on open sets and on Borel sets of finite measure, such that $L(u) = \displaystyle\int_X u\, d\mu$ for every $u \in C_c(X)$. Uniqueness is understood among measures with these regularity properties.

(b) If L is locally bounded, that is, if for each compact set $K \subseteq X$ there exists $c_K > 0$ such that $|L(u)| \leq c_K \|u\|_{K,\infty}$ for every $u \in C_c(X)$ with $\operatorname{supp}(u) \subseteq K$, where $\|u\|_{K,\infty} := \sup\limits_{x\in K} |u(x)|$, then there exists a unique local signed Radon measure ν such that $L(u) = \displaystyle\int_X u\, d\nu$ for every $u \in C_c(X)$. Moreover, there exist positive Borel measures μ_1 and μ_2, with the finiteness and regularity properties in part (a), such that

$$L(u) = \int_X u\, d\mu_1 - \int_X u\, d\mu_2, \qquad u \in C_c(X).$$

On each open set U with compact closure, we have $\nu_U = \mu_1 \restriction_U - \mu_2 \restriction_U$.

If X is σ-compact, the positive measures in both parts are Radon measures in the sense of Definition B.5.16. This applies, in particular, to open subsets of $\mathbb{R}^n$ and to second-countable smooth manifolds.

Proof. The positive representation in part (a), with its regularity and uniqueness properties, is Riesz's theorem; see [22, 40]. We shall deduce part (b) from it without assuming that one of the representing measures has finite total mass.

For $f \in C_c(X)$ with $f \geq 0$, set

$$L^+(f) := \sup\{L(g) \mid g \in C_c(X),\ 0 \leq g \leq f\}.$$

If $K = \operatorname{supp}(f)$, every admissible function g is supported in K and $\|g\|_\infty \leq \|f\|_\infty$. By local boundedness of L,

$$0 \leq L^+(f) \leq c_K \|f\|_\infty < \infty.$$

Homogeneity for nonnegative scalars follows directly from the definition, including the zero scalar.

Let $f, h \in C_c(X)$ be nonnegative. If $0 \leq g_1 \leq f$ and $0 \leq g_2 \leq h$, then $0 \leq g_1 + g_2 \leq f + h$. Taking suprema gives $L^+(f+h) \geq L^+(f) + L^+(h)$. For the reverse inequality, let $0 \leq g \leq f + h$ and define

$$g_1 := \min\{g, f\}, \qquad g_2 := g - g_1.$$

These functions are continuous and compactly supported, and satisfy $0 \leq g_1 \leq f$ and $0 \leq g_2 \leq h$. Thus $L(g) = L(g_1) + L(g_2) \leq L^+(f) + L^+(h)$. Taking the supremum over g shows that L^+ is additive on the positive

cone.

Extend L^+ to all of $C_c(X)$ by $L^+(f-h) := L^+(f) - L^+(h)$ for $f, h \geq 0$. If $f - h = f_1 - h_1$, then $f + h_1 = f_1 + h$, and the additivity just proved shows that both expressions give the same value. We thus obtain a positive linear functional. Since $g = f$ is admissible in the supremum when $f \geq 0$, we have $L^+(f) \geq L(f)$. Consequently, $L^- := L^+ - L$ is also a positive linear functional. Applying (a) to L^+ and L^- yields the measures μ_1, μ_2 with the asserted representation. Both integrals are finite for each $u \in C_c(X)$ because the measures are finite on its support.

Now let $U \subseteq X$ be open with compact closure. Then $\mu_i(U) \leq \mu_i(\overline{U}) < \infty$ for $i = 1, 2$. The restrictions of μ_i to U are regular: all their Borel sets have finite measure, and outer regularity is preserved by intersecting the approximating open sets with U. The difference

$$\nu_U := \mu_1 \restriction_U - \mu_2 \restriction_U$$

is a finite signed measure. To check that its variations are regular, set $\eta := \mu_1 \restriction_U + \mu_2 \restriction_U$. The Hahn decomposition gives $0 \leq \nu_U^{\pm} \leq \eta$. Given a Borel set $E \subseteq U$ and $\varepsilon > 0$, regularity and finiteness of η allow us to choose a compact set $K \subseteq E$ and an open set $O \supseteq E$ in U such that $\eta(O \setminus K) < \varepsilon$. Then

$$0 \leq \nu_U^{\pm}(O) - \nu_U^{\pm}(K) = \nu_U^{\pm}(O \setminus K) < \varepsilon.$$

This proves regularity of both variations. Thus ν_U is a finite signed Radon measure. These differences are compatible under restriction from one open set to another, so they define a local signed Radon measure ν.

For uniqueness, let ν and $\widetilde{\nu}$ be two representing families, and fix U as above. A function in $C_c(U)$ extends by zero to an element of $C_c(X)$, so the finite measure $\lambda := \nu_U - \widetilde{\nu}_U$ vanishes on $C_c(U)$. By Hahn–Jordan, $\int_U u \, d\lambda^+ = \int_U u \, d\lambda^-$ for every $u \in C_c(U)$. Uniqueness in (a) implies $\lambda^+ = \lambda^-$; since they are mutually singular, both vanish. Thus $\nu_U = \widetilde{\nu}_U$ for every U.

Finally, suppose that X is σ-compact, and let μ be any of the positive measures obtained. Choose increasing compact sets K_j covering X. For a Borel set E, continuity from below gives

$$\mu(E) = \lim_{j \to \infty} \mu(E \cap K_j).$$

Each $E \cap K_j$ has finite measure and is inner regular by (a). Hence $\mu(E) = \sup\{\mu(K) \mid K \subseteq E,\ K \text{ compact}\}$, also when $\mu(E) = \infty$. Outer regularity is already part of (a), and finiteness on compact sets implies local finiteness on a locally compact space. This proves the last assertion. □

The local family ν must not be confused with a difference defined on all Borel subsets of X: when $\mu_1(E) = \mu_2(E) = \infty$, the expression $\mu_1(E) - \mu_2(E)$ is not meaningful. For example, $L(u) = \int_{\mathbb{R}} xu(x)\,dx$ is locally bounded on $C_c(\mathbb{R})$ and is represented by $\max\{x, 0\}\,dx$ and $\max\{-x, 0\}\,dx$, both of infinite total mass. On each relatively compact open set, however, both measures are finite and their difference is well defined.

B.6 Lebesgue and Bochner integration

Having fixed the basic notions of measure, we recall the construction of the Lebesgue integral and its extension to Banach-valued functions through the Bochner integral. We begin with the vector-valued Riemann integral, which provides a natural way to introduce compatibility of integration with linear operators. For a more detailed

treatment of the Bochner integral, see [45].

B.6.1 The Banach-valued Riemann integral

Definition B.6.1. Let E be a Banach space, let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a,b] \longrightarrow E$. A **tagged partition** of $[a,b]$ is a pair consisting of a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ and points $\tau_i \in [t_{i-1}, t_i]$, $i \in \{1, \ldots, n\}$. Define the mesh of the partition by $|\mathcal{P}| := \max_{1 \le i \le n} (t_i - t_{i-1})$. We say that f is **Riemann integrable** if there exists $I \in E$ such that, for each $\varepsilon > 0$, there exists $\delta > 0$ with the property that $\left\| \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1}) - I \right\|_E < \varepsilon$ for every tagged partition with $|\mathcal{P}| < \delta$. In this case, I is unique and is denoted by $\int_a^b f(t)\,dt$.

Theorem B.6.2. (Graves) Let E be a Banach space. Every continuous function $f : [a,b] \longrightarrow E$ is Riemann integrable.

Proof. Since $[a,b]$ is compact, f is uniformly continuous. For a tagged partition $(\mathcal{P}, \tau)$, denote its Riemann sum by

$$S(f; \mathcal{P}, \tau) := \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1})$$

Fix $\varepsilon > 0$. Uniform continuity allows us to choose $\delta > 0$ so that

$$|s - t| < \delta \quad \Longrightarrow \quad \|f(s) - f(t)\|_E < \frac{\varepsilon}{4(b-a)}.$$

Let $(\mathcal{P}, \tau)$ and $(\mathcal{Q}, \sigma)$ be two tagged partitions with mesh less than δ, and let $\mathcal{R}$ be the partition obtained by combining and ordering the points of $\mathcal{P}$ and $\mathcal{Q}$. Choose a tag in each subinterval of $\mathcal{R}$. Each such subinterval is contained in an interval of $\mathcal{P}$; its tag and the corresponding tag of $\mathcal{P}$ are less than δ apart. By the triangle inequality,

$$\left\| S(f; \mathcal{P}, \tau) - S(f; \mathcal{R}, \rho) \right\|_E \le \frac{\varepsilon}{4(b-a)} \sum_{J \in \mathcal{R}} |J| = \frac{\varepsilon}{4}.$$

The same estimate holds with $\mathcal{Q}$ in place of $\mathcal{P}$. Consequently,

$$\left\| S(f; \mathcal{P}, \tau) - S(f; \mathcal{Q}, \sigma) \right\|_E < \frac{\varepsilon}{2}.$$

For each $k \in \mathbb{N}$, choose a tagged partition $(\mathcal{P}_k, \tau^{(k)})$ with mesh less than $1/k$, and set $S_k := S(f; \mathcal{P}_k, \tau^{(k)})$. The preceding estimate shows that (S_k) is Cauchy. Since E is complete, there exists $I \in E$ such that $S_k \to I$.
It remains to check that all sufficiently fine sums converge to I. Given $\varepsilon > 0$, repeat the preceding argument with ε in place of the quantity fixed at the beginning and obtain the corresponding δ. Choose k so large that $|\mathcal{P}_k| < \delta$ and $\|S_k - I\|_E < \varepsilon/2$. For any tagged partition $(\mathcal{P}, \tau)$ with $|\mathcal{P}| < \delta$, comparison through a common refinement gives

$$\|S(f; \mathcal{P}, \tau) - I\|_E \le \|S(f; \mathcal{P}, \tau) - S_k\|_E + \|S_k - I\|_E < \varepsilon.$$

This is precisely Riemann integrability of f. □

Proposition B.6.3. Let E be a Banach space, and let $f, g\colon [a,b] \longrightarrow E$ be Riemann integrable functions. Then $\alpha f + \beta g$ is Riemann integrable for any $\alpha, \beta \in \mathbb{K}$ and $\displaystyle\int_a^b (\alpha f + \beta g)(t)\,dt = \alpha \int_a^b f(t)\,dt + \beta \int_a^b g(t)\,dt$. Moreover, $\displaystyle\left\| \int_a^b f(t)\,dt \right\|_E \leq \int_a^b \|f(t)\|_E\,dt$. In particular, the map $\mathcal{I}\colon C([a,b],E) \longrightarrow E$, given by $\mathcal{I}(f) := \displaystyle\int_a^b f(t)\,dt$, is linear and bounded, and satisfies $\|\mathcal{I}\|_{\mathcal{L}(C([a,b],E),E)} \leq b-a$ with respect to the norm $\|f\|_{C([a,b],E)} := \max\limits_{t\in[a,b]} \|f(t)\|_E$.

Proof. Linearity follows by taking linear combinations of Riemann sums. For the inequality, every Riemann sum satisfies $\displaystyle\left\| \sum_{i=1}^n f(\tau_i)(t_i - t_{i-1}) \right\|_E \leq \sum_{i=1}^n \|f(\tau_i)\|_E (t_i - t_{i-1})$; passing to the limit gives the assertion. Finally, $\displaystyle\int_a^b \|f(t)\|_E\,dt \leq (b-a)\|f\|_{C([a,b],E)}$. □

The integral also commutes with continuous linear operators and, under an additional integrability hypothesis, with closed linear operators.

Proposition B.6.4. Let E and F be Banach spaces, and let $f\colon [a,b] \longrightarrow E$ be Riemann integrable.

(a) If $T \in \mathcal{L}(E,F)$, then $T \circ f$ is Riemann integrable and $\displaystyle T\left(\int_a^b f(t)\,dt\right) = \int_a^b T(f(t))\,dt$.

(b) If $T\colon \mathrm{Dom}(T) \subseteq E \longrightarrow F$ is a closed linear operator, $f(t) \in \mathrm{Dom}(T)$ for every $t \in [a,b]$, and $T \circ f$ is Riemann integrable, then $\displaystyle\int_a^b f(t)\,dt \in \mathrm{Dom}(T)$ and the same equality holds.

Proof. The first assertion follows by applying T to Riemann sums and using continuity. For the second, let $(S_k)_{k\in\mathbb{N}}$ be a sequence of Riemann sums of f associated with partitions whose mesh tends to zero. Since $\mathrm{Dom}(T)$ is a vector subspace, $S_k \in \mathrm{Dom}(T)$ and $T(S_k)$ is the corresponding Riemann sum for $T \circ f$. Thus $S_k \longrightarrow \displaystyle\int_a^b f(t)\,dt$ in E and $T(S_k) \longrightarrow \displaystyle\int_a^b T(f(t))\,dt$ in F. Closedness of T completes the proof. □

Theorem B.6.5. (Fundamental theorem of calculus for Banach-valued functions) Let E be a Banach space, and let $F\colon [a,b] \longrightarrow E$ be a function of class C^1. Then

$$F(b) - F(a) = \int_a^b F'(t)\,dt.$$

More generally, for any $s, t \in [a,b]$ with $s \leq t$, we have $F(t) - F(s) = \displaystyle\int_s^t F'(\tau)\,d\tau$.

Proof. Continuity of F' and Theorem B.6.2 ensure that the integral is well defined. Let $\varphi \in E'$. The scalar function $\varphi \circ F$ is of class C^1 and, by the chain rule, $(\varphi \circ F)'(t) = \varphi(F'(t))$. The fundamental theorem of calculus for real-valued functions and Proposition B.6.4 imply

$$\varphi(F(b) - F(a)) = \int_a^b \varphi(F'(t))\,dt = \varphi\left(\int_a^b F'(t)\,dt\right).$$

Thus $\varphi\left(F(b)-F(a)-\int_a^b F'(t)\,dt\right)=0$ for every $\varphi\in E'$. Since the continuous dual separates points, the identity follows. The formula on $[s,t]$ is proved in the same way. □

Proposition B.6.6. (Integration by parts for Banach-valued functions) Let E be a Banach space, let $\alpha\colon[a,b]\longrightarrow\mathbb{K}$ be of class C^1, and let $F\colon[a,b]\longrightarrow E$ be of class C^1. Then

$$\int_a^b \alpha(t)F'(t)\,dt=\alpha(b)F(b)-\alpha(a)F(a)-\int_a^b \alpha'(t)F(t)\,dt. \tag{B.7}$$

Proof. The map $B\colon\mathbb{K}\times E\longrightarrow E$, defined by $B(\lambda,v):=\lambda v$, is continuous and bilinear. The product rule gives

$$\frac{d}{dt}\big(\alpha(t)F(t)\big)=\alpha'(t)F(t)+\alpha(t)F'(t).$$

Applying Theorem B.6.5 to the function $t\longmapsto\alpha(t)F(t)$ and using linearity of the integral, we obtain

$$\alpha(b)F(b)-\alpha(a)F(a)=\int_a^b \alpha'(t)F(t)\,dt+\int_a^b \alpha(t)F'(t)\,dt.$$

Solving for the last integral gives (B.7). □

B.6.2 The Lebesgue integral

From now on, $(M,\mathcal{A},\mu)$ denotes a measure space. Measurability of a function will always be understood with respect to the σ-algebra $\mathcal{A}$ on the domain and the Borel σ-algebra on the codomain.

Definition B.6.7. Let Y be a topological space. A function $f\colon M\longrightarrow Y$ is **$\mathcal{A}$-measurable**, or simply **measurable**, if $f^{-1}(U)\in\mathcal{A}$ for every open set $U\subseteq Y$. In particular, for functions $f\colon M\longrightarrow\overline{\mathbb{R}}:=[-\infty,\infty]$, we equip $\overline{\mathbb{R}}$ with the order topology.

The Lebesgue integral is first constructed for nonnegative simple functions. We use a single notion of simple function, valid for both scalar-valued and vector-valued functions.

Definition B.6.8. Let V be a vector space. A function $s\colon M\longrightarrow V$ is **simple** if there exist pairwise disjoint measurable sets $A_1,\dots,A_r\in\mathcal{A}$ and vectors $v_1,\dots,v_r\in V$ such that $s=\sum_{i=1}^{r} v_i\mathbf{1}_{A_i}$, where $\mathbf{1}_A$ denotes the indicator function of A.

Definition B.6.9. Let $s\colon M\longrightarrow[0,\infty)$ be a simple function, and write $s=\sum_{i=1}^{r} a_i\mathbf{1}_{A_i}$, where $a_i\geq 0$ and the sets A_i are measurable and pairwise disjoint. Define $\int_M s\,d\mu:=\sum_{i=1}^{r} a_i\mu(A_i)$, with the convention $0\cdot\infty=0$. This definition is independent of the chosen representation of s.

Definition B.6.10. Let $f\colon M \longrightarrow [0,\infty]$ be measurable. Define its **Lebesgue integral** by $\int_M f\,d\mu := \sup\left\{\int_M s\,d\mu \;\middle|\; s\colon M \longrightarrow [0,\infty) \text{ is simple and } 0 \le s \le f\right\}$. If $A \in \mathcal{A}$, define $\int_A f\,d\mu := \int_M f\mathbf{1}_A\,d\mu$.

The first fundamental convergence theorem expresses continuity of the integral with respect to increasing limits.

Theorem B.6.11. (Monotone convergence theorem) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n\colon M \longrightarrow [0,\infty]$ such that $f_n \le f_{n+1}$ for every $n \in \mathbb{N}$ and $f_n(x) \longrightarrow f(x)$ for μ-almost every $x \in M$. Then f is measurable and $\int_A f\,d\mu = \lim_{n\to\infty}\int_A f_n\,d\mu$ for every $A \in \mathcal{A}$.

Proof. Modifying all the functions on a single null set, we may suppose that convergence and monotonicity hold at every point. Then $f = \sup_{n\in\mathbb{N}} f_n$, so f is measurable. It suffices to prove the equality for $A = M$, since the general case follows by applying the same argument to $f_n\mathbf{1}_A$.
By monotonicity of the integral, the sequence $\int_M f_n\,d\mu$ is increasing and

$$I := \lim_{n\to\infty}\int_M f_n\,d\mu \le \int_M f\,d\mu.$$

For the reverse inequality, let s be a nonnegative simple function with $s \le f$, and fix $c \in (0,1)$. Define

$$A_n := \{x \in M \mid f_n(x) \ge c\,s(x)\}.$$

The sets A_n are measurable, form an increasing sequence, and have union M: if $s(x) = 0$, then $x \in A_n$ for every n; if $s(x) > 0$, convergence $f_n(x) \to f(x) \ge s(x) > c\,s(x)$ implies that $x \in A_n$ for sufficiently large n. Thus

$$\int_M f_n\,d\mu \ge c\int_{A_n} s\,d\mu.$$

If $s = \sum_{j=1}^{r} a_j\mathbf{1}_{B_j}$ with $a_j \ge 0$ and the B_j pairwise disjoint, continuity from below of the measure gives

$$\lim_{n\to\infty}\int_{A_n} s\,d\mu = \sum_{j=1}^{r} a_j \lim_{n\to\infty}\mu(A_n \cap B_j) = \sum_{j=1}^{r} a_j\mu(B_j) = \int_M s\,d\mu.$$

Consequently, $I \ge c\int_M s\,d\mu$. Letting $c \uparrow 1$ and then taking the supremum over all simple functions $0 \le s \le f$ gives

$$I \ge \int_M f\,d\mu.$$

Together with the first inequality, this proves equality. □

Lemma B.6.12. (Fatou) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n\colon M \longrightarrow [0,\infty]$. Then

$$\int_M \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_M f_n \, d\mu.$$

Proof. For each $n \in \mathbb{N}$, define $g_n := \inf_{k\ge n} f_k$. The sequence (g_n) is increasing and converges pointwise to $\liminf_{n\to\infty} f_n$. By Theorem B.6.11,

$$\int_M \liminf_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_M g_n \, d\mu.$$

Since $g_n \le f_k$ for every $k \ge n$, we have $\int_M g_n \, d\mu \le \inf_{k\ge n} \int_M f_k \, d\mu$. Letting $n \to \infty$ yields the desired inequality. □

Definition B.6.13. Let $f\colon M \longrightarrow \overline{\mathbb{R}}$ be measurable, and define $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$. Whenever at least one of the quantities $\int_M f^+ \, d\mu$ and $\int_M f^- \, d\mu$ is finite, define $\int_M f \, d\mu := \int_M f^+ \, d\mu - \int_M f^- \, d\mu$. We say that f is **Lebesgue integrable**, or simply **integrable**, if $\int_M |f| \, d\mu < \infty$. Denote by $\mathcal{L}^1(M,\mu)$ the vector space of integrable functions and by $L^1(M,\mu)$ the space of their equivalence classes modulo equality almost everywhere. For $A \in \mathcal{A}$, write $\int_A f \, d\mu := \int_M f \mathbf{1}_A \, d\mu$.

Theorem B.6.14. (Lebesgue dominated convergence theorem) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n\colon M \longrightarrow \overline{\mathbb{R}}$ such that $f_n(x) \longrightarrow f(x)$ for μ-almost every $x \in M$. Suppose that there exists $g \in \mathcal{L}^1(M,\mu)$ such that $|f_n| \le g$ μ-almost everywhere for every $n \in \mathbb{N}$. Then $f \in \mathcal{L}^1(M,\mu)$, $\int_M |f_n - f| \, d\mu \longrightarrow 0$, and $\int_A f \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu$ for every $A \in \mathcal{A}$.

Proof. The inequality $|f_n| \le g$ implies $g \ge 0$ almost everywhere; if desired, g may be replaced by $|g|$. We may modify the functions on a common null set and assume that convergence and domination hold at every point. The pointwise limit f is measurable and satisfies $|f| \le g$, so $f \in \mathcal{L}^1(M,\mu)$. Moreover,

$$0 \le |f_n - f| \le 2g \quad \text{and} \quad 2g - |f_n - f| \longrightarrow 2g$$

pointwise. Applying Fatou's lemma to the nonnegative functions $2g - |f_n - f|$ gives

$$2\int_M g \, d\mu \le \liminf_{n\to\infty} \left(2\int_M g \, d\mu - \int_M |f_n - f| \, d\mu \right) = 2\int_M g \, d\mu - \limsup_{n\to\infty} \int_M |f_n - f| \, d\mu.$$

Since $\int_M g \, d\mu < \infty$, it follows that

$$\lim_{n\to\infty} \int_M |f_n - f| \, d\mu = 0.$$

Finally, for every $A \in \mathcal{A}$,

$$\left| \int_A f_n \, d\mu - \int_A f \, d\mu \right| \le \int_A |f_n - f| \, d\mu \le \int_M |f_n - f| \, d\mu \longrightarrow 0.$$

□

In Euclidean spaces, the Lebesgue integral transforms through the Jacobian of a diffeomorphism.

Theorem B.6.15. (Change-of-variables theorem) Let $\Omega, \Omega' \subseteq \mathbb{R}^n$ be open sets, and let $\varphi\colon \Omega \longrightarrow \Omega'$ be a diffeomorphism of class C^1.

(a) If $f\colon \Omega' \longrightarrow [0,\infty]$ is measurable, then $\displaystyle\int_{\Omega} f(\varphi(x))\,|\det D\varphi(x)|\,d\lambda_n(x) = \int_{\Omega'} f(y)\,d\lambda_n(y)$.

(b) A measurable function $f\colon \Omega' \longrightarrow \mathbb{R}$ is integrable if and only if $(f\circ\varphi)\,|\det D\varphi|$ is integrable on Ω, and in this case the same equality holds.

A proof based on approximation by cubes and the local behavior of a C^1 diffeomorphism can be found in [68, 22].

Corollary B.6.16. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $A \in \mathcal{L}(\mathbb{R}^n)$ satisfy $A \subseteq \Omega$ and $\lambda_n(A \setminus \operatorname{int}(A)) = 0$. Suppose that $\Phi \in C^1(\Omega, \mathbb{R}^n)$ and that $\Phi\restriction_{\operatorname{int}(A)}\colon \operatorname{int}(A) \longrightarrow \Phi(\operatorname{int}(A))$ is a diffeomorphism.

(a) If $f\colon \Phi(A) \longrightarrow [0,\infty]$ is measurable, then $\displaystyle\int_{\Phi(A)} f(y)\,d\lambda_n(y) = \int_A f(\Phi(x))\,|\det D\Phi(x)|\,d\lambda_n(x)$.

(b) A measurable function $f\colon \Phi(A) \longrightarrow \mathbb{R}$ is integrable if and only if $(f\circ\Phi)\,|\det D\Phi|$ is integrable on A, and in this case the same equality holds.

Proof. Let $N := A \setminus \operatorname{int}(A)$. By Proposition B.5.17, $\Phi(N)$ is Lebesgue measurable and $\lambda_n(\Phi(N)) = 0$, since Φ is locally Lipschitz. Moreover, $\Phi(A) \setminus \Phi(\operatorname{int}(A)) \subseteq \Phi(N)$. Thus the integrals over A and $\Phi(A)$ do not change when these sets are replaced by $\operatorname{int}(A)$ and $\Phi(\operatorname{int}(A))$, respectively. The result follows from Theorem B.6.15. □

Theorem B.6.17. (Tonelli's theorem) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. If $f\colon X \times Y \longrightarrow [0,\infty]$ is measurable, then the functions

$$x \longmapsto \int_Y f(x,y)\,d\nu(y), \qquad y \longmapsto \int_X f(x,y)\,d\mu(x)$$

are measurable and

$$\int_{X\times Y} f\,d(\mu\otimes\nu) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)\,d\mu(x)\right) d\nu(y),$$

with values in $[0,\infty]$ allowed.

Proof. The equality is first verified for indicators of measurable rectangles and, by additivity, for nonnegative simple functions. If f_k is an increasing sequence of nonnegative simple functions with $f_k \to f$, Theorem B.6.11, applied first to the inner integrals and then to the outer ones, allows passage to the limit and proves all three equalities. Measurability of the partial integrals follows in the same step as an increasing limit of measurable functions. □

Theorem B.6.18. (Fubini's theorem) Let $1 \le m < n$, and identify $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. If $f \in \mathcal{L}^1(\mathbb{R}^n, \lambda_n)$, then, for λ_{n-m}-almost every $y \in \mathbb{R}^{n-m}$, the function $x \longmapsto f(x,y)$ belongs to $\mathcal{L}^1(\mathbb{R}^m, \lambda_m)$. The function $F(y) := \int_{\mathbb{R}^m} f(x,y)\, d\lambda_m(x)$, defined arbitrarily on the exceptional set, belongs to $\mathcal{L}^1(\mathbb{R}^{n-m}, \lambda_{n-m})$, and $\int_{\mathbb{R}^n} f\, d\lambda_n = \int_{\mathbb{R}^{n-m}} F(y)\, d\lambda_{n-m}(y)$. Similarly, for λ_m-almost every $x \in \mathbb{R}^m$, the function $y \longmapsto f(x,y)$ is integrable. If $H(x) := \int_{\mathbb{R}^{n-m}} f(x,y)\, d\lambda_{n-m}(y)$, then $H \in \mathcal{L}^1(\mathbb{R}^m, \lambda_m)$ and $\int_{\mathbb{R}^n} f\, d\lambda_n = \int_{\mathbb{R}^m} H(x)\, d\lambda_m(x)$.

Proof. Apply Tonelli's theorem to the nonnegative function $|f|$. Since

$$\int_{\mathbb{R}^{n-m}} \left(\int_{\mathbb{R}^m} |f(x,y)|\, d\lambda_m(x) \right) d\lambda_{n-m}(y) = \int_{\mathbb{R}^n} |f|\, d\lambda_n < \infty,$$

the inner integral of $|f|$ is finite for λ_{n-m}-almost every y. For these y, the function $x \mapsto f(x,y)$ is integrable. Writing $f = f^+ - f^-$ and applying Tonelli separately to f^+ and f^- gives measurability of

$$F(y) = \int_{\mathbb{R}^m} f(x,y)\, d\lambda_m(x)$$

outside the exceptional set and the identity

$$\int_{\mathbb{R}^{n-m}} F(y)\, d\lambda_{n-m}(y) = \int_{\mathbb{R}^n} f^+\, d\lambda_n - \int_{\mathbb{R}^n} f^-\, d\lambda_n = \int_{\mathbb{R}^n} f\, d\lambda_n.$$

Likewise,

$$|F(y)| \le \int_{\mathbb{R}^m} |f(x,y)|\, d\lambda_m(x)$$

for almost every y, and the right-hand side is integrable; thus $F \in \mathcal{L}^1(\mathbb{R}^{n-m})$. Interchanging the roles of x and y in this argument gives the assertions about H. □

Before stating the consequence ensuring uniqueness of weak derivatives, we define local integrability.

Definition B.6.19. Let $\Omega \subseteq \mathbb{R}^n$ be open. A measurable function $f : \Omega \longrightarrow \overline{\mathbb{R}}$ is **locally integrable** if $\int_K |f|\, d\lambda_n < \infty$ for every compact set $K \subseteq \Omega$. Denote by $\mathcal{L}^1_{\mathrm{loc}}(\Omega)$ the space of such functions and by $L^1_{\mathrm{loc}}(\Omega)$ the space of their equivalence classes modulo equality almost everywhere.

The following consequence states that a locally integrable function is determined, up to a null set, by its integrals against test functions. It will be used repeatedly to justify uniqueness of weak derivatives.

Proposition B.6.20. Let $\Omega \subseteq \mathbb{R}^n$ be open. If $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfies $\int_\Omega f(x)\varphi(x)\, d\lambda_n(x) = 0$ for every $\varphi \in C_c^\infty(\Omega)$, then $f = 0$ almost everywhere on Ω.

Proof. Suppose, for a contradiction, that f does not vanish almost everywhere. An exhaustion of Ω by compact sets and the decomposition $\{|f| > 0\} = \bigcup_{j \ge 1} \{|f| \ge j^{-1}\}$ provide a compact set $K \Subset \Omega$ and a $j \ge 1$ for which $K \cap \{|f| \ge j^{-1}\}$ has positive measure. Divide the unit circle into finitely many sectors of angular width

less than $\frac{\pi}{2}$. One of them determines a measurable set $A \subseteq K$ of positive measure and a number $\omega \in \mathbb{C}$, $|\omega| = 1$, such that

$$\operatorname{Re}(\omega f) \geq c|f| \geq \frac{c}{j} \quad \text{on } A$$

for some constant $c > 0$. In the real case, it suffices to take one of the two signs.

By regularity of Lebesgue measure, choose a compact set $C \subseteq A$ of positive measure. Then $I_C := \int_C |f|\,d\lambda_n > 0$. Absolute continuity of the integral and outer regularity allow us to choose an open set U with $C \subseteq U \Subset \Omega$ and

$$\int_{U\setminus C} |f|\,d\lambda_n < \frac{c}{2} I_C.$$

By Proposition 1.3.6, there exists $\eta \in C_c^\infty(U)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on C. Applying the hypothesis to $\varphi = \omega\eta$ instead gives

$$\operatorname{Re} \int_\Omega f\varphi\,d\lambda_n \geq cI_C - \int_{U\setminus C} |f|\,d\lambda_n > \frac{c}{2} I_C > 0,$$

a contradiction. Therefore $f = 0$ almost everywhere on Ω. □

B.6.3 The Bochner integral

We now consider functions defined on $(M, \mathcal{A}, \mu)$ with values in a Banach space E. The notion of simple function introduced above remains valid upon taking $V = E$; strong measurability is obtained by pointwise approximation by these functions, whereas the Bochner integral is obtained by approximation in the L^1 norm.

Definition B.6.21. Let E be a Banach space.

(a) A function $f\colon M \longrightarrow E$ is **strongly measurable** if there exists a sequence $(s_k)_{k\in\mathbb{N}}$ of simple functions $s_k\colon M \longrightarrow E$ such that $s_k(x) \longrightarrow f(x)$ for μ-almost every $x \in M$.

(b) A simple function $s\colon M \longrightarrow E$ is **integrable** if $\int_M \|s\|_E\,d\mu < \infty$. If $s = \sum_{i=1}^r v_i \mathbf{1}_{A_i}$, where the A_i are measurable and pairwise disjoint and terms with $v_i = 0$ have been omitted, then $\mu(A_i) < \infty$ for each i, and we define $\int_M s\,d\mu := \sum_{i=1}^r v_i \mu(A_i)$.

(c) A strongly measurable function $f\colon M \longrightarrow E$ is **Bochner integrable** if there exists a sequence $(s_k)_{k\in\mathbb{N}}$ of integrable simple functions such that $\int_M \|f - s_k\|_E\,d\mu \longrightarrow 0$. In this case, define $\int_M f\,d\mu := \lim_{k\to\infty} \int_M s_k\,d\mu$. Denote by $\mathcal{L}^1(M,\mu,E)$ the space of Bochner integrable functions and by $L^1(M,\mu,E)$ the space of their equivalence classes modulo equality almost everywhere. For $A \in \mathcal{A}$, define $\int_A f\,d\mu := \int_M f\mathbf{1}_A\,d\mu$.

Lemma B.6.22. The integral of an integrable simple function $s\colon M \longrightarrow E$ is well defined, that is, independent of the representation of s, and satisfies $\left\| \int_M s\,d\mu \right\|_E \leq \int_M \|s\|_E\,d\mu$.

Proof. Two representations of s admit a common refinement consisting of intersections of their level sets, and the sum defining the integral agrees when computed on this refinement. If $s = \sum_{i=1}^{r} v_i \mathbf{1}_{A_i}$ with the A_i pairwise disjoint, then $\left\| \int_M s \, d\mu \right\|_E = \left\| \sum_{i=1}^{r} v_i \mu(A_i) \right\|_E \leq \sum_{i=1}^{r} \|v_i\|_E \mu(A_i) = \int_M \|s\|_E \, d\mu$. □

The following criterion shows that, once strong measurability has been established, Bochner integrability is characterized entirely by integrability of the norm.

Proposition B.6.23. Let E be a Banach space, and let $f : M \longrightarrow E$ be strongly measurable. Then f is Bochner integrable if and only if $\|f\|_E \in \mathcal{L}^1(M,\mu)$. In this case, the Bochner integral is well defined, and the map $\mathcal{I}_B : \mathcal{L}^1(M,\mu,E) \longrightarrow E$, given by $\mathcal{I}_B(f) := \int_M f \, d\mu$, is linear and satisfies $\left\| \int_M f \, d\mu \right\|_E \leq \int_M \|f\|_E \, d\mu$.

Proof. Since the norm is continuous, strong measurability of f implies measurability of $\|f\|_E$. First suppose that f is Bochner integrable, and let $(s_k)_{k\in\mathbb{N}}$ be a sequence of integrable simple functions such that $\int_M \|f - s_k\|_E \, d\mu \longrightarrow 0$. By Lemma B.6.22, $\left\| \int_M s_k \, d\mu - \int_M s_\ell \, d\mu \right\|_E \leq \int_M \|s_k - s_\ell\|_E \, d\mu \leq \int_M \|s_k - f\|_E \, d\mu + \int_M \|f - s_\ell\|_E \, d\mu$, so the sequence of integrals is Cauchy. The same argument applied simultaneously to two approximating sequences shows that the limit is independent of the choice of sequence. Moreover, for any k we have $\int_M \|f\|_E \, d\mu \leq \int_M \|f - s_k\|_E \, d\mu + \int_M \|s_k\|_E \, d\mu < \infty$, and $\left| \int_M \|s_k\|_E \, d\mu - \int_M \|f\|_E \, d\mu \right| \leq \int_M \|s_k - f\|_E \, d\mu \longrightarrow 0$. Consequently, $\left\| \int_M f \, d\mu \right\|_E = \lim_{k\to\infty} \left\| \int_M s_k \, d\mu \right\|_E \leq \lim_{k\to\infty} \int_M \|s_k\|_E \, d\mu = \int_M \|f\|_E \, d\mu$. Linearity follows by approximating two Bochner integrable functions by simple functions and taking linear combinations.

Conversely, suppose that $\|f\|_E \in \mathcal{L}^1(M,\mu)$. By strong measurability, there exists a sequence of simple functions $(t_k)_{k\in\mathbb{N}}$ such that $t_k(x) \longrightarrow f(x)$ for μ-almost every $x \in M$. Define $B_k := \{x \in M \mid \|t_k(x)\|_E \leq 2\|f(x)\|_E\}$ and $s_k := t_k \mathbf{1}_{B_k}$. Each B_k is measurable, s_k is simple, and $\|s_k\|_E \leq 2\|f\|_E$, so s_k is integrable. If $f(x) \neq 0$, then $\|t_k(x)\|_E \longrightarrow \|f(x)\|_E$, and hence there exists $k_x \in \mathbb{N}$ such that $x \in B_k$ for every $k \geq k_x$; if $f(x) = 0$, then $s_k(x) = 0$ for every $k \in \mathbb{N}$. Thus $s_k(x) \longrightarrow f(x)$ for μ-almost every $x \in M$. Finally, $\|s_k - f\|_E \leq 3\|f\|_E$ and Theorem B.6.14 imply that $\int_M \|s_k - f\|_E \, d\mu \longrightarrow 0$. Consequently, f is Bochner integrable. □

Stability of strong measurability under almost everywhere pointwise limits will be used in the dominated convergence theorem.

Lemma B.6.24. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of strongly measurable functions $f_k : M \longrightarrow E$. If $f_k(x) \longrightarrow f(x)$ for μ-almost every $x \in M$, then f is strongly measurable.

Proof. For each k, the essential range of f_k is contained in a separable subspace of E, since f_k is an almost everywhere pointwise limit of simple functions. Thus, outside a null set, the ranges of all the f_k and of f lie in a single closed separable subspace $E_0 \subseteq E$. Moreover, each f_k is measurable and, since E is metrizable, its almost everywhere pointwise limit f is also measurable after modification on a null set. Let $(e_j)_{j\in\mathbb{N}}$ be a dense subset of E_0. For each $r \in \mathbb{N}$, choose at every point an element of $\{e_1, \ldots, e_r\}$ at minimum distance from $f(x)$, resolving ties by choosing the smallest index. The resulting functions are simple and converge pointwise to f, since $\min_{1\leq j\leq r} \|f(x) - e_j\|_E \longrightarrow 0$. Consequently, f is strongly measurable. □

The vector-valued dominated convergence theorem is now a direct consequence of the preceding criterion.

Theorem B.6.25. (Bochner dominated convergence theorem) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of strongly measurable functions $f_k\colon M \longrightarrow E$ such that $f_k(x) \longrightarrow f(x)$ for μ-almost every $x \in M$. Suppose that there exists $g \in \mathcal{L}^1(M,\mu)$ such that $\|f_k\|_E \le g$ μ-almost everywhere for every $k \in \mathbb{N}$. Then f is strongly measurable, $f_k, f \in \mathcal{L}^1(M,\mu,E)$, and $\displaystyle\int_M \|f_k - f\|_E \, d\mu \longrightarrow 0$. In particular, $\displaystyle\int_M f_k \, d\mu \longrightarrow \int_M f \, d\mu$ in E.

Proof. By Lemma B.6.24, f is strongly measurable. Moreover, $\|f\|_E \le g$ almost everywhere and $\|f_k - f\|_E \le 2g$. Theorem B.6.14 applied to the scalar functions $\|f_k - f\|_E$ implies convergence in $\mathcal{L}^1$, while Proposition B.6.23 ensures Bochner integrability. Finally, $\displaystyle\left\| \int_M f_k \, d\mu - \int_M f \, d\mu \right\|_E \le \int_M \|f_k - f\|_E \, d\mu \longrightarrow 0$. □

Theorem B.6.26. (Fubini's theorem for Bochner integrals) Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be σ-finite measure spaces, let E be a Banach space, and let $F\colon X \times Y \longrightarrow E$ be strongly measurable. If

$$\int_{X\times Y} \|F(x,y)\|_E \, d(\mu\otimes\nu)(x,y) < \infty,$$

then $F(x,\cdot)$ is Bochner integrable for μ-almost every x, and $F(\cdot,y)$ is Bochner integrable for ν-almost every y. The functions obtained by integrating in one variable are strongly measurable and integrable in the other variable, and

$$\begin{aligned}\int_{X\times Y} F \, d(\mu\otimes\nu) &= \int_X \left(\int_Y F(x,y)\, d\nu(y) \right) d\mu(x) \\ &= \int_Y \left(\int_X F(x,y)\, d\mu(x) \right) d\nu(y).\end{aligned}$$

Proof. By Proposition B.6.23, F is Bochner integrable on $X \times Y$. Choose integrable simple functions $S_k\colon X \times Y \longrightarrow E$ such that

$$\int_{X\times Y} \|F - S_k\|_E \, d(\mu\otimes\nu) \longrightarrow 0.$$

Tonelli's theorem B.6.17, applied to $\|F - S_k\|_E$, allows us to extract a subsequence, which we do not relabel, for which

$$\int_Y \|F(x,y) - S_k(x,y)\|_E \, d\nu(y) \longrightarrow 0$$

for μ-almost every x. In particular, these sections of F are strongly measurable and Bochner integrable. For a simple function S_k, Tonelli applied to the indicators of its level sets proves directly that

$$\int_{X\times Y} S_k \, d(\mu\otimes\nu) = \int_X \left(\int_Y S_k(x,y)\, d\nu(y) \right) d\mu(x).$$

Moreover,

$$\int_X \left\| \int_Y (F - S_k)(x,y)\, d\nu(y) \right\|_E d\mu(x) \le \int_{X\times Y} \|F - S_k\|_E \, d(\mu\otimes\nu).$$

The right-hand side tends to zero. Thus the partial integrals form a convergent sequence in $L^1(X,\mu,E)$, its limit agrees almost everywhere with $x \mapsto \displaystyle\int_Y F(x,y)\, d\nu(y)$, and we may pass to the limit in the identity for S_k. Interchanging X and Y proves the second equality. □

Like the Riemann integral, the Bochner integral is compatible with continuous and closed linear operators.

Proposition B.6.27. Let E and F be Banach spaces, and let $f\colon M \longrightarrow E$ be Bochner integrable.

(a) If $T \in \mathcal{L}(E,F)$, then $T \circ f$ is Bochner integrable and $\displaystyle\int_M T(f)\,d\mu = T\left(\int_M f\,d\mu\right)$.

(b) If $T\colon \operatorname{Dom}(T) \subseteq E \longrightarrow F$ is a closed linear operator, $f(x) \in \operatorname{Dom}(T)$ for μ-almost every $x \in M$, and $T \circ f$ is Bochner integrable, then $\displaystyle\int_M f\,d\mu \in \operatorname{Dom}(T)$ and the same equality holds.

Proof. If $T \in \mathcal{L}(E,F)$ and $(s_k)_{k\in\mathbb{N}}$ approximates f in $\mathcal{L}^1$, then $T \circ s_k$ is simple, $\displaystyle\int_M \|T(f) - T(s_k)\|_F\,d\mu \le \|T\| \int_M \|f - s_k\|_E\,d\mu \longrightarrow 0$, and the equality follows first for simple functions and then by passage to the limit.

For the second assertion, modifying f and $T \circ f$ on a null set, we may assume that $f(x) \in \operatorname{Dom}(T)$ for every $x \in M$. Consider $G\colon M \longrightarrow E \times F$, given by $G(x) := (f(x), T(f(x)))$, and equip $E \times F$ with the norm $\|(u,v)\| := \|u\|_E + \|v\|_F$. The function G is strongly measurable and $\displaystyle\int_M \|G\|\,d\mu < \infty$, so it is Bochner integrable. The coordinate projections and the first part show that $\displaystyle\int_M G\,d\mu = \left(\int_M f\,d\mu, \int_M T(f)\,d\mu\right)$. Since the graph $\mathcal{G}(T)$ is a closed subspace of $E \times F$, the quotient map $Q\colon E \times F \longrightarrow (E \times F)/\mathcal{G}(T)$ is linear and continuous. Because $Q \circ G = 0$ almost everywhere, the first part implies $\displaystyle Q\left(\int_M G\,d\mu\right) = \int_M Q(G)\,d\mu = 0$. Thus $\displaystyle\int_M G\,d\mu \in \mathcal{G}(T)$, which is equivalent to the conclusion. □

Remark B.6.28. If $f\colon M \longrightarrow E$ is Bochner integrable and $\varphi \in E'$, Proposition B.6.27 applied to $\varphi\colon E \longrightarrow \mathbb{K}$ shows that $\varphi \circ f$ is integrable and $\displaystyle\varphi\left(\int_A f\,d\mu\right) = \int_A \varphi(f)\,d\mu$ for every $A \in \mathcal{A}$. In particular, every Bochner integrable function is Pettis integrable and hence Dunford integrable; all three integrals agree under the canonical inclusion $E \hookrightarrow E''$.

Theorem B.6.29. (Radon–Nikodym) Let μ and ν be σ-finite measures on $(X,\mathcal{A})$, with ν positive and $\nu \ll \mu$. There exists a measurable function $h\colon X \longrightarrow [0,\infty)$, unique μ-almost everywhere, such that

$$\nu(A) = \int_A h\,d\mu, \qquad A \in \mathcal{A}.$$

If ν is a signed or complex measure, $\nu \ll \mu$, and its total variation $|\nu|$ is σ-finite, there similarly exists a measurable density $h\colon X \to \mathbb{K}$. More precisely, one may choose a measurable exhaustion $X_k \uparrow X$ with $\mu(X_k) + |\nu|(X_k) < \infty$, and then $h\mathbf{1}_{X_k} \in L^1(X,\mu)$ and

$$\nu(A \cap X_k) = \int_{A\cap X_k} h\,d\mu \qquad (A \in \mathcal{A},\ k \in \mathbb{N}).$$

In particular, if $|\nu|(X) < \infty$, we have $h \in L^1(X,\mu)$. This formulation replaces the notation $L^1_{\mathrm{loc}}(X,\mu)$, which is not canonical on an abstract measure space without a topological notion of locality.

Proof. First suppose that $\mu(X)+\nu(X)<\infty$, and set $\lambda := \mu+\nu$. The functional

$$L(f) := \int_X f\, d\nu$$

is continuous on $L^2(X,\lambda)$, since Cauchy–Schwarz gives $|L(f)| \le \nu(X)^{\frac{1}{2}}\|f\|_{L^2(X,\lambda)}$. By Theorem B.3.12, there exists $g \in L^2(X,\lambda)$ such that

$$\int_X f\, d\nu = \int_X f g\, d\lambda$$

for every $f \in L^2(X,\lambda)$. Applying the identity to indicators gives $0 \le g \le 1$ almost everywhere and $\mu(A) = \int_A (1-g)\, d\lambda$. On the set $\{g=1\}$, μ vanishes; the hypothesis $\nu \ll \mu$ implies that ν also vanishes, so this set is λ-null. Thus

$$h := \frac{g}{1-g}$$

is defined almost everywhere, and $d\nu = h\, d\mu$.

In the σ-finite case, choose a measurable partition $X = \bigsqcup_{k\in\mathbb{N}} X_k$ with $\mu(X_k)+\nu(X_k)<\infty$ and apply the preceding case on each X_k; the densities glue by uniqueness. If two densities h_1, h_2 represent ν, integration over $\{h_1 > h_2\}$ and $\{h_2 > h_1\}$ proves that they agree almost everywhere. For a signed or complex measure, choose an exhaustion on which μ and $|\nu|$ are finite and apply the positive case to the Jordan parts of the real and imaginary components. The resulting densities are integrable on each X_k and glue by the same uniqueness argument. □

B.7 The L^p spaces

The L^p spaces will be the starting point for Sobolev spaces and several weak formulations of partial differential equations. We recall their basic properties and some inequalities used later.

Definition B.7.1. Let $(X,\mathcal{A},\mu)$ be a measure space, and fix $p \in (0,\infty)$. Define

$$\mathcal{L}^p(X) := \left\{ f : X \longrightarrow \mathbb{K} \;\middle|\; f \text{ is measurable and } \int_X |f|^p\, d\mu < \infty \right\}.$$

Define $\|\cdot\|_{\mathcal{L}^p(X)} : \mathcal{L}^p(X) \longrightarrow \mathbb{R}$ by

$$\|f\|_{\mathcal{L}^p(X)} = \left(\int_X |f|^p\, d\mu \right)^{\frac{1}{p}}.$$

If $p=\infty$, define

$$\mathcal{L}^\infty(X) := \{ f : X \longrightarrow \mathbb{K} \mid f \text{ is measurable and bounded almost everywhere on } X \},$$

and

$$\|f\|_{\mathcal{L}^\infty(X)} := \inf\{a>0 \mid \mu(\{x\in X \mid |f(x)|>a\}) = 0\}.$$

Definition B.7.2. Fix $p \in [1,\infty]$. Define a relation $\sim$ on $\mathcal{L}^p(X)$ as follows:

$$f \sim g \iff \|f-g\|_{\mathcal{L}^p(X)} = 0.$$

Note that $\sim$ is an equivalence relation; write $L^p(X) := \mathcal{L}^p(X)/\sim$. If $[f] \in L^p(X)$, define $\|[f]\|_{L^p(X)} = \|f\|_{\mathcal{L}^p(X)}$, and in practice identify $[f]$ with f.

Theorem B.7.3. (Lebesgue spaces) Let $p \in [1,\infty]$. Then:

(a) If $p < \infty$, $\|\cdot\|_{L^p(X)}$ is a norm on $L^p(X)$ and $(L^p(X), \|\cdot\|_{L^p(X)})$ is a Banach space. The triangle inequality in $L^p(X)$ is known as Minkowski's inequality.

(b) If $p = \infty$, $\|\cdot\|_{L^\infty(X)}$ is a norm on $L^\infty(X)$ and $(L^\infty(X), \|\cdot\|_{L^\infty(X)})$ is a Banach space.

Proof. First consider $1 \le p < \infty$. Homogeneity and separation of points modulo equality almost everywhere follow directly from the definition. To prove the triangle inequality when $1 < p < \infty$, first recall Hölder's inequality. If $u \in L^p(X)$, $v \in L^{p'}(X)$, and both norms are nonzero, Young's inequality applied to

$$\frac{|u|}{\|u\|_{L^p(X)}} \quad \text{and} \quad \frac{|v|}{\|v\|_{L^{p'}(X)}}$$

and integrated over X gives

$$\int_X |uv|\,d\mu \le \|u\|_{L^p(X)}\|v\|_{L^{p'}(X)}.$$

If either norm vanishes, the same inequality is immediate. Applying it with $v = |f+g|^{p-1}$ gives

$$\begin{aligned}\|f+g\|_{L^p(X)}^p &\le \int_X |f||f+g|^{p-1}\,d\mu + \int_X |g||f+g|^{p-1}\,d\mu \\ &\le \big(\|f\|_{L^p(X)} + \|g\|_{L^p(X)}\big)\big\||f+g|^{p-1}\big\|_{L^{p'}(X)} \\ &= \big(\|f\|_{L^p(X)} + \|g\|_{L^p(X)}\big)\|f+g\|_{L^p(X)}^{p-1}.\end{aligned}$$

If $\|f+g\|_{L^p(X)} = 0$, there is nothing to prove; otherwise, divide by $\|f+g\|_{L^p(X)}^{p-1}$. For $p = 1$, the inequality follows by integrating $|f+g| \le |f| + |g|$. This proves that $\|\cdot\|_{L^p(X)}$ is a norm.

We now prove completeness. Let (f_n) be a Cauchy sequence in $L^p(X)$. We may choose a subsequence (f_{n_k}) such that

$$\|f_{n_{k+1}} - f_{n_k}\|_{L^p(X)} \le 2^{-k}, \qquad k \in \mathbb{N}.$$

Set $g_k := |f_{n_{k+1}} - f_{n_k}|$. By Minkowski's inequality, already proved,

$$\left\|\sum_{k=1}^{N} g_k\right\|_{L^p(X)} \le \sum_{k=1}^{N} \|g_k\|_{L^p(X)} \le 1 \qquad (N \in \mathbb{N}).$$

The monotone convergence theorem applied to $(\sum_{k=1}^{N} g_k)^p$ shows that

$$G := \sum_{k=1}^{\infty} g_k \quad \text{belongs to } L^p(X) \quad \text{and} \quad \|G\|_{L^p(X)} \le 1.$$

Outside a null set, the series

$$\sum_{k=1}^{\infty} \bigl(f_{n_{k+1}} - f_{n_k}\bigr)$$

converges absolutely. Define there

$$f := f_{n_1} + \sum_{k=1}^{\infty} \bigl(f_{n_{k+1}} - f_{n_k}\bigr)$$

and extend f arbitrarily to the exceptional set. The function f is measurable. Moreover, for each j,

$$|f - f_{n_j}| \le \sum_{k=j}^{\infty} g_k$$

almost everywhere. Applying monotone convergence again to the tails and using Minkowski for their finite sums gives

$$\|f - f_{n_j}\|_{L^p(X)} \le \sum_{k=j}^{\infty} 2^{-k} \longrightarrow 0.$$

In particular, $f \in L^p(X)$. Since the original sequence is Cauchy and a subsequence converges to f, the entire sequence converges to f.

Now consider $p = \infty$. The norm properties follow from the pointwise inequalities

$$|\lambda f| = |\lambda||f|, \qquad |f + g| \le |f| + |g|$$

outside null sets. If (f_n) is Cauchy in $L^\infty(X)$, choose a subsequence such that

$$\|f_{n_{k+1}} - f_{n_k}\|_\infty \le 2^{-k}.$$

After modifying representatives on the union of a countable family of null sets, we may assume that

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| \le 2^{-k}$$

for every k and every x outside a single null set N. By the Weierstrass test, f_{n_k} converges uniformly on $X \setminus N$ to a measurable function f, which we extend arbitrarily to N, and

$$\|f - f_{n_j}\|_\infty \le \sum_{k=j}^{\infty} 2^{-k} \longrightarrow 0.$$

Again, the Cauchy property implies that the entire sequence converges to f in $L^\infty(X)$. Thus $L^\infty(X)$ is complete as well. □

Proposition B.7.4. (Reverse Minkowski inequality for exponents less than one) Let $(X, \mathcal{A}, \mu)$ be a measure space, let $0 < r < 1$, and let $f, g \colon X \longrightarrow [0, \infty]$ be measurable functions such that $\int_X f^r \, d\mu < \infty$ and $\int_X g^r \, d\mu < \infty$. Then

$$\left(\int_X (f+g)^r \, d\mu \right)^{\frac{1}{r}} \geq \left(\int_X f^r \, d\mu \right)^{\frac{1}{r}} + \left(\int_X g^r \, d\mu \right)^{\frac{1}{r}}.$$

Proof. Set

$$A := \left(\int_X f^r \, d\mu \right)^{\frac{1}{r}}, \qquad B := \left(\int_X g^r \, d\mu \right)^{\frac{1}{r}}.$$

If $A = 0$ or $B = 0$, one of the functions vanishes almost everywhere and the inequality is immediate. Thus suppose that $A, B > 0$ and define $\theta := \dfrac{A}{A+B}$. Concavity of $t \mapsto t^r$ on $[0, \infty)$ gives

$$\left(\theta \frac{f}{A} + (1-\theta) \frac{g}{B} \right)^r \geq \theta \left(\frac{f}{A} \right)^r + (1-\theta) \left(\frac{g}{B} \right)^r.$$

Upon integration, the right-hand side has integral one. Since

$$f + g = (A+B) \left(\theta \frac{f}{A} + (1-\theta) \frac{g}{B} \right),$$

we obtain

$$\int_X (f+g)^r \, d\mu \geq (A+B)^r.$$

Taking the rth root completes the proof. □

Proposition B.7.5. (Density of simple functions) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $1 \leq p < \infty$. Simple functions whose support has finite measure are dense in $L^p(X)$.

Proof. Choose increasing sets $X_k \in \mathcal{A}$ of finite measure with $\bigcup_{k \in \mathbb{N}} X_k = X$. For $f \in L^p(X)$, the dominated convergence theorem B.6.14 gives

$$f \mathbf{1}_{X_k \cap \{|f| \leq k\}} \longrightarrow f \quad \text{in } L^p(X).$$

Each function on the left-hand side is bounded and has support of finite measure. Approximate its positive and negative parts from below by step functions with step size 2^{-m}; the pointwise error is at most 2^{-m} on X_k, so the error in $L^p(X)$ tends to zero. For complex-valued functions, apply the argument to the real and imaginary parts. □

Lemma B.7.6. (Hölder's inequality for two factors) Let $1 \leq p \leq \infty$, and let p' be its conjugate exponent. If $f \in L^p(X)$ and $g \in L^{p'}(X)$, then $fg \in L^1(X)$ and

$$\int_X |fg| \, d\mu \leq \|f\|_{L^p(X)} \|g\|_{L^{p'}(X)}.$$

Proof. The cases $p = 1$ and $p = \infty$ follow directly from the definition of the essential supremum. If $1 < p < \infty$ and both norms are nonzero, apply Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ pointwise to $a = \dfrac{|f|}{\|f\|_{L^p(X)}}$ and $b = \dfrac{|g|}{\|g\|_{L^{p'}(X)}}$

and integrate. If either norm vanishes, the conclusion is immediate. □

Theorem B.7.7. (Duality, reflexivity, and separability of L^p) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $1 \le p < \infty$. If p' is the conjugate exponent, the map

$$\overline{L^{p'}(X)} \longrightarrow (L^p(X))', \qquad \overline{g} \longmapsto \left[f \mapsto \int_X f\,\overline{g}\,d\mu \right]$$

is a complex-linear isometric isomorphism (in the real case, the bars are omitted). Equivalently, the direct map $g \mapsto [f \mapsto \int_X f\,\overline{g}\,d\mu]$ is antilinear. Consequently, $L^p(X)$ is reflexive if $1 < p < \infty$. If, in addition, the σ-algebra modulo null sets is generated by a countable family—in particular, for an open subset of $\mathbb{R}^n$ with Lebesgue measure—then $L^p(X)$ is separable.

Proof. Lemma B.7.6 shows that each $\overline{g} \in \overline{L^{p'}(X)}$ defines a functional of norm at most $\|g\|_{L^{p'}(X)}$. The map is complex-linear because $\lambda\overline{g} = \overline{\overline{\lambda} g}$ in the conjugate space. Conversely, let $\Lambda \in (L^p(X))'$, and choose an increasing exhaustion $X = \bigcup_{k\in\mathbb{N}} X_k$, with $\mu(X_k) < \infty$. The restriction of Λ to $L^p(X_k)$ defines $\nu_k(A) := \Lambda(\mathbf{1}_A)$ for measurable $A \subset X_k$. If (A_j) is a disjoint family, the partial sums of its indicators converge in $L^p(X_k)$ to the indicator of its union; by continuity, ν_k is countably additive. For a finite partition $(A_j)_{j=1}^N$ of a subset of X_k, choose numbers θ_j of modulus one such that $\theta_j \nu_k(A_j) = |\nu_k(A_j)|$. Then

$$\sum_{j=1}^{N} |\nu_k(A_j)| = \Lambda\left(\sum_{j=1}^{N} \theta_j \mathbf{1}_{A_j} \right) \le \|\Lambda\|\, \mu(X_k)^{1/p}.$$

Thus ν_k has finite total variation; moreover, $\nu_k \ll \mu$. Theorem B.6.29 gives $d\nu_k = \overline{g}_k\, d\mu$. Uniqueness of the densities on $X_k \cap X_\ell$ allows them to be glued into a measurable function g on X.
If $1 < p < \infty$, apply Λ to $f_{N,k} := \operatorname{sgn}(g)|g|^{p'-1}\mathbf{1}_{X_k \cap \{|g|\le N\}}$ and obtain

$$\int_{X_k\cap\{|g|\le N\}} |g|^{p'}\,d\mu \le \|\Lambda\| \left(\int_{X_k\cap\{|g|\le N\}} |g|^{p'}\,d\mu \right)^{\frac{1}{p}}.$$

By monotone convergence, $g \in L^{p'}(X)$ and $\|g\|_{L^{p'}(X)} \le \|\Lambda\|$. For $p = 1$, applying Λ to indicators multiplied by the phase of g shows that $\|g\|_{L^\infty(X)} \le \|\Lambda\|$. Equality of norms and the representation for every $f \in L^p(X)$ follow by density of simple functions.
If $1 < p < \infty$, applying the preceding representation twice, while respecting the conjugate spaces, identifies the canonical inclusion $L^p(X) \to (L^p(X))''$ with an isomorphism; this proves reflexivity. Finally, simple functions with values in $\mathbb{Q} + i\mathbb{Q}$ (in the real case, in $\mathbb{Q}$) and sets belonging to the countable generating algebra form a countable dense subset, proving separability in the stated case. □

Corollary B.7.8. (L^p duality for Hermitian bundles) Let $(M, \mathbf{g})$ be a Riemannian manifold with or without boundary satisfying the second axiom of countability, and let $\mathbf{E} \to M$ be a smooth complex Hermitian bundle of finite rank. For $1 \le p < \infty$, the map

$$L^{p'}(M, \overline{\mathbf{E}}) \longrightarrow (L^p(M, \mathbf{E}))', \qquad \overline{\mathbf{v}} \longmapsto \left[u \longmapsto \int_M \mathcal{R}_{\mathbf{E}}(\overline{\mathbf{v}})(u)\, d\lambda_{\mathbf{g}} \right]$$

is a complex-linear isometric isomorphism.

Proof. A countable coordinate cover and a disjoint measurable refinement $M = \bigsqcup_{j\ge 1} A_j$ allow us to choose a measurable unitary frame of $\mathbf{E}$ on each A_j. Thus

$$L^p(M,\mathbf{E}) \cong \left(\bigoplus_{j\ge 1} L^p(A_j;\mathbb{C}^r) \right)_{\ell^p}$$

isometrically. Applying Theorem B.7.7 to each of the r components and using the discrete version of Lemma B.7.6, every element of $L^{p'}(M,\overline{\mathbf{E}})$ defines a functional of the same norm.
Conversely, the restriction of a functional to sections supported in A_j is represented by a unique $\mathbf{v}_j \in L^{p'}(A_j,\overline{\mathbf{E}})$. For every finite subset F of indices, finite-dimensional duality for $\ell^p(F)$ gives

$$\left(\sum_{j\in F} \|\mathbf{v}_j\|^{p'}_{L^{p'}(A_j,\overline{\mathbf{E}}\restriction_{A_j})} \right)^{\frac{1}{p'}} \le \|\Lambda\|$$

if $1 < p < \infty$, and $\sup_{j\in F} \|\mathbf{v}_j\|_{L^\infty(A_j,\overline{\mathbf{E}}\restriction_{A_j})} \le \|\Lambda\|$ if $p = 1$. As F increases, the sections $\mathbf{v}_j$ glue into an element $\mathbf{v} \in L^{p'}(M,\overline{\mathbf{E}})$ of norm at most $\|\Lambda\|$. Density of measurable sections supported in finitely many of the A_j proves the representation; the reverse norm inequality follows by evaluating on truncated Hölder extremizers. This proves the isometry and surjectivity. □

Proposition B.7.9. (Hölder's inequality) Let $(X,\mathcal{A},\mu)$ be a measure space, and consider $r, p_1, \dots, p_m \in [1,\infty]$ such that $\dfrac{1}{p_1} + \cdots + \dfrac{1}{p_m} = \dfrac{1}{r}$, with the convention $\dfrac{1}{\infty} = 0$. Then for any $f_j \in L^{p_j}(X)$, $1 \le j \le m$, we have $\prod_{j=1}^m f_j \in L^r(X)$ and

$$\left\| \prod_{j=1}^m f_j \right\|_{L^r(X)} \le \prod_{j=1}^m \|f_j\|_{L^{p_j}(X)}.$$

Proof. Factors with $p_j = \infty$ are taken out using their norm in $L^\infty(X)$, so it suffices to consider indices with $p_j < \infty$. If any of them satisfies $\|f_j\|_{L^{p_j}(X)} = 0$, the assertion is immediate. Otherwise, set $q_j := \dfrac{p_j}{r}$; then $q_j \ge 1$ and $\sum_{j=1}^m \dfrac{1}{q_j} = 1$. If some $q_j = 1$, it is the only finite index and the assertion is immediate. In the remaining case, the generalized numerical Young inequality,

$$\prod_j a_j \le \sum_{j=1}^m \frac{a_j^{q_j}}{q_j}, \qquad a_j \ge 0,$$

follows by iterating $ab \le \dfrac{a^q}{q} + \dfrac{b^{q'}}{q'}$. To prove the latter, with $b \ge 0$ fixed, minimize $t \mapsto \dfrac{t^q}{q} - tb + \dfrac{b^{q'}}{q'}$; its

minimum, attained when $t^{q-1} = b$, is zero. Applying the generalized inequality to

$$a_j(x) := \frac{|f_j(x)|^r}{\|f_j\|_{L^{p_j}(X)}^r}$$

and integrating gives

$$\int_X \prod_j a_j \, d\mu \le \sum_{j=1}^m \frac{1}{q_j} \int_X a_j^{q_j} \, d\mu = \sum_{j=1}^m \frac{1}{q_j} = 1.$$

Multiplying by the norms taken out and taking the rth root proves the result. The case in which all exponents are infinite is the pointwise inequality for the essential supremum. □

Proposition B.7.10. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $p \in (0, \infty)$. If $r \in (0, p)$ and $f \in \mathcal{L}^p(X)$, then $f \in \mathcal{L}^r(X)$ and

$$\|f\|_{\mathcal{L}^r(X)} \le \|f\|_{\mathcal{L}^p(X)} \mu(X)^{\frac{1}{r} - \frac{1}{p}}.$$

Proof. Apply Proposition B.7.9 to $|f|^r$ and the constant function 1, with exponents $\frac{p}{r}$ and $\frac{p}{p-r}$. Thus

$$\int_X |f|^r \, d\mu \le \left(\int_X |f|^p \, d\mu \right)^{\frac{r}{p}} \mu(X)^{1 - \frac{r}{p}}.$$

Taking the rth root gives the stated inequality. □

Proposition B.7.11. (Jensen's inequality) Let $(X, \mathcal{A}, \mu)$ be a finite measure space with $\mu(X) = 1$, let $\varphi \colon \mathbb{R} \longrightarrow \mathbb{R}$ be convex, and let $f \in \mathcal{L}^1(X)$. Then, whenever the right-hand side is well defined,

$$\varphi\left(\int_X f \, d\mu \right) \le \int_X \varphi(f) d\mu$$

Proof. Set $a := \int_X f \, d\mu$. Convexity of φ implies that there exists a supporting slope $c \in \mathbb{R}$ at a: indeed, any number between the one-sided slopes

$$\sup_{x \in (-\infty, a)} \frac{\varphi(a) - \varphi(x)}{a - x} \quad \text{and} \quad \inf_{y > a} \frac{\varphi(y) - \varphi(a)}{y - a}$$

satisfies $\varphi(t) \ge \varphi(a) + c(t - a)$ for every $t \in \mathbb{R}$. Applying this inequality to $t = f(x)$ and integrating gives

$$\int_X \varphi(f) \, d\mu \ge \varphi(a) + c \left(\int_X f \, d\mu - a\mu(X) \right) = \varphi(a).$$

□

Proposition B.7.12. Let $\omega \subseteq \Omega \subseteq \mathbb{R}^n$ be open sets, and let $p \in [1, \infty]$. Then:

(a) If $f \in L^p(\Omega)$, then $f \mathbf{1}_\omega \in L^p(\omega)$ and

$$\|f \mathbf{1}_\omega\|_{L^p(\omega)} \le \|f\|_{L^p(\Omega)}$$

(b) If $f \in L^p(\Omega)$ and X is a measurable subset of Ω with $\lambda_n(X) < \infty$, then:

(a) if $p < \infty$, we have $f\mathbf{1}_X \in L^r(\Omega)$ for every $r \in [1,p]$ and

$$\|f\mathbf{1}_X\|_{L^r(\Omega)} \le \lambda_n(X)^{\frac{p-r}{rp}} \|f\|_{L^p(\Omega)};$$

(b) if $p = \infty$, we have $f\mathbf{1}_X \in L^r(\Omega)$ for every $r \in [1,\infty)$ and

$$\|f\mathbf{1}_X\|_{L^r(\Omega)} \le \lambda_n(X)^{\frac{1}{r}} \|f\|_{L^\infty(\Omega)}.$$

(c) If $f_k \to f$ in $L^p(\Omega)$ and $\lambda_n(\omega) < \infty$, then:

(a) if $p < \infty$, $f_k\mathbf{1}_\omega \to f\mathbf{1}_\omega$ in $L^r(\omega)$ for every $r \in [1,p]$;

(b) if $p = \infty$, $f_k\mathbf{1}_\omega \to f\mathbf{1}_\omega$ in $L^r(\omega)$ for every $r \in [1,\infty)$.

Proof. Part (a) follows directly by integrating over $\omega \subseteq \Omega$. For (b), if $1 \le r < p < \infty$, apply Proposition B.7.9 on X to $|f|^r$ and 1, with exponents $\frac{p}{r}$ and $\frac{p}{p-r}$; if $r = p$, the assertion is immediate. For $p = \infty$, use $|f| \le \|f\|_{L^\infty(\Omega)}$ almost everywhere on X. This gives exactly the two bounds in (b). Finally, apply these bounds to $(f_k - f)\mathbf{1}_\omega$ to obtain (c). □

Proposition B.7.13. If $f = (f_1,\dots,f_m)$ and $f_1,\dots,f_m \in L^s(\Omega)$, define

$$\|f\|_{(L^s(\Omega))^m} := \begin{cases} \left(\sum_{j=1}^m \|f_j\|_{L^s(\Omega)}^s\right)^{\frac{1}{s}}, & 1 \le s < \infty, \\ \max\limits_{1\le j\le m} \|f_j\|_{L^\infty(\Omega)}, & s = \infty. \end{cases}$$

If $\lambda_n(\Omega) < \infty$, $1 \le p < s \le \infty$, and $f_1,\dots,f_m \in L^s(\Omega)$, then

$$\begin{cases} \|f\|_{(L^p(\Omega))^m} \le (m\lambda_n(\Omega))^{\frac{s-p}{sp}} \|f\|_{(L^s(\Omega))^m} & \text{if } s \in [1,\infty), \\ \|f\|_{(L^p(\Omega))^m} \le (m\lambda_n(\Omega))^{\frac{1}{p}} \|f\|_{(L^\infty(\Omega))^m} & \text{if } s = \infty. \end{cases}$$

Proof. Consider $\Omega \times \{1,\dots,m\}$ with the product of Lebesgue measure and counting measure, and define $F(x,j) := f_j(x)$. This space has measure $m\lambda_n(\Omega)$, and

$$\|F\|_{L^t(\Omega\times\{1,\dots,m\})} = \|f\|_{(L^t(\Omega))^m}$$

for $1 \le t \le \infty$. If $s < \infty$, Proposition B.7.9, applied to $|F|^p$ and 1 with exponents $\frac{s}{p}$ and $\frac{s}{s-p}$, gives the first inequality. The second follows by integrating the bound $|F| \le \|F\|_{L^\infty(\Omega\times\{1,\dots,m\})}$. □

Proposition B.7.14. If $0 < \lambda_n(\Omega) < \infty$ and $f \in L^\infty(\Omega)$, then

$$\|f\|_{L^\infty(\Omega)} = \lim_{p\to\infty} \lambda_n(\Omega)^{-\frac{1}{p}} \|f\|_{L^p(\Omega)}.$$

Proof. The inequality

$$\lambda_n(\Omega)^{-\frac{1}{p}}\|f\|_{L^p(\Omega)} \le \|f\|_{L^\infty(\Omega)}$$

gives the upper bound. If $0 < a < \|f\|_{L^\infty(\Omega)}$, the set $A_a := \{x \in \Omega \mid |f(x)| > a\}$ has positive measure and

$$\lambda_n(\Omega)^{-\frac{1}{p}}\|f\|_{L^p(\Omega)} \ge a\left(\frac{\lambda_n(A_a)}{\lambda_n(\Omega)}\right)^{\frac{1}{p}}.$$

The right-hand side converges to a as $p \to \infty$. Then letting $a \uparrow \|f\|_{L^\infty(\Omega)}$ gives the lower bound and hence the limit. □

Proposition B.7.15. Let (f_k) be a sequence in $L^p(\Omega)$ such that $f_k \to f$ in $L^p(\Omega)$.

(a) If $p \in [1,\infty)$, there exist a subsequence (f_{k_j}) of (f_k) and a function $g \in L^p(\Omega)$ such that $f_{k_j}(x) \to f(x)$ for almost every $x \in \Omega$, and $|f_{k_j}| \le g(x)$ for almost every $x \in \Omega$, for every $j \in \mathbb{N}$.

(b) If $p = \infty$, then $f_k(x) \to f(x)$ for almost every $x \in \Omega$, and there exists $c \in \mathbb{R}$ such that $|f_k(x)| \le c$ for almost every $x \in \Omega$ and every $k \in \mathbb{N}$.

Proof. First suppose $p < \infty$. Choosing a subsequence inductively, we may ensure that

$$\|f_{k_{j+1}} - f_{k_j}\|_{L^p(\Omega)} \le 2^{-j} \qquad (j \in \mathbb{N}).$$

Minkowski's inequality and Theorem B.6.11 show that

$$g := |f_{k_1}| + \sum_{j\in\mathbb{N}} |f_{k_{j+1}} - f_{k_j}|$$

belongs to $L^p(\Omega)$. The series of differences converges absolutely almost everywhere; hence (f_{k_j}) converges almost everywhere to a function h, and $|f_{k_j}| \le g$. Moreover, $|h| \le g$ almost everywhere. The dominated convergence theorem B.6.14, applied to $|f_{k_j} - h|^p \le (2g)^p$, shows that $f_{k_j} \to h$ in $L^p(\Omega)$. Since also $f_{k_j} \to f$ in $L^p(\Omega)$, uniqueness of the limit gives $h = f$ almost everywhere.
If $p = \infty$, choose representatives and a common null set outside which

$$|f_k(x) - f(x)| \le \|f_k - f\|_{L^\infty(\Omega)} \longrightarrow 0.$$

The sequence of norms $\|f_k - f\|_{L^\infty(\Omega)}$ is bounded; this gives a constant c dominating all the $|f_k|$ outside a null set. □

Theorem B.7.16. (Interpolation inequality) Let $1 \le p < s < r \le \infty$. If $f \in L^p(\Omega) \cap L^r(\Omega)$, then $f \in L^s(\Omega)$ and

$$\|f\|_{L^s(\Omega)} \le \|f\|^{1-\alpha}_{L^p(\Omega)}\|f\|^{\alpha}_{L^r(\Omega)},$$

where $\alpha \in (0,1)$ satisfies $\dfrac{1}{s} = \dfrac{1-\alpha}{p} + \dfrac{\alpha}{r}$ if $r < \infty$ and $\alpha := 1 - \dfrac{p}{s}$ if $r = \infty$.

Proof. If $r < \infty$, apply Proposition B.7.9 to

$$|f|^s = |f|^{(1-\alpha)s}|f|^{\alpha s}$$

with exponents $\dfrac{p}{(1-\alpha)s}$ and $\dfrac{r}{\alpha s}$, which are conjugate by the relation between p, s, r and α. Taking the sth root yields the inequality. If $r = \infty$, then $s = (1-\alpha)^{-1}p$ and

$$\int_\Omega |f|^s \, d\lambda_n \le \|f\|_{L^\infty(\Omega)}^{s-p} \int_\Omega |f|^p \, d\lambda_n,$$

which gives the same conclusion. □

Proposition B.7.17. If $\lambda_n(\Omega) < \infty$ and $1 \le p < s \le \infty$, then $L^s(\Omega) \subseteq L^p(\Omega)$ and this inclusion is continuous. Moreover, for every $f \in L^s(\Omega)$,

$$\begin{cases} \|f\|_{L^p(\Omega)} \le (\lambda_n(\Omega))^{\frac{s-p}{sp}} \|f\|_{L^s(\Omega)} & \text{if } s \in [1,\infty), \\ \|f\|_{L^p(\Omega)} \le (\lambda_n(\Omega))^{\frac{1}{p}} \|f\|_{L^\infty(\Omega)} & \text{if } s = \infty. \end{cases}$$

Proof. If $s < \infty$, apply Proposition B.7.9 to $|f|^p$ and 1 with exponents $\dfrac{s}{p}$ and $\dfrac{s}{s-p}$. If $s = \infty$, integrate the inequality $|f|^p \le \|f\|_{L^\infty(\Omega)}^p$. The resulting bounds also prove continuity of the inclusion. □

Proposition B.7.18. $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for every $p \in [1,\infty)$.

Proof. By Proposition B.7.5, it suffices to approximate in $L^p(\Omega)$ the indicator of a measurable set $A \subseteq \Omega$ of finite measure. Regularity of Lebesgue measure in Theorem B.5.15 allows us to choose a compact set $K \subseteq A$ and an open set U with $K \subseteq U \Subset \Omega$ such that

$$\lambda_n(A \setminus K) + \lambda_n(U \setminus K) < \varepsilon^p.$$

By Proposition 1.3.6, there exists $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$, $\eta = 1$ on K, and $\operatorname{supp}(\eta) \subseteq U$. Then

$$\|\mathbf{1}_A - \eta\|_{L^p(\Omega)}^p \le \lambda_n(A \setminus K) + \lambda_n(U \setminus K) < \varepsilon^p.$$

Approximating each term of a simple function and using the triangle inequality proves density. □

Proposition B.7.19. Let $p \in [1,\infty)$, let $\Omega \subseteq \mathbb{R}^n$ be open, and let $\mathcal{K} \subseteq L^p(\Omega)$ be bounded with the following properties:

(a) For each $\varepsilon > 0$ and each open set $\omega \Subset \Omega$, there exists $\delta \in (0, d(\omega, \mathbb{R}^n \setminus \Omega))$ such that $\|T_\xi f - f\|_{L^p(\omega)} < \varepsilon$ for every $\xi \in \mathbb{R}^n$ with $\|\xi\| < \delta$ and every $f \in \mathcal{K}$.

(b) For each $\varepsilon > 0$, there exists an open set $\omega \Subset \Omega$ such that $\|f\|_{L^p(\Omega \setminus \omega)} < \varepsilon$ for every $f \in \mathcal{K}$.

Here $T_\xi f(x) := f(x + \xi)$ whenever $x, x + \xi \in \Omega$. Then $\mathcal{K}$ is relatively compact in $L^p(\Omega)$.

Proof. Fix $\varepsilon > 0$. By (b), choose $\omega \Subset \Omega$ so that $\|f\|_{L^p(\Omega\setminus\omega)} < \varepsilon$ for every $f \in \mathcal{K}$, and choose ω_1 with $\omega \Subset \omega_1 \Subset \Omega$. For small $h > 0$, take a finite family of disjoint cubes $(Q_\ell)_{\ell=1}^N$, of side length h, contained in ω_1, whose union U_h contains ω. Define

$$P_h f := \sum_{\ell=1}^{N} \left(\frac{1}{|Q_\ell|} \int_{Q_\ell} f(y)\, dy \right) \mathbf{1}_{Q_\ell}.$$

Jensen's inequality and Theorem B.6.17 give

$$\|P_h f - f\|_{L^p(U_h)}^p \le \frac{1}{h^n} \sum_{\ell=1}^{N} \int_{Q_\ell} \int_{Q_\ell} |f(y) - f(x)|^p \, dy\, dx.$$

Writing $y = x + z$ and observing that $\|z\| \le \sqrt{n}h$, the right-hand side is bounded by

$$Ch^{-n} \int_{\{\|z\| \le \sqrt{n}h\}} \|T_z f - f\|_{L^p(\omega_1)}^p \, dz.$$

Condition (a) makes this quantity less than $C\varepsilon^p$ uniformly in $f \in \mathcal{K}$ if h is sufficiently small. Since $P_h f = 0$ outside U_h and $\Omega \setminus U_h \subseteq \Omega \setminus \omega$, we obtain $\|P_h f - f\|_{L^p(\Omega)} \le C\varepsilon$.

The set $P_h(\mathcal{K})$ is bounded in the finite-dimensional space spanned by $(\mathbf{1}_{Q_\ell})_{\ell=1}^N$ and is therefore totally bounded. The preceding uniform approximation shows that $\mathcal{K}$ is totally bounded in the complete space $L^p(\Omega)$; its closure is compact. □

B.8 Integral inequalities and tools from harmonic analysis

In this section we collect several tools that will appear repeatedly in the Littlewood–Paley decomposition and the study of Besov and Triebel–Lizorkin spaces. The first part contains integral and discrete inequalities allowing norms, integrals, and sums over scales to be interchanged. We then introduce Rademacher functions and the Hardy–Littlewood and Fefferman–Stein maximal estimates. Finally, we record the Calderón–Zygmund decomposition and Peetre maximal control, which allow families of frequency-localized functions to be treated uniformly.

Throughout this section, $\mathcal{M}$ denotes the uncentered Hardy–Littlewood maximal operator on $\mathbb{R}^n$.

Proposition B.8.1. (Cavalieri and Chebyshev) If $f : X \longrightarrow [0,\infty]$ is measurable and $0 < p < \infty$, then

$$\int_X f^p \, d\mu = p \int_0^\infty \lambda^{p-1} \mu(\{f > \lambda\})\, d\lambda.$$

In particular, if $f \in L^p(X)$ and $\lambda > 0$, then

$$\mu(\{|f| > \lambda\}) \le \lambda^{-p} \|f\|_{L^p(X)}^p.$$

Proof. For each $x \in X$,

$$f(x)^p = \int_0^{f(x)} p\lambda^{p-1}\, d\lambda = \int_0^\infty p\lambda^{p-1} \mathbf{1}_{\{f(x)>\lambda\}}\, d\lambda.$$

Theorem B.6.17 allows us to integrate first with respect to x and then with respect to λ, giving Cavalieri's

identity. For Chebyshev's inequality, observe that

$$\|f\|_{L^p(X)}^p \geq \int_{\{|f|>\lambda\}} |f|^p \, d\mu \geq \lambda^p \mu(\{|f| > \lambda\}).$$

□

Theorem B.8.2. (Minkowski's integral inequality) Let (X, μ) and (Y, ν) be σ-finite measure spaces, and let $1 \leq p \leq \infty$. If $F\colon X \times Y \longrightarrow \mathbb{C}$ is measurable and

$$\int_Y \|F(\cdot, y)\|_{L^p(X)} \, d\nu(y) < \infty,$$

then $x \mapsto \displaystyle\int_Y F(x,y)\, d\nu(y)$ belongs to $L^p(X)$ and

$$\left\| \int_Y F(\cdot, y)\, d\nu(y) \right\|_{L^p(X)} \leq \int_Y \|F(\cdot, y)\|_{L^p(X)} \, d\nu(y).$$

Proof. For $p = 1$, the triangle inequality and Theorem B.6.17 give

$$\int_X \left| \int_Y F(x,y)\, d\nu(y) \right| d\mu(x) \leq \int_Y \int_X |F(x,y)| \, d\mu(x) d\nu(y).$$

The case $p = \infty$ follows by taking essential suprema. Suppose $1 < p < \infty$. For $g \in L^{p'}(X)$, Fubini's theorem for Bochner integrals B.6.26, applied in the scalar case, and Hölder's inequality in Proposition B.7.9 imply

$$\begin{aligned} \left| \int_X g(x) \int_Y F(x,y)\, d\nu(y) d\mu(x) \right| &\leq \int_Y \int_X |g(x) F(x,y)| \, d\mu(x) d\nu(y) \\ &\leq \|g\|_{L^{p'}(X)} \int_Y \|F(\cdot, y)\|_{L^p(X)} \, d\nu(y). \end{aligned}$$

The dual characterization of the $L^p(X)$ norm in Theorem B.7.7 completes the proof. □

Proposition B.8.3. (Schur's test) Let (X, μ) and (Y, ν) be σ-finite measure spaces, let $r, s \in \mathbb{N}$, and let

$$K\colon X \times Y \longrightarrow \mathcal{L}(\mathbb{C}^r, \mathbb{C}^s)$$

be a measurable kernel. Suppose that there exist $A, B \geq 0$ such that

$$\sup_{x \in X} \int_Y \|K(x,y)\|_{\mathcal{L}(\mathbb{C}^r, \mathbb{C}^s)} \, d\nu(y) \leq A$$

and

$$\sup_{y \in Y} \int_X \|K(x,y)\|_{\mathcal{L}(\mathbb{C}^r, \mathbb{C}^s)} \, d\mu(x) \leq B.$$

Then, for each $f \in L^2(Y, \mathbb{C}^r)$, the integral

$$(T_K f)(x) := \int_Y K(x,y) f(y)\, d\nu(y)$$

exists for almost every $x \in X$ and defines a continuous linear operator

$$T_K \colon L^2(Y, \mathbb{C}^r) \longrightarrow L^2(X, \mathbb{C}^s)$$

satisfying

$$\|T_K\|_{\mathcal{L}(L^2(Y,\mathbb{C}^r),L^2(X,\mathbb{C}^s))} \le \sqrt{AB}.$$

Proof. Set $k(x,y) := \|K(x,y)\|_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^s)}$. For $f \in L^2(Y,\mathbb{C}^r)$, Cauchy–Schwarz with respect to the measure $k(x,y)\,d\nu(y)$ gives

$$\Big(\int_Y k(x,y)\|f(y)\|_{\mathbb{C}^r}\,d\nu(y)\Big)^2 \le A\int_Y k(x,y)\|f(y)\|_{\mathbb{C}^r}^2\,d\nu(y). \tag{B.8}$$

Tonelli's theorem B.6.17 and the second hypothesis show that

$$\begin{aligned}\int_X\int_Y k(x,y)\|f(y)\|_{\mathbb{C}^r}^2\,d\nu(y)\,d\mu(x) &= \int_Y \|f(y)\|_{\mathbb{C}^r}^2\Big(\int_X k(x,y)\,d\mu(x)\Big)\,d\nu(y)\\ &\le B\|f\|_{L^2(Y,\mathbb{C}^r)}^2.\end{aligned}$$

Thus the last factor in (B.8) is finite for almost every x; the integral defining $T_Kf(x)$ is absolutely convergent in $\mathbb{C}^s$ at these points. Measurability of T_Kf follows by integrating measurable functions componentwise. Moreover,

$$\begin{aligned}\|T_Kf\|_{L^2(X,\mathbb{C}^s)}^2 &\le A\int_X\int_Y k(x,y)\|f(y)\|_{\mathbb{C}^r}^2\,d\nu(y)\,d\mu(x)\\ &\le AB\|f\|_{L^2(Y,\mathbb{C}^r)}^2.\end{aligned}$$

This proves the estimate and, in particular, continuity. The same proof applies to a measurable kernel $K(x,y)\colon E_y \longrightarrow F_x$ between measurable Hermitian bundles of finite rank, replacing Euclidean norms by bundle norms. □

Proposition B.8.4. (Young's inequality for sequences) Let $1 \le q \le \infty$, $a = (a_j)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$, and $b = (b_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. If

$$(a*b)_k := \sum_{j\in\mathbb{Z}} a_j b_{k-j}, \qquad k \in \mathbb{Z},$$

then

$$\|a*b\|_{\ell^q(\mathbb{Z})} \le \|a\|_{\ell^q(\mathbb{Z})}\|b\|_{\ell^1(\mathbb{Z})}.$$

Proof. For $q = \infty$,

$$|(a*b)_k| \le \|a\|_{\ell^\infty(\mathbb{Z})}\sum_{j\in\mathbb{Z}}|b_{k-j}| = \|a\|_{\ell^\infty(\mathbb{Z})}\|b\|_{\ell^1(\mathbb{Z})}.$$

If $1 \le q < \infty$, write

$$|a_jb_{k-j}|^q = |a_j|^q|b_{k-j}|\,|b_{k-j}|^{q-1}$$

and apply Hölder to the sum over j with exponents q and q'. This gives

$$|(a*b)_k|^q \le \|b\|_{\ell^1(\mathbb{Z})}^{q-1}\sum_{j\in\mathbb{Z}}|a_j|^q|b_{k-j}|.$$

Summing over $k \in \mathbb{Z}$ and using Theorem B.6.17 yields

$$\|a * b\|_{\ell^q(\mathbb{Z})}^q \le \|b\|_{\ell^1(\mathbb{Z})}^q \|a\|_{\ell^q(\mathbb{Z})}^q.$$

□

Theorem B.8.5. (Discrete Hardy inequalities) Let $\alpha > 0$, $1 \le q \le \infty$, and let $(a_j)_{j\in\mathbb{Z}}$ be a nonnegative sequence. Then

$$\left\| \left(\sum_{\substack{j\in\mathbb{Z}\\ j\le k}} 2^{-\alpha(k-j)} a_j \right)_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} + \left\| \left(\sum_{\substack{j\in\mathbb{Z}\\ j\ge k}} 2^{-\alpha(j-k)} a_j \right)_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \le C_{\alpha,q} \|(a_j)_{j\in\mathbb{Z}}\|_{\ell^q(\mathbb{Z})}.$$

Proof. The first sequence is the convolution of $(a_j)_{j\in\mathbb{Z}}$ with

$$b_m := 2^{-\alpha m} \mathbf{1}_{\{m\ge 0\}}, \qquad m \in \mathbb{Z},$$

and the second with $\widetilde{b}_m := 2^{\alpha m}\mathbf{1}_{\{m\le 0\}}$. Both sequences belong to $\ell^1(\mathbb{Z})$. Proposition B.8.4 gives the two bounds. □

Theorem B.8.6. (Hardy's integral inequalities) Let $\alpha > 0$, $1 \le q \le \infty$, and let $f : (0,\infty) \longrightarrow [0,\infty]$ be measurable. Then

$$\left\| r^{-\alpha} \int_0^r t^{\alpha} f(t) \frac{dt}{t} \right\|_{L^q((0,\infty),\frac{dr}{r})} \le \frac{1}{\alpha} \|f\|_{L^q((0,\infty),\frac{dt}{t})}$$

and

$$\left\| r^{\alpha} \int_r^\infty t^{-\alpha} f(t) \frac{dt}{t} \right\|_{L^q((0,\infty),\frac{dr}{r})} \le \frac{1}{\alpha} \|f\|_{L^q((0,\infty),\frac{dt}{t})}.$$

The same assertions hold for Banach-valued functions upon replacing $f(t)$ by its norm.

Proof. Make the substitutions $r = e^x$, $t = e^y$, and set $g(y) := f(e^y)$. Since $dt/t = dy$ and $dr/r = dx$, the first expression becomes

$$\int_{-\infty}^{x} e^{-\alpha(x-y)} g(y)\, dy = \left(e^{-\alpha\cdot}\mathbf{1}_{[0,\infty)} * g\right)(x),$$

while the second is the convolution of g with $e^{\alpha\cdot}\mathbf{1}_{(-\infty,0]}$. Both kernels have $L^1(\mathbb{R})$ norm equal to $\frac{1}{\alpha}$. Young's inequality in Theorem 4.10.1 completes the proof, including the endpoints $q = 1$ and $q = \infty$. □

Rademacher functions allow a Euclidean norm of coefficients to be converted into a scalar integral. This randomization is the mechanism we shall later use to pass from multipliers with arbitrary signs to the Littlewood–Paley square function.

Lemma B.8.7. (Paley–Zygmund inequality) Let $(A, \mathcal{A}, \mu)$ be a probability space, let $X \in L^2(A)$ be nonnegative, and let $0 < \vartheta < 1$. Then

$$\mu\{X > \vartheta\, \mathbb{E}X\} \ge (1-\vartheta)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)}.$$

Here $\mathbb{E}X := \int_A X\,d\mu$; if $\mathbb{E}(X^2) = 0$, both sides are interpreted as zero.

Proof. The case $\mathbb{E}X = 0$ is immediate. Suppose $\mathbb{E}X > 0$, and let $E := \{X > \vartheta \mathbb{E}X\}$. Then

$$\mathbb{E}X = \int_{A\setminus E} X\,d\mu + \int_E X\,d\mu \le \vartheta\mathbb{E}X + (\mathbb{E}(X^2))^{\frac{1}{2}}\mu(E)^{\frac{1}{2}},$$

where we used Cauchy–Schwarz, Theorem B.3.2, in the last term. Subtracting $\vartheta\mathbb{E}X$, squaring, and dividing by $\mathbb{E}(X^2)$ gives the assertion. □

Theorem B.8.8. (Khintchine's inequality) Let $(r_j)_{j\in\mathbb{N}_0}$ be the Rademacher functions on $[0,1]$. For each $0 < p < \infty$ there exist constants $A_p, B_p > 0$ such that, for every finite set $J \subseteq \mathbb{N}_0$ and every family of scalars $(a_j)_{j\in J}$,

$$A_p \left(\sum_{j\in J} |a_j|^2\right)^{\frac{1}{2}} \le \left(\int_0^1 \left|\sum_{j\in J} a_j r_j(t)\right|^p dt\right)^{\frac{1}{p}} \le B_p \left(\sum_{j\in J} |a_j|^2\right)^{\frac{1}{2}}.$$

Proof. First suppose that the coefficients are real, and set

$$S := \sum_{j\in J} a_j r_j, \qquad \sigma^2 := \sum_{j\in J} |a_j|^2.$$

Independence of the Rademacher functions and the inequality $\cosh x \le e^{\frac{x^2}{2}}$ imply

$$\int_0^1 e^{\lambda S(t)} dt = \prod_{j\in J} \cosh(\lambda a_j) \le e^{\frac{\lambda^2\sigma^2}{2}}.$$

Apply Chebyshev, Proposition B.8.1, to $e^{\lambda S}$ and $e^{-\lambda S}$. For $u > 0$ and $\lambda > 0$, we obtain

$$|\{t \mid S(t) > u\}| \le e^{-\lambda u + \frac{\lambda^2\sigma^2}{2}},$$

and the same bound for $-S$. If $\sigma > 0$, choosing $\lambda = \frac{u}{\sigma^2}$ gives

$$|\{t \in [0,1] \mid |S(t)| > u\}| \le 2e^{-\frac{u^2}{2\sigma^2}}.$$

Cavalieri's formula then yields

$$\int_0^1 |S|^p dt = p\int_0^\infty u^{p-1} |\{|S| > u\}|\,du \le C_p \sigma^p,$$

which is the upper bound.
For the lower bound, by homogeneity we may assume $\sigma = 1$. Independence gives

$$\mathbb{E}(S^2) = 1, \qquad \mathbb{E}(S^4) = \sum_{j\in J} a_j^4 + 6\sum_{\substack{j,k\in J\\ j<k}} a_j^2 a_k^2 = 3 - 2\sum_{j\in J} a_j^4 \le 3.$$

Apply Lemma B.8.7 to $X = S^2$ with $\vartheta = \frac{1}{2}$. Thus

$$|\{|S| > 2^{-\frac{1}{2}}\}| \geq \frac{1}{12},$$

and consequently,

$$\int_0^1 |S|^p dt \geq \frac{1}{12} 2^{-\frac{p}{2}}.$$

This proves the lower bound for real coefficients. In the complex case, apply the real bounds to the real and imaginary parts. Both have sum of squares at most σ^2, and at least one has sum of squares no less than $\frac{\sigma^2}{2}$. The inequalities

$$|\operatorname{Re} S|, |\operatorname{Im} S| \leq |S| \leq |\operatorname{Re} S| + |\operatorname{Im} S|$$

complete the proof. □

Before studying the maximal operator, we record the interpolation principle used to pass from weak endpoint estimates to strong estimates at intermediate exponents.

Theorem B.8.9. (Marcinkiewicz interpolation) Let (X, μ_X) and (Y, μ_Y) be σ-finite measure spaces. Let T be a sublinear operator defined on simple functions supported on sets of finite measure in X, with values in measurable functions on Y, and suppose that it is of weak type (p_0, p_0) and (p_1, p_1), where $1 \leq p_0 < p_1 \leq \infty$. If $p_0 < p < p_1$, then T extends to a bounded operator from $L^p(X)$ to $L^p(Y)$, and

$$\|Tf\|_{L^p(Y)} \leq C\|f\|_{L^p(X)}.$$

The constant depends only on p_0, p, p_1 and the weak-type constants of T.

Proof. First suppose $p_1 < \infty$. Fix $\gamma > 0$ and, for each $\lambda > 0$, write

$$f = f_0^\lambda + f_1^\lambda, \qquad f_0^\lambda := f\mathbf{1}_{\{|f| > \gamma\lambda\}}, \qquad f_1^\lambda := f\mathbf{1}_{\{|f| \leq \gamma\lambda\}}.$$

Sublinearity implies

$$\{|Tf| > \lambda\} \subseteq \left\{|Tf_0^\lambda| > \frac{\lambda}{2}\right\} \cup \left\{|Tf_1^\lambda| > \frac{\lambda}{2}\right\}.$$

If C_0, C_1 are the weak-type constants, we obtain

$$\mu_Y(\{|Tf| > \lambda\}) \leq C_0'\lambda^{-p_0}\|f_0^\lambda\|_{L^{p_0}(X)}^{p_0} + C_1'\lambda^{-p_1}\|f_1^\lambda\|_{L^{p_1}(X)}^{p_1}.$$

Apply Cavalieri and Tonelli to the first term:

$$\begin{aligned}\int_0^\infty \lambda^{p-p_0-1} \int_{\{|f|>\gamma\lambda\}} |f(x)|^{p_0}\, d\mu_X(x) d\lambda &= \int_X |f(x)|^{p_0} \int_0^{\frac{|f(x)|}{\gamma}} \lambda^{p-p_0-1}\, d\lambda d\mu_X(x) \\ &= \frac{\gamma^{-(p-p_0)}}{p-p_0}\|f\|_{L^p(X)}^p.\end{aligned}$$

For the second term, since $p < p_1$,

$$\int_0^\infty \lambda^{p-p_1-1} \int_{\{|f|\le\gamma\lambda\}} |f(x)|^{p_1}\, d\mu_X(x) d\lambda = \int_X |f(x)|^{p_1} \int_{\frac{|f(x)|}{\gamma}}^\infty \lambda^{p-p_1-1}\, d\lambda d\mu_X(x)$$
$$= \frac{\gamma^{p_1-p}}{p_1-p} \|f\|_{L^p(X)}^p.$$

By Cavalieri, these two estimates imply $\|Tf\|_{L^p(Y)}^p \le C\|f\|_{L^p(X)}^p$.
If $p_1 = \infty$, the hypothesis at that endpoint means $\|Tg\|_{L^\infty(Y)} \le C_1\|g\|_{L^\infty(X)}$. Choose $\gamma = (2C_1)^{-1}$. Then

$$\|Tf_1^\lambda\|_{L^\infty(Y)} \le C_1\gamma\lambda = \frac{\lambda}{2},$$

so only the first term remains, which is integrated exactly as above. Density of simple functions in $L^p(X)$ provides the extension. □

Lemma B.8.10. (Vitali selection for balls) Let $\mathcal{B}$ be a family of open balls in $\mathbb{R}^n$ with radii bounded above. There exists a countable disjoint subfamily $(B_j)_{j\in\mathbb{N}} \subseteq \mathcal{B}$ such that

$$\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{j\in\mathbb{N}} 5B_j,$$

where $5B_j$ denotes the concentric ball with five times the radius.

Proof. Let $R > 0$ be an upper bound for the radii. For each $k \in \mathbb{N}_0$, consider the subfamily

$$\mathcal{B}_k := \left\{ B \in \mathcal{B} \,\middle|\, 2^{-k-1}R < r(B) \le 2^{-k}R \right\}.$$

Proceed by levels. In $\mathcal{B}_0$, choose a maximal subfamily of disjoint balls. Once the families at levels $0,\dots,k-1$ have been chosen, choose in $\mathcal{B}_k$ a maximal subfamily of mutually disjoint balls that are also disjoint from all previously selected balls. Existence of each maximal family follows from Zorn's lemma. Since every family of disjoint open balls in $\mathbb{R}^n$ is countable—each ball contains a distinct point of $\mathbb{Q}^n$—the union of all selected families is countable; denote it by $(B_j)_{j\in\mathbb{N}}$.
Let $B \in \mathcal{B}_k$ be a ball that was not selected. By maximality, B intersects a selected ball B_j from the same or an earlier level. In both cases,

$$r(B_j) \ge \frac{1}{2} r(B).$$

If $x \in B$ and c, c_j are the centers of B and B_j, respectively, choose $z \in B \cap B_j$ and obtain

$$\|x - c_j\| \le \|x - c\| + \|c - z\| + \|z - c_j\| < 2r(B) + r(B_j) \le 5r(B_j).$$

Thus $B \subseteq 5B_j$, proving that the dilates $(5B_j)_{j\in\mathbb{N}}$ cover the union of the balls in $\mathcal{B}$. □

Theorem B.8.11. (Hardy–Littlewood maximal operator) For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\mathcal{M}f(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all open balls $B \subseteq \mathbb{R}^n$ containing x. Then $\mathcal{M}$ is of weak type $(1,1)$: for $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$,

$$|\{x \in \mathbb{R}^n \mid \mathcal{M}f(x) > \lambda\}| \le \frac{C_n}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover, for $1 < p \le \infty$,

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \le C_{n,p}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Fix $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. For each $x \in E_\lambda := \{\mathcal{M}f > \lambda\}$, choose a ball $B_x \ni x$ such that

$$\frac{1}{|B_x|}\int_{B_x} |f(y)|\,dy > \lambda.$$

In particular,

$$|B_x| < \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda},$$

so the radii of the family $(B_x)_{x\in E_\lambda}$ are uniformly bounded. Lemma B.8.10 provides a disjoint family $(B_j)_{j\in\mathbb{N}}$ such that $E_\lambda \subseteq \bigcup_{j\in\mathbb{N}} 5B_j$. Consequently,

$$\begin{aligned} |E_\lambda| &\le \sum_{j\in\mathbb{N}} |5B_j| = 5^n \sum_{j\in\mathbb{N}} |B_j| \\ &< \frac{5^n}{\lambda}\sum_{j\in\mathbb{N}}\int_{B_j} |f(y)|\,dy \le \frac{5^n}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

This proves weak type $(1,1)$. On the other hand, $\mathcal{M}f \le \|f\|_{L^\infty(\mathbb{R}^n)}$ almost everywhere if $f \in L^\infty(\mathbb{R}^n)$. Theorem B.8.9, applied between the endpoints $(1,1)$ and (∞,∞), gives the strong bound for $1 < p < \infty$. □

Theorem B.8.12. (Lebesgue differentiation) If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then, for almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - f(x)|\,dy = 0.$$

Proof. Fix $R > 0$ and set $f_R := f\mathbf{1}_{B(0,R+1)} \in L^1(\mathbb{R}^n)$. For $x \in B(0,R)$ and $0 < r < 1$, the averages of f and f_R over $B(x,r)$ agree. Define

$$Dh(x) := \limsup_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} |h(y) - h(x)|\,dy.$$

If $g \in C_c^\infty(\mathbb{R}^n)$, uniform continuity of g gives $Dg(x) = 0$ for every x. Moreover, for every $h \in L^1(\mathbb{R}^n)$,

$$Dh(x) \le \mathcal{M}h(x) + |h(x)|$$

for almost every x. Now choose $g \in C_c^\infty(\mathbb{R}^n)$ and use

$$Df_R(x) \le D(f_R - g)(x) + Dg(x) \le \mathcal{M}(f_R - g)(x) + |f_R(x) - g(x)|.$$

Thus, for $\varepsilon > 0$,

$$\{x \in B(0,R) \mid Df(x) > 2\varepsilon\} \subseteq \{\mathcal{M}(f_R - g) > \varepsilon\} \cup \{|f_R - g| > \varepsilon\}.$$

Weak type $(1,1)$ from Theorem B.8.11 and Chebyshev give

$$|\{x \in B(0,R) \mid Df(x) > 2\varepsilon\}| \le \frac{C_n}{\varepsilon}\|f_R - g\|_{L^1(\mathbb{R}^n)}.$$

By Proposition B.7.18, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, so the right-hand side can be made arbitrarily small. Hence $Df = 0$ almost everywhere on $B(0,R)$. Finally, let $R \to \infty$ through the integers. □

For Triebel–Lizorkin norms, controlling each block separately does not suffice: the L^p norm is taken after the scales have been combined pointwise. We therefore need a vector-valued version of the maximal inequality. The following weighted result provides an efficient proof for part of the range of exponents and will also be useful in its own right.

Lemma B.8.13. (Weighted Fefferman–Stein inequality) If $1 < a < \infty$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and $w\colon \mathbb{R}^n \longrightarrow [0,\infty]$ is measurable, then

$$\int_{\mathbb{R}^n} (\mathcal{M}f)^a w \le C_{n,a} \int_{\mathbb{R}^n} |f|^a \mathcal{M}w.$$

The assertion is understood first for bounded compactly supported functions; the general case follows by truncation and monotone convergence.

Proof. It suffices to prove the estimate for a truncated version of the maximal operator considering only balls with radii in a compact interval of $(0,\infty)$; at the end, the truncations are removed by monotone convergence. Fix $\lambda > 0$. Applying Lemma B.8.10 to balls on which the average of $|f|$ exceeds λ gives disjoint balls $(B_j)_{j\in\mathbb{N}}$ such that

$$\{\mathcal{M}f > \lambda\} \subseteq \bigcup_{j\in\mathbb{N}} 5B_j.$$

Since the contribution of $|f|\mathbf{1}_{\{|f|\le\frac{\lambda}{2}\}}$ to the average over any ball is at most $\frac{\lambda}{2}$, for each $j \in \mathbb{N}$ we have

$$|B_j| \le \frac{2}{\lambda} \int_{B_j \cap \{|f| > \frac{\lambda}{2}\}} |f(y)|\,dy.$$

If $y \in B_j$, then $5B_j$ contains y, and the definition of $\mathcal{M}w(y)$ gives

$$\int_{5B_j} w \le |5B_j|\,\mathcal{M}w(y) = 5^n |B_j|\,\mathcal{M}w(y).$$

Taking the infimum over $y \in B_j$ and using the preceding inequality for $|B_j|$ yields

$$\begin{aligned}
\int_{\{\mathcal{M}f>\lambda\}} w &\le \sum_{j\in\mathbb{N}} \int_{5B_j} w \\
&\le \frac{C_n}{\lambda} \sum_{j\in\mathbb{N}} \int_{B_j\cap\{|f|>\frac{\lambda}{2}\}} |f(y)|\mathcal{M}w(y)\,dy \\
&\le \frac{C_n}{\lambda} \int_{\{|f|>\frac{\lambda}{2}\}} |f|\mathcal{M}w.
\end{aligned}$$

Multiply by $a\lambda^{a-1}$ and integrate in $\lambda \in (0,\infty)$. By Tonelli,

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathcal{M}f)^a w &= a\int_0^\infty \lambda^{a-1}\int_{\{\mathcal{M}f>\lambda\}} w\,d\lambda \\ &\le C_{n,a}\int_{\mathbb{R}^n} |f(y)|\mathcal{M}w(y)\left(\int_0^{2|f(y)|}\lambda^{a-2}\,d\lambda\right)dy \\ &\le C_{n,a}\int_{\mathbb{R}^n} |f|^a\mathcal{M}w.\end{aligned}$$

The inner integral is finite because $a>1$. The truncations are removed by Theorem B.6.11. □

Theorem B.8.14. (Vector-valued Fefferman–Stein inequality) Let $1<p,r<\infty$. If J is a finite set and $(f_j)_{j\in J}$ is a family of measurable functions, then

$$\left\|\left(\sum_{j\in J}|\mathcal{M}f_j|^r\right)^{\frac{1}{r}}\right\|_{L^p(\mathbb{R}^n)} \le C_{n,p,r}\left\|\left(\sum_{j\in J}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^p(\mathbb{R}^n)}.$$

The same inequality holds for countable families whenever the right-hand side is finite.

See also [146, Section 2.2.2].

Proof. First suppose $p>r$. Then $\frac{p}{r}>1$. By duality of $L^{\frac{p}{r}}(\mathbb{R}^n)$,

$$\left\|\sum_{j\in J}(\mathcal{M}f_j)^r\right\|_{L^{\frac{p}{r}}(\mathbb{R}^n)} = \sup_{\substack{h\in L^{(\frac{p}{r})'}(\mathbb{R}^n),\ h\ge 0\\ \|h\|_{L^{(\frac{p}{r})'}(\mathbb{R}^n)}\le 1}}\int_{\mathbb{R}^n}\sum_{j\in J}(\mathcal{M}f_j)^r h.$$

Apply Lemma B.8.13 to each f_j, sum, and use Tonelli:

$$\int_{\mathbb{R}^n}\sum_{j\in J}(\mathcal{M}f_j)^r h \le C\int_{\mathbb{R}^n}\sum_{j\in J}|f_j|^r\mathcal{M}h.$$

Hölder and continuity of $\mathcal{M}$ on $L^{(\frac{p}{r})'}(\mathbb{R}^n)$ give

$$\int_{\mathbb{R}^n}\sum_{j\in J}|f_j|^r\mathcal{M}h \le C\left\|\sum_{j\in J}|f_j|^r\right\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}.$$

Taking the rth root gives the inequality when $p>r$.

Now suppose $p\le r$ and fix a finite family J. The supremum defining $\mathcal{M}$ may be computed, up to a uniform dimensional constant, using a countable family of balls with rational centers and radii. For each $j\in J$ and the function $|f_j|$, measurably choose one of these balls $B_j(x)\ni x$ whose average is at least half the corresponding supremum. Once this choice has been made, define the linear operator

$$T_j g(x) := \frac{1}{|B_j(x)|}\int_{B_j(x)} g(y)\,dy.$$

The selection depends on the original family $(f_j)_{j\in J}$, but remains fixed while interpolating the diagonal operator

$$\mathcal{T}(g_j)_{j\in J} := (T_j g_j)_{j\in J}.$$

For every g_j, we have $|T_j g_j| \le \mathcal{M} g_j$. The scalar bound in Theorem B.8.11 implies

$$\begin{aligned}\|\mathcal{T}(g_j)_{j\in J}\|_{L^p(\mathbb{R}^n;\ell^p(J))}^p &= \sum_{j\in J} \|T_j g_j\|_{L^p(\mathbb{R}^n)}^p \\ &\le C \sum_{j\in J} \|g_j\|_{L^p(\mathbb{R}^n)}^p.\end{aligned}$$

Moreover,

$$\sup_{j\in J} |T_j g_j(x)| \le \mathcal{M}\left(\sup_{j\in J} |g_j|\right)(x),$$

so

$$\|\mathcal{T}(g_j)_{j\in J}\|_{L^p(\mathbb{R}^n;\ell^\infty(J))} \le C \|(g_j)_{j\in J}\|_{L^p(\mathbb{R}^n;\ell^\infty(J))}.$$

If $r = p$, the desired estimate is precisely the first of these two bounds. Thus suppose $r > p$. There exists $0 < \vartheta < 1$ such that

$$\frac{1}{r} = \frac{1-\vartheta}{p}.$$

Interpolate the estimates with values in $\ell^p(J)$ and $\ell^\infty(J)$ using Theorem 5.7.5 and Proposition 5.7.12. We obtain

$$\|\mathcal{T}(g_j)_{j\in J}\|_{L^p(\mathbb{R}^n;\ell^r(J))} \le C_{n,p,r} \|(g_j)_{j\in J}\|_{L^p(\mathbb{R}^n;\ell^r(J))}.$$

Apply this bound to $g_j = |f_j|$. By the choice of $B_j(x)$,

$$\mathcal{M} f_j(x) \le 2 T_j |f_j|(x),$$

which proves the case $p \le r$. The constant is independent of J. For a countable family, apply the result to the first N coordinates and let $N \to \infty$ using monotone convergence. □

Theorem B.8.15. (Calderón–Zygmund decomposition) Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. There exist a countable family of disjoint dyadic cubes $(Q_j)_{j\in\mathbb{N}}$ and functions $g, (b_j)_{j\in\mathbb{N}}$ such that

$$f = g + \sum_{j\in\mathbb{N}} b_j \quad \text{almost everywhere,}$$

$$\|g\|_{L^\infty(\mathbb{R}^n)} \le 2^n \lambda, \qquad \|g\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)},$$

$$\operatorname{supp}(b_j) \subseteq Q_j, \qquad \int_{Q_j} b_j = 0 \quad (j \in \mathbb{N}),$$

and

$$\sum_{j\in\mathbb{N}} |Q_j| \le \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)}, \qquad \sum_{j\in\mathbb{N}} \|b_j\|_{L^1(\mathbb{R}^n)} \le 2\|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. Consider the family of dyadic cubes Q for which

$$\frac{1}{|Q|}\int_Q |f| > \lambda.$$

For almost every x with $|f(x)| > \lambda$, Theorem B.8.12 ensures the existence of sufficiently small dyadic cubes containing x on which the average of $|f|$ exceeds λ: simply compare each cube with a concentric ball whose radius is comparable to its side length and use Lebesgue differentiation. On the other hand, since $f \in L^1(\mathbb{R}^n)$, the average over cubes whose side length tends to infinity converges to zero. We may therefore choose the maximal cubes $(Q_j)_{j\in\mathbb{N}}$ in the family. They are disjoint and cover $\{|f| > \lambda\}$ up to a null set.
If $\widetilde{Q}_j$ is the dyadic parent of Q_j, maximality implies

$$\frac{1}{|\widetilde{Q}_j|}\int_{\widetilde{Q}_j} |f| \le \lambda.$$

Since $|\widetilde{Q}_j| = 2^n|Q_j|$, we obtain

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f| \le 2^n\lambda.$$

Define

$$g(x) := \begin{cases} f(x), & x \notin \bigcup_{j\in\mathbb{N}} Q_j, \\ f_{Q_j} := \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f, & x \in Q_j, \end{cases}$$

and

$$b_j := (f - f_{Q_j})\mathbf{1}_{Q_j}.$$

Outside the union of the Q_j, we have $|f| \le \lambda$ almost everywhere, while $|f_{Q_j}| \le 2^n\lambda$. This gives $\|g\|_{L^\infty(\mathbb{R}^n)} \le 2^n\lambda$. Moreover,

$$\|g\|_{L^1(\mathbb{R}^n)} \le \int_{\mathbb{R}^n\setminus\bigcup_{j\in\mathbb{N}} Q_j} |f| + \sum_{j\in\mathbb{N}} |f_{Q_j}||Q_j| \le \|f\|_{L^1(\mathbb{R}^n)}.$$

The cancellation $\displaystyle\int_{Q_j} b_j = 0$ is immediate. Finally,

$$\lambda\sum_{j\in\mathbb{N}} |Q_j| < \sum_{j\in\mathbb{N}}\int_{Q_j} |f| \le \|f\|_{L^1(\mathbb{R}^n)}$$

and

$$\sum_{j\in\mathbb{N}} \|b_j\|_{L^1(\mathbb{R}^n)} \le \sum_{j\in\mathbb{N}} \left(\int_{Q_j} |f| + |f_{Q_j}||Q_j|\right) \le 2\|f\|_{L^1(\mathbb{R}^n)}.$$

□

The last result of this section directly connects frequency localization with the maximal operator. For a function whose spectrum is contained in a ball of radius comparable to 2^j, the maximum of its translates, penalized at scale 2^{-j}, is controlled by a Hardy–Littlewood maximal function of a power of the function. This is the form of Peetre's maximal inequality used in proofs of independence of the dyadic resolution and in almost diagonal

arguments.

Lemma B.8.16. (Peetre maximal control) Let $j \in \mathbb{Z}$, $0 < a < \infty$, $N > \frac{n}{a}$, and $A > 0$. Suppose that $f \in \mathcal{S}'(\mathbb{R}^n)$, and identify it with its canonical smooth representative. Suppose also that

$$\operatorname{supp}(\widehat{f}) \subseteq \{\xi \in \mathbb{R}^n \mid \|\xi\| \le A2^j\}.$$

Then

$$\sup_{y\in\mathbb{R}^n} \frac{|f(x-y)|}{(1+2^j\|y\|)^N} \le C\left[\mathcal{M}(|f|^a)(x)\right]^{\frac{1}{a}}, \tag{B.9}$$

where C depends only on n, a, N and A, at every point x such that $\mathcal{M}(|f|^a)(x) < \infty$. Consequently, if

$$K_j(x) := 2^{jn}K(2^jx)$$

and

$$|K(x)| \le C_K(1+\|x\|)^{-N-n-1},$$

then

$$|K_j * f(x)| \le C\left[\mathcal{M}(|f|^a)(x)\right]^{\frac{1}{a}}$$

where C also depends on C_K; at those points, the convolution converges absolutely.

The maximal method for band-limited functions is developed in [146, Sections 2.2–2.3].

Proof. Choose $\vartheta \in C_c^\infty(\mathbb{R}^n)$ such that $\vartheta = 1$ on a neighborhood of the ball $\{\|\xi\| \le A\}$, and set $\eta := \mathcal{F}^{-1}\vartheta \in \mathcal{S}(\mathbb{R}^n)$. If $\eta_j(x) := 2^{jn}\eta(2^jx)$, the convolution formula corresponding to the unitary Fourier normalization gives

$$f = (2\pi)^{-\frac{n}{2}}\eta_j * f$$

in $\mathcal{S}'(\mathbb{R}^n)$. The constant factor will be absorbed in the estimates. In particular, by Proposition 4.10.4, f has the canonical smooth representative of at most polynomial growth used in the statement.
Since η is a Schwartz function, for every $L > 0$ there exists C_L such that

$$|\eta_j(z)| \le C_L 2^{jn}(1+2^j\|z\|)^{-L}. \tag{B.10}$$

First suppose $a > 1$. For $z \in \mathbb{R}^n$, the reproducing formula and the inequality

$$1 + 2^j\|x-w\| \le (1+2^j\|x-z\|)(1+2^j\|z-w\|)$$

give, after dividing by $(1+2^j\|x-z\|)^N$,

$$\frac{|f(z)|}{(1+2^j\|x-z\|)^N} \le C_L \int_{\mathbb{R}^n} 2^{jn}(1+2^j\|z-w\|)^{-L+N}\frac{|f(w)|}{(1+2^j\|x-w\|)^N}\,dw.$$

Apply Hölder with exponents a and $a' = \frac{a}{a-1}$. If L is chosen so that $(L-N)a' > n$, the second factor remains

uniformly bounded and we obtain

$$\frac{|f(z)|}{(1+2^j\|x-z\|)^N} \le C\left(\int_{\mathbb{R}^n} 2^{jn}\frac{|f(w)|^a}{(1+2^j\|x-w\|)^{Na}}\,dw\right)^{\frac{1}{a}}.$$

Since $Na > n$, the decomposition of $\mathbb{R}^n$ into the ball $B(x,2^{-j})$ and the annuli

$$2^{\ell-j} < \|x-w\| \le 2^{\ell+1-j}, \qquad \ell \in \mathbb{N}_0,$$

proves

$$\int_{\mathbb{R}^n} 2^{jn}\frac{|f(w)|^a}{(1+2^j\|x-w\|)^{Na}}\,dw \le C\mathcal{M}(|f|^a)(x), \tag{B.11}$$

since the contribution of the ℓth annulus is bounded by $C2^{-\ell(Na-n)}\mathcal{M}(|f|^a)(x)$. Taking the supremum over $z = x - y$ proves (B.9) for $a > 1$.

Now consider $0 < a \le 1$. To avoid assuming in advance that the Peetre maximum is finite, fix $\varepsilon > 0$ and define

$$P_\varepsilon f(x) := \sup_{z\in\mathbb{R}^n} \frac{|f(z)|}{(1+2^j\|x-z\|)^N(1+\varepsilon 2^j\|x-z\|)^M},$$

where $0 < \varepsilon \le 1$ and M is chosen greater than a polynomial growth order of f. The regularity and growth established above imply $P_\varepsilon f(x) < \infty$. Using

$$|f(w)|^{1-a} \le P_\varepsilon f(x)^{1-a}(1+2^j\|x-w\|)^{N(1-a)}(1+\varepsilon 2^j\|x-w\|)^{M(1-a)}$$

in the preceding reproducing formula and applying the weight inequalities

$$1+2^j\|x-w\| \le (1+2^j\|x-z\|)(1+2^j\|z-w\|)$$

and their analogue with $\varepsilon 2^j$ gives, upon choosing L sufficiently large in (B.10),

$$P_\varepsilon f(x) \le CP_\varepsilon f(x)^{1-a}\int_{\mathbb{R}^n} 2^{jn}\frac{|f(w)|^a}{(1+2^j\|x-w\|)^{Na}}\,dw.$$

The constant C is independent of ε. If $P_\varepsilon f(x) > 0$, divide by $P_\varepsilon f(x)^{1-a}$; if it is zero, the conclusion is immediate. Using (B.11) yields

$$P_\varepsilon f(x)^a \le C\mathcal{M}(|f|^a)(x).$$

As $\varepsilon \downarrow 0$, the denominators decrease and $P_\varepsilon f(x)$ increases to the unregularized Peetre maximum. The preceding bound proves that this maximum, denoted by $Pf(x)$, is finite. In this first step, the constant may depend on M, chosen according to the growth of f.

Now repeat the estimate with the finite maximum $Pf(x)$ and without the auxiliary weight. The reproducing formula and

$$|f(w)|^{1-a} \le Pf(x)^{1-a}(1+2^j\|x-w\|)^{N(1-a)}$$

give, for $z \in \mathbb{R}^n$,

$$\frac{|f(z)|}{(1+2^j\|x-z\|)^N} \le CPf(x)^{1-a}\int_{\mathbb{R}^n} 2^{jn}\frac{|f(w)|^a}{(1+2^j\|x-w\|)^{Na}}\,dw.$$

Indeed, use $(1+2^j\|x-w\|)^N \le (1+2^j\|x-z\|)^N(1+2^j\|z-w\|)^N$ and choose $L>N$ in (B.10). The remaining factor $(1+2^j\|z-w\|)^{-L+N}$ is at most one. This time L depends only on N, and no constant depends on the growth order of f. Take the supremum over $z\in\mathbb{R}^n$, divide by $Pf(x)^{1-a}$ when it is positive, and apply (B.11). This gives (B.9) with the dependence of constants asserted in the statement.
For the second assertion, denote by $P_{j,N}f(x)$ the left-hand side of (B.9). Then

$$\begin{aligned}|K_j * f(x)| &\le \int_{\mathbb{R}^n} |K_j(y)||f(x-y)|\,dy \\ &\le P_{j,N}f(x)\int_{\mathbb{R}^n} |K_j(y)|(1+2^j\|y\|)^N\,dy.\end{aligned}$$

The change of variables $z=2^jy$ and the hypothesis on K show that the last integral is bounded uniformly in j. Finally, apply (B.9). □

Proposition B.8.17. (Continuity of translations) If $1\le p<\infty$ and $f\in L^p(\mathbb{R}^n)$, then

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^n)} \longrightarrow 0 \qquad \text{as } \|h\|\to 0, \quad \tau_h f(x) := f(x+h).$$

Proof. If $f\in C_c(\mathbb{R}^n)$, convergence follows from uniform continuity and the fact that all translates with $\|h\|\le 1$ are supported in a fixed compact set. For $f\in L^p(\mathbb{R}^n)$, choose $g\in C_c(\mathbb{R}^n)$ such that $\|f-g\|_{L^p(\mathbb{R}^n)}<\varepsilon$. Since translations are isometries of $L^p(\mathbb{R}^n)$,

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^n)} \le 2\|f-g\|_{L^p(\mathbb{R}^n)} + \|\tau_h g - g\|_{L^p(\mathbb{R}^n)}.$$

The last term tends to zero as $\|h\|\to 0$, after which we let $\varepsilon\downarrow 0$. □

B.9 Euclidean mollifiers

The local regularization used in Sobolev spaces and partial differential equations is based on a single Euclidean construction. We formulate it with the properties needed later; see also [61, Appendix C, §5].

Theorem B.9.1. (Existence and properties of Euclidean mollifiers) There exists a function $\rho\in C_c^\infty(\mathbb{R}^n)$ such that $\rho\ge 0$, $\operatorname{supp}(\rho)\subseteq \overline{B}_{\mathrm{euc}}(0,1)$, and $\displaystyle\int_{\mathbb{R}^n}\rho\,d\lambda_n = 1$. For $\varepsilon>0$, define

$$\rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right).$$

Then $\rho_\varepsilon\ge 0$, $\operatorname{supp}(\rho_\varepsilon)\subseteq\overline{B}_{\mathrm{euc}}(0,\varepsilon)$, and $\displaystyle\int_{\mathbb{R}^n}\rho_\varepsilon\,d\lambda_n = 1$. Moreover, the following properties hold.

(a) If $\Omega\subseteq\mathbb{R}^n$ is open, $u\in L^1_{\mathrm{loc}}(\Omega)$, and $\Omega_\varepsilon := \{x\in\Omega \mid \operatorname{dist}_{\mathrm{euc}}(x,\mathbb{R}^n\setminus\Omega)>\varepsilon\}$, then

$$u_\varepsilon(x) := (u*\rho_\varepsilon)(x) = \int_{\mathbb{R}^n} u(x-y)\rho_\varepsilon(y)\,dy, \qquad x\in\Omega_\varepsilon,$$

is smooth and $D^\alpha u_\varepsilon = u * D^\alpha\rho_\varepsilon$ for every multi-index α.

(b) If $1\le p<\infty$ and $u\in L^p_{\mathrm{loc}}(\Omega)$, then $u_\varepsilon\to u$ in $L^p(K)$ for every compact set $K\subseteq\Omega$ as $\varepsilon\to 0^+$.

(c) If u is continuous, then $u_\varepsilon \to u$ uniformly on each compact subset of Ω. If $u\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is uniformly continuous, convergence is uniform on $\mathbb{R}^n$.

(d) If $u\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is L-Lipschitz, then u_ε is also L-Lipschitz,

$$|Du_\varepsilon(x)| \le L$$

for every $x \in \mathbb{R}^n$, and

$$\|u_\varepsilon - u\|_{L^\infty(\mathbb{R}^n)} \le L\varepsilon m_1(\rho), \qquad m_1(\rho) := \int_{\mathbb{R}^n} \|z\|\rho(z)\,dz.$$

Proof. Define

$$\vartheta(x) := \begin{cases} \exp\left(-\frac{1}{1-\|x\|^2}\right), & \|x\| < 1, \\ 0, & \|x\| \ge 1. \end{cases}$$

The function ϑ is smooth: inside the ball this is immediate, and as one approaches $\|x\| = 1$, each derivative is the product of a rational function of $1-\|x\|^2$ and $\exp\left(-\dfrac{1}{1-\|x\|^2}\right)$, which tends to zero faster than any power of $1-\|x\|^2$. Since ϑ is nonnegative and not identically zero, the constant $c := \left(\displaystyle\int_{\mathbb{R}^n} \vartheta\,d\lambda_n\right)^{-1}$ is well defined. The function $\rho := c\vartheta$ has the required properties. The change of variables $y = \varepsilon z$ proves the assertions about ρ_ε.

Now let $u \in L^1_{\mathrm{loc}}(\Omega)$. If $x \in \Omega_\varepsilon$, the integral defining $u_\varepsilon(x)$ uses only values of u in Ω. It may also be written as

$$u_\varepsilon(x) = \int_\Omega u(z)\rho_\varepsilon(x-z)\,dz.$$

On a compact neighborhood of any point of Ω_ε, the integrand vanishes outside a fixed compact subset of Ω. Since u is integrable on that compact set and all derivatives of ρ_ε are bounded, dominated convergence allows differentiation under the integral sign any number of times. Thus $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and $D^\alpha u_\varepsilon = u * D^\alpha \rho_\varepsilon$. We now prove convergence in L^p_{loc}. First observe that, for every $v \in L^p(\mathbb{R}^n)$,

$$\|v(\cdot - h) - v\|_{L^p(\mathbb{R}^n)} \longrightarrow 0 \qquad \text{as} \qquad h \to 0.$$

Indeed, by Proposition B.7.18, choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\|v-\varphi\|_{L^p(\mathbb{R}^n)} < \delta$. Translation invariance of Lebesgue measure gives

$$\|v(\cdot - h) - v\|_{L^p(\mathbb{R}^n)} \le 2\delta + \|\varphi(\cdot - h) - \varphi\|_{L^p(\mathbb{R}^n)}.$$

The last term converges to zero by uniform continuity of φ and because, for $\|h\| \le 1$, all functions involved are supported in a fixed compact set. Since $\delta > 0$ is arbitrary, continuity of translations follows.

Let $K \subseteq \Omega$ be compact. Since K and $\mathbb{R}^n \setminus \Omega$ are disjoint and the former is compact, the number $d_K := \operatorname{dist}_{\mathrm{euc}}(K, \mathbb{R}^n \setminus \Omega)$ is positive. Set $\delta_K := \dfrac{d_K}{3}$. Then the closed set $F_K := \{x \in \mathbb{R}^n \mid \operatorname{dist}_{\mathrm{euc}}(x,K) \le 2\delta_K\}$ is contained in Ω. Proposition 1.3.6, applied to $F_K \subseteq \Omega$, provides $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on a neighborhood of F_K. Extend $v := \eta u$ by zero to $\mathbb{R}^n$. If $0 < \varepsilon < \delta_K$, then $u_\varepsilon = v * \rho_\varepsilon$ on K. Minkowski's

integral inequality and the change of variables $y = \varepsilon z$ give

$$\|v * \rho_\varepsilon - v\|_{L^p(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} \rho(z) \|v(\cdot - \varepsilon z) - v\|_{L^p(\mathbb{R}^n)} \, dz.$$

For each z, the integrand converges to zero and is dominated by $2\rho(z)\|v\|_{L^p(\mathbb{R}^n)}$. The dominated convergence theorem implies that $u_\varepsilon \to u$ in $L^p(K)$.
If u is continuous, for $x \in \Omega_\varepsilon$ we have

$$|u_\varepsilon(x) - u(x)| \le \int_{\mathbb{R}^n} \rho_\varepsilon(y) |u(x-y) - u(x)| \, dy \le \sup_{y \in \overline{B}_{\mathrm{euc}}(0,\varepsilon)} |u(x-y) - u(x)|.$$

Uniform continuity of u on a compact neighborhood of K proves uniform convergence on K. The same argument applies on all of $\mathbb{R}^n$ when u is uniformly continuous.
Finally, if u is L-Lipschitz, then

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le \int_{\mathbb{R}^n} \rho_\varepsilon(z) |u(x-z) - u(y-z)| \, dz \le L\|x - y\|.$$

Since u_ε is smooth, taking difference quotients in unit directions gives $\|Du_\varepsilon\| \le L$. Moreover,

$$|u_\varepsilon(x) - u(x)| \le \int_{\mathbb{R}^n} \rho(z) |u(x - \varepsilon z) - u(x)| \, dz \le L\varepsilon \int_{\mathbb{R}^n} \|z\| \rho(z) \, dz,$$

uniformly with respect to x. This completes the proof. □

B.10 Topological vector spaces

The function spaces arising in global analysis are often not naturally described by a single norm. This occurs, for example, with spaces of compactly supported smooth functions, spaces of test sections, and various weak topologies associated with duality problems. We therefore recall the language of topological vector spaces, which allows the linear and topological structures of a space to be studied together.

Definition B.10.1. Let X be a vector space over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ equipped with a topology τ. The pair (X, τ) is called a *topological vector space* if the functions $X \times X \longrightarrow X, \quad (x, y) \mapsto x + y$ and $\mathbb{K} \times X \longrightarrow X, \quad (\lambda, x) \mapsto \lambda x$ are continuous.

Continuity of the linear operations has immediate consequences. For example, translations and nonzero dilations are homeomorphisms. In particular, the topology of a topological vector space is determined by its behavior around the origin.

Proposition B.10.2. Let X be a topological vector space, and let $\mathcal{B}$ be a neighborhood basis at the origin. Then, for each $x \in X$, the family

$$x + \mathcal{B} := \{x + U \mid U \in \mathcal{B}\}$$

is a neighborhood basis at x. In particular, the family

$$\{x + U \mid x \in X, \ U \in \mathcal{B}\}$$

determines the open sets of X: a set $O \subseteq X$ is open if and only if, for each $x \in O$, there exists $U \in \mathcal{B}$ such that $x + U \subseteq O$. If the elements of $\mathcal{B}$ are open, their translates form a basis of open sets. Consequently, the topology of a topological vector space is uniquely determined by any neighborhood basis at the origin.

Proof. Let $x \in X$. Since the translation $\tau_x \colon X \longrightarrow X$, given by $\tau_x(y) = x + y$, is a homeomorphism, V is a neighborhood of 0 if and only if $x + V$ is a neighborhood of x. Thus, if $\mathcal{B}$ is a local basis at 0, then $x + \mathcal{B}$ is a local basis at x. If O is open and $x \in O$, there exists $U \in \mathcal{B}$ such that $x + U \subseteq O$. Conversely, if this condition holds at each $x \in O$, every point of O has a neighborhood contained in O and is an interior point. This characterizes the open sets and proves uniqueness of the topology. When the elements of $\mathcal{B}$ are open, so are their translates, which therefore form a basis of open sets. □

To describe useful local bases, we need a few elementary geometric notions in vector spaces.

Definition B.10.3. (Absorbing set) Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A set $A \subset X$ is said to be *absorbing* if, for every $x \in X$, there exists $\lambda > 0$ such that $x \in tA$, $\forall t \geq \lambda$, where $tA := \{ta \mid a \in A\}$.

Definition B.10.4. (Balanced set) Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A set $B \subset X$ is said to be *balanced* if, for every $x \in B$ and every scalar α with $|\alpha| \leq 1$, we have $\alpha x \in B$.

Definition B.10.5. (Absolutely convex set) Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A subset $A \subseteq X$ is called *absolutely convex* if it is convex and balanced.

As a first example, every neighborhood of the origin absorbs the entire space.

Lemma B.10.6. Every neighborhood of the origin in a topological vector space is absorbing.

Proof. Let U be a neighborhood of the origin, and let $x \in X$. Since the map

$$\mathbb{K} \longrightarrow X, \qquad \lambda \longmapsto \lambda x,$$

is continuous and $\lambda x \to 0$ as $\lambda \to 0$, there exists $\delta > 0$ such that $\lambda x \in U$ if $|\lambda| < \delta$. Taking $t > \delta^{-1}$ gives $t^{-1} x \in U$ and hence $x \in tU$. Since $x \in X$ was arbitrary, U is absorbing. □

Lemma B.10.7. Let X be a topological vector space. Then:

(a) for every neighborhood U of the origin, there is a neighborhood V of the origin such that $V + V \subseteq U$;

(b) for every neighborhood U of the origin, there is a balanced neighborhood B of the origin such that $B \subseteq U$;

(c) if X is Hausdorff, then, for every neighborhood U of the origin, there is a neighborhood W of the origin such that $\overline{W} \subseteq U$.

Proof. For (a), use continuity of addition at $(0,0)$. Since U is a neighborhood of 0, there are neighborhoods V_1, V_2 of the origin such that $V_1 + V_2 \subseteq U$. Taking $V = V_1 \cap V_2$ gives $V + V \subseteq U$.
For (b), use continuity of scalar multiplication at $(0,0)$. There exist $\varepsilon > 0$ and a neighborhood V of the origin

such that $\lambda V \subseteq U$ if $|\lambda| < \varepsilon$. Taking $0 < r < \varepsilon$ and defining

$$B := \bigcup_{|\lambda| \le r} \lambda V,$$

we find that B is a neighborhood of the origin, $B \subseteq U$, and B is balanced.
For (c), by part (a), choose a neighborhood V_0 of the origin such that $V_0 + V_0 \subseteq U$. The set

$$V := V_0 \cap (-V_0)$$

is a symmetric neighborhood of the origin and satisfies $V + V \subseteq U$. Let $W := \mathrm{Int}(V)$, which is an open symmetric neighborhood of the origin. If $x \in \overline{W}$, the neighborhood $x + W$ of x intersects W. Thus there exist $w_1, w_2 \in W$ such that $x + w_1 = w_2$, and

$$x = w_2 - w_1 \in W + W \subseteq V + V \subseteq U.$$

Hence $\overline{W} \subseteq U$. In particular, this argument directly proves regularity of every Hausdorff topological vector space. □

Remark B.10.8. Proposition B.10.2 shows that, in a topological vector space, it suffices to describe a local basis at the origin. Lemma B.10.7 allows neighborhoods of the origin to be refined so that they have useful properties under addition and scalar multiplication; under the Hausdorff assumption, we can find a basis of closed neighborhoods.

In a topological vector space, having a countable local basis at the origin is equivalent to being first countable. This property gives a characterization of metrizable topological vector spaces.

Theorem B.10.9. A topological vector space X is metrizable if and only if:

(a) X is Hausdorff;

(b) X has a countable local basis at 0.

The construction of a translation-invariant metric from a countable local basis, together with the immediate converse, can be found in [123, 149].

In metrizable spaces, continuity can be proved using sequences. This observation will be useful when working with Fréchet spaces and other metrizable spaces.

Proposition B.10.10. Let X be a metrizable topological vector space, and let Y be a topological vector space. A linear operator $L\colon X \longrightarrow Y$ is continuous if and only if it is sequentially continuous at the origin; that is, if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \to 0$, we have $L(x_n) \to 0$ in Y.

Proof. If L is continuous and $x_n \to 0$ in X, continuity gives $L(x_n) \to L(0) = 0$ in Y.
Conversely, suppose that L is sequentially continuous at the origin. We prove that L is continuous. Since L is linear, it suffices to prove continuity at 0. Suppose, for a contradiction, that L is not continuous at 0. There is then a neighborhood U of the origin in Y such that, for every neighborhood V of the origin in X, we have $L(V) \not\subseteq U$.

Since X is metrizable, it is first countable. Let $\{W_n\}_{n\in\mathbb{N}}$ be a countable local basis at the origin in X. Define $V_n := \bigcap_{k=1}^{n} W_k$, for each $n \in \mathbb{N}$. Then each V_n is a neighborhood of the origin, and the family $\{V_n\}_{n\in\mathbb{N}}$ is decreasing and remains a local basis at the origin.
By the choice of U, for each $n \in \mathbb{N}$ we can choose $x_n \in V_n$ such that $L(x_n) \notin U$.
Let us show that $x_n \to 0$ in X. Let W be an arbitrary neighborhood of the origin. Since $\{V_n\}_{n\in\mathbb{N}}$ is a local basis at the origin, there exists $N \in \mathbb{N}$ such that $V_N \subseteq W$. Because the family is decreasing, $V_n \subseteq V_N \subseteq W$ for every $n \geq N$. Consequently, $x_n \in W$ for every $n \geq N$. This proves that $x_n \to 0$ in X.
By sequential continuity at the origin, $L(x_n) \to 0$ in Y. This is impossible, however, because U is a neighborhood of the origin in Y and $L(x_n) \notin U$ for every $n \in \mathbb{N}$. This contradiction shows that L is continuous at 0. Finally, linearity implies that L is continuous. □

Boundedness in topological vector spaces depends on how sets are absorbed by neighborhoods of the origin, rather than on a metric.

Definition B.10.11. Let X be a topological vector space. A set $E \subseteq X$ is said to be *topologically bounded* if, for each neighborhood U of 0, there exists $t > 0$ such that $E \subseteq tU$.

When the topology τ is generated by a metric d, metric boundedness and topological boundedness may differ. The metric $d_1 = \dfrac{d}{d+1}$ generates the same topology as d, but, since $d_1 \leq 1$, every set in X is bounded with respect to d_1. Nevertheless, not every subset of X need be topologically bounded.

We will also need the sequential notion of a Cauchy sequence. In nonmetrizable topological vector spaces, general completeness is formulated using Cauchy filters or nets. In this appendix, we use the sequential version only when it suffices, particularly in metrizable spaces.

Definition B.10.12. Let X be a topological vector space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every neighborhood U of the origin, there exists $N \in \mathbb{N}$ such that

$$x_n - x_m \in U, \quad \forall n, m \geq N.$$

We say that X is *sequentially complete* if every Cauchy sequence in X converges.

Proposition B.10.13. Let X be a topological vector space, and let $\{x_n\}$ be a Cauchy sequence in X. Then the range of the sequence $\{x_n \mid n \in \mathbb{N}\}$ is topologically bounded.

Proof. Let U be a neighborhood of the origin. By Lemma B.10.7, there is a balanced neighborhood W of the origin such that $W + W \subseteq U$. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $x_n - x_N \in W$ for every $n \geq N$. Moreover, since W is absorbing, there exists $t_0 > 0$ such that $x_N \in tW$ for every $t \geq t_0$. If $t \geq \max\{1, t_0\}$ and $n \geq N$, then $x_n = x_N + (x_n - x_N) \in tW + W \subseteq t(W + W) \subseteq tU$.
On the other hand, the finite set $\{x_1, \ldots, x_{N-1}\}$ is topologically bounded because each of its points is absorbed by U. Increasing t if necessary, we find that all terms of the sequence belong to a single multiple of U. Since U was arbitrary, the range of the sequence is topologically bounded. □

The most important class of topological vector spaces for our purposes will be that of locally convex spaces. Their importance stems from the fact that their topologies can be described by seminorms, which is particularly

suitable for spaces of functions and sections.

Definition B.10.14. A topological vector space X is said to be locally convex if it has a local basis at 0 consisting of convex sets; equivalently, every neighborhood of 0 contains a convex neighborhood of 0.

Proposition B.10.15. A locally convex topological vector space admits a local basis at 0 consisting of balanced convex sets.

Proof. Let U be a neighborhood of the origin. Since X is locally convex, there is a convex neighborhood V of the origin such that $V \subseteq U$. By Lemma B.10.7, there is a balanced neighborhood B of the origin such that $B \subseteq V$. The convex hull of B, denoted by $\operatorname{conv}(B)$, is then balanced and convex. Moreover, since V is convex and $B \subseteq V$, we have $\operatorname{conv}(B) \subseteq V \subseteq U$. Thus every neighborhood of the origin contains a balanced convex neighborhood of the origin. □

Definition B.10.16. Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $p\colon X \longrightarrow [0,\infty)$ is a seminorm if

$$p(x+y) \le p(x) + p(y)$$

for any $x, y \in X$ and

$$p(tx) = |t|p(x)$$

for every $x \in X$ and any $t \in \mathbb{K}$.

Every absorbing absolutely convex set has an associated seminorm given by its Minkowski functional.

Proposition B.10.17. Let X be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $A \subseteq X$ be an absorbing set. If A is absolutely convex, its Minkowski functional

$$p_A(x) := \inf\{\lambda > 0 \mid x \in \lambda A\}, \qquad x \in X,$$

is a seminorm on X.

Proof. Since A is absorbing, for each $x \in X$ there exists $\lambda > 0$ such that $x \in \lambda A$. Thus the set in the definition of $p_A(x)$ is nonempty, and consequently $p_A(x)$ is well defined for every $x \in X$.
First consider homogeneity. Let $\alpha \in \mathbb{K}$ and $x \in X$. If $\alpha = 0$, then $p_A(\alpha x) = p_A(0) = 0$. Now suppose that $\alpha \neq 0$. Write $\alpha = |\alpha|\theta$, where $|\theta| = 1$. Since A is balanced, $\theta A = A$. Therefore, for each $\lambda > 0$,

$$\alpha x \in \lambda A \iff |\alpha|\theta x \in \lambda A \iff x \in \frac{\lambda}{|\alpha|} A.$$

It follows that

$$p_A(\alpha x) = |\alpha|\, p_A(x).$$

We now prove subadditivity. Let $x, y \in X$ and $\varepsilon > 0$. By the definition of the infimum, there exist $\lambda, \mu > 0$ such that

$$x \in \lambda A, \qquad y \in \mu A, \qquad \lambda < p_A(x) + \varepsilon, \qquad \mu < p_A(y) + \varepsilon.$$

Since A is convex, we have

$$\lambda A + \mu A \subseteq (\lambda + \mu)A.$$

If $a, b \in A$, then

$$\lambda a + \mu b = (\lambda + \mu)\left(\frac{\lambda}{\lambda+\mu}a + \frac{\mu}{\lambda+\mu}b\right),$$

and the term in parentheses belongs to A by convexity. Therefore,

$$x + y \in (\lambda + \mu)A,$$

whence

$$p_A(x+y) \le \lambda + \mu < p_A(x) + p_A(y) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$p_A(x+y) \le p_A(x) + p_A(y).$$

Thus p_A is a seminorm on X. □

Definition B.10.18. Let X be a vector space, and let $\mathcal{P}$ be a family of seminorms on X. The *topology induced by* $\mathcal{P}$ is the topology whose local basis at the origin consists of all finite intersections of sets of the form

$$\{x \in X \mid p(x) < r\}, \qquad p \in \mathcal{P}, \quad r > 0.$$

Equivalently, a neighborhood basis at a point $x \in X$ is given by sets of the form

$$x + \bigcap_{i=1}^{k} \{y \in X \mid p_i(y) < r_i\},$$

where $p_1, \dots, p_k \in \mathcal{P}$ and $r_1, \dots, r_k > 0$.

Theorem B.10.19. Let X be a locally convex topological vector space, and let $\mathcal{F}$ be a local basis at the origin consisting of open absolutely convex neighborhoods. Then the family $\{p_U \mid U \in \mathcal{F}\}$ of Minkowski functionals is a family of continuous seminorms generating the topology of X.
Conversely, given a family $\mathcal{P}$ of seminorms on a vector space X, the topology induced by $\mathcal{P}$ makes X a locally convex topological vector space, and every seminorm $p \in \mathcal{P}$ is continuous in this topology.

Proof. First suppose that $\mathcal{F}$ is a local basis at the origin consisting of open absolutely convex neighborhoods. Let $U \in \mathcal{F}$. Since U is a neighborhood of the origin in a topological vector space, Lemma B.10.6 shows that U is absorbing. Proposition B.10.17 therefore ensures that its Minkowski functional p_U is a seminorm on X.
We show that p_U is continuous. Since p_U is a seminorm, it suffices to prove continuity at the origin. Let $\varepsilon > 0$. Multiplication by the nonzero scalar $\dfrac{\varepsilon}{2}$ is a homeomorphism of X onto itself and U is a neighborhood

of the origin, so the set $\frac{\varepsilon}{2}U$ is also a neighborhood of the origin. If $x \in \frac{\varepsilon}{2}U$, then

$$p_U(x) \le \frac{\varepsilon}{2} < \varepsilon.$$

Consequently,

$$\frac{\varepsilon}{2}U \subseteq \{x \in X \mid p_U(x) < \varepsilon\},$$

which proves that p_U is continuous at 0. Finally, for any $x, y \in X$, subadditivity of p_U gives $p_U(x) \le p_U(x-y) + p_U(y)$ and $p_U(y) \le p_U(x-y) + p_U(x)$, whence $|p_U(x) - p_U(y)| \le p_U(x-y)$.
Thus continuity of p_U at the origin implies continuity at every point of X.
We now show that the family $\{p_U \mid U \in \mathcal{F}\}$ generates the original topology. For each $U \in \mathcal{F}$, since U is open, balanced, and convex, we have

$$U = \{x \in X \mid p_U(x) < 1\}.$$

If $p_U(x) < 1$, there exists $\lambda < 1$ such that $x \in \lambda U$, and balancedness of U gives $x \in U$. Conversely, if $x \in U$, openness of U and continuity of the map $t \mapsto tx$ give $r > 1$ such that $rx \in U$. Therefore $x \in \frac{1}{r}U$ and $p_U(x) \le \frac{1}{r} < 1$. Thus the sets defined by the seminorms p_U reproduce the local basis $\mathcal{F}$ and, by Proposition B.10.2, generate the topology of X.
We now prove the converse assertion. Let $\mathcal{P}$ be a family of seminorms on the vector space X, and denote by $\mathcal{B}$ the collection of all finite intersections of sets of the form

$$\{x \in X \mid p(x) < r\}, \qquad p \in \mathcal{P}, \quad r > 0.$$

Each of these sets is balanced and convex, as are their finite intersections. Define a topology τ by declaring $O \subseteq X$ open if, for each $x \in O$, there exists $B \in \mathcal{B}$ such that $x + B \subseteq O$. This makes $\mathcal{B}$ a local basis at the origin. Addition is continuous at $(0,0)$ because, given a basic element

$$B = \bigcap_{i=1}^{k} \{x \in X \mid p_i(x) < r_i\},$$

the set

$$C = \bigcap_{i=1}^{k} \left\{ x \in X \mid p_i(x) < \frac{r_i}{2} \right\}$$

satisfies $C + C \subseteq B$. Continuity of scalar multiplication follows similarly: if $\alpha_0 \in \mathbb{K}$, $x_0 \in X$, and B is as above, choose $\delta > 0$ and a basic neighborhood C of 0 such that

$$|\alpha - \alpha_0| p_i(x_0) < \frac{r_i}{2} \quad \text{and} \quad |\alpha| p_i(h) < \frac{r_i}{2}$$

for $|\alpha - \alpha_0| < \delta$, $h \in C$, and every $i \in \{1, \dots, k\}$. Then $\alpha(x_0 + h) - \alpha_0 x_0 \in B$. Thus τ makes X a topological vector space. Since $\mathcal{B}$ consists of convex sets, the space is locally convex. Finally, each seminorm $p \in \mathcal{P}$ is continuous because $\{x \in X \mid p(x) < \varepsilon\}$ is a neighborhood of the origin for every $\varepsilon > 0$. □

We now give some consequences of the description by seminorms.

Proposition B.10.20. Let X be a topological vector space. Then X is Hausdorff if and only if the set $\{0\}$ is closed.

Proof. First suppose that X is Hausdorff. Since every Hausdorff space is T_1, its singleton sets are closed. In particular, $\{0\}$ is closed.
Conversely, suppose that $\{0\}$ is closed. Let $x, y \in X$ with $x \neq y$. Then $x - y \neq 0$ and hence $x - y \in X \setminus \{0\}$. Since $X \setminus \{0\}$ is open, there is a neighborhood U of the origin such that

$$x - y + U \subseteq X \setminus \{0\}.$$

In particular, $x - y \notin -U$. Replacing U by $U \cap (-U)$, we may assume that U is symmetric, that is, $U = -U$. Thus $x - y \notin U$.
Since addition is continuous at $(0,0)$, there is a neighborhood V of the origin such that $V + V \subseteq U$. Replacing V by $V \cap (-V)$ again, we may assume that V is symmetric.
Consider the neighborhoods $x+V$ and $y+V$. We claim that they are disjoint. If there were $z \in (x+V) \cap (y+V)$, there would be $v_1, v_2 \in V$ such that $z = x + v_1 = y + v_2$. Therefore $x - y = v_2 - v_1$. Since V is symmetric, $-v_1 \in V$, and then $v_2 - v_1 \in V + V \subseteq U$, contradicting $x - y \notin U$. Therefore $(x + V) \cap (y + V) = \varnothing$.
We have found disjoint neighborhoods of x and y. Since x and y were arbitrary, X is Hausdorff. □

Proposition B.10.21. Let X be a vector space, and let $\mathcal{P}$ be a family of seminorms on X. The topology induced by $\mathcal{P}$ is Hausdorff if and only if $\mathcal{P}$ separates points, that is, if, for every $x \in X$ with $x \neq 0$, there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$. Equivalently,

$$\bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\} = \{0\}.$$

Proof. First suppose that the topology induced by $\mathcal{P}$ is Hausdorff. Let $x \in X$ with $x \neq 0$. Since the topology is Hausdorff, there are disjoint neighborhoods of the origin and of x. In particular, there is a neighborhood U of the origin such that $x \notin U$. By the definition of the induced topology, there exist $p_1, \dots, p_k \in \mathcal{P}$ and $r_1, \dots, r_k > 0$ such that

$$\bigcap_{i=1}^{k} \{y \in X \mid p_i(y) < r_i\} \subseteq U.$$

Since $x \notin U$, we also have

$$x \notin \bigcap_{i=1}^{k} \{y \in X \mid p_i(y) < r_i\}.$$

Therefore there exists $i \in \{1, \dots, k\}$ such that $p_i(x) \geq r_i > 0$. In particular, $p_i(x) \neq 0$. Thus the family $\mathcal{P}$ separates points.
Conversely, suppose that $\mathcal{P}$ separates points. Consider

$$N := \bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\}.$$

Since every seminorm satisfies $p(0) = 0$, we have $0 \in N$. On the other hand, if $x \neq 0$, the hypothesis gives $p \in \mathcal{P}$ such that $p(x) \neq 0$, so $x \notin \{y \in X \mid p(y) = 0\}$. Consequently, $x \notin N$. This proves that $N = \{0\}$.

We now show that N is closed. For each $p \in \mathcal{P}$, the set $\{x \in X \mid p(x) = 0\}$ is closed in the induced topology. If x_0 does not belong to this set, then $p(x_0) > 0$. Taking $r := \frac{p(x_0)}{2}$, the neighborhood

$$x_0 + \{y \in X \mid p(y) < r\}$$

of x_0 does not intersect $\{x \in X \mid p(x) = 0\}$. If $p(x) = 0$ and $p(x - x_0) < r$, then

$$p(x_0) = p(x_0 - x + x) \leq p(x_0 - x) + p(x) < r,$$

contradicting the choice of r. Therefore $\{x \in X \mid p(x) = 0\}$ is closed.
Since N is an intersection of closed sets, N is closed. But $N = \{0\}$, so $\{0\}$ is closed. By Proposition B.10.20, the topology induced by $\mathcal{P}$ is Hausdorff. □

Corollary B.10.22. Let $\mathcal{P}$ be a family of seminorms on a vector space X, and let $\tau_{\mathcal{P}}$ be the locally convex topology generated by $\mathcal{P}$. Then:

(a) $\tau_{\mathcal{P}}$ is Hausdorff if and only if $p(x) = 0$ for every $p \in \mathcal{P}$ implies $x = 0$.

(b) A set $A \subseteq X$ is topologically bounded if and only if the set $p(A) \subseteq \mathbb{R}$ is bounded in $\mathbb{R}$ for every $p \in \mathcal{P}$.

A topological vector space is called *seminormable* if its topology can be induced by a seminorm p. The space (X, p) is called a *seminormed* space.

Proof. The first assertion is Proposition B.10.21.
For (b), first suppose that A is topologically bounded. If $p \in \mathcal{P}$, then $\{x \in X \mid p(x) < 1\}$ is a neighborhood of the origin. There exists $t > 0$ such that $A \subseteq t\{x \in X \mid p(x) < 1\}$. Therefore, if $a \in A$, we have $p(a) < t$, so $p(A)$ is bounded.
Conversely, suppose that $p(A)$ is bounded for every $p \in \mathcal{P}$. Let U be a neighborhood of the origin. Since the topology is generated by $\mathcal{P}$, there is a basic neighborhood

$$B = \bigcap_{i=1}^{k} \{x \in X \mid p_i(x) < r_i\}$$

such that $B \subseteq U$. By hypothesis, for each i there exists $M_i > 0$ such that $p_i(a) \leq M_i$ for every $a \in A$. Choose $t > 0$ such that $M_i < t r_i$ for every i. Then $A \subseteq tB \subseteq tU$. Therefore A is topologically bounded. □

Proposition B.10.23. Let X be a vector space, and let $\mathcal{P}$ be a family, not necessarily countable, of seminorms on X. Denote by $\tau_{\mathcal{P}}$ the topology induced by $\mathcal{P}$. Then:

(a) A sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ in $\tau_{\mathcal{P}}$ if and only if $p(x_n - x) \to 0$ for every $p \in \mathcal{P}$.

(b) A sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $\tau_{\mathcal{P}}$ if and only if $p(x_n - x_m) \to 0$ as $n, m \to \infty$, for every $p \in \mathcal{P}$.

Proof. We first prove the convergence assertion. Suppose that $x_n \to x$ in $\tau_{\mathcal{P}}$. Let $p \in \mathcal{P}$ and $\varepsilon > 0$. The set

$$x + \{y \in X \mid p(y) < \varepsilon\}$$

is a neighborhood of x. By convergence of the sequence, there exists $N \in \mathbb{N}$ such that, if $n \geq N$, then x_n belongs to this neighborhood. Therefore $p(x_n - x) < \varepsilon$ for every $n \geq N$. This proves that $p(x_n - x) \to 0$.
Conversely, suppose that $p(x_n - x) \to 0$ for every $p \in \mathcal{P}$. Let U be a neighborhood of x. By the definition of the induced topology, there exist $p_1, \ldots, p_k \in \mathcal{P}$ and $r_1, \ldots, r_k > 0$ such that

$$x + \bigcap_{i=1}^{k} \{y \in X \mid p_i(y) < r_i\} \subseteq U.$$

For each $i \in \{1, \ldots, k\}$, since $p_i(x_n - x) \to 0$, there exists $N_i \in \mathbb{N}$ such that $p_i(x_n - x) < r_i$ for every $n \geq N_i$. Taking $N := \max\{N_1, \ldots, N_k\}$ gives, for every $n \geq N$,

$$x_n - x \in \bigcap_{i=1}^{k} \{y \in X \mid p_i(y) < r_i\}.$$

Therefore $x_n \in U$ for every $n \geq N$. Since U was an arbitrary neighborhood of x, we conclude that $x_n \to x$ in $\tau_{\mathcal{P}}$.
We now prove the assertion about Cauchy sequences. Suppose that $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $\tau_{\mathcal{P}}$. Let $p \in \mathcal{P}$ and $\varepsilon > 0$. The set $\{y \in X \mid p(y) < \varepsilon\}$ is a neighborhood of the origin. By the Cauchy property, there exists $N \in \mathbb{N}$ such that, if $n, m \geq N$, then $x_n - x_m \in \{y \in X \mid p(y) < \varepsilon\}$. Therefore $p(x_n - x_m) < \varepsilon$ for any $n, m \geq N$. This proves that $p(x_n - x_m) \to 0$ as $n, m \to \infty$.
Conversely, suppose that, for every $p \in \mathcal{P}$, we have $p(x_n - x_m) \to 0$ as $n, m \to \infty$. Let U be a neighborhood of the origin. By the definition of the induced topology, there exist $p_1, \ldots, p_k \in \mathcal{P}$ and $r_1, \ldots, r_k > 0$ such that

$$\bigcap_{i=1}^{k} \{y \in X \mid p_i(y) < r_i\} \subseteq U.$$

For each $i \in \{1, \ldots, k\}$, there exists $N_i \in \mathbb{N}$ such that $p_i(x_n - x_m) < r_i$ for any $n, m \geq N_i$. Taking $N := \max\{N_1, \ldots, N_k\}$, we have, for any $n, m \geq N$,

$$x_n - x_m \in \bigcap_{i=1}^{k} \{y \in X \mid p_i(y) < r_i\} \subseteq U.$$

Therefore $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $\tau_{\mathcal{P}}$. □

Proposition B.10.24. (Metrization by a countable family of seminorms) Let X be a vector space, and let $(p_n)_{n \in \mathbb{N}_0}$ be a countable family of seminorms separating points. For $x, y \in X$, define

$$d(x, y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}.$$

Then:

(a) d is a translation-invariant metric on X.

(b) The topology induced by d agrees with the locally convex topology generated by the family $(p_n)_{n \in \mathbb{N}_0}$.

Proof. Define $\theta\colon [0,\infty) \longrightarrow [0,1)$ by

$$\theta(t) := \frac{t}{1+t}.$$

This function is continuous, strictly increasing, satisfies $\theta(0) = 0$, and has $\theta(t) = 0$ if and only if $t = 0$. It is also subadditive. If $s, t \geq 0$, then

$$\theta(s+t) = \frac{s+t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = \theta(s) + \theta(t).$$

For each $n \in \mathbb{N}_0$, define

$$\delta_n(x,y) := \theta(p_n(x-y)), \qquad x, y \in X.$$

Since p_n is a seminorm, we have

$$\delta_n(x+a, y+a) = \delta_n(x,y)$$

for any $x, y, a \in X$, so δ_n is translation invariant. Moreover, since $p_n(x-y) = p_n(y-x)$, it is also symmetric. We show that δ_n satisfies the triangle inequality. If $x, y, z \in X$, subadditivity of p_n and θ gives

$$\delta_n(x,z) = \theta(p_n(x-z)) \leq \theta(p_n(x-y) + p_n(y-z)) \leq \theta(p_n(x-y)) + \theta(p_n(y-z)) = \delta_n(x,y) + \delta_n(y,z).$$

Thus each δ_n is a translation-invariant pseudometric. In general, it is not a metric, since a seminorm may vanish on nonzero vectors.

Since $0 \leq \delta_n(x,y) < 1$ for every $n \in \mathbb{N}_0$ and any $x, y \in X$, the series

$$d(x,y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \delta_n(x,y)$$

converges, being dominated by the series $\displaystyle\sum_{n=0}^{\infty} 2^{-n}$. Symmetry and translation invariance of d follow term by term. Moreover, if $x, y, z \in X$, then

$$\delta_n(x,z) \leq \delta_n(x,y) + \delta_n(y,z)$$

for every $n \in \mathbb{N}_0$. Multiplying by 2^{-n} and summing over n gives

$$d(x,z) \leq d(x,y) + d(y,z).$$

Thus d satisfies the triangle inequality.

It remains to verify that $d(x,y) = 0$ implies $x = y$. If $d(x,y) = 0$, all terms of the series are nonnegative, so necessarily $\delta_n(x,y) = 0$ for every $n \in \mathbb{N}_0$. Since $\theta(t) = 0$ if and only if $t = 0$, it follows that $p_n(x-y) = 0$ for every $n \in \mathbb{N}_0$. The family $(p_n)_{n\in\mathbb{N}_0}$ separates points, so $x - y = 0$, that is, $x = y$. Therefore d is a translation-invariant metric. This proves (a).

We now prove (b). Denote by τ the locally convex topology generated by the family $(p_n)_{n\in\mathbb{N}_0}$, and by τ_d the topology induced by the metric d. Since d is translation invariant, the topology τ_d is also translation invariant. On the other hand, τ is a topological vector space topology by Theorem B.10.19 and is therefore also translation invariant. It thus suffices to compare neighborhoods of the origin.

First, we show that every neighborhood of the origin in τ_d is a neighborhood of the origin in τ. It suffices

to prove this for the balls $B_d(0,\varepsilon)$, with $\varepsilon > 0$. Choose $N \in \mathbb{N}_0$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}.$$

Since $\theta(t) \to 0$ as $t \to 0^+$, for each $n \in \{0,\dots,N\}$ there exists $\eta_n > 0$ such that

$$\frac{1}{2^n}\theta(\eta_n) < \frac{\varepsilon}{2(N+1)}.$$

Define

$$V := \bigcap_{n=0}^{N} \{x \in X \mid p_n(x) < \eta_n\}.$$

Then V is a neighborhood of the origin in τ. If $x \in V$, then $p_n(x) < \eta_n$ for every $n \in \{0,\dots,N\}$, and monotonicity of θ gives

$$\begin{aligned} d(x,0) &= \sum_{n=0}^{N} \frac{1}{2^n}\theta(p_n(x)) + \sum_{n=N+1}^{\infty} \frac{1}{2^n}\theta(p_n(x)) \\ &< \frac{\varepsilon}{2} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon. \end{aligned}$$

Therefore $V \subseteq B_d(0,\varepsilon)$. Thus every ball centered at the origin for d contains a neighborhood of the origin in τ. Consequently, every neighborhood of the origin in τ_d is a neighborhood of the origin in τ.
Conversely, we show that every neighborhood of the origin in τ is a neighborhood of the origin in τ_d. It suffices to consider a basic neighborhood

$$W := \bigcap_{j=1}^{r} \{x \in X \mid p_{n_j}(x) < \varepsilon_j\},$$

where $n_1,\dots,n_r \in \mathbb{N}_0$ and $\varepsilon_1,\dots,\varepsilon_r > 0$. Define

$$\rho := \min_{1 \le j \le r} \frac{1}{2^{n_j}}\theta(\varepsilon_j).$$

Since each ε_j is positive and $\theta(t) > 0$ for $t > 0$, we have $\rho > 0$.
We claim that $B_d(0,\rho) \subseteq W$. If $x \in B_d(0,\rho)$, then $d(x,0) < \rho$. For each $j \in \{1,\dots,r\}$, we have

$$\frac{1}{2^{n_j}}\theta(p_{n_j}(x)) \le d(x,0) < \rho \le \frac{1}{2^{n_j}}\theta(\varepsilon_j).$$

Therefore,

$$\theta(p_{n_j}(x)) < \theta(\varepsilon_j).$$

Since θ is strictly increasing, $p_{n_j}(x) < \varepsilon_j$. This holds for every $j \in \{1,\dots,r\}$, so $x \in W$. Thus $B_d(0,\rho) \subseteq W$.
We have proved that the neighborhoods of the origin in τ and τ_d agree. Since both topologies are translation invariant, their neighborhoods at every point also agree. Consequently, $\tau = \tau_d$. □

Theorem B.10.25. Let X be a locally convex topological vector space whose topology is generated by a family of seminorms $\mathcal{P}$, and let $(Y, \|\cdot\|_Y)$ be a normed space. A linear operator $L\colon X \longrightarrow Y$ is continuous if and only if there exist a constant $C > 0$ and finitely many seminorms $p_1, \dots, p_k \in \mathcal{P}$ such that

$$\|L(x)\|_Y \le C \max_{1\le i\le k} p_i(x) \qquad \forall x \in X.$$

In particular, if the family of seminorms is increasing or directed, it suffices to find a single seminorm $p \in \mathcal{P}$ and a constant $C > 0$ such that

$$\|L(x)\|_Y \le C p(x) \qquad \forall x \in X.$$

Proof. First suppose that L is continuous. Then $L^{-1}(B_{d_Y}(0,1))$ is a neighborhood of the origin in X. Since the topology of X is generated by $\mathcal{P}$, there exist $p_1, \dots, p_k \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\{x \in X \mid p_i(x) < \varepsilon,\ i = 1, \dots, k\} \subseteq L^{-1}(B_{d_Y}(0,1)).$$

Let $q(x) := \max_{1\le i\le k} p_i(x)$. If $x \neq 0$ and $q(x) > 0$, then $\dfrac{\varepsilon}{2q(x)}x$ belongs to the preceding neighborhood, and therefore

$$\left\| L\left(\frac{\varepsilon}{2q(x)}x\right)\right\|_Y < 1.$$

This gives $\|L(x)\|_Y \le \dfrac{2}{\varepsilon}q(x)$. If $q(x) = 0$, then tx belongs to the preceding neighborhood for every $t > 0$, and linearity yields $L(x) = 0$. Thus the inequality holds for every $x \in X$ with $C = \frac{2}{\varepsilon}$.
Conversely, if there exist a constant $C > 0$ and seminorms $p_1, \dots, p_k \in \mathcal{P}$ such that

$$\|L(x)\|_Y \le C \max_{1\le i\le k} p_i(x),$$

then, given $\varepsilon > 0$, the set

$$\left\{ x \in X \,\middle|\, p_i(x) < \frac{\varepsilon}{C},\ i = 1, \dots, k \right\}$$

is a neighborhood of the origin mapped into $B_{d_Y}(0, \varepsilon)$. Thus L is continuous at 0 and, by linearity, at every point.
The final assertion follows because, if the family is increasing or directed, there exists a seminorm $p \in \mathcal{P}$ that simultaneously dominates $p_1, \dots, p_k$ up to inessential constants. □

The following proposition gives an abstract construction of locally convex topologies from a prescribed local basis.

Proposition B.10.26. Let X be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B}$ be a nonempty family of subsets of X satisfying:

(a) every $U \in \mathcal{B}$ is absolutely convex and absorbing;

(b) if $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$;

(c) for each $U \in \mathcal{B}$ and each $\lambda > 0$, there exists $W \in \mathcal{B}$ such that $W \subseteq \lambda U$.

Then there exists a unique topology τ such that (X, τ) is a locally convex topological vector space and $\mathcal{B}$ is

a neighborhood basis at the origin.

Proof. For each $U \in \mathcal{B}$, consider its Minkowski functional

$$p_U(x) := \inf\{\lambda > 0 \mid x \in \lambda U\}, \qquad x \in X.$$

Since each $U \in \mathcal{B}$ is absolutely convex and absorbing, Proposition B.10.17 ensures that p_U is a seminorm on X.
Define

$$\mathcal{P} := \{p_U \mid U \in \mathcal{B}\}.$$

By Theorem B.10.19, the family $\mathcal{P}$ determines a topology τ making X a locally convex topological vector space. We show that $\mathcal{B}$ is a local basis at the origin for this topology.
First take $U \in \mathcal{B}$. If $x \in \{y \in X \mid p_U(y) < 1\}$, there exists $\lambda \in (0,1)$ such that $x \in \lambda U$. Since U is balanced, $\lambda U \subseteq U$, and therefore

$$\{y \in X \mid p_U(y) < 1\} \subseteq U.$$

The set on the left-hand side is a neighborhood of the origin in τ, so U is also a neighborhood of the origin.
It remains to prove that every neighborhood of the origin contains an element of $\mathcal{B}$. It suffices to consider a basic neighborhood of the form

$$N = \bigcap_{i=1}^{k} \{x \in X \mid p_{U_i}(x) < r_i\},$$

where $U_1, \dots, U_k \in \mathcal{B}$ and $r_1, \dots, r_k > 0$. For each i, the third hypothesis gives $W_i \in \mathcal{B}$ such that $W_i \subseteq \frac{r_i}{2} U_i$. If $x \in W_i$, then $p_{U_i}(x) \le \frac{r_i}{2} < r_i$. Therefore $W_i \subseteq \{x \in X \mid p_{U_i}(x) < r_i\}$. Repeatedly applying the second hypothesis, we find $W \in \mathcal{B}$ such that

$$W \subseteq \bigcap_{i=1}^{k} W_i \subseteq N.$$

Thus every basic neighborhood of the origin contains an element of $\mathcal{B}$, so $\mathcal{B}$ is a local basis at the origin for τ.
Finally, uniqueness of τ follows from Proposition B.10.2, since a topological vector space topology is uniquely determined by any local basis at the origin. □

Linear operators were introduced in Definition B.2.4. If X and Y are topological vector spaces, we retain the notation $\mathcal{L}(X,Y)$ for the space of continuous linear operators from X to Y. In particular, $X' := \mathcal{L}(X,\mathbb{K})$ is the continuous dual of X, and its elements are the continuous linear functionals.

Remark B.10.27. The bilinear map

$$\langle \cdot, \cdot \rangle_{X',X} \colon X' \times X \longrightarrow \mathbb{K}, \qquad (L,x) \mapsto L(x)$$

is called the *duality pairing*. The dual space of X', denoted by X'', is called the *bidual* of X.

Definition B.10.28. Let X and Y be topological vector spaces. An operator $L \colon X \longrightarrow Y$ is said to be *bounded* if it maps topologically bounded subsets of X to topologically bounded subsets of Y.

In normed spaces, continuity and boundedness of linear operators are equivalent. In general topological vector spaces, the relationship is more subtle, although useful implications remain. In the metrizable case, continuity can be characterized by sequences.

Theorem B.10.29. Let X, Y be topological vector spaces, and let $L\colon X \longrightarrow Y$ be a linear operator. Consider the following properties:

(a) L is continuous.

(b) L is bounded.

(c) If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X and $x_n \to 0$, then $\{L(x_n) \mid n \in \mathbb{N}\} \subseteq Y$ is a topologically bounded set.

(d) If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X and $x_n \to 0$, then $L(x_n) \to 0$.

Then $(a) \Rightarrow (b) \Rightarrow (c)$. If, in addition, X is metrizable, then $(c) \Rightarrow (d) \Rightarrow (a)$ and all four properties are equivalent.

Proof. If L is continuous, it maps topologically bounded sets to topologically bounded sets. Let $A \subseteq X$ be bounded, and let U be a neighborhood of the origin in Y. By continuity of L at 0, there is a neighborhood V of the origin in X such that $L(V) \subseteq U$. Since A is bounded, there exists $t > 0$ such that $A \subseteq tV$. By linearity, $L(A) \subseteq tL(V) \subseteq tU$. Therefore $L(A)$ is bounded. This proves $(a) \Rightarrow (b)$.
If L is bounded and $x_n \to 0$ in X, the set $\{x_n \mid n \in \mathbb{N}\}$ is topologically bounded. Its image under L is therefore also topologically bounded. This proves $(b) \Rightarrow (c)$.
Now suppose that X is metrizable and (c) holds. We prove (d) by contradiction. Let $x_n \to 0$ in X, and suppose that $L(x_n)$ does not converge to 0. There is then a balanced neighborhood U of the origin in Y and a subsequence, again denoted by x_n, such that $L(x_n) \notin U$ for every n.
Since X is metrizable, we can choose the subsequence so that $nx_n \to 0$ in X. By hypothesis, the set $\{L(nx_n) \mid n \in \mathbb{N}\}$ must be topologically bounded. If this set were bounded, however, there would be $t > 0$ such that $nL(x_n) \in tU$ for every n. Taking $n > t$ and using balancedness of U would give $L(x_n) \in \frac{t}{n}U \subseteq U$, a contradiction. Therefore $L(x_n) \to 0$. This proves $(c) \Rightarrow (d)$.
Finally, if X is metrizable, the implication $(d) \Rightarrow (a)$ is precisely Proposition B.10.10. □

Theorem B.10.30. (Extension of linear operators) Let X be a metrizable topological vector space, let $D \subseteq X$ be a dense vector subspace, and let Y be a Hausdorff, sequentially complete topological vector space. If $L\colon D \longrightarrow Y$ is a continuous linear operator, where D has the topology induced by X, there is at most one continuous linear operator $\widetilde{L}\colon X \longrightarrow Y$ such that $\widetilde{L}(x) = L(x)$ for every $x \in D$. Moreover, if, for each $x \in X$ and every sequence $\{x_n\} \subseteq D$ with $x_n \to x$, the sequence $\{L(x_n)\}$ is Cauchy in Y, then this extension exists and is unique.

Proof. Uniqueness follows from density of D and the Hausdorff property of Y. If $\widetilde{L}_1$ and $\widetilde{L}_2$ are two continuous extensions of L, they agree on D. Since the set where they agree is closed and contains D, it is all of X.
Now suppose that the additional condition holds. Let $x \in X$. Since D is dense and X is metrizable, there is a sequence $\{x_n\} \subseteq D$ such that $x_n \to x$. By hypothesis, $\{L(x_n)\}$ is Cauchy in Y and, since Y is sequentially

complete, it converges to a point of Y. Define

$$\widetilde{L}(x) := \lim_{n\to\infty} L(x_n).$$

We show that this definition is independent of the chosen sequence. If $\{y_n\} \subseteq D$ also converges to x, the interleaved sequence $z_1 = x_1, z_2 = y_1, z_3 = x_2, z_4 = y_2, \dots$ converges to x. By hypothesis, $\{L(z_n)\}$ is Cauchy, so its two subsequences $\{L(x_n)\}$ and $\{L(y_n)\}$ have the same limit. Thus $\widetilde{L}$ is well defined.

Linearity follows by taking sequences in D converging to the corresponding points and using linearity of L before passing to the limit.

Finally, we prove continuity without assuming that Y is first countable. Fix a compatible translation-invariant metric d on X. Let $x_n \to 0$, and suppose, for a contradiction, that $\widetilde{L}(x_n) \not\to 0$. There are a neighborhood U of 0 in Y and a subsequence, which we do not rename, such that $\widetilde{L}(x_n) \notin U$ for every n. Choose a neighborhood V of 0 such that $V + V \subseteq U$.

For each n, the definition of $\widetilde{L}(x_n)$ gives a sequence of points of D converging to x_n whose images converge to $\widetilde{L}(x_n)$. Choose a term $d_n \in D$ of this sequence so that

$$d(d_n, x_n) < \frac{1}{n}, \qquad \widetilde{L}(x_n) - L(d_n) \in V.$$

Then $d_n \to 0$ in X. Continuity of L at D implies $L(d_n) \to 0$, so $L(d_n) \in V$ for every sufficiently large n. For these indices,

$$\widetilde{L}(x_n) \in L(d_n) + V \subseteq V + V \subseteq U,$$

a contradiction. Therefore $\widetilde{L}$ is sequentially continuous at the origin. Since X is metrizable, Proposition B.10.10 proves that $\widetilde{L}$ is continuous. □

The following geometric forms of the Hahn–Banach theorem complement the analytical form presented in the section on normed spaces and provide the separation results needed later. Their proofs, obtained by applying the analytical form to the Minkowski functional of a suitable convex open set, can be found in [123, 149].

Theorem B.10.31. (Hahn–Banach, first geometric form) Let X be a real locally convex topological vector space, and let $A, B \subseteq X$ be nonempty convex sets. Suppose that $\operatorname{Int}(A) \neq \varnothing$ and $B \cap \operatorname{Int}(A) = \varnothing$. Then there exists a continuous linear functional $L\colon X \longrightarrow \mathbb{R}$ such that $\sup_{x\in A} L(x) \leq \inf_{y\in B} L(y)$, and moreover $L(a) < \inf_{y\in B} L(y)$ for every $a \in \operatorname{Int}(A)$. In particular, $L \neq 0$.

Corollary B.10.32. Let X be a Hausdorff locally convex topological vector space. Then $X' \neq \{0\}$ if and only if X has a convex neighborhood of the origin strictly contained in X.

Theorem B.10.33. (Hahn–Banach, second geometric form) Let X be a real locally convex topological vector space, and let $C, K \subseteq X$ be disjoint nonempty convex sets, with C closed and K compact. Then there exist a continuous linear functional $L\colon X \longrightarrow \mathbb{R}$ and two numbers $a \in \mathbb{R}$ and $\varepsilon > 0$ such that $L(x) \leq a - \varepsilon$ for every $x \in C$ and $L(x) \geq a + \varepsilon$ for every $x \in K$.

Theorem B.10.34. (Mazur) Let X be a topological vector space, and let $M \subseteq X$ be a vector subspace. Let $K \subseteq X$ be a nonempty open convex subset such that $M \cap K = \varnothing$.

Then there exists a closed hyperplane $N \subseteq X$ such that

$$M \subseteq N \qquad \text{and} \qquad N \cap K = \varnothing.$$

Corollary B.10.35. (Separation of a subspace and an open convex set) Let X be a locally convex Hausdorff topological vector space. Let $M \subseteq X$ be a vector subspace, and let $U \subseteq X$ be a nonempty open convex subset such that $M \cap U = \varnothing$.
Then there exists a continuous linear functional $f \in X'$ such that

$$f(m) = 0 \text{ for every } m \in M, \text{ and } \operatorname{Re} f(x) > 0 \text{ for every } x \in U.$$

Definition B.10.36. Let (X, τ_X) and (Y, τ_Y) be topological vector spaces. We say that X *is continuously embedded in* Y, and write $X \hookrightarrow Y$, if X is a subspace of Y and the inclusion $\iota_X \colon X \longrightarrow Y$, given by $\iota_X(x) = x$, is continuous.

B.10.1 Fréchet spaces

Fréchet spaces generalize Banach spaces by allowing the topology to be determined by a countable family of seminorms instead of a single norm. This class arises naturally in the study of spaces of smooth functions and spaces of smooth sections.

Definition B.10.37. Let (X, τ_X) be a topological vector space. We say that X is a *Fréchet space* if the following properties hold:

(a) X is locally convex.

(b) The topology τ_X can be induced by a translation-invariant metric; that is, there exists a metric $d \colon X \times X \longrightarrow \mathbb{R}$ such that

$$d(x + z, y + z) = d(x, y)$$

for any $x, y, z \in X$.

(c) Some, and equivalently every, translation-invariant metric inducing τ_X is complete.

Theorem B.10.38. Let (X, τ_X) be a topological vector space. Then X is a Fréchet space if and only if the following properties hold:

(a) X is Hausdorff.

(b) The topology τ_X is induced by a countable family of seminorms $(p_j)_{j \in \mathbb{N}_0}$.

(c) X is complete with respect to some, and equivalently every, translation-invariant metric inducing this topology.

The equivalence between a complete invariant metric and a countable separating family of seminorms is proved in [123, 149].

Before turning to weak topologies, we recall two general topological constructions. The initial topology will be the natural tool for defining weak topologies, whereas the final topology will arise in the construction of inductive

limits.

B.10.2 Initial and final topologies

Definition B.10.39. (Initial topology) Let X be a set, let $\{(Y_i, \tau_i)\}_{i\in I}$ be a family of topological spaces, and, for each $i \in I$, let $f_i \colon X \longrightarrow Y_i$ be a map. The *initial topology* on X associated with the family $(f_i)_{i\in I}$ is the topology generated by the subbase

$$\mathcal{S}_{\text{in}} := \left\{ f_i^{-1}(U) \;\middle|\; i \in I,\ U \in \tau_i \right\}.$$

We denote it by τ_{in}.

Proposition B.10.40. The topology τ_{in} is the coarsest topology on X for which all the maps $f_i \colon (X, \tau_{\text{in}}) \longrightarrow (Y_i, \tau_i)$ are continuous.
Moreover, if (Z, τ_Z) is a topological space and $g \colon Z \longrightarrow X$ is a map, then $g \colon (Z, \tau_Z) \longrightarrow (X, \tau_{\text{in}})$ is continuous if and only if each composition $f_i \circ g \colon Z \longrightarrow Y_i$ is continuous.

Proof. By the definition of τ_{in}, if $i \in I$ and $U \subseteq Y_i$ is open, then $f_i^{-1}(U)$ is open in (X, τ_{in}). Therefore, each map $f_i \colon (X, \tau_{\text{in}}) \longrightarrow (Y_i, \tau_i)$ is continuous.
We show that τ_{in} is the coarsest topology with this property. Let τ be a topology on X such that all the maps $f_i \colon (X, \tau) \longrightarrow (Y_i, \tau_i)$ are continuous. Then, if U is open in Y_i, the set $f_i^{-1}(U)$ belongs to τ. Since τ_{in} is the topology generated by all these sets, it follows that $\tau_{\text{in}} \subseteq \tau$.
We now prove the characterization of continuity. First suppose that $g \colon (Z, \tau_Z) \longrightarrow (X, \tau_{\text{in}})$ is continuous. Since each f_i is continuous, the composition $f_i \circ g \colon Z \longrightarrow Y_i$ is continuous for every $i \in I$.
Conversely, suppose that each composition $f_i \circ g$ is continuous. To prove that g is continuous, it suffices to verify that the preimage under g of each element of the subbase $\mathcal{S}_{\text{in}}$ is open in Z. Let $f_i^{-1}(U) \in \mathcal{S}_{\text{in}}$, with U open in Y_i. Then

$$g^{-1}\left(f_i^{-1}(U)\right) = (f_i \circ g)^{-1}(U).$$

The set on the right-hand side is open in Z by the continuity of $f_i \circ g$. Therefore, g is continuous. □

Proposition B.10.41. Let X be a set, let $\{(Y_i, \tau_i)\}_{i\in I}$ be a family of topological spaces, and, for each $i \in I$, let $f_i \colon X \longrightarrow Y_i$ be a map. A net $(x_\alpha)_\alpha$ in X converges to $x \in X$ in the initial topology associated with $(f_i)_{i\in I}$ if and only if

$$f_i(x_\alpha) \to f_i(x)$$

in Y_i for every $i \in I$. In particular, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in the initial topology if and only if

$$f_i(x_n) \to f_i(x)$$

in Y_i for every $i \in I$.

Proof. If $x_\alpha \to x$ in the initial topology, then, since each f_i is continuous, we have $f_i(x_\alpha) \to f_i(x)$ for every $i \in I$.
Conversely, suppose that $f_i(x_\alpha) \to f_i(x)$ for every $i \in I$. Let U be a neighborhood of x in the initial topology.

By the definition of the initial topology, there exists a basic neighborhood V of x such that $x \in V \subseteq U$, where V is a finite intersection of subbasic elements. That is, there exist $i_1, \dots, i_k \in I$ and open sets $U_1 \subseteq Y_{i_1}, \dots, U_k \subseteq Y_{i_k}$ such that

$$x \in \bigcap_{j=1}^{k} f_{i_j}^{-1}(U_j) \subseteq U.$$

Since $f_{i_j}(x_\alpha) \to f_{i_j}(x)$ and U_j is a neighborhood of $f_{i_j}(x)$, for each j there exists α_j such that, if $\alpha \geq \alpha_j$, then $f_{i_j}(x_\alpha) \in U_j$. Since the index set is directed, there exists α_0 such that $\alpha_0 \geq \alpha_j$ for every j. Then, if $\alpha \geq \alpha_0$, we have $x_\alpha \in V \subseteq U$. Therefore, $x_\alpha \to x$.
The statement for sequences is the special case in which the net is indexed by $\mathbb{N}$. □

Definition B.10.42. (Final topology) Let Y be a set, let $\{(X_i, \sigma_i)\}_{i \in I}$ be a family of topological spaces, and, for each $i \in I$, let $g_i \colon X_i \longrightarrow Y$ be a map. The *final topology* on Y associated with the family $(g_i)_{i \in I}$ is the collection

$$\tau_{\text{fin}} := \left\{ U \subseteq Y \;\middle|\; g_i^{-1}(U) \in \sigma_i \text{ for every } i \in I \right\}.$$

Proposition B.10.43. The collection τ_{fin} is a topology on Y. Moreover, it is the finest topology on Y for which all the maps $g_i \colon (X_i, \sigma_i) \longrightarrow (Y, \tau_{\text{fin}})$ are continuous.
Furthermore, if (Z, τ_Z) is a topological space and $h \colon Y \longrightarrow Z$ is a map, then $h \colon (Y, \tau_{\text{fin}}) \longrightarrow (Z, \tau_Z)$ is continuous if and only if each composition $h \circ g_i \colon X_i \longrightarrow Z$ is continuous.

Proof. First we show that τ_{fin} is a topology. Since $g_i^{-1}(\varnothing) = \varnothing$ and $g_i^{-1}(Y) = X_i$ for every $i \in I$, we have $\varnothing, Y \in \tau_{\text{fin}}$.
Now let $(U_\lambda)_{\lambda \in \Lambda}$ be an arbitrary family of elements of τ_{fin}. For each $i \in I$,

$$g_i^{-1}\left(\bigcup_{\lambda \in \Lambda} U_\lambda \right) = \bigcup_{\lambda \in \Lambda} g_i^{-1}(U_\lambda).$$

Since each set $g_i^{-1}(U_\lambda)$ is open in X_i, the union on the right-hand side is open. Therefore,

$$\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau_{\text{fin}}.$$

If $U_1, \dots, U_r \in \tau_{\text{fin}}$, then, for each $i \in I$,

$$g_i^{-1}\left(\bigcap_{q=1}^{r} U_q \right) = \bigcap_{q=1}^{r} g_i^{-1}(U_q).$$

Since finite intersections of open sets are open, we conclude that

$$\bigcap_{q=1}^{r} U_q \in \tau_{\text{fin}}.$$

Thus, τ_{fin} is a topology.
By the definition of τ_{fin}, all the maps $g_i \colon (X_i, \sigma_i) \longrightarrow (Y, \tau_{\text{fin}})$ are continuous.

We show that τ_{fin} is the finest topology with this property. Let τ be a topology on Y such that all the maps $g_i\colon (X_i,\sigma_i)\longrightarrow (Y,\tau)$ are continuous. If $U\in\tau$, then $g_i^{-1}(U)$ is open in X_i for every $i\in I$. By the definition of τ_{fin}, it follows that $U\in\tau_{\mathrm{fin}}$. Therefore, $\tau\subseteq\tau_{\mathrm{fin}}$.
Finally, we prove the characterization of continuity. If $h\colon (Y,\tau_{\mathrm{fin}})\longrightarrow (Z,\tau_Z)$ is continuous, then each composition $h\circ g_i\colon X_i\longrightarrow Z$ is continuous.
Conversely, suppose that each composition $h\circ g_i$ is continuous. Let $V\subseteq Z$ be an open set. For each $i\in I$,

$$g_i^{-1}\left(h^{-1}(V)\right)=(h\circ g_i)^{-1}(V),$$

and this set is open in X_i. By the definition of τ_{fin}, we obtain $h^{-1}(V)\in\tau_{\mathrm{fin}}$. Since this holds for every open set $V\subseteq Z$, we conclude that h is continuous. □

Remark B.10.44. In some texts, the initial and final topologies associated with a family of maps are called the *weak topology* and the *strong topology*, respectively. In what follows, we will prefer the terms *initial* and *final* to avoid confusion with the weak and weak-$*$ topologies that arise in functional analysis.

B.11 Weak topologies

B.11.1 Initial topologies defined by linear operators

Weak topologies are special cases of the initial topologies introduced in Definition B.10.39. In particular, we shall use the universal property in Proposition B.10.40 and the characterization of convergence in Proposition B.10.41. When the maps determining the topology are linear, the construction can be described by seminorms.

Proposition B.11.1. Let X be a vector space over $\mathbb{K}$, and let $\{Y_i\}_{i\in I}$ be a family of locally convex topological vector spaces. For each $i\in I$, let $T_i\colon X\longrightarrow Y_i$ be a linear operator. Then the initial topology on X associated with $(T_i)_{i\in I}$ is a locally convex vector space topology. If the topology of Y_i is generated by a family of seminorms $\mathcal{P}_i$, then the initial topology on X is generated by the seminorms

$$x\longmapsto p(T_ix),\qquad i\in I,\quad p\in\mathcal{P}_i.$$

Proof. For $i\in I$ and $p\in\mathcal{P}_i$, define $q_{i,p}\colon X\longrightarrow[0,\infty)$ by $q_{i,p}(x):=p(T_ix)$. Since T_i is linear and p is a seminorm, $q_{i,p}$ is a seminorm on X. Let τ be the topology generated by the family $\{q_{i,p}\mid i\in I,\ p\in\mathcal{P}_i\}$. By Theorem B.10.19, τ makes X a locally convex topological vector space, and each $T_i\colon (X,\tau)\longrightarrow Y_i$ is continuous.
If σ is another topology on X for which all the T_i are continuous, then each seminorm $q_{i,p}=p\circ T_i$ is continuous with respect to σ. Thus $\tau\subseteq\sigma$, proving that τ is the initial topology associated with $(T_i)_{i\in I}$. □

Corollary B.11.2. Let X be a vector space, and let $\mathcal{T}$ be a family of linear operators $T\colon X\longrightarrow W_T$, where each W_T is a normed space. Then the initial topology on X associated with $\mathcal{T}$ is generated by the seminorms

$$x\longmapsto\|Tx\|_{W_T},\qquad T\in\mathcal{T}.$$

In particular, a neighborhood base at the origin consists of finite intersections of sets of the form

$$\{x \in X \mid \|T_j x\|_{W_{T_j}} < \varepsilon_j,\ j = 1, \dots, k\},$$

where $T_1, \dots, T_k \in \mathcal{T}$ and $\varepsilon_1, \dots, \varepsilon_k > 0$.

Proof. This is a special case of Proposition B.11.1, since the topology of a normed space is generated by its norm. □

Corollary B.11.3. Under the hypotheses of the preceding corollary, suppose also that each W_T is finite-dimensional. Then the initial topology is also generated by the seminorms

$$x \longmapsto |\ell(Tx)|, \qquad T \in \mathcal{T}, \quad \ell \in W_T'.$$

Proof. In finite dimensions, the topology of W_T is generated by the linear functionals in W_T'. The assertion follows from Proposition B.11.1 by composing these functionals with the operators T. □

Proposition B.11.4. Let X be a vector space, and let $\mathcal{T}$ be a family of linear operators $T\colon X \longrightarrow W_T$, where each W_T is a normed space. A net $(x_\alpha)_\alpha$ converges to $x \in X$ in the initial topology determined by $\mathcal{T}$ if and only if $\|T(x_\alpha) - T(x)\|_{W_T} \to 0$ for every $T \in \mathcal{T}$. The same characterization holds for sequences.

Proof. By the characterization of convergence in an initial topology, $x_\alpha \to x$ if and only if $T(x_\alpha) \to T(x)$ in W_T for every $T \in \mathcal{T}$. Since each W_T is normed, this is equivalent to $\|T(x_\alpha) - T(x)\|_{W_T} \to 0$ for every $T \in \mathcal{T}$. □

B.11.2 Weak topologies associated with dualities

Definition B.11.5. (Duality) Let X and Y be vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *duality* between X and Y is a bilinear map

$$\langle \cdot, \cdot \rangle\colon X \times Y \longrightarrow \mathbb{K}.$$

We say that the duality separates points of X if, for every $x \in X \setminus \{0\}$, there exists $y \in Y$ such that $\langle x, y \rangle \neq 0$. Similarly, it separates points of Y if, for every $y \in Y \setminus \{0\}$, there exists $x \in X$ such that $\langle x, y \rangle \neq 0$.

Definition B.11.6. (Weak topology associated with a duality) Let $\langle \cdot, \cdot \rangle\colon X \times Y \longrightarrow \mathbb{K}$ be a duality. The *weak topology* $\sigma(X, Y)$ on X is the initial topology associated with the linear maps

$$e_y\colon X \longrightarrow \mathbb{K}, \qquad e_y(x) := \langle x, y \rangle, \qquad y \in Y.$$

Equivalently, $\sigma(X, Y)$ is the coarsest topology on X for which all maps e_y, with $y \in Y$, are continuous.

Proposition B.11.7. (Description by seminorms) The topology $\sigma(X, Y)$ is generated by the seminorms

$$p_y(x) := |\langle x, y \rangle|, \qquad y \in Y.$$

In particular, $\sigma(X,Y)$ is locally convex and is Hausdorff if and only if the duality separates points of X.

Proof. The first assertion follows from Proposition B.11.1, applied to the family $(e_y)_{y\in Y}$. By Proposition B.10.21, the resulting topology is Hausdorff if and only if $p_y(x) = 0$ for every $y \in Y$ implies $x = 0$, which is equivalent to the duality separating points of X. □

Proposition B.11.8. Let $\langle\cdot,\cdot\rangle\colon X \times Y \longrightarrow \mathbb{K}$ be a duality. A net $(x_\alpha)_\alpha$ converges to $x \in X$ with respect to $\sigma(X,Y)$ if and only if

$$\langle x_\alpha, y\rangle \longrightarrow \langle x, y\rangle \qquad \text{for every } y \in Y.$$

The same characterization holds for sequences.

Proof. The topology $\sigma(X,Y)$ is the initial topology associated with the maps $e_y\colon X \longrightarrow \mathbb{K}$. The assertion follows from the characterization of convergence in initial topologies. □

If X is a locally convex topological vector space, we may apply the preceding construction to the natural duality between X and its continuous dual X'.

Definition B.11.9. (Weak topology of a locally convex space) Let X be a locally convex topological vector space. The *weak topology* on X is the topology $\sigma(X,X')$ associated with the duality

$$\langle x, L\rangle := L(x), \qquad x \in X, \quad L \in X'.$$

When a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ in this topology, we write $x_n \rightharpoonup x$.

Proposition B.11.10. (Description by seminorms) The weak topology $\sigma(X,X')$ is generated by the seminorms

$$p_L(x) := |L(x)|, \qquad L \in X'.$$

Proof. This is Proposition B.11.7 applied to the natural duality between X and X'. □

Proposition B.11.11. A net $(x_\alpha)_\alpha$ in X converges weakly to $x \in X$ if and only if $L(x_\alpha) \to L(x)$ for every $L \in X'$. In particular, $x_n \rightharpoonup x$ if and only if

$$L(x_n) \longrightarrow L(x) \qquad \text{for every } L \in X'.$$

Proof. This is Proposition B.11.8 applied to the natural duality between X and X'. □

The following properties allow weak convergence and weak compactness to be transported by the linear operators arising naturally in applications. The first shows that ordinary linear continuity already includes continuity with respect to weak topologies.

Proposition B.11.12. (Continuity with respect to weak topologies) Let X and Y be locally convex topological vector spaces, and let $T\colon X \longrightarrow Y$ be a continuous linear operator. Then

$$T : \big(X, \sigma(X,X')\big) \longrightarrow \big(Y, \sigma(Y,Y')\big)$$

is continuous. In particular, the image under T of a weakly compact set is weakly compact.

Proof. Let $L \in Y'$. Since T is linear and continuous, the composition $L \circ T : X \longrightarrow \mathbb{K}$ belongs to X'. By the definition of $\sigma(X, X')$, the functional $L \circ T$ is continuous on $\big(X, \sigma(X, X')\big)$. Since this holds for every $L \in Y'$ and $\sigma(Y, Y')$ is the initial topology determined by the functionals in Y', the universal property of the initial topology implies that

$$T : \big(X, \sigma(X, X')\big) \longrightarrow \big(Y, \sigma(Y, Y')\big)$$

is continuous. The last assertion follows because the continuous image of a compact set is compact. □

To relate weak convergence obtained through isometric embeddings to the weak topology of the subspace, we shall also use the following consequence of the Hahn–Banach theorem.

Proposition B.11.13. (Weak topology of a normed subspace) Let X be a normed space, and let $Y \subseteq X$ be a linear subspace equipped with the induced norm. Then the weak topology $\sigma(Y, Y')$ agrees with the topology induced by $\sigma(X, X')$ on Y. In particular, if a net $(y_\alpha)_\alpha \subseteq Y$ and an element $y \in Y$ satisfy $y_\alpha \rightharpoonup y$ in X, then $y_\alpha \rightharpoonup y$ in Y, and conversely.

Proof. The topology induced by $\sigma(X, X')$ on Y is generated by the seminorms

$$y \longmapsto |L(y)|, \qquad L \in X'.$$

Each restriction $L \restriction_Y$ belongs to Y'. Conversely, if $\ell \in Y'$, the Hahn–Banach theorem, in the form of Corollary B.2.29, provides a functional $L \in X'$ such that $L \restriction_Y = \ell$. Thus

$$\{L \restriction_Y \mid L \in X'\} = Y',$$

and the seminorms generating the induced topology are exactly those generating $\sigma(Y, Y')$. The characterization of convergence follows from equality of the two topologies. □

The following property describes the weak topology of a finite product. Finiteness is what allows a functional on the product to be recovered from a finite sum of functionals on its components.

Proposition B.11.14. (Weak topology of a finite product) Let $X_1, \dots, X_N$ be locally convex topological vector spaces, and let

$$X := \prod_{j=1}^{N} X_j$$

be equipped with the product topology. Then

$$\sigma(X, X') = \prod_{j=1}^{N} \sigma(X_j, X_j').$$

In particular, a net $x^\alpha = (x_1^\alpha, \dots, x_N^\alpha)$ converges weakly to $x = (x_1, \dots, x_N)$ in X if and only if

$$x_j^\alpha \rightharpoonup x_j \qquad \text{in } X_j$$

for every $j \in \{1,\dots,N\}$.

Proof. First suppose that $x^\alpha \rightharpoonup x$ in X. For each $j \in \{1,\dots,N\}$, the canonical projection

$$\pi_j \colon X \longrightarrow X_j$$

is linear and continuous in the original topologies. By Proposition B.11.12, it is also continuous in the weak topologies. Thus

$$x_j^\alpha = \pi_j(x^\alpha) \rightharpoonup \pi_j(x) = x_j \qquad \text{in } X_j.$$

Conversely, suppose that $x_j^\alpha \rightharpoonup x_j$ in X_j for every $j \in \{1,\dots,N\}$. Let $L \in X'$ and, for each j, let

$$\iota_j \colon X_j \longrightarrow X, \qquad \iota_j(z) := (0,\dots,0,z,0,\dots,0),$$

be the canonical inclusion into the jth component. Since ι_j is linear and continuous, $L_j := L \circ \iota_j$ belongs to X_j'. Moreover, for every $(z_1,\dots,z_N) \in X$,

$$L(z_1,\dots,z_N) = \sum_{j=1}^N L_j(z_j).$$

Consequently,

$$L(x^\alpha) = \sum_{j=1}^N L_j(x_j^\alpha) \longrightarrow \sum_{j=1}^N L_j(x_j) = L(x),$$

where passage to the limit is valid because the sum has finitely many terms. By Proposition B.11.11, $x^\alpha \rightharpoonup x$ in X. Thus the two topologies have exactly the same convergent nets and therefore agree. □

We also recall a topological characterization of lower semicontinuity that will be useful later.

Proposition B.11.15. (Characterization by sublevel sets) Let (X,τ) be a topological space, and let $F \colon X \longrightarrow (-\infty,+\infty]$. The following assertions are equivalent:

(a) F is τ-lower semicontinuous.

(b) For every $c \in \mathbb{R}$, the sublevel set

$$\{x \in X \mid F(x) \le c\}$$

is τ-closed.

Proof. By the topological characterization of lower semicontinuity, F is τ-lower semicontinuous if and only if $\{x \in X \mid F(x) > c\}$ is τ-open for every $c \in \mathbb{R}$. Since

$$\{x \in X \mid F(x) > c\} = X \setminus \{x \in X \mid F(x) \le c\},$$

this condition is equivalent to all sublevel sets of F being τ-closed. □

Theorem B.11.16. Let (X,τ) be a locally convex Hausdorff topological vector space, and let $C \subseteq X$ be convex. Then

$$\overline{C}^{\tau} = \overline{C}^{\sigma(X,X')}.$$

In particular, C is τ-closed if and only if it is weakly closed.

Proof. Since $\sigma(X,X') \subseteq \tau$, we have $\overline{C}^{\tau} \subseteq \overline{C}^{\sigma(X,X')}$. To prove the other inclusion, let $x \notin \overline{C}^{\tau}$. By the Hahn–Banach separation theorem, there exist $L \in X'$ and real numbers $\alpha < \beta$ such that

$$\operatorname{Re} L(y) \le \alpha < \beta \le \operatorname{Re} L(x) \qquad \text{for every } y \in \overline{C}^{\tau}.$$

Then $\{z \in X \mid \operatorname{Re} L(z) > \alpha\}$ is a weak neighborhood of x disjoint from C. Thus $x \notin \overline{C}^{\sigma(X,X')}$. □

Corollary B.11.17. Let (X,τ) be a locally convex Hausdorff topological vector space, and let $F\colon X \longrightarrow (-\infty,+\infty]$ be a convex, τ-lower semicontinuous function. Then F is weakly lower semicontinuous.

Proof. By Proposition B.11.15, for every $c \in \mathbb{R}$ the set $C_c := \{x \in X \mid F(x) \le c\}$ is τ-closed. Since F is convex, C_c is convex. By Theorem B.11.16, C_c is weakly closed. Applying Proposition B.11.15 again shows that F is weakly lower semicontinuous. □

The uniform boundedness principle was established in Theorem B.2.15; we now use it to relate weak boundedness to norm boundedness.

We shall also need the canonical embedding of a normed space into its bidual.

Proposition B.11.18. Let X be a normed space, and let $J\colon X \longrightarrow X''$ be the linear operator defined by

$$J(x)(L) := L(x), \qquad x \in X, \quad L \in X'.$$

Then $\|J(x)\|_{X''} = \|x\|_X$ for every $x \in X$. In particular, J is a linear isometry.

Proof. If $\|L\|_{X'} \le 1$, then $|J(x)(L)| = |L(x)| \le \|x\|_X$, so $\|J(x)\|_{X''} \le \|x\|_X$. If $x \neq 0$, Theorem B.2.29 provides $L \in X'$ such that $\|L\|_{X'} = 1$ and $L(x) = \|x\|_X$. Thus $\|J(x)\|_{X''} \ge \|x\|_X$. The case $x = 0$ is immediate. □

Theorem B.11.19. Let X be a Banach space, and let $A \subseteq X$. Then A is norm bounded if and only if it is bounded with respect to $\sigma(X,X')$.

Proof. If A is norm bounded, then $\sup_{x\in A} |L(x)| \le \|L\|_{X'} \sup_{x\in A} \|x\|_X < \infty$ for every $L \in X'$, so A is weakly bounded. Conversely, if A is weakly bounded, then $\sup_{x\in A} |L(x)| < \infty$ for every $L \in X'$. The family $\{J(x) \mid x \in A\} \subseteq X''$ is therefore pointwise bounded on X'. By Theorem B.2.15,

$$\sup_{x\in A} \|J(x)\|_{X''} < \infty.$$

Since $\|J(x)\|_{X''} = \|x\|_X$, we conclude that A is norm bounded. □

Proposition B.11.20. Let X be a Banach space. If $x_n \rightharpoonup x$ in X, then $(x_n)_{n\in\mathbb{N}}$ is bounded and

$$\|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X.$$

Proof. For each $L \in X'$, the sequence $(L(x_n))_{n\in\mathbb{N}}$ is convergent and hence bounded. Thus $\{x_n \mid n \in \mathbb{N}\}$ is weakly bounded and, by Theorem B.11.19, also norm bounded.
On the other hand, the norm is convex and continuous in the norm topology. By Corollary B.11.17, it is weakly lower semicontinuous. Consequently,

$$\|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X.$$

□

B.11.3 Weak-∗ topology and pointwise convergence

Definition B.11.21. (Weak-∗ topology) Let X be a locally convex topological vector space. The *weak-∗ topology* on X' is the initial topology $\sigma(X', X)$ associated with the maps

$$\mathrm{ev}_x \colon X' \longrightarrow \mathbb{K}, \qquad \mathrm{ev}_x(L) := L(x), \qquad x \in X.$$

When $L_n \to L$ in this topology, we write $L_n \overset{*}{\rightharpoonup} L$.

Proposition B.11.22. (Description by seminorms) The weak-∗ topology $\sigma(X', X)$ is generated by the seminorms

$$p_x(L) := |L(x)|, \qquad x \in X.$$

In particular, a net $(L_\alpha)_\alpha$ converges weakly-∗ to L if and only if

$$L_\alpha(x) \longrightarrow L(x) \qquad \text{for every } x \in X.$$

The same characterization holds for sequences.

Proof. The description by seminorms follows from Proposition B.11.7, applied to the duality between X' and X. The characterization of convergence follows from Proposition B.11.8. □

When the target space is not the scalar field, the analogous construction is called the topology of pointwise convergence.

Definition B.11.23. (Topology of pointwise convergence) Let X be a topological vector space, and let W be a finite-dimensional normed vector space. The *topology of pointwise convergence* on $\mathcal{L}(X, W)$ is the initial topology associated with the maps

$$\mathrm{ev}_x \colon \mathcal{L}(X, W) \longrightarrow W, \qquad \mathrm{ev}_x(T) := T(x), \qquad x \in X.$$

Proposition B.11.24. The topology of pointwise convergence on $\mathcal{L}(X, W)$ is generated by the seminorms

$$p_x(T) := \|T(x)\|_W, \qquad x \in X.$$

Since W is finite-dimensional, it is also generated by the seminorms

$$p_{x,\ell}(T) := |\ell(T(x))|, \qquad x \in X, \quad \ell \in W'.$$

In particular, a net $(T_\alpha)_\alpha$ converges to T in this topology if and only if

$$T_\alpha(x) \longrightarrow T(x) \qquad \text{in } W$$

for every $x \in X$. The same characterization holds for sequences.

Proof. The first description follows from Corollary B.11.2, applied to the maps ev_x. Since W is finite-dimensional, its topology is generated by the seminorms $w \mapsto |\ell(w)|$, with $\ell \in W'$, giving the second description. The characterization of convergence follows from Proposition B.11.4. □

Remark B.11.25. This construction arises naturally in spaces of distributions with values in a finite-dimensional vector space. If $\mathcal{E}$ is a test space, the space $\mathcal{L}(\mathcal{E}, W)$ is equipped with the topology of pointwise convergence, so $F_\alpha \to F$ if and only if

$$F_\alpha(\varphi) \longrightarrow F(\varphi) \qquad \text{in } W$$

for every $\varphi \in \mathcal{E}$. This is precisely weak convergence of W-valued distributions.

B.11.4 Weak compactness and reflexivity

The Banach–Alaoglu, Eberlein–Šmulian, and Kakutani theorems, together with the weak and weak-$*$ metrizability criteria stated in this subsection, can be found with proofs in [28, 123]. We include the elementary consequences used later.

Definition B.11.26. Let X be a topological vector space, and let $A \subseteq X$. The *polar* of A is the set

$$A^\circ := \{L \in X' \mid |L(x)| \le 1 \text{ for every } x \in A\}.$$

Theorem B.11.27. (Banach–Alaoglu) If U is a neighborhood of the origin in a locally convex topological vector space X, then U° is compact in the weak-$*$ topology $\sigma(X', X)$.

Corollary B.11.28. If X is a normed space, then the closed unit ball of X' is weakly-$*$ compact.

Proof. The closed unit ball of X' is the polar of the open unit ball of X. The assertion follows from Theorem B.11.27. □

Theorem B.11.29. Let X be a separable normed space, and let $K \subseteq X'$ be a bounded weakly-$*$ compact set. Then the weak-$*$ topology induced on K is metrizable. In particular, the closed unit ball of X' is weakly-$*$ compact and metrizable.

Proof. Let $\{x_j\}_{j\in\mathbb{N}}$ be a countable dense subset of X. For $L_1, L_2 \in K$, define

$$d(L_1, L_2) := \sum_{j=1}^{\infty} 2^{-j} \frac{|(L_1 - L_2)(x_j)|}{1 + |(L_1 - L_2)(x_j)|}.$$

If $d(L_1, L_2) = 0$, then $L_1 - L_2$ vanishes on the dense set $\{x_j \mid j \in \mathbb{N}\}$ and, by continuity, on all of X. Thus d is a metric.
Each summand is continuous in the weak-$*$ topology, and the series converges uniformly because it is dominated by $\sum_{j=1}^{\infty} 2^{-j}$. Consequently, the identity from K with the weak-$*$ topology to K with the topology induced by d is continuous. Since the first space is compact and the second Hausdorff, this identity is a homeomorphism. Thus d metrizes the weak-$*$ topology on K. □

Corollary B.11.30. Let X be a separable Banach space. Every bounded sequence in X' admits a weakly-$*$ convergent subsequence.

Proof. Every bounded sequence lies in a closed ball of X'. By Theorems B.11.27 and B.11.29, this ball is weakly-$*$ compact and metrizable. The conclusion follows from sequential compactness of compact metric spaces. □

Theorem B.11.31. Let X be a Banach space. The closed unit ball of X', equipped with the weak-$*$ topology, is metrizable if and only if X is separable.

Proposition B.11.32. Let X be a Banach space. If $L_n \overset{*}{\rightharpoonup} L$ in X', then $(L_n)_{n\in\mathbb{N}}$ is bounded and

$$\|L\|_{X'} \le \liminf_{n\to\infty} \|L_n\|_{X'}.$$

Proof. For each $x \in X$, the sequence $(L_n(x))_{n\in\mathbb{N}}$ is convergent and hence bounded. Theorem B.2.15 implies that $\sup_{n\in\mathbb{N}} \|L_n\|_{X'} < \infty$.
By Theorem B.11.27, closed balls in X' are weakly-$*$ compact and, since the weak-$*$ topology is Hausdorff, weakly-$*$ closed. These sets are precisely the sublevel sets of the norm. By Proposition B.11.15, the norm on X' is weakly-$*$ lower semicontinuous, giving the inequality. □

Proposition B.11.33. Let X be a Banach space. If X' is separable, then X is separable.

Proof. If $X' = \{0\}$, the dual characterization of the norm, Corollary B.2.30, implies that $X = \{0\}$. Suppose $X' \neq \{0\}$. Since X' is separable, its unit sphere contains a dense sequence $(\varphi_n)_{n\in\mathbb{N}}$. For each n, by the

definition of the dual norm, we may choose $x_n \in X$, $\|x_n\|_X = 1$, such that

$$|\varphi_n(x_n)| > \frac{1}{2}.$$

Let $Y := \overline{\operatorname{span}}\{x_n \mid n \in \mathbb{N}\}$. If $Y \neq X$, Hahn–Banach provides $\varphi \in X'$ such that $\|\varphi\| = 1$ and $\varphi \restriction_Y = 0$. Choose n with $\|\varphi_n - \varphi\| < \frac{1}{2}$. Since $x_n \in Y$,

$$|\varphi_n(x_n)| = |(\varphi_n - \varphi)(x_n)| \leq \|\varphi_n - \varphi\| \|x_n\| < \frac{1}{2},$$

which contradicts the choice of x_n. Thus $Y = X$. The subspace spanned by $\{x_n\}$ using rational coefficients—complex rational coefficients if $\mathbb{K} = \mathbb{C}$—is countable and dense in X, so X is separable. □

The converse is false. For example, $L^1(\mathbb{R}^n)$ is separable, but its dual $L^\infty(\mathbb{R}^n)$ is not.

Theorem B.11.34. Let X be a Banach space whose dual X' is separable. Then the closed unit ball of X, equipped with the weak topology $\sigma(X, X')$, is metrizable.

Definition B.11.35. Let X be a locally convex topological vector space. A set $K \subseteq X$ is *weakly sequentially compact* if every sequence in K admits a subsequence converging weakly to a point of K.

Theorem B.11.36. (Eberlein–Šmulian) Let X be a Banach space, and let $K \subseteq X$. Then K is weakly compact if and only if it is weakly sequentially compact.

Definition B.11.37. A Banach space X is *reflexive* if the canonical embedding $J : X \longrightarrow X''$ is surjective.

In this case, the canonical embedding identifies X isometrically with its bidual X''.

Theorem B.11.38. (Kakutani) A Banach space is reflexive if and only if its closed unit ball is weakly compact.

Corollary B.11.39. Let X be a reflexive Banach space. Every bounded sequence in X admits a weakly convergent subsequence.

Proof. Every bounded sequence lies in a closed ball. By Theorem B.11.38, this ball is weakly compact and, by Theorem B.11.36, weakly sequentially compact. □

Proposition B.11.40. A Banach space X is reflexive if and only if X' is reflexive.

Proof. First suppose that X is reflexive, and let $\Phi \in X'''$. Define $\varphi \in X'$ by

$$\varphi(x) := \Phi(J_X x), \qquad x \in X,$$

where $J_X : X \to X''$ is the canonical embedding. Given $x'' \in X''$, reflexivity provides $x \in X$ such that $x'' = J_X x$. Then

$$(J_{X'}\varphi)(x'') = x''(\varphi) = \varphi(x) = \Phi(J_X x) = \Phi(x'').$$

Thus $J_{X'}$ is surjective and X' is reflexive.
Conversely, suppose that X' is reflexive. The image $J_X(X)$ is a closed subspace of X'', since J_X is an isometry and X is complete. If $J_X(X) \neq X''$, the Hahn–Banach theorem provides a nonzero functional $\Phi \in X'''$ vanishing on $J_X(X)$. By reflexivity of X', there exists $\varphi \in X'$ such that $\Phi = J_{X'}\varphi$. For each $x \in X$,

$$0 = \Phi(J_X x) = (J_X x)(\varphi) = \varphi(x),$$

so $\varphi = 0$ and hence $\Phi = 0$, a contradiction. Consequently, J_X is surjective and X is reflexive. □

B.12 Topological tensor products and nuclear spaces

In this section we develop the part of topological tensor product theory needed for the Schwartz kernel theorem. The distinction between the projective, injective, and inductive topologies is essential: on the nuclear Fréchet spaces arising when supports are fixed, the projective and injective topologies agree; on the global LF spaces, however, the product representing separate continuity is the inductive one. The exposition follows Chapters 43, 50, and 51 of [149]. We include proofs of the elementary results used later and explicitly identify the single hereditary theorem on nuclearity taken from that reference.

B.12.1 The projective, injective, and inductive topologies

Let X and Y be vector spaces over $\mathbb{K}$. Recall that the algebraic tensor product $X \otimes Y$, together with the canonical bilinear map $(x, y) \mapsto x \otimes y$, is characterized by the universal property in Definition 1.10.5. When X and Y are locally convex, there are several natural topologies on this same algebraic vector space.

Definition B.12.1. (Projective tensor topology) Let X and Y be locally convex spaces. If p is a continuous seminorm on X and q is a continuous seminorm on Y, define, for $\mathbf{u} \in X \otimes Y$,

$$(p \otimes_\pi q)(\mathbf{u}) := \inf\left\{ \sum_{j=1}^{r} p(x_j) q(y_j) \;\middle|\; \mathbf{u} = \sum_{j=1}^{r} x_j \otimes y_j \right\}.$$

The locally convex topology generated by these seminorms is called the *projective tensor topology*; the resulting space is denoted by $X \otimes_\pi Y$ and its Hausdorff completion by $X \widehat{\otimes}_\pi Y$.

The infimum is taken over a nonempty set, since every element of the algebraic tensor product is a finite sum of elementary tensors. If one of the spaces is not Hausdorff, completion will always mean the completion of the quotient by the closure of $\{0\}$; all spaces in the applications will be Hausdorff.

Proposition B.12.2. (Universal property of the projective product) For every pair of continuous seminorms p and q, the map $p \otimes_\pi q$ is a seminorm and satisfies

$$(p \otimes_\pi q)(x \otimes y) = p(x) q(y).$$

Moreover, the projective topology is the finest locally convex topology on $X \otimes Y$ for which the canonical bilinear map $X \times Y \to X \otimes Y$ is continuous.
More generally, if Z is a locally convex space and $b \colon X \times Y \longrightarrow Z$ is bilinear, then b is continuous if and only

if its algebraic linearization

$$\widetilde{b}\colon X\otimes Y\longrightarrow Z,\qquad \widetilde{b}(x\otimes y):=b(x,y),$$

is continuous on $X\otimes_\pi Y$. If Z is complete, $\widetilde{b}$ extends uniquely to a continuous operator $X\widehat{\otimes}_\pi Y\to Z$.

Proof. Homogeneity of $p\otimes_\pi q$ follows by multiplying one factor in each tensor representation by the corresponding scalar. If $\mathbf{u},v\in X\otimes Y$, concatenating a representation of $\mathbf{u}$ with one of v gives a representation of $\mathbf{u}+v$; taking infima yields the triangle inequality.

For an elementary tensor, the representation $x\otimes y$ gives $(p\otimes_\pi q)(x\otimes y)\le p(x)q(y)$. For the reverse inequality, pass to the normed spaces $X_p:=X/\ker p$ and $Y_q:=Y/\ker q$. By Hahn–Banach, there exist functionals $\lambda\in X_p'$ and $\mu\in Y_q'$ of norm at most one such that $\lambda([x])=p(x)$ and $\mu([y])=q(y)$, after multiplying them by scalars of modulus one when necessary. If $x\otimes y=\sum_{j=1}^{N}x_j\otimes y_j$, then

$$p(x)q(y)=\left|\sum_{j=1}^{N}\lambda([x_j])\mu([y_j])\right|\le\sum_{j=1}^{N}p(x_j)q(y_j).$$

Taking the infimum over all representations gives equality.

The preceding equality implies that the canonical map is continuous for the projective topology. Now let τ be another locally convex topology on $X\otimes Y$ making this map continuous. If r is a continuous seminorm for τ, continuity at $(0,0)$ allows us to choose absolutely convex neighborhoods $U\subseteq X$ and $V\subseteq Y$ such that $r(x\otimes y)\le 1$ when $x\in U$ and $y\in V$. If p_U and q_V are their Minkowski functionals, homogeneity and an approximation argument give $r(x\otimes y)\le p_U(x)q_V(y)$. Thus, for $\mathbf{u}=\sum_{j=1}^{N}x_j\otimes y_j$,

$$r(\mathbf{u})\le\sum_{j=1}^{N}p_U(x_j)q_V(y_j),$$

and taking the infimum gives $r(\mathbf{u})\le(p_U\otimes_\pi q_V)(\mathbf{u})$. Hence every continuous seminorm for τ is continuous for the projective topology, proving its maximality.

If b is continuous and r is a continuous seminorm on Z, the same argument applied to $(x,y)\mapsto r(b(x,y))$ gives continuous seminorms p and q such that $r(b(x,y))\le p(x)q(y)$. Consequently,

$$r(\widetilde{b}(\mathbf{u}))\le(p\otimes_\pi q)(\mathbf{u}),$$

so $\widetilde{b}$ is continuous. The converse follows by composing $\widetilde{b}$ with the canonical bilinear map. The last assertion follows from the universal property of Hausdorff completion for continuous linear maps, which are uniformly continuous. □

Proposition B.12.3. (Projective product and finite products) Let $X_1,\dots,X_A$ and $Y_1,\dots,Y_B$ be locally convex Hausdorff spaces. If pr_a and pr_b denote the corresponding coordinate projections, the operator

$$\Theta:\left(\prod_{a=1}^{A}X_a\right)\otimes_\pi\left(\prod_{b=1}^{B}Y_b\right)\longrightarrow\prod_{a=1}^{A}\prod_{b=1}^{B}(X_a\otimes_\pi Y_b),\qquad\Theta(u):=\big((\mathrm{pr}_a\otimes\mathrm{pr}_b)u\big)_{a,b},$$

is a topological isomorphism. Consequently, it induces a topological isomorphism between the completions

$$\left(\prod_{a=1}^{A} X_a\right) \widehat{\otimes}_\pi \left(\prod_{b=1}^{B} Y_b\right) \cong \prod_{a=1}^{A}\prod_{b=1}^{B} \left(X_a \widehat{\otimes}_\pi Y_b\right).$$

Proof. Let $\iota_a \colon X_a \longrightarrow \prod_c X_c$ and $\jmath_b \colon Y_b \longrightarrow \prod_d Y_d$ be the coordinate inclusions. For each pair (a,b), the bilinear map

$$(x,y) \longmapsto \mathrm{pr}_a(x) \otimes \mathrm{pr}_b(y)$$

is continuous with values in $X_a \otimes_\pi Y_b$: it is the composition of the continuous projections with the canonical bilinear map. Proposition B.12.2 shows that $\mathrm{pr}_a \otimes \mathrm{pr}_b$ is continuous; since there are only finitely many pairs, Θ is continuous as well.

Algebraically, the inverse of Θ is given by

$$\Psi\big((u_{a,b})_{a,b}\big) := \sum_{a=1}^{A}\sum_{b=1}^{B} (\iota_a \otimes \jmath_b) u_{a,b}.$$

The same argument proves that each $\iota_a \otimes \jmath_b$ is continuous and, since the sum is finite, Ψ is continuous. For $x = (x_a)_a$ and $y = (y_b)_b$, we have

$$\Psi\Theta(x \otimes y) = \sum_{\substack{1\le a\le A\\ 1\le b\le B}} \iota_a(x_a) \otimes \jmath_b(y_b) = \left(\sum_{a=1}^{A} \iota_a(x_a)\right) \otimes \left(\sum_{b=1}^{B} \jmath_b(y_b)\right) = x \otimes y,$$

while the identity $\Theta\Psi = \mathrm{Id}$ is checked in each coordinate. Since elementary tensors span the algebraic tensor product, Θ and Ψ are inverses.

Every topological isomorphism between locally convex Hausdorff spaces extends, together with its inverse, to their completions. Finally, a net is Cauchy in a finite product if and only if each component is Cauchy; thus the completion of $\prod_{a,b}(X_a \otimes_\pi Y_b)$ is canonically identified with $\prod_{a,b}(X_a \widehat{\otimes}_\pi Y_b)$. This proves the second assertion. □

To define the injective topology, we need the notion of equicontinuity.

Definition B.12.4. (Equicontinuous family) Let X be a locally convex space. A subset $A \subseteq X'$ is *equicontinuous* if there exists a neighborhood U of the origin in X such that

$$|\lambda(x)| \le 1 \qquad \text{for every } \lambda \in A \text{ and every } x \in U.$$

Equivalently, the map $p_A(x) := \sup_{\lambda\in A} |\lambda(x)|$ is a continuous seminorm on X.

Justification of the equivalence. If A satisfies the first condition, then $p_A \le p_U$, where p_U is the Minkowski functional of an absolutely convex neighborhood contained in U; thus p_A is continuous. Conversely, if p_A is continuous, the set $\{x \in X \mid p_A(x) < 1\}$ is a neighborhood of the origin and $|\lambda(x)| < 1$ on it for every $\lambda \in A$. □

Every tensor $\mathbf{u} = \sum_{j=1}^{r} x_j \otimes y_j$ determines a bilinear form on $X' \times Y'$ by

$$\mathbf{u}(\lambda, \mu) := \sum_{j=1}^{r} \lambda(x_j)\mu(y_j).$$

The algebraic universal property shows that this expression is independent of the chosen representation.

Definition B.12.5. (Injective tensor topology) Let X and Y be locally convex Hausdorff spaces. For equicontinuous subsets $A \subseteq X'$ and $B \subseteq Y'$, define

$$\varepsilon_{A,B}(u) := \sup_{\lambda \in A,\, \mu \in B} |u(\lambda, \mu)|.$$

The locally convex topology generated by the seminorms $\varepsilon_{A,B}$ is called the *injective tensor topology*. Denote the resulting space by $X \otimes_\varepsilon Y$ and its completion by $X \widehat{\otimes}_\varepsilon Y$.

Proposition B.12.6. (Comparison between ε and π) The injective topology is coarser than the projective topology. More precisely, if $A \subseteq X'$ and $B \subseteq Y'$ are equicontinuous, there exist continuous seminorms p on X and q on Y such that

$$\varepsilon_{A,B}(\mathbf{u}) \le (p \otimes_\pi q)(\mathbf{u}) \qquad \text{for every } \mathbf{u} \in X \otimes Y.$$

In particular, the algebraic identity induces continuous operators

$$X \otimes_\pi Y \longrightarrow X \otimes_\varepsilon Y, \qquad X \widehat{\otimes}_\pi Y \longrightarrow X \widehat{\otimes}_\varepsilon Y.$$

Proof. By equicontinuity, there exist continuous seminorms p and q such that $|\lambda(x)| \le p(x)$ and $|\mu(y)| \le q(y)$ for $\lambda \in A$ and $\mu \in B$. If $\mathbf{u} = \sum_{j=1}^{N} x_j \otimes y_j$, then

$$|\mathbf{u}(\lambda, \mu)| \le \sum_{j=1}^{N} p(x_j) q(y_j).$$

Taking first the supremum over $A \times B$ and then the infimum over representations of $\mathbf{u}$ gives the inequality. The continuity assertions follow from it and from the universal property of completion. $\square$

The third topology we need represents separately continuous bilinear maps. For a subset $U \subseteq X \otimes Y$ and $x \in X$, $y \in Y$, write

$$U_x := \{v \in Y \mid x \otimes v \in U\}, \qquad {}_yU := \{u \in X \mid u \otimes y \in U\}.$$

Definition B.12.7. (Inductive tensor topology) Let X and Y be locally convex spaces. The *inductive tensor topology* ι on $X \otimes Y$ is the locally convex topology whose neighborhood base at the origin consists of absolutely convex absorbing subsets $U \subseteq X \otimes Y$ such that U_x is a neighborhood of the origin in Y for every $x \in X$ and ${}_yU$ is a neighborhood of the origin in X for every $y \in Y$. Denote the corresponding spaces by $X \otimes_\iota Y$ and $X \widehat{\otimes}_\iota Y$.

Proposition B.12.8. (Universal property of the inductive product) The topology ι is the finest locally convex topology for which the canonical map $X \times Y \to X \otimes Y$ is separately continuous. If Z is locally convex and $b\colon X \times Y \longrightarrow Z$ is bilinear, then b is separately continuous if and only if its linearization $\widetilde{b}\colon X \otimes_\iota Y \longrightarrow Z$ is continuous. Consequently,

$$\varepsilon \subseteq \pi \subseteq \iota.$$

Proof. The family described in the definition is stable under finite intersections and multiplication by nonzero scalars; moreover, for each of its elements U, the set $\frac{1}{2}U$ belongs to the same family and satisfies $\frac{1}{2}U + \frac{1}{2}U \subseteq U$. By Proposition B.10.26, it determines a locally convex topology.
For $x \in X$, the preimage of U under the linear map $y \mapsto x \otimes y$ is precisely U_x; similarly, the preimage under $x \mapsto x \otimes y$ is ${}_yU$. Thus the canonical map is separately continuous. If τ is another locally convex topology with this property and V is an absolutely convex neighborhood of the origin for τ, then V_x and ${}_yV$ are neighborhoods for every x and every y; hence V is a neighborhood for ι. This proves maximality.
Now let b be separately continuous, and let V be an absolutely convex neighborhood of the origin in Z. Then $U := \widetilde{b}^{-1}(V)$ is absolutely convex and absorbing. For each x, U_x is the preimage of V under the continuous map $y \mapsto b(x,y)$; likewise, ${}_yU$ is a neighborhood. Thus U is a neighborhood in $X \otimes_\iota Y$, and $\widetilde{b}$ is continuous. The converse follows by composition with the canonical map. Finally, every continuous bilinear map is separately continuous, so maximality of the two topologies gives $\pi \subseteq \iota$; the other inclusion was proved in Proposition B.12.6. □

Remark B.12.9. (A caution for *LF* spaces) Separate continuity does not generally imply joint continuity on the product of two *LF* spaces. For this reason, the global product of two test function spaces will be described using $\widehat{\otimes}_\iota$ rather than $\widehat{\otimes}_\pi$. The projective topology will be appropriate once supports are fixed, because the factors are then Fréchet spaces and the following proposition applies.

Proposition B.12.10. (Separate continuity on Fréchet spaces) Let X and Y be Fréchet spaces, let Z be a locally convex space, and let $b\colon X \times Y \longrightarrow Z$ be a separately continuous bilinear map. Then b is continuous. More precisely, if $(p_j)_{j\in\mathbb{N}}$ and $(q_j)_{j\in\mathbb{N}}$ are increasing families of seminorms defining the topologies of X and Y, respectively, then, for every continuous seminorm r on Z, there exist $C > 0$ and indices $k, \ell \in \mathbb{N}$ such that

$$r(b(x,y)) \le C\, p_k(x) q_\ell(y) \qquad \text{for every } x \in X \text{ and every } y \in Y.$$

Proof. Fix r. For $a, j \in \mathbb{N}$, define

$$S_{a,j} := \{x \in X \mid r(b(x,y)) \le a q_j(y) \text{ for every } y \in Y\}.$$

For each $x \in X$, continuity of the linear map $y \mapsto b(x,y)$ provides a and j such that $x \in S_{a,j}$; thus $X = \bigcup_{a,j\in\mathbb{N}} S_{a,j}$.
Each $S_{a,j}$ is closed: if $x_\nu \to x$ and $x_\nu \in S_{a,j}$, continuity of $x \mapsto b(x,y)$ for each y allows passage to the limit in the defining inequality.
Since X is a complete metric space, Baire's theorem implies that some $S_{a,j}$ has nonempty interior. The set $S_{a,j}$ is absolutely convex. If $x_0 + U \subseteq S_{a,j}$ for some neighborhood U of the origin, then $U - U \subseteq S_{a,j} - S_{a,j} \subseteq 2S_{a,j}$; consequently, $S_{a,j}$ contains a neighborhood of the origin. Thus there exist $k \in \mathbb{N}$ and $\delta > 0$ such that $p_k(x) < \delta$

implies $x \in S_{a,j}$. By homogeneity,

$$r(b(x,y)) \le \frac{2a}{\delta} p_k(x) q_j(y).$$

When $p_k(x) = 0$, we have $p_k(tx) < \delta$ for every $t > 0$; thus $tx \in S_{a,j}$ and $t\, r(b(x,y)) \le a q_j(y)$. Letting $t \to \infty$ gives $r(b(x,y)) = 0$. This proves the estimate with $C = \dfrac{2a}{\delta}$ and, by Proposition B.12.2, joint continuity. □

B.12.2 Nuclear spaces

Nuclearity allows us to compare the topologies π and ε. We begin with nuclear operators between Banach spaces.

Definition B.12.11. (Nuclear operator) Let G and H be Banach spaces. An operator $T \in \mathcal{L}(G,H)$ is *nuclear* if there exist sequences $(\lambda_j)_{j\in\mathbb{N}} \subseteq \mathbb{K}$, $(g_j')_{j\in\mathbb{N}} \subseteq G'$, and $(h_j)_{j\in\mathbb{N}} \subseteq H$ such that

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty, \qquad \|g_j'\|_{G'} \le 1, \qquad \|h_j\|_H \le 1,$$

and

$$Tg = \sum_{j=1}^{\infty} \lambda_j g_j'(g) h_j \qquad \text{for every } g \in G.$$

The series converges absolutely and uniformly for g in the unit ball of G.

Lemma B.12.12. (Ideal property) If $T\colon G \longrightarrow H$ is nuclear and $R\colon G_0 \longrightarrow G$ and $S\colon H \longrightarrow H_0$ are continuous operators between Banach spaces, then $S \circ T \circ R\colon G_0 \longrightarrow H_0$ is nuclear.

Proof. If $Tg = \displaystyle\sum_{j=1}^{\infty} \lambda_j g_j'(g) h_j$, then

$$(S \circ T \circ R) g_0 = \sum_{j=1}^{\infty} \lambda_j (g_j' \circ R)(g_0)\, S(h_j).$$

If R and S are nonzero, write $g_j' \circ R = \|R\| \widetilde{g}_j'$ and $S(h_j) = \|S\| \widetilde{h}_j$, with norms at most one, and absorb $\|R\|\|S\|$ into the coefficients. The case in which either operator is zero is immediate. □

Let X be a locally convex Hausdorff space, and let p be a continuous seminorm. Denote by X_p the completion of the normed space $X/\ker p$, whose norm is $\|[x]\|_p := p(x)$, and by $Q_p\colon X \longrightarrow X_p$ the canonical map. If $q \ge p$, the identity on X induces an operator of norm at most one

$$J_{q,p}\colon X_q \longrightarrow X_p, \qquad J_{q,p} Q_q x := Q_p x.$$

Definition B.12.13. (Nuclear space) A locally convex Hausdorff space X is *nuclear* if, for every continuous seminorm p on X, there exists a continuous seminorm $q \ge p$ such that the operator $J_{q,p}\colon X_q \longrightarrow X_p$ is nuclear.

This definition is independent of the family of seminorms used to describe the topology: if a cofinal family of seminorms satisfies the condition, every continuous seminorm is dominated, up to a positive constant, by a finite

maximum of members of that family, and the ideal property allows the condition to be transferred to it.

Theorem B.12.14. (Nuclearity and equality of the tensor product topologies) Let X be a nuclear space and let Y be a Hausdorff locally convex space. Then the projective and injective topologies coincide on $X \otimes Y$. In particular, the algebraic identity induces a canonical topological isomorphism

$$X \widehat{\otimes}_\pi Y \cong X \widehat{\otimes}_\varepsilon Y.$$

Proof. By Proposition B.12.6, we need only prove that each projective seminorm is continuous for the injective topology. Let p and r be continuous seminorms on X and Y, respectively. Choose $q \geq p$ so that $J_{q,p} \colon X_q \longrightarrow X_p$ is nuclear and write

$$J_{q,p} z = \sum_{j=1}^{\infty} \lambda_j \alpha_j(z) z_j, \qquad \sum_{j=1}^{\infty} |\lambda_j| < \infty, \quad \|\alpha_j\| \leq 1, \quad \|z_j\|_{X_p} \leq 1.$$

The functionals $a_j := \alpha_j \circ Q_q$ form an equicontinuous subset $A \subseteq X'$, since $|a_j(x)| \leq q(x)$. Let $B \subseteq Y'$ be the image, under composition with Q_r, of the unit ball of Y_r'. This set is also equicontinuous, and Hahn–Banach gives

$$r(y) = \sup_{b \in B} |b(y)|.$$

If $\mathbf{u} = \sum_{i=1}^{s} x_i \otimes y_i$, then in $X_p \widehat{\otimes}_\pi Y_r$ we have

$$\sum_{i=1}^{s} Q_p x_i \otimes Q_r y_i = \sum_{j=1}^{\infty} \lambda_j z_j \otimes \left(\sum_{i=1}^{s} a_j(x_i) Q_r y_i \right).$$

By the definition of the projective norm and the dual characterization of r,

$$\begin{aligned} (p \otimes_\pi r)(\mathbf{u}) &\leq \sum_{j=1}^{\infty} |\lambda_j| r \left(\sum_{i=1}^{s} a_j(x_i) y_i \right) \\ &\leq \left(\sum_{j=1}^{\infty} |\lambda_j| \right) \sup_{a \in A,\, b \in B} |\mathbf{u}(a,b)| \\ &= \left(\sum_{j=1}^{\infty} |\lambda_j| \right) \varepsilon_{A,B}(\mathbf{u}). \end{aligned}$$

Thus $p \otimes_\pi r$ is continuous for ε. We conclude that $\pi = \varepsilon$ on the algebraic product. Their Hausdorff completions are therefore canonically isomorphic. □

We shall need two stability properties. We state them only in the generality used below.

Proposition B.12.15. (Subspaces and finite products) Every subspace of a nuclear space, equipped with the subspace topology, is nuclear. Every finite product of nuclear spaces is nuclear. Consequently, nuclearity is preserved under topological isomorphisms and on passing to complemented or closed subspaces.

Inheritance by subspaces is the theorem in [149, (50.3)].

Proof. The assertion about subspaces is the hereditary theorem cited above. It does not follow solely from the ideal property, since the inclusion of the subspace into the codomain need not admit a continuous projection.
Now let $X = X_1 \times \cdots \times X_m$. Every continuous seminorm p on X is dominated by a seminorm of the form $p_1 + \cdots + p_m$, where p_i also denotes its pullback by the projection $X \to X_i$. For each i we choose $q_i \geq p_i$ so that J_{q_i,p_i} is nuclear and put $q := q_1 + \cdots + q_m$. The identifications

$$X_q \cong \prod_{i=1}^{m} (X_i)_{q_i}, \qquad X_{p_1+\cdots+p_m} \cong \prod_{i=1}^{m} (X_i)_{p_i}$$

show that $J_{q,p_1+\cdots+p_m}$ is the finite diagonal sum of the operators J_{q_i,p_i}. It is therefore nuclear: simply concatenate their nuclear representations. The ideal property allows us to pass from $p_1 + \cdots + p_m$ to p. This proves the assertion about finite products. Stability under topological isomorphisms follows by composing the canonical maps with the isomorphism and its inverse; complemented and closed subspaces are particular cases of the hereditary theorem. □

The basic model of a nuclear Fréchet space is the space of rapidly decreasing sequences. For $d \in \mathbb{N}$, let

$$s(\mathbb{Z}^d) := \left\{ a = (a_\nu)_{\nu \in \mathbb{Z}^d} \;\middle|\; p_k(a) := \sup_{\nu \in \mathbb{Z}^d} (1 + \|\nu\|)^k |a_\nu| < \infty \text{ for every } k \in \mathbb{N}_0 \right\}.$$

We equip it with the Fréchet topology defined by the increasing family $(p_k)_{k \in \mathbb{N}_0}$.

Theorem B.12.16. (Nuclearity of rapidly decreasing sequences) The space $s(\mathbb{Z}^d)$ is nuclear.

Proof. Fix $k \in \mathbb{N}_0$ and put $q := p_{k+d+1}$. For $\nu \in \mathbb{Z}^d$, let e_ν be the sequence equal to one at ν and zero at every other index. In the completion associated with q, the functional

$$\alpha_\nu(a) := (1 + \|\nu\|)^{k+d+1} a_\nu$$

has norm at most one, whereas $z_\nu := (1 + \|\nu\|)^{-k} e_\nu$ has norm one in the completion associated with p_k. If $\lambda_\nu := (1 + \|\nu\|)^{-d-1}$, then

$$\sum_{\nu \in \mathbb{Z}^d} \lambda_\nu < \infty$$

and, for every rapidly decreasing sequence a,

$$J_{q,p_k} a = \sum_{\nu \in \mathbb{Z}^d} \lambda_\nu \alpha_\nu(a) z_\nu.$$

Indeed, the term indexed by ν is exactly $a_\nu e_\nu$. This is a nuclear representation of J_{q,p_k}. Since k was arbitrary, $s(\mathbb{Z}^d)$ is nuclear. □

Theorem B.12.17. (Smooth Fourier series) Let $\mathbb{T}^d := \big(\mathbb{R}/(2\pi\mathbb{Z})\big)^d$. The Fourier coefficient map

$$\mathcal{F}_{\mathbb{T}} \colon C^\infty(\mathbb{T}^d) \longrightarrow s(\mathbb{Z}^d), \qquad \mathcal{F}_{\mathbb{T}}(f)_\nu := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i\langle \nu, x\rangle}\, dx,$$

is a topological isomorphism. Its inverse is given by

$$a \longmapsto \sum_{\nu \in \mathbb{Z}^d} a_\nu e^{i\langle \nu, x\rangle},$$

where the series and all its differentiated series converge uniformly. In particular, $C^\infty(\mathbb{T}^d)$ is nuclear.

Proof. If $f \in C^\infty(\mathbb{T}^d)$, integration by parts gives, for every multi-index α,

$$\nu^\alpha \widehat{f}(\nu) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} D^\alpha f(x) e^{-i\langle \nu, x\rangle}\, dx,$$

up to a complex factor of modulus one. Applying this identity to $(1-\Delta)^r f$, we obtain

$$(1 + \|\nu\|^2)^r |\widehat{f}(\nu)| \le C_{d,r} \max_{|\alpha| \le 2r} \|D^\alpha f\|_{L^\infty(\mathbb{T}^d)}.$$

Therefore $\mathcal{F}_{\mathbb{T}}(f)$ is rapidly decreasing and the Fourier map is continuous.
Conversely, let $a \in s(\mathbb{Z}^d)$. For a multi-index α, choose $N > |\alpha| + d$. Then

$$\sum_{\nu \in \mathbb{Z}^d} \|\nu\|^{|\alpha|} |a_\nu| \le p_N(a) \sum_{\nu \in \mathbb{Z}^d} \frac{\|\nu\|^{|\alpha|}}{(1+\|\nu\|)^N} < \infty.$$

The Weierstrass test shows that the series and its differentiated series of order α converge uniformly. The sum therefore defines a smooth function, and the preceding estimate proves continuity of the inverse map. Uniqueness of Fourier coefficients shows that the two maps are inverses. Nuclearity follows from Theorem B.12.16 and stability under topological isomorphisms.
The formulas were written for complex spaces. In the real case, the terms ν and $-\nu$ are grouped into sines and cosines; the same estimates prove the result and nuclearity. □

Proposition B.12.18. (Injectivity of ε) Let $X_0 \hookrightarrow X$ and $Y_0 \hookrightarrow Y$ be linear topological embeddings between Hausdorff locally convex spaces. Then the canonical algebraic map

$$X_0 \otimes_\varepsilon Y_0 \longrightarrow X \otimes_\varepsilon Y$$

is a topological embedding. The induced map between the completions is also a topological embedding.

Proof. Algebraic injectivity follows from exactness of the tensor product of vector spaces over a field. We prove the topological assertion. Let $A_0 \subseteq X_0'$ be equicontinuous. There is a continuous seminorm p_0 on X_0 such that $|\lambda_0(x)| \le p_0(x)$ for $\lambda_0 \in A_0$. Since the topology of X_0 is induced by X, there are a continuous seminorm p on X and $C > 0$ such that $p_0(x) \le C p(x)$ on X_0. By Hahn–Banach, each $\lambda_0 \in A_0$ admits an extension $\lambda \in X'$ satisfying $|\lambda(x)| \le C p(x)$. Choosing one extension for each element gives an equicontinuous

set $A \subseteq X'$ whose restriction to X_0 contains A_0. The same argument applies to every equicontinuous set $B_0 \subseteq Y_0'$.

Consequently, each seminorm ε_{A_0,B_0} is dominated by the restriction of some seminorm $\varepsilon_{A,B}$ on $X \otimes_\varepsilon Y$. The reverse inequality for restricted seminorms is immediate, since restrictions of equicontinuous sets remain equicontinuous. Thus the topology of $X_0 \otimes_\varepsilon Y_0$ coincides with that induced by $X \otimes_\varepsilon Y$.

A uniform linear embedding between locally convex spaces extends to an embedding between their completions. Indeed, if the image of a Cauchy net converges to zero, the equalities of the induced seminorms show that the net converges to zero in the original space. This proves the final assertion. □

Lemma B.12.19. (Smooth extension from half-spaces and orthants) For $d \in \mathbb{N}$ and $r \in \{0,\dots,d\}$ put $G_{d,r} := \mathbb{R}^{d-r} \times [0,\infty)^r$. A function belongs to $C^\infty(G_{d,r})$ if it is smooth in the interior and all its ordinary derivatives extend continuously to each face. There is a continuous linear operator

$$E_{d,r} \colon C^\infty(G_{d,r}) \longrightarrow C^\infty(\mathbb{R}^d) \quad \text{such that} \quad (E_{d,r} f)|_{G_{d,r}} = f.$$

Continuity refers to the derivative seminorms on compact sets. If $K \subseteq G_{d,r}$ is compact, there is a compact set $\widehat{K} \subseteq \mathbb{R}^d$, independent of f, such that

$$\operatorname{supp} f \subseteq K \implies \operatorname{supp} E_{d,r} f \subseteq \widehat{K}, \qquad \|E_{d,r} f\|_{C^m(\mathbb{R}^d)} \le C_{m,r} \|f\|_{C^m(G_{d,r})}, \quad m \in \mathbb{N}_0.$$

The operators can be chosen so that extension in different groups of variables commutes, depends continuously on smooth parameters, and satisfies

$$E_{d+e,r+s}(f \boxtimes g) = (E_{d,r} f) \boxtimes (E_{e,s} g),$$

after the coordinate permutation that brings the normal variables together.

Proof. We first construct coefficients that match all jets simultaneously. For $N \in \mathbb{N}_0$ and $\ell \in \{0,\dots,N\}$ define

$$a_{\ell,N} := \prod_{\substack{j \in \{0,\dots,N\} \\ j \neq \ell}} \frac{1 + 2^j}{2^j - 2^\ell}.$$

Lagrange interpolation at the nodes $-2^0,\dots,-2^N$, evaluated at 1, gives

$$\sum_{\ell=0}^{N} a_{\ell,N} (-2^\ell)^m = 1, \qquad m \in \{0,\dots,N\}.$$

For each fixed ℓ there exists $a_\ell := \lim_{N\to\infty} a_{\ell,N}$. Indeed, for $j > \ell$ the factor is $(1+2^{-j})/(1-2^{\ell-j})$, and the products of its numerators and denominators converge to positive numbers. Let

$$P_+ := \prod_{j=0}^{\infty} (1 + 2^{-j}) < \infty, \qquad P_- := \prod_{j=1}^{\infty} (1 - 2^{-j}) > 0.$$

These convergences follow by taking logarithms: apart from finitely many factors, $|\log(1 \pm 2^{-j})| \le 2^{1-j}$.

Separating the factors $j<\ell$ and $j>\ell$ gives

$$|a_{\ell,N}| = 2^{-\ell(\ell+1)/2}\frac{\displaystyle\prod_{\substack{0\le j\le N\\ j\neq\ell}}(1+2^{-j})}{\displaystyle\prod_{j=1}^{\ell}(1-2^{-j})\prod_{j=1}^{N-\ell}(1-2^{-j})} \le \frac{P_+}{P_-^2}\,2^{-\ell(\ell+1)/2}.$$

The same bound holds for a_ℓ. In particular,

$$\sum_{\ell\in\mathbb{N}_0}|a_\ell|2^{\ell m}<\infty, \qquad \sum_{\ell\in\mathbb{N}_0}a_\ell(-2^\ell)^m = 1, \qquad m\in\mathbb{N}_0.$$

For the second equality, extend $a_{\ell,N}$ by zero when $\ell>N$ and pass to the limit in the finite identity using the preceding summable bound, with m fixed and $N\ge m$.

Fix $R>0$ and $\chi\in C_c^\infty((-R,R))$ equal to one near zero. For $f\in C^\infty(\mathbb{R}^{d-1}\times[0,\infty))$ put

$$(E_{d,1}f)(x',t) := \begin{cases} f(x',t), & t\ge 0,\\ \displaystyle\sum_{\ell\in\mathbb{N}_0}a_\ell\chi(-2^\ell t)f(x',-2^\ell t), & t<0.\end{cases}$$

For each $t<0$ only finitely many indices contribute. More precisely, if $\alpha\in\mathbb{N}_0^{d-1}$ and $q\in\mathbb{N}_0$, the derivative of the corresponding summand is

$$a_\ell(-2^\ell)^q\left[\partial_\tau^q\left(\chi(\tau)D_{x'}^\alpha f(x',\tau)\right)\right]_{\tau=-2^\ell t}.$$

On every compact set in the variables x', its supremum norm for $t\le 0$ is bounded by $C_q|a_\ell|2^{\ell q}$ multiplied by the largest of the norms of the derivatives of f of order at most $|\alpha|+q$ on the product of that compact set with $[0,R]$. The Weierstrass test allows us to differentiate the series and extend each differentiated series continuously to $t=0$. Since $\chi=1$ near zero, its value there is

$$\left(\sum_{\ell\in\mathbb{N}_0}a_\ell(-2^\ell)^q\right)D_{x'}^\alpha\partial_t^q f(x',0) = D_{x'}^\alpha\partial_t^q f(x',0).$$

Thus the limits of all derivatives agree from both sides. To justify the gluing, apply the fundamental theorem of calculus to each normal derivative across $t=0$: the primitive of the glued continuous derivative agrees with the function on both half-lines and at zero. By induction on the order, together with the tangential derivatives, the glued function is C^∞. The same estimates prove continuity for the local seminorms and, when the global norms are finite, the C^m bound in the statement.

If $\operatorname{supp} f\subseteq K$, the part of the extension with $t<0$ is supported in $\operatorname{pr}_{x'}K\times[-R,0]$, whereas the part with $t\ge 0$ is supported in K. Their union is a fixed compact set. The construction does not change the variables x'; when parameters are included among them, all their derivatives satisfy the same estimates.

For $r>1$ we apply this formula successively to each normal variable, treating the others as parameters, and for $r=0$ we take the identity. The derivative estimates are iterated finitely many times; the support remains in a fixed compact box. If two different variables are extended, the two double series and all

their differentiated series converge absolutely on compact sets, by the sums $\sum_{\ell\in\mathbb{N}_0}|a_\ell|2^{\ell m}$. Their order may therefore be interchanged. On a product $f(x)g(y)$ each operator acts only on its own factor; this proves the asserted commutation and product identity. □

Corollary B.12.20. (Nuclearity of test function spaces with fixed support) Let $U\subseteq\mathbb{R}^d$ be open and let $K\subseteq U$ be compact. Then $\mathcal{D}_K(U)$ is a nuclear Fréchet space.

Proof. Extension by zero topologically identifies $\mathcal{D}_K(U)$ with the space of functions in $C^\infty(\mathbb{R}^d)$ whose support is contained in K. Choose an open cube containing K whose closure is contained in the interior of a larger cube. After identifying opposite faces of the larger cube, periodic extension determines a topological embedding of $\mathcal{D}_K(U)$ into $C^\infty(\mathbb{T}^d)$. Its image is the subspace

$$\{f\in C^\infty(\mathbb{T}^d)\mid f=0 \text{ on } \mathbb{T}^d\setminus K\},$$

which is closed because convergence in C^∞ implies uniform convergence. Theorem B.12.17 and Proposition B.12.15 complete the proof. □

Theorem B.12.21. (Tensor products of smooth functions on tori) The map $(f,g)\mapsto((x,y)\mapsto f(x)g(y))$ induces a topological isomorphism

$$C^\infty(\mathbb{T}^m)\widehat{\otimes}_\pi C^\infty(\mathbb{T}^n)\cong C^\infty(\mathbb{T}^{m+n}).$$

Proof. Continuity of the bilinear map follows from

$$\|D_x^\alpha D_y^\beta(f\boxtimes g)\|_{L^\infty(\mathbb{T}^{m+n})}\le\|D^\alpha f\|_{L^\infty(\mathbb{T}^m)}\|D^\beta g\|_{L^\infty(\mathbb{T}^n)};$$

by Proposition B.12.2, we obtain a continuous operator from the completed projective product.
Using the Fourier isomorphism of Theorem B.12.17, it suffices to prove that

$$s(\mathbb{Z}^m)\widehat{\otimes}_\pi s(\mathbb{Z}^n)\longrightarrow s(\mathbb{Z}^{m+n}),\qquad e_\mu\otimes e_\nu\longmapsto e_{(\mu,\nu)},$$

is an isomorphism. If $a=(a_{\mu,\nu})\in s(\mathbb{Z}^{m+n})$, consider the series

$$R(a):=\sum_{\mu\in\mathbb{Z}^m}\sum_{\nu\in\mathbb{Z}^n}a_{\mu,\nu}e_\mu\otimes e_\nu.$$

For the seminorms p_k and p_ℓ of the two factors,

$$\sum_{\substack{\mu\in\mathbb{Z}^m\\ \nu\in\mathbb{Z}^n}}|a_{\mu,\nu}|p_k(e_\mu)p_\ell(e_\nu)\le p_N(a)\sum_{\substack{\mu\in\mathbb{Z}^m\\ \nu\in\mathbb{Z}^n}}\frac{(1+\|\mu\|)^k(1+\|\nu\|)^\ell}{(1+\|\mu\|+\|\nu\|)^N},$$

and the last series is finite if $N>k+\ell+m+n$. Therefore the series defining $R(a)$ converges absolutely for each projective seminorm, and R is continuous. On finitely supported sequences, it is immediate that R is the inverse of the canonical map. These sequences are dense in $s(\mathbb{Z}^d)$: if $a^{(R)}$ is obtained from a by setting

the terms with $\|\nu\| > R$ equal to zero, then, for each k, we have

$$p_k(a - a^{(R)}) \leq \frac{1}{1+R} p_{k+1}(a) \longrightarrow 0.$$

Thus tensors of finitely supported sequences are dense in the completed product, and continuity extends both identities to the completions. □

Lemma B.12.22. (Localized tensor products in Euclidean spaces) Let $K_0 \subseteq \mathrm{Int}(K_1) \subseteq U \subseteq \mathbb{R}^m$ and $L_0 \subseteq \mathrm{Int}(L_1) \subseteq V \subseteq \mathbb{R}^n$, where K_0, K_1, L_0, L_1 are compact. The canonical map

$$T_{K_1,L_1} : \mathcal{D}_{K_1}(U) \widehat{\otimes}_\pi \mathcal{D}_{L_1}(V) \longrightarrow \mathcal{D}_{K_1 \times L_1}(U \times V)$$

is a topological embedding. Its image contains $\mathcal{D}_{K_0 \times L_0}(U \times V)$, and the inverse map restricted to this latter space is continuous.

Proof. We begin by reducing the compact sets to the fundamental cubes of two tori. Since K_1 and L_1 are bounded, there are affine diffeomorphisms $A \colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $B \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$A(K_1) \subseteq (-\pi, \pi)^m, \qquad B(L_1) \subseteq (-\pi, \pi)^n.$$

The change of variables $f \mapsto f \circ A^{-1}$ is a topological isomorphism from $\mathcal{D}_{K_1}(U)$ onto $\mathcal{D}_{A(K_1)}(A(U))$. Indeed, the chain rule expresses each derivative of $f \circ A^{-1}$ as a linear combination, with constant coefficients, of derivatives of f of the same order. The inverse change satisfies the same estimates. The corresponding assertion holds for B and for $A \times B$. We may therefore replace the open and compact sets by their images while keeping the original notation. We shall thus assume from now on that

$$K_1 \subseteq (-\pi, \pi)^m, \qquad L_1 \subseteq (-\pi, \pi)^n.$$

The inclusions $K_0 \subseteq \mathrm{Int}(K_1)$ and $L_0 \subseteq \mathrm{Int}(L_1)$ allow us to choose $\chi \in \mathcal{D}_{K_1}(U)$ and $\eta \in \mathcal{D}_{L_1}(V)$ equal to one on neighborhoods of K_0 and L_0, respectively. If a function in $\mathcal{D}_{K_1}(U)$ is extended by zero to $\mathbb{R}^m$, the extension vanishes in a neighborhood of the boundary of $[-\pi, \pi]^m$. It therefore determines a smooth function on $\mathbb{T}^m = \mathbb{R}^m/(2\pi\mathbb{Z})^m$. We thus obtain a topological embedding

$$J_m \colon \mathcal{D}_{K_1}(U) \longrightarrow C^\infty(\mathbb{T}^m).$$

The embeddings J_n and J_{m+n} for the second factor and for the product are constructed in the same way. The topology induced by C^∞ coincides with the usual topology of each space with fixed support, since in both cases it is defined by the suprema of derivatives over a fixed compact set.
If $\Phi \in \mathcal{D}_{K_0 \times L_0}(U \times V)$ and

$$\Phi(x, y) = \sum_{\mu \in \mathbb{Z}^m,\, \nu \in \mathbb{Z}^n} c_{\mu,\nu}(\Phi) e^{i\langle \mu, x \rangle} e^{i\langle \nu, y \rangle}$$

is its periodic Fourier series, define

$$R^{K_1,L_1}_{K_0,L_0}\Phi := \sum_{\substack{\mu\in\mathbb{Z}^m\\ \nu\in\mathbb{Z}^n}} c_{\mu,\nu}(\Phi)\big(\chi e^{i\langle\mu,\cdot\rangle}\big)\otimes\big(\eta e^{i\langle\nu,\cdot\rangle}\big).$$

For each $a,b\in\mathbb{N}_0$ there are constants $C_a, C_b>0$ such that

$$\|\chi e^{i\langle\mu,\cdot\rangle}\|_{C^a}\le C_a(1+\|\mu\|)^a,\qquad \|\eta e^{i\langle\nu,\cdot\rangle}\|_{C^b}\le C_b(1+\|\nu\|)^b.$$

By integration by parts, for every $N\in\mathbb{N}$ there is $C_N>0$ such that

$$|c_{\mu,\nu}(\Phi)|\le C_N(1+\|\mu\|+\|\nu\|)^{-N}\max_{|\gamma|\le 2N}\|D^\gamma\Phi\|_{L^\infty(\mathbb{T}^{m+n})}.$$

Choosing $N>a+b+m+n$, the series defining $R^{K_1,L_1}_{K_0,L_0}\Phi$ converges absolutely for the projective seminorm $\|\cdot\|_{C^a}\otimes_\pi\|\cdot\|_{C^b}$. The same estimate shows that $R^{K_1,L_1}_{K_0,L_0}$ is continuous. Since $\chi(x)\eta(y)=1$ on $K_0\times L_0$, Theorem B.12.21 gives

$$T_{K_1,L_1}R^{K_1,L_1}_{K_0,L_0}\Phi=\chi(x)\eta(y)\Phi(x,y)=\Phi(x,y).$$

It remains to prove that T_{K_1,L_1} induces the correct topology on its image. By Corollary B.12.20 and Theorem B.12.14, in the domain of T_{K_1,L_1} we may replace $\widehat{\otimes}_\pi$ by $\widehat{\otimes}_\varepsilon$. The spaces $C^\infty(\mathbb{T}^m)$ and $C^\infty(\mathbb{T}^n)$ are also nuclear, so the same replacement is valid for them. Proposition B.12.18, applied to J_m and J_n, shows that

$$J_m\widehat{\otimes}J_n:\mathcal{D}_{K_1}(U)\widehat{\otimes}_\pi\mathcal{D}_{L_1}(V)\longrightarrow C^\infty(\mathbb{T}^m)\widehat{\otimes}_\pi C^\infty(\mathbb{T}^n)$$

is a topological embedding. If $T_{\mathbb{T}}$ denotes the isomorphism of Theorem B.12.21, then

$$J_{m+n}\circ T_{K_1,L_1}=T_{\mathbb{T}}\circ(J_m\widehat{\otimes}J_n).$$

The right-hand side is a topological embedding. Since J_{m+n} is as well, the equality proves that T_{K_1,L_1} is injective and that its inverse on the image is continuous. Therefore T_{K_1,L_1} is a topological embedding. Finally, the identity obtained above shows that $R^{K_1,L_1}_{K_0,L_0}$ agrees with its inverse on $\mathcal{D}_{K_0\times L_0}(U\times V)$. □

B.13 Ordinary differential equations in Banach spaces

The local theory of ordinary differential equations in Banach spaces retains the method of successive approximations. Completeness of the model space takes the place of completeness of Euclidean spaces, but no form of local compactness is available. This distinction will be essential when studying the continuation of integral curves on Banach manifolds.

All integrals in this section are Riemann integrals with values in a Banach space. For continuous integrands on compact intervals, they coincide with the corresponding Bochner integrals.

Let E be a real Banach space, let $\Omega\subseteq\mathbb{R}\times E$ be an open subset, and let $F\colon\Omega\longrightarrow E$. We say that F is **locally Lipschitz in the state variable, uniformly in time** if, for each $(t_0,x_0)\in\Omega$, there exist $a,r,L>0$ such that

$$[t_0-a,t_0+a]\times\overline{B}_E(x_0,r)\subseteq\Omega$$

and

$$\|F(t,x)-F(t,y)\|_E \le L\|x-y\|_E$$

for all $t \in [t_0-a, t_0+a]$ and $x, y \in \overline{B}_E(x_0, r)$. When a parameter λ in a Banach space P is present, the expression *uniformly in time and in the parameter* has the analogous meaning on products of the form

$$[t_0-a, t_0+a] \times \overline{B}_E(x_0, r) \times \overline{B}_P(\lambda_0, \rho).$$

Every function of class C^1 is locally Lipschitz in the state variable. Continuity of D_2F allows its norm to be bounded on a sufficiently small product around the point under consideration, and the mean value inequalities in Theorem 23.1.10 give the Lipschitz estimate on a convex ball.

Theorem B.13.1. (Picard–Lindelöf in Banach spaces) Let E be a Banach space, let $\Omega \subseteq \mathbb{R} \times E$ be open, and let $F\colon \Omega \longrightarrow E$ be continuous and locally Lipschitz in the state variable, uniformly in time. For each $(t_0, x_0) \in \Omega$, there exist $h > 0$ and a unique function

$$x \in C^1([t_0-h, t_0+h], E)$$

whose graph is contained in Ω and which satisfies

$$x'(t) = F(t, x(t)), \qquad x(t_0) = x_0.$$

More precisely, $a, r, L, M > 0$ can be chosen so that

$$Q := [t_0-a, t_0+a] \times \overline{B}_E(x_0, r) \subseteq \Omega, \qquad \|F(t,y)\|_E \le M$$

on Q and the Lipschitz constant with respect to y is L. Any $h > 0$ such that

$$h \le a, \qquad hM \le r, \qquad hL < 1.$$

then suffices. The solution is the unique element of $C([t_0-h, t_0+h], \overline{B}_E(x_0, r))$ satisfying the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\, ds. \tag{B.12}$$

Proof. By continuity of F at (t_0, x_0) and the local Lipschitz hypothesis, we can decrease a and r to obtain the cylinder Q, the bound M, and the constant L in the statement. Let

$$\mathcal{X} := \left\{ u \in C([t_0-h, t_0+h], E) \;\middle|\; \sup_{t \in [t_0-h, t_0+h]} \|u(t) - x_0\|_E \le r \right\}.$$

The space $C([t_0-h, t_0+h], E)$, equipped with the supremum norm, is a Banach space. For a Cauchy sequence of continuous functions, completeness of E gives a pointwise limit; convergence is uniform, so the limit is continuous. Consequently, $\mathcal{X}$ is a complete metric space, since it is a closed subset.

Define $\mathcal{T}\colon \mathcal{X} \longrightarrow C([t_0-h, t_0+h], E)$ by

$$(\mathcal{T}u)(t) := x_0 + \int_{t_0}^{t} F(s, u(s))\, ds.$$

The integral is well-defined by Theorem B.6.2. For $u \in \mathcal{X}$, we have

$$\|(\mathcal{T}u)(t) - x_0\|_E \le M|t - t_0| \le Mh \le r,$$

so that $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{X}$. Similarly, for $u, v \in \mathcal{X}$,

$$\|\mathcal{T}u - \mathcal{T}v\|_\infty \le hL\|u - v\|_\infty.$$

Thus, $\mathcal{T}$ is a contraction. Theorem 23.3.1 gives a unique fixed point $x \in \mathcal{X}$, which satisfies (B.12). Since the integrand is continuous, the fundamental theorem of calculus for Banach-valued functions, Theorem B.6.5, shows that $x \in C^1$ and $x'(t) = F(t, x(t))$.
Conversely, every solution taking values in $\overline{B}_E(x_0, r)$ satisfies (B.12) by the same fundamental theorem of calculus and is therefore a fixed point of $\mathcal{T}$. If y is any other solution with the same initial datum, continuity allows both solutions to be restricted to an interval on which they remain in a common contraction cylinder. The set of times at which x and y agree is closed; repeating this local argument at each of its points shows that it is also open. Connectedness of the common interval shows that the two solutions agree wherever both are defined. This proves the asserted uniqueness. □

The preceding uniqueness result is not restricted to the particular cylinder used in the construction.

Proposition B.13.2. (Uniqueness on common domains) Under the hypotheses of Theorem B.13.1, let $x_1\colon I_1 \longrightarrow E$ and $x_2\colon I_2 \longrightarrow E$ be two solutions, and suppose that $t_* \in I_1 \cap I_2$ and $x_1(t_*) = x_2(t_*)$. Then

$$x_1(t) = x_2(t)$$

for every t in the connected component of $I_1 \cap I_2$ containing t_*. In particular, two solutions with the same initial datum agree on the intersection of their intervals of definition.

Proof. Let J be the indicated component, and let $A := \{t \in J \mid x_1(t) = x_2(t)\}$. This set is closed in J by continuity of the solutions, and it is nonempty. If $s \in A$, apply the local construction in Theorem B.13.1 at the point $(s, x_1(s))$. By continuity, both solutions remain in the uniqueness cylinder when restricted to a sufficiently small interval around s; uniqueness of the fixed point implies that they agree on that interval. Hence A is also open in J. Connectedness of J implies that $A = J$. □

Theorem B.13.3. (Maximal solution) Under the hypotheses of Theorem B.13.1, for each $(t_0, x_0) \in \Omega$ there exists a unique maximal solution

$$x_{t_0,x_0}\colon I_{t_0,x_0} \longrightarrow E,$$

where $I_{t_0,x_0} \subseteq \mathbb{R}$ is an open interval containing t_0. Every other solution with initial datum (t_0, x_0) is a restriction of this solution.

Proof. Consider all solutions defined on open intervals containing t_0 and taking the value x_0 at t_0. Proposition B.13.2 shows that any two of them agree on the intersection of their domains. Their union therefore defines a function on the union I_{t_0,x_0} of all these intervals. This union is an open interval containing t_0, and the resulting function locally agrees with one of the solutions in the family; it is therefore a solution. By construction, it extends all the others and is maximal. Uniqueness again follows from Proposition B.13.2. □

The continuation criterion must be formulated in terms of convergence rather than compactness of closed balls, since closed balls are not compact in an infinite-dimensional Banach space.

Theorem B.13.4. (Continuation criterion) Let $x\colon (a,b) \longrightarrow E$ be a maximal solution of $x' = F(t,x)$, and suppose that $b < \infty$.

(a) If there exists $x_b \in E$ such that

$$x(t) \longrightarrow x_b \qquad \text{as } t \to b^-$$

and $(b, x_b) \in \Omega$, then x extends to an interval strictly containing (a,b), contradicting maximality.

(b) If there exist $t_1 \in (a,b)$ and a constant $M > 0$ such that

$$\|F(t,x(t))\|_E \le M \qquad \text{for } t \in [t_1, b),$$

then the limit $x_b := \lim\limits_{t\to b^-} x(t)$ exists. In particular, a maximal solution with a finite right endpoint can fail to extend only if $(b, x_b) \notin \Omega$.

The symmetric statements hold at a finite left endpoint.

Proof. First suppose that x_b exists and that $(b, x_b) \in \Omega$. Define $x(b) := x_b$. Passing to the limit in the integral equation between a fixed time $t_1 < b$ and $t < b$, using continuity of F, gives

$$x_b = x(t_1) + \int_{t_1}^{b} F(s, x(s))\, ds.$$

Theorem B.13.1, applied to the datum (b, x_b), produces a solution on an interval around b. Uniqueness allows it to be glued to x, giving a continuation beyond b. This proves (a).
For (b), if $t_1 \le s < t < b$, the fundamental theorem of calculus gives

$$\|x(t) - x(s)\|_E \le \int_s^t \|F(\tau, x(\tau))\|_E \, d\tau \le M|t - s|.$$

Thus, $x(t)$ is Cauchy as $t \to b^-$. Completeness of E gives a limit $x_b \in E$, and the last assertion follows from (a). The backward argument is obtained by reversing the direction of time. □

We now record dependence on data and parameters in the form that will be used to construct flows.

Theorem B.13.5. (Dependence on initial data and parameters) Let E and P be Banach spaces, let $\Omega \subseteq \mathbb{R} \times E \times P$ be open, and let $F\colon \Omega \longrightarrow E$ be continuous and locally Lipschitz in the state variable, uniformly in time and in the parameter. Denote by

$$x(t; t_0, x_0, \lambda)$$

the maximal solution of the problem

$$x'(t) = F(t, x(t), \lambda), \qquad x(t_0) = x_0.$$

Then the natural domain

$$\mathcal{D} := \left\{ (t, t_0, x_0, \lambda) \;\middle|\; t \in I_{t_0, x_0, \lambda} \right\}$$

is open in $\mathbb{R}^2 \times E \times P$, and the solution map

$$\mathcal{D} \longrightarrow E, \qquad (t, t_0, x_0, \lambda) \longmapsto x(t; t_0, x_0, \lambda),$$

is continuous.
If F is of class C^k, with $k \in \mathbb{N} \cup \{\infty\}$, the solution map is of class C^k. In particular, for $k \geq 1$, its derivatives with respect to the initial state and the parameter satisfy the variational equations

$$D_{x_0} x(t)h = h + \int_{t_0}^{t} D_2 F(s, x(s), \lambda)\big(D_{x_0} x(s)h\big)\, ds, \tag{B.13}$$

$$D_\lambda x(t)\mu = \int_{t_0}^{t} \big[D_2 F(s, x(s), \lambda)\big(D_\lambda x(s)\mu\big) + D_3 F(s, x(s), \lambda)\mu\big]\, ds. \tag{B.14}$$

Proof. Fix a datum (t_0, x_0, λ_0) and a compact interval contained in the maximal interval of the corresponding solution. The graph of this solution over the compact interval is compact, even though no neighborhood in the Banach space is compact. A finite cover by cylinders of local existence allows the interval to be divided into finitely many subintervals on which the same Lipschitz constant works for all sufficiently nearby data. On one of these subintervals, let x_* be the reference solution and let u be the solution corresponding to a nearby datum and parameter. Subtracting their integral equations, separating the term describing variation of the parameter along x_*, and applying Grönwall's inequality from Lemma 1.8.19 gives an estimate of the form

$$\sup_{t \in J} \|u(t) - x_*(t)\|_E \leq C_J \left(\|u(t_*) - x_*(t_*)\|_E + \sup_{t \in J} \|F(t, x_*(t), \lambda) - F(t, x_*(t), \lambda_0)\|_E \right).$$

The last supremum tends to zero as $\lambda \to \lambda_0$. The image of the map $t \mapsto (t, x_*(t), \lambda_0)$ is compact, and a finite subcover of neighborhoods on which continuity gives the required bound provides an estimate uniform for $t \in J$. Iterating the estimate over the finite partition gives continuous dependence on every compact interval. The same local construction shows that nearby data have solutions on a slightly larger interval; it follows that $\mathcal{D}$ is open. The change of variable $s = t - t_0$ converts variation of the initial time into variation of a parameter in the field

$$(s, x, t_0, \lambda) \longmapsto F(t_0 + s, x, \lambda),$$

so the same argument also proves continuous dependence on t_0.
Now suppose that F is of class C^k. On a sufficiently small compact interval $J = [t_0 - h, t_0 + h]$, consider the Banach space $C(J, E)$ and the map

$$\mathscr{F}(u, a, \lambda)(t) := u(t) - a - \int_{t_0}^{t} F(s, u(s), \lambda)\, ds.$$

In a neighborhood of the reference solution, $\mathscr{F}$ is of class C^k. This follows from the chain rule, continuity

of integration, and uniformity on the compact image of the curve; its derivatives are obtained by differentiating the integrand. Moreover,

$$D_1\mathscr{F}(u,a,\lambda)w(t) = w(t) - \int_{t_0}^{t} D_2F(s,u(s),\lambda)w(s)\,ds.$$

By decreasing h, we can make the norm of the integral operator on the right-hand side less than one. Theorem B.2.10 then shows that $D_1\mathscr{F}$ is invertible. Theorem 23.3.8 implies that the solution, regarded as an element of $C(J,E)$, depends in a C^k manner on (a,λ). Evaluation at each time is linear and continuous. Furthermore, the derivatives with respect to (a,λ) obtained in $C(J,E)$ are continuous functions of time, and their evaluation depends jointly on (t,a,λ). The identity

$$\partial_t x(t;a,\lambda) = F(t,x(t;a,\lambda),\lambda)$$

and the chain rule show inductively that all mixed derivatives of total order at most k exist and are continuous. This gives joint regularity in time. Dependence on t_0 reduces to the same argument through the change of variable $s = t - t_0$ and the field $(s,x,t_0,\lambda) \mapsto F(t_0+s,x,\lambda)$.
Finally, differentiating the integral equation with respect to a and λ gives (B.13) and (B.14). Continuation over an arbitrary compact interval is obtained by composing finitely many of these local solution maps. □

Corollary B.13.6. (Global existence for bounded fields) Let E be a Banach space, and let $F\colon \mathbb{R}\times E \longrightarrow E$ be continuous and locally Lipschitz in the state variable, uniformly in time. Suppose that there exists $M \geq 0$ such that

$$\|F(t,x)\|_E \leq M$$

for every $(t,x) \in \mathbb{R}\times E$. Then every maximal solution is defined on $\mathbb{R}$ and satisfies

$$\|x(t) - x(s)\|_E \leq M|t-s|$$

for all $s,t \in \mathbb{R}$. If F has C^k regularity, the regularity of the solution map is that given by Theorem B.13.5.

Proof. Let (a,b) be the maximal interval of a solution. If $b < \infty$, the bound on F and Theorem B.13.4 give a limit $x_b \in E$ as $t \to b^-$. Since the domain of the field is all of $\mathbb{R}\times E$, the datum (b,x_b) belongs to that domain and the solution extends, contradicting maximality. Hence $b = +\infty$. The same argument backward in time shows that $a = -\infty$. The estimate follows by integrating the equation between s and t. □

C
Linear algebra

This appendix collects some results from linear algebra used in the study of metrics, bilinear forms, operators, and tensors. We mainly follow the references [83, 122]; the reader may consult them for a more extensive treatment.

C.1 Bilinear forms and matrices

Bilinear forms and their associated matrices provide the starting point for the constructions considered in this appendix.

Definition C.1.1. Let V be a real vector space of dimension n. We say that $f : V \times V \longrightarrow \mathbb{R}$ is a bilinear form on V if:

(a) $f(x+y,z) = f(x,z) + f(y,z)$ and $f(\lambda x, y) = \lambda f(x,y)$ for any $x, y, z \in V$ and $\lambda \in \mathbb{R}$.

(b) $f(x, y+z) = f(x,y) + f(x,z)$ and $f(x, \lambda y) = \lambda f(x,y)$ for any $x, y, z \in V$ and $\lambda \in \mathbb{R}$.

If $\{e_1, \ldots, e_n\}$ is a basis of V, we define the matrix associated with the bilinear form f by

$$A_{ij} = f(e_i, e_j).$$

Proposition C.1.2. If V is a real vector space of dimension n, $f : V \times V \longrightarrow \mathbb{R}$ is a bilinear form, and $\beta = \{e_1, \ldots, e_n\}$ is a basis of V, then

$$f(x,y) = [x]_\beta^T A [y]_\beta,$$

for any $x, y \in V$, where A is the matrix associated with f and $[x]_\beta, [y]_\beta$ are the coordinate representations of x and y, respectively, in the basis β.

Proof. If $x = x^i e_i$ and $y = y^j e_j$, bilinearity gives

$$f(x,y) = x^i y^j f(e_i, e_j) = x^i A_{ij} y^j = [x]_\beta^T A [y]_\beta.$$

□

Definition C.1.3. Let V be a finite-dimensional real vector space of dimension n, and let $f : V \times V \longrightarrow \mathbb{R}$ be a bilinear form. We say that it is:

(a) Symmetric if $f(x,y) = f(y,x)$ for any $x, y \in V$. In that case, the associated matrix A is symmetric.

(b) Nondegenerate if $f(x,y) = 0$ for every $y \in V$ implies $x = 0$.

Example C.1.4. Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be of class $C^2(U)$. We define the Hessian matrix of f at $x \in U$ by $H(x) = \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x) \right) \in M_{n\times n}(\mathbb{R})$. Then $H(x)$ is a symmetric matrix and is associated with the bilinear form $\nabla^2 f(x) \colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (the second total derivative of f).

Definition C.1.5. A matrix $A \in M_{n\times n}(\mathbb{C})$ is Hermitian if $A = A^*$, where A^* is the matrix obtained by conjugating the entries of the transpose of A. A Hermitian matrix is *positive semidefinite* if $x^*Ax \geq 0$ for every $x \in \mathbb{C}^n$, and *positive definite* if $x^*Ax > 0$ for every $x \neq 0$. For real matrices, we use these terms for symmetric matrices satisfying the corresponding condition on $x \in \mathbb{R}^n$.

Proposition C.1.6. Every symmetric matrix $A \in M_{n\times n}(\mathbb{R})$ is Hermitian.

Proof. Since the entries are real, conjugation leaves them unchanged. Therefore, $A^* = \overline{A^T} = A^T = A$. □

Theorem C.1.7. Every $n \times n$ Hermitian matrix is diagonalizable by a unitary matrix and has real eigenvalues.

Proof. We proceed by induction on n. The characteristic polynomial of a complex matrix has a root, so A has an eigenvector $v \neq 0$, say $Av = \lambda v$. Since $A = A^*$,

$$\lambda\langle v, v\rangle = \langle Av, v\rangle = \langle v, Av\rangle = \overline{\lambda}\langle v, v\rangle,$$

and $\lambda \in \mathbb{R}$. After normalizing v, its orthogonal complement is invariant: if $w \perp v$, then

$$\langle Aw, v\rangle = \langle w, Av\rangle = \lambda\langle w, v\rangle = 0.$$

The restriction of A to $v^\perp$ remains Hermitian. The induction hypothesis gives an orthonormal basis of eigenvectors in $v^\perp$, which together with v diagonalizes A. □

Theorem C.1.8. (Rayleigh) Let $A \in M_{n\times n}(\mathbb{C})$ be Hermitian. Then

$$\lambda_{\max} = \max_{x\neq 0} \frac{x^*Ax}{x^*x} \text{ and } \lambda_{\min} = \min_{x\neq 0} \frac{x^*Ax}{x^*x},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of A, respectively, and x^* is the matrix obtained by transposing the vector x and taking the complex conjugate of each entry.

Proof. By the preceding theorem, there is a unitary matrix Q such that

$$Q^*AQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

If $y = Q^*x$, then

$$\frac{x^*Ax}{x^*x} = \frac{\sum_{i=1}^n \lambda_i |y_i|^2}{\sum_{i=1}^n |y_i|^2}.$$

This quotient is a convex combination of the eigenvalues, so it lies between $\lambda_{\min}$ and $\lambda_{\max}$. Both extrema are attained by taking a corresponding eigenvector. □

Theorem C.1.9. (Schur decomposition) Let $A \in M_{n\times n}(\mathbb{C})$. Then $A = QUQ^{-1}$, where Q is a unitary matrix (Q is invertible and $Q^{-1} = Q^*$) and U is an upper triangular matrix whose diagonal entries are the eigenvalues of A.

Proof. The proof is by induction on n. Choose a unit eigenvector v_1 of A and extend it to an orthonormal basis. In that basis, the first column of the matrix of A is $(\lambda_1, 0, \dots, 0)^T$, so the matrix has the form

$$\begin{pmatrix} \lambda_1 & * \\ 0 & A_1 \end{pmatrix}.$$

By the induction hypothesis, a unitary change of basis in $\operatorname{span}\{v_1\}^\perp$ triangularizes A_1. Extending it by the identity on $\operatorname{span}\{v_1\}$ gives a unitary matrix Q for which Q^*AQ is upper triangular. The diagonal entries are its eigenvalues because the characteristic polynomial of a triangular matrix is the product of $\lambda - U_{ii}$. □

Theorem C.1.10. Let $A \in M_{n\times n}(\mathbb{C})$, and let λ be an eigenvalue of A. If

$$\|A\|_F := \left(\sum_{i,j=1}^n |A_{ij}|^2 \right)^{\frac{1}{2}}$$

is the Frobenius norm of A, then $|\lambda| \le \|A\|_F$.

Proof. By Theorem C.1.9, there exist a unitary matrix Q and an upper triangular matrix U such that $A = QUQ^{-1}$. The identity $\|C\|_F^2 = \operatorname{tr}(C^*C)$ and the equality $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ give $\|Q^*AQ\|_F^2 = \operatorname{tr}(Q^*A^*AQ) = \operatorname{tr}(A^*A)$; hence $\|A\|_F = \|U\|_F$. Since the diagonal entries of U are the eigenvalues of A, we have

$$\|U\|_F^2 = \sum_{i=1}^n |U_{ii}|^2 + \sum_{1\le i<j\le n} |U_{ij}|^2 \ge |\lambda|^2.$$

Therefore, $|\lambda| \le \|U\|_F = \|A\|_F$. □

Theorem C.1.11. Let A be an $n \times n$ positive semidefinite Hermitian matrix, and let $k \in \mathbb{N}$, with $k \ge 2$. Then:

(a) There exists a unique positive semidefinite Hermitian matrix B such that $A = B^k$.

(b) There exists a polynomial p with real coefficients such that $B = p(A)$. Consequently, B commutes with every matrix that commutes with A.

(c) If A is real, then B is real.

Proof. By the spectral theorem,

$$A = Q\,\mathrm{diag}(\lambda_1,\dots,\lambda_n)Q^*, \qquad \lambda_i \geq 0,$$

with Q unitary. Define

$$B := Q\,\mathrm{diag}(\lambda_1^{1/k},\dots,\lambda_n^{1/k})Q^*.$$

Then B is positive semidefinite Hermitian and $B^k = A$.
If C is another positive semidefinite Hermitian matrix with $C^k = A$, then C commutes with $A = C^k$ and preserves each eigenspace E_λ of A. The restriction of C to E_λ is positive semidefinite Hermitian and satisfies $(C\restriction_{E_\lambda})^k = \lambda I$. Its eigenvalues are nonnegative and their kth powers equal λ; hence they all equal $\lambda^{1/k}$. The spectral theorem implies $C\restriction_{E_\lambda} = \lambda^{1/k} I$, so $C = B$.
Let $\mu_1,\dots,\mu_s$ be the distinct eigenvalues of A. Lagrange interpolation gives a real polynomial p with $p(\mu_j) = \mu_j^{1/k}$. Calculation on each eigenspace shows that $B = p(A)$. If a matrix commutes with A, it commutes with $p(A)$; and if A is real, so is $p(A)$. □

Theorem C.1.12. If A is a positive semidefinite Hermitian matrix, then A is positive definite if and only if A is invertible.

Proof. By the spectral theorem, A is unitarily similar to $\mathrm{diag}(\lambda_1,\dots,\lambda_n)$ with $\lambda_i \geq 0$. It is positive definite if and only if all the λ_i are strictly positive, and it is invertible if and only if none of them is zero. The two conditions are equivalent. □

Remark C.1.13. For every real symmetric positive definite matrix A, there exists a unique real symmetric positive definite matrix B such that $A = B^2$. This matrix is called the square root of A.

Proposition C.1.14. A symmetric positive (semi)definite matrix has a positive (semi)definite square root that is also symmetric.

Proof. Apply Theorem C.1.11 with $k = 2$. Since the original matrix is real, part (c) of that theorem shows that its positive root is real; being Hermitian and real, it is symmetric. If the original matrix is positive definite, the root has no zero eigenvalues and is also positive definite. □

D

Einstein summation convention

This appendix specifies the summation convention used in expressions with indices.

D.1 Einstein summation convention

In a tensor expression, an index that appears exactly twice in the same monomial, once as a subscript and once as a superscript, is called a *dummy index*. Unless otherwise stated, summation over all possible values of that index is understood. Thus, if the indices range over $\{1,\dots,n\}$, we write

$$\lambda_i x^i := \sum_{i=1}^{n} \lambda_i x^i$$

and

$$g^{ij} F_i G_j := \sum_{i,j=1}^{n} g^{ij} F_i G_j.$$

Indices that appear only once are called *free indices*. In a tensor equality, the free indices must appear on both sides in the same positions. For example,

$$T^i{}_j X^j = Y^i$$

represents a family of equalities, one for each value of the free index i, and contains a sum over the dummy index j.

In each monomial we will avoid using the same index more than twice. When several sums are needed, we will use distinct dummy indices. This rule avoids ambiguities and makes the free indices of the expression immediately recognizable.

Local coordinates will usually be written as $(x^1,\dots,x^n)$. In particular,

$$\frac{\partial f}{\partial x^i}\,\mathbf{d}x^i := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,\mathbf{d}x^i.$$

Glossary

Banach cotangent bundle The continuous dual bundle of the tangent bundle of a Banach manifold, denoted by $T'M$; see Definition 25.2.2. 1429, *see* continuous dual of a Banach space and continuous dual Banach bundle

Banach–Finsler structure A continuous family of tangent norms that are uniformly equivalent in charts; see Definition 26.3.1. 1459, *see* strong Riemannian metric and Palais–Smale condition

Banach manifold A manifold whose charts take values in open subsets of Banach spaces and whose coordinate changes are Fréchet differentiable; see Definition 24.1.1. 1376, *see* Fréchet derivative and Banach submanifold

Banach submanifold A subset that can be locally straightened onto a closed complemented subspace of the model space; see Definition 24.3.1. 1388, *see* Banach manifold and regular level set

Bessel potential space The space $H^{s,p}$ defined by the multiplier $(1+\|\xi\|^2)^{\frac{s}{2}}$; see Definition 6.1.2. 440, *see* Sobolev–Slobodeckij space

Bochner Laplacian A second-order operator constructed from a connection and its formal adjoint; see Definition 8.2.10. 577, *see* formal adjoint operator and Hodge–de Rham Laplacian

bounded geometry A uniform geometric condition formulated in terms of the injectivity radius and bounds on the derivatives of curvature; see Definition 13.1.5. 867, *see* Riemannian manifold

bump function on a Banach space A nonzero differentiable function with bounded support; see Definition 24.8.1. 1405, *see* Banach manifold and smooth partition of unity on a Banach manifold

bundle metric A smooth choice of an inner product on each fiber of a vector bundle; see Definition 2.3.1. 131, *see* smooth vector bundle and compatible connection

Carathéodory function A function that is measurable in the geometric variable and continuous in the real variable; see Definition 29.1.1. 1554, *see* weak solution and energy functional

classical Finsler structure A finite-dimensional geometry defined by a Minkowski functional that is smooth and strongly convex away from the zero section; see Definition 26.4.2. 1464, *see* Banach–Finsler structure

coercivity The property of a functional whose value tends to $+\infty$ as the norm of its argument tends to $+\infty$. 1567, *see* direct method in the calculus of variations

compatible connection A connection that preserves the bundle metric under differentiation; see Definition 2.4.39. 168, *see* connection and bundle metric

complex interpolation Calderón's method using holomorphic functions on a strip; see Definition 5.7.4. 418, *see* real interpolation

configuration space A function space of geometric objects equipped with the topology or differentiable structure appropriate for formulating equations and symmetries. *see* manifold of Sobolev maps and ILH manifold

connection An operator that differentiates sections of a bundle in tangent directions and satisfies linearity and the Leibniz rule; see Definition 2.4.2. 136, *see* covariant derivative and compatible connection

continuous dual Banach bundle A Banach bundle whose fibers are the continuous duals of the fibers of the original bundle; see Proposition 25.1.14. 1427, *see* continuous dual of a Banach space and Banach cotangent bundle

continuous dual of a Banach space The space of continuous linear functionals, denoted by X'; see Proposition 25.1.2. 1420, *see* continuous dual Banach bundle and Banach cotangent bundle

covariant derivative The derivative of a section determined by a connection and a tangent direction. 136, *see* connection and weak covariant derivative

critical point A point at which the Fréchet derivative of the functional under consideration vanishes. 1560, *see* Fréchet derivative and energy functional

direct method in the calculus of variations An existence principle for minimizers based on weak compactness, coercivity, and lower semicontinuity; see Theorem 30.1.8. 1567, *see* lower semicontinuity and coercivity

distribution in a vector bundle A continuous linear operator defined on test sections of the dual bundle; see Definition 10.2.1. 756, *see* smooth vector bundle

energy functional A variational functional whose critical points represent weak solutions of the associated Euler–Lagrange equation; see Section 29.3. 1560, *see* critical point and variational method

essential spectrum The part of the spectrum of a self-adjoint operator that does not consist of isolated eigenvalues of finite multiplicity; see Definition 27.4.8. 1514, *see* bounded geometry

extended metric A distance function that may take the value $+\infty$ and whose restriction to each finite-distance component is a metric; see Definition 27.2.9. 1485, *see* uniform structure

formal adjoint operator A differential operator characterized by the integration-by-parts identity with respect to the metrics and the Riemann–Lebesgue measure; see Definition 8.2.1. *see* Bochner Laplacian and Hodge–de Rham Laplacian

Fourier transform An operator that decomposes a function or tempered distribution into frequencies; see Definitions 4.11.1 and 4.11.12. 393, *see* tempered distribution

Fréchet derivative The first-order continuous linear approximation to a function between Banach spaces; see Definition 23.1.1. 1342, *see* Banach manifold

Hilbert manifold A Banach manifold modeled on a Hilbert space; see Definition 26.1.1. 1447, *see* Banach manifold and strong Riemannian metric

Hilbert scale A chain of Hilbert spaces with continuous dense inclusions that records Sobolev regularity; see Definition 27.5.4. 1519, *see* ILH manifold and manifold of Sobolev maps

Hodge–de Rham Laplacian An operator on differential forms given by the sum of the compositions of the exterior derivative and its formal adjoint; see Definition 8.2.14. 578, *see* formal adjoint operator and Bochner Laplacian

ILH manifold The smooth inverse limit of a compatible tower of Hilbert manifolds; see Definition 27.5.6. 1519, *see* Hilbert scale

immersion between Banach manifolds A map whose differential admits a continuous linear left inverse; see Definition 24.5.1. 1393, *see* Banach manifold and Banach submanifold

Kato inequality An estimate comparing the weak derivative of the norm of a section with its covariant derivative; see Theorem 7.2.6. 522, *see* weak covariant derivative

Lagrange multiplier A scalar expressing the dependence between the derivative of a functional and the derivative of a constraint at a constrained critical point; see Corollary 23.4.2. 1599, *see* regular level set and Nehari manifold

lower semicontinuity The property of a functional expressing stability from below with respect to the topology under consideration; see Definitions 30.1.1 and 30.1.6. 1565, *see* direct method in the calculus of variations

manifold of Sobolev maps A component of Sobolev maps modeled on sections of f^*TN through exponential charts; see Theorem 27.3.4. 1498, *see* Banach manifold and Hilbert scale

Nehari manifold The set of nonzero elements defined by the condition $DI(u)(u) = 0$, used as a natural constraint for certain functionals; see Section 32.1. 1600, *see* Lagrange multiplier and energy functional

Palais–Smale condition A compactness condition for sequences with bounded energy whose derivative tends to zero; see Definitions 31.1.2 and 31.4.1. 1578, *see* pseudogradient vector field and Banach–Finsler structure

parametrix A two-sided inverse of an operator modulo smoothing operators. 1074, *see* pseudodifferential operator

Poincaré inequality An estimate for a function or section in terms of its derivatives, under the corresponding geometric or boundary conditions; see Theorems 3.3.7 and 9.6.3. 303, *see* Sobolev space

pseudodifferential operator An operator defined locally by quantizing a symbol and stable under the symbolic calculus modulo smoothing operators. 1054, *see* parametrix and wavefront set

pseudogradient vector field A locally Lipschitz vector field controlled by the norm of the derivative and producing uniform descent of the functional; see Definition 31.2.1. 1579, *see* Palais–Smale condition

real interpolation A method for constructing intermediate spaces using the K- and J-functionals; see Definition 5.2.3. 403, *see* complex interpolation

regular level set A level set of a functional whose derivative does not vanish at its points and which, in the setting developed here, has a submanifold structure; see Theorem 24.7.2. 1403, *see* Banach submanifold and Lagrange multiplier

Riemann–Lebesgue measure The canonical measure associated with a Riemannian metric, used to formulate global integrals without assuming orientability; see Definition 2.15.5. 261, *see* Riemannian manifold

Riemannian manifold A smooth manifold equipped with a Riemannian metric; see Definition 2.1.1. 126, *see* Riemannian metric and Riemann–Lebesgue measure

Riemannian metric A metric tensor field that determines the inner product on each tangent space; see Definition 2.1.1. 126, *see* Riemannian manifold

smooth partition of unity on a Banach manifold A locally finite family of nonnegative smooth functions subordinate to an open cover and with sum equal to one; see Theorem 24.8.9. 1409, *see* bump function on a Banach space and Banach manifold

smooth vector bundle A vector bundle whose projection, total space, and local trivializations have a smooth structure; see Definition 1.7.2. 50, *see* bundle metric and connection

Sobolev–Slobodeckij space A space of noninteger order defined using differences and the Gagliardo seminorm; see Definition 6.2.1. 449, *see* Bessel potential space

Sobolev space A function space that controls a function or section and its weak derivatives up to a given order; see Definitions 3.2.1 and 9.1.47. 290, *see* weak covariant derivative and trace operator

spaces BC^j and BUC^j Spaces of C^j functions whose derivatives are bounded and, in the second case, uniformly continuous; see Definition 6.4.1. 471, *see* uniformly regular manifold

split regular value A value whose fiber consists of points at which the differential admits a continuous linear right inverse; see Definition 24.6.4. 1400, *see* submersion between Banach manifolds and Banach submanifold

strong Riemannian metric A smooth metric whose tangent norm induces the Hilbert topology and whose musical morphism is an isomorphism; see Section 26.1. 1447, *see* Hilbert manifold and Banach–Finsler structure

submersion between Banach manifolds A map whose differential admits a continuous linear right inverse; see Definition 24.5.1. 1393, *see* Banach manifold and regular level set

tempered distribution A continuous linear functional on the Schwartz class; see Definition 4.9.1. 389, *see* Fourier transform

trace operator The continuous extension of restriction to the boundary for Sobolev classes of sufficient regularity; see Theorem 9.4.12. 705, *see* Sobolev space

uniform structure A filter of entourages of the diagonal that makes it possible to define uniform continuity, Cauchy filters, and completeness; see Definition 27.2.1. 1483, *see* uniformly regular manifold

uniformly regular manifold A manifold equipped with a class of normalized, shrinkable atlases of finite multiplicity, with uniformly controlled coordinate changes in the sense of Amann; see Definition 13.2.2. 896, *see* bounded geometry and uniform structure

variational method A procedure that translates a differential equation into a critical point problem for a functional; see Section 29.3. 1553, *see* energy functional and direct method in the calculus of variations

wavefront set A conic subset of the cotangent bundle with the zero section removed, identifying the points and codirections in which a distribution is not smooth. 1080, *see* pseudodifferential operator and tempered distribution

weak covariant derivative The distributional extension of the covariant derivative, defined through integration by parts; see Definition 9.1.5. 601, *see* covariant derivative and Sobolev space

weak Riemannian metric A smooth metric whose musical map need not be surjective onto the continuous dual; see Definition 26.1.1. 1447, *see* strong Riemannian metric and Hilbert manifold

weak solution An element of the function space that satisfies the integral identity obtained by testing the equation against admissible functions; see Definition 29.2.3. 1560, *see* energy functional and variational method

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